

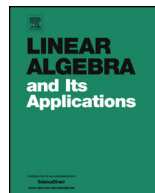


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Measure-theoretic bounds on the spectral radius of graphs from walks [☆]

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ARTICLE INFO

Article history:

Received 4 August 2020

Accepted 27 April 2021

Available online 29 April 2021

Submitted by S.M. Cioaba

MSC:

15A42

15A18

05C50

Keywords:

Walks on graphs

Spectral radius

Moment problem

ABSTRACT

Let $\mathcal{G}$ be an undirected graph with adjacency matrix A and spectral radius ρ . Let w_k, ϕ_k and $\phi_k^{(i)}$ be, respectively, the number walks of length k , closed walks of length k and closed walks starting and ending at vertex i after k steps. In this paper, we propose a measure-theoretic framework which allows us to relate walks in a graph with its spectral properties. In particular, we show that w_k, ϕ_k and $\phi_k^{(i)}$ can be interpreted as the moments of three different measures, all of them supported on the spectrum of A . Building on this interpretation, we leverage results from the classical moment problem to formulate a hierarchy of new lower and upper bounds on ρ , as well as provide alternative proofs to several well-known bounds in the literature.

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[☆] Research reported in this paper was supported by the National Science Foundation under award number NSF TRIPODS 1934876.

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1. Introduction

Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, \dots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, we define a *walk of length k* , or *k -walk*, as a sequence of vertices $(i_0, i_1, \dots, i_k)$ such that $(i_s, i_{s+1}) \in \mathcal{E}$ for $s \in \{0, \dots, k-1\}$. A walk of length k is called *closed* if $i_0 = i_k$; furthermore, we will refer to them as *closed walk from vertex i_0* when we need to distinguish them from the set of all closed k -walks. We denote the number of walks, closed walks, and closed walks from vertex i of length k by w_k , ϕ_k , and $\phi_k^{(i)}$, respectively. Denote by A the adjacency matrix of graph $\mathcal{G}$ and its eigenvalues by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The set containing these eigenvalues will be referred to as the *spectrum* of $\mathcal{G}$. From Perron-Frobenius' Theorem [1], we have that the spectral radius of A , defined by $\rho := \max_{1 \leq i \leq n} |\lambda_i|$, is equal to λ_1 .

In the literature, we find several lower bounds on ρ formulated in terms of walks in the graph. Bounds in terms of closed walks are rarer (see, e.g., [2]). Many of these bounds come from dexterous applications of the Rayleigh principle [1] or the Cauchy-Schwarz inequality. For example, making use of these tools, Collatz and Sinogowitz [3], Hoffmeister [4], Yu et al. [5], and Hong and Zhang [6] derived, respectively, the following lower bounds:

$$\rho \geq \frac{w_1}{w_0}, \quad \rho \geq \sqrt{\frac{w_2}{w_0}}, \quad \rho \geq \sqrt{\frac{w_4}{w_2}}, \quad \rho \geq \sqrt{\frac{w_6}{w_4}}. \quad (1)$$

Nikiforov¹ [7] generalized these results by expressing the number of walks w_k in terms of the eigenvalues, to obtain bounds of the form

$$\rho^r \geq \frac{w_{2s+r}}{w_{2s}}, \quad (2)$$

for $s, r \in \mathbb{N}_0$. Cioabă and Gregory [8] provide the following improvement to the first bound in (1) for the case of irregular graphs of order $n \geq 4$:

$$\rho \geq \frac{w_1}{w_0} + \frac{1}{w_0(\Delta + 2)}, \quad (3)$$

where Δ is the maximum of the vertex degrees in $\mathcal{G}$. Nikiforov [9] showed that

$$\rho > \frac{w_1}{w_0} + \frac{1}{2w_0 + w_1}, \quad (4)$$

also for irregular graphs of order $n \geq 4$. Favaron et al. [10] used the fact that there is a $K_{1,\Delta}$ subgraph in $\mathcal{G}$ to obtain:

¹ Nikiforov's notation in [7] indexes w_k in terms of the number of nodes visited by the walks instead of the number of steps, as used in our manuscript.

$$\rho \geq \sqrt{\Delta}. \quad (5)$$

There are a number of upper bounds on ρ in terms of graph invariants like the domination number [11], chromatic number [12,13], and clique number [13]. Nikiforov [7] provides a whole hierarchy of bounds in terms of the clique number $\omega(\mathcal{G})$, which for $k \in \mathbb{N}_0$ are given by

$$\rho^{k+1} \leq \left(1 - \frac{1}{\omega(\mathcal{G})}\right) w_k. \quad (6)$$

We also find in the literature several bounds in terms of the *fundamental weight* of $\mathcal{G}$, defined as $\sum_{i=1}^n u_{i1}$, where u_{i1} is the i -th entry of the *leading eigenvector*² of A denoted by $\mathbf{u}_1$. For example, Wilf [14] proved the following upper bound:

$$\rho \leq \left(1 - \frac{1}{\omega(\mathcal{G})}\right) \left(\sum_{i=1}^n u_{i1}\right)^2. \quad (7)$$

Cioabă and Gregory [15] showed that, for $k \in \mathbb{N}_0$,

$$\rho^k \leq \sqrt{w_{2k}} \max_{1 \leq i \leq n} u_{i1}. \quad (8)$$

Moreover, Van Mieghem [16] proved the bound

$$\rho^k \leq \frac{w_k}{\sum_{i=1}^n u_{i1}} \max_{1 \leq i \leq n} u_{i1}. \quad (9)$$

In this paper we provide upper and lower bounds on ρ by using classical results from probability theory that relate the moments of a measure with its support. In particular, we interpret the sequences $\{w_k\}_{k=0}^\infty$, $\{\phi_k\}_{k=0}^\infty$ and $\{\phi_k^{(i)}\}_{k=0}^\infty$ as moments of three measures supported on $[-\rho, \rho]$, and apply these results to derive a hierarchy of new bounds on the spectral radius, as well as provide alternative proofs to several existing bounds in the literature. The rest of the paper is organized as follows. Section 2 outlines the tools we will use to analyze walks on graphs using measures and moment sequences. Section 3 presents multiple lower bounds on the spectral radius derived from the moment problem, while Section 4 introduces several upper bounds.

2. Background and preliminaries

Throughout this paper, we use standard graph theory notation, as in [17]. We will use upper-case letters for matrices, calligraphic upper-case letters for sets, and bold lower-case letters for vectors. For a vector $\mathbf{v}$ or a matrix M , we denote by $\mathbf{v}^\top$ and $M^\top$ their

² The leading eigenvector of A is the eigenvector associated with the largest eigenvalue λ_1 . We assume eigenvectors to be normalized to be of unit Euclidean norm.

respective transposes. The (i, j) -th entry of a matrix M is denoted by M_{ij} . For a $n \times n$ matrix M and a set $\mathcal{J} \subseteq \{0, \dots, n-1\}$, the matrix $M_{\mathcal{J}}$ is defined to be the submatrix of M formed by columns and rows corresponding to $\mathcal{J}$; $M_{\mathcal{J}}$ is also called a *principal submatrix* of M and, if $\mathcal{J} = \{1, \dots, k\}$, $M_{\mathcal{J}}$ is called a *leading principal submatrix*. We say that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is positive semidefinite (resp. positive definite) if for every non-zero vector $\mathbf{v} \in \mathbb{R}^n$ we have $\mathbf{v}^T M \mathbf{v} \geq 0$ (resp. $\mathbf{v}^T M \mathbf{v} > 0$) and we denote this as $M \succeq 0$ (resp. $M \succ 0$). Finally, the support of a Borel measure ζ will be denoted by $\text{supp}(\zeta)$.

2.1. Spectral measures and walks

We can relate walks and closed walks on a graph $\mathcal{G}$ to its spectrum using measures, as we describe in detail below. We begin by stating the following well-known result from algebraic graph theory [18]:

Lemma 1. *For any integer k , the (i, j) -th entry of the matrix A^k is equal to the number of k -walks from vertex i to vertex j on $\mathcal{G}$.*

Since $\mathcal{G}$ is undirected, A is symmetric and admits an orthonormal diagonalization. In particular, let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a complete set of orthonormal eigenvectors of A . Hence, we have that $A^k = U \text{diag}(\lambda_1^k, \dots, \lambda_n^k) U^T$, for every $k \geq 0$, where $U := [\mathbf{u}_1 | \mathbf{u}_2 | \dots | \mathbf{u}_n]$. We denote the i -th entry of the l -th eigenvector by $u_{i\ell}$. From this factorization, we can obtain identities which will be used in the following sections.

Lemma 2. *Define $c_\ell := (\sum_{i=1}^n u_{i\ell})^2$. Then, for every $k \geq 0$, we have*

$$\phi_k = \sum_{\ell=1}^n \lambda_\ell^k, \quad \phi_k^{(i)} = \sum_{\ell=1}^n u_{i\ell}^2 \lambda_\ell^k, \quad w_k = \sum_{\ell=1}^n c_\ell \lambda_\ell^k.$$

Proof. Using Lemma 1 we have that $\phi_k = \sum_{i=1}^n (A^k)_{ii}$, $\phi_k^{(i)} = (A^k)_{ii}$, and $w_k = \sum_{i,j} (A^k)_{ij}$. Furthermore, we have that $(A^k)_{ij} = \sum_{\ell=1}^n u_{i\ell} \lambda_\ell^k u_{j\ell}$, directly from the diagonalization of A . Combining these results, and the fact that $\sum_{i=1}^n u_{i\ell}^2 = 1$, the result follows. $\square$

Next, we introduce three measures supported on the spectrum of A .

Definition 3 (Spectral measures). Let $\delta(\cdot)$ be the Dirac delta measure. For a simple graph $\mathcal{G}$ with eigenvalues $\lambda_1 \geq \lambda_2, \dots, \lambda_n$, define the *closed-walks measure* as

$$\mu_{\mathcal{G}}(x) := \sum_{\ell=1}^n \delta(x - \lambda_\ell).$$

We also define the *closed-walks measure for vertex i* as

$$\mu_{\mathcal{G}}^{(i)}(x) := \sum_{\ell=1}^n u_{i\ell}^2 \delta(x - \lambda_{\ell}),$$

and the *walks measure* as

$$\nu_{\mathcal{G}}(x) := \sum_{\ell=1}^n c_{\ell} \delta(x - \lambda_{\ell}).$$

Lemma 4. For a real measure $\zeta(x)$, define its k -th moment as $m_k(\zeta) = \int_{\mathbb{R}} x^k d\zeta(x)$. Then, the measures in Definition (3) satisfy

$$m_k(\mu_{\mathcal{G}}) = \phi_k, \quad m_k(\mu_{\mathcal{G}}^{(i)}) = \phi_k^{(i)}, \quad m_k(\nu_{\mathcal{G}}) = w_k.$$

Proof. For the case of $\mu_{\mathcal{G}}$, we evaluate k -th moment, as follows:

$$m_k(\mu_{\mathcal{G}}) = \int_{\mathbb{R}} x^k d\mu_{\mathcal{G}}(x) = \int_{\mathbb{R}} x^k \sum_{\ell=1}^n \delta(x - \lambda_{\ell}) dx = \sum_{\ell=1}^n \lambda_{\ell}^k = \phi_k.$$

The other two cases have analogous proofs. $\square$

2.2. The moment problem

In order to derive bounds on the spectral radius ρ , we will make use of results from the *moment problem* [19]. This problem is concerned with finding necessary and sufficient conditions for a sequence of real numbers to be the *moment sequence* of a measure whose support is contained in a set $\mathcal{K} \subseteq \mathbb{R}$. This is formalized below.

Definition 5 (*$\mathcal{K}$ -moment sequence*). The infinite sequence of real numbers $\mathbf{m} = (m_0, m_1, m_2, \dots)$ is called a *$\mathcal{K}$ -moment sequence* if there exists a Borel measure ζ with $\text{supp}(\zeta) \subseteq \mathcal{K} \subseteq \mathbb{R}$ such that

$$m_k = \int_{\mathcal{K}} x^k d\zeta(x), \quad \text{for all } k \in \mathbb{N}_0.$$

The following result, known as *Hamburger's theorem* [19], will be used in Sections 3 and 4.

Theorem 6 (*Hamburger's Theorem* [19]). Let $\mathbf{m} = (m_0, m_1, m_2, \dots)$ be an infinite sequence of real numbers. For $n \in \mathbb{N}_0$, define the *Hankel matrix of moments* as

$$H_n(\mathbf{m}) := \begin{bmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \dots & m_{2n} \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}. \quad (10)$$

The sequence $\mathbf{m}$ is a $\mathbb{R}$ -moment sequence if and only if, for every $n \in \mathbb{N}_0$, $H_n(\mathbf{m}) \geq 0$.

The characterizations of $\mathcal{K}$ -moment sequences where $\mathcal{K}$ is an interval of the form $(-\infty, u]$ or $[-u, u]$ are known as the Stieltjes and Hausdorff moment problems, respectively. A proof for the following theorem, known as *Stieltjes' theorem*, can be found in [19] for the case where $u = 0$, and it can be easily adapted to any $u \in \mathbb{R}$ through a simple change of variables.

Theorem 7 (*Stieltjes' theorem*). Let $\mathbf{m} = (m_0, m_1, m_2, \dots)$ be an infinite sequence of real numbers. For $n \in \mathbb{N}_0$, define the shifted Hankel matrix of moments $S_n(\mathbf{m})$ as

$$S_n(\mathbf{m}) := \begin{bmatrix} m_1 & m_2 & \dots & m_{n+1} \\ m_2 & m_3 & \dots & m_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n+1} & m_{n+2} & \dots & m_{2n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}.$$

The sequence $\mathbf{m}$ is a $(-\infty, u]$ -moment sequence if and only if, for every $n \in \mathbb{N}_0$,

$$H_n(\mathbf{m}) \geq 0, \quad \text{and} \quad uH_n(\mathbf{m}) - S_n(\mathbf{m}) \geq 0. \quad (11)$$

Similarly, the sequence $\mathbf{m}$ is a $[-u, \infty)$ -moment sequence if and only if, for every $n \in \mathbb{N}_0$,

$$H_n(\mathbf{m}) \geq 0, \quad \text{and} \quad uH_n(\mathbf{m}) + S_n(\mathbf{m}) \geq 0. \quad (12)$$

The positive (semi)definiteness of a symmetric matrix can be certified using *Sylvester's criterion*.

Theorem 8 (*Sylvester's criterion* [20]). A matrix M is positive semidefinite if and only if the determinant of every principal submatrix is non-negative. Moreover, M is positive definite if and only if the determinant of every leading principal submatrix R is positive.

3. Lower bounds on the spectral radius

The supports of the spectral measures in Definition (3) are contained in the interval $[-\rho, \rho]$ and their moments can be written in terms of walks in $\mathcal{G}$. Since the moments of a measure impose constraints on its support, the number of walks in $\mathcal{G}$ imposes constraints on ρ , as stated below. (See Table 1.)

Lemma 9. For a graph $\mathcal{G}$, let $\mathbf{m}$ be the sequence of moments of any measure supported on the spectrum of $\mathcal{G}$. Then, for any finite set $\mathcal{J} \subset \mathbb{N}_0$,

$$\rho H_{\mathcal{J}}(\mathbf{m}) - S_{\mathcal{J}}(\mathbf{m}) \geq 0, \quad (13)$$

$$\rho H_{\mathcal{J}}(\mathbf{m}) + S_{\mathcal{J}}(\mathbf{m}) \geq 0, \quad (14)$$

Table 1
Summary of lower bounds on the spectral radius ρ , obtained as corollaries of Lemma 9. The number of k -walks, closed k -walks and closed k -walks from node i , in $\mathcal{G}$, are denoted by w_k , ϕ_k and $\phi_k^{(i)}$, respectively. We write n , e and T to denote the number of nodes, edges and triangles in $\mathcal{G}$. We write d_i and T_i to denote the degree of node i and the number of triangles touching node i , respectively. The largest node degree is denoted by Δ .

General bound	Corollaries	Moment sequence	Reference
$\rho^k \geq \frac{m_{2s+k}}{m_{2s}}$	$\rho^k \geq \frac{w_{2s+k}}{w_{2s}}$ [7]	walks	Corollary 11
	$\rho^k \geq \frac{\phi_{2s+k}}{\phi_{2s}}$	closed walks	
	$\rho^k \geq \frac{\phi_{2s+k}^{(i)}}{\phi_{2s}^{(i)}}$	closed walks from node i	
$\rho \geq \frac{ \det(F^{(2s,k)}) + \sqrt{\det(F^{(2s,k)})^2 - 4 \det(H^{(2s,k)}) S^{(2s,k)}}}{2 \det(H^{(2s,k)})}$	$\rho \geq \frac{3T}{4e} + \sqrt{\left(\frac{3T}{4e}\right)^2 + \frac{e}{n}}$	closed walks	Corollary 14
	$\rho \geq \max_{i \in \{1, \dots, n\}} \frac{T_i + \sqrt{T_i^2 + d_i^3}}{d_i}$	closed walks from node i	Corollary 15
	$\rho \geq \sqrt{\Delta}$ [10]	closed walks from node i	

where $H_{\mathcal{J}}(\mathbf{m})$ and $S_{\mathcal{J}}(\mathbf{m})$ are submatrices of $H_n(\mathbf{m})$ and $S_n(\mathbf{m})$, defined in (6) and (7), respectively.

Proof. Since $\mathbf{m}$ corresponds to the sequence of moments of a measure whose support is contained in $[-\rho, \rho]$, it follows that ρ must satisfy the necessary conditions (11) and (12). Furthermore, every principal submatrix of a positive semidefinite matrix is also positive semidefinite by Sylvester's criterion; hence, the matrix inequalities (13) and (14) follow. $\square$

As stated in Lemma 4, the moments of all the three measures defined in Definition 3 can be written in terms of walks in the graph. Since the supports of these three measures are equal to the eigenvalue spectrum of $\mathcal{G}$, we can apply Lemma 9 to the moment sequences obtained by counting different types of walks in the graph. Using the above Lemma, we can use a truncated sequence of moments to find a lower bound on ρ by solving a *semidefinite program* [21], as stated below:

Theorem 10. *The solution to the following semidefinite program is a lower bound on the spectral radius of $\mathcal{G}$*

$$\begin{array}{ll} \min_u & u \\ \text{s.t.} & uH_n(\mathbf{m}) - S_n(\mathbf{m}) \succeq 0, \\ & uH_n(\mathbf{m}) + S_n(\mathbf{m}) \succeq 0, \end{array}$$

where $\mathbf{m} = (m_0, m_1, \dots, m_{2n+1})$ is a truncated sequence of moments of any measure supported on the spectrum of $\mathcal{G}$.

The above Theorem can be used to compute numerical bounds on the spectral radius by setting the moments to be one of $\phi_k, \phi_k^{(i)}$, or w_k . Moreover, we can use Lemma 9 to obtain closed-form bounds on ρ involving a small number of moments for which the semidefinite program in Theorem 10 can be solved analytically. The following corollary analyzes the case where $|\mathcal{J}| = 1$.

Corollary 11. *For an undirected graph $\mathcal{G}$, let $\mathbf{m}$ be the sequence of moments of a measure supported on the spectrum of $\mathcal{G}$. Then, for every $k, s \in \mathbb{N}_0$,*

$$\rho^k \geq \frac{m_{2s+k}}{m_{2s}}. \quad (15)$$

Proof. Let $\zeta(x)$ be a measure supported on $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, defined as $\zeta(x) := \sum_{i=1}^n z_i \delta(x - \lambda_i)$ and let $\{m_0, m_1, \dots\}$ be its moment sequence. For every $k, s \in \mathbb{N}_0$, we construct the following measure based on $\zeta(x)$:

$$\zeta_{s,k}(x) := \sum_{i=1}^n z_i \lambda_i^{2s} \delta(x - \lambda_i^k). \quad (16)$$

We see that $\zeta_{s,k}(x)$ is supported on $\{\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k\}$, and its moments are given by the sequence $\{m_{2s}, m_{2s+k}, m_{2s+2k}, \dots\}$, for $k \in \mathbb{N}_0$. We note that the support of the measure $\zeta_{s,k}(x)$ is contained in $[-\rho^k, \rho^k]$; thus, setting $\mathcal{J} = \{0\}$, we use Lemma 9 to obtain $\rho^k m_{2s} - m_{2s+k} \geq 0$, which implies (15). $\square$

If we set $\mathbf{m} = \{w_s\}_{s=0}^\infty$, this corollary gives an alternative proof for the lower bounds in (2), proven by Nikiforov [7]. It also generalizes these results to closed walks by using ϕ_k or $\phi_k^{(i)}$ as the sequence of moments.

Another interesting result comes from applying Lemma 9 to the case where $|\mathcal{J}| = 2$. Corollary 13 below provides a new lower bound in terms of the largest root of a quadratic polynomial. Its proof relies on the following lemma.

Lemma 12. *Let $\mathbf{m}$ be the sequence of moments of a measure supported on the spectrum of $\mathcal{G}$. For $s, k \geq 0$, define the following matrices:*

$$H^{(2s,k)} := \begin{bmatrix} m_{2s} & m_{2s+k} \\ m_{2s+k} & m_{2s+2k} \end{bmatrix}, \quad \text{and} \quad S^{(2s,k)} := \begin{bmatrix} m_{2s+k} & m_{2s+2k} \\ m_{2s+2k} & m_{2s+3k} \end{bmatrix}. \quad (17)$$

Whenever $\det(H^{(2s,k)}) \neq 0$, we have

$$\rho^{2k} \geq \frac{\det(S^{(2s,k)})}{\det(H^{(2s,k)})}.$$

Proof. Let $\zeta_{2s,k}(x)$ be the measure defined in (16). The support of this measure is contained in $[-\rho^k, \rho^k]$ and has moment sequence $(m_{2s}, m_{2s+k}, m_{2s+2k}, \dots)$. Applying Lemma 9 with $\mathcal{J} = \{0, 1\}$ we obtain

$$\rho^k H^{(2s,k)} \pm S^{(2s,k)} \succeq 0 \implies \rho^k \geq \frac{\mathbf{x}^\top S^{(2s,k)} \mathbf{x}}{\mathbf{x}^\top H^{(2s,k)} \mathbf{x}}, \quad (18)$$

for every non-zero $\mathbf{x} \in \mathbb{R}^2$. From Theorem 6, we know that $H^{(2s,k)} \succeq 0$; hence, its eigenvalues ξ_1 and ξ_2 satisfy $\xi_1 \geq \xi_2 \geq 0$. By Rayleigh principle, we have that

$$\frac{\mathbf{x}^\top H^{(2s,k)} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \leq \xi_1, \quad \frac{\mathbf{w}^\top H^{(2s,k)} \mathbf{w}}{\mathbf{w}^\top \mathbf{w}} = \xi_2, \quad (19)$$

for every non-zero $\mathbf{x}$ and for $\mathbf{w}$ being the eigenvector corresponding to the second eigenvalue of $H^{(2s,k)}$. Similarly, let $\gamma_1 \geq \gamma_2$ be the eigenvalues of $S^{(2s,k)}$. By Perron-Frobenius, we know that $\gamma_1 \geq 0$. If $\gamma_2 < 0$, then $\det(S^{(2s,k)}) < 0$ and the inequality (18) is trivial. If instead $\gamma_2 \geq 0$, then

$$\frac{\mathbf{x}^\top S^{(2s,k)} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \geq \gamma_2, \quad \frac{\mathbf{v}^\top S^{(2s,k)} \mathbf{v}}{\mathbf{v}^\top \mathbf{v}} = \gamma_1, \quad (20)$$

for any non-zero $\mathbf{x}$ and for $\mathbf{v}$ equal to the leading eigenvector of $S^{(2s,k)}$. We plug vectors $\mathbf{v}$ and $\mathbf{w}$ into (18) to obtain

$$\begin{aligned}\rho^k &\geq \frac{\mathbf{v}^\top S^{(2s,k)} \mathbf{v}}{\mathbf{v}^\top H^{(2s,k)} \mathbf{v}} \geq \frac{\gamma_1}{\xi_1}, \\ \rho^k &\geq \frac{\mathbf{w}^\top S^{(2s,k)} \mathbf{w}}{\mathbf{w}^\top H^{(2s,k)} \mathbf{w}} \geq \frac{\gamma_2}{\xi_2},\end{aligned}$$

where the last inequalities come from (19) and (20). Multiplying both inequalities we obtain

$$\rho^{2k} \geq \frac{\gamma_1 \gamma_2}{\xi_1 \xi_2} = \frac{\det(S^{(2s,k)})}{\det(H^{(2s,k)})}. \quad \square$$

We are now ready to prove the following corollary.

Corollary 13. *Let $\mathbf{m}$ be the sequence of moments of a measure supported on the spectrum of $\mathcal{G}$. For $s, k \in \mathbb{N}_0$, let $H^{(2s,k)}$ and $S^{(2s,k)}$ be defined as in (17) and define the following matrix:*

$$F^{(2s,k)} := \begin{bmatrix} m_{2s+k} & m_{2s+3k} \\ m_{2s} & m_{2s+2k} \end{bmatrix}.$$

Then, whenever $\det(H^{(2s,k)}) \neq 0$, we have

$$\rho^k \geq \frac{|\det(F^{(2s,k)})| + \sqrt{\det(F^{(2s,k)})^2 - 4 \det(H^{(2s,k)} S^{(2s,k)})}}{2 \det(H^{(2s,k)})}.$$

Proof. The inequality (18) implies that $\det(\rho^k H^{(2s,k)} + S^{(2s,k)}) \geq 0$. This can be expanded to

$$\det \left(\begin{bmatrix} \rho^k m_{2s} + m_{2s+k} & \rho^k m_{2s+k} + m_{2s+2k} \\ \rho^k m_{2s+k} + m_{2s+2k} & \rho^k m_{2s+2k} + m_{2s+3k} \end{bmatrix} \right) \geq 0,$$

which simplifies to

$$\det(H^{(2s,k)}) \rho^{2k} - \det(F^{(2s,k)}) \rho^k + \det(S^{(2s,k)}) \geq 0. \quad (21)$$

Similarly, Lemma 9 implies that $\det(\rho^k H^{(2s,k)} - S^{(2s,k)}) \geq 0$, which implies

$$\det(H^{(2s,k)}) \rho^{2k} + \det(F^{(2s,k)}) \rho^k + \det(S^{(2s,k)}) \geq 0. \quad (22)$$

Inequalities (21) and (22) are satisfied simultaneously if and only if

$$\det \left(H^{(2s,k)} \right) \rho^{2k} - \left| \det \left(F^{(2s,k)} \right) \right| \rho^k + \det \left(S^{(2s,k)} \right) \geq 0. \quad (23)$$

By Theorem 6 we have that $\det \left(H^{(2s,k)} \right) > 0$. Using Lemma 12, we know that $\det \left(S^{(2s,k)} \right) \leq \det \left(H^{(2s,k)} \right) \rho^{2k}$, which we substitute into (23) to yield

$$2 \det \left(H^{(2s,k)} \right) \rho^{2k} - \left| \det \left(F^{(2s,k)} \right) \right| \rho^k \geq 0.$$

Since $\rho \geq 0$, we conclude that

$$\rho^k \geq \frac{\left| \det \left(F^{(2s,k)} \right) \right|}{2 \det \left(H^{(2s,k)} \right)}. \quad (24)$$

Next, define the quadratic polynomial

$$P(r) = \det \left(H^{(2s,k)} \right) r^2 - \left| \det \left(F^{(2s,k)} \right) \right| r + \det \left(S^{(2s,k)} \right),$$

which has a positive leading coefficient. From (23) we know that $P(\rho^k) \geq 0$ and from (24) we know that ρ^k is larger than the smallest root of P . This implies that ρ^k is larger than the largest root of P and the result follows. $\square$

We can apply Corollary 13 with $s = 0$ and $k = 1$ to the closed-walks measure of a graph $\mathcal{G}$, leveraging the fact that $\phi_0 = n$, $\phi_1 = 0$, ϕ_2 is twice the number of edges, and ϕ_3 is three times the number of triangles in $\mathcal{G}$, as follows.

Corollary 14. *For a graph $\mathcal{G}$ with n vertices, e edges and T triangles, we have that*

$$\rho \geq \frac{3T}{4e} + \sqrt{\left(\frac{3T}{4e} \right)^2 + \frac{e}{n}}.$$

Similarly, because $\phi_0^{(i)} = 1$, $\phi_1^{(i)} = 0$, $\phi_2^{(i)}$ is the degree of vertex i , and $\phi_3^{(i)}$ is twice the number of triangles touching vertex i , we can apply Corollary 13 to the closed-walks measure for vertex i .

Corollary 15. *Denoting by d_i the degree of vertex i and T_i the number of triangles touching vertex i , we have*

$$\rho \geq \max_{i \in \{1, \dots, n\}} \frac{T_i + \sqrt{T_i^2 + d_i^3}}{d_i}.$$

Notice how Corollary 15 implies

$$\rho \geq \frac{T_{\Delta} + \sqrt{T_{\Delta}^2 + \Delta^3}}{\Delta} \geq \sqrt{\Delta},$$

where Δ is the maximum of the vertex degrees in $\mathcal{G}$ and T_{Δ} is the maximum triangle count amongst vertices with degree Δ , improving the bound in (5).

4. Upper bounds on the spectral radius

In this section, we make use of Theorems 6 and 7 to derive upper bounds on ρ . (See Table 2.) In particular, if $\zeta(x)$ is a spectral measure defined as $\zeta(x) := \sum_{i=1}^n z_i \delta(x - \lambda_i)$ with moment sequence $\{m_0, m_1, \dots\}$, then $\{m_0 - \rho^0, m_1 - \rho^1, \dots\}$ is the moment sequence of the measure $\sum_{i=2}^n z_i \delta(x - \lambda_i)$ whose support is also contained in $[-\rho, \rho]$. A direct application of Hamburger's Theorem thus yields the following result:

Lemma 16. *Let $\mathbf{m}$ be the sequence of moments of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of $\mathcal{G}$, and define the infinite-dimensional Hankel matrix P given by:*

$$P := \begin{bmatrix} 1 & \rho & \rho^2 & \dots \\ \rho & \rho^2 & \rho^3 & \dots \\ \rho^2 & \rho^3 & \rho^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hence, for any finite $\mathcal{J} \subset \mathbb{N}_0$,

$$H_{\mathcal{J}}(\mathbf{m}) - \alpha_1 P_{\mathcal{J}} \succeq 0,$$

where $H_{\mathcal{J}}(\mathbf{m})$ is a submatrix of H_n defined in (10).

Proof. We simply note that the matrix $H_{\mathcal{J}}(\mathbf{m}) - \alpha_1 P$ is the Hankel matrix containing the moments of the measure $\sum_{i=2}^n \alpha_i \delta(x - \lambda_i)$. The result follows directly from Theorem 6 and Sylvester's criterion. $\square$

Corollary 17. *Let $\mathbf{m}$ be the sequence of moments of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of*

$$\rho^{2k} \leq \frac{m_{2k}}{\alpha_1}.$$

Proof. For $\mathcal{J} = \{k\}$, Lemma 16 implies

$$m_{2k} - \alpha_1 \rho^{2k} \geq 0.$$

Table 2

Upper bounds on the spectral radius ρ , where m_k is the k -th moment of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of $\mathcal{G}$. The number of k -walks, closed k -walks and closed k -walks from node i are denoted by w_k , ϕ_k and $\phi_k^{(i)}$, respectively. We write $\omega(\mathcal{G})$, c_1 and x_i for the clique number of $\mathcal{G}$, the fundamental weight of A , and the i -th component of the leading eigenvector of A , respectively.

General bound	Corollaries	Moment sequence	Reference
$\rho^{2k} \leq \frac{m_{2k}}{\alpha_1}$	$\rho^{2k} \leq \phi_{2k}$	closed walks	Corollary 17
	$\rho^{2k} \leq \frac{w_{2k}}{c_1}$	all	
	$\rho^{2k+1} \leq \left(1 - \frac{1}{\omega(\mathcal{G})}\right) w_{2k}$ [7]	walks	
	$\rho^{2k} \leq \frac{\phi_{2k}}{2}$	closed walks (bipartite $\mathcal{G}$)	Corollary 20
	$\rho^{2k} \leq \frac{\phi_{2k}^{(i)}}{2}$	closed walks from node i (bipartite $\mathcal{G}$)	
$\rho^k \leq \frac{m_k + \sqrt{\left(\frac{m_0}{\alpha_1} - 1\right)(m_0 m_{2k} - m_k^2)}}{m_0}$	$\rho \leq \sqrt{\left(\frac{1}{x_i^2} - 1\right) d_i}$ [15]	closed walks from node i	Corollary 19
$\rho \leq \max_{r \text{ s.t. } m_{2k}r + m_{2k+1} - 2\alpha_1 r^{2k+1} = 0} r$	$\rho \leq \max_{r \text{ s.t. } w_{2k}r + w_{2k+1} - 2c_1 r^{2k+1} = 0} r$	walks	Corollary 23
	$\rho \leq \max_{r \text{ s.t. } w_{2k}r + w_{2k+1} - 2\frac{\omega(\mathcal{G})}{\omega(\mathcal{G}) - 1} r^{2k+2} = 0} r$	walks	Corollary 24

This finishes the proof. $\square$

Applying this corollary to the measures $\mu_{\mathcal{G}}, \mu_{\mathcal{G}}^{(i)}$ and $\nu_{\mathcal{G}}$, we obtain three different hierarchies of bounds. For example, applying Corollary 17 to the measure $\nu_{\mathcal{G}}$, which has moments $\mathbf{m} = \{w_s\}_{s=0}^{\infty}$, we obtain the bound

$$\rho^{2k} \leq \frac{w_{2k}}{c_1}, \quad (25)$$

where $c_1 = (\sum_{i=1}^n u_{i1})^2$ is the fundamental weight. One can show that the bound (6) proved by Nikiforov [7] is a corollary of (25), albeit only for even exponents. In particular, multiplying on both sides of (25) by ρ and using Wilf's inequality (7), we obtain

$$\rho^{2k+1} \leq \rho \frac{w_{2k}}{c_1} \leq \left(1 - \frac{1}{\omega(\mathcal{G})}\right) c_1 \frac{w_{2k}}{c_1} = \left(1 - \frac{1}{\omega(\mathcal{G})}\right) w_{2k},$$

where the last quantity is the upper bound in (6).

Using Lemma 16 with larger principal submatrices, we can improve these upper bounds further. The following upper bound is obtained by analyzing the case of $\mathcal{J} = \{0, k\}$.

Corollary 18. *Let $\mathbf{m}$ be the sequence of moments of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of $\mathcal{G}$. Then, for any $k \in \mathbb{N}$*

$$\rho^k \leq \frac{m_k + \sqrt{\left(\frac{m_0}{\alpha_1} - 1\right) (m_0 m_{2k} - m_k^2)}}{m_0}. \quad (26)$$

Furthermore, this bound is tighter than the one in Corollary 17.

Proof. Applying Lemma 16 with $\mathcal{J} = \{0, k\}$, we conclude that $\det(H_{\mathcal{J}}(\mathbf{m}) - \alpha_1 P_{\mathcal{J}}) \geq 0$, which simplifies to the following expression:

$$-m_0 \rho^{2k} + 2m_k \rho^k + \frac{1}{\alpha_1} ((m_0 - \alpha_1) m_{2k} - m_k^2) \geq 0.$$

Making the substitution $y = \rho^k$, we obtain the following quadratic inequality

$$-m_0 y^2 + 2m_k y + \frac{1}{\alpha_1} ((m_0 - \alpha_1) m_{2k} - m_k^2) \geq 0.$$

The quadratic on the left-hand side has a negative leading coefficient, which implies it is negative whenever y is larger than its largest root, which is given by the right hand side of (26). After substituting back ρ^k , the result follows. To see that this bound improves the one in Corollary 17, note that Corollaries 17 and 11 imply

$$\begin{aligned}
m_{2k} \geq \alpha_1 \rho^{2k} \geq \alpha_1 \frac{m_{4k}}{m_{2k}} &\implies m_{4k} \leq \frac{1}{\alpha_1} m_{2k}^2 \\
&\implies m_0 m_{4k} - m_{2k}^2 \leq \left(\frac{m_0}{\alpha_1} - 1 \right) m_{2k}^2.
\end{aligned}$$

Hence, we can use (26) to obtain

$$\begin{aligned}
\rho^{2k} &\leq \frac{m_{2k} + \sqrt{\left(\frac{m_0}{\alpha_1} - 1 \right) (m_0 m_{4k} - m_{2k}^2)}}{m_0} \\
&\leq \frac{m_{2k} + \sqrt{\left(\frac{m_0}{\alpha_1} - 1 \right) \left(\frac{m_0}{\alpha_1} - 1 \right) m_{2k}^2}}{m_0} = \frac{m_{2k}}{\alpha_1}. \quad \square
\end{aligned}$$

As with previous results, we can obtain concrete bounds from this result by substituting α_1 and m_k by either (i) 1 and ϕ_k , (ii) $c_1^{(i)}$ and $\phi_k^{(i)}$, or (iii) c_1 and w_k , respectively. For example, we can apply Corollary 18 to the closed-walks measure for node i , $\mu_{\mathcal{G}}^{(i)}$, using $\mathcal{J} = \{0, 1\}$. Since $\phi_0(i) = 1$ and $\phi_1(i) = 0$, we obtain the upper bound in the following Corollary.

Corollary 19. *For a graph $\mathcal{G}$*

$$\rho^2 \leq \left(\frac{1}{c_1^i} - 1 \right) \phi_2(i) = \left(\frac{1}{x_i^2} - 1 \right) d_i, \quad \text{for all } i \in \{1, \dots, n\}, \quad (27)$$

where d_i is the degree of vertex i and x_i is the i -th component of the leading eigenvector of A .

Notice that inequality (27) can be written as

$$x_i \leq \frac{1}{\sqrt{1 + \frac{\rho^2}{d_i}}},$$

which was first proven by Cioabă and Gregory in [15]. Our method provides an alternative proof. Furthermore, in the case of bipartite graphs, we can refine Corollary 19 as follows.

Corollary 20. *For a bipartite graph $\mathcal{G}$ and $k \in \mathbb{N}_0$, we have*

$$\rho^{2k} \leq \frac{\phi_{2k}}{2}, \quad (28)$$

$$\rho^{2k} \leq \frac{\phi_{2k}^{(i)}}{2u_{i1}^2}. \quad (29)$$

Proof. Notice that, since the eigenvalue spectrum of a bipartite graph is symmetric, we have that $\phi_{2k} = \sum_{\ell=1}^{\lceil n/2 \rceil} 2\lambda_{\ell}^{2k}$. The upper bound given by (28) then follows by a straightforward application of Corollary 17 to the measure $\sum_{\ell=1}^{\lceil n/2 \rceil} 2\delta(x - \lambda_{\ell}^2)$.

A similar application can be done for closed-walks from node i . It is easy to see that

$$\phi_{2k}^{(i)} = \sum_{\ell=1}^{\lceil n/2 \rceil} (u_{i\ell}^2 + u_{i,n-\ell}^2) \lambda_{\ell}^{2k}. \quad (30)$$

In what follows, we prove that $u_{i\ell}^2 = u_{i,n-\ell}^2$ using the *eigenvector-eigenvalue identity* [22]. We prove the equality for the case of odd n and note that for even n there is an analogous proof. Let $M_{\{-i\}}$ be the submatrix of A obtained by deleting row i and column i , and let $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{n-1}$ be the eigenvalues of $M_{\{-i\}}$. Because $M_{\{-i\}}$ is the adjacency matrix of the graph obtained by deleting node i , which is also bipartite, its spectrum is also symmetric. From the *eigenvector-eigenvalue identity* we have:

$$u_{i,\ell}^2 \prod_{j=1; j \neq \ell}^n (\lambda_{\ell} - \lambda_j) = \prod_{j=1}^{n-1} (\lambda_{\ell} - \gamma_j). \quad (31)$$

Using the symmetry of the spectrum of $M_{\{-i\}}$, the right hand side of (31) can be rewritten as

$$\prod_{j=1}^{n-1} (\lambda_{\ell} - \gamma_j) = \prod_{j=1}^{\lfloor n/2 \rfloor} (\lambda_{\ell}^2 - \gamma_j^2). \quad (32)$$

Similarly, the symmetry of the spectrum of A implies $\lambda_{n-\ell} = -\lambda_{\ell}$ and $\lambda_{\lceil n/2 \rceil} = 0$. Thus, the left hand side of (31) can be rewritten as

$$u_{i,\ell}^2 \prod_{j=1; j \neq \ell}^n (\lambda_{\ell} - \lambda_j) = u_{i,\ell}^2 (2\lambda_{\ell}) \lambda_{\ell} \prod_{\substack{j=1 \\ j \neq \ell, n-\ell}}^{\lceil n/2 \rceil} (\lambda_{\ell}^2 - \lambda_j^2), \quad (33)$$

where the factor $2\lambda_{\ell}$ on the right corresponds to the term $(\lambda_{\ell} - \lambda_{n-\ell})$ of the product on the left, the factor λ_{ℓ} corresponds to the term $(\lambda_{\ell} - \lambda_{\lceil n/2 \rceil})$ of the product on the left, and the third factor is the product of the pairs of remaining eigenvalues, multiplying each factor $(\lambda_{\ell} - \lambda_j)$ with the corresponding factor $(\lambda_{\ell} + \lambda_j)$. Because $(2\lambda_{\ell}) \lambda_{\ell} = (-2\lambda_{\ell}) (-\lambda_{\ell})$, it becomes clear that solving for $u_{i\ell}^2$ in (31) yields the same result as solving for $u_{i,n-\ell}^2$.

Hence, we can conclude from (30) that

$$\phi_{2k}^{(i)} = \sum_{\ell=1}^{\lceil n/2 \rceil} 2u_{i\ell}^2 \lambda_{\ell}^{2k}.$$

Thus, the upper bound (29) follows from applying Corollary 17 to the measure $\sum_{\ell=1}^{\lceil n/2 \rceil} 2u_{i\ell}^2 \delta(x - \lambda_{\ell}^2)$. $\square$

A more general version of Corollary 18, for general undirected graphs, is given in the following theorem.

Theorem 21. *Let $\mathbf{m}$ be the sequence of moments of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of $\mathcal{G}$. Define the infinite dimensional Hankel matrix R given by*

$$R := \begin{bmatrix} 1 & r & r^2 & \dots \\ r & r^2 & r^3 & \dots \\ r^2 & r^3 & r^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $\mathcal{J} = \{j_1, \dots, j_s\}$ and $\mathcal{J}' = \{j_1, \dots, j_{s-1}\}$ for $j_1 < j_2 < \dots < j_s \in \mathbb{N}_0$, such that $H_{\mathcal{J}'}(\mathbf{m}) \succ 0$. Then, the largest root r^ of the polynomial:*

$$Q(r) := \det(H_{\mathcal{J}}(\mathbf{m}) - \alpha_1 R_{\mathcal{J}}),$$

is an upper bound on the spectral radius.

Proof. We will prove that $Q(r)$ has a negative leading coefficient equal to $-\alpha_1 \det(H(\mathbf{m})_{\mathcal{J}'})$. It is well known (see for example [23]) that if $A \in \mathbb{R}^{n \times n}$ and $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^n$ then

$$\det(A + \mathbf{u}\mathbf{v}^\top) = \det(A) + \mathbf{v}^\top \text{adj}(A)\mathbf{u}, \quad (34)$$

where $\text{adj}(A)$ is the cofactor matrix of A . Let $\mathbf{r} := (r^{j_1-1}, \dots, r^{j_s-1})$. We note that

$$-\alpha_1 R_{\mathcal{J}} = (\alpha_1 (r^{j_1-1}, \dots, r^{j_s-1}))^\top (- (r^{j_1-1}, \dots, r^{j_s-1})) = (\alpha_1 \mathbf{r})(-\mathbf{r})^\top.$$

Using (34), we obtain

$$\det(H_{\mathcal{J}}(\mathbf{m}) - \alpha_1 R_{\mathcal{J}}) = \det(H_{\mathcal{J}}(\mathbf{m})) - \alpha_1 \mathbf{r}^\top \text{adj}(H_{\mathcal{J}}(\mathbf{m}))\mathbf{r}.$$

It follows that the leading coefficient of $Q(r)$ is $-\alpha \det(H_{\mathcal{J}'}) < 0$. By Lemma 16 we have $Q(\rho) \geq 0$ and, therefore, $\rho \leq r^*$. $\square$

Lemma 16 was proved applying Hamburger's Theorem to the moment sequence $\{m_s - \alpha_1 r^s\}_{s=0}^\infty$. We can also apply Stieltjes' Theorem to the same moment sequence to obtain a different hierarchy of upper bounds. This is stated in the following theorem.

Theorem 22. *Let $\mathbf{m}$ be the sequence of moments of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of $\mathcal{G}$. Then, for any $\mathcal{J} \in \mathbb{N}_0$,*

$$\rho(H_{\mathcal{J}}(\mathbf{m}) - \alpha P_{\mathcal{J}}) + (S_{\mathcal{J}}(\mathbf{m}) - \alpha_1 \rho P_{\mathcal{J}}) \succeq 0,$$

where $H(\mathbf{m})$ and $S(\mathbf{m})$ are the Hankel matrices of moments defined in Theorems 6 and 7, respectively.

Proof. Recall from Lemma 16 that the sequence $\{m_s - \alpha_1 \rho^s\}_{s=0}^{\infty}$ corresponds to the moment sequence of a measure whose support is contained in $[-\rho, \rho]$ and, therefore, the result follows from Theorem 7. $\square$

Theorem 22 can be used to obtain bounds that improve on those of Corollary 17, as shown below.

Corollary 23. Let $\mathbf{m}$ be the sequence of moments of a measure $\mu(x) = \sum_{i=1}^n \alpha_i \delta(x - \lambda_i)$ supported on the spectrum of $\mathcal{G}$. Then, the largest root r^* of the following polynomial:

$$Q(r) := m_{2k}r + m_{2k+1} - 2\alpha_1 r^{2k+1},$$

is an upper bound on the spectral radius. Furthermore, this bound is tighter than the bound in Corollary 17.

Proof. Applying Theorem 22 with $\mathcal{J} = \{k\}$, we obtain

$$\rho[m_{2k}] + [m_{2k+1}] - 2\alpha_1 \rho[\rho^{2k}] \succeq 0 \implies \rho m_{2k} + m_{2k+1} - 2\alpha_1 \rho^{2k+1} \geq 0,$$

and, thus, $Q(\rho) \geq 0$. Since the leading coefficient of $Q(r)$ is negative, it follows that $\rho \leq r^*$. The upper bound in Corollary 17 can be written as $(m_{2k}/\alpha_1)^{1/2k}$, hence, proving that r^* is a tighter bound amounts to proving $r^* \leq (m_{2k}/\alpha_1)^{1/2k}$. We first prove that r^* is the unique root of $Q(r)$ in the region defined by

$$r \geq \left(\frac{m_{2k}}{2\alpha_1(2k+1)} \right)^{1/2k}. \quad (35)$$

This is indeed the case, because the derivative of $Q(r)$, given by $Q'(r) = m_{2k} - 2(2k+1)\alpha_1 r^{2k}$, is negative in the interval defined in (35). Also, note that

$$\left(\frac{m_{2k}}{\alpha_1} \right)^{1/2k} \geq \left(\frac{m_{2k}}{2\alpha_1(2k+1)} \right)^{1/2k}.$$

Therefore, it suffices to show that

$$Q\left(\left(\frac{m_{2k}}{\alpha_1} \right)^{1/2k} \right) \leq 0.$$

To this end, we evaluate and obtain

$$\begin{aligned}
& Q \left(\left(\frac{1}{\alpha_1} m_{2k} \right)^{1/2k} \right) \leq 0 \\
& \iff \left(\frac{1}{\alpha_1} m_{2k} \right)^{1/2k} m_{2k} + m_{2k+1} - 2\alpha_1 \left(\left(\frac{1}{\alpha_1} m_{2k} \right)^{1/2k} \right)^{2k+1} \leq 0 \\
& \iff \left(\frac{1}{\alpha_1} m_{2k} \right)^{1/2k} m_{2k} + m_{2k+1} - 2m_{2k} \left(\frac{1}{\alpha_1} m_{2k} \right)^{1/2k} \leq 0 \\
& \iff \frac{m_{2k+1}}{m_{2k}} \leq \left(\frac{1}{\alpha_1} m_{2k} \right)^{1/2k},
\end{aligned}$$

where the last inequality is true since the left-hand side is a lower bound of ρ by Corollary 11, and the right hand side is an upper bound of ρ by Corollary 17. This finishes the proof. $\square$

The implicit bound in Corollary 23, when applied to the moment sequence $\mathbf{m} = \{w_s\}_{s=0}^\infty$, provides an improvement on the bound given in Corollary 11. Using inequality (7), we can also obtain a bound in terms of the clique number instead of the fundamental weight, which also improves on (2), as we show below.

Corollary 24. *The largest root r^* of the following polynomial:*

$$Q(r) := m_{2k}r + m_{2k+1} - 2 \frac{\omega(\mathcal{G})}{\omega(\mathcal{G}) - 1} r^{2k+2},$$

is an upper bound on the spectral radius. Furthermore, this bound is an improvement on (6).

Proof. The proof is very similar to that of Corollary 23. Using (7), we have that

$$\rho w_{2k} + w_{2k+1} - \frac{\omega(\mathcal{G})}{\omega(\mathcal{G}) - 1} \rho^{2k+1} \geq \rho w_{2k} + w_{2k+1} - 2\alpha_1 \rho^{2k+1} \geq 0.$$

We omit the details to avoid repetitions. $\square$

Declaration of competing interest

The authors declare no competing interests.

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