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The Brezis–Nirenberg problem for systems involving divergence-form operators

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Abstract. We study a system of nonlinear elliptic partial differential equations involving divergence-form operators. The problem under consideration is a natural generalization of the classical Brezis–Nirenberg problem. We find conditions on the domain, the coupling coefficients and the coefficients of the differential operator under which positive solutions are guaranteed to exist and conditions on these objects under which no positive solution exists.

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1. Introduction

In 1983 Brezis and Nirenberg [5] determined conditions on $\lambda \in \mathbb{R}$ and the bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ for which the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{\frac{4}{n-2}} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

admits a positive solution and conditions on these objects under which problem (1.1) does not admit a positive solution. They established the following theorem. In the statement of the theorem, $\lambda_1 = \lambda_1(-\Delta) > 0$ is the first eigenvalue of the Dirichlet Laplacian.

Theorem A. *(a) If $n = 3$ then there are constants $\lambda_*(\Omega) \leq \lambda^*(\Omega)$ satisfying $0 < \lambda_* \leq \lambda^* < \lambda_1$ such that (1.1) admits a positive solution if $\lambda \in (\lambda^*, \lambda_1)$ and (1.1) does not admit a positive solution if $\lambda \in (0, \lambda_*]$.*

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(b) If $n \geq 4$ then problem (1.1) admits a positive solution if and only if $\lambda \in (0, \lambda_1)$.

The subtleties on the conditions under which problem (1.1) is solvable, at least in the case that one is interested in positive solutions, are already apparent in the statement of Theorem A–the solvability depends on the the dimension n, the domain Ω and the value of λ. These subtleties make problem (1.1) a natural candidate for further investigation and indeed, these particular subtleties were investigated in the works that followed [5]. For example, the regime $\lambda > \lambda_1$ was considered in [6], where it was shown that if $n \geq 4$ then problem (1.1) admits a non-trivial solution. In [17], problem (1.1) was realized as a member of a more general family of problems and the local L^2-summability (or lack thereof) of the fundamental solution for $-\Delta$ was identified as a reason for the fact that, in Theorem A, the conditions on λ under which problem (1.1) admits a positive solution depend on n. See also [14] where this n-dependence was linked to the ability to improve sharp inequalities of Sobolev-type.

Even in the present day, extensions and variants of problem (1.1) continue to be posed and investigated. Let us discuss some general themes present in the literature regarding extensions of problem (1.1) that are particularly relevant for this work. One theme for extending problem (1.1) concerns the replacement of the operator $-\Delta$ with a different (and often times more general) operator. Works in this family include the extension to the p-Laplacian [13], the extension to an operator of Hardy type [11,14,17], the extension to the fractional Laplacian [20] and the extension to more general divergence-form operators [10,15,16,18]. A second family of extensions of problem (1.1) concerns analogous problems having vector-valued unknown functions. Results in this direction can be found in [1,12].

In this work we consider an extension of problem (1.1) having vector-valued unknown function and second-order divergence form operator. The particular choice of problem we consider is motivated by primarily by [1,18]. For $n \geq 3$ and for a bounded domain $\Omega \subset \mathbb{R}^n$ we consider the problem

$$\begin{cases} -\mathcal{L}u_1 = au_1 + bu_2 + \alpha |u_1|^{\alpha-2}|u_2|^{\beta}u_1 & \text{in } \Omega \\ -\mathcal{L}u_2 = bu_1 + cu_2 + \beta |u_1|^{\alpha}|u_2|^{\beta-2}u_2 & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{L}$ is the divergence-form operator

$$\mathcal{L}u = \operatorname{div}\,(A(x)\nabla u) \tag{1.3}$$

and $A : \overline{\Omega} \to M(n;\mathbb{R})$ is a matrix-valued function. In order that $\mathcal{L}$ retain many of the essential properties of the Laplacian, we will assume that A satisfies

A1. $A : \overline{\Omega} \to M(n;\mathbb{R})$ is continuous
A2. $A(x) = A(x)^{\top}$ for all $x \in \overline{\Omega}$
A3. A is uniformly positive definite in the sense that there is a constant $\tau > 0$ such that

$$\tau|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \qquad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Here and throughout this article we use $\langle \cdot, \cdot \rangle$ to denote the usual real Euclidean inner product. Conditions A1, A2 and A3 ensure that $\mathcal{L}$ is symmetric on $H_0^1(\Omega)$ and that the first eigenvalue $\lambda_1(-\mathcal{L})$ of $-\mathcal{L}$ is positive. To retain the character of the nonlinearity in problem (1.1), the exponents α and β in (1.2) will be assumed to satisfy

$$\alpha + \beta = 2^*, \tag{1.4}$$

where $2^* = 2n/(n-2)$ is the critical exponent for the embedding of $H_0^1(\Omega)$ into Lebesgue's spaces. The matrix

$$\Lambda = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{1.5}$$

consisting of the coupling coefficients of the linear terms on the right-hand side of (1.2) will play the role of the parameter λ in (1.1). Our aim in this work is to determine conditions on n, Λ, Ω and A that guarantee the existence of a positive solution to (1.2) and conditions on these objects that guarantee that problem (1.2) does not have a positive solution. Here and throughout the article, when we use the adjective *positive* to describe a vector-valued function on Ω, we mean that all coordinate functions are strictly positive in Ω. Our conditions on Λ will be expressed in terms of the eigenvalues of Λ which, in view of the symmetry of Λ are necessarily real. Letting $\mu_1 \le \mu_2$ denote these eigenvalues, we have

$$\mu_1|\xi|^2 \le \langle \Lambda\xi, \xi\rangle \le \mu_2|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^2. \tag{1.6}$$

Generally, both the location of a global minimizer x_0 of $\det A$ and the behavior of A near x_0 play roles in our formulations of sufficient conditions for existence of a positive solution to (1.2). The statements of our existence theorems (Theorems 1.1, 1.3 and 1.6 below) assume minimal regularity assumptions on A and $\partial\Omega$. A standard iteration argument shows that under these assumptions, any weak solution $\boldsymbol{u}$ to (1.2) satisfies $\boldsymbol{u} \in L^\infty(\Omega) \times L^\infty(\Omega)$. Thus, the standard elliptic theory guarantees that any weak solution $\boldsymbol{u}$ to (1.2) possesses as much regularity as A and $\partial\Omega$ permit. The first of our existence theorems concerns the case where A is not too flat near a global minimizer of $\det A$ and is as follows:

Theorem 1.1. *Let $n \ge 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $A : \overline{\Omega} \to M(n;\mathbb{R})$ satisfies A1, A2 and A3 and that $\alpha, \beta \in \mathbb{R}$ satisfy both $1 < \min\{\alpha, \beta\}$ and (1.4). If there is $x_0 \in \Omega$, $C_0 > 0$ and $\gamma \in (0, 2]$ such that*

$$A(x) \ge A(x_0) + C_0|x - x_0|^\gamma I_n \qquad \text{for all } x \in \Omega, \tag{1.7}$$

where the inequality is understood in the sense of bilinear forms, then there exists a constant $\lambda^* \in (0, \lambda_1(-\mathcal{L}))$ such that problem (1.2) has a nontrivial weak solution whenever $\lambda^* < \mu_1 \le \mu_2 < \lambda_1(-\mathcal{L})$. If, in addition to the above hypotheses, $b > 0$ then then problem (1.2) has a positive weak solution.

Remark 1.2. Condition (1.7) implies that $A(x) \ge A(x_0)$ in the sense of bilinear forms for all $x \in \Omega$, so any $x_0 \in \overline{\Omega}$ for which (1.7) holds is necessarily a global minimizer of $\det A$.

Our next two existence theorems concern the case where A satisfies a flatness condition near a minimizer of $\det A$. One of these theorems concerns the case where a minimizer occurs in the interior of Ω while the other theorem concerns the case where a minimizer occurs on $\partial\Omega$.

Theorem 1.3. *Let $n \geq 4$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let A be a matrix-valued function on $\overline{\Omega}$ satisfying A1, A2 and A3. Suppose α and β satisfy both $1 < \min\{\alpha, \beta\}$ and (1.4). Suppose further that the eigenvalues μ_1, μ_2 of Λ satisfy $0 < \mu_1 \leq \mu_2 < \lambda_1(-\mathcal{L})$. If $\det A$ attains its minimum value at $x_0 \in \Omega$ and if there are constants $C_0 > 0$ and $\gamma > 2$ such that*

$$A(x) \leq A(x_0) + C_0|x - x_0|^\gamma I_n \tag{1.8}$$

locally near x_0 in the sense of bilinear forms, then problem (1.2) admits a nontrivial weak solution. If, in addition to the above hypotheses, $b > 0$ then problem (1.2) admits a positive weak solution.

Theorem 1.6 below provides an existence result for the case where there is a minimizer x_0 of $\det A$ on $\partial\Omega$ and the boundary of Ω has favorable geometry near x_0. The following definition and example describe this geometry.

Definition 1.4. The boundary of $\Omega \subset \mathbb{R}^n$ is said to be *interior θ-singular* at $x_0 \in \partial\Omega$ with $\theta \geq 1$ if there is a constant $\delta > 0$ and a sequence $(x_i) \subset \Omega$ such that $x_i \to x_0$ as $i \to \infty$ and $B(x_i, \delta|x_i - x_0|^\theta) \subset \Omega$.

Example 1.5. For $\theta \geq 1$, the set $\Omega = \{(x, y) \in \mathbb{R}^2 : y > |x|^{1/\theta}\}$ is interior θ-singular at the origin. Indeed, there is $\delta > 0$ such that for any $0 < r < 1$, $B((0, r), \delta r^\theta) \subset \Omega$.

Theorem 1.6. *Let $n \geq 5$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $A : \overline{\Omega} \to M(n; \mathbb{R})$ satisfies A1, A2 and A3; suppose $\alpha, \beta \in \mathbb{R}$ satisfy $1 < \min\{\alpha, \beta\}$ and (1.4); and suppose the eigenvalues $\mu_1 \leq \mu_2$ of Λ satisfy $0 < \mu_1 \leq \mu_2 < \lambda_1(-\mathcal{L})$. If $\det A$ attains its global minimum at a point $x_0 \in \partial\Omega$ such that (1.8) is satisfied for some $\gamma > \frac{2n-4}{n-4}$, and if $\partial\Omega$ is interior θ-singular at x_0 for some $\theta \in [1, \frac{\gamma(n-4)}{2n-4})$, then problem (1.2) admits a nontrivial weak solution. If, in addition to the above hypotheses, $b > 0$ then problem (1.2) admits a positive weak solution.*

Our nonexistence results are in Theorems 1.7 and 1.8 below.

Theorem 1.7. *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose α and β satisfy both $\min\{\alpha, \beta\} \geq 1$ and $\alpha + \beta \leq 2^*$ and suppose $A : \overline{\Omega} \to M(n; \mathbb{R})$ satisfies A1, A2 and A3. If $b \geq 0$ and $\mu_2 \geq \lambda_1(-\mathcal{L})$ then (1.2) has no positive weak solution.*

Now we consider nonexistence results for star-shaped domains. Define $B : \overline{\Omega} \to M(n; \mathbb{R})$ by

$$b_{ij}(x) = \langle \nabla a_{ij}(x), x - x_0 \rangle,$$

where $a_{ij}(x)$ are the entries of $A(x)$. Clearly $B(x)$ is symmetric for each $x \in \Omega$.

Theorem 1.8. *Let $n \geq 3$ and assume $\Omega \subset \mathbb{R}^n$ is of class C^1 and star-shaped with respect to $x_0 \in \Omega$. Let $A : \overline{\Omega} \to M_n(\mathbb{R})$ satisfy A1, A2 and A3 and have entries $a_{ij} \in C^1(\overline{\Omega}\backslash\{x_0\})$ for which $x \mapsto b_{ij}(x) = \langle \nabla a_{ij}(x), x - x_0\rangle$ extends continuously to x_0 for all $(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$. Assume further that α and β satisfy both $\min\{\alpha,\beta\} > 1$ and (1.4) and that there is $0 < \gamma \leq 2$ and a positive constant C_0 for which*

$$B(x) \geq \gamma C_0 |x - x_0|^\gamma I_n \tag{1.9}$$

for all $x \in \Omega$ in the sense of bilinear forms. There is a constant $\lambda_* = \lambda_*(n, \gamma, \Omega, x_0, C_0) > 0$ such that if $\mu_2 \leq \lambda_*$ then problem (1.2) has no positive solution $\boldsymbol{u} \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$.

Remark 1.9. In (1.9), the factor γ in the multiplicative constant γC_0 plays no essential role in the proof of Theorem 1.8. However, writing the multiplicative constant as such facilitates comparison between Theorems 1.1 and 1.8. Indeed, condition (1.9) implies condition (1.7) so, under the hypotheses of Theorem 1.8 and the additional assumption that $b > 0$, there are $0 < \lambda_* \leq \lambda^*$ such that a positive C^1 solution of (1.2) exists for any $\lambda^* < \mu_1 \leq \mu_2 < \lambda_1(-\mathcal{L})$ but there is no such solution for $\mu_2 \in (-\infty, \lambda_*]$. Estimates for λ_* and λ^* can be found in Sect. 4 of [18].

This paper is organized as follows. In Sect. 2, some mathematical preliminaries will be discussed and some notational conventions will be established. In Sect. 3 a sharp inequality of Sobolev type will be established. The sharp constant in this inequality will be used in Sect. 4 to establish a sufficient condition for existence of nontrivial solutions that arise as minimizers for a certain constrained minimization problem. Section 5 is devoted to establishing the positivity of minimizing solutions. In Sect. 6 we prove Theorems 1.1, 1.3 and 1.6, all of which are established by verifying that the infimum of a suitable constrained energy functional is sufficiently small. Section 7 contains the proofs of the non-existence assertions of Theorems 1.7 and 1.8. Finally, Sect. 8 is an "Appendix" where we have collected some computations whose inclusion in the main body of the manuscript would detract from the presentation.

2. Preliminaries

We assume throughout this section that A satisfies A1, A2 and A3. It is routine to verify that for any such A the map $H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$(u, v) \mapsto \int_\Omega \langle A(x)\nabla u, \nabla v\rangle \,\mathrm{d}x$$

is an inner product on $H_0^1(\Omega)$ and that the corresponding norm is equivalent to the usual norm on $H_0^1(\Omega)$. In particular, denoting this inner product by $\langle\cdot,\cdot\rangle_{\mathcal{X}_A(\Omega)}$, we have that $H_0^1(\Omega)$, when equipped with $\langle\cdot,\cdot\rangle_{\mathcal{X}_A(\Omega)}$, is also a Hilbert space. We will denote this Hilbert space by $\mathcal{X}_A(\Omega)$. With $\mathcal{L}$ as in (1.3), $-\mathcal{L}$

is symmetric and positive definite and the variational characterization of the minimal eigenvalue $\lambda_1(-\mathcal{L})$, which is necessarily positive, is

$$\lambda_1(-\mathcal{L}) = \inf\{\|u\|^2_{\mathcal{X}_A(\Omega)} : u \in H^1_0(\Omega) \text{ and } \|u\|_{L^2(\Omega)} = 1\}. \tag{2.1}$$

For $\mathbb{R}^2$-valued functions $\boldsymbol{u} = (u_1, u_2)$ we consider the product norms

$$\|\boldsymbol{u}\|^2_{\mathcal{H}_A} = \|\boldsymbol{u}\|^2_{\mathcal{X}_A(\Omega)\times\mathcal{X}_A(\Omega)} = \|u_1\|^2_{\mathcal{X}_A(\Omega)} + \|u_2\|^2_{\mathcal{X}_A(\Omega)}$$
$$\|\boldsymbol{u}\|^p_{L^p(\Omega;\mathbb{R}^2)} = \|\boldsymbol{u}\|^p_{L^p(\Omega)\times L^p(\Omega)} = \|u_1\|^p_{L^p(\Omega)} + \|u_2\|^p_{L^p(\Omega)},$$

where, for ease of notation, we set

$$\mathcal{H}_A = \mathcal{X}_A(\Omega) \times \mathcal{X}_A(\Omega)$$
$$\mathcal{H} = \mathcal{H}_I = H^1_0(\Omega) \times H^1_0(\Omega).$$

Although the equivalence of the norms $\|\cdot\|_{H^1_0(\Omega)}$ and $\|\cdot\|_{\mathcal{X}_A(\Omega)}$ ensures that $\mathcal{H} = \mathcal{H}_A$ as sets, we retain the notational distinction as doing so will be convenient for expressing the norms and inner products. For Λ as in (1.5), we define the functional $\Phi_\Lambda : \mathcal{H} \to \mathbb{R}$ by

$$\Phi_\Lambda(\boldsymbol{u}) = \|\boldsymbol{u}\|^2_{\mathcal{H}_A} - \int_\Omega \langle \Lambda\boldsymbol{u}, \boldsymbol{u}\rangle \,\mathrm{d}x. \tag{2.2}$$

In view of (2.1), if $\mu_2 \geq 0$ then

$$\Phi_\Lambda(\boldsymbol{u}) \geq \left(1 - \frac{\mu_2}{\lambda_1(-\mathcal{L})}\right) \|\boldsymbol{u}\|^2_{\mathcal{H}_A}.$$

In particular, if $0 \leq \mu_2 < \lambda_1(-\mathcal{L})$ then Φ_Λ is coercive. A *weak solution* to problem (1.2) is a vector-valued function $\boldsymbol{u} = (u_1, u_2) \in \mathcal{H}$ for which

$$\int_\Omega \langle A(x)\nabla u_j, \nabla\varphi_j\rangle \,\mathrm{d}x = \int_\Omega f_j(\boldsymbol{u})\varphi_j \,\mathrm{d}x \qquad \text{for all } \boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{H},\ j = 1, 2,$$

where

$$\begin{aligned} f_1(\boldsymbol{u}) &= au_1 + bu_2 + \alpha|u_1|^{\alpha-2}|u_2|^\beta u_1 \\ f_2(\boldsymbol{u}) &= bu_1 + cu_2 + \beta|u_1|^\alpha|u_2|^{\beta-2}u_2. \end{aligned} \tag{2.3}$$

If $\alpha + \beta \leq 2^*$ then the functional $Q_{\Lambda;\alpha,\beta} : \mathcal{H}\backslash\{\boldsymbol{0}\} \to \mathbb{R}$ given by

$$Q_{\Lambda;\alpha,\beta}(\boldsymbol{u}) = \frac{\Phi_\Lambda(\boldsymbol{u})}{\left(\int_\Omega G(\boldsymbol{u}) \,\mathrm{d}x\right)^{2/2^*}}, \tag{2.4}$$

where

$$G(\boldsymbol{u}) = G_{(\alpha,\beta)}(\boldsymbol{u}) = |u_1|^\alpha|u_2|^\beta \tag{2.5}$$

is of class C^0 whenever $0 < \min\{\alpha, \beta\}$ and $Q_{\Lambda;\alpha,\beta}$ is of class C^1 whenever $1 < \min\{\alpha, \beta\}$. In the latter case, weak solutions to problem (1.2) can be realized as critical points of $Q_{\Lambda;\alpha,\beta}$. We will not consider critical points of $Q_{\Lambda;\alpha,\beta}$ in full generality. Instead we will consider only minimizers of $Q_{\Lambda;\alpha,\beta}$. To see that $Q_{\Lambda;\alpha,\beta}$ is bounded below, note that Young's inequality ensures that $\|G(\boldsymbol{u})\|^{2/2^*}_{L^1(\Omega)} \leq C(\alpha, \beta, |\Omega|) \sum_{j=1}^2 \|u_j\|^2_{L^{2^*}(\Omega)}$ whenever α and β are non-negative numbers for which $0 < \alpha + \beta \leq 2^*$. Thus, if $0 < \mu_2 < \lambda_1(-\mathcal{L})$ then from the equivalence of $\|\cdot\|_{\mathcal{X}_A(\Omega)}$ and $\|\cdot\|_{H^1_0(\Omega)}$ and the Sobolev inequality one easily deduces that $Q_{\Lambda;\alpha,\beta}$ is bounded below by a positive constant. In fact, since $Q_{\Lambda;\alpha,\beta}$ is invariant under the scaling $\boldsymbol{u} \mapsto \delta\boldsymbol{u}$ for any $\delta \in \mathbb{R}\backslash\{0\}$, any minimizer of $Q_{\Lambda;\alpha,\beta}$ can be realized as a minimizer of Φ_Λ subject to the constraint $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$. In the following section we will discuss some quantities that will allow us to formulate a sufficient condition for the restriction of Φ_Λ to $\{\boldsymbol{u} \in \mathcal{H} : \|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1\}$ to attain a minimum.

3. Sharp inequalities of Sobolev type

In this section we formulate some sharp inequalities of Sobolev type. The sharp constants in these inequalities will be used in Sect. 4 to formulate a sufficient condition for the existence of a constrained minimizer of the functional Φ_Λ given in equation (2.2), see Proposition 4.3.

The sharp constant in the classical Sobolev inequality is

$$\mathcal{S}^{-1} = \inf\left\{\|\nabla u\|^2_{L^2(\Omega)} : u \in C_c^\infty(\Omega) \text{ and } \|u\|_{L^{2^*}(\Omega)} = 1\right\}. \tag{3.1}$$

It is well-known that $\mathcal{S}$ depends only on n [2,21]. In particular, $\mathcal{S}$ is independent of Ω and the infimum in (3.1) is not attained unless $\Omega = \mathbb{R}^n$. In this case, the infimum in (3.1) is attained by the Aubin-Talenti bubbles; the nonzero constant multiples of the functions

$$U_{x_0,\epsilon}(x) = \epsilon^{-(n-2)/2}U((x-x_0)/\epsilon), \tag{3.2}$$

where

$$U(x) = (1+|x|^2)^{-(n-2)/2} \qquad \text{for } x \in \mathbb{R}^n \tag{3.3}$$

and $(x_0,\epsilon) \in \mathbb{R}^n \times (0,\infty)$. For any symmetric positive definite matrix M (having constant entries) we have the following generalization of the Sobolev constant:

$$\mathcal{S}(M)^{-1} = \inf\left\{\|u\|^2_{\mathcal{X}_M(\Omega)} : u \in H_0^1(\Omega) \text{ and } \|u\|_{L^{2^*}(\Omega)} = 1\right\}. \tag{3.4}$$

In this notation, the usual Sobolev constant in (3.1) is equal to $\mathcal{S}(I_n)$, where I_n is the $n \times n$ identity matrix. The following lemma relates the values of $\mathcal{S}$ and $\mathcal{S}(M)$. The proof follows from a routine computation using the change of variable $x \mapsto P^{-1}y$, where $P \in GL(n;\mathbb{R})$ satisfies $PMP^\top = I_n$, see Appendix A of [18] for details.

Lemma 3.1. *Let M be a symmetric positive definite matrix. The Sobolev-type constants in (3.1) and (3.4) are related via the equality*

$$\mathcal{S}(M) = (\det M)^{-1/n}\mathcal{S}. \tag{3.5}$$

In the subsequent sections of the paper we will need an analogue of (3.4) that compares the greatest lower bound of $\|\boldsymbol{u}\|^2_{\mathcal{H}_A}$ (together with a lower-order term) as $\boldsymbol{u}$ varies among functions satisfying $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$, where $G(\boldsymbol{u})$

is as in (2.5). We will develop a more general inequality that holds for $\mathbb{R}^m$-valued functions under relaxed assumptions on G. These developments may be of independent interest. In the confines of this section let us use the notation

$$\mathcal{H} = H_0^1(\Omega;\mathbb{R}^m) = H_0^1(\Omega) \times \cdots \times H_0^1(\Omega)$$
$$\mathcal{H}_A = \mathcal{X}_A(\Omega;\mathbb{R}^m) = \mathcal{X}_A(\Omega) \times \cdots \times \mathcal{X}_A(\Omega)$$

and let the corresponding norms be denoted by

$$\|\boldsymbol{u}\|_{\mathcal{H}}^2 = \sum_{j=1}^m \|u_j\|_{H_0^1(\Omega)}^2$$
$$\|\boldsymbol{u}\|_{\mathcal{H}_A}^2 = \sum_{j=1}^m \|u_j\|_{\mathcal{X}_A(\Omega)}^2.$$

Still in the confines of this section, we will relax the assumptions on G. Specifically, rather than assuming G has the explicit form (2.5), we will only assume that $G : \mathbb{R}^m \to \mathbb{R}$ satisfies

G1. $G \in C(\mathbb{R}^m;\mathbb{R})$ and $G(\tau) \geq 0$ for all $\tau \in \mathbb{R}^m$.
G2. G is homogeneous of degree 2^* in the sense that $G(\lambda\tau) = |\lambda|^{2^*} G(\tau)$ for all $\lambda \in \mathbb{R}$ and all $\tau \in \mathbb{R}^m$.

Of course, for $m = 2$, the function G in (2.5) satisfies both G1 and G2. To describe the vector-valued analogue of (3.4), note that for any $\tau \in \mathbb{R}^m \setminus \{0\}$ we have

$$G(\tau) = |\tau|^{2^*} G(|\tau|^{-1}\tau) \leq |\tau|^{2^*} M_G,$$

where

$$M_G := \max_{\tau \in \mathbb{S}^{m-1}} G(\tau). \tag{3.6}$$

Therefore, for a symmetric positive definite constant matrix M and for $\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathcal{H} \setminus \{\boldsymbol{0}\}$, writing $|\boldsymbol{u}|^2 = \sum_{j=1}^m u_j^2$ we have

$$\begin{aligned} \left(\int_\Omega G(\boldsymbol{u})\, \mathrm{d}x\right)^{2/2^*} &\leq M_G^{2/2^*} \left(\int_\Omega \left(\sum_{j=1}^m u_j^2\right)^{2^*/2} \mathrm{d}x\right)^{2/2^*} \\ &\leq M_G^{2/2^*} \sum_{j=1}^m \left(\int_\Omega |u_j|^{2^*}\, \mathrm{d}x\right)^{2/2^*} \\ &\leq M_G^{2/2^*} \mathcal{S}(M) \sum_{j=1}^m \|u_j\|_{\mathcal{X}_M(\Omega)}^2 \\ &= M_G^{2/2^*} \mathcal{S}(M) \|\boldsymbol{u}\|_{\mathcal{H}_M}^2. \end{aligned} \tag{3.7}$$

This computation shows that the quantity

$$\mathcal{S}(M;G)^{-1} := \inf\left\{\|\boldsymbol{u}\|_{\mathcal{H}_M}^2 : \boldsymbol{u} \in \mathcal{H} \text{ and } \int_\Omega G(\boldsymbol{u})\, \mathrm{d}x = 1\right\}, \tag{3.8}$$

is well-defined and satisfies $\mathcal{S}(M;G) \leq M_G^{2/2^*}\mathcal{S}(M)$. The following lemma shows that equality holds.

Lemma 3.2. *Let $M \in M(n;\mathbb{R})$ be a symmetric positive definite matrix. If G satisfies G1 and G2 then $\mathcal{S}(M;G) = M_G^{2/2^*}\mathcal{S}(M)$.*

Proof. In view of (3.7), we only need to show that $\mathcal{S}(M;G) \geq M_G^{2/2^*}\mathcal{S}(M)$. To do so, choose functions $(\varphi_i)_{i=1}^{\infty} \subset H_0^1(\Omega)$ for which both $\|\varphi_i\|_{L^{2^*}(\Omega)} = M_G^{-1/2^*}$ for all i and

$$\frac{\|\varphi_i\|^2_{\mathcal{X}_M(\Omega)}}{\|\varphi_i\|^2_{L^{2^*}(\Omega)}} \to \mathcal{S}(M)^{-1}.$$

Let $\tau \in \mathbb{S}^{m-1}$ satisfy $G(\tau) = M_G$. The functions $\boldsymbol{u}_i$ defined by $\boldsymbol{u}_i = \varphi_i\tau$ satisfy both $\|G(\boldsymbol{u}_i)\|_{L^1(\Omega)} = 1$ and

$$\|\boldsymbol{u}_i\|^2_{\mathcal{H}_M} = \sum_{j=1}^{m} \|\varphi_i\tau_j\|^2_{\mathcal{X}_M(\Omega)} = \|\varphi_i\|^2_{\mathcal{X}_M(\Omega)} \to M_G^{-2/2^*}\mathcal{S}(M)^{-1}.$$

This establishes the desired inequality. □

Remark 3.3. A simple rescaling argument shows that, under the hypotheses of Lemma 3.2, for all $(x_0,\delta) \in \overline{\Omega} \times (0,\infty)$, the quantity

$$\mathcal{S}_{x_0,\delta}(M;G)^{-1} = \inf\left\{\|\boldsymbol{u}\|^2_{\mathcal{H}_M} : \boldsymbol{u} \in \mathcal{H}(x_0,\delta) \text{ and } \int_\Omega G(\boldsymbol{u})\,\mathrm{d}x = 1\right\}$$

satisfies $\mathcal{S}_{x_0,\delta}(M;G) = \mathcal{S}(M;G)$, where

$$\begin{aligned}\mathcal{H}(x_0,\delta) &= H_0^1(\Omega \cap B(x_0,\delta);\mathbb{R}^m)\\ &= H_0^1(\Omega \cap B(x_0,\delta)) \times \ldots \times H_0^1(\Omega \cap B(x_0,\delta)).\end{aligned} \tag{3.9}$$

If we have a non-constant matrix $A : \overline{\Omega} \to M(n;\mathbb{R})$ in place of the constant matrix M, we consider the following inequality for scalar-valued functions:

$$\|u\|^2_{L^{2^*}(\Omega)} \leq C_1\|u\|^2_{\mathcal{X}_A(\Omega)} + C_2\|u\|^2_{L^2(\Omega)}. \tag{3.10}$$

In this case, the analogue of the sharp constant $\mathcal{S}(M)$ in Eq. (3.4) is

$$\mathcal{N}(A) = \inf\{C_1 : \text{ there exists } C_2 > 0 \text{ for which(3.10) holds for all } u \in H_0^1(\Omega)\}. \tag{3.11}$$

It was shown in Proposition A.1 of [18] that $\mathcal{N}(A) = m_A^{-1/n}\mathcal{S}$, where $m_A = \min\{\det A(x) : x \in \overline{\Omega}\}$. To describe the vector-valued analog of (3.11) we consider in place of (3.10) the inequality

$$\left(\int_\Omega G(\boldsymbol{u})\,\mathrm{d}x\right)^{2/2^*} \leq C_1\|\boldsymbol{u}\|^2_{\mathcal{H}_A} + C_2\|\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{R}^m)} \tag{3.12}$$

and the corresponding sharp constant

$$\mathcal{N}(A;G) = \inf\{C_1 : \text{ there exists } C_2 > 0 \text{ for which (3.12) holds for all } \boldsymbol{u} \in \mathcal{H}\}. \tag{3.13}$$

The value of $\mathcal{N}(A;G)$ is given explicitly in the following proposition.

Proposition 3.4. *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If $A : \overline{\Omega} \to M(n;\mathbb{R})$ is a matrix-valued function satisfying A1, A2 and A3 and if G satisfies both G1 and G2 then*

$$\mathcal{N}(A;G) = m_A^{-1/n} M_G^{2/2^*} \mathcal{S}, \tag{3.14}$$

where m_A is the minimum value of $\det A : \overline{\Omega} \to \mathbb{R}$*, M_G is as in (3.6) and $\mathcal{S}$ is the sharp Sobolev constant in given in Eq. (3.1).*

Remark 3.5. Under the hypotheses of Proposition 3.4, for any $x_0 \in \overline{\Omega}$, applying Lemma 3.2 with $M = A(x_0)$ gives

$$\mathcal{S}(A(x_0);G) = (\det A(x_0))^{-1/n} M_G^{2/2^*} \mathcal{S}.$$

By choosing x_0 to be a minimizer of $\det A$, we see that the assertion in Proposition 3.4 is precisely the assertion that

$$\mathcal{N}(A;G) = \max\{\mathcal{S}(A(x_0);G) : x_0 \in \overline{\Omega}\} =: \overline{\mathcal{N}}(A;G). \tag{3.15}$$

Proof of Proposition 3.4. In view of Remark 3.5, to establish Proposition 3.4 it is sufficient to show that $\mathcal{N}(A;G) \geq \overline{\mathcal{N}}(A;G)$ and that $\mathcal{N}(A;G) \leq \overline{\mathcal{N}}(A;G)$. These inequalities are established separately in Lemmas 3.6 and 3.7 below. ☐

Lemma 3.6. *Under the hypotheses of Proposition 3.4, if C_1 and C_2 are constants for which inequality (3.12) holds for all $\boldsymbol{u} \in \mathcal{H}$ then $C_1 \geq \overline{\mathcal{N}}(A;G)$. In particular, $\overline{\mathcal{N}}(A;G) \leq \mathcal{N}(A;G)$.*

Proof of Lemma 3.6. We proceed by way of contradiction. Suppose $C_1 \in (0, \overline{\mathcal{N}}(A;G))$ and $C_2 \in (0,\infty)$ are constants for which (3.12) holds for all $\boldsymbol{u} \in \mathcal{H}$. By the definition of $\overline{\mathcal{N}}(A;G)$, there is $x_0 \in \overline{\Omega}$ for which $C_1 < \mathcal{S}(A(x_0);G)$. Let us fix any such x_0 and, for ease of notation, set $M = A(x_0)$. Since A satisfies A1, A2 and A3, for any $\epsilon > 0$ there is $\delta > 0$ such that

$$(1-\epsilon)\langle M\xi,\xi\rangle \leq \langle A(x)\xi,\xi\rangle \leq (1+\epsilon)\langle M\xi,\xi\rangle \quad \text{for all } (x,\xi) \in (\overline{\Omega} \cap B(x_0,\delta)) \times \mathbb{R}^n.$$

Fixing $\epsilon > 0$, choosing $\delta \in (0,\sqrt{\epsilon})$ as such, one finds that for any $\boldsymbol{u} \in \mathcal{H}(x_0,\delta)$,

$$\|\boldsymbol{u}\|_{\mathcal{H}_A}^2 \leq (1+\epsilon)\sum_{j=1}^{m} \int_\Omega \langle M\nabla u_j, \nabla u_j\rangle \,\mathrm{d}x = (1+\epsilon)\|\boldsymbol{u}\|_{\mathcal{H}_M}^2, \tag{3.16}$$

where $\mathcal{H}(x_0,\delta)$ is as in (3.9). Moreover, still for $\boldsymbol{u} \in \mathcal{H}(x_0,\delta)$, using Hölder's inequality, the Sobolev inequality and the fact that $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{\mathcal{X}_M(\Omega)}$ are equivalent norms, we find that

$$\|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^m)}^2 \leq C\delta^2\|\boldsymbol{u}\|_{\mathcal{H}_M}^2 \leq C\epsilon\|\boldsymbol{u}\|_{\mathcal{H}_M}^2 \tag{3.17}$$

for some constant C that is independent of both $\boldsymbol{u}$ and ϵ. In view of estimates (3.16) and (3.17), for any $\epsilon > 0$ and any $\boldsymbol{u} \in \mathcal{H}(x_0,\delta)$ satisfying $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$ we have

$$1 \leq C_1\|\boldsymbol{u}\|_{\mathcal{H}_A}^2 + C_2\|\boldsymbol{u}\|_{L^2(\Omega;\mathbb{R}^m)}^2 \leq C_1(1+C\epsilon)\|\boldsymbol{u}\|_{\mathcal{H}_M}^2. \tag{3.18}$$

Choosing $\epsilon > 0$ sufficiently small so that $2C_1(1 + C\epsilon) < \mathcal{S}(M; G) + C_1$ and choosing $\delta = \delta(\epsilon) \in (0, \sqrt{\epsilon})$ small enough to ensure that (3.18) holds for all $\boldsymbol{u} \in H(x_0, \delta)$ for which $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$ gives

$$1 < \frac{\mathcal{S}(M; G) + C_1}{2} \|\boldsymbol{u}\|^2_{\mathcal{H}_M}$$

for all such $\boldsymbol{u}$. This implies that $2/(\mathcal{S}(M; G) + C_1) \leq \mathcal{S}(M; G)^{-1}$ and thus contradicts the assumption $C_1 < \mathcal{S}(M; G)$. □

Lemma 3.7. (ϵ-sharp inequality) *Under the hypotheses of Proposition 3.4, for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that the estimate*

$$\left(\int_\Omega G(\boldsymbol{u})\,\mathrm{d}x\right)^{2/2^*} \leq (\overline{\mathcal{N}}(A; G) + \epsilon)\|\boldsymbol{u}\|^2_{\mathcal{H}_A} + C_\epsilon \|\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{R}^m)} \tag{3.19}$$

holds for all $\boldsymbol{u} \in \mathcal{H}$. In particular, $\overline{\mathcal{N}}(A; G) \geq \mathcal{N}(A; G)$.

Proof of Lemma 3.7. The proof is a standard partition of unity argument. Since A satisfies A1, A2 and A3, and since $\overline{\Omega}$ is compact, for all $\epsilon_0 > 0$ there is $\delta > 0$, $N \in \mathbb{N}$ and $\{x^i\}_{i=1}^N \subset \overline{\Omega}$ such that $\overline{\Omega} \subset \bigcup_{i=1}^N B(x^i, \delta)$ and

$$\begin{aligned}(1 - \epsilon_0)\langle A(x^i)\xi, \xi\rangle \leq \langle A(x)\xi, \xi\rangle \leq (1 + \epsilon_0)\langle A(x^i)\xi, \xi\rangle \\ \text{for all } (x, \xi) \in (\overline{\Omega} \cap B_i) \times \mathbb{R}^n,\end{aligned} \tag{3.20}$$

where, for ease of notation we set $B_i = B(x^i, \delta)$. Let $\{\eta_i\}_{i=1}^N$ be a partition of unity subordinate to the open cover $\{B_i\}_{i=1}^N$ for which $\eta_i^{1/2} \in C_c^\infty(B_i)$. For any $u \in \mathcal{H}\backslash\{\mathbf{0}\}$, performing routine estimates gives

$$\begin{aligned}\left(\int_\Omega G(\boldsymbol{u})\,\mathrm{d}x\right)^{2/2^*} &= \left(\int_\Omega \left(\sum_i \eta_i |\boldsymbol{u}|^2 G(|\boldsymbol{u}|^{-1}\boldsymbol{u})^{2/2^*}\right)^{\frac{2^*}{2}} \mathrm{d}x\right)^{2/2^*} \\ &\leq \sum_i \left(\int_\Omega (\sqrt{\eta_i}|\boldsymbol{u}|)^{2^*} G(|\boldsymbol{u}|^{-1}\boldsymbol{u})\,\mathrm{d}x\right)^{2/2^*} \\ &= \sum_i \left(\int_\Omega G(\sqrt{\eta_i}\boldsymbol{u})\,\mathrm{d}x\right)^{2/2^*} \\ &\leq \sum_i \mathcal{S}(A(x^i); G)\|\sqrt{\eta_i}\boldsymbol{u}\|^2_{\mathcal{H}_{A(x^i)}} \\ &\leq \overline{\mathcal{N}}(A; G) \sum_i \|\sqrt{\eta_i}\boldsymbol{u}\|^2_{\mathcal{H}_{A(x^i)}}.\end{aligned} \tag{3.21}$$

We estimate the sum on the right-most side of (3.21) by using property A2, the left-most inequality in (3.20) and the equivalence of the norms $\|\cdot\|_{\mathcal{H}}$ and

$\|\cdot\|_{\mathcal{H}_A}$ as follows:

$$\begin{aligned}
&\sum_i \|\sqrt{\eta_i}\boldsymbol{u}\|^2_{\mathcal{H}_{A(x^i)}}\\
&\quad = \sum_i \sum_{j=1}^m \int_\Omega \big(\eta_i\langle A(x^i)\nabla u_j, \nabla u_j\rangle + 2\langle A(x^i)\sqrt{\eta_i}\nabla u_j, u_j\nabla\sqrt{\eta_i}\rangle\\
&\qquad + u_j^2\langle A(x^i)\nabla\sqrt{\eta_i}, \nabla\sqrt{\eta_i}\rangle\big)\ \mathrm{d}x \qquad (3.22)\\
&\quad \leq \frac{1}{1-\epsilon_0}\|\boldsymbol{u}\|^2_{\mathcal{H}_A} + \epsilon_0\|\boldsymbol{u}\|^2_{\mathcal{H}} + C\|\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{R}^m)}\\
&\quad \leq \left(\frac{1}{1-\epsilon_0} + C_0^2\epsilon_0\right)\|\boldsymbol{u}\|^2_{\mathcal{H}_A} + C\|\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{R}^m)},
\end{aligned}$$

where C_0 is any constant for which $\|\cdot\|_{\mathcal{H}} \leq C_0\|\cdot\|_{\mathcal{H}_A}$ and $C > 0$ depends on A, ϵ_0, Ω and $\{\eta_i\}$. Since $\epsilon_0 > 0$ is arbitrary, the asserted estimate follows from using estimate (3.22) in estimate (3.21). □

With Proposition 3.4 in hand, combining (3.15) and Lemma 3.7 gives the following ϵ-sharp inequality.

Corollary 3.8. *Under the hypotheses of Proposition 3.4, for every $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that the estimate*

$$\left(\int_\Omega G(\boldsymbol{u})\ \mathrm{d}x\right)^{2/2^*} \leq (\mathcal{N}(A;G)+\epsilon)\|\boldsymbol{u}\|^2_{\mathcal{H}_A} + C_\epsilon\|\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{R}^m)} \tag{3.23}$$

holds for all $\boldsymbol{u} \in \mathcal{H}$.

4. A sufficient condition for existence

We assume throughout this section that G is as in (2.5) and we use the notation

$$\mathcal{N}(A;\alpha,\beta) := \mathcal{N}(A;G), \tag{4.1}$$

where $\mathcal{N}(A;G)$ is defined (for more general G) in (3.13). We start by stating a lemma that carries the same sentiment as the classical Brezis–Lieb lemma [3], but is suitable for application to functionals of the form $\boldsymbol{u} \mapsto \|G(\boldsymbol{u})\|_{L^1(\Omega)}$. A similar lemma has been used previously in [1], see also Proposition A.1 of [4]. For the convenience of the reader, we provide a proof in the "Appendix".

Lemma 4.1. *For $q = (q_1,\ldots,q_m) \in (1,\infty)^m$ and $\xi = (\xi_1,\ldots,\xi_m) \in \mathbb{R}^m$ let*

$$H(\xi) = |\xi_1|^{q_1}|\xi_2|^{q_2}\ldots|\xi_m|^{q_m} = \prod_{j=1}^m |\xi_j|^{q_j}$$

and set $p = \sum_j q_j$. If $\boldsymbol{u}^k = (u_1^k,\ldots,u_m^k)$ is a bounded sequence in $L^p(\Omega;\mathbb{R}^m)$, and if there is $\boldsymbol{u} = (u_1,\ldots,u_m) \in L^p(\Omega;\mathbb{R}^m)$ for which $\boldsymbol{u}^k \to \boldsymbol{u}$ a.e. in Ω then

$$\lim_k \int_\Omega |H(\boldsymbol{u}^k) - H(\boldsymbol{u}^k - \boldsymbol{u}) - H(\boldsymbol{u})|\ \mathrm{d}x = 0.$$

By choosing $m = 2$ and $q = (\alpha, \beta)$ in Lemma 4.1 we obtain the following corollary. It is the version of Lemma 4.1 that will be of use to us in the sequel.

Corollary 4.2. *Let $\alpha, \beta \in \mathbb{R}$ satisfy $1 < \min\{\alpha, \beta\}$. If $(\boldsymbol{u}^k)_{k=1}^{\infty}$ is a bounded sequence in $L^{\alpha+\beta}(\Omega) \times L^{\alpha+\beta}(\Omega)$ and if there is $\boldsymbol{u} \in L^{\alpha+\beta}(\Omega) \times L^{\alpha+\beta}(\Omega)$ for which $\boldsymbol{u}^k \to \boldsymbol{u}$ a.e in Ω, then with G as in (2.5),*

$$\int_\Omega G(\boldsymbol{u}^k - \boldsymbol{u})\, \mathrm{d}x = \int_\Omega G(\boldsymbol{u}^k)\, \mathrm{d}x - \int_\Omega G(\boldsymbol{u})\, \mathrm{d}x + \circ(1).$$

With Φ_Λ as in (2.2) and G as in (2.5) we define $K_\Lambda(A; \alpha, \beta)$ by

$$K_\Lambda(A; \alpha, \beta)^{-1} = \inf\left\{\Phi_\Lambda(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{H} \text{ and } \int_\Omega G(\boldsymbol{u})\, \mathrm{d}x = 1\right\}. \tag{4.2}$$

By making cosmetic changes to the argument in the final paragraph of Sect. 2 one finds that $K_\Lambda(A; \alpha, \beta)^{-1}$ is strictly positive whenever α and β are non-negative, $0 < \alpha + \beta \leq 2^*$ and the largest eigenvalue μ_2 of Λ satisfies $0 \leq \mu_2 < \lambda_1(-\mathcal{L})$. The following proposition provides a sufficient condition for the attainment of the infimum in (4.2).

Proposition 4.3. *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose $A : \overline{\Omega} \to M(n; \mathbb{R})$ satisfies A1, A2 and A3. Let α, β satisfy both $1 < \min\{\alpha, \beta\}$ and (1.4) and let G be as in (2.5). With Λ as in (1.5), suppose the maximum eigenvalue μ_2 of Λ satisfies $0 < \mu_2 < \lambda_1(-\mathcal{L})$. If*

$$K_\Lambda(A; \alpha, \beta)^{-1} < \mathcal{N}(A; \alpha, \beta)^{-1} \tag{4.3}$$

then the restriction of the functional Φ_Λ to the set $\{\boldsymbol{u} \in \mathcal{H} : \|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1\}$ attains its minimum.

Proof of Proposition 4.3. For ease of notation we write $K = K_\Lambda(A; \alpha, \beta)$ and $\mathcal{N} = \mathcal{N}(A; \alpha, \beta)$. Let $(\boldsymbol{u}^k)_{k=1}^{\infty} \subset \mathcal{H}$ satisfy both

$$\int_\Omega G(\boldsymbol{u}^k)\, \mathrm{d}x = 1 \qquad \text{for } k = 1, 2, \ldots \tag{4.4}$$

and

$$K^{-1} + \circ(1) = \Phi_\Lambda(\boldsymbol{u}^k) = \|\boldsymbol{u}^k\|_{\mathcal{H}_A}^2 - \int_\Omega \langle \Lambda \boldsymbol{u}^k, \boldsymbol{u}^k \rangle\, \mathrm{d}x. \tag{4.5}$$

The assumption $0 < \mu_2 < \lambda_1(-\mathcal{L})$ ensures the coercivity of Φ_Λ so (4.5) ensures that $(\boldsymbol{u}^k)_{k=1}^{\infty}$ is bounded in $\mathcal{H}_A$. In view of the reflexivity of $\mathcal{H}_A$ there is $\boldsymbol{u} \in \mathcal{H}$ and a subsequence of $(\boldsymbol{u}^k)_{k=1}^{\infty}$ (still denoted $\boldsymbol{u}^k$) along which $\boldsymbol{u}^k \rightharpoonup \boldsymbol{u}$ weakly in $\mathcal{H}_A$. The equivalence of the norms $\|\cdot\|_{\mathcal{H}_A}$ and $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{H}$ ensures that $(\boldsymbol{u}^k)_{k=1}^{\infty}$ is also bounded in $\mathcal{H}$, so the compactness of the subcritical Sobolev embedding guarantees the existence of a subsequence of $\boldsymbol{u}^k$ along which $\boldsymbol{u}^k \to \boldsymbol{u}$ in $L^2(\Omega) \times L^2(\Omega)$ and $\boldsymbol{u}^k \to \boldsymbol{u}$ a.e. in Ω. Since $\boldsymbol{u}^k \rightharpoonup \boldsymbol{u}$ weakly in $\mathcal{H}_A$ we have

$$\begin{aligned}\|\boldsymbol{u}^k - \boldsymbol{u}\|_{\mathcal{H}_A}^2 &= \|\boldsymbol{u}^k\|_{\mathcal{H}_A}^2 - 2\sum_{j=1}^{2} \int_\Omega \langle A(x)\nabla u_j^k, \nabla u_j \rangle\, \mathrm{d}x + \|\boldsymbol{u}\|_{\mathcal{H}_A}^2 \\ &= \|\boldsymbol{u}^k\|_{\mathcal{H}_A}^2 - \|\boldsymbol{u}\|_{\mathcal{H}_A}^2 + \circ(1).\end{aligned}$$

Using this equality in Eq. (4.5) and in view of the $L^2 \times L^2$-convergence $\boldsymbol{u}^k \to \boldsymbol{u}$ we obtain

$$\begin{aligned} K^{-1} + \circ(1) &= \|\boldsymbol{u}\|_{\mathcal{H}_A}^2 - \int_\Omega \langle \Lambda \boldsymbol{u}, \boldsymbol{u}\rangle \, \mathrm{d}x + \|\boldsymbol{u}^k - \boldsymbol{u}\|_{\mathcal{H}_A}^2 + \circ(1) \\ &\geq K^{-1} \left(\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} + \|\boldsymbol{u}^k - \boldsymbol{u}\|_{\mathcal{H}_A}^2 + \circ(1). \end{aligned} \tag{4.6}$$

Moreover, for each $k \in \mathbb{N}$ and for any $\epsilon > 0$, applying Corollaries 3.8 and 4.2, then using (4.4) gives

$$\begin{aligned} &\|\boldsymbol{u}^k - \boldsymbol{u}\|_{\mathcal{H}_A} \\ &\quad \geq (\mathcal{N} + \epsilon)^{-1} \left(\int_\Omega G(\boldsymbol{u}^k - \boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} - C_\epsilon \|\boldsymbol{u}^k - \boldsymbol{u}\|_{L^2(\Omega)\times L^2(\Omega)}^2 \\ &\quad = (\mathcal{N} + \epsilon)^{-1} \left(\int_\Omega G(\boldsymbol{u}^k) \mathrm{d}x - \int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} + \circ(1) \\ &\quad = (\mathcal{N} + \epsilon)^{-1} \left(1 - \int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} + \circ(1) \\ &\quad \geq (\mathcal{N} + \epsilon)^{-1} \left(1 - \left(\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} \right) + \circ(1). \end{aligned} \tag{4.7}$$

Using (4.7) in (4.6) and letting $k \to \infty$ gives

$$(K^{-1} - (\mathcal{N} + \epsilon)^{-1}) \left(1 - \left(\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} \right) \geq 0. \tag{4.8}$$

Fatou's lemma and (4.4) give

$$1 = \liminf_k \int_\Omega G(\boldsymbol{u}^k) \, \mathrm{d}x \geq \int_\Omega \liminf_k G(\boldsymbol{u}^k) \, \mathrm{d}x = \int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x,$$

so by choosing $\epsilon > 0$ small enough so that $K > \mathcal{N} + \epsilon$, inequality (4.8) implies that $\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x = 1$. Using this in (4.6) gives $\|\boldsymbol{u}^k - \boldsymbol{u}\|_{\mathcal{H}_A}^2 \to 0$. Finally, from (4.5) and the continuity of Φ_Λ on $\mathcal{H}$ we obtain $K^{-1} = \Phi_\Lambda(\boldsymbol{u})$. ☐

Since minimizers of Φ_Λ constrained to the set $\{\boldsymbol{u} \in \mathcal{H} : \|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1\}$ are nontrivial weak solutions to (1.2), we obtain the following corollary.

Corollary 4.4. *Under the hypotheses of Proposition 4.3, problem (1.2) has a nontrivial weak solution.*

5. Positivity of minimizing solutions

We assume throughout this section that G is as in (2.5) and we use the notation (4.1). We establish conditions that guarantee that if $\boldsymbol{u} \in \mathcal{H}$ minimizes the functional Φ_Λ given in (2.2) subject to the constraint $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$, then one of $\boldsymbol{u}$ or $-\boldsymbol{u}$ is a positive solution to (1.2).

Proposition 5.1. *Suppose* $0 < \mu_1 \le \mu_2 < \lambda_1(-\mathcal{L})$ *and suppose* α *and* β *satisfy* $1 < \min\{\alpha, \beta\}$ *and* (1.4). *If* $b > 0$ *and if* $\boldsymbol{u} \in \mathcal{H}$ *is a minimizer of the restriction of* Φ_Λ *to the set* $\{\boldsymbol{u} \in \mathcal{H} : \|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1\}$, *where* G *is as in* (2.5), *then either* $u_j > 0$ *in* Ω *for* $j = 1, 2$ *or* $u_j < 0$ *in* Ω *for* $j = 1, 2$. *In either case we can arrange that* $\boldsymbol{u} > 0$ *in* Ω *by considering* $-\boldsymbol{u}$ *in place of* $\boldsymbol{u}$ *if necessary.*

Proof of Proposition 5.1. For any $\boldsymbol{u} \in \mathcal{H}$, since $\nabla u_j = 0$ a.e. on $\{u_j = 0\}$ we have $\langle A(x)\nabla(|u_j|), \nabla(|u_j|)\rangle = \langle A(x)\nabla u_j, \nabla u_j\rangle$ on $\{\nabla u_j \neq 0\}$. Moreover, writing $|\boldsymbol{u}| = (|u_1|, |u_2|)$, we have $G(|\boldsymbol{u}|) = G(\boldsymbol{u})$ and

$$\langle \Lambda|\boldsymbol{u}|, |\boldsymbol{u}|\rangle = \langle \Lambda\boldsymbol{u}, \boldsymbol{u}\rangle + 2b(|u_1||u_2| - u_1u_2).$$

Therefore,

$$\Phi_\Lambda(|\boldsymbol{u}|) = \Phi_\Lambda(\boldsymbol{u}) - 2b\int_\Omega (|u_1||u_2| - u_1u_2)\,\mathrm{d}x. \tag{5.1}$$

Now suppose $\boldsymbol{u}$ minimizes Φ_Λ subject to the constraint $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$. For such $\boldsymbol{u}$ we have $\Phi_\Lambda(\boldsymbol{u}) \le \Phi_\Lambda(|\boldsymbol{u}|)$, so Eq. (5.1) gives

$$b\int_\Omega (|u_1||u_2| - u_1u_2)\,\mathrm{d}x \le 0.$$

The non-negativity of the integrand together with the assumption that $b > 0$ gives

$$(u_1^+ + u_1^-)(u_2^+ + u_2^-) = |u_1||u_2| = u_1u_2 = (u_1^+ - u_1^-)(u_2^+ - u_2^-) \qquad \text{a.e. } x \in \Omega,$$

from which we deduce that $u_1^+u_2^- + u_1^-u_2^+ = 0$. Since both summands in this equality are non-negative we obtain both

$$u_1^+u_2^- = 0 \qquad \text{and} \qquad u_1^-u_2^+ = 0 \qquad \text{a.e. in } \Omega. \tag{5.2}$$

In particular, using the notational conventions $\boldsymbol{u}^+ = (u_1^+, u_2^+)$ and $\boldsymbol{u}^- = (u_1^-, u_2^-)$ we have $\langle \Lambda\boldsymbol{u}^+, \boldsymbol{u}^-\rangle = 0$ a.e. in Ω and therefore

$$\Phi_\Lambda(\boldsymbol{u}) = \Phi_\Lambda(\boldsymbol{u}^+) + \Phi_\Lambda(\boldsymbol{u}^-). \tag{5.3}$$

With $K_\Lambda(A; \alpha, \beta)$ defined in (4.2) we write $K = K_\Lambda(A; \alpha, \beta)$ and we observe that the assumptions on μ_2, α and β ensure that $K > 0$. The assumption that $\boldsymbol{u}$ is a constrained minimizer and decomposition (5.3) give

$$\begin{aligned} 1 &= K\Phi_\Lambda(\boldsymbol{u}) \\ &= K(\Phi_\Lambda(\boldsymbol{u}^+) + \Phi_\Lambda(\boldsymbol{u}^-)) \\ &\ge \left(\int_\Omega G(\boldsymbol{u}^+)\,\mathrm{d}x\right)^{2/2^*} + \left(\int_\Omega G(\boldsymbol{u}^-)\,\mathrm{d}x\right)^{2/2^*}. \end{aligned} \tag{5.4}$$

Using this estimate, together with the fact that (5.2) guarantees that $G(\boldsymbol{u}) = G(\boldsymbol{u}^+) + G(\boldsymbol{u}^-)$ we have

$$\begin{aligned} 1 &\ge \left(\int_\Omega G(\boldsymbol{u}^+)\,\mathrm{d}x\right)^{2/2^*} + \left(\int_\Omega G(\boldsymbol{u}^-)\,\mathrm{d}x\right)^{2/2^*} \\ &\ge \left(\int_\Omega (G(\boldsymbol{u}^+) + G(\boldsymbol{u}^-))\,\mathrm{d}x\right)^{2/2^*} \\ &= 1, \end{aligned} \tag{5.5}$$

where the second inequality holds in view of the elementary inequality $(a+b)^s \le a^s + b^s$ for $a \ge 0$, $b \ge 0$ and $0 < s < 1$. Since equality holds in this elementary inequality if and only if at least one of a or b is zero and since we have equality throughout (5.5), we deduce that either $u_1^+ u_2^+ = 0$ or $u_1^- u_2^- = 0$. Since $\Phi_\Lambda(\boldsymbol{u}) = \Phi_\Lambda(-\boldsymbol{u})$ we may assume with no loss of generality that $u_1^- u_2^- = 0$. In particular we have $G(\boldsymbol{u}) = G(\boldsymbol{u}^+)$ so $\|G(\boldsymbol{u}^+)\|_{L^1(\Omega)} = 1$. Now, the coercivity of Φ_Λ ensures that $\Phi_\Lambda(\boldsymbol{u}^-) \ge 0$, so we have

$$\begin{aligned} 1 &= K(\Phi_\Lambda(\boldsymbol{u}^+) + \Phi_\Lambda(\boldsymbol{u}^-)) \\ &\ge K\Phi_\Lambda(\boldsymbol{u}^+) \\ &\ge \left(\int_\Omega G(\boldsymbol{u}^+)\,\mathrm{d}x\right)^{2/2^*} \\ &= 1, \end{aligned}$$

from which we deduce both $\Phi_\Lambda(\boldsymbol{u}^+) = K^{-1}\|G(\boldsymbol{u}^+)\|_{L^1(\Omega)}^{2/2^*}$ and $\Phi_\Lambda(\boldsymbol{u}^-) = 0$. In view of the coercivity of Φ_Λ, the second of these equalities ensures that $\boldsymbol{u}^- \equiv 0$. In particular, $\boldsymbol{u}^+ = \boldsymbol{u}$ is a constrained minimizer of Φ_Λ for which $\boldsymbol{u} \ge 0$ in Ω. Up to a positive constant multiple, $\boldsymbol{u}$ is a non-negative weak solution to (1.2):

$$\begin{cases} -\mathcal{L}u_1 = au_1 + bu_2 + \alpha u_1^{\alpha-1}u_2^{\beta} & \text{in } \Omega \\ -\mathcal{L}u_2 = bu_1 + cu_2 + \beta u_1^{\alpha}u_2^{\beta-1} & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

The positivity assumption on μ_1 and μ_2 (the eigenvalues of Λ) ensures that the diagonal entries of Λ satisfy $a > 0$ and $c > 0$. Combining these inequalities with the assumption $b > 0$ we find that $\mathcal{L}u_j \le 0$ in Ω for $j = 1, 2$. The strong maximum principle together with the fact that $\boldsymbol{u} \ge 0$ in Ω shows that either $\boldsymbol{u} > 0$ in Ω or $\boldsymbol{u} \equiv 0$ in Ω. The latter of these possibilities is ruled out by the fact that $\|G(\boldsymbol{u})\|_{L^1(\Omega)} = 1$. □

6. Proofs of existence theorems

In this section we prove Theorems 1.1, 1.3 and 1.6. In view of Proposition 4.3 and Corollary 4.4, in order to establish existence of a nontrivial solution to (1.2), it is sufficient to construct a test function $\boldsymbol{u} \in \mathcal{H}$ for which $\Phi_\Lambda(\boldsymbol{u}) < \mathcal{N}(A;\alpha,\beta)^{-1}$, where $\mathcal{N}(A;\alpha,\beta)$ is as in (4.1) with G given by (2.5). For this choice of G one easily verifies that M_G as defined in (3.6) satisfies

$$M_G^{-2/2^*} = \left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\alpha}{\beta}\right)^{-\alpha/2^*}$$

so from Eq. (3.14), the explicit value of $\mathcal{N}(A;\alpha,\beta)$ is

$$\mathcal{N}(A;\alpha,\beta) = m_A^{-1/n}\left(\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\alpha}{\beta}\right)^{-\alpha/2^*}\right)^{-1}\mathcal{S}. \tag{6.1}$$

6.1. Construction of test function when $0 < \gamma \leq 2$

Proof of Theorem 1.1. In view of the boundedness of Ω and the assumption that $0 < \gamma \leq 2$, the following Hardy–Sobolev type inequality holds for all $u \in H_0^1(\Omega)$

$$\int_\Omega u^2 \, \mathrm{d}x \leq K_0(n, \gamma, \Omega, x_0)^2 \int_\Omega |x - x_0|^\gamma |\nabla u|^2 \, \mathrm{d}x. \tag{6.2}$$

This inequality is a special case of the Caffarelli–Kohn–Nirenberg inequalities established in [7]. For any $\boldsymbol{u} \in \mathcal{H}$, assumption (1.7) and inequality (6.2) give

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathcal{H}_A}^2 &\geq \sum_{j=1}^2 \int_\Omega \left(\langle A(x_0)\nabla u_j, \nabla u_j \rangle + C_0 |x - x_0|^\gamma |\nabla u_j|^2 \right) \, \mathrm{d}x \\ &\geq \mathcal{S}(A(x_0); \alpha, \beta)^{-1} \left(\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} + C_0 K_0(n, \gamma, \Omega, x_0)^{-2} \sum_{j=1}^2 \int_\Omega u_j^2 \, \mathrm{d}x, \end{aligned} \tag{6.3}$$

where $\mathcal{S}(A(x_0); \alpha, \beta)$ is as in (3.8) with $G(\boldsymbol{u}) = |u_1|^\alpha |u_2|^\beta$ as in (2.5). Define Θ to be the set of $\lambda > 0$ such that the inequality

$$\|\boldsymbol{u}\|_{\mathcal{H}_A}^2 \geq \mathcal{S}(A(x_0); \alpha, \beta)^{-1} \left(\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} + \lambda \sum_{j=1}^2 \int_\Omega u_j^2 \, \mathrm{d}x$$

holds for all $\boldsymbol{u} \in \mathcal{H}$. Estimate (6.3) ensures that $C_0 K_0(n, \gamma, \Omega, x_0)^{-2} \in \Theta$, so $\Theta \neq \emptyset$. An elementary argument shows that Θ is both closed and bounded above. Setting $\lambda^* = \sup \Theta$ we have both $\lambda^* \geq C_0 K_0(n, \gamma, \Omega, x_0)^{-2}$ and, for every $\boldsymbol{u} \in \mathcal{H}$,

$$\sum_{j=1}^2 \int_\Omega \langle A(x)\nabla u_j, \nabla u_j \rangle \, \mathrm{d}x \geq \mathcal{S}(A(x_0); \alpha, \beta)^{-1} \left(\int_\Omega G(\boldsymbol{u}) \, \mathrm{d}x \right)^{2/2^*} + \lambda^* \sum_{j=1}^2 \int_\Omega u_j^2 \, \mathrm{d}x.$$

Moreover, the following short computation using the variational characterization of $\lambda_1(-\mathcal{L})$ shows that $\lambda^* < \lambda_1(-\mathcal{L})$. Let $\psi > 0$ be an eigenfunction for $-\mathcal{L}$ corresponding to $\lambda_1(-\mathcal{L})$. For $\boldsymbol{\varphi} = (\psi, \psi)$ we have

$$\begin{aligned} 2\lambda_1(-\mathcal{L}) \|\psi\|_{L^2(\Omega)}^2 &= \|\boldsymbol{\varphi}\|_{\mathcal{H}_A}^2 \\ &\geq \mathcal{S}(A(x_0); \alpha, \beta)^{-1} \left(\int_\Omega G(\boldsymbol{\varphi}) \, \mathrm{d}x \right)^{2/2^*} + \lambda^* \sum_{j=1}^2 \|\psi\|_{L^2(\Omega)}^2 \\ &> 2\lambda^* \|\psi\|_{L^2(\Omega)}^2, \end{aligned}$$

from which we deduce $\lambda^* < \lambda_1(-\mathcal{L})$. Next we show that the assumption $\mu_1 > \lambda^*$ ensures that $K_\Lambda(A; \alpha, \beta)^{-1} < \mathcal{S}(A(x_0); \alpha, \beta)^{-1}$, where K_Λ is defined in

(4.2). By definition of λ^*, if $\mu_1 > \lambda^*$ then there is $\boldsymbol{u} \in \mathcal{H}$ such that

$$\begin{aligned} \Phi_\Lambda(\boldsymbol{u}) &= \sum_{j=1}^{2} \int_\Omega \langle A(x)\nabla u_j, \nabla u_j\rangle \,\mathrm{d}x - \int_\Omega \langle \Lambda \boldsymbol{u}, \boldsymbol{u}\rangle \,\mathrm{d}x \\ &\le \sum_{j=1}^{2} \int_\Omega \left(\langle A(x)\nabla u_j, \nabla u_j\rangle - \mu_1 u_j^2\right) \,\mathrm{d}x \\ &< \mathcal{S}(A(x_0);\alpha,\beta)^{-1} \left(\int_\Omega G(\boldsymbol{u}) \,\mathrm{d}x\right)^{2/2^*}. \end{aligned} \tag{6.4}$$

Since the hypotheses of the theorem ensure the coercivity of Φ_Λ, any $\boldsymbol{u}$ satisfying (6.4) must also satisfy $\int_\Omega G(\boldsymbol{u}) \,\mathrm{d}x > 0$. From inequality (6.4) and the fact that Φ_Λ is homogeneous of degree two, we find that $\boldsymbol{v} = \left(\int_\Omega G(\boldsymbol{u}) \,\mathrm{d}x\right)^{-1/2^*} \boldsymbol{u}$ satisfies $\int_\Omega G(\boldsymbol{v}) \,\mathrm{d}x = 1$ and $\Phi_\Lambda(\boldsymbol{v}) < \mathcal{S}(A(x_0);\alpha,\beta)^{-1}$. Now Remark 3.5 together with the fact that x_0 is a minimizer of $\det A$ (see Remark 1.2) implies that $\mathcal{S}(A(x_0);\alpha,\beta) = \mathcal{N}(A;\alpha,\beta)$. Therefore, Corollary 4.4 ensures the existence of a nontrivial solution to (1.2) whenever $\lambda^* < \mu_1 \le \mu_2 < \lambda_1(-\mathcal{L})$. Finally, if $b > 0$ then an application of Proposition 5.1 ensures the existence of a positive solution to problem (1.2). □

6.2. Construction of test function when $\gamma > 2$

In this subsection we construct a test function $\boldsymbol{u} \in \mathcal{H}$ for which

$$Q_{\Lambda;\alpha,\beta}(\boldsymbol{u}) < \mathcal{N}(A;\alpha,\beta)^{-1}.$$

Having constructed such a $\boldsymbol{u}$, and for c defined by $c^{-2^*} = \int_\Omega G(\boldsymbol{u}) \,\mathrm{d}x$, the function $c\boldsymbol{u}$ demonstrates that the hypotheses of Proposition 4.3 are satisfied so Corollary 4.4 ensures the existence of a nontrivial solution to (1.2). Since the method of constructing $\boldsymbol{u}$ in the case that a minimizer of $\det A$ is located in the interior of Ω differs from the method of construction in the case that a minimizer of $\det A$ is located on $\partial\Omega$, we consider these two cases separately.

6.2.1. Case 1: There is a minimizer of det A in Ω. In this case we show that estimate (4.3) holds by constructing a test function from the Aubin-Talenti bubbles given in (3.2), (3.3).

Proposition 6.1. *Under the hypotheses of Theorem 1.3, but without the assumption that $b > 0$, there is $\boldsymbol{u} \in \mathcal{H}\backslash\{\mathbf{0}\}$ for which $Q_{\Lambda;\alpha,\beta}(\boldsymbol{u}) < \mathcal{N}(A;\alpha,\beta)^{-1}$.*

Proof. We assume with no loss of generality that $\det A$ is minimized at $x_0 = 0 \in \Omega$ and we set

$$m_A = \min_{\overline{\Omega}} \det A = \det A(0).$$

Since $A(0)$ is symmetric and positive definite there is an orthogonal matrix $Q \in M(n;\mathbb{R})$ such that $QA(0)Q^\top = D$, where $D = \mathrm{diag}(a_1,\ldots,a_n)$ and $0 < a_1 \le a_2 \le \cdots \le a_n$ are the eigenvalues of $A(0)$. Setting $P = D^{-1/2}Q$, we have $PA(0)P^\top = I_n$, we have $a_1|y|^2 \le |P^{-1}y|^2 \le a_n|y|^2$ for all $y \in \mathbb{R}^n$ and we have $|P^\top\xi|^2 \le a_1^{-1}|\xi|^2$ for all $\xi \in \mathbb{R}^n$. For any $\boldsymbol{u} \in \mathcal{H}$, using the change of variable $x = P^{-1}y$, setting $v_j(y) = u_j(P^{-1}y)$, and using the above-mentioned properties of P as well as assumption (1.8) yields

$$\begin{aligned}\Phi_\Lambda(\boldsymbol{u}) &\le \sum_{j=1}^{2}\int_\Omega \left(\langle A(0)\nabla u_j, \nabla u_j\rangle + C_0|x|^\gamma|\nabla u_j|^2\right)\,\mathrm{d}x - \int_\Omega \langle \Lambda \boldsymbol{u}, \boldsymbol{u}\rangle\,\mathrm{d}x\\ &= (\det P)^{-1}\Big[\sum_{j=1}^{2}\int_{P\Omega}\left(\langle PA(0)P^\top\nabla v_j, \nabla v_j\rangle + C_0|P^{-1}y|^\gamma|P^\top\nabla v_j|^2\right)\,\mathrm{d}y\\ &\quad - \int_{P\Omega}\langle \Lambda\boldsymbol{v}, \boldsymbol{v}\rangle\,\mathrm{d}y\Big]\\ &\le (\det A(0))^{1/2}\left[\sum_{j=1}^{2}\int_{P\Omega}\left(|\nabla v_j|^2 + C_0 a_n^{\gamma/2}a_1^{-1}|y|^\gamma|\nabla v_j|^2\right)\,\mathrm{d}y - \int_{P\Omega}\langle\Lambda\boldsymbol{v}, \boldsymbol{v}\rangle\,\mathrm{d}y\right].\end{aligned} \tag{6.5}$$

Similarly, using the same change of variable, we have

$$\int_\Omega G(\boldsymbol{u})\,\mathrm{d}x = (\det A(0))^{1/2}\int_{P\Omega} G(\boldsymbol{v})\,\mathrm{d}y. \tag{6.6}$$

For $\epsilon > 0$ consider the Aubin-Talenti bubble

$$U_{0,\epsilon}(x) = c_n\left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^{\frac{n-2}{2}},$$

where c_n is a normalization constant chosen so that $\|U_{0,\epsilon}\|_{L^{2^*}(\mathbb{R}^n)} = 1$ (independently of ϵ). For convenience we will write U in place of $U_{0,1}$ and U_ϵ in place of $U_{0,\epsilon}$. Let $\delta > 0$ satisfy $B(0, 2\delta) \subset\subset P\Omega$ and let $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfy $0 \le \eta \le 1$, $\eta \equiv 1$ in $B(0,\delta)$, $\eta \equiv 0$ in $\mathbb{R}^n \setminus B(0, 2\delta)$ and $|\nabla\eta| \le C\delta^{-1}$. Set

$$w_\epsilon(y) = \eta(y)U_\epsilon(y). \tag{6.7}$$

$$\boldsymbol{v} = \boldsymbol{v}_\epsilon = (\sqrt{\alpha}w_\epsilon, \sqrt{\beta}w_\epsilon), \tag{6.8}$$

and for $\boldsymbol{u}_\epsilon(x) = \boldsymbol{v}_\epsilon(Px)$, estimate (6.5) gives

$$\begin{aligned}\frac{\Phi_\Lambda(\boldsymbol{u}_\epsilon)}{(\alpha+\beta)(\det A(0))^{1/2}} - \int_{\mathbb{R}^n}\left(|\nabla w_\epsilon|^2 + \frac{C_0 a_n^{\gamma/2}}{a_1}|y|^\gamma|\nabla w_\epsilon|^2\right)\,\mathrm{d}y\\ \le -\frac{a\alpha + 2b\sqrt{\alpha}\sqrt{\beta} + c\beta}{\alpha+\beta}\int_{\mathbb{R}^n} w_\epsilon^2\,\mathrm{d}y\\ \le -\mu_1\int_{\mathbb{R}^n} w_\epsilon^2\,\mathrm{d}y\end{aligned} \tag{6.9}$$

and (6.6) gives

$$\left(\int_\Omega G(\boldsymbol{u}_\epsilon)\,\mathrm{d}x\right)^{2/2^*} = (\alpha^\alpha\beta^\beta\det A(0))^{1/2^*}\left(\int_{\mathbb{R}^n}|w_\epsilon|^{2^*}\,\mathrm{d}y\right)^{2/2^*}. \tag{6.10}$$

Therefore, using (6.9), (6.10) and the explicit value of $\mathcal{N}(A;\alpha,\beta)$ given in (6.1) we have

$$\mathcal{N}(A;\alpha,\beta)\frac{\Phi_\Lambda(\boldsymbol{u}_\epsilon)}{\left(\int_\Omega G(\boldsymbol{u}_\epsilon)\,\mathrm{d}x\right)^{2/2^*}} \le \mathcal{S}\tilde{Q}(w_\epsilon), \tag{6.11}$$

where $\tilde{Q}$ is defined for $\varphi \in C_c^\infty(\mathbb{R}^n)\backslash\{0\}$ by

$$\tilde{Q}(\varphi)\left(\int_{\mathbb{R}^n} |\varphi|^{2^*}\,\mathrm{d}x\right)^{2/2^*} = \int_{\mathbb{R}^n}\left(|\nabla\varphi|^2 + C_0 a_n^{\gamma/2} a_1^{-1}|y|^\gamma|\nabla\varphi|^2\right)\,\mathrm{d}y - \mu_1\int_{\mathbb{R}^n}\varphi^2\,\mathrm{d}y. \tag{6.12}$$

By performing standard computations, one can show that

$$\begin{aligned} \|\nabla w_\epsilon\|^2_{L^2(\mathbb{R}^n)} &= \mathcal{S}^{-1} + O(\epsilon^{n-2}) \\ \|w_\epsilon\|^2_{L^{2^*}(\mathbb{R}^n)} &= 1 + O(\epsilon^n) \end{aligned} \tag{6.13}$$

and

$$\|w_\epsilon\|^2_{L^2(\mathbb{R}^n)} = \begin{cases} b_4\epsilon^2|\log\epsilon| + O(\epsilon^2) & \text{if } n = 4 \\ b_n\epsilon^2 + O(\epsilon^{n-2}) & \text{if } n \geq 5, \end{cases} \tag{6.14}$$

where $b_4 = c_4^2|\mathbb{S}^3|$ and $b_n = c_n^2|\mathbb{S}^{n-1}|\frac{\sqrt{\pi}\Gamma(n/2)}{2^{n-3}(n-4)\Gamma((n-1)/2)}$ for $n \geq 5$. Moreover, using the assumption $\gamma > 2$ and by performing elementary computations one can show that

$$\int_{\mathbb{R}^n} |y|^\gamma|\nabla w_\epsilon|^2\,\mathrm{d}y = \begin{cases} O(\epsilon^2) & \text{if } n = 4 \\ \circ(\epsilon^2) & \text{if } n \geq 5. \end{cases} \tag{6.15}$$

If $n = 4$ then estimates (6.13), (6.14) and (6.15) give

$$\mathcal{S}\tilde{Q}(w_\epsilon) = \frac{1 - \mathcal{S}\mu_1 b_4\epsilon^2|\log\epsilon| + O(\epsilon^2)}{1 + O(\epsilon^4)} < 1,$$

where the final estimate holds provided $\epsilon > 0$ is sufficiently small. Using this in (6.11) shows that if $\epsilon > 0$ is sufficiently small then

$$\frac{\Phi_\Lambda(\boldsymbol{u}_\epsilon)}{\left(\int_\Omega G(\boldsymbol{u}_\epsilon)\,\mathrm{d}x\right)^{2/2^*}} < \mathcal{N}(A;\alpha,\beta)^{-1}. \tag{6.16}$$

Similarly if $n \geq 5$ and $\epsilon > 0$ is sufficiently small then estimates (6.13), (6.14) and (6.15) give

$$\mathcal{S}\tilde{Q}(w_\epsilon) = \frac{1 - \mathcal{S}\mu_1 b_n\epsilon^2 + \circ(\epsilon^2)}{1 + O(\epsilon^n)} < 1,$$

from which estimate (6.16) follows. □

Proof of Theorem 1.3. Since the hypotheses of Propositions 6.1 and 4.3 are satisfied, the infimum in (4.2) is obtained by some $\boldsymbol{u} \in \mathcal{H}\backslash\{\mathbf{0}\}$. Any such $\boldsymbol{u}$ is a nontrivial solution to (1.2). If $b > 0$ then Proposition 5.1 ensures that one of $\boldsymbol{u}$ or $-\boldsymbol{u}$ is a positive solution to (1.2). □

6.2.2. Case 2: There is a minimizer of det A on $\partial\Omega$.

Proposition 6.2. *Under the hypotheses of Theorem* 1.6*, except for the positivity of* b*, there is* $\boldsymbol{u} \in \mathcal{H}$ *for which* $\int_\Omega G(\boldsymbol{u})\,\mathrm{d}x = 1$ *and* $\Phi_\Lambda(\boldsymbol{u}) < \mathcal{N}(A;\alpha,\beta)^{-1}$*.*

Proof. We assume with no loss of generality that $x_0 = 0$. For $\boldsymbol{u} \in \mathcal{H}$, using the change of variable $x = P^{-1}y$ and setting $\boldsymbol{v}(y) = \boldsymbol{u}(P^{-1}y)$ as in the proof of Proposition 6.1, we still have estimate (6.5) and Eq. (6.6). It is routine to check that the interior θ-singularity assumption on $\partial\Omega$ at $x_0 = 0$ guarantees the interior θ-singularity of $\partial P\Omega$ at $y_0 = Px_0 = 0$. Indeed, if $(x_i) \subset \Omega$ with $x_i \to 0$ and $B(x_i, \delta|x_i|^\theta) \subset \Omega$, then for $y_i = Px_i$ we have $B(y_i, \tilde{\delta}|y_i|^\theta) \subset P\Omega$, where $\tilde{\delta} = a_n^{-1/2} a_1^{\theta/2}\delta$. For x_i and y_i as such, let $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfy $0 \le \eta \le 1$, $\eta \equiv 1$ in $B(0, \tilde{\delta}/2)$ and $\eta \equiv 0$ on $\mathbb{R}^n \backslash B(0, \tilde{\delta})$ and consider the test function

$$\varphi^i(y) = \eta\left(\frac{y - y_i}{\epsilon_i^\theta}\right) U_{\epsilon_i^\rho}(y - y_i),$$

where and $\epsilon_i = |y_i|$ and ρ satisfies

$$\frac{2(n-2)\theta}{n-4} < 2\rho < \gamma. \tag{6.17}$$

We have

$$\varphi^i(y) = \epsilon_i^{-\theta(n-2)/2} w_{\epsilon_i^{\rho-\theta}}\left(\frac{y - y_i}{\epsilon_i^\theta}\right),$$

where w_ϵ is given in (6.7), so Eqs. (6.13) and (6.14) yield the following estimates

$$\begin{aligned} \|\nabla\varphi^i\|_{L^2(\mathbb{R}^n)}^2 &= \|\nabla w_{\epsilon_i^{\rho-\theta}}\|_{L^2(\mathbb{R}^n)}^2 = \mathcal{S}^{-1} + O(\epsilon_i^{(n-2)(\rho-\theta)}) \\ \|\varphi^i\|_{L^{2^*}(\mathbb{R}^n)}^2 &= \|w_{\epsilon_i^{\rho-\theta}}\|_{L^{2^*}(\mathbb{R}^n)}^2 = 1 + O(\epsilon_i^{n(\rho-\theta)}) \\ \|\varphi^i\|_{L^2(\mathbb{R}^n)}^2 &= b_n\epsilon_i^{2\rho} + O(\epsilon_i^{(n-2)\rho-(n-4)\theta}). \end{aligned} \tag{6.18}$$

Moreover, since $\theta \ge 1$, for any $y \in \operatorname{supp}\varphi^i$ we have $|y| \le C\epsilon_i$ and therefore

$$\int_{\mathbb{R}^n} |y|^\gamma |\nabla\varphi^i|^2 \, \mathrm{d}y \le C\epsilon_i^\gamma \int_{\mathbb{R}^n} |\nabla\varphi^i|^2 \, \mathrm{d}y = O(\epsilon_i^\gamma). \tag{6.19}$$

Using estimates (6.18) and (6.19) we find that for i sufficiently large,

$$\mathcal{S}\tilde{Q}(\varphi^i) = \frac{1 - \mu_1\mathcal{S}b_n\epsilon_i^{2\rho} + O(\epsilon_i^{(n-2)(\rho-\theta)}) + O(\epsilon_i^\gamma)}{1 + O(\epsilon_i^{n(\rho-\theta)})} < 1,$$

where $\tilde{Q}$ is as in (6.12) and the inequality holds by (6.17). Finally, for $i = 1, 2, \ldots$, setting $\boldsymbol{v}^i = (\sqrt{\alpha}\varphi^i, \sqrt{\beta}\varphi^i)$ and $\boldsymbol{u}^i(x) = \boldsymbol{v}^i(Px)$, if i is sufficiently large then we have

$$\mathcal{N}(A; \alpha, \beta)\frac{\Phi_\Lambda(\boldsymbol{u}^i)}{\left(\int_\Omega G(\boldsymbol{u}^i) \, \mathrm{d}x\right)^{2/2^*}} \le \mathcal{S}\tilde{Q}(\varphi^i) < 1.$$

For any such i, the function $\left(\int_\Omega G(\boldsymbol{u}^i) \, \mathrm{d}x\right)^{-1/2^*} \boldsymbol{u}^i$ satisfies the assertion of the proposition. □

Proof of Theorem 1.6. Since the hypotheses of Propositions 6.2 and 4.3 are satisfied, the infimum in (4.2) is obtained by some $\boldsymbol{u} \in \mathcal{H}\backslash\{\mathbf{0}\}$. Any such $\boldsymbol{u}$ is a nontrivial solution to (1.2). If $b > 0$ then Proposition 5.1 ensures that one of $\boldsymbol{u}$ or $-\boldsymbol{u}$ is a positive solution to (1.2). □

7. Non-existence results

In this section we provide proofs for the non-existence results in Theorems 1.7 and 1.8. We start with the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $\lambda_1 = \lambda_1(-\mathcal{L}) > 0$ be the first eigenvalue of $-\mathcal{L}$ with homogeneous Dirichlet boundary data on Ω and let $\varphi > 0$ be a corresponding eigenfunction. The assumption $b \geq 0$ and the Perron-Frobenius theorem (applied to $\Lambda + kI_2$ for suitably large $k \geq 0$) ensure the existence of an eigenvector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ for Λ corresponding to eigenvalue μ_2 such that $\xi_j \geq 0$ for $j = 1, 2$ and at least one of ξ_1 or ξ_2 positive. If $\boldsymbol{u} \in \mathcal{H}$ is a positive weak solution to (1.2) then testing the first equation of (1.2) against $\xi_1\varphi$ and using A2 we have

$$\begin{aligned}\int_\Omega (au_1 + bu_2 + \alpha u_1^{\alpha-1}u_2^{\beta})\xi_1\varphi \,\mathrm{d}x &= \xi_1 \int_\Omega \langle A(x)\nabla u_1, \nabla\varphi\rangle \,\mathrm{d}x \\ &= \xi_1 \int_\Omega \langle A(x)\nabla\varphi, \nabla u_1\rangle \,\mathrm{d}x \\ &= -\xi_1 \int_\Omega \mathrm{div}\, (A(x)\nabla\varphi)u_1 \,\mathrm{d}x \\ &= \lambda_1\xi_1 \int_\Omega \varphi u_1 \,\mathrm{d}x.\end{aligned}$$

Similarly, testing the second equation in (1.2) against $\xi_2\varphi$ gives

$$\int_\Omega (bu_1 + cu_2 + \beta u_1^{\alpha}u_2^{\beta-1})\xi_2\varphi \,\mathrm{d}x = \lambda_1\xi_2 \int_\Omega \varphi u_2 \,\mathrm{d}x.$$

Summing these two equalities and using both the symmetry of Λ and the assumption that $\Lambda\xi = \mu_2\xi$ gives

$$\begin{aligned}\int_\Omega \left(\alpha u_1^{\alpha-1}u_2^{\beta}\xi_1 + \beta u_1^{\alpha}u_2^{\beta-1}\xi_2\right)\varphi \,\mathrm{d}x &= \lambda_1 \int_\Omega (u_1\xi_1 + u_2\xi_2)\varphi \,\mathrm{d}x - \int_\Omega \langle\Lambda u, \xi\rangle\varphi \,\mathrm{d}x \\ &= \lambda_1 \int_\Omega (u_1\xi_1 + u_2\xi_2)\varphi \,\mathrm{d}x - \int_\Omega \langle\Lambda\xi, u\rangle\varphi \,\mathrm{d}x \\ &= (\lambda_1 - \mu_2) \int_\Omega (u_1\xi_1 + u_2\xi_2)\varphi \,\mathrm{d}x.\end{aligned}$$

Since both of $\int_\Omega \left(\alpha u_1^{\alpha-1}u_2^{\beta}\xi_1 + \beta u_1^{\alpha}u_2^{\beta-1}\xi_2\right)\varphi \,\mathrm{d}x$ and $\int_\Omega (u_1\xi_1 + u_2\xi_2)\varphi \,\mathrm{d}x$ are positive, we conclude that $\mu_2 < \lambda_1$. □

The following Pohozaev-type identity for vector-valued functions is a special case of the general variational identity established for C^2 solutions in Proposition 3 of [19]. The C^2 assumption was relaxed in [9] where an approximation technique was used to prove the identity for C^1 solutions.

Lemma 7.1. *Suppose $a_{ij} \in C^1(\overline{\Omega} \setminus \{x_0\})$ and that $b_{ij}(x) = \langle\nabla a_{ij}(x), x - x_0\rangle$ extends continuously to x_0. If $\boldsymbol{u} \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is a weak solution to (1.2) and if α, β satisfy both $\min\{\alpha, \beta\} > 1$ and (1.4) then the following identity holds*

$$\frac{1}{2}\sum_{j=1}^{2}\int_{\partial\Omega}\langle A(x)\nabla u_j,\nabla u_j\rangle\langle x,\nu\rangle \, \mathrm{d}S_x$$
$$= \int_{\Omega}\langle \Lambda \boldsymbol{u},\boldsymbol{u}\rangle \, \mathrm{d}x - \frac{1}{2}\sum_{j=1}^{2}\int_{\Omega}\langle B(x)\nabla u_j,\nabla u_j\rangle \, \mathrm{d}x. \tag{7.1}$$

Proof of Theorem 1.8. Assume with no loss of generality that $x_0 = 0$. Since $u_j \equiv 0$ on $\partial\Omega$ we have $\nabla u_j = \frac{\partial u_j}{\partial\nu}\nu$ so the integrand on the boundary integral on the left-hand side of (7.1) becomes

$$\langle A(x)\nabla u_j,\nabla u_j\rangle\ \langle x,\nu\rangle = \langle A(x)\nu,\nu\rangle\ \langle x,\nu\rangle\sum_{j=1}^{2}\left(\frac{\partial u_j}{\partial\nu}\right)^2.$$

Since Ω is star shaped with respect to $x_0 = 0$, we have $\langle x,\nu\rangle \geq 0$ and this quantity is not identically zero on $\partial\Omega$. By assumption A3 we have $\langle A(x)\nu,\nu\rangle > 0$ and from the Hopf Lemma for C^1 subsolutions for operators in divergence form (see [8]) we have $(\frac{\partial u_j}{\partial\nu})^2 > 0$ on $\partial\Omega$ for $j = 1, 2$. In particular the left-hand side of (7.1) is strictly positive. Therefore, the Pohozaev-type identity (7.1) together with the aid of the assumption $0 < \gamma \leq 2$, the assumption (1.9) and inequality (6.2) we obtain

$$\begin{aligned}\mu_2\sum_{j=1}^{2}\int_{\Omega}u_j^2 \, \mathrm{d}x &\geq \int_{\Omega}\langle \Lambda\boldsymbol{u},\boldsymbol{u}\rangle \, \mathrm{d}x \\ &> \frac{1}{2}\sum_{j=1}^{2}\int_{\Omega}\langle B(x)\nabla u_j,\nabla u_j\rangle \, \mathrm{d}x \\ &\geq \frac{\gamma C_0}{2}\sum_{j=1}^{2}\int_{\Omega}|x|^{\gamma}|\nabla u_j|^2 \, \mathrm{d}x \\ &\geq \frac{\gamma C_0}{2}K_0(n,\gamma,\Omega,x_0)^{-2}\sum_{j=1}^{2}\int_{\Omega}u_j^2 \, \mathrm{d}x.\end{aligned}$$

In particular, setting $\lambda_* = \frac{\gamma C_0}{2}K_0(n,\gamma,\Omega,x_0)^{-2}$ we find that if $\mu_2 \leq \lambda_*$ then there is no positive solution $\boldsymbol{u} \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ to (1.2). □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

8. Appendix

Here we record the details of some computations whose inclusion in the main body of the text would have distracted from the story.

Proof of Lemma 4.1. For any $\xi, \zeta \in \mathbb{R}^m$ we have

$$H(\xi) - H(\xi - \zeta) = \sum_{i=1}^{m} (A_i(\xi,\zeta) + B_i(\xi,\zeta)), \tag{8.1}$$

where

$$A_i(\xi,\zeta) = (|\xi_i|^{q_i} - |\xi_i - \zeta_i|^{q_i} - |\zeta_i|^{q_i}) P_i(\xi,\zeta), \qquad B_i(\xi,\zeta) = |\zeta_i|^{q_i} P_i(\xi,\zeta),$$

and

$$P_i(\xi,\zeta) = \prod_{j=1}^{i-1} |\xi_j - \zeta_j|^{q_j} \cdot \prod_{\ell=i+1}^{m} |\xi_\ell|^{q_\ell}.$$

In our notation for $P_i(\xi,\zeta)$ empty products (for example products of the form $\prod_{j=1}^{0} c_j$ or of the form $\prod_{j=m+1}^{m} c_j$) are understood to equal 1. For any $1 < r < \infty$ and any $\epsilon > 0$, the inequality

$$||a+b|^r - |a|^r - |b|^r| \leq \epsilon |a|^r + C_\epsilon |b|^r$$

holds for all $a, b \in \mathbb{R}$. For each $i = 1, \ldots, m$, applying this inequality to to the first factor of $A_i(\xi,\zeta)$ (with $a = \xi_i - \zeta_i$, $b = \zeta_i$ and $r = q_i$) gives

$$||\xi_i|^{q_i} - |\xi_i - \zeta_i|^{q_i} - |\zeta_i|^{q_i}| \leq \epsilon |\xi_i - \zeta_i|^{q_i} + C_\epsilon |\zeta_i|^{q_i}.$$

Therefore, using (8.1) we obtain

$$\begin{aligned} &|H(\xi) - H(\xi - \zeta) - H(\zeta)| \\ &\quad \leq \sum_{i=1}^{m} \left(\epsilon |\xi_i - \zeta_i|^{q_i} + C_\epsilon |\zeta_i|^{q_i}\right) P_i(\xi,\zeta) + |\zeta_1|^{q_1} \left| P_1(\xi,\zeta) \right. \\ &\qquad \left. - \prod_{j=2}^{m} |\zeta_j|^{q_j} \right| + \sum_{i=2}^{m} B_i(\xi,\zeta) \end{aligned} \tag{8.2}$$

for all $\epsilon > 0$ and all $(\xi,\zeta) \in \mathbb{R}^m \times \mathbb{R}^m$. For $\epsilon > 0$, and with $\boldsymbol{u}^k$ as in the hypotheses of the lemma, setting

$$f_\epsilon^k = \left(\left| H(\boldsymbol{u}^k) - H(\boldsymbol{u}^k - \boldsymbol{u}) - H(\boldsymbol{u}) \right| - \epsilon \sum_{i=1}^{m} |u_i^k - u_i|^{q_i} P_i(\boldsymbol{u}^k, \boldsymbol{u}) \right)^+$$

we have $f_\epsilon^k \to 0$ a.e. in Ω. Moreover, in view of (8.2) we have $0 \le f_\epsilon^k \le g_\epsilon^k$, where

$$g_\epsilon^k = C_\epsilon \sum_{i=1}^m |u_i|^{q_i} P_i(\boldsymbol{u}^k, \boldsymbol{u}) + |u_1|^{q_1} \left| P_1(\boldsymbol{u}^k, \boldsymbol{u}) - \prod_{j=2}^m |u_j|^{q_j} \right| + \sum_{i=2}^m B_i(\boldsymbol{u}^k, \boldsymbol{u}).$$

For every $i = 1, \dots, m$, $P_i(\boldsymbol{u}^k, \boldsymbol{u})$ is bounded in $L^{p/(p-q_i)}(\Omega)$ and

$$P_i(\boldsymbol{u}^k, \boldsymbol{u}) \to \begin{cases} \prod_{\ell=2}^m |u_\ell|^{q_\ell} & \text{if } i = 1 \\ 0 & \text{if } i \in \{2, \dots, m\} \end{cases} \qquad \text{a.e. } x \in \Omega.$$

Therefore, $P_1(\boldsymbol{u}^k, \boldsymbol{u}) \rightharpoonup \prod_{\ell=2}^m |u_\ell|^{q_\ell}$ weakly in $L^{p/(p-q_1)}(\Omega)$ and, for $i \in \{2, \dots, m\}$, $P_i(\boldsymbol{u}^k, \boldsymbol{u}) \rightharpoonup 0$ weakly in $L^{p/(p-q_i)}(\Omega)$. Consequently,

$$\sum_{i=1}^m \int_\Omega |u_i|^{q_i} P_i(\boldsymbol{u}^k, \boldsymbol{u})\, \mathrm{d}x \to \int_\Omega H(\boldsymbol{u})\, \mathrm{d}x \qquad \text{as } k \to \infty.$$

By a similar argument, we find both that $\left| P_1(\boldsymbol{u}^k, \boldsymbol{u}) - \prod_{j=2}^m |u_j|^{q_j} \right| \rightharpoonup 0$ weakly in $L^{p/(p-q_1)}(\Omega)$ and that $B_i(\boldsymbol{u}^k, \boldsymbol{u}) \rightharpoonup 0$ weakly in $L^{p/(p-q_i)}(\Omega)$ for $i \in \{2, \dots, m\}$, so we deduce that $\int_\Omega g_\epsilon^k\, \mathrm{d}x \to C_\epsilon \int_\Omega H(\boldsymbol{u})\, \mathrm{d}x$ as $k \to \infty$. The (generalized) Dominated Convergence Theorem now guarantees that $\int_\Omega f_\epsilon^k\, \mathrm{d}x \to 0$. Finally,

$$\begin{aligned} &\int_\Omega |H(\boldsymbol{u}^k) - H(\boldsymbol{u}^k - \boldsymbol{u}) - H(\boldsymbol{u})|\, \mathrm{d}x \\ &\quad \le \int_\Omega f_\epsilon^k\, \mathrm{d}x + \epsilon \sum_{i=1}^m \int_\Omega |u_i^k - u_i|^{q_i} P_i(\boldsymbol{u}^k, \boldsymbol{u})\, \mathrm{d}x \\ &\quad \le C\epsilon + \circ(1), \end{aligned}$$

where $\circ(1) \to 0$ as $k \to \infty$ and C depends on m, p, $\|\boldsymbol{u}\|_{L^p(\Omega;\mathbb{R}^m)}$, and an upper bound for $\{\|\boldsymbol{u}^k\|_{L^p(\Omega;\mathbb{R}^m)}\}_{k=1}^\infty$, but is independent of k. Since $\epsilon > 0$ is arbitrary, the lemma is established. □

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