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Spectral bounds for the quantum chromatic number of quantum graphs



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ABSTRACT

Quantum graphs are an operator space generalization of classical graphs that have emerged in different branches of mathematics including operator theory, non-commutative topology and quantum information theory. In this paper, we obtain lower bounds for the classical, quantum and quantum-commuting chromatic number of a quantum graph using eigenvalues of the quantum adjacency matrix. In particular, we prove a quantum generalization of Hoffman's bound and introduce quantum analogues for the edge number, Laplacian and signless Laplacian. We generalize all the spectral bounds of Elphick & Wocjan [14] to the quantum graph setting and demonstrate the tightness of these bounds in the case of complete quantum graphs. Our results are achieved using techniques from linear algebra and a combinatorial definition of quantum graph coloring, which is obtained from the winning strategies of a quantum-to-classical nonlocal graph coloring game [5].

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1. Introduction

Graph coloring has been well-studied in mathematics since the eighteenth century and has widespread applications in day-to-day life, including scheduling problems, register allocation, radio frequency assignments and sudoku solutions [25]. Traditionally, the coloring of a graph refers to an assignment of labels (called colors) to the vertices of a graph such that no two adjacent vertices share the same color. The chromatic number of a graph is defined to be the minimum number of colors for which such an assignment is possible.

More recently, a quantum generalization of the chromatic number was introduced within the framework of non-local games in quantum information theory [6]. The quantum chromatic number of a graph is defined as the minimal number of colors necessary in a quantum protocol in which two separated players, who cannot communicate with each other but share an entangled quantum state, try to convince an interrogator with certainty that they have a coloring for the given graph. There are known examples of graphs whose quantum chromatic number is strictly smaller than its classical chromatic number [22,6], thus exhibiting the power of quantum entanglement. Quantum chromatic number of classical graphs have close connections to Tsirelson's conjecture and the Connes embedding problem and have been extensively studied in the past decade [28,22,27,26]. In general, computing the chromatic number of a graph is an NP-hard problem. However, several lower bounds on the (quantum) chromatic number have been shown using spectral graph theory [14]. In this paper, we are interested in generalizing these spectral bounds to the setting of *quantum graphs* and estimating the quantum chromatic number of a quantum graph.

Quantum graphs are a non-commutative generalization of classical graphs that have attracted significant attention in recent years due to their intriguing connections with several areas of mathematics, physics and computer science. They first appeared in [11], and have independently emerged in other disguises thereafter. In information theory, quantum graphs were introduced as a quantum analogue of the confusability graph of classical channels [10]. Another definition was proposed in the context of quantum relations [33], which describes a quantum graph as a reflexive and symmetric quantum relation on a finite dimensional von-Neumann algebra. In [23], an equivalent perspective on quantum graphs was developed in a categorical framework for quantum functions, using a quantum adjacency matrix, and was further generalized in [3]. In recent years, research in quantum graph theory has undergone vast developments and quantum graphs have been explored in the context of zero-error quantum information theory, quantum error correction, operator algebras, non-local games, quantum symmetries, non-commutative topology and other fields [4,7,32,17,21,15,8]. In particular, there have been multiple studies on the coloring of quantum graphs [24,18,29,5,30] leading to different variants of the chromatic number of a quantum graph, in both the classical and quantum sense.

One recent approach developed in [5] defines the coloring of a quantum graph using a two-player non-local game involving quantum inputs and classical outputs. This game

generalizes the coloring game for classical graphs [6] and introduces chromatic numbers of a quantum graph in different mathematical models: *loc*, *q*, *qa*, *qc*, *C**, *hered*, *alg*. It was shown that the chromatic numbers defined in this framework agree nicely with other versions in the literature [22,29,18], and also lead to a four-coloring theorem for quantum graphs in the algebraic model. We adopt this formalism of quantum graph coloring in the present paper.

The goal of this paper is to obtain bounds for the quantum chromatic number of quantum graphs. Chromatic numbers of quantum graphs are closely related to the zero-error capacity of quantum channels [10]. Hence, estimating these numbers is useful for the development of zero-error quantum communication. In [14], the authors proved many lower bounds on the quantum chromatic number of *classical graphs* using an algebraic characterization of graph coloring. We extend their results to the setting of quantum graphs using a combinatorial definition of quantum graph coloring developed in [5]. Our approach uses the quantum adjacency matrix, defined in [23,3], to associate a spectrum with the given quantum graph. We use this spectrum and techniques adapted from [14] to achieve spectral estimates for the quantum chromatic number of quantum graphs. Our methods can also be readily adapted to the infinite dimensional setting, and yield lower bounds for the chromatic number in the quantum commuting model. Along the process, we naturally get lower bounds for the classical chromatic numbers of quantum graphs since the classical chromatic number is greater than or equal to the quantum chromatic number.

Our main result can be summarized as follows:

Theorem 1.1. *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph, and let $\chi_q(\mathcal{G})$ denote the quantum chromatic number of \mathcal{G} . Then,*

$$1 + \max \left\{ \frac{\lambda_{\max}}{|\lambda_{\min}|}, \frac{\dim(S)}{\dim(S) - \dim(\mathcal{M})\gamma_{\min}}, \frac{s^\pm}{s^\mp}, \frac{n^\pm}{n^\mp}, \frac{\lambda_{\max}}{\lambda_{\max} - \gamma_{\max} + \theta_{\max}} \right\} \leq \chi_q(\mathcal{G}).$$

Specifically, we prove that Hoffman's bound [16] holds in the case of quantum graphs. We also introduce quantum analogues for the edge number, inertia, Laplacian and signless Laplacian of a quantum graph along the way. Further, we demonstrate the tightness of all the bounds in the case of irreflexive complete quantum graphs.

Our paper is organized as follows: Section §2 provides the necessary background on quantum graphs and the connections between different perspectives. We also review the notion of quantum graph coloring and chromatic numbers in this section. Section §3 introduces the spectrum of a quantum graph and develops algebraic results connecting the quantum adjacency operator to quantum graph coloring. In section §4, we prove the spectral lower bounds listed in Theorem 1.1 for the quantum chromatic numbers of quantum graphs. We conclude with an illustration of the bounds in the case of complete quantum graphs in section §5. The appendix §A presents a table translating the different definitions of quantum graphs.

2. Preliminaries

In this section, we review some definitions and results on quantum graphs and quantum coloring which will be required for our discussion. We begin by listing some notations used in the paper.

2.1. Notations

- $[n]$ denotes the discrete set $\{1, 2, \dots, n\}$.
- $|\cdot\rangle$ denotes a column vector, while $\langle\cdot|$ denotes its conjugate transpose.
- M_n denotes the set of all $n \times n$ complex matrices.
- e_{ij} denotes the matrix whose i^{th} row - j^{th} column has entry 1 and all other entries are 0.
- Tr denotes the natural trace, given by summing all diagonal terms of a matrix.
- $B(\mathcal{H})$ denotes the algebra of bounded linear operators on a Hilbert space \mathcal{H} .
- If T is a subset of an algebra \mathcal{A} , then the commutant of T is denoted by $T' = \{a \in \mathcal{A} : at = ta, \forall t \in T\}$.
- The spectrum of an operator A is denoted by $\sigma(A)$.
- G denotes a classical graph, $\chi(G)$ denotes the classical chromatic number of G and $\chi_q(G)$ denotes the quantum chromatic number of G .

2.2. Quantum graphs as operator spaces

Quantum graphs can be defined in different ways, as mentioned in the introduction. One way to describe them is as operator spaces satisfying a certain bimodule property [33]. This approach is commonly used for studying quantum coloring problems. We discuss this formalism here:

Definition 2.1. Let \mathcal{M} be a finite dimensional (non-degenerate) von Neumann algebra, represented on a finite dimensional Hilbert space \mathcal{H} . A *quantum graph* on $\mathcal{M} \subseteq B(\mathcal{H})$ is an operator space $S \subseteq B(\mathcal{H})$ that is closed under adjoint and is a bimodule over the commutant of \mathcal{M} , that is $\mathcal{M}'S\mathcal{M}' \subseteq S$. We denote this quantum graph by the tuple $\mathcal{G} = (S, \mathcal{M}, B(\mathcal{H}))$.

Motivated by confusability graphs in information theory, quantum graphs are generally assumed to be reflexive ($I \in S$) and hence, S is an *operator system* in $B(\mathcal{H})$. But for the purposes of graph coloring, we will only consider irreflexive quantum graphs, that is quantum analogues of graphs without loops. Before we begin, recall that

Definition 2.2. A quantum graph $(S, \mathcal{M}, B(\mathcal{H}))$ is said to be *irreflexive* if $S \subseteq (\mathcal{M}')^\perp$. Here \perp denotes the orthogonal complement with respect to the (unnormalized) trace inner product $\langle x, y \rangle = \text{Tr}(y^*x)$ on $B(\mathcal{H})$.

In particular, an irreflexive quantum graph on M_n (with the standard representation $M_n = B(\mathbb{C}^n)$) is simply a self-adjoint traceless operator subspace in M_n . This is sometimes used as the definition of non-commutative graphs in the literature [29].

It can be shown that the operator space S associated to a quantum graph $(S, \mathcal{M}, B(\mathcal{H}))$ is essentially independent of the representation of \mathcal{M} [31]. The intuition is that S contains operators that represent edges in the graph, as illustrated by the following example.

Example 2.3. Let G be a classical graph on n vertices. One can identify the vertex set of G with the algebra of diagonal matrices $D_n \subseteq M_n$, by identifying each vertex i with the diagonal matrix $e_{ii} \in D_n$. Then, $S_G = \text{span}\{e_{ij} : (i, j) \text{ is an edge in } G\} \subseteq M_n$ is a quantum graph over D_n .

Remark 2.4. Any quantum graph over D_n is necessarily of the form S_G for some classical graph G [33]. Also, two reflexive classical graphs G_1, G_2 are isomorphic if and only if their corresponding operator systems S_{G_1}, S_{G_2} are unitally completely order isomorphic [24].

A “purely quantum” example is the following one:

Example 2.5. Let $\mathcal{M} = M_2$ and $S = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$. Then $(S, M_2, B(\mathbb{C}^2))$ is a quantum graph on M_2 that doesn’t arise from any classical graph.

2.3. Quantum graphs with a quantum adjacency matrix

In this paper, we take advantage of an alternate (but equivalent) definition of a quantum graph, which involves quantizing the vertex set and the adjacency matrix. This formalism was first introduced in [23] using the language of special symmetric dagger Frobenius algebras, and was later generalized to the non-tracial case in [3,21]. In this perspective, the non-commutative analogue of a vertex set is played by a C^* -algebra, which also carries the structure of a Hilbert space. It is defined as follows:

Definition 2.6 (Quantum set). A quantum set is a pair (\mathcal{M}, ψ) , where \mathcal{M} is a finite dimensional C^* -algebra and $\psi : \mathcal{M} \rightarrow \mathbb{C}$ is a faithful state.

Using ψ , one can view \mathcal{M} as a Hilbert space $L^2(\mathcal{M}) = L^2(\mathcal{M}, \psi)$ obtained from the GNS representation of \mathcal{M} with respect to ψ . That is, $L^2(\mathcal{M})$ is the vector space \mathcal{M} equipped with the inner product $\langle x, y \rangle = \psi(y^* x)$.

Let $m : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M}$ denote the multiplication map and m^* denote the adjoint of m when viewed as a linear operator from $L^2(\mathcal{M}) \otimes L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$. Further, we denote the unit of \mathcal{M} by $\mathbb{1}$ and let $\eta : \mathbb{C} \rightarrow \mathcal{M}$ be the unit map $\lambda \mapsto \lambda \mathbb{1}$. The adjoint of η (as an operator on Hilbert spaces) is denoted by η^* and is equal to ψ . While there are many

choices for a faithful state ψ on \mathcal{M} , we will restrict our attention to δ -forms, as done in [3].

Definition 2.7. For $\delta > 0$, a state $\psi : \mathcal{M} \rightarrow \mathbb{C}$ is called a δ -form if $mm^* = \delta^2 I$.

Example 2.8. Let X be a finite set and $\mathcal{M} = C(X)$ be the algebra of complex valued functions on X . Then the uniform measure $\psi(f) = \frac{1}{|X|} \sum_{x \in X} f(x)$ is a δ -form on $C(X)$ with $\delta^2 = |X|$. In this case, m^* is given by $e_i \mapsto |X|(e_i \otimes e_i)$, where e_i is the characteristic function on the set $\{i\} \subseteq X$.

Example 2.9. Let \mathcal{M} be M_n equipped with the canonical normalized trace $\psi = \frac{1}{n} \text{Tr}$. Then $m^*(e_{ij}) = n \sum_{k=1}^n e_{ik} \otimes e_{kj}$, and ψ is an n -form on M_n .

The δ -forms in the above examples are tracial, that is $\psi(xy) = \psi(yx)$ for all $x, y \in \mathcal{M}$. A tracial δ -form on a finite dimensional C^* -algebra is unique and has a nice form, which will be used in later sections. We recall this now:

Proposition 2.10 ([2]). Let \mathcal{M} be a finite dimensional C^* -algebra, decomposed as $\mathcal{M} \cong \bigoplus_{i=1}^N M_{n_i}$, where N, n_1, n_2, \dots, n_N are some positive integers. Then, there exists a unique tracial δ -form on \mathcal{M} given by

$$\psi = \frac{1}{\dim(\mathcal{M})} \bigoplus_{i=1}^N n_i \text{Tr}(\cdot) \quad (2.3.1)$$

In this case, $\delta^2 = \dim(\mathcal{M})$ and the state ψ is called the *Plancheral trace*. Moreover, $\psi = \frac{1}{\dim(\mathcal{M})} \text{Tr}|_{\mathcal{M}}$, where $\text{Tr} : B(L^2(\mathcal{M})) \rightarrow \mathbb{C}$ is the canonical trace.

A quantum set endowed with an additional structure of a quantum adjacency matrix yields a quantum graph.

Definition 2.11 ([3]). Let \mathcal{M} be a finite dimensional C^* -algebra equipped with a δ -form ψ . A self-adjoint linear map $A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ is called a *quantum adjacency matrix* if it satisfies the following conditions:

1. $m(A \otimes A)m^* = \delta^2 A$,
2. $(I \otimes \eta^* m)(I \otimes A \otimes I)(m^* \eta \otimes I) = A$.

The tuple $\mathcal{G} = (\mathcal{M}, \psi, A)$ is called an (undirected) quantum graph.

The quantum graph (\mathcal{M}, ψ, A) is said to be *reflexive* if it further satisfies the condition $m(A \otimes I)m^* = \delta^2 I$ or is said to be *irreflexive* if it satisfies the condition $m(A \otimes I)m^* = 0$.

The motivation for the above definition comes from the commutative setting where $\mathcal{M} = C(X)$ and ψ is the uniform measure on X . In this case, the quantum adjacency

matrix $A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ can be identified with a matrix in $M_{|\mathcal{X}|}(\mathbb{C})$, and the operation $\delta^{-2}m(P \otimes Q)m^*$ is simply the schur product of the matrices P and Q , given by entrywise multiplication. So, the first condition in Definition 2.11 says that A must be an idempotent with respect to Schur multiplication, which is equivalent to saying that A has entries in $\{0, 1\}$. The second condition says $A = A^T$. If we drop the second condition in Definition 2.11, it is called a *directed* quantum graph [4].

Remark 2.12. The self-adjointness of A along with condition (2) in Definition 2.11 implies that A is also $*$ -preserving [21], that is $Ax^* = (Ax)^*$ for all $x \in \mathcal{M}$.

Every quantum set can be easily equipped with an adjacency operator to obtain a quantum graph. An example is the complete quantum graph.

Definition 2.13. Let (\mathcal{M}, ψ) be a quantum set. A reflexive *complete quantum graph* on \mathcal{M} is defined by $A = \delta^2\psi(\cdot)\mathbb{1}$. In the classical case, this gives the all 1s matrix and corresponds to the reflexive complete graph on $\dim(\mathcal{M})$ vertices.

An *irreflexive* complete quantum graph on (\mathcal{M}, ψ) is defined by $A = \delta^2\psi(\cdot)\mathbb{1} - I$.

There are several non-trivial examples of quantum graphs. In particular, [21] gives a concrete classification of all undirected reflexive quantum graphs on M_2 , and [15] gives an example of a quantum graph, which is not quantum isomorphic to any classical graph.

2.4. Translation between different perspectives of quantum graphs

The two definitions of quantum graphs given in 2.1 and 2.11 can be shown to be equivalent [23], using a bijective correspondence between linear operators on $L^2(\mathcal{M})$ and elements in $\mathcal{M} \otimes \mathcal{M}^{op}$. A detailed proof for the correspondence between different definitions of quantum graphs may be found in [23,20,8]. We summarize this connection below:

- Given a quantum graph (\mathcal{M}, ψ, A) , define a linear operator P on $B(L^2(\mathcal{M}))$ as

$$P(X) = \delta^{-2}m(A \otimes X)m^*. \quad (2.4.1)$$

Then, $S := \text{range}(P)$ is a self-adjoint operator subspace in $B(L^2(\mathcal{M}))$ that is a bimodule over \mathcal{M}' .

- Given a quantum graph $(S, (\mathcal{M}, \psi), B(L^2(\mathcal{M})))$, let $P : B(L^2(\mathcal{M})) \rightarrow B(L^2(\mathcal{M}))$ denote a self-adjoint $\mathcal{M}' - \mathcal{M}'$ bimodule projection on $B(L^2(\mathcal{M}))$ with $\text{range}(P) = S$. That is, $P(axb) = aP(x)b$, for all $x \in B(L^2(\mathcal{M}))$, $a, b \in \mathcal{M}'$ and $P^2 = P = P^*$, where the adjoint is taken with respect to the trace inner product on $B(L^2(\mathcal{M}))$. (Such a P always exists and is unique for the given S [31].)

Then, $A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ defined by

$$A(x) = \delta^2(\psi \otimes I)P(x \otimes \mathbb{1}) \quad (2.4.2)$$

is a quantum adjacency matrix on (\mathcal{M}, ψ) .

Remark 2.14. In (2.4.2) P is interpreted as an element of $\mathcal{M} \otimes \mathcal{M}^{op}$ using the following well-known $*$ -isomorphism in finite dimensions [12]:

$$\begin{aligned} \mathcal{M} \otimes \mathcal{M}^{op} &\cong {}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M}))), \text{ given by} \\ x \otimes y^{op} &\longleftrightarrow x(\cdot)y. \end{aligned}$$

Here, \mathcal{M}^{op} denotes the opposite algebra of \mathcal{M} and ${}_{\mathcal{M}'}CB_{\mathcal{M}'}(B(L^2(\mathcal{M})))$ denotes the set of completely bounded maps P on $B(L^2(\mathcal{M}))$ with the property $P(axb) = aP(x)b$, for all $x \in B(L^2(\mathcal{M})), a, b \in \mathcal{M}'$. An infinite dimensional version of this result can be found in [12].

The expressions (2.4.1) and (2.4.2) are inverses of each other, which will be discussed in more detail in Proposition 3.1.

In general, the correspondence between the self-adjoint operator space S and linear operators A is not one-one since there are several different $\mathcal{M}' - \mathcal{M}'$ bimodule idempotents P with the same range S . However, there is a *unique* self-adjoint quantum adjacency matrix A for a given S , which corresponds to the unique *orthogonal* bimodule projection onto S . In this case, A is also completely positive, which was used as an alternate definition of quantum adjacency matrix in [7].

2.5. Chromatic number of quantum graphs

In this section, we review a notion of quantum graph coloring that was developed in [5] using a two-player quantum-to-classical nonlocal game, generalizing the coloring game for classical graphs [6]. The inputs for the quantum graph coloring game are elements from a suitably chosen basis for the graph operator space (known as quantum edge basis) and the outputs are classical color assignments. The inputs are quantum in the sense that they are tensor product states, where one player receives the left leg of the tensor and the other player receives the right leg. The players win the game if their responses jointly satisfy a synchronicity condition and respect the adjacency structure of the quantum graph. We refer the reader to [5] for more details on the game, and for the results presented in this section.

Using the winning strategies for the quantum graph coloring game, the chromatic number of a quantum graph can be defined in different mathematical models: *loc*, *q*, *qa*, *qc*, *C**, *hered*, *alg*. In this paper, we will restrict our discussion to the classical

(*loc*), quantum (*q*) and quantum commuting (*qc*) chromatic numbers. We begin with recalling an algebraic definition of quantum graph coloring that arises from the non-local game in [5].

Definition 2.15 ([5]). Let $\mathcal{G} = (S, \mathcal{M}, B(\mathcal{H}))$ be an irreflexive quantum graph. We say that there is a c -coloring of \mathcal{G} if there exists a finite von-Neumann algebra \mathcal{N} with a faithful normal trace and projections $\{P_a\}_{a=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ such that

1. $P_a^2 = P_a = P_a^*$, for $1 \leq a \leq c$,
2. $\sum_{i=1}^c P_a = I_{\mathcal{M} \otimes \mathcal{N}}$,

satisfying the following condition:

$$P_a(X \otimes I_{\mathcal{N}})P_a = 0, \quad \forall X \in S \text{ and } 1 \leq a \leq c. \quad (2.5.1)$$

If $\dim(\mathcal{N}) = 1$, we call it a *classical (loc)* coloring of \mathcal{G} and if $\dim(\mathcal{N}) < \infty$, we call it a *quantum (q)* coloring of \mathcal{G} . More generally, when \mathcal{N} is a finite von-Neumann algebra (possibly infinite dimensional), it is called a *quantum commuting (qc)* coloring of \mathcal{G} .

The projections $\{P_a\}_{a=1}^c$ are obtained from the winning strategies of the non-local quantum graph coloring game. In particular, when $\mathcal{M} = D_n$, we recover the usual classical and quantum coloring of classical graphs on n vertices.

Definition 2.16 ([5]). Let $\mathcal{G} = (S, \mathcal{M}, B(\mathcal{H}))$ be an irreflexive quantum graph. The *quantum chromatic number* of \mathcal{G} is defined to be the least c such that there exists a c -coloring of \mathcal{G} in the sense of Definition 2.15, with $\dim(\mathcal{N}) < \infty$. We denote this quantum chromatic number by $\chi_q(\mathcal{G})$. Moreover, when $\dim(\mathcal{N}) = 1$, it is called the *classical chromatic number* $\chi(\mathcal{G}) = \chi_{loc}(\mathcal{G})$ and when $\dim(\mathcal{N}) = \infty$, it is called the *quantum commuting chromatic number* $\chi_{qc}(\mathcal{G})$.

It was shown in [5] that every quantum graph $\mathcal{G} = (S, \mathcal{M}, M_n)$ has a finite quantum coloring and $\chi_q(\mathcal{G}) \leq \dim(\mathcal{M})$. Further, for all quantum graphs \mathcal{G} , we have

$$\chi_{qc}(\mathcal{G}) \leq \chi_q(\mathcal{G}) \leq \chi(\mathcal{G}). \quad (2.5.2)$$

Also, if (S, \mathcal{M}, M_n) and (T, \mathcal{M}, M_n) are two quantum graphs such that $S \subseteq T$, then $\chi_t(S, \mathcal{M}, M_n) \leq \chi_t(T, \mathcal{M}, M_n)$, where $t \in \{\text{loc, q, qc}\}$.

Example 2.17. Let G be a classical graph on n vertices and $\mathcal{G} = (S_G, D_n, M_n)$ be the quantum graph associated with G , as in Example 2.3. Then,

$$\chi(\mathcal{G}) = \chi(G) \text{ and } \chi_q(\mathcal{G}) = \chi_q(G). \quad (2.5.3)$$

Example 2.18. For complete quantum graphs, the quantum chromatic number is the full dimension of the quantum vertex set. That is, $\chi_q(M_n, \mathcal{M}, M_n) = \dim(\mathcal{M})$.

Remark 2.19. Indeed, $\chi(M_n, \mathcal{M}, M_n) < \infty$ if and only if \mathcal{M} is abelian. In particular, if \mathcal{M} is non-abelian, then $\chi(M_n, \mathcal{M}, M_n) \neq \chi_q(M_n, \mathcal{M}, M_n)$.

It is also useful to note that Definition 2.16 is a special case of Stahlke's entanglement-assisted chromatic number [29]. Also, when $\mathcal{N} = \mathbb{C}$, it is equivalent to Kim & Mehta's strong chromatic numbers of non-commutative graphs [18].

3. Use of quantum adjacency matrix in coloring

While Definition 2.1 was used in [5] for developing the chromatic number of quantum graphs, Definition 2.11 offers the advantage of associating a spectrum to the quantum graph, which is useful for estimating these chromatic numbers. We begin with the following result, which will be used to obtain a quantum adjacency operator for defining the spectrum.

Proposition 3.1. *Let \mathcal{M} be a finite dimensional C^* -algebra equipped with its tracial δ -form ψ . There is a bijective correspondence between linear operators A on $L^2(\mathcal{M})$ and $\mathcal{M}' - \mathcal{M}'$ bimodule linear operators P on $B(L^2(\mathcal{M}))$, given as follows:*

$$\begin{aligned} A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M}) &\longleftrightarrow P : B(L^2(\mathcal{M})) \rightarrow B(L^2(\mathcal{M})) \\ A(\cdot) = \delta^2(\psi \otimes I)P((\cdot) \otimes 1) &\longleftrightarrow P(\cdot) = \frac{1}{\delta^2}m(A \otimes (\cdot))m^*. \end{aligned} \quad (3.0.1)$$

This correspondence has the following property:

$$\frac{1}{\delta^2}m(A \otimes A)m^* = A \iff P^2 = P.$$

Proof. Let \mathcal{M}^{op} denote the opposite algebra of \mathcal{M} . It was shown in [23] that there is a bijective correspondence between linear operators $A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ and elements $p = \sum_{i=1}^t x_i \otimes y_i \in \mathcal{M} \otimes \mathcal{M}^{op}$, given by

$$A(\cdot) = \delta^2(\psi \otimes I)p((\cdot) \otimes 1) = \delta^2 \sum_{i=1}^t \psi(x_i(\cdot))y_i, \text{ and } p = \frac{1}{\delta^2}(A \otimes I)m^*(1), \quad (3.0.2)$$

with the property that $\frac{1}{\delta^2}m(A \otimes A)m^* = A \iff p^2 = p$.

Since \mathcal{M} is finite dimensional, elements $p = \sum_{i=1}^t x_i \otimes y_i \in \mathcal{M} \otimes \mathcal{M}^{op}$ are $*$ -isomorphic to $\mathcal{M}' - \mathcal{M}'$ bimodule maps P_p on $B(L^2(\mathcal{M}))$, given by $P_p(\cdot) = \sum_{i=1}^t x_i(\cdot)y_i$ discussed in Remark 2.14.

So, the bijective correspondence between A and p in (3.0.2) may be rewritten as a bijective correspondence between A and P_p as follows:

$$A(\cdot) = \delta^2(\psi \otimes I)P_p((\cdot) \otimes \mathbb{1}), \text{ and } P_p(\cdot) = \frac{1}{\delta^2} \sum_{i=1}^N (Au_i)(\cdot)(u_i^*), \quad (3.0.3)$$

where $\{u_1, u_2, \dots, u_N\}$ is a basis for $L^2(\mathcal{M})$ such that $m^*(\mathbb{1}) = \sum_{i=1}^N u_i \otimes u_i^*$.

The expression for P_p in (3.0.3) can be reformulated as $P(\cdot) = \frac{1}{\delta^2}m(A \otimes (\cdot))m^*$. To see this, take any $X \in B(L^2(\mathcal{M}))$ and $\xi \in L^2(\mathcal{M})$. Then,

$$\begin{aligned} P_p(X)\xi &= \frac{1}{\delta^2} \sum_{i=1}^N (Au_i)X(u_i^*)\xi = \frac{1}{\delta^2} \sum_{i=1}^N (Au_i)X(u_i^*\xi) \\ &= \frac{1}{\delta^2} \sum_{i=1}^N m(Au_i \otimes X(u_i^*\xi)) \\ &= \frac{1}{\delta^2} \sum_{i=1}^N m(A \otimes X)(u_i \otimes (u_i^*\xi)) \\ &= \frac{1}{\delta^2} m(A \otimes X)(\sum_{i=1}^N u_i \otimes (u_i^*\xi)) \\ &= \frac{1}{\delta^2} m(A \otimes X)(m^*\xi) \\ &= \frac{1}{\delta^2} m(A \otimes X)m^*(\xi) \\ &= P(X)(\xi) \end{aligned}$$

Thus, we get the desired bijective correspondence between linear operators $A : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$, elements $p \in \mathcal{M} \otimes \mathcal{M}^{op}$ and $\mathcal{M}' - \mathcal{M}'$ bimodule linear maps $P : B(L^2(\mathcal{M})) \rightarrow B(L^2(\mathcal{M}))$, with the property that

$$\frac{1}{\delta^2}m(A \otimes A)m^* = A \iff p^2 = p \iff P^2 = P. \quad \square$$

We now introduce the spectrum of a quantum graph as follows:

Definition 3.2. Let \mathcal{M} be a finite dimensional C*-algebra equipped with its tracial δ -form ψ , and let $\mathcal{G} = (S, \mathcal{M}, B(L^2(\mathcal{M}, \psi)))$ be an undirected¹ quantum graph on (\mathcal{M}, ψ) . The *spectrum* of \mathcal{G} is defined to be the spectrum of the quantum adjacency operator A , defined by

¹ Here, undirected simply means S is closed under adjoint. Alternately, P_S is *-preserving.

$$A = \delta^2(\psi \otimes I)P_S(I \otimes \mathbb{1}), \quad (3.0.4)$$

where P_S is the orthogonal $\mathcal{M}' - \mathcal{M}'$ bimodule projection onto S .

Note that A is *self-adjoint* and so, the spectrum of an undirected quantum graph is real.

Convention 1. For the remainder of this paper, \mathcal{M} denotes a finite dimensional C^* -algebra equipped with its *tracial* δ -form ψ , as given in 2.10. We assume that our quantum graph $(S, \mathcal{M}, B(L^2(\mathcal{M}, \psi)))$ is irreflexive. Further, A always refers to the unique self-adjoint quantum adjacency matrix associated with S . We denote this quantum graph by $\mathcal{G} = (\mathcal{M}, \psi, A, S)$.

We now show the connection between quantum adjacency matrix and quantum graph coloring by generalizing some algebraic results in [14] to the quantum graph setting. The following lemma proves that “pinching” operation annihilates the quantum adjacency matrix and leaves the commutant of the quantum vertex set invariant.

Lemma 3.3. *Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph. If $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ is an arbitrary c -coloring of \mathcal{G} in the sense of Definition 2.15, then*

$$\sum_{k=1}^c P_k(A \otimes I_{\mathcal{N}})P_k = 0, \quad (3.0.5)$$

$$\sum_{k=1}^c P_k(E \otimes I_{\mathcal{N}})P_k = E \otimes I_{\mathcal{N}}, \quad \forall E \in \mathcal{M}'. \quad (3.0.6)$$

Proof. We first show that $A \in S$. Recall that A is given by (3.0.4), using the orthogonal $\mathcal{M}' - \mathcal{M}'$ bimodule projection P_S onto S . By Proposition 3.1, $P_S(\cdot) = \delta^{-2}m(A \otimes (\cdot))m^*$. Since $P_S^2 = P_S$, in particular $P_S(A) = \delta^{-2}m(A \otimes A)m^* = A$ by Proposition 3.1. So, $A \in \text{range}(P_S) = S$. Now, by (2.5.1), we get that $\sum_{k=1}^c P_k(A \otimes I_{\mathcal{N}})P_k = 0$. Equation (3.0.6) follows from the fact that the projections $P_k \in \mathcal{M} \otimes \mathcal{N}$ commute with $E \otimes I_{\mathcal{N}} \in \mathcal{M}' \otimes \mathcal{N}'$, and $\sum_{k=1}^c P_k = I_{\mathcal{M} \otimes \mathcal{N}}$. \square

The next lemma is a corresponding result for the “twirling” operation.

Lemma 3.4. *Suppose $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ is an irreflexive quantum graph and $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ is a c -coloring of \mathcal{G} in the sense of Definition 2.15. Define $U := \sum_{l=1}^c \omega^l P_l$, where $\omega = e^{2\pi i/c}$ is a c^{th} root of unity. Then,*

$$\sum_{k=1}^c P_k(X \otimes I_{\mathcal{N}})P_k = \frac{1}{c} \sum_{k=1}^c U^k (X \otimes I_{\mathcal{N}})(U^*)^k, \quad \forall X \in B(L^2(\mathcal{M})). \quad (3.0.7)$$

In particular,

$$\sum_{k=1}^c U^k (A \otimes I_{\mathcal{N}}) (U^*)^k = 0, \quad (3.0.8)$$

$$\sum_{k=1}^c U^k (E \otimes I_{\mathcal{N}}) (U^*)^k = c (E \otimes I_{\mathcal{N}}), \quad \forall E \in \mathcal{M}'. \quad (3.0.9)$$

Proof. Note that $U^* = \sum_{l=1}^c \omega^{-l} P_l$ since $\{P_l\}_{l=1}^c$ are self-adjoint. Also, the k^{th} power of U is given by

$$U^k = \sum_{l=1}^c \omega^{lk} P_l$$

as the projections $\{P_l\}_{l=1}^c$ are mutually orthogonal, that is $P_i P_j = 0$ if $i \neq j$. Now, for $X \in B(L^2(\mathcal{M}))$, we obtain:

$$\begin{aligned} \sum_{k=1}^c U^k (X \otimes I_{\mathcal{N}}) (U^*)^k &= \sum_{k=1}^c \sum_{l,l'=1}^c \omega^{(l-l')k} P_l (X \otimes I_{\mathcal{N}}) P_{l'} \\ &= \sum_{l,l'=1}^c \left(\sum_{k=1}^c \omega^{(l-l')k} \right) P_l (X \otimes I_{\mathcal{N}}) P_{l'} \\ &= \sum_{l,l'=1}^c (c \delta_{l,l'}) P_l (X \otimes I_{\mathcal{N}}) P_{l'}, \\ &\quad \text{where } \delta_{l,l'} \text{ denotes the Krönecker delta} \\ &= c \sum_{l=1}^c P_l (X \otimes I_{\mathcal{N}}) P_l \end{aligned}$$

Hence, we get the result. The rest follows from Lemma 3.3. \square

Next, we note some obvious properties of $A \otimes I_{\mathcal{N}}$ for future reference.

Proposition 3.5. Suppose $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ is an irreflexive quantum graph and $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ is an arbitrary c -quantum coloring of \mathcal{G} . Assume that $2 \leq \dim(\mathcal{M}) < \infty$ and $\mathcal{N} \subseteq B(\mathcal{H})$ for some Hilbert space \mathcal{H} , with $\dim(\mathcal{H}) = d$.

Define $\tilde{A} = A \otimes I_{\mathcal{N}}$. Then

1. \tilde{A} is self-adjoint and has real eigenvalues.
2. The spectrum of \tilde{A} has the same elements as the spectrum of A , but each with a multiplicity of d . In particular, the largest and smallest eigenvalue of \tilde{A} coincide with the largest and smallest eigenvalue of A , respectively.
3. $\tilde{A} = \sum_{a,b=1}^c P_a \tilde{A} P_b$.

4. \tilde{A} can be expressed as a block partitioned matrix

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \dots & \hat{A}_{1c} \\ \hat{A}_{21} & \hat{A}_{22} & \dots & \hat{A}_{2c} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{A}_{c1} & \hat{A}_{c2} & \dots & \hat{A}_{cc} \end{bmatrix},$$

$$\hat{A}_{ii} = 0 \text{ for all } i \in [c]. \text{ In particular, } \text{Tr}(A) = \frac{1}{d} \text{Tr}(\tilde{A}) = 0.$$

Proof. The first two statements are evident since A is self-adjoint and tensoring with identity only produces more copies of the same eigenvalues. The third statement follows from the fact that $\sum_{k=1}^c P_k = I_{\mathcal{M} \otimes \mathcal{N}}$.

To see the last statement, note that \tilde{A} can be interpreted as a giant matrix over complex numbers as \mathcal{M} and \mathcal{N} are finite dimensional. Choose an orthonormal basis for $L^2(\mathcal{M}) \otimes \mathcal{H}$ such that all the projections P_k are represented as diagonal matrices. Identify \hat{A}_{ab} with the matrix $P_a \tilde{A} P_b$. Then, we get the desired block partition. From (3.0.5), it follows that $\hat{A}_{ii} = 0$ for $1 \leq i \leq c$. \square

4. Spectral lower bounds for $\chi_q(\mathcal{G})$ and $\chi(\mathcal{G})$

In this section, we obtain spectral lower bounds for the quantum chromatic number of quantum graphs, generalizing results from [14]. Since $\chi_{qc}(\mathcal{G}) \leq \chi_q(\mathcal{G}) \leq \chi(\mathcal{G})$ [5], our estimates are also lower bounds for the classical chromatic number of quantum graphs. Our spectral bounds for an undirected quantum graph $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ can be summarized as follows:

$$\begin{aligned} 1 + \max \left\{ \frac{\lambda_{\max}}{|\lambda_{\min}|}, \frac{\dim(S)}{\dim(S) - \dim(\mathcal{M})\gamma_{\min}}, \frac{s^{\pm}}{s^{\mp}}, \frac{n^{\pm}}{n^{\mp}}, \frac{\lambda_{\max}}{\lambda_{\max} - \gamma_{\max} + \theta_{\max}} \right\} \\ \leq \chi_{qc}(\mathcal{G}) \leq \chi_q(\mathcal{G}) \leq \chi(\mathcal{G}). \end{aligned} \quad (4.0.1)$$

Here, $\lambda_{\max}, \lambda_{\min}$ denote the maximum and minimum eigenvalues of A ; s^+, s^- denote the sum of the squares of the positive and negative eigenvalues of A respectively; n^+, n^- are the number of positive and negative eigenvalues of A including multiplicities; $\gamma_{\max}, \gamma_{\min}$ denote the maximum and minimum eigenvalues of the signless Laplacian operator (Definition 4.6); and θ_{\max} denotes the maximum eigenvalue of the Laplacian operator (Definition 4.6).

The key ingredient in proving these bounds is Lemma 3.3 and 3.4, which holds true across the classical (*loc*), quantum (*q*) and quantum-commuting (*qc*) coloring models. With these, the proof of the corresponding bounds for classical graphs can essentially be adapted to our setting. For the sake of concreteness, we prove our results in the quantum coloring framework ($\dim(\mathcal{N}) < \infty$). However, by using an infinite dimensional version of Proposition 3.5 ($\dim(\mathcal{N}) = \infty$), all the bounds can be directly transferred to the quantum commuting chromatic numbers as well.

Throughout our discussion, we follow convention 1. So, A always refers to the unique self-adjoint quantum adjacency matrix associated with $(S, \mathcal{M}, B(L^2(\mathcal{M}, \psi)))$, as in (3.0.4).

4.1. Hoffman's bound

One of the well-known spectral bounds in graph theory is the Hoffman's bound [16]. This is a lower bound on the chromatic number of a graph using the largest and smallest eigenvalues of the adjacency matrix. The classical bound is as follows: If G is an irreflexive classical graph whose adjacency matrix A has eigenvalues $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = \lambda_{\min}$, then

$$1 + \frac{\lambda_{\max}}{|\lambda_{\min}|} \leq \chi(G). \quad (4.1.1)$$

We can prove a quantum version of this bound using the following result from linear algebra.

Lemma 4.1. *Let A be a self-adjoint matrix, block partitioned as*
$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}.$$
 Then,

$$(n-1)\lambda_{\min}(A) + \lambda_{\max}(A) \leq \sum_{i=1}^n \lambda_{\max}(A_{ii}),$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and minimum eigenvalues of that matrix.

Proof. We start with the case $n = 2$.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a normalized eigenvector ($\|x_1\|^2 + \|x_2\|^2 = 1$) corresponding to $\lambda_{\max}(A)$. Define $y = \begin{bmatrix} \frac{\|x_2\|}{\|x_1\|}x_1 \\ -\frac{\|x_1\|}{\|x_2\|}x_2 \end{bmatrix}$. Then, we have

$$\begin{aligned} \lambda_{\max}(A) + \lambda_{\min}(A) &\leq \langle x | A | x \rangle + \langle y | A | y \rangle = \frac{\langle x_1 | A_{11} | x_1 \rangle}{\|x_1\|^2} + \frac{\langle x_2 | A_{22} | x_2 \rangle}{\|x_2\|^2} \\ &\leq \lambda_{\max}(A_{11}) + \lambda_{\max}(A_{22}). \end{aligned}$$

The general case follows by induction on n . \square

The generalization of Hoffman's bound to quantum graphs is as follows:

Theorem 4.2. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph and $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\dim(\mathcal{M})} = \lambda_{\min}$ be all the eigenvalues of A . Then

$$1 + \frac{\lambda_{\max}}{|\lambda_{\min}|} \leq \chi_q(\mathcal{G}). \quad (4.1.2)$$

Proof. Let $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ be a c -quantum coloring of \mathcal{G} and $\tilde{A} = A \otimes I_{\mathcal{N}}$. Partition \tilde{A} as $[\tilde{A}_{ab}]_{a,b=1}^c$, as in Proposition 3.5. Applying Lemma 4.1, we get

$$(c-1)\lambda_{\min}(\tilde{A}) + \lambda_{\max}(\tilde{A}) \leq \sum_{i=1}^c \lambda_{\max}(\tilde{A}_{ii}). \quad (4.1.3)$$

But $\tilde{A}_{ii} = 0$ for all $1 \leq i \leq c$. Hence equation (4.1.3) reduces to

$$(c-1)\lambda_{\min}(\tilde{A}) + \lambda_{\max}(\tilde{A}) \leq 0.$$

By Proposition 3.5, we have $\lambda_{\min}(\tilde{A}) = \lambda_{\min}(A)$ and $\lambda_{\max}(\tilde{A}) = \lambda_{\max}(A)$. So, we get $(c-1)\lambda_{\min}(A) + \lambda_{\max}(A) \leq 0$. On rearranging and taking minimum over all c , we get

$$1 + \frac{\lambda_{\max}(A)}{|\lambda_{\min}(A)|} \leq \chi_q(\mathcal{G}). \quad \square$$

4.2. Lower bound using edge number

In this section, we prove a spectral lower bound on the quantum chromatic number using a quantum analogue for the number of edges in the graph.

For a classical graph G with n vertices and m edges, it was shown [9] that

$$1 + \frac{2m}{2m - n\gamma_{\min}} \leq \chi(G), \quad (4.2.1)$$

where γ_{\min} is the minimum eigenvalue of the signless Laplacian of G . To prove a generalization of this bound to arbitrary quantum graphs $(\mathcal{M}, \psi, A, S)$, we first introduce a quantum analogue for m, n and γ_{\min} .

Recall that the degree matrix for classical graphs is a diagonal matrix obtained from the action of the adjacency matrix on the all 1s vector. This can be extended to quantum graphs as follows:

Definition 4.3. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be a quantum graph and $\mathbb{1}$ denote the unit in \mathcal{M} . Then the *quantum degree matrix* of \mathcal{G} is a linear operator $D \in B(L^2(\mathcal{M}))$ given by

$$D : \mathcal{M} \longrightarrow \mathcal{M} \text{ as } x \mapsto x(A\mathbb{1}), \forall x \in \mathcal{M}.$$

In other words, D can be interpreted as $A\mathbb{1} \in \mathcal{M}$ viewed as an element of $B(L^2(\mathcal{M}))$ under the *right* regular representation.

Remark 4.4. The definition $D = A\mathbb{1}$ was also used in [7] and [21]. The only difference in our case is that we view D under the *right* regular representation, instead of the usual left regular representation of \mathcal{M} . The advantage of using right regular representation is that D then belongs to \mathcal{M}' .

Our next goal is to define a quantum analogue for the “number of edges” in the graph. To do that, we need the following result:

Proposition 4.5. *Let \mathcal{M} be a finite dimensional C^* -algebra, equipped with its tracial δ -form ψ . If $(\mathcal{M}, \psi, A, S)$ is a quantum graph with degree matrix D , then,*

$$\mathrm{Tr}(D) = \delta^2 \psi(A\mathbb{1}) = \dim(S). \quad (4.2.2)$$

Proof. Let $P_S : B(L^2(\mathcal{M})) \rightarrow B(L^2(\mathcal{M}))$ denote the orthogonal bimodule projection onto S . We can express P_S as an element $\sum_{i=1}^t x_i \otimes y_i^{op} \in \mathcal{M} \otimes \mathcal{M}^{op}$, such that $P_S(a \otimes b) = \sum_{i=1}^t x_i a \otimes b y_i$, for all $a, b \in \mathcal{M}$ using the correspondence mentioned in Remark 2.14. Now, $A = \delta^2(\psi \otimes I)P_S(I \otimes \eta)$ implies

$$A(\mathbb{1}) = \delta^2(\psi \otimes I)P_S(\mathbb{1} \otimes \mathbb{1}) = \delta^2(\psi \otimes I)\left(\sum_{i=1}^t x_i \otimes y_i\right) = \delta^2 \sum_{i=1}^t \psi(x_i)y_i. \quad (4.2.3)$$

Thus,

$$\begin{aligned} \psi(A\mathbb{1}) &= \psi(\delta^2 \sum_{i=1}^t \psi(x_i)y_i) = \delta^2 \sum_{i=1}^t \psi(x_i)\psi(y_i) \\ &= \delta^2 \sum_{i=1}^t \langle x_i, \mathbb{1} \rangle \langle y_i, \mathbb{1} \rangle \\ &= \delta^2 \sum_{i=1}^t \langle x_i \otimes y_i, \mathbb{1} \otimes \mathbb{1} \rangle = \delta^2 \left\langle \sum_{i=1}^t x_i \otimes y_i, \mathbb{1} \otimes \mathbb{1} \right\rangle \\ &= \delta^2 \langle P_S, I \rangle, \text{ when viewed as operators on } B(L^2(\mathcal{M})) \\ &= \delta^2 \frac{\mathrm{Tr}(P_S)}{\dim(B(L^2(\mathcal{M})))} \\ &= \dim(\mathcal{M}) \frac{\dim(S)}{\dim(\mathcal{M})^2} = \frac{\dim(S)}{\dim(\mathcal{M})} \end{aligned}$$

where we have used the fact that ψ is a tracial state and $\delta^2 = \dim(\mathcal{M})$. Also, the trace on $B(L^2(\mathcal{M}))$ restricted to \mathcal{M} (or \mathcal{M}' by symmetry) is just $\dim(\mathcal{M})\psi$. So,

$$\mathrm{Tr}(D) = \dim(\mathcal{M}) \psi(A\mathbb{1}).$$

Hence, $\mathrm{Tr}(D) = \dim(S)$. \square

We now define quantum analogues of some classical quantities:

Definition 4.6. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph with degree matrix D .

1. The *quantum vertex number* for \mathcal{G} is defined to be $\dim(\mathcal{M})$.
2. The *quantum edge number* for \mathcal{G} is defined to be $\frac{\text{Tr}(D)}{2} = \frac{\dim(S)}{2}$.
3. The *Laplacian* of \mathcal{G} is the linear operator $L = D - A \in B(L^2(\mathcal{M}))$.
4. The *signless Laplacian* of \mathcal{G} is the linear operator $Q = D + A \in B(L^2(\mathcal{M}))$.

For a classical irreflexive graph $G = (V, E)$, these definitions clearly coincide with the usual values. In particular, if $\mathcal{G} = (S_G, D_{|V|}, M_{|V|})$, then the quantum vertex number is $|V|$ and the quantum edge number is $|E|$ since $2|E| = \sum_{v \in V} \deg(v) = \text{Tr}(D)$.

Remark 4.7. The quantum edge number need not be an integer in general. But for most purposes, we will only need $2m = \text{Tr}(D) = \dim(S)$.

We are now ready to prove a quantum version of the spectral bound in (4.2.1).

Theorem 4.8. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph. Then

$$1 + \frac{2m}{2m - n\gamma_{\min}} \leq \chi_q(\mathcal{G}), \quad (4.2.4)$$

where m is the quantum edge number, n is the quantum vertex number and γ_{\min} is the minimum eigenvalue of the signless Laplacian of \mathcal{G} , in the sense of Definition 4.6. More precisely,

$$1 + \frac{\dim(S)}{\dim(S) - \dim(\mathcal{M})\gamma_{\min}} \leq \chi_q(\mathcal{G}). \quad (4.2.5)$$

Proof. Let $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ be a c -quantum coloring of \mathcal{G} and let U be defined as in Lemma 3.4. Then, (3.0.8) can be rearranged as $U^c(A \otimes I_{\mathcal{N}})(U^*)^c = -\sum_{k=1}^{c-1} U^k(A \otimes I_{\mathcal{N}})(U^*)^k$. Using $D - Q = -A$ and $U^c = I_{\mathcal{M} \otimes \mathcal{N}}$, we get

$$\begin{aligned} A \otimes I_{\mathcal{N}} &= \sum_{k=1}^{c-1} U^k((D - Q) \otimes I_{\mathcal{N}})(U^*)^k \\ &= \sum_{k=1}^{c-1} U^k(D \otimes I_{\mathcal{N}})(U^*)^k - \sum_{k=1}^{c-1} U^k(Q \otimes I_{\mathcal{N}})(U^*)^k \\ &= (D \otimes I_{\mathcal{N}}) \sum_{k=1}^{c-1} U^k(U^*)^k - \sum_{k=1}^{c-1} U^k(Q \otimes I_{\mathcal{N}})(U^*)^k \end{aligned}$$

$$= (c-1)(D \otimes I_{\mathcal{N}}) - \sum_{k=1}^{c-1} U^k (Q \otimes I_{\mathcal{N}})(U^*)^k$$

where we have used the fact that $D \in \mathcal{M}'$ and hence $D \otimes I_{\mathcal{N}}$ commutes with $U \in \mathcal{M} \otimes \mathcal{N}$. Let \mathcal{N} be represented in some $B(\mathcal{H})$ and let u denote a unit vector in \mathcal{H} such that $\langle u, u \rangle = 1$. Further, let $|\xi\rangle = \mathbb{1} \otimes u$ denote a column vector in $L^2(\mathcal{M}) \otimes \mathcal{H}$ and $\langle \xi|$ denote its corresponding conjugate row vector. Multiplying the left and right most sides of the above equation by $\langle \xi|$ from the left and by $|\xi\rangle$ from the right, we obtain

$$\langle \xi| A \otimes I_{\mathcal{N}} |\xi\rangle = (c-1) \langle \xi| D \otimes I_{\mathcal{N}} |\xi\rangle - \sum_{k=1}^{c-1} \langle \xi| U^k (Q \otimes I_{\mathcal{N}})(U^*)^k |\xi\rangle. \quad (4.2.6)$$

Now, $\langle \xi| A \otimes I_{\mathcal{N}} |\xi\rangle = \langle \mathbb{1}, A\mathbb{1} \rangle \langle u, u \rangle = \psi((A\mathbb{1})^*) = \psi(A\mathbb{1}) = \frac{\dim(S)}{\dim(\mathcal{M})}$, where we use the $*$ -preserving property of A (Remark 2.12) and Proposition 4.5. Similarly, $\langle \xi| D \otimes I_{\mathcal{N}} |\xi\rangle = \frac{\dim(S)}{\dim(\mathcal{M})}$. To estimate the last term, recall that eigenvalues are invariant under unitary conjugation and tensoring with identity only changes their multiplicity. So,

$$\begin{aligned} \gamma_{\min} &= \min \{ \langle w| Q |w\rangle : w \in L^2(\mathcal{M}), \langle w, w \rangle = 1 \} \\ &= \min \{ \langle v| Q \otimes I_{\mathcal{N}} |v\rangle : v \in L^2(\mathcal{M}) \otimes \mathcal{H}, \langle v, v \rangle = 1 \} \\ &= \min \{ \langle v| U^k (Q \otimes I_{\mathcal{N}})(U^*)^k |v\rangle : v \in L^2(\mathcal{M}) \otimes \mathcal{H}, \langle v, v \rangle = 1 \} \\ &\leq \langle \xi| U^k (Q \otimes I_{\mathcal{N}})(U^*)^k |\xi\rangle, \quad \forall k \in [c]. \end{aligned}$$

Hence, (4.2.6) leads to

$$\frac{\dim(S)}{\dim(\mathcal{M})} \leq (c-1) \frac{\dim(S)}{\dim(\mathcal{M})} - (c-1)\gamma_{\min}, \quad (4.2.7)$$

which upon rearranging yields $1 + \frac{\dim(S)}{\dim(S) - \dim(\mathcal{M})\gamma_{\min}} \leq c$. Taking minimum over all c , we get the desired bound. \square

4.3. Bound using the sum of square of eigenvalues

In [1], it was proved that for a classical graph G ,

$$1 + \max \left\{ \frac{s^+}{s^-}, \frac{s^-}{s^+} \right\} \leq \chi(G), \quad (4.3.1)$$

where s^+ is the sum of the squares of the positive eigenvalues of the adjacency matrix and s^- is the sum of the squares of its negative eigenvalues. In this section, we show that the above bound also works in the setting of quantum graphs. We first recall the following result from linear algebra, whose proof can be found in [1].

Lemma 4.9. Let $X = [X_{ij}]_{i,j}^r$ and $Y = [Y_{ij}]_{i,j}^r$ be two positive semidefinite matrices conformally partitioned. If $X_{ii} = Y_{ii}$ for $1 \leq i \leq r$ and $XY = 0$, then $\text{Tr}(X^*X) \leq (r-1)\text{Tr}(Y^*Y)$.

We now adapt the proof of the classical bound in [1] to the quantum case.

Theorem 4.10. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\dim(\mathcal{M})}$ be all the eigenvalues of A . Let $s^+ = \sum_{\lambda_i > 0} (\lambda_i)^2$ and $s^- = \sum_{\lambda_i < 0} (\lambda_i)^2$. Then,

$$1 + \max \left\{ \frac{s^+}{s^-}, \frac{s^-}{s^+} \right\} \leq \chi_q(\mathcal{G}). \quad (4.3.2)$$

Proof. Let $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ be a c -quantum coloring of \mathcal{G} . Further, let $\tilde{A} = A \otimes I_{\mathcal{N}}$ and let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_t$ be all the eigenvalues of \tilde{A} . Consider a spectral decomposition of \tilde{A} ,

$$\tilde{A} = \sum_{i=1}^t \mu_i (v_i v_i^*), \text{ where } v_i \in L^2(\mathcal{M}) \otimes \mathcal{H}, \quad (4.3.3)$$

and write $\tilde{A} = \tilde{B} - \tilde{C}$, where

$$\tilde{B} = \sum_{\mu_i > 0} \mu_i (v_i v_i^*) \quad \tilde{C} = \sum_{\mu_i < 0} -\mu_i (v_i v_i^*). \quad (4.3.4)$$

Suppose $N \subseteq B(\mathcal{H})$ for some Hilbert space \mathcal{H} . Then,

$$\text{Tr}(\tilde{B}^* \tilde{B}) = \sum_{\mu_i > 0} \mu_i^2 = \dim(\mathcal{H}) s^+ \text{ and } \text{Tr}(\tilde{C}^* \tilde{C}) = \sum_{\mu_i < 0} \mu_i^2 = \dim(\mathcal{H}) s^-. \quad (4.3.5)$$

Partition \tilde{A} as $[\hat{A}_{ab}]_{a,b=1}^c$ as in Proposition 3.5. Similarly, let

$$\tilde{B} = [\hat{B}_{ab}]_{a,b=1}^c = \sum_{a,b=1}^c P_a \tilde{B} P_b \text{ and } \tilde{C} = [\hat{C}_{ab}]_{a,b=1}^c = \sum_{a,b=1}^c P_a \tilde{C} P_b.$$

Now, B and C are positive semidefinite matrices that are conformally partitioned. Further, $\hat{B}_{ii} = \hat{C}_{ii}$ since $0 = P_i \tilde{A} P_i = P_i \tilde{B} P_i - P_i \tilde{C} P_i$ for all $1 \leq i \leq c$. Also $\tilde{B} \tilde{C} = \tilde{C} \tilde{B} = 0$. So, by Lemma 4.9 and (4.3.5), it follows that $\frac{s^+}{s^-} \leq c-1$ and $\frac{s^-}{s^+} \leq c-1$. Taking minimum over all c , we get $1 + \max \left\{ \frac{s^+}{s^-}, \frac{s^-}{s^+} \right\} \leq \chi_q(\mathcal{G})$. \square

4.4. Inertial lower bound

In this section, our goal is to generalize the following inertial bound [13] to quantum graphs:

$$1 + \max \left\{ \frac{n^+}{n^-}, \frac{n^-}{n^+} \right\} \leq \chi(G), \quad (4.4.1)$$

where (n^+, n^0, n^-) is the inertia of G . We begin with defining the inertia of a quantum graph:

Definition 4.11. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be a quantum graph and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\dim(\mathcal{M})}$ denote the eigenvalues of A . The *inertia* of \mathcal{G} is the ordered triple (n^+, n^0, n^-) , where n^+ , n^0 and n^- are the numbers of positive, zero and negative eigenvalues of A including multiplicities.

Theorem 4.12. Let $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ be an irreflexive quantum graph with inertia (n^+, n^0, n^-) . Then,

$$1 + \max \left\{ \frac{n^+}{n^-}, \frac{n^-}{n^+} \right\} \leq \chi_q(\mathcal{G}). \quad (4.4.2)$$

Proof. Let $\{P_k\}_{k=1}^c \subseteq \mathcal{M} \otimes \mathcal{N}$ be a c -quantum coloring of \mathcal{G} . Let U be defined as in Lemma 3.4 and \tilde{A} , \tilde{B} and \tilde{C} be defined as in the proof of Theorem 4.10. Then, we have

$$\sum_{k=1}^{c-1} U^k \tilde{B}(U^*)^k - \sum_{k=1}^{c-1} U^k \tilde{C}(U^*)^k = \sum_{k=1}^{c-1} U^k \tilde{A}(U^*)^k = -\tilde{A} = \tilde{C} - \tilde{B}. \quad (4.4.3)$$

Note that \tilde{B} and \tilde{C} are positive definite operators with $\text{rank}(\tilde{B}) = n^+$ and $\text{rank}(\tilde{C}) = n^-$. Further let

$$P^+ = \sum_{\mu_i > 0} v_i v_i^* \text{ and } P^- = \sum_{\mu_i < 0} v_i v_i^*$$

denote the orthogonal projectors onto the subspaces spanned by the eigenvectors corresponding to the positive and negative eigenvalues of \tilde{A} respectively. Observe that $\tilde{B} = P^+ \tilde{A} P^+$ and $\tilde{C} = -P^- \tilde{A} P^-$. Multiplying (4.4.3) by P^- on both sides, we obtain:

$$P^- \sum_{k=1}^{c-1} U^k \tilde{B}(U^*)^k P^- - P^- \sum_{k=1}^{c-1} U^k \tilde{C}(U^*)^k P^- = C \quad (4.4.4)$$

Now we use the fact that if X, Y are two positive definite matrices such that $X - Y$ is positive definite, then $\text{rank}(X) \geq \text{rank}(Y)$. By applying this to (4.4.4), we get

$$\text{rank}(P^- \sum_{k=1}^{c-1} U^k \tilde{B}(U^*)^k P^-) \geq \text{rank}(C).$$

Recall that the rank of a sum is less than or equal to the sum of the ranks of the summands, and that the rank of a product is less than or equal to the minimum of the ranks of the factors. So, we get $(c-1)n^+ \geq n^-$. Similarly, it can be shown that $(c-1)n^- \geq n^+$. Hence, $\max \left\{ \frac{n^+}{n^-}, \frac{n^-}{n^+} \right\} \leq c-1$. Taking minimum over all c , we get the desired bound. \square

4.5. Bound using maximum eigenvalue of the Laplacian and signless Laplacian

Let L and Q denote the Laplacian and signless Laplacian of $\mathcal{G} = (\mathcal{M}, \psi, A, S)$ in the sense of Definition 4.6. Further, let $\lambda_{\max}, \theta_{\max}$ and γ_{\max} denote the largest eigenvalue of A, L and Q respectively. Then

$$1 + \frac{\lambda_{\max}}{\lambda_{\max} - \gamma_{\max} + \theta_{\max}} \leq \chi_q(\mathcal{G}). \quad (4.5.1)$$

Like the previous cases, this bound can also be shown by adapting the classical proof [19] and applying Lemma 3.4.

5. Illustration

In this section, we illustrate the tightness of these bounds in the case of complete quantum graphs. Let $K_{\mathcal{M}}$ denote the irreflexive complete quantum graph on (\mathcal{M}, ψ) . The quantum adjacency matrix in this case is given by $A = \delta^2 \psi(\cdot) \mathbb{1} - I$. For $x \in \mathcal{M}$, we have

$$\begin{aligned} A(x) &= \delta^2 \psi(x) \mathbb{1} - I \\ &= (\dim \mathcal{M}) \langle x, \mathbb{1} \rangle \mathbb{1} - I \\ &= (\dim \mathcal{M}) P_{\mathbb{1}}(x) - I, \end{aligned}$$

where $P_{\mathbb{1}} : \mathcal{M} \rightarrow \mathcal{M}$ denotes the orthogonal projection onto $\mathbb{1}$, given by $x \mapsto \langle x, \mathbb{1} \rangle \mathbb{1}$. Since $P_{\mathbb{1}}$ is a rank-1 projection, its spectrum is precisely $\{0, 1\}$, where 0 has a multiplicity of $\dim(\mathcal{M}) - 1$. Using functional calculus, we get

$$\sigma(A) = \{\dim(\mathcal{M}) - 1, -1\}, \quad (5.0.1)$$

where -1 has a multiplicity of $\dim(\mathcal{M}) - 1$. Similarly, we get

$$\sigma(Q) = \{2 \dim(\mathcal{M}) - 2, \dim(\mathcal{M}) - 2\}, \quad (5.0.2)$$

where $\dim(\mathcal{M}) - 2$ has a multiplicity of $\dim(\mathcal{M}) - 1$, and

$$\sigma(L) = \{\dim(\mathcal{M}), 0\}, \quad (5.0.3)$$

where $\dim(\mathcal{M})$ has a multiplicity of $\dim(\mathcal{M}) - 1$.

Thus, for an irreflexive complete quantum graph, we have:

- $\lambda_{\max} = \dim \mathcal{M} - 1, \lambda_{\min} = -1$
- $\gamma_{\max} = 2\dim(\mathcal{M}) - 2, \gamma_{\min} = \dim \mathcal{M} - 2$
- $\theta_{\max} = \dim \mathcal{M}$
- $s^+ = (\dim(\mathcal{M}) - 1)^2, s^- = \dim(\mathcal{M}) - 1$
- $n^+ = 1, n^- = \dim(\mathcal{M}) - 1$
- $2m = \dim(\mathcal{M})^2 - \dim(\mathcal{M})$

On applying these to Theorem 1.1, we see that all the five spectral bounds give the same result, namely:

$$\dim(\mathcal{M}) \leq \chi_q(K_{\mathcal{M}}). \quad (5.0.4)$$

The reverse inequality $\chi_q(K_{\mathcal{M}}) \leq \dim(\mathcal{M})$ was proved in [5], and $\chi_q(K_{\mathcal{M}}) = \dim(\mathcal{M})$. So, we conclude that all the bounds in Theorem 1.1 are tight in the case of complete quantum graphs.

6. Conclusion and future directions

In this work, we have shown that several spectral lower bounds for the chromatic number of classical graphs are also lower bounds for the classical, quantum and quantum-commuting chromatic numbers of quantum graphs. We believe that quantum graph spectral theory is a promising field of study. As a next step, it would be interesting to find bounds that exhibit a separation between the different variants of chromatic numbers of quantum graphs. Alternatively, investigating examples of quantum graphs that show a separation between these spectral bounds would also be helpful. We hope to explore these in a future work.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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Appendix A

Let \mathcal{M} be a finite dimensional C^* -algebra, equipped with its tracial δ -form ψ . The properties of a quantum graph on $\mathcal{M} \subseteq B(L^2(\mathcal{M}, \psi))$ in the different perspectives is summarized in the following table.

Here, $p = \sum_{i=1}^t a_i \otimes b_i \in \mathcal{M} \otimes \mathcal{M}^{op}$ and m, σ denote the multiplication map and swap map on $\mathcal{M} \otimes \mathcal{M}^{op}$ respectively. Further, $\mathcal{H} = L^2(\mathcal{M}, \psi)$, $T \in B(L^2(\mathcal{M}))$, $\xi \in L^2(\mathcal{M})$ and $x, y \in \mathcal{M}'$.

PROPERTY	CLASSICAL GRAPH	$S \subseteq B(\mathcal{H})$	$p \in \mathcal{M} \otimes \mathcal{M}^{op}$	$A : \mathcal{M} \rightarrow \mathcal{M}$	$P : B(\mathcal{H}) \rightarrow B(\mathcal{H})$
Bimodule structure	Relations on a set	$\mathcal{M}' S \mathcal{M}' \subseteq S$	$\sum_i a_i (x T y) b_i = x (\sum_i a_i T b_i) y$	$m(A \otimes x T y) m^* = x (m(A \otimes T) m^*) y$	$P(x T y) = x P(T) y$
Schur idempotent	$A \in M_n(\{0, 1\})$	$A \in S$	$p^2 = p$	$m(A \otimes A) m^* = \delta^2 A$	$P^2 = P$
Reflexive	$I \in S_G$	$\mathcal{M}' \subseteq S$	$m(p) = \mathbb{1}$	$m(A \otimes I) m^* = \delta^2 I$	$P(I) = I$
Irreflexive	$I \notin S_G$	$\mathcal{M}' \perp S$	$m(p) = 0$	$m(A \otimes I) m^* = 0$	$P(I) = 0$
Undirected	$A = A^T$	$S = S^*$	$\sigma(p) = p$	$(I \otimes \eta^* m)(I \otimes A \otimes I)$ $(m^* \eta \otimes I) = A$ Alternatively, $A(\xi^*) = [A^*(\xi)]^*$	$P^*(T) = P(T^*)^*$, (* denotes adjoint as an operator on Hilbert spaces)
Self adjoint	$A = A^*$		$\sigma(p) = p^*$	$A(\xi) = A^*(\xi)$	$P(T^*) = P(T)^*$
Real	$A = \bar{A}$		$p^* = p$	$A(\xi^*) = (A(\xi))^*$	$P^*(T) = P(T)$
Positivity	A is C.P		p is positive (i.e. $p = g^* g$)	A is completely positive (C.P)	P is positive (i.e. $P = G^* G$)

In particular, for undirected quantum graphs:

$$\begin{aligned}
 P^2 = P = P^* &\iff p^2 = p = p^* \\
 &\iff A \text{ is Schur-idempotent and real} \\
 &\iff A \text{ is Schur-idempotent and self-adjoint.}
 \end{aligned}$$

References

[1] Tsuyoshi Ando, Minghua Lin, Proof of a conjectured lower bound on the chromatic number of a graph, *Linear Algebra Appl.* 485 (2015) 480–484.

- [2] Teodor Banica, Symmetries of a generic coaction, *Math. Ann.* 314 (4) (1999) 763–780.
- [3] Michael Brannan, Alexandru Chirvasitu, Kari Eifler, Samuel Harris, Vern Paulsen, Xiaoyu Su, Mateusz Wasilewski, Bigalois extensions and the graph isomorphism game, *Commun. Math. Phys.* 375 (3) (2020) 1777–1809.
- [4] Michael Brannan, Kari Eifler, Christian Voigt, Moritz Weber, Quantum cuntz-kiyger algebras, *Trans. Amer. Math. Soc. Ser. B* 9 (26) (2022) 782–826.
- [5] Michael Brannan, Priyanga Ganesan, Samuel J. Harris, The quantum-to-classical graph homomorphism game, *J. Math. Phys.* 63 (11) (2022) 112204.
- [6] Peter J. Cameron, Ashley Montanaro, Michael W. Newman, Simone Severini, Andreas Winter, On the quantum chromatic number of a graph, *Electron. J. Comb.* 14 (1) (2007) 81, 15. MR2365980.
- [7] Alexandru Chirvasitu, Mateusz Wasilewski, Random quantum graphs, *Trans. Am. Math. Soc.* 375 (05) (2022) 3061–3087.
- [8] Matthew Daws, Quantum graphs: different perspectives, homomorphisms and quantum automorphisms, preprint, arXiv:2203.08716, 2022.
- [9] Leonardo Silva de Lima, Carla Silva Oliveira, Nair Maria Maia de Abreu, Vladimir Nikiforov, The smallest eigenvalue of the signless Laplacian, *Linear Algebra Appl.* 435 (10) (2011) 2570–2584. MR2811139.
- [10] Runyao Duan, Simone Severini, Andreas Winter, Zero-error communication via quantum channels, noncommutative graphs, and a quantum lovász number, *IEEE Trans. Inf. Theory* 59 (2) (2012) 1164–1174.
- [11] J.A. Erdos, A. Katavolos, V.S. Shulman, Rank one subspaces of bimodules over maximal abelian selfadjoint algebras, *J. Funct. Anal.* 157 (2) (1998) 554–587.
- [12] E.G. Effros, Zh-J. Ruan, *Operator Spaces*, The Clandon Press, vol. 23, Oxford University Press, New York, 2000.
- [13] Clive Elphick, Paweł Wocjan, An inertial lower bound for the chromatic number of a graph, *Electron. J. Comb.* 24 (1) (2017) 1.58, 9. MR3651940.
- [14] Clive Elphick, Paweł Wocjan, Spectral lower bounds for the quantum chromatic number of a graph, *J. Comb. Theory, Ser. A* 168 (2019) 338–347.
- [15] Daniel Gromada, Some examples of quantum graphs, *Lett. Math. Phys.* 112 (6) (2022) 122.
- [16] Alan J. Hoffman, On eigenvalues and colorings of graphs, in: *Graph Theory and Its Applications*, Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969, 1970, pp. 79–91. MR0284373.
- [17] Matthew Kennedy, Taras Kolomatski, Daniel Spivak, An infinite quantum Ramsey theorem, *J. Oper. Theory* 84 (1) (2020) 49–65. MR4157353.
- [18] Se-Jin Kim, Arthur Mehta, Chromatic numbers, Sabidussi’s theorem and Hedetniemi’s conjecture for non-commutative graphs, *Linear Algebra Appl.* 582 (2019) 291–309. MR3992428.
- [19] L. Kolotilina, Inequalities for the extreme eigenvalues of block-partitioned hermitian matrices with applications to spectral graph theory, *J. Math. Sci.* 176 (1) (2011).
- [20] Larissa Kroell, Quantum graphs, 2021, Master’s thesis.
- [21] Junichiro Matsuda, Classification of quantum graphs on m_2 and their quantum automorphism groups, preprint, arXiv:2110.09085, 2021.
- [22] Laura Mančinska, David E. Roberson, Oddities of quantum coloring, *Baltic J. Modern Comput.* 36 (4) (2016) 846–859.
- [23] Benjamin Musto, David Reutter, Dominic Verdon, A compositional approach to quantum functions, *J. Math. Phys.* 59 (8) (2018) 081706.
- [24] Carlos M. Ortiz, Vern I. Paulsen, Quantum graph homomorphisms via operator systems, *Linear Algebra Appl.* 497 (2016) 23–43.
- [25] M. Panos Pardalos, Thelma Mavridou, Jue Xue, The graph coloring problem: a bibliographic survey, in: *Handbook of Combinatorial Optimization*, 1998, pp. 1077–1141.
- [26] Vern I. Paulsen, Simone Severini, Daniel Stahlke, Ivan G. Todorov, Andreas Winter, Estimating quantum chromatic numbers, *J. Funct. Anal.* 270 (6) (2016) 2188–2222. MR3460238.
- [27] Vern I. Paulsen, Ivan G. Todorov, Quantum chromatic numbers via operator systems, *Q. J. Math.* 66 (2) (2015) 677–692. MR3356844.
- [28] Giannicola Scarpa, Simone Severini, Kochen-Specker sets and the rank-1 quantum chromatic number, *IEEE Trans. Inf. Theory* 58 (4) (2012) 2524–2529. MR2934082.
- [29] Dan Stahlke, Quantum zero-error source-channel coding and non-commutative graph theory, *IEEE Trans. Inf. Theory* 62 (1) (2016) 554–577. MR3447998.
- [30] Ivan G. Todorov, Lyudmila Turowska, Quantum no-signalling correlations and non-local games, preprint, arXiv:2009.07016, 2020.

- [31] Nik Weaver, Quantum relations, *Mem. Am. Math. Soc.* 215 (v-vi) (2012) 81–140.
- [32] Nik Weaver, A “quantum” ramsey theorem for operator systems, *Proc. Am. Math. Soc.* 145 (11) (2017) 4595–4605.
- [33] Nik Weaver, Quantum graphs as quantum relations, *J. Geom. Anal.* (2021) 1–23.