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Structural Target Controllability of Undirected Networks

Jingqi Li, Ximing Chen, Sérgio Pequito, George J. Pappas, and Victor M. Preciado

Abstract—**In this paper, we study the *target controllability problem* of networked dynamical systems, in which we are tasked to steer a subset of network states towards a desired objective. More specifically, we derive necessary and sufficient conditions for the *structural target controllability problem* of linear time-invariant (LTI) systems with symmetric state matrices, such as undirected dynamical networks with unknown link weights. To achieve our goal, we first characterize the generic rank of *symmetrically structured matrices*, as well as the modes of any numerical realization. Subsequently, we provide a graph-theoretic necessary and sufficient condition for the structural controllability of undirected networks with multiple control nodes. Finally, we derive a graph-theoretic necessary and sufficient condition for structural target controllability of undirected networks. Remarkably, apart from the standard reachability condition, only local topological information is needed for the verification of structural target controllability.**

I. Introduction

Complex networks have been shown to be a powerful tool for modeling dynamical systems [1–3]. In particular, when analyzing and designing networked systems, it is crucial to verify their controllability, i.e., a property ensuring the existence of an input sequence allowing us to drive the states of the system towards arbitrary states within finite time. Nonetheless, verifying such a property requires full knowledge of the parameters describing the system's dynamics [4]. In applications involving large-scale networks, those parameters are difficult, or even impossible, to obtain [5]. Alternatively, it is more viable to identify the presence of dynamical interconnections among the states of a network. Subsequently, it is of interest to analyze system properties such as controllability using topological information of the system dynamics, which led to the development of system analysis tools using graph theory [6].

Seminal work on graph-theoretic analysis of controllability can be found in [7], in which the notion of *structural controllability* was stated. Following this seminal work, the authors in [8–11] provided necessary and sufficient conditions for structural controllability of multi-input linear time-invariant (LTI) systems using various graph-theoretic notions. Nonetheless, existing results on structural controllability assumed implicitly that the parameters are either fixed zeros or independent free variables. Such an assumption is often violated in practical scenarios, for instance, when the system is characterized by undirected networks [12], or when different interconnections in the system are strongly correlated [13]. Consequently, it is of interest to provide necessary and sufficient conditions for structural systems characterized by graph with special weight constraints. Such problems are considered in [14] and [15]. However, the result in [14] is not applicable to systems modeled by undirected graph, whereas the matrix net approach in [15] may suffer from computational complexity in large-scale systems. Recently, the authors in [16] and [17] proposed graph-theoretic necessary and sufficient conditions for structural controllability of dynamical systems modeled by symmetric graph. Different from their approaches, in this paper, we provide a full characterization of the controllable modes using structural information of an undirected network, which facilitates a deeper understanding of structural controllability for systems involving symmetric parameter constraints.

However, in certain scenarios, we are only concerned about our ability to steer a collection of states, which can be captured by the notion of *(structural) target controllability* [18, 19]. The target controllability problem is a particular case of output controllability problem [20], while the necessary and sufficient condition of structural output controllability is still unknown [21]. Recently, the authors in [22, 23] proposed conditions for strong target controllability using zero-forcing sets. Nonetheless, to the best of our knowledge, providing necessary and sufficient conditions for structural target controllability remains an unsolved problem.

In this paper, we derive necessary and sufficient conditions for the problem of structural target controllability of LTI systems with symmetric state matrices, such as undirected dynamical networks with unknown link weights. Our contribution is three-fold: First, we introduce the concept of symmetrically structured matrix. We then characterize the generic rank of symmetrically structured matrices, as well as generic spectral properties of any numerical realization. Secondly, we propose graph-theoretic necessary and sufficient conditions for structural controllability of undirected networks with multiple inputs. Finally, we derive a necessary and sufficient condition for structural target controllability of undirected networks.

The rest of the paper is organized as follows. In Section II, we introduce preliminaries in algebra and graph theory. We formulate the problem under consideration in Section III. In Section IV, we present our main results. In Section V,

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we present an example to illustrate our results. Finally, we conclude the paper in Section VI. Due to page limitations, we only present the proofs of the main theorems in the Appendix. The proofs of other results can be found in the full version of this paper [24].

II. Notation and Preliminaries

We denote the cardinality of a set $\mathcal{S}$ by $|\mathcal{S}|$. We adopt the notation $[n]$ to represent the set of integers $\{1,\ldots,n\}$. Let $\mathbf{0}_{n\times m}\in\mathbb{R}^{n\times m}$ be the matrix with all entries equals to zero. Whenever clear from the context, $\mathbf{0}_{n\times m}$ is abbreviated as $\mathbf{0}$.

Given $M_1\in\mathbb{R}^{n\times m_1}$ and $M_2\in\mathbb{R}^{n\times m_2}$, we let $[M_1,M_2]\in\mathbb{R}^{n\times(m_1+m_2)}$ be the concatenation of M_1 and M_2. The (i,j)-th entry of $M\in\mathbb{R}^{n\times n}$ is denoted by $[M]_{ij}$. Moreover, we let $[M]_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}$ be the $k\times k$ submatrix of M formed by collecting $i_1,\ldots,i_k$-th rows and $j_1,\ldots,j_k$-th columns of M.

A matrix $\bar{M}\in\{0,\star\}^{n\times m}$ is called a *structured matrix*, if $[\bar{M}]_{ij}$ is either a fixed zero or an independent free parameter denoted by $\star$. In particular, we define a matrix $\bar{M}\in\{0,\star\}^{n\times n}$ to be *symmetrically structured*, if the value of the free parameter associated with $[\bar{M}]_{ij}$ is constrained to be the same as the value of the free parameter associated with $[\bar{M}]_{ji}$, for all j and i. For example, consider $\bar{M}$ and $\bar{A}$ be specified by

$$\bar{M}=\begin{bmatrix}0 & m_{12}\\ m_{21} & m_{22}\end{bmatrix} \text{ and } \bar{A}=\begin{bmatrix}0 & a_{12}\\ a_{12} & a_{22}\end{bmatrix},$$

where m_{12},m_{21},m_{22} and a_{12},a_{22} are independent free parameters. In this case, $\bar{M}$ is a structured matrix whereas $\bar{A}$ is symmetrically structured.

In the rest of the paper, we refer to $\tilde{M}$ as a *numerical realization* of a (symmetrically) structured matrix $\bar{M}$, i.e., $\tilde{M}$ is a matrix obtained by independently assigning real numbers to each independent free parameter in $\bar{M}$. In addition, we say that the structured matrix $\bar{M}\in\{0,\star\}^{n\times m}$ is the *structural pattern* of the matrix $M\in\mathbb{R}^{n\times m}$, where $[\bar{M}]_{ij}=\star$ if and only if $[M]_{ij}\neq 0$, for $\forall i\in[n],\forall j\in[m]$.

Given a (symmetrically) structured matrix $\bar{M}$, we let $n_{\bar{M}}$ be the number of its independent free parameters and we associate with $\bar{M}$ a parameter space $\mathbb{R}^{n_{\bar{M}}}$. Furthermore, we use vector $\mathbf{p}_{\tilde{M}}=(p_1,\ldots,p_{n_{\bar{M}}})^\top\in\mathbb{R}^{n_{\bar{M}}}$ to encode the value of independent free entries of $\bar{M}$ of a numerical realization $\tilde{M}$.

In what follows, a set $V\subseteq\mathbb{R}^n$ is called a *variety* if there exist polynomials $\varphi_1,\ldots,\varphi_k$, such that $V=\{x\in\mathbb{R}^n\colon\varphi_i(x)=0,\forall i\in\{1,\ldots,k\}\}$, and V is a *proper variety* when $V\neq\mathbb{R}^n$. We denote by $V^c:=\mathbb{R}^n\setminus V$ its complement.

The *term rank* [6] of a (symmetrically) structured matrix $\bar{M}$, denoted as t–rank$(\bar{M})$, is the largest integer k such that, for some suitably chosen distinct rows $i_1,\ldots,i_k$ and distinct columns $j_1,\ldots,j_k$, all of the entries $\{[\bar{M}]_{i_\ell j_\ell}\}_{\ell=1}^k$ are $\star$-entries. Additionally, a (symmetrically) structured matrix $\bar{M}\in\{0,\star\}^{n\times m}$ is said to have *generic rank* k, denoted as g–rank$(\bar{M})=k$, if there exists a numerical realization $\tilde{M}$ of $\bar{M}$, such that rank$(\tilde{M})=k$. If g–rank$(\bar{M})>0$, it is worth noting that the set of parameters describing all possible realizations when rank$(\tilde{M})<$ g–rank$(\bar{M})$ forms a proper variety, [11].

In the rest of the paper, we let $\mathcal{D}=(\mathcal{V},\mathcal{E})$ denote a directed graph whose vertex-set and edge-set are denoted by $\mathcal{V}=\{v_1,\ldots,v_n\}$ and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$, respectively. A *path* $\mathcal{P}$ in $\mathcal{D}$ is defined as an ordered sequence of distinct vertices $\mathcal{P}=(v_1,\ldots,v_k)$ with $\{v_1,\ldots,v_k\}\subseteq\mathcal{V}$ and $(v_i,v_{i+1})\in\mathcal{E}$ for all $i=1,\ldots,k-1$. Given a set $\mathcal{S}\subseteq\mathcal{V}$, we define the *in-neighbour set* of $\mathcal{S}$ as $\mathcal{N}(\mathcal{S})=\{v_i\in\mathcal{V}:(v_i,v_j)\in\mathcal{E},v_j\in\mathcal{S}\}$. We say a vertex v_i is *reachable* from vertex v_j in $\mathcal{D}(\mathcal{V},\mathcal{E})$, if there exists a path from vertex v_j to vertex v_i.

Given a directed graph $\mathcal{D}=(\mathcal{V},\mathcal{E})$ and two sets $\mathcal{S}_1,\mathcal{S}_2\subseteq\mathcal{V}$, we define the *bipartite graph* $\mathcal{B}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ as an undirected graph, whose vertex set is $\mathcal{S}_1\cup\mathcal{S}_2$ and edge set[1] $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}=\{\{s_1,s_2\}\colon(s_1,s_2)\in\mathcal{E},s_1\in\mathcal{S}_1,s_2\in\mathcal{S}_2\}$. Given $\mathcal{B}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, and a set $\mathcal{S}\subseteq\mathcal{S}_1$ or $\mathcal{S}\subseteq\mathcal{S}_2$, we define the *bipartite neighbor set* of $\mathcal{S}$ as $\mathcal{N}_{\mathcal{B}}(\mathcal{S})=\{j\colon\{j,i\}\in\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2},i\in\mathcal{S}\}$. A *matching* $\mathcal{M}$ is a set of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ that do not share vertices, i.e., given edges $e=\{s_1,s_2\}$ and $e'=\{s_1',s_2'\}$, $e,e'\in\mathcal{M}$ only if $s_1\neq s_1'$ and $s_2\neq s_2'$. The vertex v is said to be *right-unmatched* with respect to a matching $\mathcal{M}$ associated with $\mathcal{B}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ if $v\in\mathcal{S}_2$, and v does not belong to an edge in the matching $\mathcal{M}$.

III. Problem Statements

We consider a linear time-invariant system whose dynamics is captured by

$$\dot{x}=Ax+Bu,\quad y=Cx, \tag{1}$$

where $x\in\mathbb{R}^n$, $y\in\mathbb{R}^k$ and $u\in\mathbb{R}^m$ are the state, the output and the input vectors, respectively. In addition, the matrix $A\in\mathbb{R}^{n\times n}$ is the state matrix, $B\in\mathbb{R}^{n\times m}$ is the input matrix and $C\in\mathbb{R}^{k\times n}$ is the output matrix. In this paper, we consider the following assumption:

Assumption 1. *The state matrix $A\in\mathbb{R}^{n\times n}$ is symmetric, i.e., $A=A^\top$.*

This symmetry assumption is motivated by control problems arising in undirected networked dynamical systems. Furthermore, this assumption will be crucial when establishing graph-theoretic results characterizing structural controllability problems in undirected networks.

Hereafter, we use the 3-tuple (A,B,C) to represent the system (1). In particular, we use the pair (A,B) to denote a system without a measured output. A pair (A,B) is called *reducible* if there exists a permutation matrix P, such that

$$PAP^{-1}=\begin{bmatrix}A_{11} & \mathbf{0}\\ A_{21} & A_{22}\end{bmatrix},\quad PB=\begin{bmatrix}\mathbf{0}\\ B_2\end{bmatrix}, \tag{2}$$

where $A_{11}\in\mathbb{R}^{q\times q}$ and $B_2\in\mathbb{R}^{(n-q)\times m}$, $1\leq q<n$. The pair (A,B) is called *irreducible* otherwise. Furthermore, we use $\bar{A}$ and $\bar{B}$ to represent the structural pattern of A and B, respectively. In particular, by Assumption 1, we consider

[1] We denote undirected edges using curly brackets $\{v_i,v_j\}$, in contrast with directed edges, for which we use parenthesis.

$\bar{A}$ to be symmetrically structured. Thus, $(\bar{A}, \bar{B})$ is referred to as the *structural pair* of the system (A, B). Given a structured matrix $\bar{A}$, we associate it with a directed graph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, which we refer to as the *state digraph*, where $\mathcal{X} = \{x_1, \ldots, x_n\}$ is the state vertex set, and $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_j, x_i) : [\bar{A}]_{ij} = \star\}$ is the set of edges. Similarly, we associate a directed graph $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ with the structural pair $(\bar{A}, \bar{B})$, where $\mathcal{U} = \{u_1, \ldots, u_m\}$ is the set of input vertices and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_j, x_i) \colon [\bar{B}]_{ij} = \star\}$ is the set of edges from input vertices to state vertices. We refer to $\mathcal{D}(\bar{A}, \bar{B})$ as the *system digraph*.

Definition 1 (Structural Controllability [7]). *A structural pair $(\bar{A}, \bar{B})$ is structurally controllable if there exists a numerical realization $(\tilde{A}, \tilde{B})$, such that the controllability matrix $Q(\tilde{A}, \tilde{B}) := [\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B}]$ has full row rank.*

While controllability is concerned about the ability to steer all the states of a system to a desired final state, under certain circumstances, it is more preferred to control the behavior of only a subset of states. More specifically, given a set $\mathcal{T} \subseteq [n]$, which we refer to as the *target set*, it is of interest to consider whether the set of selected states can be steered arbitrarily. If so, we say that the pair (A, B) is *target controllable* with respect to $\mathcal{T}$ [18]. Notice that this does not exclude the possibility of some other states indexed by $[n] \setminus \mathcal{T}$ being controllable as well. Similarly, we introduce the notion of *structural target controllability* in the context of structural pairs.

Definition 2 (Structural Target Controllability [19]). *Given a structural pair $(\bar{A}, \bar{B})$, and a target set $\mathcal{T} = \{i_1, \ldots, i_k\} \subseteq [n]$, let $\mathcal{X}_\mathcal{T}$ be the set of state vertices corresponding to $\mathcal{T}$ in $\mathcal{D}(\bar{A}, \bar{B})$. We define a matrix $C_\mathcal{T} \in \mathbb{R}^{k \times n}$ by*

$$[C_\mathcal{T}]_{\ell j} = \begin{cases} 1, & \text{if } j = i_\ell,\ i_\ell \in \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The structural pair $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$ if there exists a numerical realization $(\tilde{A}, \tilde{B})$, such that the target controllability matrix $Q_\mathcal{T}(\tilde{A}, \tilde{B}) := C_\mathcal{T}[\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B}]$ has full row rank.

Note that structural controllability is equivalent to structural target controllability when $\mathcal{T} = [n]$. Therefore, the necessary and sufficient conditions for structurally target controllable undirected networks can be applied to characterize structural controllability. Subsequently, in this paper, we consider the following problem:

Problem 1. *Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured and $\bar{B}$ is a structured matrix, and a target set $\mathcal{T} \subseteq [n]$, find a necessary and sufficient condition for $(\bar{A}, \bar{B})$ to be structurally target controllable with respect to $\mathcal{T}$.*

IV. Main Results

In this section, we first introduce a proposition that is crucial for developing our solution to Problem 1. Then, we characterize the generic rank of symmetrically structured matrices in Lemma 1. Subsequently, we characterize the relationship between the term-rank of a symmetrically structured matrix and the presence of non-zero simple eigenvalues in a numerical realization in Lemma 2. This allows us to obtain a result characterizing the relationship between irreducibility and structural controllability of a structural pair involving symmetrically structured matrix (see Lemma 3). Based on these results, we propose graph-theoretic necessary and sufficient conditions for structural controllability and structural target controllability in Theorems 1 and 2, respectively.

Proposition 1 (Popov-Belevitch-Hautus (PBH) test [25]). *The pair (A, B) is uncontrollable if and only if there exists a $\lambda \in \mathbb{C}$ and a nontrivial vector $e \in \mathbb{C}^n$, such that $e^\top A = \lambda e^\top$ and $e^\top B = 0$.*

Given a pair (A, B), where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, we say that the mode (λ, e) of A, where $\lambda \in \mathbb{C}$ and $e \in \mathbb{C}^n$, is an *uncontrollable mode* if $e^\top A = \lambda e^\top$ and $e^\top B = 0$.

A. Generic Properties

If a symmetrically structured matrix is generically full rank, then any numerical realization has almost surely no zero eigenvalue. In this subsection, we characterize the generic rank of a symmetrically structured matrix in Lemma 1, which lays the foundation for a further characterization of spectral properties of numerical realizations.

Lemma 1. *Consider an $n \times n$ symmetrically structured matrix $\bar{A}$, and a set $\mathcal{T} = \{i_1, \ldots, i_k\} \subseteq [n]$. Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the digraph representation of $\bar{A}$, $\mathcal{X}_\mathcal{T} \subseteq \mathcal{X}$ be the set of vertices indexed by $\mathcal{T}$, and $C_\mathcal{T}$ be defined as in* (3). *The generic-rank of $C_\mathcal{T}\bar{A}$ equals to k if and only if $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$.*

Lemma 1 establishes a relationship between the generic rank of a submatrix of a symmetrically structured matrix and the topology of its corresponding digraph. Subsequently, Corollary 1 follows, which characterizes the generic rank of the concatenation of a symmetrically structured matrix $\bar{A} \in \{0, \star\}^{n \times n}$ and a structured matrix $\bar{B} \in \{0, \star\}^{n \times m}$.

Corollary 1. *Consider a structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \star\}^{n \times n}$ is symmetrically structured, and a set $\mathcal{T} = \{i_1, \ldots, i_k\} \subseteq [n]$. Let $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ be the digraph representation of $(\bar{A}, \bar{B})$, and $\mathcal{X}_\mathcal{T} \subseteq \mathcal{X}$ be the set of vertices indexed by $\mathcal{T}$. If $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_\mathcal{T}$, then* g–rank$(C_\mathcal{T}[\bar{A}, \bar{B}]) = k$.

In the remaining subsections, we aim to provide necessary and sufficient conditions for structural controllability. To achieve this goal, we notice that the eigenvalues of the state matrix are closely related to controllability, as indicated by Proposition 1. Besides, the approach in [10] shows that for an irreducible structural pair with no symmetric parameter dependencies, all the nonzero modes of its numerical realization are almost surely simple and controllable. Similarly, to characterize structural controllability of undirected networks, we will provide characterizations of the modes in the numerical realization of a structural pair involving symmetrically structured matrix. Instead of using the maximum order of principle minor as in [10], we derive below a condition based on the term rank to ensure that generically the numerical realization of a symmetrically structured matrix has k nonzero simple eigenvalues.

Lemma 2. *Given an $n \times n$ symmetrically structured matrix $\bar{A}$, if $\text{t–rank}(\bar{A}) = k$, then there exists a proper variety $V_1 \subset \mathbb{R}^{n_{\bar{A}}}$, such that for any numerical realization $\tilde{A}$, where the numerical values assigned to free parameters of $\bar{A}$ are encoded in the vector $\mathbf{p}_{\tilde{A}} \in \mathbb{R}^{n_{\bar{A}}} \setminus V_1$, $\tilde{A}$ has k nonzero simple eigenvalues.*

Remark 1. *The challenge in the proof of Lemma 2, which can be found in the full version of this paper [24], is to construct a finite number of nonzero polynomials, i.e., the polynomials of where not every coefficient is zero, such that the numerical values assigned to free parameters of $\bar{A}$ in a numerical realization $\tilde{A}$, where $\tilde{A}$ does not have k nonzero simple eigenvalues, are the zeros of those polynomials. Since the set of zeros of a nonzero polynomial has Lebesgue measure zero [26], it follows that for any numerical realization $\tilde{A}$, $\tilde{A}$ has almost surely k nonzero simple eigenvalues.*

Remark 2. *Lemma 2 generally is not true for a structured matrix. For example, consider $\bar{M} = \left[\begin{smallmatrix} 0,\star \\ 0,0 \end{smallmatrix}\right]$, $\text{t–rank}(\bar{M}) = 1$, but for any numerical realization, $\tilde{M}$ has no nonzero mode.*

As shown in [7, 10], irreducibility is a necessary condition for structural controllability. We can expect that irreducibility also plays a similar role in symmetrically structured systems. Moreover, we show below that irreducibility ensures that all nonzero simple modes of $\tilde{A}$ are controllable, generically.

Lemma 3. *Given a structural pair $(\bar{A}, \bar{B})$, where $\bar{A}$ is symmetrically structured and t–rank$(\bar{A}) = k$, if $(\bar{A}, \bar{B})$ is irreducible, then there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that for any numerical realization $(\tilde{A}, \tilde{B})$ with $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}} \setminus V$, $\tilde{A}$ has k nonzero, simple and controllable modes.*

B. Structural Controllability

We have shown that irreducibility guarantees that generically all non-zero simple modes of $(\tilde{A}, \tilde{B})$ are controllable. In this subsection, Theorem 1 proposes conditions guaranteeing that generically both the nonzero and zero modes of $(\tilde{A}, \tilde{B})$ are controllable, therefore establishes a graph-theoretic necessary and sufficient condition for structural controllability in symmetrically structured system.

Theorem 1. *Let $(\bar{A}, \bar{B})$ be a structural pair, with $\bar{A}$ being a symmetrically structured matrix, and let $\mathcal{X}$ be the set of state vertices in $\mathcal{D}(\bar{A}, \bar{B})$. The structural pair $(\bar{A}, \bar{B})$ is structurally controllable, if and only if, the following conditions hold simultaneously in $\mathcal{D}(\bar{A}, \bar{B})$:*

1) all the state vertices are input-reachable;
2) $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}$.

Notice that Conditions *1)* and *2)* in Theorem 1 admits a similar form as the conditions for structural controllability (see, for example [7]). Subsequently, if a structural pair with symmetric parameter dependencies is structurally controllable, then the structural pair with the same structural pattern without symmetric parameter dependencies, will also be structurally controllable. However, the converse cannot be trivially derived due to symmetric parameter dependencies in Assumption 1.

C. Structural Target Controllability

We now extend the solution approach in Theorem 1 to establish graph-theoretic necessary and sufficient conditions for structural target controllability of the given structural pair $(\bar{A}, \bar{B})$ and target set $\mathcal{T}$.

Theorem 2. *Consider a structural pair $(\bar{A}, \bar{B})$, with $\bar{A}$ being symmetrically structured, and a target set $\mathcal{T} \subseteq [n]$. Let $\mathcal{X}_{\mathcal{T}}$ be the set of state vertices corresponding to $\mathcal{T}$ in $\mathcal{D}(\bar{A}, \bar{B})$. The structural pair $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$, if and only if, the following conditions hold simultaneously in $\mathcal{D}(\bar{A}, \bar{B})$:*

1) all the states vertices in $\mathcal{X}_{\mathcal{T}}$ are input-reachable;
2) $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$.

Remark 3. *Condition 2) in Theorem 2 can be verified using local topological information in a network. In particular, this condition is satisfied if there exists a matching in the bipartite graph $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ associated with $\mathcal{D}(\bar{A}, \bar{B})$, where $\mathcal{S}_1 = \mathcal{X} \cup \mathcal{U}$ and $\mathcal{S}_2 = \mathcal{X}_{\mathcal{T}}$, such that all vertices in $\mathcal{S}_2$ are right-matched. The existence of such a matching can be verified in $\mathcal{O}(\sqrt{|\mathcal{S}_1 \cup \mathcal{S}_2|}|\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}|)$ time [27, §23.6].*

Through the proof of Theorem 2, we notice that the characterization of structural target controllability relies on the assumption that the state matrix is symmetric. More specifically, since the state matrix is symmetric, the eigenvectors of the state matrix form a complete basis of the state space, which allows us to generalize the PBH test in the context of target controllability problems. On the contrary, when the system is characterized by a directed network, the state matrix A is, in general, non-diagonalizable, which prevents us from generalizing PBH test to characterize the target controllability problems - see [21, Example 3] for a reference.

In addition, the proof of Theorem 2 suggests that, when $\bar{A}$ is not symmetrically structured, if either Conditions *1)* or *2)* fails to hold, then $(\bar{A}, \bar{B})$ is not structurally target controllable. Therefore, in general, when the structured matrix $\bar{A} \in \{0, \star\}^{n \times n}$ is not symmetrically structured, the Conditions *1)* and *2)* in Theorem 2 are necessary but not sufficient conditions for structural target controllability of the pair $(\bar{A}, \bar{B})$.

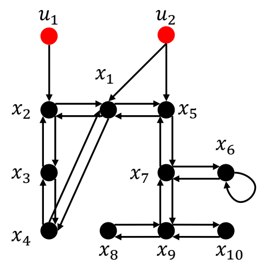

Figure 1: the digraph representation of the structural pair $(\bar{A}, \bar{B})$, where the red and black vertices represent input and state vertices, respectively. The black arrows represent edges in $\mathcal{D}(\bar{A}, \bar{B})$.

V. Illustrative Examples

In this section, we provide an example to illustrate our necessary and sufficient conditions in Theorem 1 and Theorem 2. We consider a symmetrically structured system with 10 states and 2 inputs modeled by an undirected network with unknown link weights. The structural representations of its state and input matrix are denoted by $\bar{A} \in \{0, \star\}^{10\times 10}$ and $\bar{B} \in \{0, \star\}^{10\times 2}$, as follows.

$$\bar{A} = \begin{bmatrix} 0 & a_{12} & 0 & a_{14} & a_{15} & 0 & 0 & 0 & 0 & 0 \\ a_{12} & 0 & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{23} & 0 & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{14} & 0 & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{15} & 0 & 0 & 0 & 0 & 0 & a_{57} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & a_{67} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{57} & a_{67} & 0 & 0 & a_{79} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{89} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{79} & a_{89} & 0 & a_{910} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{910} & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0,b_{12} \\ b_{21},0 \\ 0,0 \\ 0,0 \\ 0,b_{52} \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \end{bmatrix}.$$

In addition, we let the target set be $\mathcal{T} = \{2, 6, 8\}$. Subsequently, $C_{\mathcal{T}}$, defined according to (3), equals to

$$C_{\mathcal{T}} = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \end{bmatrix}.$$

We also associate the structural pair $(\bar{A}, \bar{B})$ with the digraph $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ as depicted in Figure 1, where $\mathcal{X} = \{x_1, \ldots, x_{10}\}$, $\mathcal{U} = \{u_1, u_2\}$ and $\mathcal{X}_{\mathcal{T}} = \{x_2, x_6, x_8\}$.

Notice that by letting $\mathcal{S} = \{x_8, x_{10}\}$, we have $\mathcal{N}(\mathcal{S}) = \{x_9\}$. As a result, according to Theorem 1, $\exists \mathcal{S} \subseteq \mathcal{X}$, $|\mathcal{N}(\mathcal{S})| < |\mathcal{S}|$ implies that the system is not structurally controllable. However, since all the vertices in $\mathcal{X}_{\mathcal{T}}$ are input-reachable, and $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$, by Theorem 2, $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$. This example also shows that, if the input-reachability of the vertices in $\mathcal{X}_{\mathcal{T}}$ is guaranteed, then the structural target controllability in undirected networks can be verified by only local topological information.

VI. Conclusions

In this paper, we study the problem of characterizing structural target controllability in undirected networks with unknown link weights. We achieved this goal by first characterizing the generic properties of symmetrically structured matrices. We then proposed a necessary and sufficient condition for structural controllability of undirected networks with multiple control inputs. Finally, we provided a graph-theoretic necessary and sufficient condition for structural target controllability of undirected networks. In the future, we will use the conditions to implement algorithms for designing minimum number of input actuations to ensure the structural target controllability of undirected networks.

Appendix

A. Proof of Theorem 1

We first introduce Lemma 4, which lays the foundation for the proof of Theorem 1.

Lemma 4. *Consider a structural pair $(\bar{A}, \bar{B})$, and a target set $\mathcal{T}$ with the corresponding state vertex set $\mathcal{X}_{\mathcal{T}}$ in $\mathcal{D}(\bar{A}, \bar{B})$. We define $C_{\mathcal{T}}$ according to (3). Given a numerical realization $(\tilde{A}, \tilde{B})$, we define controllability matrix $Q(\tilde{A}, \tilde{B})$ as in Definition 1. Then, for any numerical realization $(\tilde{A}, \tilde{B})$, we have that* $\text{rank}(C_{\mathcal{T}} Q(\tilde{A}, \tilde{B})) \leq |\mathcal{N}(\mathcal{X}_{\mathcal{T}})|$.

Proof of Lemma 4. Suppose we have a numerical realization $(\tilde{A}, \tilde{B})$. By Cayley-Hamilton theorem,

$$\begin{aligned} \text{rank}(C_{\mathcal{T}}[\tilde{B}, \tilde{A}Q(\tilde{A}, \tilde{B})]) &= \text{rank}(C_{\mathcal{T}}[\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{n-1}\tilde{B}, \tilde{A}^{n}\tilde{B}]) \\ &= \text{rank}([C_{\mathcal{T}}Q(\tilde{A}, \tilde{B}), C_{\mathcal{T}}\tilde{A}^{n}\tilde{B}]) \\ &= \text{rank}(C_{\mathcal{T}}Q(\tilde{A}, \tilde{B})). \end{aligned} \tag{4}$$

In $\mathcal{D}(\bar{A}, \bar{B})$, let m_1, m_2 be the number of input, state vertices in $\mathcal{N}(\mathcal{X}_{\mathcal{T}})$, respectively. Then, (4) yields,

$$\begin{aligned} \text{rank}(C_{\mathcal{T}}Q(\tilde{A}, \tilde{B})) &= \text{rank}(C_{\mathcal{T}}[\tilde{B}, \tilde{A}Q(\tilde{A}, \tilde{B})]) \\ &\leq \text{rank}(C_{\mathcal{T}}\tilde{B}) + \text{rank}(C_{\mathcal{T}}\tilde{A}Q(\tilde{A}, \tilde{B})) \\ &\leq m_1 + \min(\text{rank}(C_{\mathcal{T}}\tilde{A}),\ \text{rank}(Q(\tilde{A}, \tilde{B}))) \\ &\leq m_1 + m_2 \\ &= |\mathcal{N}(\mathcal{X}_{\mathcal{T}})|. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1. To show the necessity of the theorem, suppose that there exists a vertex $v_i \in \mathcal{X}$ that is not input-reachable, then the i-th row of controllability matrix will be zero row, which implies that $\text{rank}(Q(\tilde{A}, \tilde{B})) < n$, for any numerical realization of the pair $(\bar{A}, \bar{B})$. On the other hand, suppose there exists a set $\mathcal{S} \subseteq \mathcal{X}$, such that $|\mathcal{N}(\mathcal{S})| < |\mathcal{S}|$, then by Lemma 4, $\text{rank}(Q(\tilde{A}, \tilde{B})) < n$, for any numerical realization of the pair $(\bar{A}, \bar{B})$. Hence, the necessity is proved.

To show the sufficiency, we proceed as follows. First, since $|\mathcal{N}(S)| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}$, it follows from Corollary 1 that g–rank$([\bar{A}, \bar{B}]) = n$. Because all the state vertices are input-reachable, $(\bar{A}, \bar{B})$ is irreducible. If we denote the term-rank of $\bar{A}$ as k, then by Lemma 3, there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$ such that, if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$ then $\tilde{A}$ has k nonzero, simple and controllable modes. Let λ be an eigenvalue of $\tilde{A}$. On one hand, if $\lambda \neq 0$, then λ is controllable by Lemma 3. On the other hand, if $\lambda = 0$, since g–rank$([\bar{A}, \bar{B}]) = n$, then there exists a proper variety $W \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in W^c \cap V^c$, then $\text{rank}([\tilde{A}, \tilde{B}]) = n$. As a result, $\lambda = 0$ is controllable by the eigenvalue PBH test. Since all the modes of $\tilde{A}$ are controllable generically, $(\bar{A}, \bar{B})$ is structurally controllable. □

B. Proof of Theorem 2

Proof. The necessity of Conditions *1)* and *2)* can be proved in a similar approach as the proof in Theorem 1. What remains to be shown is their sufficiency. It suffices to show that Conditions *1)* and *2)* result in that generically the left null space of target controllability matrix is trivial.

Suppose there exists an input-unreachable state vertex $v_{x_i} \in \mathcal{X} \setminus \mathcal{X}_{\mathcal{T}}$. Since all the vertices in $\mathcal{X}_{\mathcal{T}}$ are input-reachable, for $\forall v_{x_j} \in \mathcal{X}_{\mathcal{T}}$, there is no path from v_{x_j} to v_{x_i}, and there is also no path from v_{x_i} to v_{x_j} due to the symmetry in $\mathcal{D}(\bar{A})$. This implies in model (1) that the ith state has no impact on the dynamics of $\mathcal{T}$ corresponding states. Omitting the ith state from the system will not change the dynamics of $\mathcal{T}$ corresponding states. Hence, we could assume that $(\bar{A}, \bar{B})$ is irreducible. By Lemma 3, there exists a proper variety $V \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$, then all the nonzero modes of $\tilde{A}$ are controllable. In the rest of the proof, we assume $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$. Denote by $e_1, \ldots, e_l$ the left eigenvectors corresponding to zero modes of $\tilde{A}$, and $e_{l+1}, \ldots, e_n$ the left eigenvectors for nonzero modes. Denote the left null space of a matrix M as $\boldsymbol{N}(M^\top)$.

From Lemma 3, we have that if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c$, then $\boldsymbol{N}((Q(\tilde{A}, \tilde{B}))^\top) \subseteq \text{span}\{e_1^\top, \ldots, e_l^\top\}$. For the target set $\mathcal{T}$, define the matrix $C_{\mathcal{T}}$ according to (3). By the assumption $|\mathcal{N}(\mathcal{S})| \geq |\mathcal{S}|$, $\forall \mathcal{S} \subseteq \mathcal{X}_{\mathcal{T}}$, and Corollary 1, we have that g–rank$(C_{\mathcal{T}}[\bar{A}, \bar{B}]) = |\mathcal{T}|$, which implies that there exists a proper variety $W \subset \mathbb{R}^{n_{\bar{A}}+n_{\bar{B}}}$, such that if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c \cap W^c$, then rank$(C_{\mathcal{T}}[\tilde{A}, \tilde{B}]) = |\mathcal{T}|$, i.e., $\boldsymbol{N}((C_{\mathcal{T}}[\tilde{A}, \tilde{B}])^\top) = \mathbf{0}$. Define $\hat{I} \in \mathbb{R}^{n \times n}$ as

$$[\hat{I}]_{ij} = \begin{cases} 1, & \text{if } j = i,\ i \in \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

We claim that there does not exist a nontrivial vector $e \in \mathbb{C}^n$ such that $\hat{I}e = e$, $e^\top \tilde{A} = 0e^\top$ and $e^\top \tilde{B} = \mathbf{0}$. Otherwise, $e^\top[\tilde{A}, \tilde{B}] = \mathbf{0}$, which contradicts $\boldsymbol{N}((C_{\mathcal{T}}[\tilde{A}, \tilde{B}])^\top) = \mathbf{0}$.

Hence, if $[\mathbf{p}_{\tilde{A}}, \mathbf{p}_{\tilde{B}}] \in V^c \cap W^c$, then there is no nontrivial vector $v \in \mathbb{C}^{|\mathcal{T}|}$, such that $v^\top C_{\mathcal{T}} \in \text{span}\{e_1^\top, \ldots, e_l^\top\}$. Thus, generically, $\boldsymbol{N}((C_{\mathcal{T}} Q(\tilde{A}, \tilde{B}))^\top) = \mathbf{0}$. The $(\bar{A}, \bar{B})$ is structurally target controllable with respect to $\mathcal{T}$. ☐

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