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Reconstructing braided subcategories of $SU(N)_k$

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ABSTRACT

Ocneanu rigidity implies that there are finitely many (braided) fusion categories with a given set of fusion rules. While there is no method for determining all such categories up to equivalence, there are a few cases for which one can. For example, Kazhdan and Wenzl described all fusion categories with fusion rules isomorphic to those of $SU(N)_k$. In this paper we extend their results to a statement about braided fusion categories, and obtain similar results for certain subcategories of $SU(N)_k$.

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1. Introduction

The purpose of this article is to study certain subtleties on the problem of classifying braided fusion categories with a fixed set of fusion rules. Some of our results are possibly well-known to experts, but have not been carefully written down. We assume the reader is familiar with the basic notions of the theory of fusion categories, taking [10] as a basic reference.

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In explicit classifications of braided fusion categories (e.g. [16,3]) one is often confronted with the following question: if $\mathcal{C}$ and $\mathcal{D}$ have the same fusion rules (i.e. are *Grothendieck equivalent* [17]), are they related in some explicit way? By (braided) Ocneanu rigidity [11, Theorem 2.31] there are finitely many (braided) fusion categories with the same fusion rules as a given one, but this does not provide a classification or even an enumerative bound up to equivalence. Often it is desirable to have such an enumeration, for example in categories appearing in applications to condensed matter physics and quantum computation [18].

There are two straightforward ways to construct potentially inequivalent fusion categories from a given category $\mathcal{C}$.

Firstly, it is a consequence of results of [11] that any fusion category over $\mathbb{C}$ can be defined over an algebraic extension K of $\mathbb{Q}$. In [5] it is carefully shown that the axioms of a (braided, ribbon) fusion category can be expressed as structure constants satisfying algebraic equations so that for any Galois automorphism σ one may define a (braided, ribbon) fusion category $\sigma(\mathcal{C})$ by applying σ to the structure constants. Since the fusion coefficients are rational integers, the fusion rules of $\mathcal{C}$ and $\sigma(\mathcal{C})$ are the same.

If $\mathcal{C}$ is a faithfully G-graded fusion category with associativity constraint α then for any 3-cocycle $\omega \in Z^3(G, U(1))$ we may obtain a new fusion category $\mathcal{C}^\omega$ by twisting α by ω on homogeneous components:

$$\alpha^\omega_{X,Y,Z} = \omega(\deg(X), \deg(Y), \deg(Z))\alpha_{X,Y,Z}.$$

Indeed, the pentagon axioms correspond exactly to the cocycle condition and twisting by cohomologous 3-cocycles yield monoidally equivalent categories.

In some situations these two constructions suffice to describe all categories with given fusion rules. For example, any fusion category with fusion rules like Vec_G for a finite group G is of the form Vec^ω_G. The results of Kazhdan and Wenzl [14] show that the same is true for fusion categories with the same fusion rules as the $\mathbb{Z}_N$-graded fusion categories $SU(N)_k$ obtained from quantum groups $U_q\mathfrak{sl}_N$ for $q = e^{\pi i/(N+k)}$. They show that if $\mathcal{C}$ has fusion rules like $SU(N)_k$ then $\mathcal{C}$ is a twist of the fusion category $\mathrm{Fus}(\mathcal{C}(\mathfrak{sl}_N, N+k, \tilde{q}))$ underlying the fusion category $\mathcal{C}(\mathfrak{sl}_N, N+k, \tilde{q})$ obtained from $U_{\tilde{q}}\mathfrak{sl}_N$ where $\tilde{q}$ is another root of unity of the same order as q. The results mentioned above make it clear that $q \to \tilde{q}$ can be implemented by a Galois automorphism. The approach of [14] is fairly technical and uses the relationship between the Hecke algebras $\mathcal{H}_n(q)$ and the centralizer algebras in $SU(N)_k$ in an essential way.

The categories $SU(N)_k$ admit further structure and properties: they are non-degenerate braided fusion categories. Moreover $SU(N)_k$ has a well-studied factorization into braided subcategories $\mathcal{M}SU(N)_k \boxtimes \mathcal{C}(\mathbb{Z}_m, P)$: here $\mathcal{C}(\mathbb{Z}_m, P)$ is a pointed modular category with fusion rules like $\mathbb{Z}_m$ where m is the largest factor of N coprime to k, and $\mathcal{M}SU(N)_k$ is the centralizer of $\mathcal{C}(\mathbb{Z}_m, P)$. In the case $\gcd(N, k) = 1$ the category $\mathcal{M}SU(N)_k$ is often denoted $PSU(N)_k$, see e.g. [2].

The motivating question for this paper is:

Question 1.1. Can we classify braided fusion categories with the same fusion rules as $SU(N)_k$ or $\mathcal{M}SU(N)_k$ up to braided equivalence?

We shall be particularly interested in describing all *non-degenerate* braidings on such categories.

For a fixed braided fusion category $\mathcal{C}$ one can also change the braiding in a number of ways. Firstly, one may always reverse the braiding to obtain a (potentially) new braided fusion category $\mathcal{C}^{rev}$ with the same underlying fusion category: the braiding on $\mathcal{C}^{rev}$ is defined to be $\tilde{c}_{X,Y} := (c_{Y,X})^{-1}$ where the braiding on $\mathcal{C}$ is given by c. If $\mathcal{C}$ is a braided faithfully G-graded fusion category and $\chi : G \times G \to U(1)$ is a bicharacter then we can equip the underlying fusion category $\mathrm{Fus}(\mathcal{C})$ with a (potentially) new braiding by defining on homogeneous objects X, Y

$$c^{\chi}_{X,Y} = \chi(\deg(X), \deg(Y))c_{X,Y}.$$

The proof that this is valid is essentially by inspecting the hexagon equations, and goes back to Joyal and Street at least in some cases [13]. Finally, for any tensor autoequivalence ϕ of $\mathcal{C}$, the image of the braiding on $\mathcal{C}$ under ϕ is a braiding on the underlying fusion category of $\mathcal{C}$. In particular if ϕ is not braided (recall [10] that being braided is a property autoequivalences may or may not have) then we obtain a (potentially new) braiding on the fusion category $\mathrm{Fus}(\mathcal{C})$.

It is natural to ponder the possibility of first twisting the associativity on a G-graded braided fusion category $\mathcal{C}$ and then changing the braiding correspondingly (first done in [13]). This leads to the notion of *abelian* 3*-cocycles* (ω, χ) (see [10, Exercise 8.4.3]), where ω is a 3-cocycle as above, and $\chi : G \times G \to U(1)$ is a function (not necessarily a bicharacter, unless ω is trivial). The pentagon and hexagon axioms constrain ω and χ significantly–for example, if $|G|$ is odd then the only abelian 3-cocycles have ω trivial. This can be regarded as a special case of (braided) *zesting* introduced in [6].

For completeness, we mention that there is an additional structure on $SU(N)_k$: they are spherical (non-degenerate) braided fusion categories with canonical spherical structure coming from the standard ribbon twist. This structure may also be changed–for a non-degenerate braided fusion category that admits a spherical structure other spherical structures are in one-to-one correspondence with self-dual invertible objects [3]. The same is true for the non-degenerate subcategories $\mathcal{M}SU(N)_k$.

We fully answer the question above. In particular, we describe all *braided* fusion categories with the fusion rules like those of the modular categories $SU(N)_k$ and of the modular subcategories $\mathcal{M}SU(N)_k$.

We summarize our main results: we denote a pointed braided fusion category associated with the metric group (A, Q) by $\mathcal{C}(A, Q)$.

Theorem.

(1) Let $\mathcal{C}$ be a braided fusion category with the same fusion rules as $SU(N)_k$. Then there exist exactly $2N$ different braidings over the underlying fusion category $\mathrm{Fus}(\mathcal{C})$.
(2) $SU(N)_k \cong \mathcal{C}(\mathbb{Z}_m, Q) \boxtimes \mathcal{M}SU(N)_k$ as braided fusion categories where m is the maximal divisor of N coprime to k.
(3) Let $\mathcal{D}$ be a braided fusion category with the same fusion rules as $MSU(N)_k$. There are at most two different braid structures on $\mathcal{D}$ up to the object relabeling and braid reversing (for details, see Theorem 5.9).

The structure of this paper is as follows. In section 2, we lay out the basic definitions and general results about braided tensor categories. We also fix the notations that will be used in the sequel sections. In section 3, we introduce basic properties of the categories $\mathcal{C}(\mathfrak{sl}_N, N+k, q)$. In section 4, we classify all possible braidings over $\mathcal{C}(\mathfrak{sl}_N, N+k, q)$. In section 5, we classify all possible braidings over categories obtained from $\mathcal{C}(\mathfrak{sl}_N, N+k, q)$ by twisting the associativity constraints, we also classify all possible braidings over fusion categories with the same fusion rules as certain subcategories of $SU(N)_k$.

After completing this paper we were made aware of [4, Section 23], which includes some overlap with our results, and we thank C. Pinzari for bringing this to our attention.

2. Preliminaries

In this section, we fix notations coming from the general theory of braided fusion categories. We refer readers to [10] for more details.

A fusion category over $\mathbb{C}$ is a $\mathbb{C}$-linear semisimple rigid monoidal category with finitely many isomorphism classes of simple objects and finite-dimensional Hom-spaces. We denote by $\mathcal{O}(\mathcal{C})$ the set of all isomorphism classes of simple objects of $\mathcal{C}$. An object X is said to be pointed if the evaluation morphism $X^* \otimes X \to \mathbf{1}$ is an isomorphism. A fusion category is said to be pointed if all simple objects are pointed. Given a fusion category $\mathcal{C}$, the fusion subcategory generated by pointed objects is a pointed fusion category.

We denote the pointed fusion category of all $\mathbb{C}$-vector spaces by **Vec**.

Let G be a finite group. A monoidal category $\mathcal{C}$ is G-graded if $\mathcal{C} \cong \bigoplus_{g\in G} \mathcal{C}^g$ as abelian categories and $\mathcal{C}^g \otimes \mathcal{C}^h \subset \mathcal{C}^{gh}$. In this case there is a function $\deg : \mathcal{O}(\mathcal{C}) \to G$ given by $\deg(X) = g$ if $X \in \mathcal{C}^g$. In particular, if an object Z is a subobject of the tensor product of simple objects $X \otimes Y$ then $\deg(Z) = \deg(X)\deg(Y)$. We say the grading is faithful if deg is surjective. Notice that the grading only depends on the fusion rules.

Example 2.1. A pointed fusion category is automatically G-graded, where G is the group of isomorphism classes of simple objects with product induced by $\otimes$. The category of G-graded vector spaces is a pointed fusion category which we denote by $\mathbf{Vec}_G$. The related pointed fusion category $\mathbf{Vec}_G^\omega$ is obtained by twisting the associativity on $\mathbf{Vec}_G$ by a

3-cocycle ω. Pointed braided fusion categories are classified by pre-metric groups and the underlying group is abelian, see [10].

Let $\mathcal{C}$ be a braided fusion category with braiding $c_{X,Y}$. We say two objects X and Y centralize each other if $c_{Y,X} \circ c_{X,Y} = \mathbf{id}_{X\otimes Y}$, and projectively centralize each other if

$$c_{Y,X} \circ c_{X,Y} = b_{X,Y}\mathbf{id}_{X\otimes Y}$$

for some scalar $b_{X,Y}$. Moreover, pointed objects always projectively centralize simple objects.

3. Universal grading and decomposition of $SU(N)_k$

We briefly describe some of the relevant notation for the categories $Rep(SL(N))$ of complex $SL(N)$ representations and $SU(N)_k$ the modular fusion category associated with quantum groups of Lie type A_{N-1} the specific root of unity $q = e^{\pi i/(N+k)}$. For more complete details we refer to [1,14,20].

3.1. Combinatorial data

The monoidal category of complex $SL(N)$-representations is semisimple: the isomorphism classes of simple objects are parametrized by the set Λ_N of Young diagrams λ with at most $N-1$ rows. These are either written row-wise as $(m_1, \dots, m_{N-1})$ where the weakly decreasing m_i represent the number of boxes in the ith row, or as $[\lambda_1, \dots, \lambda_j]$ where λ_i represents the number of boxes in the ith column, with λ_j the last non-empty column, unless $\lambda_1 = 0$. For instance $X_{[0]} = X_{(0,\dots,0)}$ denotes the unit object corresponding to the trivial representation. The object labeled by a single box $X_{[1]} = X_{\square}$ is the generating object which corresponds to the N-dimensional fundamental representation. In the rest of the paper, we will use X to denote the generating object $X_{[1]}$ for simplicity.

The fusion rules of $Rep(SL(N))$ satisfy

$$X \otimes X_\mu \cong \bigoplus_{\lambda=\mu+\square} X_\lambda,$$

where $\lambda = \mu+\square$ indicates that λ is obtained from μ by adding one box to any row/column of μ, with the convention that if μ has $N-1$ rows, then instead of adding one box to the first column one deletes the first column. For example the object $X_{[1^{N-1}]}$ labeled by a column of $N-1$ boxes is the dual object to X, since $X_{[0]} \subset X \otimes X_{[1^{N-1}]}$. The Grothendieck semi-ring of this fusion category is a based $\mathbb{Z}_+$-ring with basis parametrized by Young diagrams with at most $N-1$ rows, and the product is the obvious one coming from the tensor product.

3.2. Fusion categories

The braided fusion categories $SU(N)_k$ are obtained as a subquotients of the categories $Rep(U_q\mathfrak{sl}_N)$ with $q = e^{\pi i/(N+k)}$, see e.g., [1] for details. The fusion rules of $SU(N)_k$ are truncated versions of those presented above for representations of $SL(N)$. To be precise, the fusion ring of $SU(N)_k$ is the quotient of the fusion ring of $SU(N)$, with the kernel generated by objects corresponding to those Young diagrams with more than k columns.

3.3. Universal grading

The universal grading group of a fusion category $\mathcal{C}$ depends only on the fusion rules and is isomorphic to the group $Aut_\otimes(id_\mathcal{C})$ of tensor automorphisms of the identity functor [10, Proposition 4.14.3]. If $\mathcal{C}$, or another category with the same fusion rules as $\mathcal{C}$, admits the structure of a non-degenerate braided fusion category then [10, Lemma 8.22.9(iii)] implies that the universal grading group is isomorphic to the group of invertible objects in $\mathcal{C}$. This implies that the universal grading group on the fusion categories $SU(N)_k$ is isomorphic to $\mathbb{Z}_N$. This can be described explicitly by counting the number of boxes of the Young diagrams mod N. For instance, the generating object X is of grade 1 and the trivial object $\mathbf{1} = X_{[0]}$ is of grade 0. Indeed, the trivial object $\mathbf{1}$ appears in $X^{\otimes N}$ and N is the smallest positive t so that $\mathbf{1} \subset X^{\otimes t}$. Now since every object in $SU(N)_k$ appears in some tensor power of X this shows that there can be no larger grading group. It can also be shown that the universal grading group of $Rep(SU(N))$ is $\mathbb{Z}_N$, by considering $SU(N)_k$ and sending k to infinity (we do not need this fact, but it is interesting nonetheless).

There are (braided) fusion subcategories coming from the universal grading. Suppose H is a subgroup of $\mathbb{Z}_N$, then $\bigoplus_{h\in H}(SU(N)_k)^h$ is a (braided) fusion subcategory.

3.4. Pointed subcategory

There are exactly N invertible objects in $SU(N)_k$. The corresponding Young diagrams are those of rectangular shape $i \times k$, where $0 \le i \le N-1$. If $\mathcal{C}$ is a braided fusion category with the same fusion rules as $SU(N)_k$ the braided monoidal structure restricts to the pointed subcategory $\mathcal{P}(\mathcal{C})$ of $\mathcal{C}$. The braiding and monoidal structures of pointed categories are completely classified using quadratic forms and we denote them by $\mathcal{C}(A, Q)$ where A is an abelian group and Q is a quadratic form, see [10, Section 8.4]. In the present case the corresponding abelian group is $\mathbb{Z}_N$. Since $\mathcal{C}$ is braided, $\mathcal{P}(\mathcal{C})$ must be monoidally equivalent to $\mathbf{Vec}^\omega_{\mathbb{Z}_N}$, where

(1) for N even either

$$\omega(i,j,\ell) = \begin{cases} 1 & i+j < N \\ (-1)^\ell & i+j \ge N \end{cases}$$

or $[\omega]$ is cohomologically trivial, and
(2) $[\omega]$ is cohomologically trivial if N is odd.

A direct computation by solving the hexagon equations is found in the appendix. Similar computations appear in constructing new monoidal and braid structures from old ones (see Section 5).

In $SU(N)_k$. We denote the pointed simple object $X_{(k)}$ by g. Notice that g appears as a subobject of $X^{\otimes k}$, so the grading of g is $k \pmod N$, of course).

3.5. Decomposition formula

One can derive fusion subcategories of $SU(N)_k$ from both the grading and the pointed objects. However, not all of them split as a direct (Deligne) product of braided fusion categories, or even as fusion categories. The following proposition shows that any braided fusion category $\mathcal{C}$ with the same fusion rules as $SU(N)_k$ does have such a decomposition, which is maximal in a certain sense. For such a $\mathcal{C}$, let m be the largest divisor of N that is relatively prime to k, and set $n = N/m$. Some authors (e.g., [8]), define $n = \gcd(N, k^\infty) := \lim_{i\to\infty} \gcd(N, k^i)$. We denote by $\mathcal{MC} = \bigoplus_{i=0}^{n-1} \mathcal{C}^{im}$ the fusion subcategory of $\mathcal{C}$ generated by the mj-graded components (i.e. corresponding to the subgroup $m\mathbb{Z}_N < \mathbb{Z}_N$), and by $\mathcal{C}(\mathbb{Z}_m, P)$ the pointed subcategory generated by g^n. $\mathcal{C}(\mathbb{Z}_m, P)$ has rank m since $ni < N$ for $i < m$. Notice that since g lies in the k-graded component, g^n lies in the $nk \pmod N$ component. The intersection of the two fusion subcategories is trivial since n and m are relatively prime.

Proposition 3.1. *A braided monoidal category $\mathcal{C}$ with the same fusion rules as $SU(N)_k$ admits a braided tensor decomposition (in the notation above):*

$$\mathcal{C} = \mathcal{MC} \boxtimes \mathcal{C}(\mathbb{Z}_m, P). \tag{3.1}$$

Proof. It is clear by the construction above that $\mathcal{MC}$ and $\mathcal{C}(\mathbb{Z}_m, P)$ are fusion subcategories. Since $\mathcal{C}$ is braided, one has a monoidal functor

$$F : \mathcal{MC} \boxtimes \mathcal{C}(\mathbb{Z}_m, P) \to \mathcal{C}$$

given by the tensor product. We will verify F is an equivalence of fusion categories by showing that F induces a bijection on the set of simple objects.

By the Chinese remainder theorem, F induces a group isomorphism $f : \mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{Z}_N$ on the universal grading groups of $\mathcal{MC}, \mathcal{C}(\mathbb{Z}_m, P)$ and $\mathcal{C}$. Suppose $F(U \boxtimes J) \cong 1$, then U and J both must lie in grade 0, so $J \cong 1$ and $U \cong 1$ by the injectivity of f. This proves F is an injection. For the surjection, let V be a simple object in $\mathcal{C}$ of grade ℓ, and $f(i, j) = \ell$. In other words,

$$im + jn = \ell \pmod N.$$

Then $V = (V \otimes g^{-jn}) \otimes g^{jn}$ where $V \otimes g^{-jn}$ is of grade im, and thus an object in $\mathcal{MC}$, and g^{jn} is an object in $\mathcal{C}(\mathbb{Z}_m, P)$.

To show F is an equivalence of *braided* tensor categories, we need to show the pointed factor $\mathcal{C}(\mathbb{Z}_m, P)$ centralizes $\mathcal{MC}$.

Let X and g be the simple objects defined above. Since the pointed objects centralize all simple objects projectively, the quantity b_{Y,g^i} characterizes the braiding of a pointed object g^i and a simple object Y, where b_{Y,g^i} is defined via

$$c_{g^i,Y} \circ c_{Y,g^i} = b_{Y,g^i}\mathbf{id}_{Y\otimes g^i}.$$

Note that the value of b_{Y,g^i} only depends on the grade of Y, not the object itself [10, Lemma 8.22.9]. By the compatibility with the tensor product, the pointed objects projectively centralize $X^{\otimes j}$ for all j, with $b_{X^{\otimes j},g^i} = b_{X,g}^{ij}$. Since X is a generating object, the quantity b_{Y,g^i} is determined by $b_{X,g}$ for all simple objects Y. To be specific, suppose $b_{X,g} = t$ and $Y \in \mathcal{C}^j$, then $b_{Y,g^i} = t^{ij}$.

Since the identity object $\mathbf{1}$ centralizes all object in $\mathcal{C}$, we have

$$b_{X,\mathbf{1}} = b_{X,g^N} = t^N = 1.$$

Now we prove $\mathcal{C}(\mathbb{Z}_m, P)$ centralizes all objects in $\mathcal{MC}$. The pointed subcategory $\mathcal{C}(\mathbb{Z}_m, P)$ is generated by g^n. Let Y be a simple object in $\mathcal{C}^{im}$. Then

$$b_{Y,g^n} = t^{(im)(n)} = t^{Ni} = 1.$$

Thus the generating object g^n centralizes all objects in $M\mathcal{C}$, therefore the same holds all other objects in $\mathcal{C}(\mathbb{Z}_m, P)$. □

With the notation established above we immediately have the following:

Corollary 3.2. *The braiding over $\mathcal{C}$ is uniquely determined by a braiding over $\mathcal{MC}$ and a braiding over $\mathcal{C}(\mathbb{Z}_m, P)$.*

The following is of independent interest:

Lemma 3.3. *Let $\mathcal{C}$ be a braided fusion category with the same fusion rules as $SU(N)_k$ and let m be chosen to be maximal among divisors of N coprime to k. Then m is maximal among all divisors t of N such that $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_t, Q) \boxtimes \mathcal{F}$.*

Proof. Suppose that t is chosen to be maximal such that there exists a braided tensor decomposition $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_t, Q)\boxtimes\mathcal{D}$. Without loss of generality we may assume that $\mathcal{C}(\mathbb{Z}_t, Q)$ and $\mathcal{D}$ are subcategories of $\mathcal{C}$. It is enough to show that $\gcd(t, k) = 1$.

First note that $\mathcal{D}$ is faithfully $\mathbb{Z}_{N/t}$-graded since $\mathcal{C}(\mathbb{Z}_t, Q)$ is faithfully $\mathbb{Z}_t$-graded and $\mathcal{C}$ is $\mathbb{Z}_N$-graded. Indeed, we have an isomorphism of grading groups $\mathbb{Z}_t \times \mathbb{Z}_{N/t} \cong \mathbb{Z}_N$

Table 1
Simple current autoequivalences of $SU(N)_k$.

$SU(N)_k$	ScEq	Braided ScEq
$k=2, N=2$	$\{e\}$	$\{e\}$
2 exactly divides $\gcd(N,k)$	$\mathbb{Z}_m^* \times \mathbb{Z}_2 \times \mathbb{Z}_{\frac{n}{2}}$	$\mathbb{Z}_2^{p+t}$
otherwise	$\mathbb{Z}_m^* \times \mathbb{Z}_n$	$\mathbb{Z}_2^{p+t}$

given by $(i,j) \to \frac{N}{t}i + tj \pmod{N}$. In particular $\langle \mathbf{1} \rangle \boxtimes \mathcal{D}_0 \cong \mathcal{C}_0$ since the preimage of $0 \in \mathbb{Z}_N$ is $(0,0)$. In particular, $\mathcal{C}_0 \subset \mathcal{D}$. Notice also that $\mathcal{C}(\mathbb{Z}_t, Q)$ is generated by $a = g^{\frac{N}{t}}$, since all pointed objects of order t lie in $\mathcal{C}(\mathbb{Z}_t, Q)$. In particular, by the decomposition above we must have $\mathcal{C}(\mathbb{Z}_t, Q) \cap \mathcal{C}_0 = \mathbf{Vec} = \langle \mathbf{1} \rangle$.

Now suppose that $p \mid \gcd(t,k)$. Consider $a^{\frac{t}{p}} = g^{\frac{Nt}{pt}} = g^{\frac{N}{p}}$. Now g lies in $\mathcal{C}_k$ so that $a^{\frac{t}{p}}$ lies in grade $\frac{Nk}{p} \pmod{N} \equiv 0 \pmod{N}$. Since the only tensor power of a in $\mathcal{C}_0$ is $\mathbf{1}$ and a has order t, we conclude that $p = 1$. □

Notation 3.4. In the rest of the paper, we will denote by $\mathcal{MC}$ the non-pointed factor in the decomposition of a braided fusion category with the same fusion rules as $SU(N)_k$ and $\mathcal{C}(\mathbb{Z}_m, P)$ to denote the (maximal) pointed factor for convenience.

3.6. Autoequivalences of $SU(N)_k$

The (braided) monoidal autoequivalences of the category $SU(N)_k$ are classified by Edie-Michell [9,8] and Gannon [12]. Gannon first classified all the automorphisms of the fusion rings of $SU(N)_k$. They are generated by two types of automorphisms, namely:

(1) *charge conjugation* that interchanges the classes $[X]$ and $[X^*]$
(2) *simple current automorphisms* that sends the generating object class $[X]$ to $[X \otimes g^a]$ for any a such that $1 + ka$ is coprime to N.

Then Edie-Michell [9] showed that all such fusion ring autoequivalences can be realized uniquely as a monoidal equivalence of $SU(N)_k$, and determined when they are braided. In particular, charge conjugation always induces a braided monoidal equivalence of $SU(N)_k$. The simple current autoequivalences, denoted ScEq, may change the braiding or not. In particular, if we apply an autoequivalence that is not braided we obtain a new braiding on our category. The group of (braided) simple current autoequivalences is given in Table 1, in which m is the largest factor of N coprime to k and $n = N/m$, p is the number of distinct odd primes that divide N but not k, and

$$t = \begin{cases} 0, & N \text{ is odd}, \\ 0 & N \text{ is even and } k \equiv 0 \pmod{4}, \text{ or if } k \text{ is odd and } N \equiv 2 \pmod{4}, \\ 1, & \text{otherwise}. \end{cases}$$

Part of the results in [8] applies to any braided fusion category $\mathcal{C}$ with the same fusion rules as $SU(N)_k$, namely, any simple current automorphism of the fusion ring lifts to an autoequivalence of $\mathcal{C}$. We do not know if it lifts uniquely: the issue is that we do not know that the trivial simple current automorphism is only realized by the identity autoequivalence: there could be non-trivial gauge autoequivalences. On the other hand, we show that all such autoequivalences are braided, see 5.5, they can be ignored from the perspective of counting distinct braidings. We end this section with the following.

Lemma 3.5. *Suppose $\mathcal{C}$ is a braided fusion category with the same fusion rules as $SU(N)_k$, then the ring autoequivalence*

$$\begin{aligned} f : \mathcal{K}(\mathcal{C}) &\to \mathcal{K}(\mathcal{C}) \\ [X] &\mapsto [X \otimes g^a] \end{aligned}$$

lifts to a monoidal autoequivalence.

Proof. Since $\mathcal{C}$ admits a braiding, the braiding equips $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ with the structure of a monoidal functor. Since $[X] \mapsto [X \otimes g^a]$ is a ring automorphism, we get a monoidal equivalence of the subcategory of $\mathcal{C} \boxtimes \mathcal{C}$ tensor generated by $X \otimes g^a \boxtimes g^{-a}$ and $\mathcal{C}$ by restricting $\otimes$ to the subcategory. Taking the inverse, we get a section functor $s : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ sending X to $X \otimes g^a \boxtimes g^{-a}$.

Then we define the lifting of f to be the composition of the following functors:

$$\mathcal{C} \xrightarrow{s} \mathcal{C} \boxtimes \langle g^a \rangle \xrightarrow{\text{Forget}} \mathcal{C} \boxtimes \mathbf{Vec} \xrightarrow{p_1} \mathcal{C}.$$

The first and third functors are always monoidal. Since we have $ka+1$ coprime to N, the monoidal struture on the pointed category generated by $\langle g^a \rangle$ has the trivial associator, thus the forgetful functor in the middle is also monoidal. Therefore, their composition is a monoidal equivalence sending X to $X \otimes g^a$. □

4. Main results

In this section, we let $\text{Fus}(\mathcal{C}(\mathfrak{sl}_N, k+N, q))$ denote the monoidal category underlying $\mathcal{C}(\mathfrak{sl}_N, k+N, q)$ equipped with the standard (untwisted) associativity constraints. We only discuss the cases with $k \geq 2$, so the category is not pointed. Since $\text{Fus}(\mathcal{C}(\mathfrak{sl}_N, k+N, q))$ admits a non-degenerate braiding (Galois conjugation does not change the degeneracy of the S-matrix), the results of [10] and [15] on classifying braidings over fusion categories can be applied.

Recall that if $\mathcal{C}$ is a non-degenerate braided fusion category then the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is braided equivalent to $\mathcal{C} \boxtimes \mathcal{C}^{rev}$, where $\mathcal{C}^{rev}$ denotes the braided fusion category $\mathcal{C}$ equipped with the reversed braiding $c^{rev}_{X,Y} := c^{-1}_{Y,X}$. The forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is identified as tensor functor $\otimes$ (see Proposition 3.7 of [7]). It is well-known that braidings over the underlying fusion category $\mathcal{C}$ are in bijection with sections of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Here sections are tensor functors s such that $F \circ s \cong Id_{\mathcal{C}}$. Since $\mathcal{C}$ is non-degenerate we have:

$$F : \mathcal{C} \boxtimes \mathcal{C}^{rev} \to \mathcal{C}$$
$$X \boxtimes Y \mapsto X \otimes Y,$$

where the associated braiding is defined as

$$\tilde{c}_{X_1 \boxtimes Y_1, X_2 \boxtimes Y_2} = c_{X_1,X_2} \boxtimes c^{-1}_{Y_2,Y_1}.$$

With the above identification, we can classify all braidings over $\mathrm{Fus}(SU(N)_k)$ when $k \geq 2$. We remark that the classification of braidings over $\mathrm{Fus}(SU(N)_1)$ is well-known as they are all pointed categories with the same fusion rules as $\mathbf{Vec}_{\mathbb{Z}_N}$, see [10] and the Appendix for details.

Theorem 4.1. *For $k \geq 2$, there are exactly $2N$ different braid structures over fusion category $\mathrm{Fus}(\mathcal{C}(\mathfrak{sl}_N, k+N, q))$. In addition, $\mathrm{Fus}(\mathcal{C}(\mathfrak{sl}_N, k+N, q))$ admits a degenerate braiding if and only if N has an odd prime factor which is relatively prime to k.*

Proof. Let $\mathcal{C} = \mathrm{Fus}(\mathcal{C}(\mathfrak{sl}_N, k+N, q))$ equipped with the nondegenerate braiding coming from the Galois conjugation of $SU(N)_k$. It suffices to classify all sections of $\mathcal{C}$ in its center $\mathcal{C} \boxtimes \mathcal{C}^{rev}$.

Suppose $s : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}^{rev}$ is a section. Since $\mathcal{C}$ is tensor generated by X, the section s is uniquely determined by its image $s(X)$ in $SU(N)_k \boxtimes SU(N)_k^{rev}$. Since X is simple, $s(X)$ is also simple in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$, say $s(X) = Y \boxtimes Z$ where Y and Z are simple objects. Moreover, $s(X^*) \cong s(X)^*$ since s is monoidal. In $SU(N)_k$, $X \otimes X^* = \mathbf{1} \oplus B$ where B is a simple object. Thus $s(X) \otimes s(X^*) \cong \mathbf{1} \boxtimes \mathbf{1} \oplus s(B)$ with $s(B)$ simple since s is monoidal. Thus $(Y \otimes Y^*) \boxtimes (Z \otimes Z^*) \cong \mathbf{1} \boxtimes \mathbf{1} \oplus s(B)$, and since $s(B)$ is simple, one of $(Z \otimes Z^*)$ or $(Y \otimes Y^*)$ is simple since otherwise we would have at least 4 simple summands. Now they each contain $\mathbf{1}$ and therefore one of Y or Z has to be pointed, i.e. of the form g^r for some r. In order to make s a section on the level of fusion rings, $s(X)$ is either of the form $X \otimes g^i \boxtimes g^{-i}$ or the opposite $g^{-i} \boxtimes X \otimes g^i$ where $0 \leq i \leq N-1$. We hence obtain in total $2N$ choices of sections on the level of fusion rings.

In order to prove the first part of the theorem, it remains to show all these assignments lift to monoidal functors. This can be seen in a similar way as in the proof of Lemma 3.5. We restrict the $\otimes$ functor to the subcategory of $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ generated by $s(X)$. The functor is a monoidal equivalence and by taking the inverse, we lift s to a monoidal functor.

Next we check the symmetric center of the corresponding braiding. We only consider the case when $s(X) = X \otimes g^i \boxtimes g^{-i}$, the other case is identical. Since the symmetric center of the induced braiding remains pointed (see Corollary 4.5 of [15]), it suffices to compute $\tilde{b}_{s(X),s(g)}$, where $\tilde{b}_{Y,h}$ is defined such that

$$\tilde{b}_{Y,h} \mathbf{id}_{Y \otimes h} = \tilde{c}_{g,Y} \circ \tilde{c}_{Y,g}$$

in $\mathcal{C} \boxtimes \mathcal{C}^{rev}$.

Let $b_{X,g} = t$ in $\mathcal{C}$. Since the symmetric center of $\mathcal{C}$ is trivial, t is a primitive N-th root of unity. Otherwise, there exists $g^i \neq \mathbf{1}$ that centralizes X, and consequently, centralizes the whole category. Notice g appears in $X^{\otimes k}$, then $s(g) = g \otimes g^{ik} \boxtimes g^{-ik}$. Thus we have

$$\begin{aligned}\tilde{b}_{s(X),s(g)} &= \tilde{b}_{X\otimes g^i \boxtimes g^{-i}, g^{ik+1}\boxtimes g^{-ik}} \\ &= b_{X\otimes g^i, g^{ik+1}} b^{-1}_{g^{-i}, g^{-ik}} \\ &= b_{X,g^{ik+1}} b_{g^i, g^{ik+1}} b^{-1}_{g^{-i}, g^{-ik}} \\ &= t^{ik+1} t^{ik(ik+1)} t^{-(-ik)(-ik)} \\ &= t^{2ik+1}\end{aligned} \tag{4.1}$$

Since t is a primitive N-th root of unity, g^j is in the symmetric center if and only if

$$(2ik+1)j \equiv 0 \mod N. \tag{4.2}$$

Now we are ready to prove the second part of Theorem 4.1. Notice that $2ik + 1$ is relatively prime to 2 and all the common prime factors of N and k. This proves the 'only if' part. To prove the 'if' part, we construct a degenerate braiding over the underlying fusion category. Let m be the maximal *odd* divisor of N that relatively prime to k as in Section 3.5. According to Proposition 3.1, $\mathcal{C}$ has a braided factor $\mathcal{C}(\mathbb{Z}_m, P)$ with trivial associativity constraint. In particular this factor admits a symmetric braiding. By choosing this symmetric braiding over $\mathrm{Fus}(\mathcal{C}(\mathbb{Z}_m, P))$, we construct a degenerate braiding over $\mathcal{C}$. We hence finish the proof of Theorem 4.1. □

5. Generalizations

5.1. Group cohomology

In [14], Kazhdan and Wenzl classified all monoidal structures over categories with the same fusion rules as $SU(N)_k$. Different monoidal structures can be obtained by twisting the associators by (a cocycle representative of) a class in the third cohomology group of $\mathbb{Z}_N$ with coefficients in $U(1)$ and/or changing the choice of a primitive root of unity.

It is well-known that the monoidal structures over the category of G-graded vector spaces are in one-to-one correspondence with the classes in $H^3(G, U(1))$. We briefly recall the construction, see [10] for details.

Let $\mathbf{Vec}_G$ be the skeletal category with associativity constraints equal to the identity morphisms. Suppose ω is a 3-cocycle representing $[\omega] \in H^3(G, U(1))$. We denote by $\mathbf{Vec}_G^\omega$ the category with the same fusion rules as $\mathbf{Vec}_G$, with associativity constraints replaced by $a_{g_1,g_2,g_3} = \omega(g_1, g_2, g_3)\mathbf{id}$. It is easy to check that the pentagon axiom is equivalent to the condition that ω is a cocycle

$$\omega(g_1g_2, g_3, g_4)\omega(g_1, g_2, g_3g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2g_3, g_4)\omega(g_2, g_3, g_4).$$

The category $\mathbf{Vec}_G^\omega$ is also called G-graded vector spaces twisted by ω. The construction above applies to all G-graded categories in a straightforward way, one replaces the associativity constraint $a_{X,Y,Z}$ by $\omega(g_X, g_Y, g_Z)a_{X,Y,Z}$ if X, Y, Z belong to grade g_X, g_Y, g_Z respectively. The theorem below by Kazhdan and Wenzl says that all monoidal categories with fusion rules the same as $SU(N)_k$ are obtained from $\text{Fus}(SU(N)_k)$ by such cocycle twists and/or by changing the quantum parameter q.

Theorem 5.1 *([14]). Let $\mathcal{C}$ be a fusion $\mathbb{C}$-category such that the fusion ring is isomorphic to the fusion ring of $SU(N)_k$. Then $\mathcal{C}$ is monoidally equivalent to $\mathcal{C}(\mathfrak{sl}_N, k+N, q)^\omega$ for some primitive root of unity q of order $2(k+N)$ uniquely determined up to $q \to q^{-1}$, and some 3-cocycle $\omega \in Z^3(\mathbb{Z}_N, U(1))$.*

To characterize such twists, we give explicit representatives of elements in $H^3(\mathbb{Z}_N, U(1))$.

Proposition 5.2. *Let $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$. The third cohomology group $H^3(\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$, with elements represented by cocycles*

$$\omega_\eta(i,j,k) = \begin{cases} 1 & i+j < N \\ \eta^k & i+j \geq N \end{cases}$$

where η is an N-th root of unity.

The twisted fusion categories admit a braiding if and only if $\eta = \pm 1$ (see the erratum of [19]). In particular, η can take the value -1 only if N is an even number. In this section, we will classify all braidings over the underlying fusion category obtain by twisting the $\mathcal{C}(\mathfrak{sl}_N, k+N, q)$ by -1.

Before we state our classification theorem, we first fix some notations. Let us denote the twisted category by $\mathcal{C}(\mathfrak{sl}_N, k+N, q)^-$, and equip it with a braiding as follows. Let $\mathcal{C}(\mathfrak{sl}_N, k+N, q)$ be a category with untwisted associative constraint $a_{-,-,-}$ and braiding $c_{-,-}$. Then one can get a monoidal category that is equivalent to $\mathcal{C}(\mathfrak{sl}_N, k+N, q)^-$ by replacing $a_{W,Y,Z}$ by $a'_{W,Y,Z} = \omega_{-1}(g_W, g_Y, g_Z)a_{W,Y,Z}$. We further replace the braiding morphism by $c'_{Y,Z} = s^{g_Y+g_Z}c(Y,Z)$. It is easy to verify a' and c' satisfies the hexagon equations if $s^N = -1$ (see Appendix). Suppose $b_{X,g} = t$ in $\mathcal{C}(\mathfrak{sl}_N, k+N, q)$ and $N = 2^p r$ where r is odd. We choose $s = \sqrt{t^r}$ for the braiding of $\mathcal{C}(\mathfrak{sl}_N, k+N, q)^-$.

5.2. Braidings over $Fus(\mathcal{C}(\mathfrak{sl}_N, k+N, q)^-)$

Theorem 5.3. *For $k \geq 2$,*

(1) If k is odd, there are exactly $2N$ different braid structures over $Fus(\mathcal{C}(\mathfrak{sl}_N, k+N, q)^-)$, all of which are degenerate.

(2) If k is even, there are exactly $2N$ different braid structures over $Fus(\mathcal{C}(\mathfrak{sl}_N, k + N, q)^-)$. Moreover, the category admits a non-degenerate braiding.

Proof. Let $\mathcal{C} = \mathcal{C}(\mathfrak{sl}_N, k+N, q)$ as in (4.1) and let $\mathcal{C}^-$ denote the braided fusion category $\mathcal{C}(\mathfrak{sl}_N, k + N, q)^-$ constructed as above.

Case 1: k is odd.

Since $\mathcal{C}$ and $\mathcal{C}^-$ are both braided, we write the decomposition of $\mathcal{C}$ and $\mathcal{C}^-$ as in Proposition 3.1

$$\mathcal{C} = \mathcal{MC} \boxtimes \mathcal{C}(\mathbb{Z}_m, P)$$
$$\mathcal{C}^- = \mathcal{MC}^- \boxtimes \mathcal{C}(\mathbb{Z}_m, P^-).$$

Notice that the grading of objects in $\mathcal{MC}^-$ are all multiples of 2^p, thus $\mathcal{MC}^-$ is braided equivalent to $\mathcal{MC}$ and hence nondegenerate. On the other hand, the total number of braidings over $\mathcal{C}$ is $2N = 2mn$, while the number of braidings over $\mathcal{C}(\mathbb{Z}_m, P)$ is m (see Appendix). Thus there are $2n$ different braidings over Fus($\mathcal{MC}$).

The second factor is equivalent as a fusion category to the category of $\mathbb{Z}_m$-graded vector spaces with trivial associativity constraint. The braidings on it are uniquely determined by $c_{1,1} = \xi$, where ξ is a m-th root of unity. Thus we have m different braidings over the second factor. Notice that among all braidings, $b_{i,m/2} = (\xi^{im/2})^2 = \xi^{im} = 1$, so the object labeled by $[2^{r-1}]$ is in the symmetric center (recall that r is the maximal odd factor of N).

According to Corollary 3.2, a braiding over Fus($\mathcal{C}^-$) is uniquely determined by braidings over the two factors. Therefore, the number of different braidings over Fus($\mathcal{C}^-$) is $2nm = 2N$.

Case 2: k is even.

We only show that the braiding over $\mathcal{C}^-$ is nondegenerate. Then we can count the number of different braidings using the same argument in the proof of Theorem 4.1.

We prove the non-degeneracy by checking the symmetric center. Let Y be a simple object in the symmetric center of $\mathcal{C}^-$. Notice that the fusion rings of $\mathcal{C}^-$ and $\mathcal{C}$ are isomorphic. The corresponding object Y in $\mathcal{C}$ centralizes $\mathcal{C}$ projectively. In the following, we first show that Y is pointed, then we examine all the pointed objects in $\mathcal{C}$ to show that Y can only be the unit object.

Since Y centralizes $\mathcal{C}$ projectively, then we can define $b_{Y,Z}$ to be the quantity such that

$$c_{Y,Z} \circ c_{Z,Y} = b_{Y,Z}\mathbf{id}_{Z\otimes Y}$$

for Z simple. We may also use this notation for non-simple Z if Y and Z projectively centralize each other. Notice

$$b_{X,Y^{\otimes N}} = b_{X,Y}^N = b_{X^{\otimes N},Y} = b_{\mathbf{1},Y} = 1.$$

Thus $Y^{\otimes N}$ centralizes X. Because $\mathcal{C}$ is nondegenerate, the only simple object in $\mathcal{C}$ that centralizes X is the unit object. Therefore $Y^{\otimes N}$ is a direct sum of unit objects. Let Z be a simple summand of $Y^{\otimes N-1}$. Then $Y \otimes Z$ is again a direct sum of unit objects. Moreover, $\mathcal{C}$ is rigid, so that $\mathrm{Hom}(Y \otimes Z, \mathbf{1}) = \mathrm{Hom}(Z, Y^*)$. The latter space is the hom space of two simple objects, thus of dimension at most 1. Therefore Z must be Y^* and Y is pointed.

Next we compute $b'_{X,g}$ in $\mathcal{C}^-$. Since g is in grade k, $c'_{X,g} = s^k c_{X,g}$ and $c'_{g,X} = s^k c_{g,X}$. Then

$$b'_{X,g} = s^{2k} b_{X,g} = t^{1+2kr}.$$

Notice that $2kr$ divides all prime factors of N, thus $1 + 2kr$ is relatively prime to N, and $b'_{X,g}$ is still a primitive N-th root of unity. Therefore, none of the non-trivial pointed objects are in the symmetric center, hence the braiding over $\mathcal{C}^-$ is nondegenerate. □

5.3. Braidings on subcategories

In this subsection, we classify all braided tensor categories $\mathcal{D}$ with the same fusion rule as the first factor $\mathcal{MC}$ in the decomposition formula (3.1). We also show the braiding is unique up to certain symmetries.

Theorem 5.4. *Let $N = mn$ where $n = gcd(N, k^\infty)$. Suppose $\mathcal{D}$ is a braided fusion category with the same fusion rules as*

$$\mathcal{M}SU(N)_k = \bigoplus_{i=0}^{n-1} (SU(N)_k)^{im}.$$

Then

(1) The braiding over $\mathcal{D}$ must be nondegenerate.
(2) The underlying monoidal category Fus($\mathcal{D}$) admits $2n$ different braidings.

Proof. Let $\mathcal{P}(\mathbb{Z}_m)$ be a pointed fusion category with the same fusion rules as $\mathbf{Vec}_{\mathbb{Z}_m}$. Then, the fusion ring of $\mathcal{D} \boxtimes \mathcal{P}(\mathbb{Z}_m)$ is isomorphic to the fusion ring of $SU(N)_k$. Thus $\mathcal{D} \boxtimes \mathcal{P}(\mathbb{Z}_m)$ is monoidal equivalent to the monoidal category $\mathcal{C}(\mathfrak{sl}_N, N+k, q))$ or the twisted monoidal category $\mathcal{C}(\mathfrak{sl}_N, N+k, q)^-$. Both underlying fusion categories admit a nondegenerate braiding on the nonpointed factor in the decomposition formula (3.1). On the other hand, we may choose a monoidal structure on the pointed factor $\mathcal{P}(\mathbb{Z}_m)$, so that $\mathcal{C} := \mathcal{D} \boxtimes \mathcal{P}(\mathbb{Z}_m)$ admits a nondegenerate braiding. As a result, all braidings over the underlying fusion category of $\mathrm{Fus}(\mathcal{D})$ come from the restriction of a braiding over $\mathrm{Fus}(\mathcal{C})$. Now we show for all the braidings over $\mathcal{D}$, the symmetric center must be trivial.

Since $\mathcal{D}$ admits a nondegenerate braiding, we only need to examine the pointed objects for categories equipped with other braidings. Due to the decomposition formula in (3.1), all such objects can be written in the form g^{jm}, where $0 \leq j \leq n-1$. Notice that g^{jm} centralizes $\mathcal{D}$ if and only if g^{jm} centralizes $X \in \mathcal{O}(\mathcal{C})$. Let us choose a nondegenerate braiding $c_{-,-}$ over the underlying fusion category $\mathrm{Fus}(\mathcal{C})$. Then we have

(1) the invariant $b_{X,g} = t$ is a primitive N-th root of unity;
(2) the braiding over $\mathcal{C}$ is in one to one correspondence with sections of $\mathcal{C} \boxtimes \mathcal{C}^{rev} \to \mathcal{C}$.

Pick one braiding over $\mathcal{C}$ denoted by $\bar{c}$ that corresponding to the section s. We denote by $\bar{b}_{X,Y}$ the quantity such that

$$\bar{c}_{Y,X} \circ \bar{c}_{X,Y} = \bar{b}_{X,Y} \mathbf{id}_{X \otimes Y}$$

if X and Y centralize each other.

Recall in (4.1), we have computed $\bar{b}_{X,g^{jm}} = t^{(2ki+1)jm}$. Thus, g^{jm} is in the symmetric center if and only if

$$(2ki+1)jm \equiv 0 \mod N,$$

which is equivalent to

$$(2ki+1)j \equiv 0 \mod n.$$

Notice that n is a divisor of k^M for large M. This implies that $2ki+1$ is invertible in ring $\mathbb{Z}_n$ since $2ki$ is nilpotent. Thus g^{jm} is in the symmetric center if and only if $j = 0$. This proves the first statement of Theorem 5.4.

The second statement can be proved by counting the number of braidings on $\mathcal{C} = \mathcal{D} \boxtimes \mathbb{Z}_m$ and the pointed factor $\mathcal{P}(\mathbb{Z}_m)$. There are $2N$ different braidings over $\mathcal{C}$, and there are m different braidings over $\mathbb{Z}_m$, see Appendix. Thus there are exactly $2N/m = 2n$ distinct braidings over $\mathrm{Fus}(\mathcal{D})$. $\square$

5.4. *Autoequivalences and braidings over* $\mathrm{Fus}(\mathcal{D})$

In this subsection, we consider the action of the group of autoequivalences of $\mathrm{Fus}(\mathcal{D})$ on the set of braidings. We denoted this group by $\mathrm{Eq}(\mathrm{Fus}(\mathcal{D}))$. Notice that all the autoequivalences of $\mathcal{C}$ for $\mathcal{C}$ Grothendieck equivalent to $SU(N)_k$ preserve the factorization $\mathcal{C} = \mathcal{MC} \boxtimes \mathcal{C}(\mathbb{Z}_m, P)$, since the factorization is preserved at the level of fusion rings. Consequently, $\mathrm{Eq}(\mathcal{C}) = \mathrm{Eq}(\mathcal{MC}) \times \mathrm{Eq}(\mathcal{C}(\mathbb{Z}_m, P))$. This allows us to study the group of autoequivalences of $\mathcal{D}$ as a subgroup of the group of autoequivalences of $\mathcal{C}$ as we constructed in the last subsection.

On the other hand, for the autoequivalence group of $\mathcal{C}$, we have the following exact sequence.

$$0 \to \mathrm{Gauge}(\mathcal{C}) \to \mathrm{Eq}(\mathcal{C}) \xrightarrow{\text{Forget}} \mathrm{Aut}(\mathcal{K}_0(\mathcal{C}))$$

where $\mathrm{Gauge}(\mathcal{C})$ is the subgroup of autoequivalences of $\mathcal{C}$ that fix the simple objects, and $\mathrm{Aut}(\mathcal{K}_0(\mathcal{C}))$ is the group of automorphisms of the fusion ring of $\mathcal{C}$. By the following lemma, we show in our case, the Gauge equivalences are all braided.

Lemma 5.5. *Let $\mathcal{C}$ be a braided fusion category and $(Id_{\mathcal{C}}, J)$ be a monoidal endofunctor. Suppose $\mathcal{C}$ is tensor generated by a single object X and $X \otimes X$ decomposes into distinct simple objects. Then $(Id_{\mathcal{C}}, J)$ is braided.*

Proof. Since $(\mathrm{Id}_{\mathcal{C}}, J)$ is an monoidal autoequivalence of $\mathcal{C}$, we denote the original braiding by c and the braiding induced by $(\mathrm{Id}_{\mathcal{C}}, J)$ by c'. Notice that the induced braiding $c'_{Y,Z}$ is induced by the following diagram.

$$\begin{array}{ccc} Y \otimes Z & \xrightarrow{c_{Y,Z}} & Z \otimes Y \\ {\scriptstyle J_{Y,Z}}\uparrow & & \downarrow{\scriptstyle J^{-1}_{Z,Y}} \\ Y \otimes Z & \xrightarrow{c'_{Y,Z}} & Z \otimes Y \end{array}$$

In particular, $(\mathrm{Id}_{\mathcal{C}}, J)$ is braided if the induced braiding $c' = c$. On the other hand, since our category is tensor generated by a single object X, in order to show $(\mathrm{Id}_{\mathcal{C}}, J)$ is braided we only need to show $c_{X,X} = c'_{X,X}$.

$$\begin{array}{ccc} X \otimes X & \xrightarrow{c_{X,X}} & X \otimes X \\ {\scriptstyle J_{X,X}}\uparrow & & \downarrow{\scriptstyle J^{-1}_{X,X}} \\ X \otimes X & \xrightarrow{c'_{X,X}} & X \otimes X \end{array}$$

Since $X \otimes X$ decomposes into distinct simple objects, $\mathrm{End}(X \otimes X)$ is commutative. Therefore, the composition

$$c'_{X,X} := J^{-1}_{X,X} c_{X,X} J_{X,X} = c_{X,X}.$$

This proves the lemma. $\square$

With the lemma above, the group action of $\mathrm{Eq}(\mathcal{C})$ on the set of braidings descends to an action of the group of its image in $\mathrm{Aut}(\mathcal{K}_0(\mathcal{C}))$. In particular, the simple current autoequivalences we constructed in Section 3 descend to a subgroup of $\mathrm{Aut}(\mathcal{K}_0(\mathcal{C}))$. We denote this group by $\mathrm{ScEq}(\mathcal{C})$, it acts on the set of braidings.

In the rest of the section, we consider the group $\mathrm{ScEq}(\mathcal{D})$, acting on the set of braidings over the underlying fusion category.

Lemma 5.6. *Let $\mathcal{C}$ be a nondegenerate braided fusion category with the fusion rules of $SU(N)_k$. Suppose F is a simple current autoequivalence of $\mathcal{C}$ sending X to $X \otimes g^a$. Then*

(i) $F(c_{X,X})$ and $c_{X,X}$ have the same eigenvalue ratio.
(ii) F is braided if and only if

$$a + \frac{ka^2}{2} \equiv 0 \ mod\ N. \tag{5.1}$$

Proof. Suppose $c_{X,X} = \lambda_1 p_1 + \lambda_2 p_2$, where p_1 and p_2 are two idempotents. In order to prove the lemma, we would like to decompose $c_{F(X),F(X)}$ into two idempotents. Notice

$$\tilde{p_i} = \frac{1}{b_{X,g^a} c_{g^a,g^a}} \mathrm{id}_X \otimes c_{X,g^a} \otimes \mathrm{id}_{g^a} \circ p_i \otimes c_{g^a,g^a} \circ \mathrm{id}_X \otimes c_{g^a,X} \otimes \mathrm{id}_{g^a}$$

is an idempotent, where $i = 1, 2$. On the other hand,

$$c_{F(X),F(X)} = b_{X,g^a} c_{g^a,g^a} (\lambda_1 \tilde{p_1} + \lambda_2 \tilde{p_2}).$$

The computation above proves (i).

To prove (ii), we assume $t = b_{X,g}$ is a primitive N-th root of unity, since g is in grade k, we have $c_{g^a,g^a} = t^{\frac{ka^2}{2}}$, since the braiding of $\mathcal{C}$ is generated by $c_{X,X}$, F is braided if and only if $c_{X,X}$ and $c_{F(X),F(X)}$ have the same eigenvalues. □

Remark 5.7. Part (ii) of the above lemma also appears in [8, Lemma 2.4], we provide a proof above for completeness.

Let $\mathcal{D}$ be a braided fusion category as in the theorem, the braiding is always nondegenerate by Theorem 5.4. Let $\mathcal{C} = \mathcal{D} \boxtimes \mathcal{P}(\mathbb{Z}_m)$ where $\mathcal{P}(\mathbb{Z}_m)$ is a pointed fusion category equipped with a nondegenerate braiding. Now $\mathcal{C}$ is a nondegenerate braided fusion category with fusion rules of $SU(N)_k$. Autoequivalences of $\mathcal{D}$ obviously extend to autoequivalences of $\mathcal{C}$.

Let F_a be the simple current autoequivalence of $\mathcal{C}$ sending X to $X \otimes g^a$, we would like to know when F_a is braided after restricting on $\mathcal{D}$. We first notice F_a and F_{a+n} restrict to the same autoequivalence of $\mathcal{D}$: Let Y be a simple object in $\mathcal{D}$, then Y lies in grade im for some i, then

$$F_{a+n}(Y) = Y \otimes g^{im(a+n)} = Y \otimes g^{ima+iN} = Y \otimes g^{ima} = F_a.$$

Thus F_a restricts to a nontrivial autoequivalence on $\mathcal{D}$ if and only if $n \nmid a$.

Lemma 5.8. *Let F be a braided autoequivalence of $\mathcal{D}$ that extend to a simple current braided autoequivalence F_a of $\mathcal{C}$. Then*

(1) $a \equiv \frac{n}{2}$ *or* $0 \mod n$ *if* $N \equiv 2 \mod 4$ *and* $k \equiv 2 \mod 4$.
(2) a *can take two distinct value modulo* n *if* $N \equiv 0 \mod 4$ *and* $k \equiv 2 \mod 4$.
(3) $a \equiv 0 \mod n$ *otherwise.*

Proof. F_a is braided if and only if a solves equation (5.1). Thus $n | a(1 + \frac{ka}{2})$. We first examine in the case (3), $a = 0$ modulo n.

a. If k or N is odd, then $n = \gcd(N, k^\infty)$ is odd. In this case, $(1 + \frac{ka}{2})$ and n are coprime, so $n \mid a$.
b. If $4 \mid k$, then $1 + \frac{ka}{2}$ is odd. In this case, $(1 + \frac{ka}{2})$ and n are coprime, so $n \mid a$.

Then we examine case (1). Assume $k = 2q$ and $n = 2q'$ where q and q' are odd numbers. Since $n = \gcd(N, k^\infty)$, q and q' share the same set of odd prime factors. Therefore, $\frac{ka}{2} + 1 = q'a + 1$ and q are coprime. This implies $q|a$. On the other hand, $a = q$ and $a = 0$ solves equation (5.1). This proves the first case.

For case (2), we assume $n = 2^p q$ and $k = 2q'$, since q, q' and m are odd numbers, thus invertible in $\mathbb{Z}_{2^p}$. We claim $a_0 = jmq$ is a solution of equation (5.1), where $j = (-mqq')^{-1}$ in $\mathbb{Z}_{2^p}$. The claim follows from

$$mq \mid a \text{ and } 2^p \mid 1 + mq'$$

In order to finish the proof of case (2). It remains to show if a is another solution such that $n \nmid a$, then $a - a_0$ is a multiple of n. Notice $1 + \frac{ak}{2}$ is coprime to q, so $q \mid a$ and $q|a - a_0$. On the other hand, a must be an odd number, otherwise $1 + \frac{ak}{2}$ will be an odd number, and $2^p \mid a$, leads to a contradiction. In order to make equation (5.1) hold, we need $2^p \mid 1 + aq$. Since $2^p \mid 1 + a_0 q$, we conclude $2^p \mid (a - a_0)q$ and $2^p \mid a - a_0$. The conclusion follows. □

Given a braided fusion category $\mathcal{D}$, we may get a new braided structure with the following actions:

(a) Reverse the braiding;
(b) Relabel objects by an simple current autoequivalence F_a.

If n is even, one can also

(c) replace $c_{X,Y}$ by $-c_{X,Y}$ if both X and Y are of odd grade.

Notice action (a) changes the eigenvalue ratio of $c_{X,X}$. By Lemma 5.6, (a) always induces a different braiding from (b) and (c). In the next theorem, we discuss how these actions change the set of braidings of $\mathrm{Fus}(\mathcal{D})$.

Theorem 5.9. *Let $\mathcal{D}$ be a braided fusion category Grothendieck equivalent to $\mathcal{M}SU(N)_k$, and $N = mn$ where $n = gcd(N, k^\infty)$. Then*

(1) In case (1) of Lemma 5.8, (a)(b)(c) act transversely on the set of braidings.
(2) In case (2) of Lemma 5.8, action (c) can be realized by (b), and there are two orbits under the action of (a) and (b).
(3) In case (3), (a)(b) act transversely on the set of braidings.

Proof. By Lemma 5.8, the subgroup of braided autoequivalences is

(i) $\mathbb{Z}_2$ in case (1) and (2);
(ii) trivial in case (3).

Notice action (a) generates a group of order 2, simple current autoequivalences generate a group of order n and the action (c) generates a group of order 2.

The case (3) follows directly by the orbit-stabilizer theorem.

For the other two cases, we only need to determine if action (c) can be achieved by action (b). In other words, we solve

$$\tilde{c}_{X,X} = b_{X,g^a} c_{g^a,g^a} c_{X,X} = -c_{X,X}.$$

Observe that $b_{X,g} = t$ is a primitive N-th root of unity. The above equation is simplified to

$$t^{a+\frac{ka^2}{2}} = -1,$$

or equivalently

$$a + \frac{ka^2}{2} \equiv \frac{N}{2} \mod N.$$

The above equation does not have a solution if $N \equiv 2 \mod 4$ and $k \equiv 2 \mod 4$. The three actions generate a group of order $2n$, case (1) follows from the orbit-stabilizer theorem.

To prove case (2), we only need to provide a solution to the equation above, $a = \frac{N}{2}$ solves the equation. □

Remark 5.10. In the case $N \equiv 0 \mod 4$ and $k \equiv 2 \mod 4$, assume $N = 2^p N'$ such that N' is odd. There are two orbits under the group action described in Theorem 5.9. Suppose we equipped $\mathcal{D}$ with a braiding that extends to a braiding $c_{-,-}$ over $\mathcal{C}$. One can achieve a braiding in the other orbit by replacing $c_{Y,Z}$ by $c'_{Y,Z} = t^{N' \deg Y \deg Z} c_{Y,Z}$ if Y and Z are homogeneous objects in $\mathcal{C}$. This can be seen as follows. Notice $t^{N'}$ is an N-th root of unity, thus $c'_{-,-}$ satisfies the hexagon equations. On the other hand, $X^{\otimes N'}$

is an object in $\mathcal{D}$ and $c'_{X^{\otimes N'},X^{\otimes N'}} = t^{N'^3} c_{X^{\otimes N'},X^{\otimes N'}}$, thus c' and c restrict to different braidings in $\mathcal{D}$. Lastly, to see c' cannot be achieved by a simple current autoequivalence, it suffices to show the equation below does not have a solution.

$$t^{a+\frac{ka^2}{2}} = t^{N'},$$

or equivalently

$$a + \frac{ka^2}{2} = N' \pmod N$$

Since $k \equiv 2 \mod 4$, the left hand side is always even and N' is an odd number.

Data availability

No data was used for the research described in the article.

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Appendix A. Braidings over $\mathbb{Z}_N$

In this appendix, we compute all possible twists of $\mathbb{Z}_N$ that admit a braid structure, as a special case of [10, Exercise 8.4.3], the insight of which comes from [13].

Proposition A.1. *The category* $\textbf{Vec}^{\omega}_{\mathbb{Z}_N}$ *has a braid structure if and only if*

(1) ω *is trivial for* N *is odd;*
(2) ω^2 *is trivial for* N *is even.*

In addition, ω *is trivial if and only if the following conditions hold: For all objects* l *with order* 2^p*, the quantity* $b_{l,l}$ *has order less than* 2^p*.*

Proof. Without loss of generality, we assume the category is skeletal with simple objects $\{0, 1, 2, \dots, N-1\}$. We take the following representative 3-cocycles

$$\omega(i,j,k) = \begin{cases} 1 & i+j < N \\ \eta^k & i+j \geq N \end{cases}$$

where η is an N-th root of unity.

The category admits a braid structure if $c_{i,j} := c(i,j)\mathbf{id}_{i\otimes j}$ satisfies the (hexagon) equations

$$\omega(j,k,i)c(i,j+k)\omega(i,j,k) = c(i,k)\omega(j,i,k)c(i,j) \tag{A.1}$$

$$\omega(k,i,j)^{-1}c(i+j,k)\omega(i,j,k)^{-1} = c(i,k)\omega(i,k,j)^{-1}c(j,k). \tag{A.2}$$

With our choice of the 3-cocycles, $\omega(i,j,k) = \omega(j,i,k)$ so the equations simplify to

$$\omega(j,k,i)c(i,j+k) = c(i,k)c(i,j) \tag{A.3}$$

$$c(i+j,k)\omega(i,j,k)^{-1} = c(i,k)c(j,k). \tag{A.4}$$

Setting $c(1,1) = s$ and applying equation (A.3) inductively, we get $c(i,j) = s^{ij}$ for $0 < i < N$ and $0 < j < N$.

For j,k such that $j+k = N$, equation (A.3) becomes

$$\eta^i = s^{ik}s^{ij} = s^{iN}.$$

For i,j such that $i+j = N$, equation (A.4) becomes

$$\eta^{-k} = s^{ik}s^{jk} = s^{kN}.$$

Thus $\eta^{2i} = 1$ for all i so $\eta = 1$ or -1. Since η is an N-th root of unity, $\eta = -1$ only if N is even.

To finish the proof of the first part of the proposition we construct braidings in each case. If $\eta = 1$, taking $s = 1$ solves the hexagon equation. If $\eta = -1$, taking s to be a primitive $2N$-th root of unity solves the hexagon equation.

In order to prove the second part, we may assume $N = 2^q$, as the category factors otherwise. Notice $\eta = s^{2^q}$ and $b_{l,l} = s^{2l^2}$. $\eta = 1$ if and only if s has order less than 2^q. Since order of s must be power of 2, the later condition is equivalent to s^{2l^2} has order less than $2^{N-(2N-2p+1)} = 2^{2p-1-N}$. The "only if" part is obvious and the "if" part comes from plugging in $p = N$. $\square$

We get the following corollary immediately:

Corollary A.2. *Suppose $\mathcal{P}$ is a pointed fusion category with $\mathbb{Z}_N$ fusion rules that admits a braiding. Then its full subcategory $\mathcal{P}^2$ generated by simple objects in even grades must have trivial associativity constraints.*

Corollary A.3. *Let $\mathcal{P}$ be a braided fusion category Grothendieck equivalent to $\mathbf{Vec}_{\mathbb{Z}_N}$. Then $\mathcal{P}$ admits N braid structures.*

Proof. As a monoidal category, $\mathcal{P}$ is monoidal equivalent to $\mathbf{Vec}_{\mathbb{Z}_N}^{\omega}$ for some 3-cocycle ω by [10, Proposition 2.6.1(iii)]. As in the proof of the proposition, we let $\mathbf{1}$ be the generating object, and $c(1,1) = s$. We compute the possible braidings case by case. Suppose $N = 2^r q$ where q is an odd number.

Case 1 $r = 0$. Since N is odd, we have $s^N = 1$, there are exactly N different choices of s.

Case 2 $r \geq 0$ with trivial ω. By the proof of the proposition we have $s^N = 1$ thus there are exactly N different choices of s.

Case 3 $r \geq 0$ with nontrivial ω. By the proof of the proposition we have $s^N = -1$ which has N solutions. □

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