Received: 30 January 2023 | Revised: 10 August 2023 | Accepted: 29 August 2023

DOI: 10.1112/plms.12562

RESEARCH ARTICLE

Proceedings of the London Mathematical Society

An intrinsic approach to relative braid group symmetries on ıquantum groups

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Funding information
NSF, Grant/Award Number: DMS-2001351; University of Virginia

Abstract

We initiate a general approach to the relative braid group symmetries on (universal) ıquantum groups, arising from quantum symmetric pairs of arbitrary finite types, and their modules. Our approach is built on new intertwining properties of quasi K-matrices which we develop and braid group symmetries on (Drinfeld double) quantum groups. Explicit formulas for these new symmetries on ıquantum groups are obtained. We establish a number of fundamental properties for these symmetries on ıquantum groups, strikingly parallel to their well-known quantum group counterparts. We apply these symmetries to fully establish rank 1 factorizations of quasi K-matrices, and this factorization property, in turn, helps to show that the new symmetries satisfy relative braid relations. As a consequence, conjectures of Kolb–Pellegrini and Dobson–Kolb are settled affirmatively. Finally, the above approach allows us to construct compatible relative braid group actions on modules over quantum groups for the first time.

MSC 2020
20G42, 17B37 (primary)

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1 | INTRODUCTION

1.1 | Background

Braid group symmetries have played an essential role in understanding the structures of Drinfeld–Jimbo quantum groups $\mathbf{U}$ and have found applications in geometric representation theory and categorification among others. These symmetries were constructed by Lusztig and used in first constructions of PBW bases and canonical bases in ADE type [26]. They have further been generalized to nonsimply laced types and beyond [27, 28]. Another crucial property is that there exists a compatible braid group action on integrable $\mathbf{U}$-modules. A systematic exposition on the braid group actions on quantum groups and their modules forms a significant portion of Lusztig's book [28, Ch. 5, Part VI].

Let $\widetilde{\mathbf{U}} = \langle E_i, F_i, K_i, K'_i \mid i \in \mathbb{I} \rangle$ be the Drinfeld double quantum group, where $K_iK'_i$ are central. The quantum group $\mathbf{U} = \langle E_i, F_i, K_i^{\pm 1} \mid i \in \mathbb{I} \rangle$ is recovered from $\widetilde{\mathbf{U}}$ by a central reduction:

$$\mathbf{U} = \widetilde{\mathbf{U}}/(K_iK'_i - 1 \mid i \in \mathbb{I}).$$

The Drinfeld doubles naturally arise from the Hall algebra construction of Bridgeland [3], and it is shown in [31] that reflection functors provide braid group actions on the Drinfeld doubles; see Proposition 2.3. As a straightforward generalization for Lusztig's symmetries on $\mathbf{U}$ [28, 37.2.4], there are four variants of braid group operators $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ on $\widetilde{\mathbf{U}}$, for $e \in \{\pm 1\}$ and $i \in \mathbb{I}$, which are related to each other by conjugations of certain (anti-) involutions [31]; see (2.10):

$$\widetilde{T}'_{i,-e} = \sigma \circ \widetilde{T}''_{i,+e} \circ \sigma, \qquad \widetilde{T}''_{i,-e} := \psi \circ \widetilde{T}''_{i,+e} \circ \psi, \qquad \widetilde{T}'_{i,+e} := \psi \circ \widetilde{T}'_{i,-e} \circ \psi. \tag{1.1}$$

Here, ψ is the bar involution and σ is an anti-involution on $\widetilde{\mathbf{U}}$; see Proposition 2.2.

Associated with any Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath_\varsigma)$ was introduced by Gail Letzter in finite type [22, 23] as a q-deformation of the usual symmetric pair; here, $\mathbf{U}^\imath_\varsigma$ is a coideal subalgebra of $\mathbf{U}$ depending on parameters $\boldsymbol{\varsigma} = (\varsigma_i)_{i\in\mathbb{I}_\circ}$. *Universal* quantum symmetric pairs $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ (of quasi-split type) were formulated in [32], where the parameters are replaced by suitable central elements in $\widetilde{\mathbf{U}}^\imath$, and $\mathbf{U}^\imath_\varsigma$ is recovered from $\widetilde{\mathbf{U}}^\imath$ by a central reduction. ($\mathbf{U}^\imath_\varsigma, \widetilde{\mathbf{U}}^\imath$ will be referred to as ıquantum groups, and they are called *quasi-split* if $\mathbb{I}_\bullet = \emptyset$ and *split* if in addition $\tau = \mathrm{Id}$.) Several fundamental constructions on quantum groups, including (quasi) R-matrix, canonical bases, and Hall algebra realization have been generalized to the setting of quantum symmetric pairs in recent years; see [5, 7, 8, 32].

Lusztig's braid group actions on $\mathbf{U}$ do not preserve the subalgebra $\mathbf{U}^\imath_\varsigma$ in general. Kolb–Pellegrini [20] proposed that there should be relative braid group symmetries on ıquantum groups corresponding to the relative (or restricted) Weyl groups for the underlying symmetric pairs. For a class of ıquantum groups of finite type (including all quasi-split types and type AII) with some specific parameters, formulas for such braid group actions were found and verified *loc. cit.* via computer computation. The relative braid group action for type AI appeared earlier in [10] and [33].

There has been some limited progress on relative braid group action on $\mathbf{U}^\imath_\varsigma$ in the last decade; for type AIII, see Dobson [14]. An ıHall algebra approach has been developed to realize the universal *quasi-split* ıquantum groups $\widetilde{\mathbf{U}}^\imath$ [32]. As a generalization of Ringel's construction [35], reflection functors [30, 31] are used to construct relative braid group actions on $\widetilde{\mathbf{U}}^\imath$ of quasi-split type, where the braid group operators act on the central elements in $\widetilde{\mathbf{U}}^\imath$ nontrivially. For $\widetilde{\mathbf{U}}^\imath$ or $\mathbf{U}^\imath_\varsigma$ in general beyond quasi-split type, no conjectural formulas or conceptual explanations for relative braid group actions were available.

There are braid group actions on $\mathbf{U}$-modules that are compatible with braid group actions on quantum groups, cf. [28]. In contrast, no relative braid group action on $\mathbf{U}^\imath_\varsigma$-modules has been known to date. The Hall algebra approach does not help providing any clue on such action at the module level.

1.2 | Goal

Our goal is to develop a conceptual and general approach to relative braid group actions on ıquantum groups, arising from (universal) quantum symmetric pairs of arbitrary finite type, and on their modules for the first time. This, in particular, settles the longstanding conjecture of Kolb and Pellegrini [20] in a constructive manner.

It is crucial for us to work with universal ıquantum groups. We shall formulate relative braid group symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath$, for $e \in \{\pm 1\}$ and $i \in \mathbb{I}_{\circ,\tau}$, which are related to each other via conjugations by a bar involution $\psi^\imath$ and an anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$; compare (1.1):

$$\widetilde{\mathbf{T}}'_{i,-e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,+e} \circ \sigma^\imath, \qquad \widetilde{\mathbf{T}}''_{i,-e} := \psi^\imath \circ \widetilde{\mathbf{T}}''_{i,+e} \circ \psi^\imath, \qquad \widetilde{\mathbf{T}}'_{i,+e} := \psi^\imath \circ \widetilde{\mathbf{T}}'_{i,-e} \circ \psi^\imath.$$

By central reductions and rescaling automorphisms, these symmetries descend to relative braid group actions on ıquantum groups with parameters $\mathbf{U}^\imath_\varsigma$. Moreover, we are able to formulate compatible relative braid group actions on integrable **U**-modules. We further establish a number of basic properties of these new symmetries that are natural $\imath$-counterparts of well-known properties for Lusztig's braid group symmetries.

1.3 The basic idea

Various constructions for quantum groups can be regarded as constructions for quantum symmetric pairs of diagonal type $(\mathbf{U} \otimes \mathbf{U}, \mathbf{U})$, and hence, $\imath$quantum groups can be viewed as a vast generalization of quantum groups. This simple observation can be instrumental on determining what form a suitable $\imath$-generalization should take; for example, this view was applied successfully in the developments of $\imath$canonical bases arising from quantum symmetric pairs in [8] and $\imath$Hall algebras that realize universal $\imath$quantum groups [32]; see also the recent development of $\imath$crystal bases by Watanabe [37].

Denote by $\mathbf{L}''_i$ the rank 1 quasi R-matrix associated to $i \in \mathbb{I}$, and let $\mathbf{L}'_i$ be its inverse. The following formula in [28, 37.3.2]

$$(T'_{i,-1} \otimes T'_{i,-1})\Delta(T''_{i,+1}u) = \mathbf{L}'_i\Delta(u)\mathbf{L}''_i \tag{1.2}$$

provides a relation between braid group actions on $\mathbf{U}$ and $\mathbf{U} \otimes \mathbf{U}$; a formula similar to (1.2) via a different formulation of braid operators appeared in [24] and [17]. A starting point of this paper is to view a variant of the identity (1.2) as a formula in the setting of (universal) quantum symmetric pairs of diagonal type $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$; see §4.4.

Now, let $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ be a general universal quantum symmetric pair. Inspired by the relation (1.2), we aim at formulating a relation between braid group action on the Drinfeld double $\widetilde{\mathbf{U}}$ and the desired relative braid group action on the universal $\imath$quantum group $\widetilde{\mathbf{U}}^\imath$ through conjugations of rank 1 quasi K-matrices $\widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$.

Quasi K-matrices were originally formulated in [7] as an intertwiner between the embedding $\imath : \mathbf{U}^\imath_{\boldsymbol{\varsigma}} \to \mathbf{U}$ and a bar-involution conjugated embedding (for parameters $\boldsymbol{\varsigma}$ satisfying strong constraints); a proof in greater generality was given in [5] under a technical assumption (which was removed later in [9]). A reformulation by Appel and Vlaar [2] (also see [21]) bypassed a direct use of the bar maps, allowing more general parameters $\boldsymbol{\varsigma}$. In this paper, we upgrade these constructions by formulating the quasi K-matrices $\widetilde{\Upsilon}$ for universal quantum symmetric pairs, and, in particular, the rank 1 quasi K-matrices $\widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$.

Dobson and Kolb [15] proposed (conjectural) factorizations of quasi K-matrices in finite types into products of rank 1 quasi K-matrices, analogous to factorizations of quasi R-matrices [17, 24]. In their formulation, a certain scaling twist shows up. In this paper, we upgrade the formulation of the factorization together with the corresponding scaling twist to quasi K-matrices $\widetilde{\Upsilon}$ in the universal setting.

Examples indicate that our basic idea of constructing the desired relative braid group action on $\widetilde{\mathbf{U}}^\imath$ via quasi K-matrix and braid group action on $\widetilde{\mathbf{U}}$ (viewed as a generalization of (1.2)) basically works — up to a simple twist: it is necessary to use *suitably rescaled* braid group operators on $\widetilde{\mathbf{U}}$. Remarkably, this scaling turns out to coincide with the aforementioned scaling which appears in the factorizations of a quasi K-matrix $\widetilde{\Upsilon}$. We are able to explore this compatibility to draw strong consequences on the seemingly unrelated topics: relative braid group actions and the factorization of quasi K-matrices.

1.4 | Main results

1.4.1 | New intertwining properties of quasi K-matrices

We formulate universal quantum symmetric pairs $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ associated to arbitrary Satake diagrams and their basic properties in Section 2.4, following and generalizing the quasi-split setting in [32]. The algebra $\widetilde{\mathbf{U}}^\imath$ contains $\widetilde{\mathbf{U}}^{\imath 0}$ and $\widetilde{\mathbf{U}}_\bullet$ naturally as subalgebras, where $\widetilde{\mathbf{U}}_\bullet$ is the Drinfeld double associated to $\mathbb{I}_\bullet$ and $\widetilde{\mathbf{U}}^{\imath 0}$ is a Cartan subalgebra generated by $\widetilde{k}_i = K_i K'_{\tau i}$, for $i \in \mathbb{I}_\circ$.

We recall the recent somewhat technical formulation of a quasi K-matrix Υ_ς for $(\mathbf{U}, \mathbf{U}^\imath_\varsigma)$ from [2] (cf. [5, 7, 8] for earlier constructions) in Theorem 3.1 and upgrade it to a universal version $\widetilde{\Upsilon}$ for $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ in Theorem 3.2. It turns out that $\widetilde{\Upsilon}$ admits a more conceptual and simpler characterization in terms of the anti-involution σ on $\widetilde{\mathbf{U}}$ as follows.

Theorem A (Theorem 3.6). *The quasi K-matrix $\widetilde{\Upsilon} = \sum_{\mu \in \mathbb{N}\mathbb{I}} \widetilde{\Upsilon}^\mu$, for $\widetilde{\Upsilon}^\mu \in \widetilde{\mathbf{U}}^+_\mu$, is uniquely characterized by $\widetilde{\Upsilon}^0 = 1$ and the following intertwining relations:*

$$B_i \widetilde{\Upsilon} = \widetilde{\Upsilon} B_i^\sigma \quad (i \in \mathbb{I}_\circ), \qquad x\widetilde{\Upsilon} = \widetilde{\Upsilon} x \quad (x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet).$$

This characterization of $\widetilde{\Upsilon}$ plays a basic role in producing explicit formulas for relative braid group actions on $\widetilde{\mathbf{U}}^\imath$; see the proof of Theorem 5.5 in §5.4. There is a similar simple characterization of Υ_ς for $\mathbf{U}^\imath_\varsigma$ in terms of the anti-involution $\sigma\tau$ on $\mathbf{U}$; see Theorem 3.16. (It is tempting to regard this as a new definition of Υ_ς.)

We use a distinguished scaling automorphism $\widetilde{\Psi}_{\varsigma_\star}$ to define a rescaled bar involution $\psi_\star$ on $\widetilde{\mathbf{U}}$ (by twisting the bar involution ψ on $\widetilde{\mathbf{U}}$). By exploring further intertwining properties via $\widetilde{\Upsilon}$ as in [19], we establish in Kac–Moody generality a bar involution $\psi^\imath$ (see Proposition 3.4) and an anti-involution $\sigma^\imath$ (see Proposition 3.12) from $\psi_\star$ and σ, respectively. These (anti-)involutions $\psi^\imath$ and $\sigma^\imath$ were known in some quasi-split cases; see [12].

Denote by $\widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$, the quasi K-matrix associated to the rank 1 Satake subdiagram $(\mathbb{I}_\bullet \cup \{i, \tau i\}, \tau)$.

1.4.2 | New symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$

Associated to a Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, one has the (absolute) Weyl group W generated by the simple reflections s_i, for $i \in \mathbb{I}$, and a finite parabolic subgroup $W_\bullet = \langle s_i \mid i \in \mathbb{I}_\bullet \rangle$ with the longest element $w_\bullet$. Given $i \in \mathbb{I}_\circ$, one has a rank 1 Satake subdiagram $(\mathbb{I}_{\bullet,i} = \mathbb{I}_\bullet \cup \{i, \tau i\}, \tau)$, and define $\mathbf{r}_i \in W$ as in (2.14). As $\mathbf{r}_i = \mathbf{r}_{\tau i}$, it suffices to restrict to $\mathbf{r}_i$, for $i \in \mathbb{I}_{\circ,\tau}$ (here $\mathbb{I}_{\circ,\tau}$ is a set of fixed representatives of τ-orbits on $\mathbb{I}$). The relative Weyl group W° is a subgroup of W generated by $\mathbf{r}_i$, for $i \in \mathbb{I}_{\circ,\tau}$; abstractly, W° is a Weyl group with $\mathbf{r}_i$ $(i \in \mathbb{I}_{\circ,\tau})$ as simple reflections [25]; also see [15, 29, 34].

Let $\widetilde{T}''_{i,+1}$ and $\widetilde{T}'_{i,-1}$, for $i \in \mathbb{I}$, be the braid group operators on $\widetilde{\mathbf{U}}$ [31]; see Proposition 2.3. Let $\widetilde{\mathscr{T}}''_{i,+1}$ and $\widetilde{\mathscr{T}}'_{i,-1}$ be the rescaled version of $\widetilde{T}''_{i,+1}$ and $\widetilde{T}'_{i,-1}$ via conjugation by a scaling automorphism $\widetilde{\Psi}_{\varsigma_\diamond}$; see (4.2)–(4.3). As $\widetilde{\mathscr{T}}'_{j,-1}$, for $j \in \mathbb{I}$, satisfy the braid relations, we can make sense of $\widetilde{\mathscr{T}}'_{w,-1}$, for $w \in W$, and, in particular, $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}$, for $i \in \mathbb{I}_\circ$, as automorphisms of $\widetilde{\mathbf{U}}$.

Theorem B (Theorem 4.7, Proposition 4.11, Theorem 4.14, Theorem 5.5). *Let $i \in \mathbb{I}_\circ$. There exists a unique automorphism $\widetilde{\mathbf{T}}'_{i,-1}$ of $\widetilde{\mathbf{U}}^\imath$ such that the following intertwining relation holds:*

$$\widetilde{\mathbf{T}}'_{i,-1}(x)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(x), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{1.3}$$

More precisely, the action of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ is given as follows:

(1) $\widetilde{\mathbf{T}}'_{i,-1}(x) = (\widehat{\tau}_{\bullet,i} \circ \widehat{\tau})(x)$, *and* $\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}$, *for all $x \in \widetilde{\mathbf{U}}_\bullet$, $j \in \mathbb{I}_\circ$.*
(2) $\widetilde{\mathbf{T}}'_{i,-1}(B_i) = -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\widetilde{\mathscr{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}$.
(3) *The formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ ($i \neq j \in \mathbb{I}_{\circ,\tau}$) are listed in Table 3.*

See (2.15) and (4.15) for notation $\tau_{\bullet,i}$ and $\widetilde{k}_{\lambda,\diamond}$; also see (4.5) and Remark 4.3 for the braid group operator $\widetilde{\mathscr{T}}_{w_\bullet}$. By definition, we have $\mathbf{r}_i = \mathbf{r}_{\tau i}$, $\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_{\tau i}$, and $\widetilde{\mathbf{T}}'_{i,-1} = \widetilde{\mathbf{T}}'_{\tau i,-1}$; thus, we only need to consider $\widetilde{\mathbf{T}}'_{i,-1}$, for $i \in \mathbb{I}_{\circ,\tau}$.

In the same spirit of (1.3) in Theorem B, the identity (1.2) for the Drinfeld double quantum group $\widetilde{\mathbf{U}}$ can be reformulated as the intertwining relation (4.8) for quantum symmetric pair $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$ of diagonal type.

Another symmetry $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, is formulated in Theorem 6.1 that satisfies the following intertwining relation in (6.1), similar to (1.3):

$$\widetilde{\mathbf{T}}''_{i,+1}(x)\,\widetilde{\mathscr{T}}''_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathscr{T}}''_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1})\,\widetilde{\mathscr{T}}''_{\mathbf{r}_i,+1}(x), \quad \text{for all } x \in \widetilde{\mathbf{U}}^\imath.$$

We further define two more symmetries $\widetilde{\mathbf{T}}'_{i,+1}$ and $\widetilde{\mathbf{T}}''_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ by conjugating $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ via the involution $\psi^\imath$; see (6.11). These symmetries are related to each other as follows; compare [28, Chap. 37].

Theorem C (Theorem 6.7). *Let $e = \pm 1$ and $i \in \mathbb{I}_\circ$. The symmetries $\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathbf{T}}''_{i,-e}$ are mutual inverses. Moreover, we have $\widetilde{\mathbf{T}}'_{i,e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,-e} \circ \sigma^\imath$.*

Actually, part of the proof of Theorem B (i.e., the invertibility of $\widetilde{\mathbf{T}}'_{i,-1}$) is completed only when it is established in Theorem C that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses. This is one main reason why we have formulated $\widetilde{\mathbf{T}}''_{i,+1}$ separately in spite of its many similarities with the properties for $\widetilde{\mathbf{T}}'_{i,-1}$ which we already established.

Here is an outline of proofs of Theorems B–C. We first establish the existence of an endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ which satisfies the intertwining relation (1.3), by proving Properties (1)–(3) in Theorem B one-by-one. Properties (1)–(2) are established uniformly in Proposition 4.11 and Theorem 4.14. We formulate a structural result in Proposition 5.11 as a main step toward a uniform proof of the rank 2 formulas in (3) (see Theorem 5.5); Proposition 5.11 is then verified by a type-by-type computation in the Appendix. In order to prove the invertibility of $\widetilde{\mathbf{T}}'_{i,-1}$, we establish another endomorphism $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$ which satisfies the intertwining relation (6.1) in Theorem 6.1; the existence for $\widetilde{\mathbf{T}}''_{i,+1}$ is proved by a strategy similar to the one for $\widetilde{\mathbf{T}}'_{i,-1}$. Finally, we show in Theorem 6.7 that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses by invoking the uniqueness of elements satisfying an intertwining relation.

The formulas for actions of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ on generators of $\widetilde{\mathbf{U}}^\imath$ are mostly new. In quasi-split types, up to some twistings, we recover the formulas obtained by Hall algebra computation in [30], and by central reductions to $\mathbf{U}^\imath_{\varsigma}$, we recover formulas obtained by computer computation in [20].

1.4.3 | A basic property of braid symmetries

The following theorem is a generalization of a well-known basic property of braid group action on quantum groups; see [28].

Theorem D (See Theorem 7.13). *Suppose that $wi \in \mathbb{I}_\circ$, for $w \in W^\circ$ and $i \in \mathbb{I}_\circ$. Then we have $\widetilde{\mathbf{T}}''_{w,+1}(B_i) = B_{wi}$.*

The dependence in the formulation of Theorem 7.13 on reduced expressions $\underline{w}$ of w can be removed, once Theorem F on braid relations for $\widetilde{\mathbf{T}}''_{j,+1}$ is established. We reduce the proof of Theorem D to the rank 2 cases. The proofs in rank 2 cases are largely uniform (avoiding type-by-type computation), based on the counterpart results in quantum group setting, the defining intertwining property of $\widetilde{\mathbf{T}}''_{w,+1}$, and some weight arguments.

1.4.4 | Factorizations of a quasi K-matrix

It is well known that a quasi R-matrix admits a factorization into a product of rank 1 R-matrices parametrized by positive roots; see [17, 24]; also cf. [16].

Dobson and Kolb [15] proposed a conjecture on an analogous factorization of a quasi K-matrix into a product, denoted by $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, of rank 1 factors parametrized by restricted positive roots; see (8.1) for notation. They established a reduction from a general finite type to the rank 2 Satake diagrams. In addition, they established the rank 2 cases of *split* types and type AII/AIII, via a type-by-type lengthy computation based on several explicit formulas for rank 1 quasi K-matrices which they computed.

Exploring (the rank 2 cases of) Theorem D and some of its consequences, we provide a uniform and concise proof that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the same defining intertwining relations for $\widetilde{\Upsilon}$. Then, the factorization property for arbitrary finite types follows by the uniqueness of $\widetilde{\Upsilon}$.

Theorem E (Dobson–Kolb Conjecture, Theorem 8.1). *The quasi K-matrix $\widetilde{\Upsilon}$ for $\widetilde{\mathbf{U}}^\imath$ of finite type admits a factorization $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$.*

1.4.5 | Relative braid group relations

Recall Lusztig's symmetries $T'_{i,e}, T''_{i,e}$ on a quantum group $\mathbf{U}$ satisfy braid group relations associated to the (absolute) Weyl group W [28]; see [31] for analogous statements on a Drinfeld double $\widetilde{\mathbf{U}}$. We have the following generalization in the setting of ıquantum groups. Denote by $\mathrm{Br}(W^\circ)$ the braid group associated to W°.

Theorem F (Theorem 9.1). *Fix $e \in \{\pm 1\}$. The symmetries $\widetilde{\mathbf{T}}'_{i,e}$ (and respectively, $\widetilde{\mathbf{T}}''_{i,e}$) of $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_{\circ,\tau}$, satisfy the relative braid group relations in* $\mathrm{Br}(W^\circ)$.

With the help of the intertwining relation (1.3), the proof of Theorem F is built on the braid group relations for $\widetilde{\mathscr{T}}_i$ ($i \in \mathbb{I}$) and the factorization properties of rank 2 quasi K-matrices established in Theorem E.

It was shown in [8] that Lusztig's symmetries $T'_{i,e}$ and $T''_{i,e}$ on $\mathbf{U}$, for $i \in \mathbb{I}_\bullet$, preserve the subalgebra $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (under some constraints on $\boldsymbol{\varsigma}$). We easily upgrade this statement to the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$, providing a braid group action of $\mathrm{Br}(W_\bullet)$ on $\widetilde{\mathbf{U}}^\imath$; see Proposition 4.5. Actually, we obtain four variants of actions of $\mathrm{Br}(W_\bullet)$ on $\widetilde{\mathbf{U}}^\imath$ generated by $\widetilde{\mathscr{T}}'_{j,e}$ or $\widetilde{\mathscr{T}}''_{j,e}$, for $j \in \mathbb{I}_\bullet$, respectively.

It is further established that the two ("black and white") braid group actions on $\widetilde{\mathbf{U}}^\imath$ combine neatly into an action of a semidirect product $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$.

Theorem G (Theorem 9.3, Corollary 9.7). *Let $e = \pm 1$.*

(1) *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathscr{T}}'_{j,e}$ ($j \in \mathbb{I}_\bullet$) and $\widetilde{\mathbf{T}}'_{i,e}$ ($i \in \mathbb{I}_{\circ,\tau}$).*
(2) *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathscr{T}}''_{j,e}$ ($j \in \mathbb{I}_\bullet$) and $\widetilde{\mathbf{T}}''_{i,e}$ ($i \in \mathbb{I}_{\circ,\tau}$).*

Theorem G (or more precisely, its $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$-counterpart in Theorem 9.10; see §1.4.6 below) confirms an old conjecture of Kolb and Pellegrini [20, Conjecture 1.2] in full generality, and moreover, we have provided precise formulas for the braid group actions.

1.4.6 | Relative braid group symmetries on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$

By central reductions, the symmetries $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$ on the universal $\imath$quantum group $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, descend naturally to the $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ with the distinguished parameter $\boldsymbol{\varsigma}_\diamond$. On the other hand, the symmetries $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ naturally descend to $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$; see the commutative diagrams in §9.4. We then transport the relative braid group symmetries from $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond}$ and $\mathbf{U}^\imath_{\overline{\boldsymbol{\varsigma}}_{\star\diamond}}$ to the $\imath$quantum groups $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (see Theorems 9.9–9.10), for an arbitrary parameter $\boldsymbol{\varsigma}$, thanks to the isomorphism $\mathbf{U}^\imath_{\boldsymbol{\varsigma}_\diamond} \cong \mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ given in Proposition 2.7.

1.4.7 | Relative braid group actions on $\mathbf{U}$-modules

Let $i \in \mathbb{I}_\circ$, $e = \pm 1$, and $\boldsymbol{\varsigma}$ be a balanced parameter (see the line below (2.18)). We show that the symmetries $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on the $\imath$quantum group $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (defined by central reductions) satisfy natural intertwining relations with the usual braid group symmetries on $\mathbf{U}$. These intertwining properties allow us to formulate automorphisms (denoted again by the same notations $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$) on an arbitrary finite-dimensional $\mathbf{U}$-module M of type $\mathbf{1}$; see (10.12). These operators on M admit favorable properties parallel to those satisfied by Lusztig's braid group actions on modules.

Theorem H (Theorem 10.5, Theorem 10.6). *Let $i \in \mathbb{I}_\circ$ and $e = \pm 1$, and let M be any finite-dimensional* $\mathbf{U}$*-module of type* **1***. The automorphisms $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on M are compatible with the corresponding automorphisms on $\mathbf{U}^\imath_\varsigma$, that is,*

$$\mathbf{T}'_{i,e}(xv) = \mathbf{T}'_{i,e}(x)\mathbf{T}'_{i,e}(v), \qquad \mathbf{T}''_{i,e}(xv) = \mathbf{T}''_{i,e}(x)\mathbf{T}''_{i,e}(v),$$

for any $x \in \mathbf{U}^\imath_\varsigma, v \in M$. Moreover, the operators $\mathbf{T}'_{i,e}$ (respectively, $\mathbf{T}''_{i,e}$) on M, for $i \in \mathbb{I}_\circ$, satisfy the relative braid group relations in $\mathrm{Br}(W^\circ)$.

In this paper, we have assumed that a ground field $\mathbb{F}$ is the algebraic closure of $\mathbb{Q}(q)$ partly due to uses of rescaling automorphisms, though often it suffices to work with the field $\mathbb{Q}(q^{\frac{1}{2}})$ if we choose the parameters $\boldsymbol{\varsigma}$ suitably. There is a $\mathbb{Q}(q)$-form ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$ of $\widetilde{\mathbf{U}}^\imath$ such that $\widetilde{\mathbf{U}}^\imath = \mathbb{F} \otimes_{\mathbb{Q}(q)} {}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$; see (5.17). The symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ indeed preserve the $\mathbb{Q}(q)$-subalgebra ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$; see Proposition 5.9. Theorems A–G remain valid for ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$.

1.5 | Future works and applications

The formulations of the main results (Theorems A–H), up to some reasonable rephrasing, make sense for universal quantum symmetric pairs of arbitrary Kac–Moody type (cf. [18]), and we conjecture they are valid in this great generality. For example, the symmetries $\widetilde{\mathbf{T}}'_{i,-1}$, for $i \in \mathbb{I}_\circ$, for $\widetilde{\mathbf{U}}^\imath$ of Kac-Moody type will follow once Conjecture 5.13 is confirmed. The main reason on the restriction to finite types in this paper is that we rely on the classification of Satake diagrams to explicitly compute the rank 2 formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ and $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$, which, in particular, verify that they lie in $\widetilde{\mathbf{U}}^\imath$. Section 3 is valid in Kac–Moody generality. Steps (1)–(2) in Theorem B (which occupy most of Section 4) are also valid in the Kac–Moody setting.

Some further developments will be carried out in future works. We shall extend the constructions of relative braid group actions to (universal) ıquantum groups of affine type. We plan to use the new tools developed in this paper to attack the conjectures in [11, 12] on relative braid group actions on quasi-split universal ıquantum groups of Kac–Moody type. We also plan to understand the relative braid group action on $\mathbf{U}^\imath_\varsigma$-modules more explicitly, and this may serve as a starting point for a new approach toward relative braid group action on ıquantum groups; compare [28].

The relative braid group symmetries of this paper (and their affine generalization) will be used crucially in the Drinfeld-type presentation of quasi-split affine ıquantum groups in an upcoming work joint with Ming Lu. It is expected that they will continue to play a key role for Drinfeld-type presentations of general affine ıquantum groups.

One may hope that these new braid group symmetries preserve the integral $\mathbb{Z}[q, q^{-1}]$-form on (modified) ıquantum groups in [8, 9]. (This will be highly nontrivial to verify, as the ıdivided powers are much more sophisticated than the divided powers.) It will be interesting to develop further connections among relative braid group actions, PBW bases, and ıcanonical bases; compare [28]. They may help to stimulate further Khovanov–Lauda–Rouquier (KLR)-type categorification of ıquantum groups as well as ıHall algebra realization of ıquantum groups beyond quasi-split type.

Kolb and Yakimov [21] extended the construction of quantum symmetric pairs to the setting of Nichols algebras of diagonal type. The new intertwining properties of quasi K-matrices and the relative braid group actions established in this paper seem well suited for generalizations in this direction.

The notion of relative Coxeter groups, which is valid in a more general setting than symmetric pairs, admits a geometric interpretation [25, 29]. It will be exciting to realize relative braid group action in geometric and categorical frameworks, and develop possible connections to the representation theory of real groups (cf. [6] and references therein). It will be very interesting to explore more general braid group actions associated to relative Coxeter groups.

1.6 | Organization

The paper is organized into Sections 2–10 and the Appendix. Below we provide a detailed description section by section.

In Section 2, we review and set up the basics and notations on quantum groups $\mathbf{U}$ and Drinfeld doubles $\widetilde{\mathbf{U}}$, including several (anti-) involutions and a rescaling automorphism $\widetilde{\Psi}_{\mathbf{a}}$ on $\widetilde{\mathbf{U}}$. We recall explicit formulas for braid group actions on $\widetilde{\mathbf{U}}$. Associated to a Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, we form a relative Weyl group $W^\circ = \langle \mathbf{r}_i \mid i \in \mathbb{I}_\circ \rangle$. Then, we formulate (universal) quantum symmetric pairs $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ and $(\mathbf{U}, \mathbf{U}^\imath_{\boldsymbol{\varsigma}})$.

In Section 3, we formulate quasi K-matrix $\widetilde{\Upsilon}$ in the universal quantum symmetric pair setting, and establish a new intertwining property via the anti-involution σ on $\widetilde{\mathbf{U}}$. We establish an anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ via σ and an intertwining property of $\widetilde{\Upsilon}$. We formulate a rescaled bar involution $\psi_\star$ on $\widetilde{\mathbf{U}}$, and then establish a bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$ via $\psi_\star$ and an intertwining property of $\widetilde{\Upsilon}$. An anti-involution σ_τ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ for an arbitrary parameter $\boldsymbol{\varsigma}$ is also established.

In Section 4, we formulate rescaled braid group symmetries $\widetilde{\mathscr{T}}'_{w,-1}$, for $w \in W$, on $\widetilde{\mathbf{U}}$ via a rescaling automorphism $\Psi_{\boldsymbol{\varsigma}_\diamond}$. We define $\widetilde{\mathbf{T}}'_{i,-1}$ in terms of an intertwining property involving $\widetilde{\Upsilon}$ and the rescaled braid group symmetries $\widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_i} \equiv \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}$; see Theorem 4.7. We then formulate additional symmetries $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,\pm1}$ on $\widetilde{\mathbf{U}}^\imath$ via conjugations of $\widetilde{\mathbf{T}}'_{i,-1}$ by an anti-involution $\sigma^\imath$ and a bar involution $\psi^\imath$. We obtain explicit formulas for the actions of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ in Proposition 4.11 and on B_i in Theorem 4.14.

In Section 5, we formulate a general structural result that relates formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ and $\widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_i}(F_j)$; see Proposition 5.11. The explicit formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in each rank 2 universal $\imath$quantum group are collected in Table 3. The type-by-type verification of these formulas is postponed to the Appendix.

In Section 6.1, we formulate another symmetry $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$ using a different intertwining property. Then we formulate the counterparts of the results in Sections 4–5. We collect all rank 2 formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ in Table 4, whose proofs similar to the Appendix will be skipped (the detail can be found in Appendix B in an arXiv version).

We then show that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses, completing the proofs that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are automorphisms of $\widetilde{\mathbf{U}}^\imath$. The property $\widetilde{\mathbf{T}}'_{i,e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,-e} \circ \sigma^\imath$ follows by inspection from the explicit formulas for the actions of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$.

In Section 7, we establish a basic formula $\widetilde{\mathbf{T}}_{\underline{w}}(B_i) = B_j$, for $i, j \in \mathbb{I}_\circ$ and $w \in W^\circ$ such that $w\alpha_i = \alpha_j$, generalizing a well-known formula in quantum groups. We reduce the proof of the formula to the rank 2 Satake diagrams. We then provide uniform proofs in the rank 2 cases.

In Section 8, we prove uniformly the factorization property of quasi K-matrices in all rank 2 quantum symmetric pairs, completing the proof of Dobson–Kolb's conjecture in arbitrary finite types. This is an application of the formula established in Section 7.

In Section 9, we verify that the symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ satisfy the braid group relations in $\mathrm{Br}(W^\circ)$. Together with the braid group action given by $\widetilde{\mathscr{T}}'_{j,e}, \widetilde{\mathscr{T}}''_{j,e}$, for $j \in \mathbb{I}_\bullet$, we obtain four braid group actions of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$. By taking central reductions and using the isomorphism $\phi_\varsigma : \mathbf{U}^\imath_{\varsigma_\diamond} \cong \mathbf{U}^\imath_\varsigma$, we construct relative braid group symmetries $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath_\varsigma$ for general parameters ς, confirming the main conjecture in [20].

In Section 10, we formulate linear operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on any finite-dimensional $\mathbf{U}$-module. We show that they are compatible with corresponding automorphisms on $\widetilde{\mathbf{U}}^\imath$, and that they satisfy the relative braid group relations.

1.7 | Notations

We list the notations that are often used throughout the paper.

- ▷ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ — sets of nonnegative integers, integers, rational, and complex numbers
- ▷ $\mathcal{R}, \mathcal{R}^\vee$ — systems of roots and coroots with simple systems $\Pi = \{\alpha_i | i \in \mathbb{I}\}$ and $\Pi^\vee = \{\alpha_i^\vee | i \in \mathbb{I}\}$, respectively
- ▷ $W, \ell(\cdot)$ — the Weyl group and its length function
- ▷ w_0, τ_0 — the longest element in W and its associated diagram involution
- ▷ $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$ — braid group symmetries on $\widetilde{\mathbf{U}}$
- ▷ $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ — admissible pairs (aka Satake diagrams)
- ▷ $W_\bullet, \mathcal{R}_\bullet$ — the Weyl group and root system associated to the subdiagram $\mathbb{I}_\bullet$
- ▷ $w_\bullet$ — the longest element in $W_\bullet$
- ▷ $W_{\bullet,i}$ — the parabolic subgroup of W generated by s_k, for $k \in \mathbb{I}_{\bullet,i} := \mathbb{I}_\bullet \cup \{i, \tau i\}$
- ▷ $w_{\bullet,i}, \tau_{\bullet,i}$ — the longest element of $W_{\bullet,i}$ and its associated diagram involution
- ▷ $W^\circ, \ell_\circ(\cdot)$ — the relative Weyl group generated by $\mathbf{r}_i := w_{\bullet,i} w_\bullet$, for $i \in \mathbb{I}_\circ$, and its length function such that $\ell_\circ(\mathbf{r}_i) = 1$
- ▷ $\boldsymbol{w}_\circ$ — the longest element in W°
- ▷ $\mathbf{U}, \widetilde{\mathbf{U}}$ — quantum group and Drinfeld double
- ▷ $\widehat{\tau}, \widehat{\tau}_0$ — involutions on $\widetilde{\mathbf{U}}$ induced by the diagram involutions τ, τ_0
- ▷ $\widetilde{\mathbf{U}}^\imath, \mathbf{U}^\imath_\varsigma$ — universal ıquantum group and ıquantum group with parameter ς
- ▷ $\widetilde{\Upsilon}$ — quasi K-matrix for universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$
- ▷ $\varsigma_\diamond, \varsigma_\star$ — two distinguished parameters; see (2.21) and (3.8)
- ▷ $\widetilde{\Psi}_{\mathbf{a}}$ — a rescaling automorphism of $\widetilde{\mathbf{U}}$; see (2.7)
- ▷ $\Phi_{\mathbf{a}}$ — a rescaling automorphism of $\mathbf{U}$; see (2.8)
- ▷ π_ς — a central reduction from $\widetilde{\mathbf{U}}$ to $\mathbf{U}$; see (2.6)
- ▷ $\pi^\imath_\varsigma$ — a central reduction from $\widetilde{\mathbf{U}}^\imath$ to $\mathbf{U}^\imath_\varsigma$; see Proposition 2.8
- ▷ $\psi^\imath$ — a bar involution on $\widetilde{\mathbf{U}}^\imath$; see (3.10)
- ▷ $\sigma^\imath$ — an anti-involution on $\widetilde{\mathbf{U}}^\imath$; see (3.24)
- ▷ σ_τ — an anti-involution on $\mathbf{U}^\imath_\varsigma$; see (3.26)
- ▷ $\widetilde{\mathscr{T}}'_{i,e}, \widetilde{\mathscr{T}}''_{i,e}$ — rescaled (via $\widetilde{\Psi}_{\mathbf{a}}$) braid group symmetries on $\widetilde{\mathbf{U}}$; see (4.2)–(4.3)
- ▷ $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ — braid group symmetries on $\widetilde{\mathbf{U}}^\imath$
- ▷ $\widetilde{\mathscr{T}}_i, \widetilde{\mathscr{T}}_i^{-1}, \widetilde{\mathbf{T}}_i, \widetilde{\mathbf{T}}_i^{-1}$ — shorthand notations for $\widetilde{\mathscr{T}}''_{i,+1}, \widetilde{\mathscr{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}, \widetilde{\mathbf{T}}'_{i,-1}$
- ▷ $\mathscr{T}'_{i,e;\varsigma}, \mathscr{T}''_{i,e;\varsigma}$ — rescaled braid group symmetries on $\mathbf{U}$; see (10.1), (10.7)

2 | DRINFELD DOUBLES AND QUANTUM SYMMETRIC PAIRS

In this section, we set up notations for quantum groups, Drinfeld doubles, and quantum symmetric pairs. We review the relative Weyl and braid groups associated to Satake diagrams. Several basic properties of (universal) $\imath$quantum groups are presented.

2.1 | Quantum groups and Drinfeld doubles

We set up notations for a quantum group $\mathbf{U}$ of finite type and its Drinfeld double $\widetilde{\mathbf{U}}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with a symmetrizable Cartan matrix $C = (c_{ij})_{i,j\in\mathbb{I}}$. Let $D = \mathrm{diag}(\epsilon_i \mid \epsilon_i \in \mathbb{Z}_{\geqslant 1},\ i \in \mathbb{I})$ be a symmetrizer, that is, DC is symmetric, such that $\gcd\{\epsilon_i \mid i \in \mathbb{I}\} = 1$. Fix a simple system $\Pi = \{\alpha_i | i \in \mathbb{I}\}$ of $\mathfrak{g}$ and a set of simple coroots $\Pi^\vee = \{\alpha_i^\vee | i \in \mathbb{I}\}$. Let $\mathcal{R}$ and $\mathcal{R}^\vee$ be the corresponding root and coroot systems. Denote the root lattice by $\mathbb{Z}\mathbb{I} := \bigoplus_{i\in\mathbb{I}}\mathbb{Z}\alpha_i$. Let $(\cdot,\cdot)$ be the normalized Killing form on $\mathbb{Z}\mathbb{I}$ so that the short roots have squared length 2. The Weyl group W is generated by the simple reflections $s_i : \mathbb{Z}\mathbb{I} \to \mathbb{Z}\mathbb{I}$, for $i \in \mathbb{I}$, such that $s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$. Set w_0 to be the longest element of W.

Let q be an indeterminate and $\mathbb{Q}(q)$ be the field of rational functions in q with coefficients in $\mathbb{Q}$, the field of rational numbers. Set $\mathbb{F}$ to be the algebraic closure of $\mathbb{Q}(q)$ and $\mathbb{F}^\times := \mathbb{F} \setminus \{0\}$. We denote

$$q_i := q^{\epsilon_i}, \qquad \forall i \in \mathbb{I}.$$

Denote, for $r, m \in \mathbb{N}$,

$$[r]_t = \frac{t^r - t^{-r}}{t - t^{-1}}, \quad [r]_t! = \prod_{i=1}^{r}[i]_t, \quad \begin{bmatrix} m \\ r \end{bmatrix}_t = \frac{[m]_t[m-1]_t \dots [m-r+1]_t}{[r]_t!}.$$

We mainly take $t = q, q_i$.

Then $\widetilde{\mathbf{U}} := \widetilde{\mathbf{U}}_q(\mathfrak{g})$ is defined to be the $\mathbb{F}$-algebra generated by E_i, F_i, K_i, K_i', $i \in \mathbb{I}$, where K_i, K_i' are invertible, subject to the following relations: K_i, K_j' commute with each other, for all $i, j \in \mathbb{I}$,

$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i'}{q - q^{-1}}, \quad K_iE_j = q_i^{c_{ij}}E_jK_i, \quad K_iF_j = q_i^{-c_{ij}}F_jK_i, \tag{2.1}$$

$$K_i'E_j = q_i^{-c_{ij}}E_jK_i', \qquad K_i'F_j = q_i^{c_{ij}}F_jK_i', \tag{2.2}$$

and the quantum Serre relations, for $i \neq j \in \mathbb{I}$,

$$\sum_{s=0}^{1-c_{ij}} (-1)^s \begin{bmatrix} 1-c_{ij} \\ s \end{bmatrix}_{q_i} E_i^sE_jE_i^{1-c_{ij}-s} = 0, \tag{2.3}$$

$$\sum_{s=0}^{1-c_{ij}} (-1)^s \begin{bmatrix} 1-c_{ij} \\ s \end{bmatrix}_{q_i} F_i^sF_jF_i^{1-c_{ij}-s} = 0. \tag{2.4}$$

Note that K_iK_i' are central in $\widetilde{\mathbf{U}}$, for all $i \in \mathbb{I}$.

The comultiplication $\Delta : \widetilde{\mathbf{U}} \to \widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}$ is defined as follows:

$$\begin{aligned} \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i', \\ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i') = K_i' \otimes K_i'. \end{aligned} \tag{2.5}$$

Let $\mathbf{U} = \mathbf{U}_q(\mathfrak{g})$ be the Drinfeld–Jimbo quantum group associated to $\mathfrak{g}$ over $\mathbb{F}$ with Chevalley generators $\{E_i, F_i, K_i^{\pm 1} | i \in \mathbb{I}\}$, whose relations can be obtained from $\widetilde{\mathbf{U}}$ above by simply replacing K_i' by K_i^{-1}, for all i; that is, one identifies $\mathbf{U} = \widetilde{\mathbf{U}}/(K_iK_i' - 1 \mid i \in \mathbb{I})$. Both $\widetilde{\mathbf{U}}$ and $\mathbf{U}$ admit standard triangular decompositions, $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}^-\widetilde{\mathbf{U}}^0\widetilde{\mathbf{U}}^+$ and $\mathbf{U} = \mathbf{U}^-\mathbf{U}^0\mathbf{U}^+$; we identify $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+ = \langle E_i \mid i \in \mathbb{I}\rangle$ and $\widetilde{\mathbf{U}}^- = \mathbf{U}^-$.

For any scalars $\mathbf{a} = (a_i)_{i\in\mathbb{I}} \in \mathbb{F}^{\times,\mathbb{I}}$, one has a isomorphism

$$\widetilde{\mathbf{U}}/(K_iK_i' - a_i \mid i \in \mathbb{I}) \xrightarrow{\cong} \mathbf{U}$$

through the central reduction

$$\begin{aligned} \pi_{\mathbf{a}} : \widetilde{\mathbf{U}} \longrightarrow \mathbf{U}, \\ F_i \mapsto F_i, \quad E_i \mapsto \sqrt{a_i}E_i, \quad K_i \mapsto \sqrt{a_i}K_i, \quad K_i' \mapsto \sqrt{a_i}K_i^{-1}. \end{aligned} \tag{2.6}$$

The canonical identification uses $\pi_{\mathbf{1}}$, for $\mathbf{1} = \{1\}_{i\in\mathbb{I}}$.

Proposition 2.1. *Let $\mathbf{a} = (a_i)_{i\in\mathbb{I}} \in (\mathbb{F}^\times)^{\mathbb{I}}$. We have an automorphism $\widetilde{\Psi}_{\mathbf{a}}$ on the $\mathbb{F}$-algebra $\widetilde{\mathbf{U}}$ such that*

$$\widetilde{\Psi}_{\mathbf{a}} : K_i \mapsto a_i^{1/2}K_i, \quad K_i' \mapsto a_i^{1/2}K_i', \quad E_i \mapsto a_i^{1/2}E_i, \quad F_i \mapsto F_i. \tag{2.7}$$

We have an automorphism $\Phi_{\mathbf{a}}$ on the $\mathbb{F}$-algebra $\mathbf{U}$ such that

$$\Phi_{\mathbf{a}} : K_i \mapsto K_i, \quad E_i \mapsto a_i^{1/2}E_i, \quad F_i \mapsto a_i^{-1/2}F_i. \tag{2.8}$$

We have

$$\pi_{\mathbf{a}} = \pi_{\mathbf{1}} \circ \widetilde{\Psi}_{\mathbf{a}}. \tag{2.9}$$

A $\mathbb{Q}$-linear operator on a $\mathbb{F}$-algebra is *antilinear* if it sends $q^m \mapsto q^{-m}$, for $m \in \mathbb{Z}$.

Proposition 2.2.

(1) *There exists an antilinear involution ψ on $\widetilde{\mathbf{U}}$, which fixes E_i, F_i and swaps $K_i \leftrightarrow K_i'$, for $i \in \mathbb{I}$.*
(2) *There exists an antilinear involution on $\mathbf{U}$, also denoted by ψ, which fixes E_i, F_i and swaps $K_i \leftrightarrow K_i^{-1}$, for $i \in \mathbb{I}$.*
(3) *There exists an anti-involution σ on $\widetilde{\mathbf{U}}$ that fixes E_i, F_i and swaps $K_i \leftrightarrow K_i'$, for $i \in \mathbb{I}$.*
(4) *There exists a Chevalley involution ω on $\widetilde{\mathbf{U}}$ that swaps E_i and F_i and swaps $K_i \leftrightarrow K_i'$, for $i \in \mathbb{I}$.*

Let $\widetilde{\mathbf{U}} = \bigoplus_{\nu\in\mathbb{Z}\mathbb{I}} \widetilde{\mathbf{U}}_\nu$ be the weight decomposition of $\widetilde{\mathbf{U}}$ such that $E_i \in \widetilde{\mathbf{U}}_{\alpha_i}, F_i \in \widetilde{\mathbf{U}}_{-\alpha_i}, K_i, K_i' \in \widetilde{\mathbf{U}}_0$. Write $\widetilde{\mathbf{U}}_\nu^+ := \widetilde{\mathbf{U}}_\nu \cap \widetilde{\mathbf{U}}^+$.

2.2 | Braid group action on the Drinfeld double $\widetilde{\mathbf{U}}$

Lusztig introduced braid group symmetries $T'_{i,e}, T''_{i,e}$, for $i \in \mathbb{I}$ and $e = \pm 1$, on a quantum group $\mathbf{U}$ [28, §37.1.3]. Analogous braid group symmetries $\widetilde{T}'_{i,e}, \widetilde{T}''_{i,e}$, for $i \in \mathbb{I}$ and $e = \pm 1$, exist on the Drinfeld double $\widetilde{\mathbf{U}}$; see [31, Propositions 6.20–6.21]. (Our notations $\widetilde{T}'_{i,e}$, $\widetilde{T}''_{i,e}$ here correspond to $\mathbf{T}'_{i,e}$, $\mathbf{T}''_{i,e}$ therein.) We recall the formulation of $\widetilde{T}''_{i,+1}$ below.

Proposition 2.3 [31, Proposition 6.21]. *Set $r = -c_{ij}$. There exist an automorphism $\widetilde{T}''_{i,+1}$, for $i \in \mathbb{I}$, on $\widetilde{\mathbf{U}}$ such that*

$$\widetilde{T}''_{i,+1}(K_j) = K_j K_i^{-c_{ij}}, \qquad \widetilde{T}''_{i,+1}(K'_j) = K'_j K_i'^{-c_{ij}},$$
$$\widetilde{T}''_{i,+1}(E_i) = -F_i K_i'^{-1}, \qquad \widetilde{T}''_{i,+1}(F_i) = -K_i^{-1} E_i,$$
$$\widetilde{T}''_{i,+1}(E_j) = \sum_{s=0}^{r} (-1)^s q_i^{-s} E_i^{(r-s)} E_j E_i^{(s)}, \qquad j \neq i,$$
$$\widetilde{T}''_{i,+1}(F_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} F_i^{(s)} F_j F_i^{(r-s)}, \qquad j \neq i.$$

Moreover, the $\widetilde{T}''_{i,+1}$, for $i \in \mathbb{I}$, satisfy the braid relations.

We sometimes use the following conventional short notations:

$$\widetilde{T}_i := \widetilde{T}''_{i,+1}, \quad \widetilde{T}_i^{-1} := \widetilde{T}'_{i,-1}, \quad T_i := T''_{i,+1}, \quad T_i^{-1} := T'_{i,-1}.$$

Hence, we can define

$$\widetilde{T}_w \equiv \widetilde{T}''_{w,+1} := \widetilde{T}_{i_1} \cdots \widetilde{T}_{i_r} \in \mathrm{Aut}(\widetilde{\mathbf{U}}),$$

where $w = s_{i_1} \cdots s_{i_r}$ is any reduced expression of $w \in W$. Similarly, one defines T_w for $w \in W$.

The symmetries $\widetilde{T}'_{i,e}$ and $\widetilde{T}''_{i,e}$, for $i \in \mathbb{I}$, satisfy the following identities in $\widetilde{\mathbf{U}}$ [31] (analogous to [28, 37.2.4] in $\mathbf{U}$):

$$\widetilde{T}'_{i,-1} = \sigma \circ \widetilde{T}''_{i,+1} \circ \sigma,$$
$$\widetilde{T}''_{i,-e} = \psi \circ \widetilde{T}''_{i,+e} \circ \psi, \qquad \widetilde{T}'_{i,+e} = \psi \circ \widetilde{T}'_{i,-e} \circ \psi. \tag{2.10}$$

The automorphism $\widetilde{T}''_{i,+1}$ descends to Lusztig's automorphisms $T''_{i,+1}$ on $\mathbf{U}$:

$$\pi_{\mathbf{1}} \circ \widetilde{T}''_{i,+1} = T''_{i,+1} \circ \pi_{\mathbf{1}}. \tag{2.11}$$

2.3 | Satake diagrams and relative Weyl/braid groups

Given a subset $\mathbb{I}_\bullet \subset \mathbb{I}$, denote by $W_\bullet$ the parabolic subgroup of W generated by $s_i, i \in \mathbb{I}_\bullet$. Set $w_\bullet$ to be the longest element of $W_\bullet$. Let $\mathcal{R}_\bullet$ be the set of roots that lie in the span of $\alpha_i, i \in \mathbb{I}_\bullet$. Similarly,

$\mathcal{R}^\vee_\bullet$ is the set of coroots that lie in the span of $\alpha_i^\vee, i \in \mathbb{I}_\bullet$. Let $\rho_\bullet$ be the half sum of positive roots in the root system $\mathcal{R}_\bullet$, and $\rho^\vee_\bullet$ be the half sum of positive coroots in $\mathcal{R}^\vee_\bullet$.

An *admissible pair* $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ (cf. [4, 18]) consists of a partition $\mathbb{I}_\bullet \cup \mathbb{I}_\circ$ of $\mathbb{I}$, and a Dynkin diagram involution τ of $\mathfrak{g}$ (where $\tau = \mathrm{Id}$ is allowed) such that

(1) $w_\bullet(\alpha_j) = -\alpha_{\tau j}$ for $j \in \mathbb{I}_\bullet$,
(2) If $j \in \mathbb{I}_\circ$ and $\tau j = j$, then $\alpha_j(\rho^\vee_\bullet) \in \mathbb{Z}$.

The diagrams associated to admissible pairs are known as Satake diagrams. We shall use the terms between admissible pairs and Satake diagrams interchangeably. Throughout the paper, we shall always work with admissible pairs $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$. A symmetric pair $(\mathfrak{g}, \theta)$ (of finite type) consists of a semisimple Lie algebra $\mathfrak{g}$ and an involution θ on $\mathfrak{g}$; the irreducible symmetric pairs (of finite type) are classified by Satake diagrams.

Given an admissible pair $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, the corresponding involution θ (acting on the weight lattice) is recovered as

$$\theta = -w_\bullet \circ \tau. \tag{2.12}$$

Set $\mathbb{I}_{\circ,\tau}$ to be a (fixed) set of representatives of τ-orbits in $\mathbb{I}_\circ$. The (real) rank of a Satake diagram is the cardinality of $\mathbb{I}_{\circ,\tau}$. We call a Satake diagram $(\mathbb{I}^1 = \mathbb{I}^1_\bullet \cup \mathbb{I}^1_\circ, \tau^1)$ a subdiagram of another Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$, if $\mathbb{I}^1_\circ \subset \mathbb{I}_\circ$, $\mathbb{I}^1_\bullet \subset \mathbb{I}_\bullet$, $\tau^1|_{\mathbb{I}^1_\circ} = \tau|_{\mathbb{I}^1_\circ}$, and $\mathbb{I}^1_\bullet$ contains all black nodes in $\mathbb{I}$ which lie in the connected components of $\mathbb{I}^1_\circ$ in $\mathbb{I}_\bullet \cup \mathbb{I}^1_\circ$.

Given an admissible pair $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ and $i \in \mathbb{I}_\circ$, we set

$$\mathbb{I}_{\bullet,i} := \mathbb{I}_\bullet \cup \{i, \tau i\}. \tag{2.13}$$

Let $W_{\bullet,i}$ be the parabolic subgroup of W generated by $s_i, i \in \mathbb{I}_{\bullet,i}$. Let $w_{\bullet,i}$ the longest element of $W_{\bullet,i}$. The following constructions are a special case of those by Lusztig [25]; also cf. [15, 29]. Define $\mathbf{r}_i \in W_{\bullet,i}$ such that

$$w_{\bullet,i} = \mathbf{r}_i w_\bullet (= w_\bullet \mathbf{r}_i), \qquad \text{where } \ell(w_{\bullet,i}) = \ell(\mathbf{r}_i) + \ell(w_\bullet). \tag{2.14}$$

(It follows from the admissible pair requirement that $w_{\bullet,i}, \mathbf{r}_i$, and $w_\bullet$ commute with each other.) Then the subgroup of W,

$$W^\circ := \langle \mathbf{r}_i | i \in \mathbb{I}_{\circ,\tau} \rangle,$$

is a Weyl group by itself with its simple reflections identified with $\{\mathbf{r}_i \mid i \in \mathbb{I}_{\circ,\tau}\}$. Denote by $\ell_\circ$ the length function of the Coxeter systerm $(W^\circ, \mathbb{I}_{\circ,\tau})$ and by $\boldsymbol{w}_\circ$ its longest element.

Proposition 2.4 ([25, Theorem 5.9]). *Let $w_1, w_2 \in W^\circ$. Then $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ if and only if $\ell_\circ(w_1 w_2) = \ell_\circ(w_1) + \ell_\circ(w_2)$.*

Hence, there is no ambiguity to refer to the Coxeter system W° or W when we talk about reduced expressions of an element $w \in W^\circ \subset W$. By definition, we have identifications $\mathbb{I}_{\bullet,i} = \mathbb{I}_{\bullet,\tau i}, W_{\bullet,i} = W_{\bullet,\tau i}, w_{\bullet,i} = w_{\bullet,\tau i}$, and $\mathbf{r}_i = \mathbf{r}_{\tau i}$. Denote by $\tau_{\bullet,i}$ the diagram involution on $\mathbb{I}_{\bullet,i}$ such that

$$w_{\bullet,i}(\alpha_j) = -\alpha_{\tau_{\bullet,i} j}, \qquad \forall j \in \mathbb{I}_{\bullet,i}. \tag{2.15}$$

The *relative Weyl group* associated to the Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ can be identified with W°. Let $\{\overline{\alpha}_i | i \in \mathbb{I}_{\circ,\tau}\}$ be the simple system of the relative (or restricted) root system, where $\overline{\alpha}_i$ is identified with the following element (cf. [15, §2.3]):

$$\overline{\alpha}_i := \frac{\alpha_i - \theta(\alpha_i)}{2}, \qquad (i \in \mathbb{I}_\circ). \tag{2.16}$$

Note that $\overline{\alpha}_i = \overline{\alpha}_{\tau i}$.

We introduce a subgroup of W:

$$W^\theta = \{w \in W \mid w\theta = \theta w\}.$$

It is well known that (see, e.g., [15, §2.2])

$$W_\bullet \rtimes W^\circ \cong W^\theta.$$

We shall refer to the braid group associated to the relative Weyl group W° as the *relative braid group* and denote it by $\mathrm{Br}(W^\circ)$. Accordingly, we denote the braid group associated to $W_\bullet$ by $\mathrm{Br}(W_\bullet)$.

2.4 | Universal ıquantum groups

We set up some basics for the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$, following and somewhat generalizing [32].

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a Satake diagram. Define $\widetilde{\mathbf{U}}_\bullet$ to be the subalgebra of $\widetilde{\mathbf{U}}$ with the set of Chevalley generators

$$\widetilde{\mathcal{G}}_\bullet := \{E_j, F_j, K_j, K_j' \mid j \in \mathbb{I}_\bullet\}.$$

The universal ıquantum group associated to the Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ is defined to be the $\mathbb{F}$-subalgebra of $\widetilde{\mathbf{U}}$

$$\widetilde{\mathbf{U}}^\imath = \langle B_i, \widetilde{k}_i, g \mid i \in \mathbb{I}_\circ, g \in \widetilde{\mathcal{G}}_\bullet \rangle$$

via the embedding $\imath : \widetilde{\mathbf{U}}^\imath \to \widetilde{\mathbf{U}}, u \mapsto u^\imath$, with

$$B_i \mapsto F_i + \widetilde{T}_{w_\bullet}(E_{\tau i})K_i', \quad \widetilde{k}_i \mapsto K_i K_{\tau i}', \quad g \mapsto g, \quad \text{for } i \in \mathbb{I}_\circ,\ g \in \widetilde{\mathcal{G}}_\bullet. \tag{2.17}$$

(The notation $u^\imath$ for $u \in \widetilde{\mathbf{U}}^\imath$, for example, $B_i^\imath$, is mainly used when we need to apply braid group operators on $\widetilde{\mathbf{U}}$ to $u^\imath$.) By definition, $\widetilde{\mathbf{U}}^\imath$ contains the Drinfeld double $\widetilde{\mathbf{U}}_\bullet$ associated to $\mathbb{I}_\bullet$ as a subalgebra.

Let $\widetilde{\mathbf{U}}^{\imath 0}$ denote the subalgebra of $\widetilde{\mathbf{U}}^\imath$ generated by $\widetilde{k}_i, K_j, K_j'$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$. The following lemma is clear.

Lemma 2.5. *If $i = \tau i, i \in \mathbb{I}_\circ$, then $\widetilde{k}_i$ is central in $\widetilde{\mathbf{U}}^\imath$. If $\tau i \neq i \in \mathbb{I}_\circ$, then $\widetilde{k}_i \widetilde{k}_{\tau i}$ is central in $\widetilde{\mathbf{U}}^\imath$.*

Following [22] and [18, §6.2], we formulate a monomial basis for $\widetilde{\mathbf{U}}^\imath$. Denote $B_j = F_j$, for $j \in \mathbb{I}_\bullet$. For a multi-index $J = (j_1, j_2, \dots, j_n) \in \mathbb{I}^n$, we define $F_J := F_{j_1} F_{j_2} \cdots F_{j_n}$ and $B_J := B_{j_1} B_{j_2} \cdots B_{j_n}$. Let $\mathcal{J}$ be a fixed subset of $\bigcup_{n \geqslant 0} \mathbb{I}^n$ such that $\{F_J | J \in \mathcal{J}\}$ forms a basis of $\widetilde{\mathbf{U}}$ as a $\widetilde{\mathbf{U}}^+ \widetilde{\mathbf{U}}^0$-module.

Proposition 2.6 (cf. [18, Proposition 6.2]). *The set* $\{B_J | J \in \mathcal{J}\}$ *is a basis of the left (or right)* $\widetilde{\mathbf{U}}^+_\bullet \widetilde{\mathbf{U}}^{\imath 0}$*-modules* $\widetilde{\mathbf{U}}^\imath$.

2.5 | ıQuantum group $\mathbf{U}^\imath_\varsigma$ via central reduction

We recall some basics for quantum symmetric pairs $(\mathbf{U}, \mathbf{U}^\imath_\varsigma)$, cf. [18, 22], where the parameter $\boldsymbol{\varsigma} = (\varsigma_i)_{i \in \mathbb{I}_\circ} \in \mathbb{F}^{\times, \mathbb{I}_\circ}$ is always assumed to satisfy the following conditions (cf. [22] [18, Section 5.1]):

$$\varsigma_i = \varsigma_{\tau i}, \qquad \text{if } \tau i \neq i \text{ and } (\alpha_i, w_\bullet \alpha_{\tau i}) = 0. \tag{2.18}$$

We call $\boldsymbol{\varsigma}$ a *balanced parameter*, if $\varsigma_i = \varsigma_{\tau i}$ for any $i \in \mathbb{I}_\circ$. For an arbitrary parameter $\boldsymbol{\varsigma}$, we define an associated balanced parameter $\boldsymbol{\varsigma}^e$ such that

$$\varsigma_i^e = \varsigma_{\tau i}^e = \sqrt{\varsigma_i \varsigma_{\tau i}}. \tag{2.19}$$

Define $\mathbf{U}_\bullet$ to be the subalgebra of $\mathbf{U}$ with the set of Chevalley generators

$$\mathcal{G}_\bullet := \{E_j, F_j, K_j^{\pm 1} \mid j \in \mathbb{I}_\bullet\}.$$

The ıquantum group associated to the Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ with parameter $\boldsymbol{\varsigma}$ is defined to be the $\mathbb{F}$-subalgebra of $\mathbf{U}$

$$\mathbf{U}^\imath_\varsigma = \langle B_i, k_j, g \mid i \in \mathbb{I}_\circ, j \in \mathbb{I} \setminus \mathbb{I}_{\circ,\tau}, g \in \mathcal{G}_\bullet \rangle$$

via the embedding $\imath : \mathbf{U}^\imath_\varsigma \to \mathbf{U}$ with

$$B_i \mapsto F_i + \varsigma_i T_{w_\bullet}(E_{\tau i}) K_i^{-1}, \qquad k_j \mapsto K_j K_{\tau j}^{-1}, \qquad \forall i \in \mathbb{I}_\circ, j \in \mathbb{I} \setminus \mathbb{I}_{\circ,\tau}. \tag{2.20}$$

Note that $\mathbf{U}^\imath$ contains $\mathbf{U}_\bullet$ as a subalgebra. For $i \in \mathbb{I}_{\circ,\tau}$, we set $k_i = 1$ if $i = \tau i$ and $k_i = k_{\tau i}^{-1}$ if $i \neq \tau i$. Similarly, we denote by $\mathbf{U}^{\imath 0}$ the subalgebra of $\mathbf{U}^\imath$ generated by k_i, K_j, for $i \in \mathbb{I}_\circ$, $j \in \mathbb{I}_\bullet$.

Recall from (2.16) that $\overline{\alpha}_i = (\alpha_i + w_\bullet \alpha_{\tau i})/2$. Define a distinguished balanced parameter $\boldsymbol{\varsigma}_\diamond = (\varsigma_{i,\diamond})_{i \in \mathbb{I}_\circ}$ such that

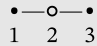
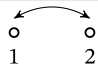
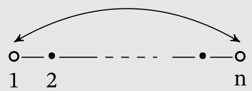
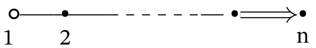
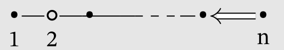
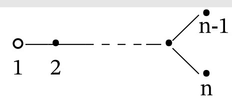
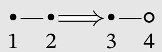
$$\varsigma_{i,\diamond} = -q^{-(\alpha_i, \alpha_i + w_\bullet \alpha_{\tau i})/2} = -q^{-(\overline{\alpha}_i, \overline{\alpha}_i)}, \quad \text{for } i \in \mathbb{I}_\circ. \tag{2.21}$$

The parameter $\boldsymbol{\varsigma}_\diamond$ will play a basic role in this paper; also cf. [15].

Letzter [23] and Kolb [18, Proposition 9.2, Theorem 9.7] raised and addressed the question on when ıquantum groups for different parameters are related by Hopf algebra automorphisms of $\widetilde{\mathbf{U}}$. Watanabe [36, Lemma 2.5.1] showed that the ıquantum groups for *arbitrarily* different parameters are all isomorphic (not necessarily by Hopf algebra automorphisms); we recall the following special case of Watanabe's result.

Proposition 2.7 [36, Lemma 2.5.1]. *For any parameter* $\boldsymbol{\varsigma}$*, there exists an algebra isomorphism* ϕ_ς : $\mathbf{U}^\imath_{\varsigma_\diamond} \to \mathbf{U}^\imath_\varsigma$ *which sends* $B_i \mapsto \sqrt{\varsigma_{i,\diamond}(\varsigma_i \varsigma_{\tau i})^{-1/2}} B_i, E_j \mapsto E_j, F_j \mapsto F_j, K_j \mapsto K_j, k_r \mapsto \sqrt{\varsigma_r^{-1} \varsigma_{\tau r}} k_r, i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet, r \in \mathbb{I} \setminus \mathbb{I}_{\circ,\tau}$.

TABLE 1 Rank 1 Satake diagrams and local datum.

Type	Satake diagram	$\varsigma_{i,\diamond}$	$\mathbf{r}_i$
AI_1		$\varsigma_{1,\diamond} = -q^{-2}$	$\mathbf{r}_1 = s_1$
AII_3		$\varsigma_{2,\diamond} = -q^{-1}$	$\mathbf{r}_2 = s_{2132}$
AIII_{11}		$\varsigma_{1,\diamond} = -q^{-1}$	$\mathbf{r}_1 = s_1 s_2$
AIV, n≥2		$\varsigma_{1,\diamond} = -q^{-1/2}$	$\mathbf{r}_1 = s_{1\cdots n\cdots 1}$
BII, n≥ 2		$\varsigma_{1,\diamond} = -q_1^{-1}$	$\mathbf{r}_1 = s_{1\cdots n\cdots 1}$
CII, n≥3		$\varsigma_{2,\diamond} = -q_2^{-1/2}$	$\mathbf{r}_2 = s_{2\cdots n\cdots 212\cdots n\cdots 2}$
DII, n≥4		$\varsigma_{1,\diamond} = -q^{-1}$	$\mathbf{r}_1 = s_{1\cdots n-2\cdot n-1\cdot n\cdot n-2\cdots 1}$
FII		$\varsigma_{4,\diamond} = -q_4^{-1/2}$	$\mathbf{r}_4 = s_{432312343231234}$

It follows that there is an algebra isomorphism

$$\phi_\varsigma \phi_{\varsigma^e}^{-1} : \mathbf{U}^\imath_{\varsigma^e} \longrightarrow \mathbf{U}^\imath_\varsigma \tag{2.22}$$

which sends $B_i \mapsto B_i, E_j \mapsto E_j, F_j \mapsto F_j, K_j \mapsto K_j, k_r \mapsto \sqrt{\varsigma_r^{-1}\varsigma_{\tau r}}k_r$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet, r \in \mathbb{I} \setminus \mathbb{I}_{\circ,\tau}$.

We have the following central reduction $\pi^\imath_\varsigma : \widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_\varsigma$, generalizing [32, Proposition 6.2] in the quasi-split setting.

Proposition 2.8. *There exists a quotient morphism $\pi^\imath_\varsigma : \widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_\varsigma$ sending*

$$B_i \mapsto B_i, \quad \widetilde{k}_j \mapsto \varsigma_{\tau j} k_j, \quad \widetilde{k}_{\tau j} \mapsto \varsigma_j k_{\tau j}, \qquad (i \in \mathbb{I}_\circ, j \in \mathbb{I}_{\circ,\tau}),$$

and $\pi^\imath_\varsigma|_{\widetilde{\mathbf{U}}_\bullet} = \pi_{\mathbf{1}}|_{\widetilde{\mathbf{U}}_\bullet}$. The kernel of $\pi^\imath_\varsigma$ is generated by

$$\widetilde{k}_i - \varsigma_i \quad (i = \tau i, i \in \mathbb{I}_\circ), \qquad \widetilde{k}_i\widetilde{k}_{\tau i} - \varsigma_i\varsigma_{\tau i} \quad (i \neq \tau i, i \in \mathbb{I}_\circ), \quad K_j K_j' - 1 \quad (j \in \mathbb{I}_\bullet).$$

Proof. By (2.6), the restriction of π_ς on $\widetilde{\mathbf{U}}^\imath$ sends

$$B_i \mapsto F_i + \sqrt{\varsigma_i\varsigma_{\tau i}} T_{w_\bullet}(E_{\tau i})K_i^{-1}, \qquad \widetilde{k}_i \mapsto \sqrt{\varsigma_i\varsigma_{\tau i}} k_i, \qquad i \in \mathbb{I}_\circ,$$
$$K_j \mapsto K_j, \qquad E_j \mapsto E_j, \qquad F_j \mapsto F_j, \qquad j \in \mathbb{I}_\bullet.$$

Since the images generate $\mathbf{U}^\imath_{\varsigma^e}$ (see (2.19) for the definition of $\boldsymbol{\varsigma}^e$), π_ς restricts to a surjective homomorphism $\widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_{\varsigma^e}$, and we denote it by $\pi^\imath_{\varsigma^e}$. Moreover, we have $\ker \pi^\imath_{\varsigma^e} = \ker \pi_\varsigma \cap \widetilde{\mathbf{U}}^\imath$. Since

$\ker \pi_{\varsigma}$ is generated by elements $K_i K_i' - \varsigma_i$ for $i \in \mathbb{I}$, we conclude that $\ker \pi^{\imath}_{\varsigma^e}$ is generated by

$$\widetilde{k}_i - \varsigma_i \quad (i = \tau i, i \in \mathbb{I}_\circ), \qquad \widetilde{k}_i \widetilde{k}_{\tau i} - \varsigma_i \varsigma_{\tau i} \quad (i \neq \tau i, i \in \mathbb{I}_\circ), \quad K_j K_j' - 1 \quad (j \in \mathbb{I}_\bullet).$$

The composition with the isomorphism $\phi_{\varsigma} \phi_{\varsigma^e}^{-1}$ from (2.22), $\pi^{\imath}_{\varsigma} := \phi_{\varsigma} \phi_{\varsigma^e}^{-1} \circ \pi^{\imath}_{\varsigma^e}$, defines a surjective homomorphism $\widetilde{\mathbf{U}}^{\imath} \to \mathbf{U}^{\imath}_{\varsigma}$. Finally, it is clear from Proposition 2.7 that $\ker \pi^{\imath}_{\varsigma}$ is generated by the desired elements. □

Remark 2.9. For a balanced parameter $\boldsymbol{\varsigma}$, $\pi^{\imath}_{\varsigma}$ coincides with the restriction of π_{ς} on $\widetilde{\mathbf{U}}^{\imath}$. However, this is not the case for an unbalanced parameter.

3 | QUASI *K*-MATRIX AND INTERTWINING PROPERTIES

In this section, we establish the quasi K-matrix $\widetilde{\Upsilon}$ for the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^{\imath})$, and a new characterization of $\widetilde{\Upsilon}$ in terms of an anti-involution σ. Then using suitable intertwining properties with the quasi K-matrix, we establish an anti-involution $\sigma^{\imath}$ and a bar involution $\psi^{\imath}$ on $\widetilde{\mathbf{U}}^{\imath}$ from the anti-involution σ and a rescaled bar involution $\psi_{\star}$ on $\widetilde{\mathbf{U}}$. We also establish an anti-involution σ_{τ} on $\mathbf{U}^{\imath}_{\varsigma}$ for an arbitrary parameter $\boldsymbol{\varsigma}$.

3.1 | Quasi *K*-matrix

The quasi K-matrix was introduced in [7, §2.3] as the intertwiner between the embedding $\imath : \mathbf{U}^{\imath}_{\varsigma} \to \mathbf{U}$ and its bar-conjugated embedding (where some constraints on $\boldsymbol{\varsigma}$ are imposed); this was expected to be valid for general quantum symmetric pairs early on. A proof for the existence of the quasi K-matrix was given in [5] in greater generality (modulo a technical assumption, which was later removed in [9]). Appel-Vlaar [2, Theorem 7.4] reformulated the definition of quasi K-matrix Υ_{ς} associated to $(\mathbf{U}, \mathbf{U}^{\imath}_{\varsigma})$ without reference to the bar involution on $\mathbf{U}^{\imath}_{\varsigma}$; this somewhat technical (see (3.1)) reformulation removes constraints on the parameter $\boldsymbol{\varsigma}$ for quasi K-matrix. Recall the bar involution ψ on $\mathbf{U}$.

Theorem 3.1 (cf. [2]). *There exists a unique element* $\Upsilon_{\varsigma} = \sum_{\mu \in \mathbb{N}\mathbb{I}} \Upsilon^{\mu}_{\varsigma}$, *for* $\Upsilon^{\mu}_{\varsigma} \in \mathbf{U}^{+}_{\mu}$, *such that* $\Upsilon^{0}_{\varsigma} = 1$ *and the following identities hold:*

$$B_i \Upsilon_{\varsigma} = \Upsilon_{\varsigma} \Big(F_i + (-1)^{\alpha_i(2\rho^{\vee}_{\bullet})} q^{(\alpha_i, w_{\bullet}(\alpha_{\tau i}) + 2\rho_{\bullet})} \varsigma_{\tau i} \psi \Big(T_{w_{\bullet}} E_{\tau i} \Big) K_i \Big), \tag{3.1}$$

$$x \Upsilon_{\varsigma} = \Upsilon_{\varsigma} x, \tag{3.2}$$

for $i \in \mathbb{I}_\circ$ *and* $x \in \mathbf{U}^{\imath 0} \mathbf{U}_{\bullet}$. *Moreover,* $\widetilde{\Upsilon}^{\mu} = 0$ *unless* $\theta(\mu) = -\mu$.

Recall the bar involution ψ on $\widetilde{\mathbf{U}}$ from Proposition 2.2. The quasi K-matrix $\widetilde{\Upsilon}$ associated to $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^{\imath})$ is defined in a similar way as in Theorem 3.1.

Theorem 3.2. *There exists a unique element* $\widetilde{\Upsilon} = \sum_{\mu\in\mathbb{N}\mathbb{I}} \widetilde{\Upsilon}^\mu$ *such that* $\widetilde{\Upsilon}^0 = 1, \widetilde{\Upsilon}^\mu \in \widetilde{\mathbf{U}}^+_\mu$ *and the following identities hold:*

$$B_i\widetilde{\Upsilon} = \widetilde{\Upsilon}\Big(F_i + (-1)^{\alpha_i(2\rho^\vee_\bullet)} q^{(\alpha_i, w_\bullet\alpha_{\tau i}+2\rho_\bullet)}\psi\Big(\widetilde{T}_{w_\bullet}E_{\tau i}\Big)K_i\Big), \tag{3.3}$$

$$x\widetilde{\Upsilon} = \widetilde{\Upsilon}x, \tag{3.4}$$

for $i \in \mathbb{I}_\circ$ *and* $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. *Moreover,* $\widetilde{\Upsilon}^\mu = 0$ *unless* $\theta(\mu) = -\mu$.

Proof. Follows by a rerun of the proof of Theorem 3.1 as in [2] or in [19]. (The strategy of the proof does not differ substantially from the one given in [7].) □

Remark 3.3. Applying the central reduction π_ς in (2.6) to (3.3) gives us

$$\begin{aligned}&\Big(F_i + \sqrt{\varsigma_i\varsigma_{\tau i}}T_{w_\bullet}(E_{\tau i})K_i^{-1}\Big)\pi_\varsigma(\widetilde{\Upsilon})\\ &= \pi_\varsigma(\widetilde{\Upsilon})\Big(F_i + (-1)^{\alpha_i(2\rho^\vee_\bullet)} q^{(\alpha_i, w_\bullet(\alpha_{\tau i})+2\rho_\bullet)}\sqrt{\varsigma_i\varsigma_{\tau i}}\psi\Big(T_{w_\bullet}(E_{\tau i})\Big)K_i\Big),\end{aligned} \tag{3.5}$$

$$x\pi_\varsigma(\widetilde{\Upsilon}) = \pi_\varsigma(\widetilde{\Upsilon})x, \tag{3.6}$$

for $i \in \mathbb{I}_\circ, x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$. Comparing (3.5) with (3.1), we obtain by the uniqueness of the quasi K-matrix that (see (2.19) for $\boldsymbol{\varsigma}^e$)

$$\pi_\varsigma(\widetilde{\Upsilon}) = \Upsilon_{\varsigma^e}. \tag{3.7}$$

In particular, $\pi_\varsigma(\widetilde{\Upsilon}) = \Upsilon_\varsigma$ if and only if $\boldsymbol{\varsigma}$ is a balanced parameter.

3.2 | A bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$

Introduce a balanced parameter $\boldsymbol{\varsigma}_\star = (\varsigma_{i,\star})_{i\in\mathbb{I}_\circ}$ by letting

$$\varsigma_{i,\star} = (-1)^{\alpha_i(2\rho^\vee_\bullet)} q^{(\alpha_i, w_\bullet\alpha_{\tau i}+2\rho_\bullet)}, \qquad (i \in \mathbb{I}_\circ). \tag{3.8}$$

Note that $\varsigma_{i,\star}$ are exactly the scalars appearing on the RHS (3.3). We extend $\boldsymbol{\varsigma}_\star$ trivially to an $\mathbb{I}$-tuple, again denoted by $\boldsymbol{\varsigma}_\star$ by abuse of notation, by setting

$$\varsigma_{j,\star} = 1 \qquad (j \in \mathbb{I}_\bullet).$$

Recall the scaling automorphism $\widetilde{\Psi}_{\varsigma_\star}$ from (2.7) and the bar involution ψ on $\widetilde{\mathbf{U}}$ from Proposition 2.2. The composition

$$\psi_\star := \widetilde{\Psi}_{\varsigma_\star} \circ \psi \tag{3.9}$$

is an antilinear involutive automorphism of $\widetilde{\mathbf{U}}$.

Let Ad_y be the operator such that $\mathrm{Ad}_y(u) := yuy^{-1}$ for y invertible.

Proposition 3.4. *There exists a unique antilinear involution $\psi^\imath$ of $\widetilde{\mathbf{U}}^\imath$ such that*

$$\psi^\imath(B_i) = B_i, \qquad \psi^\imath(x) = \psi_\star(x), \qquad \textit{for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet. \tag{3.10}$$

Moreover, $\psi^\imath$ satisfies the following intertwining relation:

$$\psi^\imath(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\psi_\star(x), \qquad \textit{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{3.11}$$

($\psi^\imath$ is called a bar involution on $\widetilde{\mathbf{U}}^\imath$.)

Proof. We follow the same strategy in [19] who established a bar involution on $\mathbf{U}^\imath_\varsigma$ (for suitable ς) without using a Serre presentation.

By definition of $\psi_\star$, we have, for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$,

$$\begin{aligned}\psi_\star(B_i) &= F_i + (-1)^{\alpha_i(2\rho^\vee_\bullet)} q^{(\alpha_i, w_\bullet \alpha_{\tau i} + 2\rho_\bullet)} \psi\Big(\widetilde{T}_{w_\bullet} E_{\tau i}\Big) K_i,\\ \psi_\star(x) &\in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet.\end{aligned} \tag{3.12}$$

The composition $\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$ is an antilinear homomorphism from $\widetilde{\mathbf{U}}$ to a completion of $\widetilde{\mathbf{U}}$. Then, the image of $\widetilde{\mathbf{U}}^\imath$ under $\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$ is a subalgebra generated by

$$(\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(B_i), \quad (\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(x), \qquad \text{for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet.$$

By Theorem 3.2 and the identities (3.12), we have, for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$,

$$(\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(B_i) = B_i, \quad (\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star)(x) = \psi_\star(x). \tag{3.13}$$

Since each element in (3.13) lies in $\widetilde{\mathbf{U}}^\imath$, $\mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$ restricts to an antilinear endomorphism on $\widetilde{\mathbf{U}}^\imath$, which we shall denote by $\psi^\imath : \widetilde{\mathbf{U}}^\imath \to \widetilde{\mathbf{U}}^\imath$.

By construction, $\psi^\imath$ satisfies (3.10)–(3.11). Finally, $\psi^\imath$ is unique and is an involutive automorphism of $\widetilde{\mathbf{U}}^\imath$ since it satisfies (3.10). □

Proposition 3.5. *We have*

$$\psi_\star(\widetilde{\Upsilon})\widetilde{\Upsilon} = 1. \tag{3.14}$$

Proof. Applying $\psi_\star$ to (3.11) results the identity $\psi_\star(y)\psi_\star(\widetilde{\Upsilon}) = \psi_\star(\widetilde{\Upsilon})\psi^\imath(y)$, for $y \in \widetilde{\mathbf{U}}^\imath$. We rewrite this identity as

$$\psi^\imath(y)\psi_\star(\widetilde{\Upsilon})^{-1} = \psi_\star(\widetilde{\Upsilon})^{-1}\psi_\star(y). \tag{3.15}$$

Using (3.12) and Proposition 3.4, the above identity (3.15) implies following relations:

$$\begin{aligned}B_i\psi_\star(\widetilde{\Upsilon})^{-1} &= \psi_\star(\widetilde{\Upsilon})^{-1}\Big(F_i + (-1)^{\alpha_i(2\rho^\vee_\bullet)} q^{(\alpha_i, w_\bullet \alpha_{\tau i} + 2\rho_\bullet)} \psi\Big(\widetilde{T}_{w_\bullet} E_{\tau i}\Big) K_i\Big),\\ x\psi_\star(\widetilde{\Upsilon})^{-1} &= \psi_\star(\widetilde{\Upsilon})^{-1} x,\end{aligned} \tag{3.16}$$

for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. Hence, $\psi_\star(\widetilde{\Upsilon})^{-1}$ satisfies (3.3)–(3.4) as well. Clearly, $\psi_\star(\widetilde{\Upsilon})^{-1}$ has constant term 1. Thanks to the uniqueness of $\widetilde{\Upsilon}$ in Theorem 3.2, we have $\psi_\star(\widetilde{\Upsilon})^{-1} = \widetilde{\Upsilon}$. □

3.3 | Quasi K-matrix and anti-involution σ

We provide a new characterization for $\widetilde{\Upsilon}$ in terms of the anti-involution σ (see Proposition 2.2), which turns out to be much cleaner than Theorem 3.2. Denote

$$B_i^\sigma = \sigma(B_i) = F_i + K_i \widetilde{T}_{w_\bullet}^{-1}(E_{\tau i}), \tag{3.17}$$

where the second identity above follows by noting $\widetilde{T}_{w_\bullet}^{-1} = \sigma \widetilde{T}_{w_\bullet} \sigma$; see (2.10). The following characterization of a quasi K-matrix $\widetilde{\Upsilon}$ is valid for $\widetilde{\mathbf{U}}^\imath$ of arbitrary Kac–Moody type.

Theorem 3.6. *There exists a unique element* $\widetilde{\Upsilon} = \sum_{\mu \in \mathbb{N}\mathbb{I}} \widetilde{\Upsilon}^\mu$ *such that* $\widetilde{\Upsilon}^0 = 1, \widetilde{\Upsilon}^\mu \in \widetilde{\mathbf{U}}^+_\mu$ *and the following intertwining relations hold:*

$$\begin{aligned} B_i \widetilde{\Upsilon} &= \widetilde{\Upsilon} B_i^\sigma, \qquad (i \in \mathbb{I}_\circ), \\ x \widetilde{\Upsilon} &= \widetilde{\Upsilon} x, \qquad (x \in \widetilde{\mathbf{U}}^{\imath 0} \widetilde{\mathbf{U}}_\bullet). \end{aligned} \tag{3.18}$$

Moreover, $\widetilde{\Upsilon}^\mu = 0$ *unless* $\theta(\mu) = -\mu$.

Proof. We show that the identity (3.18) is equivalent to (3.3), for any fixed $i \in \mathbb{I}_\circ$. Since $\psi(\widetilde{T}_{w_\bullet}(E_{\tau i}))$ has weight $w_\bullet \alpha_{\tau i}$, the identity (3.3) is equivalent to

$$B_i \widetilde{\Upsilon} = \widetilde{\Upsilon}\Big(F_i + (-1)^{\alpha_i(2\rho_\bullet^\vee)} q^{(\alpha_i, 2\rho_\bullet)} K_i \psi\Big(\widetilde{T}_{w_\bullet}(E_{\tau i})\Big)\Big). \tag{3.19}$$

Moreover, by [8, Lemma 4.17] and $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+$, we have

$$(-1)^{\alpha_i(2\rho_\bullet^\vee)} q^{(\alpha_i, 2\rho_\bullet^\vee)} \psi\Big(\widetilde{T}_{w_\bullet}(E_{\tau i})\Big) = \widetilde{T}_{w_\bullet}^{-1}(E_{\tau i}),$$

and hence, the identity (3.19) is equivalent to (3.18) as desired. □

Remark 3.7. By abuse of notation, we denote again by σ the anti-involution on $\mathbf{U}$ that fixes E_i, F_i and sends $K_i \mapsto K_i^{-1}$ for $i \in \mathbb{I}$. For a balanced parameter $\boldsymbol{\varsigma}$, we obtain the intertwining relation for $\mathbf{U}^\imath_\varsigma$, $B_i \Upsilon_\varsigma = \Upsilon_\varsigma B_i^\sigma$ $(i \in \mathbb{I}_\circ)$, by applying the central reduction π_ς to (3.18), thanks to (3.7). Here, $B_i^\sigma = \sigma(B_i) = F_i + \varsigma_i K_i T_{w_\bullet}^{-1}(E_{\tau i})$.

On the other hand, for (not necessarily balanced) parameter $\boldsymbol{\varsigma}$, we have

$$B_i \Upsilon_\varsigma = \Upsilon_\varsigma B_{\tau i}^{\sigma\tau}. \tag{3.20}$$

Note that the involution τ induces an involution $\widehat{\tau} \in \mathrm{Aut}(\widetilde{\mathbf{U}})$ that preserves $\widetilde{\mathbf{U}}^\imath$. For $i \in \mathbb{I}_\circ$, the rank 1 quasi K-matrix

$$\widetilde{\Upsilon}_i \in \widetilde{\mathbf{U}}^+_{\mathbb{I}_\bullet, i} (\subset \widetilde{\mathbf{U}}^+)$$

is defined to be the quasi K-matrix associated to the rank 1 Satake subdiagram $(\mathbb{I}_\bullet \cup \{i, \tau i\}, \tau)$; cf. (2.13). Clearly, we have $\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_{\tau i}$.

Proposition 3.8. *We have* $\sigma(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$ *and* $\widehat{\tau}(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$*, In addition, for* $i \in \mathbb{I}_\circ$*, we have*

$$\sigma(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i, \qquad \widehat{\tau}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i.$$

In addition, $\widehat{\tau}_{\bullet, i}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$.

Proof. By applying the anti-involution σ to the identities in Theorem 3.6, we have

$$\sigma(\widetilde{\Upsilon})B_i^{\sigma} = B_i\sigma(\widetilde{\Upsilon}), \qquad (i \in \mathbb{I}_\circ), \tag{3.21}$$

$$\sigma(\widetilde{\Upsilon})y = y\sigma(\widetilde{\Upsilon}), \qquad (x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet), \tag{3.22}$$

where $y = \sigma(x) \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. This means that $\sigma(\widetilde{\Upsilon})$ satisfies the same characterization in Theorem 3.6 as $\widetilde{\Upsilon}$, and hence by uniqueness, we have $\sigma(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$.

Noting that $\sigma\widehat{\tau} = \widehat{\tau}\sigma$ and $\widehat{\tau}$ preserves $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, then the identity $\widehat{\tau}(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$ follows by the same type argument as above.

The identities $\sigma(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ and $\widehat{\tau}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ are immediate by restricting σ and $\widehat{\tau}$ to the Drinfeld double associated to rank 1 Satake subdiagram $(\mathbb{I}_{\bullet,i}, \mathbb{I}_\bullet, \tau|_{\mathbb{I}_{\bullet,i}})$.

According to the rank 1 Table 1, $\tau_{\bullet,i} = 1$ except in type AIV when $\tau_{\bullet,i}$ coincides with the restriction of τ to the rank 1 Satake diagram. In either case, we have $\widehat{\tau}_{\bullet,i}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$. □

Remark 3.9. For balanced parameters $\boldsymbol{\varsigma}$, by taking a central reduction π_{ς}, the property $\tau(\Upsilon_{i,\varsigma}) = \Upsilon_{i,\varsigma}$ remains valid. However, for unbalanced parameters $\boldsymbol{\varsigma}$, we do not necessarily have $\tau(\Upsilon_{i,\varsigma}) = \Upsilon_{i,\varsigma}$; instead, we have $\tau(\Upsilon_{i,\varsigma}) = \Upsilon_{i,\tau\varsigma}$, which can be proved by Theorem 3.1. The property $\Upsilon_{i,\varsigma} = \Upsilon_{\tau i,\varsigma}$ is true, regardless of balanced or unbalanced parameters.

Remark 3.10. It follows by Theorem 3.2 that the rank 1 quasi K-matrix $\widetilde{\Upsilon}_i$ has the form $\widetilde{\Upsilon}_i = \sum_{m\geqslant 0}\widetilde{\Upsilon}_{i,m}$, for $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}_{m(\alpha_i+w_\bullet\alpha_{\tau i})}$.

3.4 | An anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$

Define $\mathcal{K}_i \in \widetilde{\mathbf{U}}^\imath$ by

$$\mathcal{K}_i = K_i K'_{w_\bullet\alpha_{\tau i}}, \qquad \text{for } i \in \mathbb{I}_\circ. \tag{3.23}$$

Lemma 3.11. *Let $i \in \mathbb{I}_\circ$. We have $\mathcal{K}_i \in \widetilde{\mathbf{U}}^{\imath 0}$.*

Proof. By definition, the element $\mathcal{K}_i$ is a product of $\widetilde{k}_i = K_i K'_{\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}$ and an element in $\widetilde{\mathbf{U}}^0_\bullet$, and hence, $\mathcal{K}_i \in \widetilde{\mathbf{U}}^{\imath 0}$. □

Recall the anti-involution σ on $\widetilde{\mathbf{U}}$ from Proposition 2.2.

Proposition 3.12. *There exists a unique anti-involution $\sigma^\imath$ of $\widetilde{\mathbf{U}}^\imath$ such that*

$$\sigma^\imath(B_i) = B_i, \qquad \sigma^\imath(x) = \sigma(x), \qquad \textit{for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet. \tag{3.24}$$

Moreover, $\sigma^\imath$ satisfies the following intertwining relation:

$$\sigma^\imath(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x), \qquad \textit{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{3.25}$$

Proof. Given $x \in \widetilde{\mathbf{U}}^\imath$, an element $\widehat{x} \in \widetilde{\mathbf{U}}^\imath$ (if it exists) such that $\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$ must be unique due to the invertibility of $\widetilde{\Upsilon}$.

Claim (∗). Suppose that there exist $\widehat{x}, \widehat{y} \in \widetilde{\mathbf{U}}^\imath$ that $\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$ and $\widehat{y}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(y)$, for given $x, y \in \widetilde{\mathbf{U}}^\imath$. Then we have

$$\widehat{y}\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(xy).$$

Indeed, the Claim holds since $\widehat{y}\widehat{x}\widetilde{\Upsilon} = \widehat{y}\widetilde{\Upsilon}\sigma(x) = \widetilde{\Upsilon}\sigma(y)\sigma(x) = \widetilde{\Upsilon}\sigma(xy)$.

Observe that σ preserves the subalgebra $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ of $\widetilde{\mathbf{U}}^\imath$. Hence, by Theorem 3.6, we have $\sigma(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$, for all $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. By Theorem 3.6 again, we have $B_i\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(B_i)$, for all $i \in \mathbb{I}_\circ$. Since the assumption for Claim (∗) holds for a generating set $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet \cup \{B_i | i \in \mathbb{I}_\circ\}$ of $\widetilde{\mathbf{U}}^\imath$, we conclude by Claim (∗) that there exists a (unique) $\widehat{x} \in \widetilde{\mathbf{U}}^\imath$ such that $\widehat{x}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma(x)$, for any $x \in \widetilde{\mathbf{U}}^\imath$, and moreover, sending $x \mapsto \widehat{x}$ defines an antiendmorphism of $\widetilde{\mathbf{U}}^\imath$ (which will be denoted by $\sigma^\imath$).

Clearly, by construction, $\sigma^\imath$ satisfies (3.24) and the identity (3.25). Finally, $\sigma^\imath$ is an involutive antiautomorphism of $\widetilde{\mathbf{U}}^\imath$ since it satisfies (3.24). □

Remark 3.13. The strategy in establishing a bar involution on $\mathbf{U}^\imath_\varsigma$ without use of a Serre presentations appeared first in [19]. For quasi-split $\imath$quantum groups, that is, $\mathbb{I}_\bullet = \emptyset$, our $\psi^\imath$ coincides with the bar involution in [12, Lemma 2.4(a)] (see also [31, Lemma 6.9]). Unlike the proof *loc. cit.*, our proofs of Propositions 3.4 and 3.12 do not use a Serre presentation of $\widetilde{\mathbf{U}}^\imath$. Hence, the (anti-) involutions $\sigma^\imath$ and $\psi^\imath$ are valid for $\widetilde{\mathbf{U}}^\imath$ of arbitrary Kac–Moody type.

3.5 | An anti-involution σ_τ on $\mathbf{U}^\imath_\varsigma$

The anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ in Proposition 3.12 can descend to an $\imath$quantum group $\mathbf{U}^\imath_\varsigma$, only for any *balanced* parameter $\boldsymbol{\varsigma}$. It turns out that the anti-involution $\sigma^\imath\tau$ on $\widetilde{\mathbf{U}}^\imath$ can descend to an $\imath$quantum group $\mathbf{U}^\imath_\varsigma$, for an *arbitrary* parameter $\boldsymbol{\varsigma}$.

Proposition 3.14. *Let $\boldsymbol{\varsigma}$ be an arbitrary parameter. There exists a unique anti-involution σ_τ of $\mathbf{U}^\imath_\varsigma$ such that*

$$\sigma_\tau(B_i) = B_{\tau i}, \qquad \sigma_\tau(x) = \sigma\tau(x), \quad \textit{for } i \in \mathbb{I}_\circ, x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet. \tag{3.26}$$

Moreover, σ_τ satisfies the following intertwining relation:

$$\sigma_\tau(x)\Upsilon_\varsigma = \Upsilon_\varsigma\sigma\tau(x), \qquad \textit{for all } x \in \mathbf{U}^\imath_\varsigma. \tag{3.27}$$

Proof. A proof similar to the one for Proposition 3.12 works here, and we outline it.

We claim that, for any $x \in \mathbf{U}^\imath_\varsigma$, there exists $\widehat{x} \in \mathbf{U}^\imath_\varsigma$ such that

$$\widehat{x}\Upsilon_\varsigma = \Upsilon_\varsigma\sigma\tau(x). \tag{3.28}$$

As argued in the proof of Proposition 3.12, it suffices to show that (3.28) holds for x in a generating set $\{B_i | i \in \mathbb{I}_\circ\} \cup \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$ of $\mathbf{U}^\imath_\varsigma$. Indeed, by (3.20), we have $B_{\tau i}\Upsilon_\varsigma = \Upsilon_\varsigma\sigma\tau(B_i)$. For $x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$, note that $\sigma\tau(x) \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet$, and then by Theorem 3.1, we have $\sigma\tau(x)\Upsilon_\varsigma = \Upsilon_\varsigma\sigma\tau(x)$. This proves (3.28).

Now sending $x \mapsto \widehat{x}$ defines an antiendomorphism σ_τ, which satisfies (3.26) and (3.27) by construction above. Finally, σ_τ is involutive since it satisfies (3.26). □

Remark 3.15. Our construction of σ_τ generalizes the $\sigma_\imath$ in [9, Proposition 3.13], which is constructed via bar involutions under certain restrictions on parameters.

Thanks to Proposition 3.14, we have a conceptual formulation of the quasi K-matrix Υ_ς for $\mathbf{U}^\imath_\varsigma$ below, which is a variant of Theorem 3.6; compare Theorem 3.1 (see [2]). This new formulation can also be proved directly.

Theorem 3.16. *Let $\boldsymbol{\varsigma}$ be an arbitrary parameter. There exists a unique element $\Upsilon_\varsigma = \sum_{\mu\in\mathbb{N}\mathbb{I}} \Upsilon^\mu$ such that $\Upsilon^0 = 1, \Upsilon^\mu \in \widetilde{\mathbf{U}}^+_\mu$ and the following intertwining relations hold:*

$$B_{\tau i}\widetilde{\Upsilon} = \widetilde{\Upsilon}\sigma\tau(B_i), \qquad (i \in \mathbb{I}_\circ),$$

$$x\widetilde{\Upsilon} = \widetilde{\Upsilon}x, \qquad (x \in \mathbf{U}^{\imath 0}\mathbf{U}_\bullet).$$

4 | NEW SYMMETRIES $\widetilde{\mathbf{T}}'_{i,-1}$ ON $\widetilde{\mathbf{U}}^\imath$

In this section, we define explicitly certain rescaled braid group actions $\widetilde{\mathscr{T}}'_{j,-1}$ on a Drinfeld double $\widetilde{\mathbf{U}}$. We then formulate the new symmetries $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, via an intertwining property using the quasi K-matrix $\widetilde{\Upsilon}$ and a rescaled braid automorphism $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}$; the proof will be completed in the coming sections. We show that $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}$ on $\widetilde{\mathbf{U}}$ preserves the subalgebra $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, and that the actions of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ coincide. Explicit formulas for the action of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ are presented. Then, we obtain a compact close rank 1 formula for $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$.

4.1 | Rescaled braid group action on $\widetilde{\mathbf{U}}$

Recall the distinguished parameter $\boldsymbol{\varsigma}_\diamond$ from (2.21). Extend $\boldsymbol{\varsigma}_\diamond$ trivially to an $\mathbb{I}$-tuple of scalars $(\varsigma_{i,\diamond})_{i\in\mathbb{I}}$ by setting

$$\varsigma_{j,\diamond} = 1, \qquad \text{for } j \in \mathbb{I}_\bullet. \tag{4.1}$$

Then, we have the scaling automorphism $\widetilde{\Psi}_{\varsigma_\diamond}$ on $\widetilde{\mathbf{U}}$ by Proposition 2.1. We define symmetries $\widetilde{\mathscr{T}}''_{i,+1}$ and $\widetilde{\mathscr{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}$ by rescaling $\widetilde{T}''_{i,+1}$ and $\widetilde{T}'_{i,-1}$ in Proposition 2.3 and (2.10) via the rescaling automorphism $\widetilde{\Psi}_{\varsigma_\diamond}$:

$$\widetilde{\mathscr{T}}''_{i,+1} := \widetilde{\Psi}^{-1}_{\varsigma_\diamond} \circ \widetilde{T}''_{i,+1} \circ \widetilde{\Psi}_{\varsigma_\diamond}, \tag{4.2}$$

$$\widetilde{\mathscr{T}}'_{i,-1} := \widetilde{\Psi}^{-1}_{\varsigma_\diamond} \circ \widetilde{T}'_{i,-1} \circ \widetilde{\Psi}_{\varsigma_\diamond}. \tag{4.3}$$

Since $\widetilde{T}''_{i,+1}, \widetilde{T}'_{i,-1}$ are mutually inverses, $\widetilde{\mathscr{T}}''_{i,+1}, \widetilde{\mathscr{T}}'_{i,-1}$ are also mutually inverses. We shall often use the shorthand notation

$$\widetilde{\mathscr{T}}_i = \widetilde{\mathscr{T}}''_{i,+1}, \qquad \widetilde{\mathscr{T}}_i^{-1} = \widetilde{\mathscr{T}}'_{i,-1}. \tag{4.4}$$

Remark 4.1. These rescaled symmetries $\widetilde{\mathscr{T}}_i^{-1}$ will play a central role in our construction of symmetries on $\widetilde{\mathbf{U}}^\imath$; see Theorem 4.7. Our rescaling twist using $\widetilde{\Psi}_{\varsigma_\diamond}$ is compatible with the rescaling twist in [15, (3.45), Remark 3.16].

We write down the explicit actions for $\widetilde{\mathscr{T}}_i$ and $\widetilde{\mathscr{T}}_i^{-1}$ for later use.

Proposition 4.2. *Set $r = -c_{ij}$, for $i, j \in \mathbb{I}$. The automorphism $\widetilde{\mathscr{T}}_i \in \mathrm{Aut}(\widetilde{\mathbf{U}})$ defined in (4.2) is given by*

$$\widetilde{\mathscr{T}}_i(K_j) = \varsigma_{i,\diamond}^{c_{ij}/2} K_j K_i^{-c_{ij}}, \qquad \widetilde{\mathscr{T}}_i(K_j') = \varsigma_{i,\diamond}^{c_{ij}/2} K_j' K_i'^{-c_{ij}},$$

$$\widetilde{\mathscr{T}}_i(E_i) = -\varsigma_{i,\diamond} F_i K_i'^{-1}, \qquad \widetilde{\mathscr{T}}_i(F_i) = -K_i^{-1} E_i,$$

$$\widetilde{\mathscr{T}}_i(E_j) = \varsigma_{i,\diamond}^{-r/2} \sum_{s=0}^{r} (-1)^s q_i^{-s} E_i^{(r-s)} E_j E_i^{(s)}, \qquad j \neq i,$$

$$\widetilde{\mathscr{T}}_i(F_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} F_i^{(s)} F_j F_i^{(r-s)}, \qquad j \neq i.$$

The inverse of $\widetilde{\mathscr{T}}_i$ (see (4.3)) is given by

$$\widetilde{\mathscr{T}}_i^{-1}(K_j) = \varsigma_{i,\diamond}^{c_{ij}/2} K_j K_i^{-c_{ij}}, \qquad \widetilde{\mathscr{T}}_i^{-1}(K_j') = \varsigma_{i,\diamond}^{c_{ij}/2} K_j' K_i'^{-c_{ij}},$$

$$\widetilde{\mathscr{T}}_i^{-1}(E_i) = -\varsigma_{i,\diamond} K_i^{-1} F_i, \qquad \widetilde{\mathscr{T}}_i^{-1}(F_i) = -E_i K_i'^{-1},$$

$$\widetilde{\mathscr{T}}_i^{-1}(E_j) = \varsigma_{i,\diamond}^{-r/2} \sum_{s=0}^{r} (-1)^s q_i^{-s} E_i^{(s)} E_j E_i^{(r-s)}, \qquad j \neq i,$$

$$\widetilde{\mathscr{T}}_i^{-1}(F_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} F_i^{(r-s)} F_j F_i^{(s)}, \qquad j \neq i.$$

Moreover, $\widetilde{\mathscr{T}}_i$, for $i \in \mathbb{I}$, satisfy the braid group relations.

Hence, we obtain

$$\widetilde{\mathscr{T}}_w = \widetilde{\mathscr{T}}''_{w,+1} := \widetilde{\mathscr{T}}_{i_1} \cdots \widetilde{\mathscr{T}}_{i_r} \in \mathrm{Aut}(\widetilde{\mathbf{U}}), \quad \text{for } w \in W, \tag{4.5}$$

where $w = s_{i_1} \cdots s_{i_r}$ is any reduced expression. Similarly, we have $\widetilde{\mathscr{T}}'_{w,-1} \in \mathrm{Aut}(\widetilde{\mathbf{U}})$.

Remark 4.3. Let $i \in \mathbb{I}_\bullet$. The rescaling for $\widetilde{\mathscr{T}}_i^{\pm 1}$ is trivial, thanks to $\varsigma_{\diamond,i} = 1$; that is, $\widetilde{\mathscr{T}}_i = \widetilde{T}_i$. In particular, $\widetilde{\mathscr{T}}_{w_\bullet} = \widetilde{T}_{w_\bullet}$. Moreover, $\widetilde{T}_{w_\bullet}(E_{\tau i}) = \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau i}) = T_{w_\bullet}(E_{\tau i})$ in $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+$; cf. the formula for B_i in (2.17).

Let τ_0 be the diagram automorphism associated to the longest element w_0 of the Weyl group W. The following fact is well known (up to the rescaling via $\boldsymbol{\varsigma}_\diamond$); cf. for example, [18, Lemma 3.4].

Lemma 4.4. *We have, for* $j \in \mathbb{I}$,

$$\widetilde{\mathscr{T}}_{w_0}(F_j) = -K_{\tau_0 j}^{-1} E_{\tau_0 j}, \qquad \widetilde{\mathscr{T}}_{w_0}(E_j) = -\varsigma_{j,\diamond} F_{\tau_0 j} K_{\tau_0 j}^{\prime -1},$$

$$\widetilde{\mathscr{T}}_{w_0}^{-1}(E_j) = -\varsigma_{j,\diamond} K_{\tau_0 j}^{-1} F_{\tau_0 j}, \qquad \widetilde{\mathscr{T}}_{w_0}^{-1}(F_j) = -E_{\tau_0 j} K_{\tau_0 j}^{\prime -1}.$$

4.2 | Symmetries $\widetilde{\mathscr{T}}''_{j,+1}$, for $j \in \mathbb{I}_\bullet$

It is known [8] that Lusztig's operators $T'_{j,\pm 1}, T''_{j,\pm 1}$ on $\mathbf{U}$, for $j \in \mathbb{I}_\bullet$, restrict to automorphisms of $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ (where the $\boldsymbol{\varsigma}$ satisfies certain constraints); moreover, these operators fix Υ. In this subsection, we formulate analogous statements for the universal quantum symmetric pair $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ while skipping the identical proofs.

Recall the automorphisms $\widetilde{T}''_{i,+1}$ on the Drinfeld double $\widetilde{\mathbf{U}}$, for $i \in \mathbb{I}$, from Proposition 2.3, and recall Remark 4.3.

Proposition 4.5 (cf. [8, Theorem 4.2]). *Let* $j \in \mathbb{I}_\bullet$. *The automorphism* $\widetilde{\mathscr{T}}''_{j,+1} = \widetilde{T}''_{j,+1}$ *on* $\widetilde{\mathbf{U}}$ *restricts to an automorphism of* $\widetilde{\mathbf{U}}^\imath$. *Moreover, the action of* $\widetilde{\mathscr{T}}''_{j,+1}$ *on* B_i $(i \in \mathbb{I}_\circ)$ *is given by*

$$\widetilde{\mathscr{T}}''_{j,+1}(B_i) = \sum_{s=0}^{r} (-1)^s q_j^s F_j^{(s)} B_i F_j^{(r-s)}, \qquad \textit{for } r = -c_{ij}. \tag{4.6}$$

Proposition 4.6 (cf. [8, Proposition 4.13]). *Let* $j \in \mathbb{I}_\bullet$. *Then* $\widetilde{\mathscr{T}}''_{j,+1}(\widetilde{\Upsilon}) = \widetilde{\Upsilon}$, *and* $\widetilde{\mathscr{T}}''_{j,+1}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$, *for* $i \in \mathbb{I}_\circ$.

4.3 | Characterization of $\widetilde{\mathbf{T}}'_{i,-1}$

Let $(\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}^\imath)$ be the quantum symmetric pair associated to an arbitrary Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$. Recall that $\widetilde{\Upsilon}_i$, for $i \in \mathbb{I}_\circ$, are the quasi K-matrix associated to the rank 1 Satake subdiagram $(\mathbb{I}_\bullet \cup \{i, \tau i\}, \tau|_{\mathbb{I}_\bullet \cup \{i,\tau i\}})$. Recall $\mathbf{r}_i \in W$ from (2.14) and $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1} \in \mathrm{Aut}(\widetilde{\mathbf{U}})$ from (4.5) whose definition uses (4.2). We now formulate our first main result.

Theorem 4.7. *Let* $i \in \mathbb{I}_\circ$.

(1) *For any* $x \in \widetilde{\mathbf{U}}^\imath$, *there is a unique element* $\hat{x} \in \widetilde{\mathbf{U}}^\imath$ *such that* $\hat{x}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(x^\imath)$.
(2) *The map* $x \mapsto \hat{x}$ *is an automorphism of the algebra* $\widetilde{\mathbf{U}}^\imath$, *denoted by* $\widetilde{\mathbf{T}}'_{i,-1}$.

Therefore, we have

$$\widetilde{\mathbf{T}}'_{i,-1}(x)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(x^\imath), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{4.7}$$

Proof. A complete proof of this theorem requires the developments in the coming Sections 4–6.1. Let us outline the main steps below.

For a given $x \in \widetilde{\mathbf{U}}^\imath$, the element $\hat{x} \in \widetilde{\mathbf{U}}^\imath$ satisfying the identity in (1) is clearly unique (if it exists) since $\widetilde{\Upsilon}_i$ is invertible.

The explicit formulas of $\hat{x}$ associated to generators x of $\widetilde{\mathbf{U}}^\imath$, for each of (ranks 1 and 2) Satake diagrams, are given in the forthcoming Sections 4–5. The formulas therein show manifestly that $\hat{x} \in \widetilde{\mathbf{U}}^\imath$; see Proposition 4.11 on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, Theorem 4.14 for rank 1, and Theorem 5.5 for rank 2.

Assume that $\hat{x}, \hat{y} \in \widetilde{\mathbf{U}}^\imath$ satisfy (1), for $x, y \in \widetilde{\mathbf{U}}^\imath$; that is, $\hat{x}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(x^\imath)$, and $y'\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(y^\imath)$. Then it follows readily that $\hat{x}\hat{y} \in \widetilde{\mathbf{U}}^\imath$ satisfies the identity in (1) for xy; that is, $\hat{x}\hat{y}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}((xy)^\imath)$. Hence, we have obtained a well-defined endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ that sends $x \mapsto \hat{x}$.

To complete the proof of the theorem, it remains to show that $\widetilde{\mathbf{T}}'_{i,-1}$ is surjective. To this end, we introduce and study in depth a variant of $\widetilde{\mathbf{T}}'_{i,-1}$, a second endomorphism $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$ in Section 6.1. The bijectivity of $\widetilde{\mathbf{T}}'_{i,-1}$ follows by Theorem 6.7 that shows that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses. □

Remark 4.8. By Proposition 3.8 and the definition (2.14) of $\mathbf{r}_i$, we have $\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_{\tau i}$, $\mathbf{r}_i = \mathbf{r}_{\tau i}$, and hence $\widetilde{\mathbf{T}}'_{i,-1} = \widetilde{\mathbf{T}}'_{\tau i,-1}$. Thus, we may label $\widetilde{\mathbf{T}}'_{i,-1}$ by $\mathbb{I}_{\circ,\tau}$ instead of $\mathbb{I}_\circ$.

In this and later sections, we shall construct four variants of symmetries of $\widetilde{\mathbf{U}}^\imath$ (denoted by $\widetilde{\mathbf{T}}'_{i,e}$, $\widetilde{\mathbf{T}}''_{i,e}$) via (4.7) and three additional intertwining relations and the rescaled braid group symmetries $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,\pm1}$, $\widetilde{\mathscr{T}}''_{\mathbf{r}_i,\pm1}$ of $\widetilde{\mathbf{U}}^\imath$. We choose to start with the (simplest) intertwining relation (4.7) for $\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}$. From now on, following (4.4), we often write

$$\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1} = \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}, \qquad \widetilde{\mathscr{T}}_{\mathbf{r}_i} = \widetilde{\mathscr{T}}''_{\mathbf{r}_i,+1}.$$

4.4 | Quantum symmetric pairs of diagonal type

Recall from Proposition 2.2 the Chevalley involution ω and the comultiplication Δ (2.5) on $\widetilde{\mathbf{U}}$. Denote ${}^\omega\mathbf{L}''_i := (\omega \otimes 1)\mathbf{L}''_i$ for $i \in \mathbb{I}$, where $\mathbf{L}''_i, i \in \mathbb{I}$ is the rank 1 quasi R-matrix for $\widetilde{\mathbf{U}}$ (same as for $\mathbf{U}$); see [28]. We regard $\widetilde{\mathbf{U}}$ as a coideal subalgebra of $\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}$ via the embedding ${}^\omega\Delta := (\omega \otimes 1)\Delta$, and then, $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$ is a universal quantum symmetric pair of diagonal type; cf. [8, Remark 4.10]. In this way, the rank 1 quasi K-matrices for quantum symmetric pairs of diagonal type are given by ${}^\omega\mathbf{L}''_i$.

In this subsection, we shall reformulate the identity [28, 37.3.2] (= (1.2)) as an intertwining relation in the framework of quantum symmetric pairs of diagonal type.

Proposition 4.9. *For the quantum symmetric pair of diagonal type* $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$, *the following intertwining relation holds:*

$${}^\omega\Delta(\widetilde{T}'_{i,-1}u)\,{}^\omega\mathbf{L}''_i = {}^\omega\mathbf{L}''_i\,(\widetilde{\mathscr{T}}''_{i,-1} \otimes \widetilde{\mathscr{T}}'_{i,-1})\,{}^\omega\Delta(u), \qquad \forall u \in \widetilde{\mathbf{U}}. \tag{4.8}$$

Proof. Recall from [28, 37.2.4] that

$$\omega \circ \widetilde{T}'_{i,-1} \circ \omega = \widetilde{T}''_{i,-1}. \tag{4.9}$$

The identity (1.2) for $\mathbf{U}$ admits an identical version for $\widetilde{\mathbf{U}}$. Applying $\omega \otimes 1$ to this identity, we obtain

$${}^\omega\Delta(\widetilde{T}'_{i,-1}u)\,{}^\omega\mathbf{L}''_i = {}^\omega\mathbf{L}''_i\,(\widetilde{T}''_{i,-1} \otimes \widetilde{T}'_{i,-1})\,{}^\omega\Delta(u), \qquad \forall u \in \widetilde{\mathbf{U}}. \tag{4.10}$$

To prove (4.8), it suffices to prove the following identity:

$$(\widetilde{\mathscr{T}}''_{j,-1} \otimes \widetilde{\mathscr{T}}'_{j,-1})\,{}^{\omega}\Delta(u) = (\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})\,{}^{\omega}\Delta(u), \qquad \forall u \in \widetilde{\mathbf{U}}. \tag{4.11}$$

Clearly, it suffices to prove (4.11) when u is the generator of $\widetilde{\mathbf{U}}$. We have the following formulas:

$$\begin{aligned} {}^{\omega}\Delta(E_j) &= F_j \otimes 1 + K'_j \otimes E_j, \qquad & {}^{\omega}\Delta(F_j) &= 1 \otimes F_j + E_j \otimes K'_j, \\ {}^{\omega}\Delta(K_j) &= K'_j \otimes K_j, \qquad & {}^{\omega}\Delta(K'_j) &= K_j \otimes K'_j. \end{aligned} \tag{4.12}$$

Recall $\widetilde{\mathscr{T}}'_{j,-1} = \widetilde{\Psi}_{\varsigma_\diamond}^{-1} \widetilde{T}'_{j,-1} \widetilde{\Psi}_{\varsigma_\diamond}$ from (4.3). By Lemma 9.5 and noting that $\varsigma_{\star\diamond} = \varsigma_\diamond$ in our case, the twisting for $\widetilde{\mathscr{T}}''_{j,-1}$ is opposite to the one on $\widetilde{\mathscr{T}}'_{j,-1}$, that is, $\widetilde{\mathscr{T}}''_{j,-1} = \widetilde{\Psi}_{\varsigma_\diamond} \widetilde{T}''_{j,-1} \widetilde{\Psi}_{\varsigma_\diamond}^{-1}$. By Proposition 2.1, we see that the RHS of each formula in (4.12) is fixed by $\widetilde{\Psi}_{\varsigma_\diamond}^{-1} \otimes \widetilde{\Psi}_{\varsigma_\diamond}$. The formulas for $\widetilde{T}''_{i,+1}$ is given in Proposition 2.3, and the formulas for $\widetilde{T}'_{i,-1}, \widetilde{T}''_{i,-1}$ can be obtained from there by suitable twisting; using these formulas, we observe that $(\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})\,{}^{\omega}\Delta(u)$ is fixed by $\widetilde{\Psi}_{\varsigma_\diamond} \otimes \widetilde{\Psi}_{\varsigma_\diamond}^{-1}$ for $u = E_j, F_j, K_j, K'_j$. Hence, for $u = E_j, F_j, K_j, K'_j$, $j \in \mathbb{I}$,

$$\begin{aligned} (\widetilde{\mathscr{T}}''_{j,-1} \otimes \widetilde{\mathscr{T}}'_{j,-1})\,{}^{\omega}\Delta(u) &= (\widetilde{\Psi}_{\varsigma_\diamond} \otimes \widetilde{\Psi}_{\varsigma_\diamond}^{-1})(\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})(\widetilde{\Psi}_{\varsigma_\diamond}^{-1} \otimes \widetilde{\Psi}_{\varsigma_\diamond})^{\omega}\Delta(u) \\ &= (\widetilde{T}''_{j,-1} \otimes \widetilde{T}'_{j,-1})\,{}^{\omega}\Delta(u), \end{aligned}$$

which implies the desired identity (4.11). □

In this way, the intertwining relation (4.8) (reformulated from (4.10) via (4.9)) can be viewed as a variant of the intertwining relation (4.7) in the setting of quantum symmetric pair $(\widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}})$, where the coideal subalgebra is identified with the image of the embedding ${}^{\omega}\Delta : \widetilde{\mathbf{U}} \to \widetilde{\mathbf{U}} \otimes \widetilde{\mathbf{U}}$.

4.5 | Action of $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$

We formulate $\widetilde{\mathbf{T}}'_{i,-1}(x)$, for $i \in \mathbb{I}_\circ, x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ in this subsection. We will show that $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}$ preserves both $\widetilde{\mathbf{U}}^{\imath 0}$ and $\widetilde{\mathbf{U}}_\bullet$; hence, by Theorem 3.6, the element $\widetilde{\mathbf{T}}'_{i,-1}(x) := \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(x)$ satisfies (4.7) for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$.

Recall that the diagram involution associated to $w_{\bullet,i}$ is denoted by $\tau_{\bullet,i}$. By definition of admissible pairs, the diagram involution associated to $w_\bullet$ is $\tau|_{\mathbb{I}_\bullet}$. Both $\tau_{\bullet,i}$ and τ induce (commuting) involutive automorphisms, denoted by $\widehat{\tau}_{\bullet,i}$ and $\widehat{\tau}$, on $\widetilde{\mathbf{U}}_\bullet$.

We first calculated $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(x)$ for $x \in \widetilde{\mathbf{U}}_\bullet$. By applying Lemma 4.4 twice, we obtain

$$\widetilde{\mathscr{T}}_{w_\bullet}^{-1}\widehat{\tau}(x) = \widetilde{\mathscr{T}}_{w_{\bullet,i}}^{-1}\widehat{\tau}_{\bullet,i}(x) = \widetilde{\mathscr{T}}_{w_\bullet}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\widehat{\tau}_{\bullet,i}(x);$$

note that the second identity above holds since $\widetilde{\mathscr{T}}_{w_{\bullet,i}}^{-1} = \widetilde{\mathscr{T}}_{w_\bullet}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}$ by (2.14). Hence, we have $\widehat{\tau}(x) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\widehat{\tau}_{\bullet,i}(x)$, which implies that

$$\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(x) = \widehat{\tau}_{\bullet,i} \circ \widehat{\tau}(x) \in \widetilde{\mathbf{U}}_\bullet, \qquad \text{for all } x \in \widetilde{\mathbf{U}}_\bullet. \tag{4.13}$$

We next formulate the actions of $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}$ on $\widetilde{\mathbf{U}}^{\imath 0}$, for $i \in \mathbb{I}_\circ$. Recall $\boldsymbol{\varsigma}_\diamond$ from (2.21) and (4.1), and $\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$ from Proposition 2.1. Denote

$$\widetilde{k}_{i,\diamond} := \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1}(\widetilde{k}_i) = \varsigma_{i,\diamond}^{-1} K_i K'_{\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}. \tag{4.14}$$

Note that $\widetilde{k}_{j,\diamond} = \widetilde{k}_j = K_j K'_{\tau j}$, for $j \in \mathbb{I}_\bullet$. We shall denote

$$\widetilde{k}_{\lambda,\diamond} := \prod_{i \in \mathbb{I}} \widetilde{k}_{i,\diamond}^{m_i} \in \widetilde{\mathbf{U}}^{\imath 0}, \qquad \text{for } \lambda = \sum_{i \in \mathbb{I}} m_i \alpha_i \in \mathbb{Z}\mathbb{I}. \tag{4.15}$$

Lemma 4.10. *Let $w \in W$ be such that $w\tau = \tau w$. Then $\widetilde{\mathscr{T}}'_{w,-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{w\alpha_j,\diamond}$, for $j \in \mathbb{I}_\circ$.*

Proof. By Proposition 2.3, we have

$$\widetilde{T}'_{w,-1}(\widetilde{k}_j) = \widetilde{T}'_{w,-1}(K_j K'_{\tau j}) = K_{w\alpha_j} K'_{w\alpha_{\tau j}} = K_{w\alpha_j} K'_{\tau w \alpha_j} = \widetilde{k}_{w\alpha_j}.$$

By (4.14)–(4.15), we have $\widetilde{k}_{\lambda,\diamond} = \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1}(\widetilde{k}_\lambda)$, for $\lambda \in \mathbb{Z}\mathbb{I}$. By (4.2) and (4.5), we have $\widetilde{\mathscr{T}}'_{w,-1} = \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1} \circ \widetilde{T}'_{w,-1} \circ \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}$, and hence,

$$\widetilde{\mathscr{T}}'_{w,-1}(\widetilde{k}_{j,\diamond}) = (\widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1} \circ \widetilde{T}'_{w,-1})(\widetilde{k}_j) = \widetilde{\Psi}_{\boldsymbol{\varsigma}_\diamond}^{-1}(\widetilde{k}_{w\alpha_j}) = \widetilde{k}_{w\alpha_j,\diamond}.$$

The lemma is proved. □

In particular, setting $w = \mathbf{r}_i$ ($i \in \mathbb{I}_\circ$) in Lemma 4.10 gives us

$$\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}.$$

Summarizing the above discussion, we have obtained the following.

Proposition 4.11. *Let $i \in \mathbb{I}_\circ$. There exists element $\widetilde{\mathbf{T}}'_{i,-1}(x) := \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(x)$, which satisfies the intertwining relation (4.7), for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. More explicitly, we have*

$$\widetilde{\mathbf{T}}'_{i,-1}(u) = (\widehat{\tau}_{\bullet,i} \circ \widehat{\tau})(u), \qquad \textit{for } u \in \widetilde{\mathbf{U}}_\bullet, \tag{4.16}$$

$$\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}, \qquad \textit{for } j \in \mathbb{I}_\circ. \tag{4.17}$$

4.6 | Integrality of $\widetilde{\mathbf{T}}'_{i,-1}$

The formula (4.16) clearly preserves the Lusztig integral $\mathbb{Z}[q, q^{-1}]$-form on $\widetilde{\mathbf{U}}_\bullet$. We shall explain below that our braid group action is also integral on the Cartan part, even though the definition (4.14) of $\widetilde{k}_{j,\diamond}$ may involve $q^{1/2}$.

Lemma 4.12. *We have*

$$\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_j) = \varsigma_{\mathbf{r}_i\alpha_j-\alpha_j,\diamond}^{-1} \widetilde{k}_{\mathbf{r}_i\alpha_j}, \tag{4.18}$$

where $\varsigma_{\mathbf{r}_i\alpha_j-\alpha_j,\diamond}^{-1} \in \mathbb{Z}[q, q^{-1}]$, for all $i, j \in \mathbb{I}_\circ$.

Proof. Formula (4.18) follows from (4.17) by unraveling the notation $\widetilde{k}_{j,\diamond}, \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}$ in (4.14)–(4.15).

It remains to show that $\varsigma^{-1}_{\mathbf{r}_i\alpha_j-\alpha_j,\diamond} \in \mathbb{Z}[q,q^{-1}]$. Recall from the definition (2.21), we have $\varsigma_{j,\diamond} \in -q^{\mathbb{Z}/2}$, for all $j \in \mathbb{I}_\circ$.

For $j = i$, since $\mathbf{r}_i(\alpha_i) = -\alpha_i + \alpha_\bullet$ for some $\alpha_\bullet \in \mathbb{Z}\mathbb{I}_\bullet$, we have

$$\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_i) = \varsigma^2_{i,\diamond}\widetilde{k}_{\mathbf{r}_i\alpha_i}.$$

where $\varsigma^2_{i,\diamond} \in q^{\mathbb{Z}}$. The integrality for $\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_{\tau i})$ can be then obtained by applying $\widehat{\tau}$ to the above formula.

For $j \neq i, \tau i$, we only need to consider the case $\varsigma_{i,\diamond} = -q^{-1/2}$. In this case, by (2.21), $\overline{\alpha}_i$ is a short root. Moreover, due to the classification of Satake diagrams and the corresponding restricted root systems [1], we have $\frac{(\overline{\alpha}_j,\overline{\alpha}_i)}{(\overline{\alpha}_i,\overline{\alpha}_i)} = -2$ or 0. It remains to consider the nontrivial case $\frac{(\overline{\alpha}_j,\overline{\alpha}_i)}{(\overline{\alpha}_i,\overline{\alpha}_i)} = -2$. It follows that $\mathbf{r}_i\overline{\alpha}_j - \overline{\alpha}_j = 2\overline{\alpha}_i$, which implies $\mathbf{r}_i\alpha_j \in \alpha_j + k\alpha_i + l\alpha_{\tau i} + \mathbb{Z}\mathbb{I}_\bullet$, for some $k, l \geqslant 0, k + l = 2$. Since $\varsigma_{i,\diamond} = \varsigma_{\tau i,\diamond}$, the formula (4.18) is unraveled as the following integral formula $\widetilde{\mathbf{T}}'_{i,-1}(\widetilde{k}_j) = \varsigma^{-2}_{i,\diamond}\widetilde{k}_{\mathbf{r}_i\alpha_j}$.

Therefore, the integrality of (4.18) holds in all cases. □

4.7 | A uniform formula for $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$

In this subsection, we introduce a uniform method to calculate $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$. Note that $\widetilde{\mathbf{T}}_i = \widetilde{\mathbf{T}}_{\tau i}$ and this takes care of $\widetilde{\mathbf{T}}'_{i,-1}(B_{\tau i})$. To that end, without loss of generality, we can restrict ourselves to a Satake diagram $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ of real rank 1; that is, $\mathbb{I}_\circ = \{i, \tau i\}$ for some $i \in \mathbb{I}_\circ$.

Recall the diagram involution $\tau_{\bullet,i}$ associated to the longest element $w_{\bullet,i}$ in the Weyl group $W_{\mathbb{I}_\bullet\cup\{i,\tau i\}}$. By definition of admissible pairs, the diagram involution associated to $w_\bullet$ is τ. Observe that $\tau_{\bullet,i}\tau i \in \{i, \tau i\}$, by Table 1 on rank 1 Satake diagrams.

Recall $\mathcal{K}_i, \mathcal{K}_{\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}$ from (3.23).

Lemma 4.13. *We have*

$$\mathscr{T}^{-1}_{\mathbf{r}_i}(B_i) = -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\mathscr{T}^2_{w_\bullet}(B^\sigma_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}, \tag{4.19}$$

where B^σ_i is given in (3.17).

Proof. Recall from (2.21) and (4.1) that $\varsigma_{i,\diamond} = -q^{-(\alpha_i,\alpha_i+w_\bullet\alpha_{\tau i})/2}$, for $i \in \mathbb{I}_\circ$, and $\varsigma_{j,\diamond} = 1$, for $j \in \mathbb{I}_\bullet$. By (2.14), we have $\mathscr{T}_{w_{\bullet,i}} = \mathscr{T}_{\mathbf{r}_i}\mathscr{T}_{w_\bullet}$. By Lemma 4.4, we compute

$$\begin{aligned}
\mathscr{T}^{-1}_{\mathbf{r}_i}(B_i) &= \mathscr{T}^{-1}_{\mathbf{r}_i}\Big(F_i + \mathscr{T}_{w_\bullet}(E_{\tau i})K'_i\Big) \\
&= \mathscr{T}_{w_\bullet}\mathscr{T}^{-1}_{w_{\bullet,i}}\Big(F_i + \mathscr{T}_{w_\bullet}(E_{\tau i})K'_i\Big) \\
&= \mathscr{T}^2_{w_\bullet}\Big(\mathscr{T}^{-1}_{w_\bullet}\mathscr{T}^{-1}_{w_{\bullet,i}}(F_i) + \mathscr{T}^{-1}_{w_{\bullet,i}}(E_{\tau i})\mathscr{T}^{-1}_{w_\bullet}\mathscr{T}^{-1}_{w_{\bullet,i}}(K'_i)\Big) \\
&= \mathscr{T}^2_{w_\bullet}\Big(-\mathscr{T}^{-1}_{w_\bullet}(E_{\tau_{\bullet,i}i}K'^{-1}_{\tau_{\bullet,i}i}) - q^{-(\alpha_i,\alpha_i+w_\bullet\alpha_{\tau i})}K^{-1}_{\tau_{\bullet,i}\tau i}F_{\tau_{\bullet,i}\tau i}\mathscr{T}^{-1}_{w_\bullet}(K'^{-1}_{\tau_{\bullet,i}i})\Big)
\end{aligned}$$

$$= -\widetilde{\mathscr{T}}^2_{w_\bullet}\Big(\widetilde{\mathscr{T}}^{-1}_{w_\bullet}(E_{\tau_{\bullet,i}i})K_{\tau_{\bullet,i}\tau i} + q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}F_{\tau_{\bullet,i}\tau i}\Big)K^{-1}_{\tau_{\bullet,i}\tau i}\widetilde{\mathscr{T}}_{w_\bullet}(K'_{\tau_{\bullet,i}i})^{-1}$$

$$= -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\widetilde{\mathscr{T}}^2_{w_\bullet}\Big(K_{\tau_{\bullet,i}\tau i}\widetilde{\mathscr{T}}^{-1}_{w_\bullet}(E_{\tau_{\bullet,i}i}) + F_{\tau_{\bullet,i}\tau i}\Big)\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}$$

$$= -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\widetilde{\mathscr{T}}^2_{w_\bullet}(B^\sigma_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}.$$

This proves the lemma. □

Theorem 4.14. *Let $i \in \mathbb{I}_\circ$. There exists a unique element $\widetilde{\mathbf{T}}'_{i,-1}(B_i) \in \widetilde{\mathbf{U}}^\imath$ which satisfies the following intertwining relation (see (4.7))*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_i)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_i}(B_i).$$

More explicitly, we have

$$\widetilde{\mathbf{T}}'_{i,-1}(B_i) = -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\widetilde{\mathscr{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}. \tag{4.20}$$

Proof. Recall $\tau_{\bullet,i}\tau i \in \{i, \tau i\}$; see Table 1. By Theorem 3.6, we have $\widetilde{\Upsilon}_i B^\sigma_{\tau_{\bullet,i}\tau i} = B_{\tau_{\bullet,i}\tau i}\widetilde{\Upsilon}_i$. By Proposition 4.6, we have $\widetilde{\mathscr{T}}_{w_\bullet}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$, and hence $\widetilde{\Upsilon}_i\widetilde{\mathscr{T}}^2_{w_\bullet}(B^\sigma_{\tau_{\bullet,i}\tau i}) = \widetilde{\mathscr{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\widetilde{\Upsilon}_i$. By Lemma 3.11, we have $\mathcal{K}_{\tau_{\bullet,i}\tau i} \in \widetilde{\mathbf{U}}^{\imath 0}$, and hence, $\mathcal{K}_{\tau_{\bullet,i}\tau i}$ commutes with $\widetilde{\Upsilon}_i$. Putting these together with (4.19), we have

$$-q^{-(\alpha_i, w_\bullet(\alpha_{\tau i}))}\widetilde{\mathscr{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_i}(B_i). \tag{4.21}$$

It follows by Proposition 4.5 that $-q^{-(\alpha_i, w_\bullet(\alpha_{\tau i}))}\widetilde{\mathscr{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i} \in \widetilde{\mathbf{U}}^\imath$. Hence, setting $\widetilde{\mathbf{T}}'_{i,-1}(B_i) = -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})}\widetilde{\mathscr{T}}^2_{w_\bullet}(B_{\tau_{\bullet,i}\tau i})\mathcal{K}^{-1}_{\tau_{\bullet,i}\tau i}$, we have proved the theorem. □

5 | RANK 2 FORMULAS FOR $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a rank 2 irreducible Satake diagram. Fix $i, j \in \mathbb{I}_{\circ,\tau}$ such that $i \neq j$, such that $\mathbb{I}_\circ = \{i, \tau i, j, \tau j\}$. A complete list of formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ is formulated in Table 3 (listed after §10.3). We show that the formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Table 3 satisfy the intertwining relation (4.7); see Theorem 5.5. Together with the formulas in the previous section, we have established the existence of an endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ satisfying (4.7).

5.1 | Some commutator relations with $\widetilde{\Upsilon}$

For $w \in W$, let $\mathbf{U}^+[w]$ be the well-known subalgebra of $\mathbf{U}^+$ spanned by PBW basis elements generated by certain q-root vectors so that $\mathbf{U}^+[w_0] = \mathbf{U}^+$; see [16, 8.24]. As we identify $\widetilde{\mathbf{U}}^+ = \mathbf{U}^+$, we denote by $\widetilde{\mathbf{U}}^+[w]$ the subalgebra of $\widetilde{\mathbf{U}}^+$ corresponding to $\mathbf{U}^+[w]$. The next lemma is valid for all Satake diagrams.

Lemma 5.1. *For $i \neq j \in \mathbb{I}_{\circ,\tau}$, we have*

$$F_j \widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i F_j, \tag{5.1}$$

$$\mathscr{T}_{\mathbf{r}_i}^{-1}\Big(\mathscr{T}_{w_\bullet}(E_{\tau j})K_j'\Big) \cdot \widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \cdot \mathscr{T}_{\mathbf{r}_i}^{-1}\Big(\mathscr{T}_{w_\bullet}(E_{\tau j})K_j'\Big). \tag{5.2}$$

Proof. Write $\widetilde{\Upsilon}_i = \sum_{m\geqslant 0} \widetilde{\Upsilon}_{i,m}$, where $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}^+_{m(\alpha_i+w_\bullet\alpha_{\tau i})}$. By [8, Proposition 4.5], we have $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}^+[\mathbf{r}_i]$, for $m \geqslant 0$. Since the simple reflection s_j does not appear in any reduced expression of $\mathbf{r}_i$, F_j commutes with any element in $\widetilde{\mathbf{U}}^+[\mathbf{r}_i]$; in particular, F_j commutes with $\widetilde{\Upsilon}_i$. This proves the identity (5.1).

By Proposition 3.8, $\widetilde{\Upsilon}$ is fixed by $\widehat{\tau}_{\bullet,i}$ (which is equal to either Id or $\widehat{\tau}$). Hence, by Lemma 4.4 and the fact that $\widetilde{\Upsilon}_{i,m} \in \widetilde{\mathbf{U}}^+_{m(\alpha_i+w_\bullet\alpha_{\tau i})}$, we have

$$\mathscr{T}_{w_{\bullet,i}}(\widetilde{\Upsilon}_{i,m}) = \mathscr{T}_{w_{\bullet,i}}\widehat{\tau}_{\bullet,i}(\widetilde{\Upsilon}_{i,m}) \in \widetilde{\mathbf{U}}^-_{-m(\alpha_i+w_\bullet\alpha_{\tau i})}K'^{-m}_{\alpha_i+w_\bullet\alpha_{\tau i}},$$

or equivalently,

$$\mathcal{F} := \mathscr{T}_{w_{\bullet,i}}(\widetilde{\Upsilon}_{i,m})K'^{m}_{\alpha_i+w_\bullet\alpha_{\tau i}} \in \widetilde{\mathbf{U}}^-_{-m(\alpha_i+w_\bullet\alpha_{\tau i})}. \tag{5.3}$$

Since $w_\bullet\alpha_{\tau i} = \alpha_{\tau i} + \sum_{r\in\mathbb{I}_\bullet} a_r\alpha_r$ for some $a_r \in \mathbb{N}$, the eigenspace $\widetilde{\mathbf{U}}^-_{-m(\alpha_i+w_\bullet\alpha_{\tau i})}$ lies in the subalgebra of $\widetilde{\mathbf{U}}^-$ generated by $F_i, F_{\tau i}, F_r, r \in \mathbb{I}_\bullet$; clearly, $E_{\tau j}$ commutes with any of these elements, and hence, we have by (5.3) that $[E_{\tau j}, \mathcal{F}] = 0$. For each m, we compute

$$\begin{aligned}
&\Big[E_{\tau j}\mathscr{T}_{w_\bullet}(K_j'), \mathscr{T}_{w_{\bullet,i}}(\widetilde{\Upsilon}_{i,m})\Big]\\
&= \Big[E_{\tau j}\mathscr{T}_{w_\bullet}(K_j'), \mathcal{F}K'^{-m}_{\alpha_i+w_\bullet(\alpha_{\tau i})}\Big]\\
&= q^{m(w_\bullet\alpha_j,\alpha_i+w_\bullet\alpha_{\tau i})}E_{\tau j}\mathcal{F}\mathscr{T}_{w_\bullet}(K_j')K'^{-m}_{\alpha_i+w_\bullet\alpha_i} - q^{m(\alpha_{\tau j},\alpha_i+w_\bullet\alpha_{\tau i})}\mathcal{F}E_{\tau j}\mathscr{T}_{w_\bullet}(K_j')K'^{-m}_{\alpha_i+w_\bullet\alpha_i}\\
&= q^{m(\alpha_{\tau j},\alpha_i+w_\bullet\alpha_{\tau i})}[E_{\tau j},\mathcal{F}]\cdot\mathscr{T}_{w_\bullet}(K_j')K'^{-m}_{\alpha_i+w_\bullet\alpha_i} = 0.
\end{aligned}$$

Hence, we obtain an identity

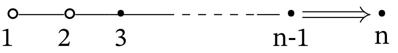
$$E_{\tau j}\mathscr{T}_{w_\bullet}(K_j')\cdot\mathscr{T}_{w_{\bullet,i}}(\widetilde{\Upsilon}_i) = \mathscr{T}_{w_{\bullet,i}}(\widetilde{\Upsilon}_i)\cdot E_{\tau j}\mathscr{T}_{w_\bullet}(K_j'). \tag{5.4}$$

The desired identity (5.2) now follows by applying $\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{w_\bullet}$ to (5.4). Indeed, we have $\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{w_\bullet}\mathscr{T}_{w_{\bullet,i}}(\widetilde{\Upsilon}_i) = \mathscr{T}_{w_\bullet}^2(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ since $\mathscr{T}_{w_{\bullet,i}} = \mathscr{T}_{\mathbf{r}_i}\mathscr{T}_{w_\bullet} = \mathscr{T}_{w_\bullet}\mathscr{T}_{\mathbf{r}_i}$ by (2.14) and $\mathscr{T}_{w_\bullet}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ by Proposition 4.6. Also, we clearly have $\mathscr{T}_{w_\bullet}^2(K_j') = K_j'$. □

5.2 | Motivating examples: Types BI, DI, DIII$_4$

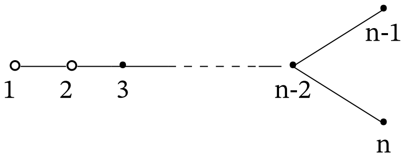
We provide examples in this subsection to motivate how we obtain the general rank 2 formulas $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Theorem 5.5 below. The three examples are of types BI$_n$ ($n \geqslant 3$), DI$_n$ ($n \geqslant 5$), DIII$_4$, and they will be treated uniformly.

The Satake diagrams of these types are listed below. For each type, we define elements $t_j \in W_\bullet$ for $j \in \mathbb{I}_\bullet$ following each diagram; these notations t_j allow a uniform proof of Lemma 5.2 thanks to the properties (5.5) below.



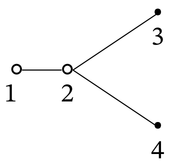
$\mathrm{BI}_n, n \geq 3$

$$t_a = s_a \cdots s_n \cdots s_a, \qquad (3 \leq a \leq n).$$



$\mathrm{DI}_n, n \geq 5$

$$t_a = s_{a\cdots n-2} s_{n-1} s_n s_{n-2\cdots a}, \qquad (3 \leq a \leq n-2), \qquad t_{n-1} = t_n = s_{n-1} s_n.$$



DIII_4

$$t_3 = t_4 = s_3 s_4.$$

Note that, for each of the three types, we always have

$$\mathbf{r}_2 = s_2 t_3 s_2, \qquad \ell(\mathbf{r}_2) = \ell(t_3) + 2, \qquad B_1 = F_1 + E_1 K_1'. \tag{5.5}$$

Recall the notation B_i^σ from (3.17).

Lemma 5.2. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) = \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2^\sigma), [B_2^\sigma, F_1]_{q_2}\Big]_{q_2} - q_2 F_1 \widetilde{\mathscr{T}}_{w_\bullet}(K_2) K_2'. \tag{5.6}$$

Proof. By Lemma A.1, $[B_2^\sigma, F_1]_{q_2} = [F_2, F_1]_{q_2}$, and RHS (5.6) is simplified as follows:

$$\begin{aligned}
\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2^\sigma), [B_2^\sigma, F_1]_{q_2}\Big]_{q_2} &= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2^\sigma), [F_2, F_1]_{q_2}\Big]_{q_2} \\
&= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(F_2), [F_2, F_1]_{q_2}\Big]_{q_2} + \Big[E_2 \widetilde{\mathscr{T}}_{w_\bullet}(K_2), [F_2, F_1]_{q_2}\Big]_{q_2} \\
&= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(F_2), [F_2, F_1]_{q_2}\Big]_{q_2} + q_2 F_1 \widetilde{\mathscr{T}}_{w_\bullet}(K_2) K_2'.
\end{aligned} \tag{5.7}$$

On the other hand, by a direct computation using (5.5) and Proposition 4.2, we have

$$\begin{aligned}
\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) &= \widetilde{\mathscr{T}}_2^{-1} \widetilde{\mathscr{T}}_{t_3}^{-1}([F_2, F_1]_{q_2}) \\
&= \Big[\widetilde{\mathscr{T}}_2^{-1} \widetilde{\mathscr{T}}_{t_3}^{-1}(F_2), [F_2, F_1]_{q_2}\Big]_{q_2} \\
&= \Big[\widetilde{\mathscr{T}}_{t_3}(F_2), [F_2, F_1]_{q_2}\Big]_{q_2}.
\end{aligned} \tag{5.8}$$

The desired formula (5.6) follows from (5.7)–(5.8) by noting that $\widetilde{\mathscr{T}}_{w_\bullet}(F_2) = \widetilde{\mathscr{T}}_{t_3}(F_2)$. □

Note that $q_1 = q_2$ in all three types.

Lemma 5.3. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1') = \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, E_1K_1']_{q_2}\Big]_{q_2} - q_2E_1K_1'\widetilde{\mathscr{T}}_{w_\bullet}(K_2)K_2'. \tag{5.9}$$

Proof. We shall establish the identity (5.9) by applying the operator $\ni := \widetilde{\mathscr{T}}_{w_\bullet}\widetilde{\mathscr{T}}_{w_0}$ to (5.6) as follows.

Recall $\mathcal{K}_i \in \widetilde{\mathbf{U}}^{\imath 0}$ from (3.23). By (4.19) and noting $(\alpha_2, w_\bullet\alpha_{\tau 2}) = 0$ in each of the three types, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(B_2^\imath) = -\widetilde{\mathscr{T}}_{w_\bullet}^2(B_2^\sigma)\mathcal{K}_2^{-1}$, or equivalently,

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(B_2^\imath)\mathcal{K}_2 = -\widetilde{\mathscr{T}}_{w_\bullet}^2(B_2^\sigma). \tag{5.10}$$

By Lemma 4.4, we have $\widetilde{\mathscr{T}}_{w_0}(B_2^\sigma) = \widetilde{\mathscr{T}}_{w_{\bullet,2}}(B_2^\sigma)$. Hence, applying $\widetilde{\mathscr{T}}_{\mathbf{r}_2}$ to both sides of (5.10), we obtain

$$B_2\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\mathcal{K}_2^\imath) = -\widetilde{\mathscr{T}}_{w_\bullet}\widetilde{\mathscr{T}}_{w_{\bullet,2}}(B_2^\sigma) = -\widetilde{\mathscr{T}}_{w_\bullet}\widetilde{\mathscr{T}}_{w_0}(B_2^\sigma). \tag{5.11}$$

Moreover, by Lemma 4.4, we have $\ni(F_1) = -K_1^{-1}E_1 = -q_2{}^{-2}E_1K_1'\widetilde{k}_1^{-1}$. Note also that $\ni$ commutes with both $\widetilde{\mathscr{T}}_{w_\bullet}$ and $\widetilde{\mathscr{T}}_{\mathbf{r}_2}$. Hence, by applying $\ni$ to (5.6) and then using (5.11), we have

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1')\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = &\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2)\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\mathcal{K}_2), [B_2\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\mathcal{K}_2), E_1K_1'\widetilde{k}_1^{-1}]_{q_2}\Big]_{q_2}\\ &- q_2E_1K_1'\widetilde{k}_1^{-1}\widetilde{\mathscr{T}}_{w_\bullet}\ni(\mathcal{K}_2).\end{aligned} \tag{5.12}$$

For weight reason, (5.12) is simplified as

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1')\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = q_2{}^2&\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, E_1K_1']_{q_2}\Big]_{q_2}\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\mathcal{K}_2)\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\mathcal{K}_2)\widetilde{k}_1^{-1}\\ &- q_2E_1K_1'\widetilde{k}_1^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_2)K_{w_\bullet\alpha_2-\alpha_2}.\end{aligned} \tag{5.13}$$

By definition (3.23), we have $\mathcal{K}_2 = \widetilde{k}_2K'_{w_\bullet\alpha_2-\alpha_2}$; in addition, by (4.13), $K'_{w_\bullet\alpha_2-\alpha_2}$ is fixed by $\widetilde{\mathscr{T}}_{\mathbf{r}_2}$. We also have $\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\mathcal{K}_2) = q_2{}^{-2}\mathcal{K}_2^{-1}$. Hence, (5.13) is further simplified as

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1')\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = &\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, E_1K_1']_{q_2}\Big]_{q_2}\widetilde{k}_2^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_2)\widetilde{k}_1^{-1}\\ &- q_2E_1K_1'K_{w_\bullet(\alpha_2)}K_2'\widetilde{k}_1^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_2)\widetilde{k}_2^{-1}.\end{aligned} \tag{5.14}$$

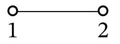
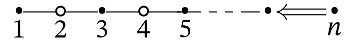
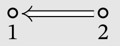
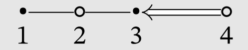
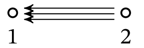
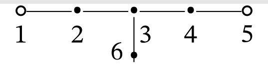
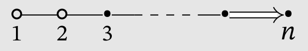
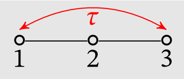
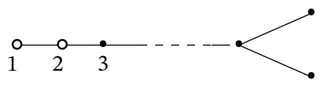
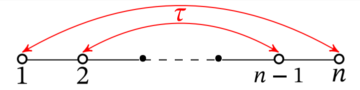
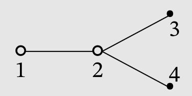
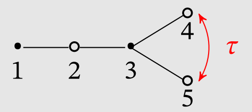
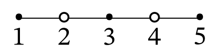
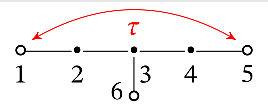
Finally, by Lemma 4.10, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_1^{-1}) = \widetilde{k}_1^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_2}(\widetilde{k}_2)\widetilde{k}_2^{-1}$, and then, the identity (5.14) can be transformed into an equivalent form (5.9). ☐

Proposition 5.4. *The following element*

$$\widetilde{\mathbf{T}}'_{2,-1}(B_1) := \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, B_1]_{q_2}\Big]_{q_2} - q_2B_1\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2) \in \widetilde{\mathbf{U}}^\imath \tag{5.15}$$

satisfies the intertwining relation $\widetilde{\mathbf{T}}'_{2,-1}(B_1)\widetilde{\Upsilon}_2 = \widetilde{\Upsilon}_2\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(B_1^\imath)$ *(i.e., (4.7), for* $i = 2, x = B_1$*).*

TABLE 2 Rank 2 Satake diagrams.

SP	Satake diagrams	RS		Satake diagrams	RS
AI_2		A_2	CII_n		BC_2
CI_2		C_2	CII_4		C_2
G_2		G_2	EIV		A_2
BI_n		B_2	$AIII_3$		C_2
DI_n		B_2	$AIII_n$		BC_2
$DIII_4$		C_2	$DIII_5$		BC_2
AII_5		A_2	EIII		BC_2

RS, relative root system; SP, symmetric pair.

Proof. The intertwining relation follows by the following computation:

$$
\begin{aligned}
&\widetilde{\Upsilon}_2 \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(B_1^\imath)\widetilde{\Upsilon}_2^{-1} \\
&= \widetilde{\Upsilon}_2\Big(\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) + \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1')\Big)\widetilde{\Upsilon}_2^{-1} \\
&\overset{(5.2)}{=} \widetilde{\Upsilon}_2 \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1)\widetilde{\Upsilon}_2^{-1} + \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1') \\
&\overset{(5.6)}{=} \widetilde{\Upsilon}_2\Big(\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2^\sigma), [B_2^\sigma, F_1]_{q_2}\Big]_{q_2} - q_2 F_1 \widetilde{\mathscr{T}}_{w_\bullet}(K_2)K_2'\Big)\widetilde{\Upsilon}_2^{-1} + \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1') \\
&\overset{(*)}{=} \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, F_1]_{q_2}\Big]_{q_2} - q_2 F_1 \widetilde{\mathscr{T}}_{w_\bullet}(K_2)K_2' + \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_1K_1') \\
&\overset{(5.9)}{=} \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, F_1]_{q_2}\Big]_{q_2} - q_2 F_1 \widetilde{\mathscr{T}}_{w_\bullet}(K_2)K_2' \\
&\qquad + \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, E_1K_1']_{q_2}\Big]_{q_2} - q_2 E_1K_1' \widetilde{\mathscr{T}}_{w_\bullet}(K_2)K_2' \\
&= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, B_1]_{q_2}\Big]_{q_2} - q_2 B_1 \widetilde{\mathscr{T}}_{w_\bullet}(K_2)K_2' = \widetilde{\mathbf{T}}_{2,-1}'(B_1),
\end{aligned}
$$

where the equality (*) follows from Theorem 3.6 and Lemma 5.1. ☐

5.3 | Formulation for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$

Theorem 5.5. *The elements $\widetilde{\mathbf{T}}'_{i,-1}(B_j) \in \widetilde{\mathbf{U}}^\imath$ listed in Table 3 satisfy the following intertwining relation (see (4.7)):*

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(B_j). \tag{5.16}$$

We clarify a few points regarding Table 3 in the following remarks.

Remark 5.6. Recall that $\widetilde{\mathscr{T}}_s$ $(s \in \mathbb{I}_\bullet)$ restrict to automorphisms on $\widetilde{\mathbf{U}}^\imath$ by Proposition 4.5; hence, the use of $\widetilde{\mathscr{T}}_s$ $(s \in \mathbb{I}_\bullet)$ in the formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ is legitimate; see (4.6).

Remark 5.7. Let ρ be a diagram involution on the underlying Dynkin diagram (ρ is not necessarily equal to τ). By the intertwining relation (4.7), the formula of $\widetilde{\mathbf{T}}'_{\rho i,-1}(B_{\rho j})$ can be obtained from $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ via

$$\widetilde{\mathbf{T}}'_{\rho i,-1}(B_{\rho j}) = \rho\Big(\widetilde{\mathbf{T}}'_{i,-1}(B_j)\Big).$$

In particular, when $\rho = \tau$, we have $\widetilde{\mathbf{T}}'_{i,-1}(B_{\tau j}) = \widehat{\tau}(\widetilde{\mathbf{T}}'_{i,-1}(B_j))$ by Remark 4.8. Accordingly, only one formula of $\widetilde{\mathbf{T}}'_{\rho i,-1}(B_{\rho j})$ and $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ is included in the table; see types AII$_5$, EIV, and all types with $\tau \neq \mathrm{Id}$.

Remark 5.8. The formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ only depend on the subdiagram generated by nodes $i, \tau i, j$ and the component of black nodes which is connected to either i or τi. For example, the formula for $\widetilde{\mathbf{T}}'_{2,-1}(B_4)$ in type DIII$_5$ is formally identical to the formula for $\widetilde{\mathbf{T}}'_{2,-1}(B_4)$ in type AII$_5$. (Note that such a subdiagram may not be a Satake subdiagram as the vertex τj is not included.)

Recall that $\widetilde{\mathbf{U}}^\imath$ is defined over an extension field $\mathbb{F}$ of $\mathbb{Q}(q)$. Denote

$${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath := \mathbb{Q}(q)\text{-subalgebra of } \widetilde{\mathbf{U}}^\imath \text{ generated by } B_i, \widetilde{k}_i, x \text{ for } i \in \mathbb{I}_\circ, x \in \widetilde{\mathcal{G}}_\bullet. \tag{5.17}$$

Proposition 5.9. *The symmetries $\widetilde{\mathbf{T}}'_{i,-1}$ $(i \in \mathbb{I}_\circ)$ preserve the $\mathbb{Q}(q)$-algebra ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$.*

Proof. This follows by the formula for $\widetilde{\mathbf{T}}'_{i,-1}$ acting on the Cartan part in Proposition 4.11 (see Lemma 4.12), the rank 1 formulas in (4.20), and the rank 2 formulas in Table 3. □

Remark 5.10. It would cause no difficulty if we have replaced $\widetilde{\mathbf{U}}^\imath$ (over $\mathbb{F}$) by ${}_{\mathbb{Q}}\widetilde{\mathbf{U}}^\imath$ over $\mathbb{Q}(q)$ throughout the paper. We need to work with $\widetilde{\mathbf{U}}$ over $\mathbb{Q}(q^{\frac{1}{2}})$ in several places. The results for $\mathbf{U}^\imath_{\varsigma_\diamond}$ will be valid over $\mathbb{Q}(q)$, while some results over $\mathbf{U}^\imath_{\varsigma}$, for ς over $\mathbb{Q}(q)$, are valid over $\mathbb{Q}(q^{\frac{1}{2}})$.

5.4 | Proof of Theorem 5.5

Proposition 5.11. *Let $i, j \in \mathbb{I}_{\circ,\tau}$ be such that $j \notin \{i, \tau i\}$. Then there exists a noncommutative polynomial $R_{ij}(x_i, x_{\tau i}, y_i, y_{\tau i}, z; \widetilde{\mathcal{G}}_\bullet)$, which is linear in z, such that*

(1) $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(F_j) = R_{ij}(B_i^\sigma, B_{\tau i}^\sigma, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)$;

(2) $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K_j') = R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K_j'; \widetilde{\mathcal{G}}_\bullet)$.

Remark 5.12. In case $\tau i = i$, the polynomials R_{ij} depend only on x_i, y_i, z and $\widetilde{\mathcal{G}}_\bullet$. In this case, it is understood in Proposition 5.11 that $R_{ij}(x_i, x_{\tau i}, y_i, y_{\tau i}, z; \widetilde{\mathcal{G}}_\bullet)$ is replaced by $R_{ij}(x_i, y_i, z; \widetilde{\mathcal{G}}_\bullet)$ (which is linear in z), and $R_{ij}(B_i^\sigma, B_{\tau i}^\sigma, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)$ is replaced by $R_{ij}(B_i^\sigma, \mathcal{K}_i, F_j; \widetilde{\mathcal{G}}_\bullet)$, and so on.

The proof of Proposition 5.11 will be carried out through type-by-type computation in the Appendix.

We define

$$\widetilde{\mathbf{T}}'_{i,-1}(B_j) := \begin{cases} R_{ij}(B_i, \mathcal{K}_i, B_j; \widetilde{\mathcal{G}}_\bullet), & \text{if } i = \tau i, \\ R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet) & \text{if } i \neq \tau i. \end{cases} \tag{5.18}$$

Clearly, we have $\widetilde{\mathbf{T}}'_{i,-1}(B_j) \in \widetilde{\mathbf{U}}^\imath$; see Table 3.

The polynomials R_{ij} in all types can be read off from Table 3. For instance, in type AII_5 it reads as follows:

$$R_{ij}(x, y, z; \widetilde{\mathcal{G}}_\bullet) = [[x, F_3], z]_q.$$

In order to read R_{ij} off from Table 3, one first needs to unravel $\widetilde{\mathscr{T}}_w$, for $w \in W_\bullet$, appearing in those formulas in terms of $E_j, F_j, K_j, K_j', j \in \mathbb{I}_\bullet$.

Proof of Theorem 5.5. We start with a general comment. Originally, we computed the explicit formulas in Table 3 type by type; see §5.2 for examples in types BI, DI, and DIII_4. In the process, we observed that parts of the arguments can be streamlined a uniform formulation in Proposition 5.11, even though its proof requires quite some computations. We hope that this uniform formulation helps to conceptualize the structures of the formulas for $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(B_j)$.

We now prove Theorem 5.5 using Proposition 5.11. Recall by Theorem 3.6 that $\widetilde{\Upsilon}_i B_i^\sigma \widetilde{\Upsilon}_i^{-1} = B_i$ and $\widetilde{\Upsilon}_i x \widetilde{\Upsilon}_i^{-1} = x$ for $x \in \widetilde{\mathbf{U}}^{\imath 0} \widetilde{\mathbf{U}}_\bullet$.

For definiteness, let us assume that $i \neq \tau i$. (The case when $i = \tau i$ is similar using the interpretation of notation in Remark 5.12.) By Lemma 5.1, Proposition 5.11, and definition of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in (5.18), we have

$$\begin{aligned}
\widetilde{\Upsilon}_i \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(B_j) &= \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(F_j) + \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}\big(E_{\tau j})K_j'\big) \\
&= \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(F_j) + \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}\big(E_{\tau j})K_j'\big)\widetilde{\Upsilon}_i \\
&= \widetilde{\Upsilon}_i R_{ij}(B_i^\sigma, B_{\tau i}^\sigma, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet) + R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K_j'; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i \\
&= R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i + R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K_j'; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i \\
&= R_{ij}(B_i, B_{\tau i}, \mathcal{K}_i, \mathcal{K}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet)\widetilde{\Upsilon}_i = \widetilde{\mathbf{T}}'_{i,-1}(B_j)\widetilde{\Upsilon}_i,
\end{aligned}$$

where the second last step follows form the linearity of R_{ij} in its fifth component. This proves the desired identity (5.16), whence the theorem. □

Conjecture 5.13. *For $\widetilde{\mathbf{U}}^\imath$ of Kac–Moody type, Proposition 5.11 remains valid.*

Assume Conjecture 5.13 holds. Then $\widetilde{\mathbf{T}}'_{i,-1}(B_j) \in \widetilde{\mathbf{U}}^\imath$ defined in (5.18) satisfies the intertwining relation (5.16), and hence, $\widetilde{\mathbf{T}}'_{i,-1}$ is a symmetry of $\widetilde{\mathbf{U}}^\imath$ of Kac–Moody type.

5.5 | A comparison with earlier results

We compare our formulas with some special cases obtained in the literature.

By choosing a reduced expression of $w_\bullet$, we can write out the formula (4.20) explicitly for rank 1 Satake diagrams in Table 1. We list some explicit formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_i)$ and compare them with braid group actions obtained earlier in [14, 20, 30]. (The index i is specified in each case.) In some rank 2 cases, our formulas differ from those in [30] and they can be matched by some twisting. As noted in [30, Remark 7.4], the formulas for braid operators in [20] may involve $\sqrt{v}$ and are related to those in [30] by some other twisting.

5.5.1 | Type AI_1

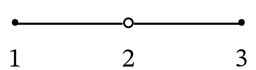
We shall label the single white node in rank 1 type AI by 1. In this case, the formula (4.20) reads as follows:

$$\widetilde{\mathbf{T}}'_{1,-1}(B_1) = -q^{-2}B_1\mathcal{K}_1^{-1} = -q^{-2}B_1\widetilde{k}_1^{-1}. \tag{5.19}$$

Note also that $\varsigma_{1,\diamond} = -q^{-2}$. Applying the central reduction $\pi^\imath_{\varsigma_\diamond}$ to (5.19), we have $\mathbf{T}^{-1}_{1,\diamond}(B_1) = B_1 \in \mathbf{U}^\imath_{\varsigma_\diamond}$. Our formula (5.19) of $\widetilde{\mathbf{T}}'_{1,-1}(B_1)$ coincides with the formula $\mathrm{T}_i^{-1}(B_i)$ in [30, Lemma 5.1]. Our formulation of $\mathbf{T}^{-1}_{1,\diamond}(B_1)$ coincides with the formula $\tau_i^{-1}(B_i)$ given in [20, (3.1)] for $(\mathbf{U}, \mathbf{U}^\imath_{q^{-2}})$.

5.5.2 | Type AII_3

The rank 1 Satake diagram of type AII is given by

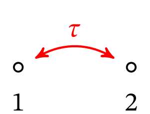


By Table 1, $\mathbf{r}_2 = s_{2132}$, and the formula (4.20) reads as follows:

$$\begin{aligned}\widetilde{\mathbf{T}}'_{2,-1}(B_2) = &-q^{-2}(q-q^{-1})^2\big[[B_2,F_3]_q,F_1\big]_q E_3E_1\widetilde{k}_2^{-1}\\ &+(q-q^{-1})\big([B_2,F_3]_qK_1E_3+[B_2,F_1]_qK_3E_1\big)\widetilde{k}_2^{-1} - q^2B_2K_3K_1\widetilde{k}_2^{-1}.\end{aligned}$$

5.5.3 | Type AIII_{11}

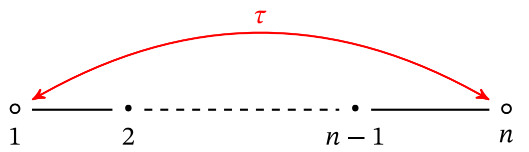
The AIII_{11} Satake diagram is given by



In this case, the formula (4.20) reads as $\widetilde{\mathbf{T}}'_{1,-1}(B_1) = -B_2\mathcal{K}_2^{-1} = -B_2\widetilde{k}_2^{-1}$.

5.5.4 | Type AIII_{11}

The rank 1 AIV Satake diagram is given by



In this case, the formula (4.20) reads as $\widetilde{\mathbf{T}}'_{1,-1}(B_1) = -q\widetilde{\mathscr{T}}^2_{w_\bullet}(B_1)\mathcal{K}_1^{-1}\prod_{j\in\mathbb{I}_\bullet} K_j'^{-1}$.

Remark 5.14. For type AIV, Dobson [14, Theorem 3.4] obtained a different automorphism $\mathcal{T}_1$ on $\mathbf{U}^\imath_\varsigma$ such that $\mathcal{T}_1^{-1}(B_1) = qB_1k_nK_{\varpi_{n-1}-\varpi_2}$. Here, ϖ_j are the fundamental weights and k_i is denoted by L_i *loc. cit.*

5.5.5 | Split type

The formulas of $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in the split types AI_2, CI_2, and G_2 are identical to the braid group operators obtained using the ıHall algebra approach, cf. [30, Lemma 5.1].

5.5.6 | Formulas on $\mathbf{U}^\imath_{\varsigma_\diamond}$

Applying central reductions and isomorphisms $\phi_\varsigma : \mathbf{U}^\imath_{\varsigma_\diamond} \cong \mathbf{U}^\imath_\varsigma$ (see §9.4 below) to our formulas, we recover various formulas obtained for $\mathbf{U}^\imath_\varsigma$ in [20] in split types and type AII.

6 | NEW SYMMETRIES $\widetilde{\mathbf{T}}''_{i,+1}$ ON $\widetilde{\mathbf{U}}^\imath$

In this section, we introduce new symmetries $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^\imath$, for $i \in \mathbb{I}_\circ$, via a new intertwining property using the quasi K-matrix, and establish explicit formulas of $\widetilde{\mathbf{T}}''_{i,+1}$ acting on the generators of $\widetilde{\mathbf{U}}^\imath$. Then, we show that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutual inverses. (This in particular completes the proof of Theorem 4.7 that $\widetilde{\mathbf{T}}'_{i,-1}$ is an automorphism.)

6.1 | Characterization of $\widetilde{\mathbf{T}}''_{i,+1}$

We formulate $\widetilde{\mathbf{T}}''_{i,+1}$ below, as a variant of $\widetilde{\mathbf{T}}'_{i,-1}$ introduced in Theorem 4.7.

Theorem 6.1. *Let $i \in \mathbb{I}_\circ$.*

(1) *For any $x \in \widetilde{\mathbf{U}}^\imath$, there is a unique element $x'' \in \widetilde{\mathbf{U}}^\imath$ such that $x''\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathscr{T}}_{\mathbf{r}_i}(x)$.*
(2) *The map $x \mapsto x''$ defines an automorphism of the algebra $\widetilde{\mathbf{U}}^\imath$, denoted by $\widetilde{\mathbf{T}}''_{i,+1}$.*

The strategy of proving Theorem 6.1 is largely parallel to that of Theorem 4.7 given in the previous sections. We shall prove Theorem 6.1(1) and a weaker version of Part (2) that $x \mapsto x''$ defines an endomorphism $\widetilde{\mathbf{T}}''_{i,+1}$ of the algebra $\widetilde{\mathbf{U}}^\imath$, by combining Proposition 6.2, Proposition 6.3, and Theorem 6.6. Finally, we show that $\widetilde{\mathbf{T}}''_{i,+1}$ is an automorphism of $\widetilde{\mathbf{U}}^\imath$ in Theorem 6.7.

Hence, $\widetilde{\mathbf{T}}''_{i,+1}$ satisfies the following intertwining relation:

$$\widetilde{\mathbf{T}}''_{i,+1}(x)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathscr{T}}_{\mathbf{r}_i}(x), \qquad \text{for all } x \in \widetilde{\mathbf{U}}^\imath. \tag{6.1}$$

6.2 | Action of $\widetilde{\mathbf{T}}''_{i,+1}$ on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$

Just as for Proposition 4.11, we can prove the following.

Proposition 6.2. *Let $i \in \mathbb{I}_\circ$. For each $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, there is a unique element $\widetilde{\mathbf{T}}''_{i,+1}(x) \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$ such that the intertwining relation $\widetilde{\mathbf{T}}''_{i,+1}(x)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathscr{T}}_{\mathbf{r}_i}(x)$ holds; see (6.1). More explicitly,*

$$\widetilde{\mathbf{T}}''_{i,+1}(u) = (\widehat{\tau}_{\bullet,i} \circ \widehat{\tau})(u), \quad \widetilde{\mathbf{T}}''_{i,+1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}, \qquad \textit{for } u \in \widetilde{\mathbf{U}}_\bullet \textit{ and } j \in \mathbb{I}_\circ.$$

It follows by Propositions 4.11 and 6.2 that $\widetilde{\mathbf{T}}'_{i,-1}$, $\widetilde{\mathbf{T}}''_{i,+1}$, and $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{\pm 1}$ coincide on $\widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$. In particular, we have

$$\widetilde{\mathbf{T}}''_{i,+1}(x) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \sigma^\imath)(x), \qquad \text{for } x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet. \tag{6.2}$$

6.3 | Rank 1 formula for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$

We shall establish a uniform formula for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$, for $i \in \mathbb{I}_\circ$, a counterpart of Theorem 4.14. Recall the anti-involution $\sigma^\imath$ of $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.12.

Proposition 6.3. *Let $i \in \mathbb{I}_\circ$. There exists a unique element $\widetilde{\mathbf{T}}''_{i,+1}(B_i) \in \widetilde{\mathbf{U}}^\imath$ which satisfies the following intertwining relation (see (6.1))*

$$\widetilde{\mathbf{T}}''_{i,+1}(B_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(B_i). \tag{6.3}$$

More explicitly, we have

$$\widetilde{\mathbf{T}}''_{i,+1}(B_i) = -q^{-(\alpha_i,\alpha_i)}\widetilde{\mathscr{T}}_{w_\bullet}^{-2}(B_{\tau_{\bullet,i}\tau i})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}^{-1}_{\tau_{\bullet,i}i}). \tag{6.4}$$

In particular, we have $\widetilde{\mathbf{T}}''_{i,+1}(B_i) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i)$.

Proof. By Theorem 3.6 (applied to the rank 1 setting), we have $B_i\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i B_i^\sigma$, which can be rewritten as

$$\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(B_i^\sigma) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(B_i)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i). \tag{6.5}$$

Hence, by comparing (6.3) and (6.5) and then applying (2.10), we obtain that

$$\widetilde{\mathbf{T}}''_{i,+1}(B_i) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(B_i^\sigma) = (\sigma \circ \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1})(B_i). \tag{6.6}$$

We now convert the formula (6.6) to the desired formula (6.4) for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$, which particularly shows that $\widetilde{\mathbf{T}}''_{i,+1}(B_i) \in \widetilde{\mathbf{U}}^\imath$. To that end, note that $\sigma(\mathcal{K}_{\tau_\bullet,i\tau i}) = \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{\tau_\bullet,i}i)$, by Proposition 2.2 and definition (3.23) of $\mathcal{K}_i$. Applying σ to the identity $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(B_i) = -q^{-(\alpha_i, w_\bullet\alpha_{\tau i})}\widetilde{\mathscr{T}}_{w_\bullet}^2(B^\sigma_{\tau_\bullet,i\tau i})\mathcal{K}^{-1}_{\tau_\bullet,i\tau i}$ in (4.19) and using (6.6), we have established the formula (6.4) for $\widetilde{\mathbf{T}}''_{i,+1}(B_i)$.

It remains to show that $\widetilde{\mathbf{T}}''_{i,+1}(B_i) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i)$. Recall $(\sigma^\imath)^2 = 1$. Indeed, we have

$$
\begin{aligned}
\widetilde{\mathbf{T}}''_{i,+1}(B_i) &\overset{(6.6)}{=} (\sigma \circ \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1})(B_i) \\
&\overset{(*)}{=} (\sigma \circ \mathrm{Ad}_{\widetilde{\Upsilon}_i^{-1}} \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i) \\
&\overset{(\dagger)}{=} \Big(\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1}\Big)(B_i),
\end{aligned}
$$

where (*) follows by Theorem 4.14, and (†) follows by applying (3.25) to the rank 1 Satake subdiagram associated with i. □

6.4 | Rank 2 formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$

The following lemma is a reformulation of Lemma 5.1.

Lemma 6.4. *We have*

(1) $\widetilde{\mathscr{T}}_{\mathbf{r}_i}(F_j)$ *commutes with* $\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)$;
(2) $\widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K'_j$ *commutes with* $\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)$.

Introduce a shorthand notation

$$\widehat{B}_i := \widetilde{\mathscr{T}}_{\mathbf{r}_i}\Big(\widetilde{\mathbf{T}}'_{i,-1}(B_i)\Big). \tag{6.7}$$

We reformulate the intertwining relation (5.16) as

$$\widehat{B}_i \cdot \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i) \cdot B_i. \tag{6.8}$$

Proposition 6.5. *Let $i \neq j \in \mathbb{I}_{\circ,\tau}$ be such that $j \notin \{i, \tau i\}$. Then there exists a noncommutative polynomial $P_{ij}(x_i, x_{\tau i}, y_i, y_{\tau i}, z; \widetilde{\mathcal{G}}_\bullet)$, which is linear in z, such that*

(1) $\widetilde{\mathscr{T}}_{\mathbf{r}_i}(F_j) = P_{ij}(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, F_j; \widetilde{\mathcal{G}}_\bullet)$,
(2) $\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K'_j) = P_{ij}(\widehat{B}_i, \widehat{B}_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K'_j; \widetilde{\mathcal{G}}_\bullet)$.

The proof of Proposition 6.5 is carried out through a type-by-type computation similar to the Appendix (the detail can be found in Appendix B in an arXiv version).

We set

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j) := P_{ij}(B_i, B_{\tau i}, \widetilde{k}_i, \widetilde{k}_{\tau i}, B_j; \widetilde{\mathcal{G}}_\bullet). \tag{6.9}$$

Clearly, we have $\widetilde{\mathbf{T}}''_{i,+1}(B_j) \in \widetilde{\mathbf{U}}^\imath$.

Theorem 6.6. *Let $i \neq j \in \mathbb{I}_{\circ,\tau}$. The elements $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ listed in Table 4 satisfy the following intertwining relation (see (6.1)):*

$$\widetilde{\mathbf{T}}''_{i,+1}(B_j)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1} = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}(B_j). \tag{6.10}$$

Proof. Recall $B_j = F_j + \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K'_j$. By Lemma 6.4, (6.8), and (6.9), we have

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\cdot\widetilde{\mathscr{T}}_{\mathbf{r}_i}(B_j)\cdot\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
&=\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\Big(\widetilde{\mathscr{T}}_{\mathbf{r}_i}(F_j)+\widetilde{\mathscr{T}}_{\mathbf{r}_i}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K'_j\Big)\Big)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
&=\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}(F_j)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)+\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j})K'_j\Big)\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i)\\
&=P_{ij}(B_i,B_{\tau i},\widetilde{k}_i,\widetilde{k}_{\tau i},F_j;\widetilde{\mathcal{G}}_\bullet)+P_{ij}(B_i,B_{\tau i},\widetilde{k}_i,\widetilde{k}_{\tau i},\widetilde{\mathscr{T}}_{w_\bullet}(E_j)K'_j;\widetilde{\mathcal{G}}_\bullet)\\
&=P_{ij}(B_i,B_{\tau i},\widetilde{k}_i,\widetilde{k}_{\tau i},B_j;\widetilde{\mathcal{G}}_\bullet)\\
&=\widetilde{\mathbf{T}}''_{i,+1}(B_j),
\end{aligned}$$

where the linearity of the polynomial P_{ij} with respect to the fifth variable is used in the last step. This proves the desired intertwining property (6.10) and whence the theorem. □

6.5 | $\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathbf{T}}''_{i,-e}$ as inverses

Recall the automorphisms $\widetilde{\mathbf{T}}'_{i,-1} \in \mathrm{Aut}(\widetilde{\mathbf{U}}^\imath)$ by Theorem 4.7. Recalling the bar involution $\psi^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.4, we define two more automorphisms $\widetilde{\mathbf{T}}''_{i,-1}, \widetilde{\mathbf{T}}'_{i,+1} \in \mathrm{Aut}(\widetilde{\mathbf{U}}^\imath)$ via

$$\widetilde{\mathbf{T}}''_{i,-1} := \psi^\imath \circ \widetilde{\mathbf{T}}''_{i,+1} \circ \psi^\imath, \qquad \widetilde{\mathbf{T}}'_{i,+1} := \psi^\imath \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \psi^\imath. \tag{6.11}$$

Recall that Lusztig's symmetries $\widetilde{T}'_{i,e}$ and $\widetilde{T}''_{i,-e}$ are mutually inverses, for $i \in \mathbb{I}, e = \pm 1$; see [28, 37.1.2]. They in addition satisfy the relation $\widetilde{T}'_{i,-1} = \sigma \circ \widetilde{T}''_{i,+1} \circ \sigma$; see (2.10). We prove the following $\imath$-analog of Lusztig's symmetries.

Theorem 6.7. *$\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathbf{T}}''_{i,-e}$ are mutually inverse automorphisms on $\widetilde{\mathbf{U}}^\imath$, for $e = \pm 1, i \in \mathbb{I}_\circ$. Moreover, we have*

$$\widetilde{\mathbf{T}}'_{i,e} = \sigma^\imath \circ \widetilde{\mathbf{T}}''_{i,-e} \circ \sigma^\imath. \tag{6.12}$$

Proof. By definition (6.11), $\widetilde{\mathbf{T}}''_{i,-1} = \psi^\imath \widetilde{\mathbf{T}}''_{i,+1}\psi^\imath$, and $\widetilde{\mathbf{T}}'_{i,+1} = \psi^\imath \widetilde{\mathbf{T}}'_{i,-1}\psi^\imath$. Hence, it suffices to show that $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses.

We already knew that $\widetilde{\mathbf{T}}'_{i,-1} : \widetilde{\mathbf{U}}^\imath \to \widetilde{\mathbf{U}}^\imath$ is an injective endomorphism. Let us now prove that this endomorphism $\widetilde{\mathbf{T}}'_{i,-1}$ is surjective. More precisely, we shall show the following.

Claim. For any $z \in \widetilde{\mathbf{U}}^\imath$, set $y := \widetilde{\mathbf{T}}''_{i,+1}(z)$. Then we have $z = \widetilde{\mathbf{T}}'_{i,-1}(y)$.

Let us prove the claim. The identity (6.1) reads in our setting as $y^\imath\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1}) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_i^{-1})\widetilde{\mathscr{T}}_{\mathbf{r}_i}(z)$. Applying $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}$ to both sides of this identity, we obtain $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(y^\imath)\widetilde{\Upsilon}_i^{-1} = \widetilde{\Upsilon}_i^{-1} z$, which can be rewritten as $z\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(y^\imath)$. By (4.7) and the uniqueness in Theorem 4.7(1), we conclude that $z = \widetilde{\mathbf{T}}'_{i,-1}(y)$.

By an entirely similar argument as above (switching the role of $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$) and using the uniqueness in Theorem 6.1(1), we show that, for any $y_1 \in \widetilde{\mathbf{U}}^\imath$, we have $y_1 = \widetilde{\mathbf{T}}''_{i,+1}(z_1)$, where $z_1 := \widetilde{\mathbf{T}}'_{i,-1}(y_1)$.

Hence, $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses. As $\widetilde{\mathbf{T}}'_{i,-1}$ is an endomorphism, we see that both $\widetilde{\mathbf{T}}'_{i,-1}$ and $\widetilde{\mathbf{T}}''_{i,+1}$ are automorphisms of $\widetilde{\mathbf{U}}^\imath$.

Recall the anti-involution $\sigma^\imath$ on $\widetilde{\mathbf{U}}^\imath$ from Proposition 3.12. It remains to prove that $\widetilde{\mathbf{T}}''_{i,+1} = \sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1} \circ \sigma^\imath$. This follows from the identity (6.2), the identity $\widetilde{\mathbf{T}}''_{i,+1}(B_i) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_i)$ from Proposition 6.3, and $\widetilde{\mathbf{T}}''_{i,+1}(B_j) = (\sigma^\imath \circ \widetilde{\mathbf{T}}'_{i,-1})(B_j)$, for $i \neq j \in \mathbb{I}_{\circ,\tau}$; the last identity follows by comparing the rank 2 formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Table 3 and for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ in Table 4. □

In particular, Theorem 6.7 above completes the proof of Theorem 4.7 that $\widetilde{\mathbf{T}}'_{i,-1}$ are automorphisms of $\widetilde{\mathbf{U}}^\imath$. From now on, thanks to Theorem 6.7, we shall denote

$$\widetilde{\mathbf{T}}_i := \widetilde{\mathbf{T}}''_{i,+1}, \qquad \widetilde{\mathbf{T}}_i^{-1} := \widetilde{\mathbf{T}}'_{i,-1}.$$

7 | A BASIC PROPERTY OF NEW SYMMETRIES

In this section, we establish a basic property that $\widetilde{\mathbf{T}}_{\underline{w}}$, for $w \in W^\circ$, sends B_i to B_j, if $w\alpha_i = \alpha_j$; see Theorem 7.13. This is a generalization of a well-known property of braid group action on Chevalley generators in the setting of quantum groups.

We shall first study the rank 2 cases separately, depending on whether $\ell_\circ(\boldsymbol{w}_\circ) = 3, 4$, or 6. Then we deal with the general cases.

7.1 | Rank 2 cases with $\ell_\circ(\boldsymbol{w}_\circ) = 3$

Assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 3$; in this case, according to Table 2, we must have $\tau = \mathrm{Id}$, and hence, we identify $\mathbb{I}_\circ = \{i, j\}$ as well.

Lemma 7.1. *We have* $\widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j}(B_i) = B_j$.

Proof. Noting that $\ell(\mathbf{r}_i\mathbf{r}_j) = \ell(\mathbf{r}_i) + \ell(\mathbf{r}_j)$, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j} = \widetilde{\mathscr{T}}_{\mathbf{r}_i\mathbf{r}_j}$. Noting that $\mathbf{r}_i\mathbf{r}_j(\alpha_i) = \alpha_j$, we have that $\widetilde{\mathscr{T}}_{\mathbf{r}_i\mathbf{r}_j}(X_i) = X_j$, for $X = F, E$ or K'; cf. [28, 39.2] or [16, Proposition 8.20].

Recall $\tau = \mathrm{Id}$, and $B_i = F_i + \widetilde{\mathscr{T}}_{w_\bullet}(E_i)K'_i$. Thanks to (2.14), $\widetilde{\mathscr{T}}_{w_\bullet}$ commutes with both $\widetilde{\mathscr{T}}_{\mathbf{r}_i}$ and $\widetilde{\mathscr{T}}_{\mathbf{r}_j}$. Therefore, we have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j}(B_i) = \widetilde{\mathscr{T}}_{\mathbf{r}_i\mathbf{r}_j}(F_i + \widetilde{\mathscr{T}}_{w_\bullet}(E_i)K'_i) = F_j + \widetilde{\mathscr{T}}_{w_\bullet}(E_j)K'_j = B_j.$$

The lemma is proved. □

Proposition 7.2. *We have* $\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = B_j$*; or equivalently,* $\widetilde{\mathbf{T}}_j\widetilde{\mathbf{T}}_i(B_j) = B_i$.

Proof. Since $\widetilde{\mathbf{T}}_i^{-1}$ and $\widetilde{\mathbf{T}}_j^{-1}$ are automorphism of $\widetilde{\mathbf{U}}^\imath$, we have $\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_j \in \widetilde{\mathbf{U}}^\imath$. Then, we can write this element in terms of monomial basis of $\widetilde{\mathbf{U}}^\imath$ (see Proposition 2.6):

$$\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_j = \sum_{J\in\mathcal{J}} A_J B_J, \qquad \text{for some } A_J \in \widetilde{\mathbf{U}}^+_\bullet\widetilde{\mathbf{U}}^{\imath 0}. \tag{7.1}$$

On the other hand, using the intertwining relation (4.7) twice, we have

$$\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = \widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(B_i)\cdot\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\widetilde{\Upsilon}_i^{-1}$$

By Lemma 7.1, we rewrite the above identity as

$$\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = \widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot B_j\cdot\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\widetilde{\Upsilon}_i^{-1}. \tag{7.2}$$

By the equality (7.2), we rewrite (7.1) in the following form:

$$\widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot B_j\cdot\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\widetilde{\Upsilon}_i^{-1} - B_j = \sum_{J\in\mathcal{J}} A_J B_J. \tag{7.3}$$

Now we claim $A_J B_J = 0$, for each $J \in \mathcal{J}$, by comparing the weights in $\mathbb{Z}\mathbb{I}$. Recall from Remark 3.10 that $\widetilde{\Upsilon}_i = \sum_{m\geqslant 0}\widetilde{\Upsilon}_i^m$ where $\mathrm{wt}(\widetilde{\Upsilon}_i^m) = m(\alpha_i + w_\bullet\alpha_{\tau i})$ and then weights of $\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)$ lie in $\mathbb{N}(\mathbf{r}_i\alpha_j + \mathbf{r}_i w_\bullet\alpha_{\tau j})$. Hence, the weights appearing on LHS (7.3) must belong to the set Q_{ij}, where

$$\begin{aligned} Q_{ij} &= Q_{ij}^- \cup Q_{ij}^+, \\ Q_{ij}^- &:= -\alpha_j + \mathbb{N}(\alpha_i + w_\bullet\alpha_{\tau i}) + \mathbb{N}(\mathbf{r}_i\alpha_j + \mathbf{r}_i w_\bullet\alpha_{\tau j}), \\ Q_{ij}^+ &:= w_\bullet(\alpha_j) + \mathbb{N}(\alpha_i + w_\bullet\alpha_{\tau i}) + \mathbb{N}(\mathbf{r}_i\alpha_j + \mathbf{r}_i w_\bullet\alpha_{\tau j}). \end{aligned}$$

On the other hand, note that the weight of the lowest weight component of $A_J B_J$ lies in $Q_J := -\mathrm{wt}(J) + \mathbb{N}\mathbb{I}_\bullet$. Then $A_J B_J \neq 0$ only if $Q_J \cap Q_{ij} \neq \emptyset$. It immediately follows that $A_J B_J = 0$ unless $\mathrm{wt}(J) \in \alpha_j + \mathbb{N}\mathbb{I}_\bullet$. Moreover, when $\mathrm{wt}(J) \in \alpha_j + \mathbb{N}\mathbb{I}_\bullet$, the only possible element in the intersection $Q_J \cap Q_{ij}$ is $-\alpha_j$.

However, since $\widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)$ has constant term 1, the weight $(-\alpha_j)$ component for LHS (7.3) is 0. This implies that $A_J B_J = 0$, for each $J \in \mathcal{J}$, and then the desired identity follows by (7.1). □

Corollary 7.3. *We have*

$$\widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)B_j = B_j\widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j). \tag{7.4}$$

Proof. One reads off from the proof of Proposition 7.2 that $A_J B_J = 0$, for $J \in \mathcal{J}$, and hence, the corollary follows from the relation (7.3). □

Corollary 7.4. *We have*

$$B_i\widetilde{\Upsilon}_j\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)B_i, \tag{7.5}$$

$$B_j^\sigma\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_i = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_i B_j^\sigma. \tag{7.6}$$

Proof. Switching i, j in (7.4), we obtain

$$\widetilde{\Upsilon}_j \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i) B_i = B_i \widetilde{\Upsilon}_j \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i). \tag{7.7}$$

By Proposition 8.3, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)$. Hence, (7.7) implies the desired identity (7.5).

Recall from Proposition 3.8 that $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$ are both fixed by the anti-involution σ. Recall also that $\sigma \widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1} \sigma = \widetilde{\mathscr{T}}_{\mathbf{r}_i}$. Applying the anti-involution σ to the identity (7.4), we have proved (7.6). □

7.2 | Rank 2 cases with $\ell_\circ(\boldsymbol{w}_\circ) = 4$

In this subsection, we assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 4$. Let $\{i, \tau i\}$ and $\{j, \tau j\}$ be the corresponding two distinct τ-orbits of $\mathbb{I}_\circ$.

Lemma 7.5. *Denote the diagram involution $\varrho := \tau_0 \tau_{\bullet,i}$. Then we have*

$$\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_i) = \alpha_{\varrho i}, \quad \text{and} \quad \widetilde{\mathscr{T}}_{\mathbf{r}_j} \widetilde{\mathscr{T}}_{\mathbf{r}_i} \widetilde{\mathscr{T}}_{\mathbf{r}_j}(B_i) = B_{\varrho i}.$$

(Moreover, a nontrivial ϱ can occur only in type AIII, and in this case, $\varrho = \tau$.)

Proof. As before, set w_0 to be the longest element of the Weyl group W and $w_{\bullet,i} = \mathbf{r}_i w_\bullet$; set τ_0 and $\tau_{\bullet,i}$ to be the diagram automorphisms corresponding to w_0 and $w_{\bullet,i}$, respectively. In this case, w_0 satisfies the relation $w_0 = \boldsymbol{w}_\circ w_\bullet = \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i w_\bullet = \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j w_{\bullet,i}$. Then we have

$$\tau_0(\alpha_i) = -w_0(\alpha_i) = -\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j w_{\bullet,i}(\alpha_i) = \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j \tau_{\bullet,i}(\alpha_i).$$

Setting $\varrho := \tau_0 \tau_{\bullet,i}$, we have obtained $\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_i) = \alpha_{\varrho i}$. (We thank Stefan Kolb for providing the above conceptual argument that replaces our earlier case-by-case proof of the existence of ϱ; moreover, his argument produces a precise formula for ϱ.)

In particular, we observe that a nontrivial ϱ occurs only in type AIII (for some particular i), and in this case, $\varrho = \tau$.

Recalling $\mathbf{r}_i = \mathbf{r}_{\tau i}$, we also have $\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_{\tau i}) = \alpha_{\varrho \tau i}$.

We have $\ell(\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j) = \ell(\mathbf{r}_j) + \ell(\mathbf{r}_i) + \ell(\mathbf{r}_j)$, by Proposition 2.4. Therefore, it follows from $\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j(\alpha_i) = \alpha_{\varrho i}$ that $\widetilde{\mathscr{T}}_{\mathbf{r}_j} \widetilde{\mathscr{T}}_{\mathbf{r}_i} \widetilde{\mathscr{T}}_{\mathbf{r}_j}(X_i) = X_{\varrho i}$, for $X = F, K'$; cf. [28, 39.2] or [16, Proposition 8.20]. Similarly, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_j} \widetilde{\mathscr{T}}_{\mathbf{r}_i} \widetilde{\mathscr{T}}_{\mathbf{r}_j}(E_{\tau i}) = E_{\varrho \tau i}$.

Recall $B_i = F_i + \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau i}) K_i'$. Thanks to (2.14), $\widetilde{\mathscr{T}}_{w_\bullet}$ commutes with both $\widetilde{\mathscr{T}}_{\mathbf{r}_i}$ and $\widetilde{\mathscr{T}}_{\mathbf{r}_j}$. Therefore, we have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_j} \widetilde{\mathscr{T}}_{\mathbf{r}_i} \widetilde{\mathscr{T}}_{\mathbf{r}_j}(B_i) = \widetilde{\mathscr{T}}_{\mathbf{r}_j \mathbf{r}_i \mathbf{r}_j}(F_i + \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau i}) K_i') = F_{\varrho i} + \widetilde{\mathscr{T}}_{w_\bullet}(E_{\varrho \tau i}) K_{\varrho i}' = B_{\varrho i}.$$

The lemma is proved. □

Proposition 7.6. *Retain the notation in Lemma 7.5. Then $\widetilde{\mathbf{T}}_j^{-1} \widetilde{\mathbf{T}}_i^{-1} \widetilde{\mathbf{T}}_j^{-1}(B_i) = B_{\varrho i}$; or equivalently, $\widetilde{\mathbf{T}}_j \widetilde{\mathbf{T}}_i \widetilde{\mathbf{T}}_j(B_i) = B_{\varrho i}$.*

Proof. Since $\widetilde{\mathbf{T}}_i^{-1}$ and $\widetilde{\mathbf{T}}_j^{-1}$ are automorphism of $\widetilde{\mathbf{U}}^\imath$, we have $\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_{\varrho i} \in \widetilde{\mathbf{U}}^\imath$. Then we can write this element in terms of monomial basis of $\widetilde{\mathbf{U}}^\imath$ (see Proposition 2.6):

$$\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) - B_{\varrho i} = \sum_{J\in\mathcal{J}} A_J B_J, \qquad \text{for some } A_J \in \widetilde{\mathbf{U}}^+_\bullet \widetilde{\mathbf{U}}^{\imath 0}. \tag{7.8}$$

On the other hand, using the intertwining relation (4.7) of $\widetilde{\mathbf{T}}_i^{-1}$, we have

$$\begin{aligned}&\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i)\\ &= \widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(B_i)\cdot \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i^{-1})\, \widetilde{\Upsilon}_j^{-1}.\end{aligned}$$

Since $\widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(B_i) = B_{\varrho i}$ by Lemma 7.5, we rewrite the above identity as

$$\widetilde{\mathbf{T}}_j^{-1}\widetilde{\mathbf{T}}_i^{-1}\widetilde{\mathbf{T}}_j^{-1}(B_i) = \widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot B_{\varrho i}\cdot \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i^{-1})\, \widetilde{\Upsilon}_j^{-1}. \tag{7.9}$$

By the identity (7.9), we rewrite (7.8) in the following form:

$$\widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)\cdot B_{\varrho i}\cdot \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j^{-1})\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i^{-1})\, \widetilde{\Upsilon}_j^{-1} - B_{\varrho i} = \sum_{J\in\mathcal{J}} A_J B_J. \tag{7.10}$$

By a weight argument entirely similar to the proof of Proposition 7.2, we obtain $\sum_{J\in\mathcal{J}} A_J B_J = 0$. Thus, the proposition follows by (7.8). □

Corollary 7.7. *We have*

$$B_{\varrho i}\widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)B_{\varrho i}. \tag{7.11}$$

Proof. Since $\sum_{J\in\mathcal{J}} A_J B_J = 0$, as shown in the proof of Proposition 7.6, the corollary follows from the relation (7.10). □

Corollary 7.8. *We have*

$$B_i\widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)B_i, \tag{7.12}$$

$$B_j^\sigma\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)\, \widetilde{\Upsilon}_i = \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)\, \widetilde{\Upsilon}_i B_j^\sigma. \tag{7.13}$$

Proof. We prove (7.12). Noting that ϱ equals either Id or τ, we have by Remark 4.8 that ϱ commutes with $\widetilde{\mathscr{T}}_{\mathbf{r}_i}, \widetilde{\mathscr{T}}_{\mathbf{r}_j}$, and by Proposition 3.8 that ϱ fixes $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$. Hence, applying ϱ to both sides of (7.11), we have

$$B_i\widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j)B_i. \tag{7.14}$$

By Proposition 8.3, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(\widetilde{\Upsilon}_j) = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j)$. Hence, the desired relation (7.12) follows by (7.14).

We next show (7.13). Recall from Proposition 3.8 that $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$ are both fixed by the anti-involution σ. Recall also that $\sigma\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\sigma = \widetilde{\mathscr{T}}_{\mathbf{r}_i}$. Switching i, j in (7.12) and then applying σ to it, we obtain (7.13). □

7.3 | Rank 2 case with $\ell_\circ(\boldsymbol{w}_\circ) = 6$

The rank 2 case with $\ell_\circ(\boldsymbol{w}_\circ) = 6$ occurs only in split G_2 type. Let $(\mathbb{I} = \mathbb{I}_\circ, \mathrm{Id})$ be a Satake diagram of split type G_2. In this case, the relative Weyl group W° is identified with W and $\mathbf{r}_a = s_a$ for $a \in \mathbb{I} = \mathbb{I}_\circ = \{i, j\}$. We do not specify which root i or j is long.

Set $\underline{w_i} = s_j s_i s_j s_i s_j$ and $\widetilde{\mathscr{T}}_{\underline{w_i}} = \widetilde{\mathscr{T}}_j \widetilde{\mathscr{T}}_i \widetilde{\mathscr{T}}_j \widetilde{\mathscr{T}}_i \widetilde{\mathscr{T}}_j$. Then we have $\underline{w_i}(\alpha_i) = \alpha_i$.

Lemma 7.9. *We have* $\widetilde{\mathscr{T}}_{\underline{w_i}}^{-1}(B_i) = B_i$.

Proof. Follows by [28, 39.2] and the same type of arguments as for Lemmas 7.1 and 7.5. □

Proposition 7.10. *We have* $\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) = B_i$*; or equivalently,* $\widetilde{\mathbf{T}}_{\underline{w_i}}(B_i) = B_i$.

Proof. Since $\widetilde{\mathbf{T}}_i^{-1}$ and $\widetilde{\mathbf{T}}_j^{-1}$ are automorphism of $\widetilde{\mathbf{U}}^\imath$, we have $\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) - B_i \in \widetilde{\mathbf{U}}^\imath$. Then we can write this element in terms of monomial basis of $\widetilde{\mathbf{U}}^\imath$ (see Proposition 2.6):

$$\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) - B_i = \sum_{J \in \mathcal{J}} A_J B_J, \qquad \text{for some } A_J \in \widetilde{\mathbf{U}}^+_\bullet \widetilde{\mathbf{U}}^{\imath 0}. \tag{7.15}$$

On the other hand, using the intertwining relation (4.7) of $\widetilde{\mathbf{T}}_i^{-1}$, we have

$$\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) = \Omega_i \widetilde{\mathscr{T}}_{\underline{w_i}}^{-1}(B_i) \Omega_i^{-1}, \tag{7.16}$$

where

$$\Omega_i = \widetilde{\Upsilon}_j\, \widetilde{\mathscr{T}}_j^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_j^{-1} \widetilde{\mathscr{T}}_i^{-1}(\widetilde{\Upsilon}_j)\, \widetilde{\mathscr{T}}_j^{-1} \widetilde{\mathscr{T}}_i^{-1} \widetilde{\mathscr{T}}_j^{-1}(\widetilde{\Upsilon}_i)\, \widetilde{\mathscr{T}}_j^{-1} \widetilde{\mathscr{T}}_i^{-1} \widetilde{\mathscr{T}}_j^{-1} \widetilde{\mathscr{T}}_i^{-1}(\widetilde{\Upsilon}_j). \tag{7.17}$$

By Lemma 7.9, we rewrite the identity (7.16) as

$$\widetilde{\mathbf{T}}_{\underline{w_i}}^{-1}(B_i) = \Omega_i B_i \Omega_i^{-1}. \tag{7.18}$$

By the identity (7.18), we rewrite (7.15) in the following form:

$$\Omega_i B_i \Omega_i^{-1} - B_i = \sum_{J \in \mathcal{J}} A_J B_J. \tag{7.19}$$

By a weight argument entirely similar to the proof of Proposition 7.2, we obtain $\sum_{J \in \mathcal{J}} A_J B_J = 0$. Thus, the proposition follows by (7.15). □

Corollary 7.11. *Let Ω_i be as in (7.17). We have*

$$B_i \Omega_i = \Omega_i B_i. \tag{7.20}$$

Proof. Since $\sum_{J \in \mathcal{J}} A_J B_J = 0$, as shown in the proof of Proposition 7.10, the corollary follows from the formula (7.19). □

Corollary 7.12. *We have the following intertwining relations:*

$$B_i\widetilde{\Upsilon}_j\widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_i(\widetilde{\Upsilon}_j)$$
$$= \widetilde{\Upsilon}_j\widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_i(\widetilde{\Upsilon}_j)B_i, \tag{7.21}$$

$$B_j^\sigma\widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_i(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_i$$
$$= \widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_i(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_iB_j^\sigma. \tag{7.22}$$

Proof. By Proposition 8.3, we have $\widetilde{\mathscr{T}}_{s_js_is_js_is_j}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$ and $\widetilde{\mathscr{T}}_{s_is_js_is_js_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j$. Then we have

$$\Omega_i = \widetilde{\Upsilon}_j\widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_i(\widetilde{\Upsilon}_j).$$

Hence, the desired identity (7.21) follows by (7.20).

We next prove (7.22). Switching i, j in (7.20), we have

$$B_j\Omega_j = \Omega_jB_j, \tag{7.23}$$

where Ω_j is defined by switching i, j in (7.17).

Recall from Proposition 3.8 that $\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_j$ are both fixed by σ. Then by the definition of Ω_j, we have

$$\sigma(\Omega_j) = \widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j)\widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i)\widetilde{\mathscr{T}}_i(\widetilde{\Upsilon}_j)\widetilde{\Upsilon}_i.$$

Hence, applying σ to (7.23) and then using this formula of $\sigma(\Omega_j)$, we obtain (7.22). □

7.4 | The general identity $\widetilde{\mathbf{T}}_w(B_i) = B_{wi}$

Let $w \in W^\circ$. Given a reduced expression $\underline{w} = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\dots\mathbf{r}_{i_k}$ for w, we shall denote $\widetilde{\mathbf{T}}_{\underline{w}} = \widetilde{\mathbf{T}}_{i_1}\widetilde{\mathbf{T}}_{i_2}\dots\widetilde{\mathbf{T}}_{i_k}$.

Theorem 7.13. *Suppose that $wi \in \mathbb{I}_\circ$, for $w \in W^\circ$ and $i \in \mathbb{I}_\circ$. Then $\widetilde{\mathbf{T}}_{\underline{w}}(B_i) = B_{wi}$, for* some *reduced expression $\underline{w}$ of w.*

(Once Theorem 9.1 on braid relation for $\widetilde{\mathbf{T}}_i$ is proved, we can replace $\widetilde{\mathbf{T}}_{\underline{w}}$ in Theorem 7.13 by $\widetilde{\mathbf{T}}_w$, which depends only on w, not on a reduced expression $\underline{w}$ of w.)

Proof. The strategy of the proof is modified from a well-known quantum group counterpart, cf. [16, Lemma 8.20]. We shall reduce the proof to the rank 2 cases which were established earlier and finish the proof by induction on $\ell_\circ(w)$.

The statement holds for arbitrary rank 2 Satake (sub) diagrams $(\mathbb{I}_\bullet \cup \{i, \tau i, j, \tau j\}, \tau)$. Indeed, in case when $\ell(\boldsymbol{w}_\circ) = 2$, the claim is trivial. In case when $\ell(\boldsymbol{w}_\circ) = 3, 4$ or 6, the claim has been established in Propositions 7.2, 7.6, and 7.10, respectively.

In general, we use an induction on $l_\circ(w)$, for $w \in W^\circ$, where $l_\circ$ is the length function for the relative Weyl group W°. Recall the simple system $\{\overline{\alpha}_i | i \in \mathbb{I}_{\circ,\tau}\}$ for the relative root system from

(2.16). Since $w\theta = \theta w$ and $wi \in \mathbb{I}_\circ$ by assumption, we have $w(\overline{\alpha}_i) = \overline{\alpha}_{wi}$. We denote a positive (and negative) root in the relative root system by $\beta > 0$ (and respectively, $\beta < 0$).

Suppose that $l_\circ(w) > 0$. Then there exists $j \in \mathbb{I}_{\circ,\tau}$ such that $w(\overline{\alpha}_j) < 0$; clearly $j \neq i$ since $w(\overline{\alpha}_i) > 0$. Consider the minimal length representatives of W° with respect to the rank 2 parabolic subgroup $\langle \mathbf{r}_i, \mathbf{r}_j \rangle$. We have a decomposition $w = w'w''$ in W° such that $w'(\overline{\alpha}_i) > 0, w'(\overline{\alpha}_j) > 0$ and w'' lies in the subgroup $\langle \mathbf{r}_i, \mathbf{r}_j \rangle$; moreover, $l_\circ(w) = l_\circ(w') + l_\circ(w'')$. Now $w(\overline{\alpha}_i) > 0$ and $w(\overline{\alpha}_j) < 0$ implies that $w''(\overline{\alpha}_i) > 0$ and $w''(\overline{\alpha}_j) < 0$ (since w' preserves the signs of the roots $w''(\overline{\alpha}_i)$ and $w''(\overline{\alpha}_j)$). It follows that

$$w''(\alpha_i) > 0, \qquad w''(\alpha_j) < 0, \qquad w'(\alpha_i) > 0, \qquad w'(\alpha_j) > 0.$$

(The positive system of the restricted root system is compatible with the positive system of $\mathcal{R}$.) Moreover, since $\mathbf{r}_s$, for any $s \in \mathbb{I}_\circ$, acts on $\mathbb{I}_\bullet$ as the involution $\tau_{\bullet,s}\tau$, we must have $w'(\alpha_a) > 0$, for any $a \in \mathbb{I}_\bullet$; see also Proposition 4.11.

We show that $w''i \in \mathbb{I}_\circ$. Since $w''(\alpha_i) > 0$ and $w''(\alpha_i) \in \mathcal{R} \cap (\mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j + \mathbb{Z}\mathbb{I}_\bullet)$, we can write $w''(\alpha_i) \in \mathcal{R}$ in the following form:

$$w''(\alpha_i) = r\alpha_i + s\alpha_j + \alpha_\bullet$$

for some $r, s \geqslant 0, \alpha_\bullet \in \mathbb{N}\mathbb{I}_\bullet$. We consider the following cases.

(1) At least two of $r, s, \alpha_\bullet$ are nonzero. Then $w'w''(\alpha_i) = rw'(\alpha_i) + sw'(\alpha_j) + w'(\alpha_\bullet)$ cannot be simple for $w'(\alpha_i) > 0, w'(\alpha_j) > 0, w'(\alpha_\bullet) > 0$; this contradicts that $w(\alpha_i) = w'w''(\alpha_i)$ is simple.
(2) $r = 0, \alpha_\bullet = 0$, and $s > 0$. Then $s = 1$ and $w''(\alpha_i) = \alpha_j$ is simple. A similar argument applying to the case $s = 0, \alpha_\bullet = 0$ and $r > 0$ shows that $w''(\alpha_i) = \alpha_i$ is simple.
(3) $r = s = 0, \alpha_\bullet \neq 0$. We show that this case cannot occur. Indeed, we have $\theta w''(\alpha_i) = \theta(\alpha_\bullet) = \alpha_\bullet = w''(\alpha_i)$. Since $w''\theta = \theta w''$, the above identity implies that α_i is fixed by θ, which is impossible for $i \in \mathbb{I}_\circ$.

Therefore, we have shown $w''i \in \mathbb{I}_\circ$ and $w''(\alpha_i) = \alpha_{w''i}$. By the rank 2 results in Propositions 7.2 and 7.6, we have $\widetilde{\mathbf{T}}_{\underline{w''}}(B_i) = B_{w''i}$, for any reduced expression $\underline{w''}$ of w''. Now using the induction hypothesis, there exists a reduced expression $\underline{w'}$ such that $\underline{w} = \underline{w'} \cdot \underline{w''}$ is a reduced expression for w and

$$\widetilde{\mathbf{T}}_{\underline{w}}(B_i) = \widetilde{\mathbf{T}}_{\underline{w'}}\widetilde{\mathbf{T}}_{\underline{w''}}(B_i) = \widetilde{\mathbf{T}}_{\underline{w'}}(B_{w''i}) = B_{wi}.$$

The theorem is proved. □

8 | FACTORIZATION OF QUASI K-MATRICES

It is conjectured by Dobson and Kolb [15] that quasi K-matrices admit factorization into products of rank 1 quasi K-matrices analogous to the factorization properties of quasi R-matrices. They showed that the factorization of quasi K-matrices for arbitrary finite types reduces to the rank 2 cases. In this section, using (the rank 2 cases of) Theorem 7.13, we provide a uniform proof of the factorization of quasi K-matrices for all rank 2 Satake diagrams, hence completing the proof of Dobson–Kolb conjecture in all finite types.

8.1 | Factorization of $\widetilde{\Upsilon}$

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a Satake diagram of arbitrary finite type. Let w be any element in the relative Weyl group W° with a reduced expression

$$\underline{w} = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_m};$$

here $m = \ell_\circ(w)$, the length of $w \in W^\circ$ (not to be confused as the length $\ell(w)$ in W).

Following [15] (who worked in the setting of $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$), we define, for $1 \leqslant k \leqslant m$,

$$\begin{aligned}\widetilde{\Upsilon}^{[k]} &= \widetilde{\mathscr{T}}_{\mathbf{r}_{i_1}}\widetilde{\mathscr{T}}_{\mathbf{r}_{i_2}}\cdots\widetilde{\mathscr{T}}_{\mathbf{r}_{i_{k-1}}}(\widetilde{\Upsilon}_{i_k}),\\ \widetilde{\Upsilon}_{\underline{w}} &= \widetilde{\Upsilon}^{[m]}\widetilde{\Upsilon}^{[m-1]}\cdots\widetilde{\Upsilon}^{[1]}.\end{aligned} \tag{8.1}$$

(In the notation $\widetilde{\Upsilon}^{[k]}$ above, we have suppressed the dependence on $\underline{w}$.)

The goal of this section is to establish Theorem 8.1, which is a $\widetilde{\mathbf{U}}^\imath$-variant of (and implies) [15, Conjecture 3.22] for $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ with general parameters $\boldsymbol{\varsigma}$. The restriction on parameters $\boldsymbol{\varsigma}$ in [15] can be removed in light of the development in [2, 21], which allows more general parameters in quasi K-matrices. Recall that $\boldsymbol{w}_\circ$ is the longest element in the relative Weyl group W°.

Theorem 8.1.

(1) *For any $w \in W^\circ$, the partial quasi K-matrix $\widetilde{\Upsilon}_{\underline{w}}$ is independent of the choice of reduced expressions of w (and hence can be denoted by $\widetilde{\Upsilon}_w$).*
(2) *The quasi K-matrix $\widetilde{\Upsilon}$ for $\widetilde{\mathbf{U}}^\imath$ of any finite type admits a factorization $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$.*

8.2 | Reduction to rank 2

Let us recall some partial results from [15] in this direction (which can be adapted from $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ to $\widetilde{\mathbf{U}}^\imath$ without difficulties).

Theorem 8.2 [15, Theorems 3.17 and 3.20]. *Theorem 8.1 holds for $\widetilde{\mathbf{U}}^\imath$ of a given finite type if it holds for all its rank 2 Satake subdiagrams.*

The arguments for Theorem 8.2 are largely formal once the following crucial result (see [15, Proposition 3.18]) is in place. We provide a short new proof below. Recall that $\boldsymbol{w}_\circ$ is the longest element in W°. Recall also the diagram involution τ_0 such that $w_0(\alpha_i) = -\alpha_{\tau_0\alpha_i}$, for all i, where w_0 is the longest element in W.

Proposition 8.3 [15, Proposition 3.18]. *Let $\boldsymbol{w}_\circ = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_m}$ be a reduced expression of $\boldsymbol{w}_\circ$. Then we have $\widetilde{\mathscr{T}}_{\mathbf{r}_{i_1}}\widetilde{\mathscr{T}}_{\mathbf{r}_{i_2}}\cdots\widetilde{\mathscr{T}}_{\mathbf{r}_{i_{m-1}}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\Upsilon}_{\tau_0 i_m}$.*

Proof. We have $w_0 = \boldsymbol{w}_\circ w_\bullet$, and hence, $\widetilde{\mathscr{T}}_{w_0} = \widetilde{\mathscr{T}}_{\boldsymbol{w}_\circ}\widetilde{\mathscr{T}}_{w_\bullet}$. It follows by Lemma 4.4 that $\widetilde{\mathscr{T}}_{w_0}^{-1}\widehat{\tau}_0 = \widetilde{\mathscr{T}}_{w_{\bullet,i_m}}^{-1}\widehat{\tau}_{\bullet,i_m}$ when acting on $\widetilde{\mathbf{U}}_{\mathbb{I}_{\bullet,i_m}}$. Thus,

$$\widetilde{\mathscr{T}}_{w_0}^{-1}\widehat{\tau}_0(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}_{w_{\bullet,i_m}}^{-1}\widehat{\tau}_{\bullet,i_m}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}_{w_{\bullet,i_m}}^{-1}(\widetilde{\Upsilon}_{i_m}),$$

since the quasi K-matrix $\widetilde{\Upsilon}_{i_m}$ lies in a completion of $\widetilde{\mathbf{U}}^+_{\mathbb{I}_\bullet, i_m}$ and $\widehat{\tau}_{\bullet, i_m}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\Upsilon}_{i_m}$ (see Proposition 3.8). Then, we obtain

$$\widetilde{\mathscr{T}}^{-1}_{w_0}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathscr{T}}^{-1}_{w_0}\widehat{\tau}_0(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}^{-1}_{w_{\bullet, i_m}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_{i_m}}\widetilde{\mathscr{T}}^{-1}_{w_\bullet}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_{i_m}}(\widetilde{\Upsilon}_{i_m}),$$

where the last equality follows by Proposition 4.6. By Proposition 4.6 again, we have

$$\widetilde{\mathscr{T}}^{-1}_{\boldsymbol{w}_\circ}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathscr{T}}^{-1}_{w_0}\widetilde{\mathscr{T}}_{w_\bullet}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathscr{T}}^{-1}_{w_0}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_{i_m}}(\widetilde{\Upsilon}_{i_m}).$$

Hence, $\widetilde{\mathscr{T}}_{\mathbf{r}_{i_1}}\widetilde{\mathscr{T}}_{\mathbf{r}_{i_2}}\cdots\widetilde{\mathscr{T}}_{\mathbf{r}_{i_{m-1}}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}_{\boldsymbol{w}_\circ}\widetilde{\mathscr{T}}^{-1}_{\mathbf{r}_{i_m}}(\widetilde{\Upsilon}_{i_m}) = \widetilde{\mathscr{T}}_{\boldsymbol{w}_\circ}\widetilde{\mathscr{T}}^{-1}_{\boldsymbol{w}_\circ}(\widetilde{\Upsilon}_{\tau_0 i_m}) = \widetilde{\Upsilon}_{\tau_0 i_m}$. □

Remark 8.4. It was verified in [15] that Theorem 8.1 holds in all type A rank 2 and all split rank 2 cases. The long computational proof therein is carried out case-by-case based on several explicit rank 1 formulas which they also computed.

We note that in the rank 2 setting, the first statement in Theorem 8.1 is nontrivial only when $w = \boldsymbol{w}_\circ$, the longest element in W°. Hence, in the remainder of this section, to prove Theorem 8.1, we can and shall assume that

$$(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau) \text{ is any } \textit{rank 2} \text{ Satake diagram of finite type, and } w = \boldsymbol{w}_\circ.$$

Moreover, we denote $\mathbb{I}_\circ = \{i, \tau i, j, \tau j\}$.

Let $\boldsymbol{w}_\circ = \mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_m}$ be a reduced expression. Theorem 8.1 in the case for $\ell_\circ(\boldsymbol{w}_\circ) = 2$, that is, $\boldsymbol{w}_\circ = \mathbf{r}_i\mathbf{r}_j = \mathbf{r}_j\mathbf{r}_i$, trivially holds. The next proposition reduces the proof of Theorem 8.1 in the remaining nontrivial cases into verifying its assumption.

Proposition 8.5. *Assume that* $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B^\sigma_p$*, for* $p = i, j$*. Then, we have* $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$*, for any reduced expression of* $\boldsymbol{w}_\circ$*.*

Proof. The identity $x\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}x$, for $x \in \widetilde{\mathbf{U}}^{\imath 0}\widetilde{\mathbf{U}}_\bullet$, holds by (3.4), Proposition 4.11, and (8.1). Together with the assumption that $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B^\sigma_p$ $(p = i, j)$, we conclude that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the same intertwining relations in Theorem 3.6 as for $\widetilde{\Upsilon}$. Note also that clearly, we have the constant term $(\widetilde{\Upsilon}_{\boldsymbol{w}_\circ})^0 = 1$. Therefore, the desired identity $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ follows by the uniqueness in Theorem 3.6. □

8.3 | Factorizations in rank 2

The verification that $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B^\sigma_p$ in the three cases $\ell_\circ(\boldsymbol{w}_\circ) = 3, 4$, or 6, is based on the same idea, though the notations are a little different. In the subsections below, we shall consider the three cases separately.

8.3.1 | Factorization for $\ell_\circ(w_\circ) = 3$

In this subsection, we deal with the rank 2 cases for $\ell_\circ(\boldsymbol{w}_\circ) = 3$, with the help of Proposition 7.2 and Corollary 7.4.

Assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 3$; in this case, only $\tau = \mathrm{Id}$ and hence we identify $\mathbb{I}_\circ = \{i, j\}$ as well. The longest element $\boldsymbol{w}_\circ$ of the relative Weyl group has a reduced expression

$$\boldsymbol{w}_\circ = \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i. \tag{8.2}$$

By definition (8.1) of $\widetilde{\Upsilon}^{[k]}$ and $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, we have

$$\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}, \tag{8.3}$$

where by Proposition 8.3, $\widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_j$ and $\widetilde{\mathscr{T}}_{\mathbf{r}_j}\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_i$, and hence,

$$\widetilde{\Upsilon}^{[3]} = \widetilde{\Upsilon}_j, \qquad \widetilde{\Upsilon}^{[2]} = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j), \qquad \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}_i. \tag{8.4}$$

By Corollary 7.4, we have

$$B_i\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}B_i, \tag{8.5}$$

$$B_j^\sigma\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_j^\sigma. \tag{8.6}$$

It follows by Theorem 3.6 that, for $p = i, j$,

$$B_p\widetilde{\Upsilon}_p = \widetilde{\Upsilon}_p B_p^\sigma. \tag{8.7}$$

Now we show that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the following intertwining relations:

$$B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma, \qquad (p = i, j).$$

Indeed, $B_i\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = B_i\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}B_i\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_i^\sigma$, by (8.3), (8.5), and (8.7). Also, $B_j\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = B_j\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]}B_j^\sigma\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_j^\sigma$, by (8.4), (8.7), and (8.6).

It follows by Proposition 8.5 (whose assumption is verified above), we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$. Using the other reduced expression for $\boldsymbol{w}_\circ$ amounts to switching notations i, j above. Hence, $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ is independent of the choice of a reduced expression for $\boldsymbol{w}_\circ$.

8.3.2 | Factorization for $\ell_\circ(w_\circ) = 4$

In this subsection, we deal with the rank 2 cases for $\ell_\circ(\boldsymbol{w}_\circ) = 4$, with the help of Proposition 7.6 and Corollary 7.8.

Assume that $\mathbb{I}_{\circ,\tau} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 4$. Let $\{i, \tau i\}$ and $\{j, \tau j\}$ be the corresponding two distinct τ-orbits of $\mathbb{I}_\circ$. The longest element $\boldsymbol{w}_\circ$ of the relative Weyl group has a reduced expression

$$\boldsymbol{w}_\circ = \mathbf{r}_i \mathbf{r}_j \mathbf{r}_i \mathbf{r}_j. \tag{8.8}$$

By definition (8.1) of $\widetilde{\Upsilon}^{[k]}$ and $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, we have

$$\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}, \tag{8.9}$$

where by Proposition 8.3, $\widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j}\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j$ and $\widetilde{\mathscr{T}}_{\mathbf{r}_j}\widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j}(\widetilde{\Upsilon}_i) = \widetilde{\Upsilon}_i$, and hence,

$$\widetilde{\Upsilon}^{[4]} = \widetilde{\Upsilon}_j, \quad \widetilde{\Upsilon}^{[3]} = \widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_{\mathbf{r}_j}(\widetilde{\Upsilon}_i) = \widetilde{\mathscr{T}}_{\mathbf{r}_j}^{-1}(\widetilde{\Upsilon}_i), \quad \widetilde{\Upsilon}^{[2]} = \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\Upsilon}_j), \quad \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}_i. \tag{8.10}$$

By Corollary 7.8, we have

$$B_i\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]} = \widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}B_i, \tag{8.11}$$

$$B_j^\sigma\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_j^\sigma. \tag{8.12}$$

Just as in §8.3.1, using the identities (8.11)–(8.12), we can show that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the following intertwining relations $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma$, for $p = i, j$. It follows by Proposition 8.5 (whose assumption is verified above), we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, which is independent of the choice of a reduced expression for $\boldsymbol{w}_\circ$.

8.3.3 | Factorization for $\ell_\circ(w_\circ) = 6$

The case for $\ell_\circ(\boldsymbol{w}_\circ) = 6$ occurs only in split G_2 type. We shall prove this using Proposition 7.10 and Corollary 7.12.

Let $(\mathbb{I} = \mathbb{I}_\circ, \tau = \mathrm{Id})$ be the Satake diagram of split type G_2. In this case, $W^\circ = W$ and $\mathbf{r}_a = s_a$. Assume that $\mathbb{I} = \{i, j\}$ such that $\ell_\circ(\boldsymbol{w}_\circ) = 6$. The longest element $\boldsymbol{w}_\circ$ of the relative Weyl group has a reduced expression

$$\boldsymbol{w}_\circ = s_is_js_is_js_is_j. \tag{8.13}$$

By definition (8.1) of $\widetilde{\Upsilon}^{[k]}$ and $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, we have

$$\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}^{[6]}\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}. \tag{8.14}$$

where by Proposition 8.3, $\widetilde{\mathscr{T}}_{s_is_js_is_js_i}(\widetilde{\Upsilon}_j) = \widetilde{\Upsilon}_j$, and hence,

$$\begin{aligned} &\widetilde{\Upsilon}^{[6]} = \widetilde{\Upsilon}_j, \quad \widetilde{\Upsilon}^{[5]} = \widetilde{\mathscr{T}}_{s_is_js_is_j}(\widetilde{\Upsilon}_i), \quad \widetilde{\Upsilon}^{[4]} = \widetilde{\mathscr{T}}_{s_is_js_i}(\widetilde{\Upsilon}_j), \\ &\widetilde{\Upsilon}^{[3]} = \widetilde{\mathscr{T}}_{s_is_j}(\widetilde{\Upsilon}_i), \quad \widetilde{\Upsilon}^{[2]} = \widetilde{\mathscr{T}}_{s_i}(\widetilde{\Upsilon}_j), \quad \widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}_i. \end{aligned} \tag{8.15}$$

By Corollary 7.12, we have

$$B_i\widetilde{\Upsilon}^{[6]}\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]} = \widetilde{\Upsilon}^{[6]}\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}B_i, \tag{8.16}$$

$$B_j^\sigma\widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]} = \widetilde{\Upsilon}^{[5]}\widetilde{\Upsilon}^{[4]}\widetilde{\Upsilon}^{[3]}\widetilde{\Upsilon}^{[2]}\widetilde{\Upsilon}^{[1]}B_j^\sigma. \tag{8.17}$$

Just as in §8.3.1, using the identities (8.16)–(8.17), we can show that $\widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$ satisfies the following intertwining relations $B_p\widetilde{\Upsilon}_{\boldsymbol{w}_\circ} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}B_p^\sigma$, for $p = i, j$. It follows by Proposition 8.5 (whose assumption is verified above), we have $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{\boldsymbol{w}_\circ}$, which is independent of the choice of a reduced expression for $\boldsymbol{w}_\circ$.

Remark 8.6. A different and more computational proof of the factorization of the quasi K-matrix in split type G_2 was given earlier in Dobson's thesis [13].

9 | RELATIVE BRAID GROUP ACTIONS ON ıQUANTUM GROUPS

In this section, we show that $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$, where $e = \pm 1$ and $i \in \mathbb{I}_{\circ,\tau}$, satisfy the relative braid group relations in $\mathrm{Br}(W^\circ)$. An action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ is then established. Moreover, we show that, by central reductions and isomorphisms among ıquantum groups with different parameters, the symmetries $\widetilde{\mathbf{T}}'_{i,e}, \widetilde{\mathbf{T}}''_{i,e}$ on $\widetilde{\mathbf{U}}^\imath$ descend to $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on the ıquantum groups $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, inducing relative braid group actions on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, for an arbitrary parameter $\boldsymbol{\varsigma}$.

9.1 | Braid group relations among $\widetilde{\mathbf{T}}_i$

For $i \neq j \in \mathbb{I}_{\circ,\tau}$, let m_{ij} be the order of $\mathbf{r}_i\mathbf{r}_j$ in W°, with $m_{ij} \in \{2,3,4,6\}$. Then the following braid relation is satisfied in $\mathrm{Br}(W^\circ)$:

$$\underbrace{\mathbf{r}_i\mathbf{r}_j\mathbf{r}_i\cdots}_{m_{ij}} = \underbrace{\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j\cdots}_{m_{ij}}. \tag{9.1}$$

Theorem 9.1. *For* $i \neq j \in \mathbb{I}_{\circ,\tau}, e = \pm 1$, *we have*

$$\begin{aligned} \underbrace{\widetilde{\mathbf{T}}'_{i,e}\widetilde{\mathbf{T}}'_{j,e}\widetilde{\mathbf{T}}'_{i,e}\cdots}_{m_{ij}} &= \underbrace{\widetilde{\mathbf{T}}'_{j,e}\widetilde{\mathbf{T}}'_{i,e}\widetilde{\mathbf{T}}'_{j,e}\cdots}_{m_{ij}}, \\ \underbrace{\widetilde{\mathbf{T}}''_{i,e}\widetilde{\mathbf{T}}''_{j,e}\widetilde{\mathbf{T}}''_{i,e}\cdots}_{m_{ij}} &= \underbrace{\widetilde{\mathbf{T}}''_{j,e}\widetilde{\mathbf{T}}''_{i,e}\widetilde{\mathbf{T}}''_{j,e}\cdots}_{m_{ij}}. \end{aligned} \tag{9.2}$$

Proof. By Theorem 6.7, $\widetilde{\mathbf{T}}''_{i,+1}$ is the inverse of $\widetilde{\mathbf{T}}'_{i,-1}$. Moreover, by definition (6.11), $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ are conjugations of $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$, respectively. Hence. it suffices to prove the identity (9.2) for $\widetilde{\mathbf{T}}'_{i,-1}$.

Set $m = m_{ij}$. Let $\boldsymbol{w}_\circ = \mathbf{r}_i\mathbf{r}_j\mathbf{r}_i\cdots$ be a reduced expression of length m. Define $\boldsymbol{w}_k$, for $1 \leqslant k \leqslant m$, to be

$$\boldsymbol{w}_1 = \mathbf{r}_i, \quad \boldsymbol{w}_2 = \mathbf{r}_i\mathbf{r}_j, \quad \boldsymbol{w}_3 = \mathbf{r}_i\mathbf{r}_j\mathbf{r}_i, \quad \dots, \quad \boldsymbol{w}_m = \boldsymbol{w}_\circ.$$

Write $\boldsymbol{w}'_\circ$ for the other reduced expression $\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j\cdots$, and define $\boldsymbol{w}'_k$, for $1 \leqslant k \leqslant m$, accordingly. Let r denote the last index in the reduced expression of $\boldsymbol{w}_\circ$; that is, $r = j$ if $m = 2,4,6$ and $r = i$ if $m = 3$. Similarly, we define r' for $\boldsymbol{w}'_\circ$.

Applying the intertwining property (4.7) for m times, we obtain the following two identities:

$$\begin{aligned} &\underbrace{\widetilde{\mathbf{T}}'_{i,-1}\widetilde{\mathbf{T}}'_{j,-1}\widetilde{\mathbf{T}}'_{i,-1}\cdots(u)}_{m}\cdot\widetilde{\Upsilon}_i\cdot\mathscr{T}'_{\boldsymbol{w}_1,-1}(\widetilde{\Upsilon}_j)\cdots\mathscr{T}'_{\boldsymbol{w}_{m-1},-1}(\widetilde{\Upsilon}_r) \\ &\qquad= \widetilde{\Upsilon}_i\cdot\mathscr{T}'_{\boldsymbol{w}_1,-1}(\widetilde{\Upsilon}_j)\cdots\mathscr{T}'_{\boldsymbol{w}_{m-1},-1}(\widetilde{\Upsilon}_r)\cdot\underbrace{\mathscr{T}'_{\mathbf{r}_i,-1}\mathscr{T}'_{\mathbf{r}_j,-1}\cdots(u^\imath)}_{m}, \end{aligned} \tag{9.3}$$

$$\underbrace{\widetilde{\mathbf{T}}'_{j,-1}\widetilde{\mathbf{T}}'_{i,-1}\widetilde{\mathbf{T}}'_{j,-1}\cdots(u)}_{m}\cdot\widetilde{\mathrm{Y}}_j\cdot\mathscr{T}'_{\boldsymbol{w}'_1,-1}(\widetilde{\mathrm{Y}}_i)\cdots\mathscr{T}'_{\boldsymbol{w}'_{m-1},-1}(\widetilde{\mathrm{Y}}_{r'})$$

$$=\widetilde{\mathrm{Y}}_j\cdot\mathscr{T}'_{\boldsymbol{w}'_1,-1}(\widetilde{\mathrm{Y}}_i)\cdots\mathscr{T}'_{\boldsymbol{w}'_{m-1},-1}(\widetilde{\mathrm{Y}}_{r'})\cdot\underbrace{\mathscr{T}'_{\mathbf{r}_j,-1}\mathscr{T}'_{\mathbf{r}_i,-1}\cdots(u^\imath)}_{m}, \tag{9.4}$$

for all $u\in\widetilde{\mathbf{U}}^\imath$.

By Proposition 4.2, the $\mathscr{T}'_{k,-1}$'s satisfy braid relations. As $\ell(\mathbf{r}_i\mathbf{r}_j\mathbf{r}_i\cdots)=\ell(\boldsymbol{w}_\circ)=\ell(\mathbf{r}_j\mathbf{r}_i\mathbf{r}_j\cdots)$, we have

$$\underbrace{\mathscr{T}'_{\mathbf{r}_i,-1}\mathscr{T}'_{\mathbf{r}_j,-1}\mathscr{T}'_{\mathbf{r}_i,-1}\cdots}_{m}=\underbrace{\mathscr{T}'_{\mathbf{r}_j,-1}\mathscr{T}'_{\mathbf{r}_i,-1}\mathscr{T}'_{\mathbf{r}_j,-1}\cdots}_{m}. \tag{9.5}$$

Hence, by a comparison of (9.3)–(9.4), we reduce the proof of the desired identity (9.2) to showing that

$$\widetilde{\mathrm{Y}}_i\cdot\mathscr{T}'_{\boldsymbol{w}_1,-1}(\widetilde{\mathrm{Y}}_j)\cdots\mathscr{T}'_{\boldsymbol{w}_{m-1},-1}(\widetilde{\mathrm{Y}}_r)=\widetilde{\mathrm{Y}}_j\cdot\mathscr{T}'_{\boldsymbol{w}'_1,-1}(\widetilde{\mathrm{Y}}_i)\cdots\mathscr{T}'_{\boldsymbol{w}'_{m-1},-1}(\widetilde{\mathrm{Y}}_{r'}). \tag{9.6}$$

By definition (8.1), $\widetilde{\mathrm{Y}}_{\boldsymbol{w}_\circ}=\widetilde{\mathscr{T}}_{\boldsymbol{w}_{m-1}}(\widetilde{\mathrm{Y}}_r)\cdots\widetilde{\mathscr{T}}_{\boldsymbol{w}_1}(\widetilde{\mathrm{Y}}_j)\widetilde{\mathrm{Y}}_i$. Applying σ to this identity and then using Proposition 3.8, we obtain

$$\sigma(\widetilde{\mathrm{Y}}_{\boldsymbol{w}_\circ})=\widetilde{\mathrm{Y}}_i\cdot\mathscr{T}'_{\boldsymbol{w}_1,-1}(\widetilde{\mathrm{Y}}_j)\cdots\mathscr{T}'_{\boldsymbol{w}_{m-1},-1}(\widetilde{\mathrm{Y}}_r). \tag{9.7}$$

We have a similar formula for $\sigma(\widetilde{\mathrm{Y}}_{\boldsymbol{w}'_\circ})$ as well. It follows by Theorem 8.1 that $\sigma(\widetilde{\mathrm{Y}}_{\boldsymbol{w}_\circ})=\sigma(\widetilde{\mathrm{Y}}_{\boldsymbol{w}'_\circ})$. The identity (9.6) now follows by the formula (9.7) and its $\boldsymbol{w}'_\circ$-counterpart.

This completes the proof of the theorem. □

For $w\in W^\circ$, take a reduced expression $w=\mathbf{r}_{i_1}\mathbf{r}_{i_2}\cdots\mathbf{r}_{i_k}$ and define

$$\widetilde{\mathbf{T}}'_{w,e}:=\widetilde{\mathbf{T}}'_{i_1,e}\widetilde{\mathbf{T}}'_{i_2,e}\cdots\widetilde{\mathbf{T}}'_{i_k,e},\qquad \widetilde{\mathbf{T}}'_{w,e}:=\widetilde{\mathbf{T}}''_{i_1,e}\widetilde{\mathbf{T}}''_{i_2,e}\cdots\widetilde{\mathbf{T}}''_{i_k,e}. \tag{9.8}$$

By Theorem 9.1, these are independent of the choice of reduced expressions for w.

9.2 | Action of the braid group $\mathrm{Br}(W_\bullet)\rtimes\mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$

We first establish a commutator relation between $\widetilde{\mathbf{T}}'_{i,-1}$ $(i\in\mathbb{I}_\circ)$ and $\mathscr{T}_j^{-1}\equiv\mathscr{T}'_{j,-1}$ $(j\in\mathbb{I}_\bullet)$.

Lemma 9.2. *We have* $\mathscr{T}_j^{-1}\widetilde{\mathbf{T}}'_{i,-1}(x)=\widetilde{\mathbf{T}}'_{i,-1}\mathscr{T}^{-1}_{\tau_{\bullet,i}\tau j}(x)$, *for* $i\in\mathbb{I}_{\circ,\tau}$, $j\in\mathbb{I}_\bullet$, *and* $x\in\widetilde{\mathbf{U}}^\imath$.

Proof. Note that $\tau(j),\tau_{\bullet,i}(j),\tau_{\bullet,i}\tau(j)\in\mathbb{I}_\bullet$, for $j\in\mathbb{I}_\bullet$. Since $w_\bullet s_j=s_{\tau j}w_\bullet$, for $j\in\mathbb{I}_\bullet$, and $w_{\bullet,i}s_j=s_{\tau_{\bullet,i}j}w_{\bullet,i}$, for $i\in\mathbb{I}_\circ$, we have

$$\mathbf{r}_is_j=w_{\bullet,i}w_\bullet^{-1}s_j=s_{\tau_{\bullet,i}\tau j}w_{\bullet,i}w_\bullet^{-1}=s_{\tau_{\bullet,i}\tau j}\mathbf{r}_i. \tag{9.9}$$

Since $\ell(\mathbf{r}_i s_j) = \ell(\mathbf{r}_i) + 1$, it follows by (9.9) that

$$\widetilde{\mathscr{T}}_{\tau_{\bullet,i}\tau(j)}\widetilde{\mathscr{T}}_{\mathbf{r}_i} = \widetilde{\mathscr{T}}_{\mathbf{r}_i}\widetilde{\mathscr{T}}_j. \tag{9.10}$$

By Proposition 4.6, $\widetilde{\Upsilon}_i$ is fixed by $\widetilde{\mathscr{T}}_j^{-1}$. Hence, applying $\widetilde{\mathscr{T}}_j^{-1}$ to the intertwining relation (4.7) in Theorem 4.7 and then using (9.10), we obtain, for $x \in \widetilde{\mathbf{U}}^\imath$,

$$\begin{aligned}\widetilde{\mathscr{T}}_j^{-1}\widetilde{\mathbf{T}}'_{i,-1}(x)\widetilde{\Upsilon}_i &= \widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_j^{-1}\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}(x)\\ &= \widetilde{\Upsilon}_i\widetilde{\mathscr{T}}_{\mathbf{r}_i}^{-1}\widetilde{\mathscr{T}}_{\tau_{\bullet,i}\tau j}^{-1}(x) = \widetilde{\mathbf{T}}'_{i,-1}\widetilde{\mathscr{T}}_{\tau_{\bullet,i}\tau j}^{-1}(x)\widetilde{\Upsilon}_i,\end{aligned} \tag{9.11}$$

where the last step uses Theorem 4.7 and the fact that $\widetilde{\mathscr{T}}_{\tau_{\bullet,i}\tau j}^{-1}(x) \in \widetilde{\mathbf{U}}^\imath$ by Proposition 4.5. The identity (9.11) clearly implies the identity in the lemma. □

Let $\mathrm{Br}(W_\bullet)$ and $\mathrm{Br}(W^\circ)$ be the braid groups associated to $W_\bullet$ and W°, respectively.

Theorem 9.3. *There exists a braid group action of* $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ *on* $\widetilde{\mathbf{U}}^\imath$ *as automorphisms of algebras generated by* $\widetilde{\mathscr{T}}'_{j,-1}$ $(j \in \mathbb{I}_\bullet)$ *and* $\widetilde{\mathbf{T}}'_{i,-1}$ $(i \in \mathbb{I}_{\circ,\tau})$.

Proof. By Remark 4.8, $\widetilde{\mathbf{T}}'_{i,-1}$ is independent of the choice of representatives in a τ-orbit. The defining relations of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ consist of braid relations for $\mathrm{Br}(W_\bullet)$, the braid relations for $\mathrm{Br}(W^\circ)$, and relations (9.9). The braid relations for $\widetilde{\mathscr{T}}'_{j,-1}, j \in \mathbb{I}_\bullet$ are verified in Proposition 4.2. The braid relations for $\widetilde{\mathbf{T}}'_{i,-1}, i \in \mathbb{I}_{\circ,\tau}$ are verified in Theorem 9.1. The commutator relation for $\widetilde{\mathscr{T}}'_{j,-1}, \widetilde{\mathbf{T}}'_{i,-1}$ corresponding to (9.9) is verified in Lemma 9.2. □

Remark 9.4. Since $\widetilde{\mathbf{T}}'_{i,-1}, \widetilde{\mathbf{T}}''_{i,+1}$ are mutually inverses and $\widetilde{\mathscr{T}}'_{j,-1}, \widetilde{\mathscr{T}}''_{j,+1}$ are mutually inverses, there also exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathscr{T}}''_{j,+1}$ $(j \in \mathbb{I}_\bullet)$ and $\widetilde{\mathbf{T}}''_{i,+1}$ $(i \in \mathbb{I}_{\circ,\tau})$.

Recall the remaining two symmetries $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ from (6.11). We shall establish a variant of Theorem 9.3 for $\widetilde{\mathbf{T}}'_{i,e}$ and $\widetilde{\mathscr{T}}'_{j,e}$ (and, respectively, $\widetilde{\mathbf{T}}''_{i,e}$ and $\widetilde{\mathscr{T}}''_{j,e}$).

Let $j \in \mathbb{I}$. Recall $\widetilde{\mathscr{T}}''_{j,+1}$ and $\widetilde{\mathscr{T}}'_{j,-1}$ from (4.2)–(4.3). Recalling $\psi_\star = \widetilde{\Psi}_{\varsigma_\star} \circ \psi$ from (3.9), we define

$$\widetilde{\mathscr{T}}''_{j,-1} := \psi_\star \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi_\star, \qquad \widetilde{\mathscr{T}}'_{j,+1} := \psi_\star \circ \widetilde{\mathscr{T}}'_{j,-1} \circ \psi_\star. \tag{9.12}$$

Let $\boldsymbol{\varsigma}_{\star\diamond} := (\varsigma_{j,\star}\varsigma_{j,\diamond})_{j\in\mathbb{I}_\circ}$ be the parameter obtained as the componentwise product of parameters $\boldsymbol{\varsigma}_\diamond$ and $\boldsymbol{\varsigma}_\star$ from (2.21) and (3.8).

Lemma 9.5. *The* $\widetilde{\mathscr{T}}''_{j,-1}, \widetilde{\mathscr{T}}'_{j,+1}$ *are related to* $\widetilde{T}''_{j,-1}, \widetilde{T}'_{j,+1}$ *via a rescaling automorphism:*

$$\widetilde{\mathscr{T}}''_{j,-1} = \widetilde{\Psi}_{\varsigma_{\star\diamond}}\widetilde{T}''_{j,-1}\widetilde{\Psi}_{\varsigma_{\star\diamond}}^{-1}, \qquad \widetilde{\mathscr{T}}'_{j,+1} = \widetilde{\Psi}_{\varsigma_{\star\diamond}}\widetilde{T}'_{j,+1}\widetilde{\Psi}_{\varsigma_{\star\diamond}}^{-1}.$$

Proof. Recall $\widetilde{\mathscr{T}}''_{j,+1} = \widetilde{\Psi}_{\varsigma_\diamond}^{-1} \circ \widetilde{T}''_{j,+1} \circ \widetilde{\Psi}_{\varsigma_\diamond}$ and $\widetilde{\mathscr{T}}'_{j,-1} = \widetilde{\Psi}_{\varsigma_\diamond}^{-1} \circ \widetilde{T}'_{j,-1} \circ \widetilde{\Psi}_{\varsigma_\diamond}$ from (4.2)–(4.3).

Recall from (2.10) that $\widetilde{T}''_{i,-1} = \psi \circ \widetilde{T}''_{i,+1} \circ \psi$ and $\widetilde{T}'_{i,+1} = \psi \circ \widetilde{T}'_{i,-1} \circ \psi$. Then, we have

$$\begin{aligned}\widetilde{\mathscr{T}}''_{j,-1} &= \psi_\star \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi_\star \\ &= \widetilde{\Psi}_{\varsigma_\star} \psi \circ \widetilde{\Psi}^{-1}_{\varsigma_\diamond} \widetilde{T}''_{j,+1} \widetilde{\Psi}_{\varsigma_\diamond} \circ \widetilde{\Psi}_{\varsigma_\star} \psi = \widetilde{\Psi}_{\varsigma_{\star\diamond}} \widetilde{T}''_{j,-1} \widetilde{\Psi}^{-1}_{\varsigma_{\star\diamond}},\end{aligned}$$

where we used $\psi \circ \widetilde{\Psi}^{-1}_{\varsigma_\diamond} = \widetilde{\Psi}_{\varsigma_\diamond} \circ \psi$. The proof for the other formula is similar. □

By Proposition 4.5, the automorphisms $\widetilde{\mathscr{T}}''_{j,+1}, \widetilde{\mathscr{T}}'_{j,-1}$ for $j \in \mathbb{I}_\bullet$ restrict to automorphisms on $\widetilde{\mathbf{U}}^\imath$.

Lemma 9.6. *The automorphisms $\widetilde{\mathscr{T}}''_{j,e}, \widetilde{\mathscr{T}}'_{j,e}$, for $j \in \mathbb{I}_\bullet$ and $e = \pm 1$, restrict to automorphisms on $\widetilde{\mathbf{U}}^\imath$. Moreover, the following identities hold:*

$$\widetilde{\mathscr{T}}''_{j,-1} := \psi^\imath \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi^\imath, \qquad \widetilde{\mathscr{T}}'_{j,+1} := \psi^\imath \circ \widetilde{\mathscr{T}}'_{j,-1} \circ \psi^\imath. \tag{9.13}$$

Proof. As $\widetilde{\mathscr{T}}_j \equiv \widetilde{\mathscr{T}}''_{j,+1}$ restricts to an automorphism on $\widetilde{\mathbf{U}}^\imath$ by Proposition 4.5, it suffices to prove (9.13).

By Proposition 3.4, we have $\psi_\star = \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} \circ \psi^\imath$ when acting on $\widetilde{\mathbf{U}}^\imath$. By Proposition 4.6, $\mathrm{Ad}_{\widetilde{\Upsilon}^{-1}}$ commutes with $\widetilde{\mathscr{T}}_j$. By Proposition 3.5, we have $\psi_\star \circ \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} = \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star$. Using these properties and (9.12), we have, for $x \in \widetilde{\mathbf{U}}^\imath$,

$$\begin{aligned}\widetilde{\mathscr{T}}''_{j,-1}(x) &= \psi_\star \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi_\star(x) = \psi_\star \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} \circ \psi^\imath(x) \\ &= \psi_\star \circ \mathrm{Ad}_{\widetilde{\Upsilon}^{-1}} \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi^\imath(x) = \mathrm{Ad}_{\widetilde{\Upsilon}} \circ \psi_\star \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi^\imath(x) = \psi^\imath \circ \widetilde{\mathscr{T}}''_{j,+1} \circ \psi^\imath(x),\end{aligned}$$

where the last equality uses (3.18).

The proof of the other formula for $\widetilde{\mathscr{T}}'_{j,+1}$ is similar and hence skipped. □

The next result follows from (6.11), Theorem 9.3, Remark 9.4, and Lemma 9.6.

Corollary 9.7. *Let $e = \pm 1$.*

(1) *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathscr{T}}'_{j,e}$ $(j \in \mathbb{I}_\bullet)$ and $\widetilde{\mathbf{T}}'_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.*
(2) *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\widetilde{\mathbf{U}}^\imath$ as automorphisms of algebras generated by $\widetilde{\mathscr{T}}''_{j,e}$ $(j \in \mathbb{I}_\bullet)$ and $\widetilde{\mathbf{T}}''_{i,e}$ $(i \in \mathbb{I}_{\circ,\tau})$.*

9.3 | Intertwining properties of $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$

The automorphisms $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ on $\widetilde{\mathbf{U}}^\imath$ also satisfy intertwining relations similar to those satisfied by $\widetilde{\mathbf{T}}'_{i,-1}$ in (4.7) and $\widetilde{\mathbf{T}}''_{i,+1}$ in (6.1). These relations on $\widetilde{\mathbf{U}}^\imath$ will descend to $\mathbf{U}^\imath_\varsigma$ (see Proposition 10.2) and will then be used to define the relative braid operators on module level (see Definition 10.4).

Proposition 9.8. *The automorphisms* $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ *satisfy the following intertwining relations:*

$$\widetilde{\mathbf{T}}'_{i,+1}(x)\mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1}) = \mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1})\mathscr{T}'_{\mathbf{r}_i,+1}(x), \tag{9.14}$$

$$\widetilde{\mathbf{T}}''_{i,-1}(x)\widetilde{\Upsilon}_i = \widetilde{\Upsilon}_i\mathscr{T}''_{\mathbf{r}_i,-1}(x). \tag{9.15}$$

Proof. We prove the first identity (9.14); the second identity (9.15) can be derived from the first one by noting that $\widetilde{\mathbf{T}}'_{i,+1}, \widetilde{\mathbf{T}}''_{i,-1}$ are inverses and $\mathscr{T}'_{\mathbf{r}_i,+1}, \mathscr{T}''_{\mathbf{r}_i,-1}$ are inverses.

We claim the following identity holds:

$$\widetilde{\mathbf{T}}'_{i,+1}(x)\cdot\widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i)\mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) = \widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i)\mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1})\cdot\mathscr{T}'_{\mathbf{r}_i,+1}(x). \tag{9.16}$$

Let us prove (9.16). Recall from (6.11) that $\widetilde{\mathbf{T}}'_{i,+1} = \psi^\imath\widetilde{\mathbf{T}}'_{i,-1}\psi^\imath$ and from (3.11) that $\widetilde{\Upsilon}^{-1}\psi^\imath(u)\widetilde{\Upsilon} = \psi_\star(u)$. Hence,

$$\widetilde{\Upsilon}^{-1}\widetilde{\mathbf{T}}'_{i,+1}(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}^{-1}\psi^\imath(\widetilde{\mathbf{T}}'_{i,-1}(\psi^\imath x))\widetilde{\Upsilon} = \psi_\star(\widetilde{\mathbf{T}}'_{i,-1}(\psi^\imath x)).$$

By (4.7), $\widetilde{\Upsilon}_i^{-1}\widetilde{\mathbf{T}}'_{i,-1}(\psi^\imath x)\widetilde{\Upsilon}_i = \mathscr{T}'_{\mathbf{r}_i,-1}(\psi^\imath x)$. Hence,

$$\psi_\star(\widetilde{\Upsilon}_i)^{-1}\widetilde{\Upsilon}^{-1}\widetilde{\mathbf{T}}'_{i,+1}(x)\widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i) = \psi_\star(\mathscr{T}'_{\mathbf{r}_i,-1}(\psi^\imath(x))).$$

This allows us to write (9.16) as an equivalent identity

$$\psi_\star(\mathscr{T}'_{\mathbf{r}_i,-1}(\psi^\imath(x)))\mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) = \mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1})\mathscr{T}'_{\mathbf{r}_i,+1}(x). \tag{9.17}$$

Recalling by (9.12) that $\mathscr{T}'_{\mathbf{r}_i,+1} = \psi_\star\mathscr{T}'_{\mathbf{r}_i,-1}\psi_\star$, we reduce the proof of (9.17) to verifying that $\psi^\imath(x)\psi_\star(\widetilde{\Upsilon})^{-1} = \psi_\star(\widetilde{\Upsilon})^{-1}\psi_\star(x)$, which by Proposition 3.5 is equivalent to $\psi^\imath(x)\widetilde{\Upsilon} = \widetilde{\Upsilon}\psi_\star(x)$. This last identity holds by (3.11). Therefore, (9.16) is proved.

Observe that if we define $\widetilde{\Upsilon}_{[w]}$ by replacing $\mathscr{T}_{\mathbf{r}_i} \equiv \mathscr{T}''_{\mathbf{r}_i,+1}$ in the definition (8.1) of $\widetilde{\Upsilon}_w$ by $\mathscr{T}'_{\mathbf{r}_i,+1}$, then we still have a factorization $\widetilde{\Upsilon} = \widetilde{\Upsilon}_{[\boldsymbol{w}_\circ]}$, for any reduced expression of $\boldsymbol{w}_\circ$. Below we shall use this version of factorization.

Let $\boldsymbol{w}'_\circ$ be a reduced expression of $\boldsymbol{w}_\circ$ starting with $\mathbf{r}_i$, and $\boldsymbol{w}''_\circ(= w_0\boldsymbol{w}'_\circ w_0)$ be a reduced expression of $\boldsymbol{w}_\circ$ ending with $\mathbf{r}_{\tau_0 i}$. It follows by definition that

$$\widetilde{\Upsilon} = \widetilde{\Upsilon}_{[\boldsymbol{w}'_\circ]} = \mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_{[\mathbf{r}_i\boldsymbol{w}'_\circ]})\widetilde{\Upsilon}_i. \tag{9.18}$$

Since $w_0\mathbf{r}_{\tau_0 i} = \mathbf{r}_i w_0$ and $w_0 = \boldsymbol{w}_\circ w_\bullet$, we have $\boldsymbol{w}_\circ\mathbf{r}_{\tau_0 i} = \mathbf{r}_i\boldsymbol{w}_\circ$. By definition and Proposition 8.3, we obtain

$$\widetilde{\Upsilon} = \widetilde{\Upsilon}_{[\boldsymbol{w}''_\circ]} = \widetilde{\Upsilon}_i\widetilde{\Upsilon}_{[\boldsymbol{w}_\circ\mathbf{r}_{\tau_0 i}]} = \widetilde{\Upsilon}_i\widetilde{\Upsilon}_{[\mathbf{r}_i\boldsymbol{w}_\circ]}. \tag{9.19}$$

Now, using (9.18)–(9.19), we can simplify a key component appearing in (9.16) as follows:

$$\begin{aligned}\widetilde{\Upsilon}\psi_\star(\widetilde{\Upsilon}_i)\mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1}) &= \widetilde{\Upsilon}\widetilde{\Upsilon}_i^{-1}\mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}^{-1})\\ &= \mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_{[\mathbf{r}_i\boldsymbol{w}_\circ]}\widetilde{\Upsilon}^{-1}) = \mathscr{T}'_{\mathbf{r}_i,+1}(\widetilde{\Upsilon}_i^{-1}).\end{aligned}$$

Hence, the identity (9.14) follows from (9.16). □

9.4 | Braid group action on $\mathbf{U}^\imath_\varsigma$

Recall from (2.20) the $\imath$quantum group $\mathbf{U}^\imath_\varsigma$ with parameter ς satisfying (2.18) (à la Letzter), and recall a central reduction $\pi^\imath_\varsigma : \widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_\varsigma$ from Proposition 2.8.

We first construct the braid group action on $\mathbf{U}^\imath_{\varsigma_\diamond}$ for the distinguished parameter $\varsigma_\diamond$ (2.21). By the definition (4.14) of $\widetilde{k}_{j,\diamond}$ and Proposition 2.8, the kernel $\ker \pi^\imath_{\varsigma_\diamond}$ is generated by

$$\widetilde{k}_{j,\diamond} - 1 \quad (\tau j = j \in \mathbb{I}_\circ), \quad \widetilde{k}_{j,\diamond}\widetilde{k}_{\tau j,\diamond} - 1 \quad (\tau j \neq j \in \mathbb{I}_\circ), \quad K_j K'_j - 1 \quad (j \in \mathbb{I}_\bullet).$$

In addition, by Proposition 4.11, we have $\widetilde{\mathbf{T}}''_{i,+1}(\widetilde{k}_{j,\diamond}) = \widetilde{k}_{\mathbf{r}_i\alpha_j,\diamond}$. Hence, the kernel of $\pi_{\varsigma_\diamond}$ is preserved by $\widetilde{\mathbf{T}}''_{i,+1}$. Therefore, $\widetilde{\mathbf{T}}''_{i,+1}$ induces a automorphism $\mathbf{T}''_{i,+1;\varsigma_\diamond}$ on $\mathbf{U}^\imath_{\varsigma_\diamond}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\mathbf{U}}^\imath & \xrightarrow{\widetilde{\mathbf{T}}''_{i,+1}} & \widetilde{\mathbf{U}}^\imath \\
\downarrow{\scriptstyle \pi^\imath_{\varsigma_\diamond}} & & \downarrow{\scriptstyle \pi_{\varsigma^\imath_\diamond}} \\
\mathbf{U}^\imath_{\varsigma_\diamond} & \xrightarrow{\mathbf{T}''_{i,+1;\varsigma_\diamond}} & \mathbf{U}^\imath_{\varsigma_\diamond}
\end{array}$$

It follows from Theorem 9.1 that $\mathbf{T}''_{i,+1;\varsigma_\diamond}$ satisfy the braid relations. By definition, $\widetilde{\mathscr{T}}_j$ $(j \in \mathbb{I}_\bullet)$ descends to Lusztig's automorphism T_j under the central reduction $\pi^\imath_{\varsigma_\diamond}$. It then follows by Theorem 9.3 and Remark 9.4 that there exists an action of the braid group $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_{\varsigma_\diamond}$ generated by $T_j, \mathbf{T}''_{i,+1;\varsigma_\diamond}$, for $j \in \mathbb{I}_\bullet, i \in \mathbb{I}_{\circ,\tau}$.

We now consider the symmetries on $\mathbf{U}^\imath_\varsigma$, for an arbitrary parameter ς satisfying (2.18).

Via the isomorphism $\phi_\varsigma : \mathbf{U}^\imath_{\varsigma_\diamond} \to \mathbf{U}^\imath_\varsigma$ constructed in Proposition 2.7, we transport the relative braid group action on $\mathbf{U}^\imath_{\varsigma_\diamond}$ to a relative braid group action on $\mathbf{U}^\imath_\varsigma$. More precisely, there exist automorphisms $\mathbf{T}''_{i,+1;\varsigma}$ on $\mathbf{U}^\imath_\varsigma$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{U}^\imath_{\varsigma_\diamond} & \xrightarrow{\mathbf{T}''_{i,+1;\varsigma_\diamond}} & \mathbf{U}^\imath_{\varsigma_\diamond} \\
\downarrow{\scriptstyle \phi_\varsigma} & & \downarrow{\scriptstyle \phi_\varsigma} \\
\mathbf{U}^\imath_{\varsigma} & \xrightarrow{\mathbf{T}''_{i,+1}} & \mathbf{U}^\imath_{\varsigma}
\end{array}$$

Our convention here and below is that we suppress the dependence on a general parameter ς for the symmetries $\mathbf{T}''_{i,+1}$ (and $\mathbf{T}'_{i,-1}$, $\mathbf{T}''_{i,-1}$, and $\mathbf{T}'_{i,+1}$ below) on $\mathbf{U}^\imath_\varsigma$.

In addition, T_j commutes with ϕ_ς for $j \in \mathbb{I}_\bullet$. Summarizing we have obtained the following braid group action on $\mathbf{U}^\imath_\varsigma$ (from Theorem 9.1, Theorem 9.3, and Remark 9.4).

Theorem 9.9. *For an arbitrary parameter ς satisfying (2.18), there exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_\varsigma$ as automorphisms of algebras generated by T_j $(j \in \mathbb{I}_\bullet)$ and $\mathbf{T}''_{i,+1}$ $(i \in \mathbb{I}_{\circ,\tau})$.*

We next construct $\mathbf{T}'_{i,+1}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ for general parameters $\boldsymbol{\varsigma}$. By a similar argument as in §4.5, we have $\widetilde{\mathbf{T}}'_{i,+1} = \widetilde{\mathscr{T}}'_{\mathbf{r}_i,+1}$ on $\widetilde{\mathbf{U}}^{\imath 0}$ and both are given by

$$\varsigma_{j,\star\diamond}\widetilde{k}_j \mapsto \varsigma_{\mathbf{r}_i\alpha_j,\star\diamond}\widetilde{k}_{\mathbf{r}_i\alpha_j}. \tag{9.20}$$

Denote the parameter $\bar{\boldsymbol{\varsigma}}_{\star\diamond} := (\bar{\varsigma}_{j,\star\diamond})_{j\in\mathbb{I}_\circ}$. Then by (9.20), $\widetilde{\mathbf{T}}'_{i,+1}$ preserves the kernel of $\pi^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$, and hence, it induces an automorphism $\mathbf{T}'_{i,+1;\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ on $\mathbf{U}^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\mathbf{U}}^\imath & \xrightarrow{\ \widetilde{\mathbf{T}}'_{i,+1}\ } & \widetilde{\mathbf{U}}^\imath \\
\Big\downarrow \pi^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}} & & \Big\downarrow \pi^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}} \\
\mathbf{U}^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}} & \xrightarrow{\ \mathbf{T}'_{i,+1;\bar{\boldsymbol{\varsigma}}_{\star\diamond}}\ } & \mathbf{U}^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}
\end{array}$$

On the other hand, by Lemma 9.5, $\widetilde{\mathscr{T}}'_{j,+1}$ descends to Lusztig's automorphism $T'_{j,+1}$ under the central reduction $\pi^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$. Hence, by Corollary 9.7, there exits an action of the braid group $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ generated by $T'_{j,+1}$ ($j \in \mathbb{I}_\bullet$) and $\mathbf{T}'_{i,+1;\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ ($i \in \mathbb{I}_{\circ,\tau}$).

Now, for an arbitrary parameter $\boldsymbol{\varsigma}$, we can use the isomorphism $\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ to translate this action on $\mathbf{U}^\imath_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$ to an action on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, that is, there exists automorphisms $\mathbf{T}'_{i,+1}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ such that

$$\mathbf{T}'_{i,+1}\circ\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}} = \phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}} \circ \mathbf{T}'_{i,+1;\boldsymbol{\varsigma}_{\star\diamond}}.$$

In addition, $\widetilde{\mathscr{T}}'_{j,+1}$ commutes with $\phi_{\boldsymbol{\varsigma}}\phi^{-1}_{\bar{\boldsymbol{\varsigma}}_{\star\diamond}}$.

Similarly, we can formulate the automorphisms $\mathbf{T}'_{i,-1}, \mathbf{T}''_{i,-1}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, which are inverses to $\mathbf{T}''_{i,+1}, \mathbf{T}'_{i,+1}$; the detail is skipped. Summarizing, we have established the following theorem, which was conjectured in [20, Conjecture 1.2].

Theorem 9.10. *Let $e = \pm 1$, and $\boldsymbol{\varsigma}$ be an arbitrary parameter satisfying (2.18).*

(1) *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ as automorphisms of algebras generated by $T'_{j,e}$ ($j \in \mathbb{I}_\bullet$) and $\mathbf{T}'_{i,e}$ ($i \in \mathbb{I}_{\circ,\tau}$).*
(2) *There exists a braid group action of $\mathrm{Br}(W_\bullet) \rtimes \mathrm{Br}(W^\circ)$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$ as automorphisms of algebras generated by $T''_{j,e}$ ($j \in \mathbb{I}_\bullet$) and $\mathbf{T}''_{i,e}$ ($i \in \mathbb{I}_{\circ,\tau}$).*

10 | RELATIVE BRAID GROUP ACTIONS ON U-MODULES

Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be a Satake diagram of arbitrary type and $(\mathbf{U}, \mathbf{U}_{\boldsymbol{\varsigma}})$ be the associated quantum symmetric pair. We set $\boldsymbol{\varsigma}$ to be a balanced parameter throughout this section. Based on the intertwining properties of $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on $\mathbf{U}^\imath_{\boldsymbol{\varsigma}}$, we formulate the compatible action of corresponding operators on an arbitrary finite-dimensional $\mathbf{U}$-module M. We then show that these operators on M satisfy relative braid group relations.

10.1 | Intertwining relations on $\mathbf{U}^\imath_\varsigma$

Recall that the symmetries $\mathbf{T}'_{i,e}$ and $\mathbf{T}''_{i,e}$ on $\mathbf{U}^\imath_\varsigma$, for $e = \pm 1$, were defined in §9.4. In this subsection, we formulate the intertwining properties of these symmetries.

Recall ϕ_ς from Proposition 2.7. Since ς is a balanced parameter, ϕ_ς is the restriction of $\Phi_{\bar{\varsigma}_\diamond\varsigma}$, where $\bar{\varsigma}_\diamond\varsigma$ is defined by componentwise multiplication with $\bar{\varsigma}_\diamond = (\overline{\varsigma_{j,\diamond}})_{j\in\mathbb{I}_\circ}$; cf. also Proposition 2.7. Define

$$\mathscr{T}''_{i,+1;\varsigma} := \Phi_{\bar{\varsigma}_\diamond\varsigma} T''_{i,+1} \Phi^{-1}_{\bar{\varsigma}_\diamond\varsigma}, \qquad \mathscr{T}'_{i,-1;\varsigma} := \Phi_{\bar{\varsigma}_\diamond\varsigma} T'_{i,-1} \Phi^{-1}_{\bar{\varsigma}_\diamond\varsigma}. \tag{10.1}$$

Proposition 10.1. *Let ς be a balanced parameter. The automorphisms $\mathbf{T}'_{i,-1}$ and $\mathbf{T}''_{i,+1}$ on $\mathbf{U}^\imath_\varsigma$ satisfy the following intertwining relations:*

$$\mathbf{T}'_{i,-1}(x)\mathrm{Y}_{i,\varsigma} = \mathrm{Y}_{i,\varsigma}\mathscr{T}'_{\mathbf{r}_i,-1;\varsigma}(x), \tag{10.2}$$

$$\mathbf{T}''_{i,+1}(x)\,\mathscr{T}''_{\mathbf{r}_i,+1;\varsigma}(\mathrm{Y}^{-1}_{i,\varsigma}) = \mathscr{T}''_{\mathbf{r}_i,+1;\varsigma}(\mathrm{Y}^{-1}_{i,\varsigma})\,\mathscr{T}''_{\mathbf{r}_i,+1;\varsigma}(x), \tag{10.3}$$

for $x \in \mathbf{U}^\imath_\varsigma$.

Proof. By Theorems 4.7 and 6.1, we have, for any $x \in \widetilde{\mathbf{U}}^\imath$,

$$\begin{aligned}\widetilde{\mathbf{T}}'_{i,-1}(x)\,\widetilde{\mathrm{Y}}_i &= \widetilde{\mathrm{Y}}_i\,\widetilde{\mathscr{T}}'_{\mathbf{r}_i,-1}(x),\\ \widetilde{\mathbf{T}}''_{i,+1}(x)\,\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\mathrm{Y}}^{-1}_i) &= \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\widetilde{\mathrm{Y}}^{-1}_i)\,\widetilde{\mathscr{T}}''_{\mathbf{r}_i,+1}(x).\end{aligned} \tag{10.4}$$

Let $T'_{i,e}, T''_{i,e}$ be Lusztig's automorphisms on $\mathbf{U}$. Recall the central reduction $\pi_{\varsigma_\diamond} : \widetilde{\mathbf{U}} \to \mathbf{U}$ from (2.6). By (2.9) (with $\mathbf{a} = \varsigma_\diamond$) and (2.11), we have

$$\pi_{\varsigma_\diamond}\circ\widetilde{\mathscr{T}}''_{i,+1} = T''_{i,+1}\circ\pi_{\varsigma_\diamond}, \qquad \pi_{\varsigma_\diamond}\circ\widetilde{\mathscr{T}}'_{i,-1} = T'_{i,-1}\circ\pi_{\varsigma_\diamond}.$$

Hence, $\pi^\imath_{\varsigma_\diamond}\circ\widetilde{\mathbf{T}}''_{i,+1} = \mathbf{T}''_{i,+1;\varsigma_\diamond}\circ\pi^\imath_{\varsigma_\diamond}$. Since the parameter $\varsigma_\diamond$ is balanced, $\pi^\imath_{\varsigma_\diamond}$ is the restriction of $\pi_{\varsigma_\diamond}$ to $\mathbf{U}^\imath_{\varsigma_\diamond}$. Applying $\pi_{\varsigma_\diamond}$ to the intertwining relations (10.4), we obtain, for any $x \in \mathbf{U}^\imath_{\varsigma_\diamond}$,

$$\begin{aligned}\mathbf{T}'_{i,-1;\varsigma_\diamond}(x)\,\mathrm{Y}_{i,\varsigma_\diamond} &= \mathrm{Y}_{i,\varsigma_\diamond}\,T'_{\mathbf{r}_i,-1}(x),\\ \mathbf{T}''_{i,+1;\varsigma_\diamond}(x)\,T''_{\mathbf{r}_i,+1}(\mathrm{Y}^{-1}_{i,\varsigma_\diamond}) &= T''_{\mathbf{r}_i,+1}(\mathrm{Y}^{-1}_{i,\varsigma_\diamond})\,T''_{\mathbf{r}_i,+1}(x).\end{aligned} \tag{10.5}$$

Recall ϕ_ς from Proposition 2.7. As we have seen in §9.4, we have $\phi_\varsigma\circ\mathbf{T}''_{i,+1;\varsigma_\diamond} = \mathbf{T}''_{i,+1}\circ\phi_\varsigma$, and $\phi_\varsigma\circ\mathbf{T}'_{i,-1;\varsigma_\diamond} = \mathbf{T}'_{i,-1}\circ\phi_\varsigma$. Therefore, applying ϕ_ς to the identities (10.5) gives us the desired intertwining relations in the proposition. □

We next formulate intertwining relations for the other two automorphisms $\mathbf{T}'_{i,+1}$ and $\mathbf{T}''_{i,-1}$.

Recall the central reductions $\pi_\varsigma : \widetilde{\mathbf{U}} \to \mathbf{U}$ from (2.6) and $\pi^\imath_\varsigma : \widetilde{\mathbf{U}}^\imath \to \mathbf{U}^\imath_\varsigma$ from Proposition 2.8. By Lemma 9.5, we have $\pi_{\bar{\varsigma}_{\star\diamond}}\circ\widetilde{\mathscr{T}}'_{i,+1} = T'_{i,+1}\circ\pi_{\bar{\varsigma}_{\star\diamond}}$ and $\pi^\imath_{\bar{\varsigma}_{\star\diamond}}\circ\widetilde{\mathbf{T}}'_{i,+1} = \mathbf{T}'_{i,+1;\bar{\varsigma}_{\star\diamond}}\circ\pi^\imath_{\bar{\varsigma}_{\star\diamond}}$. Since the parameter $\bar{\varsigma}_{\star\diamond}$ is balanced, $\pi^\imath_{\bar{\varsigma}_{\star\diamond}}$ is the restriction of $\pi_{\bar{\varsigma}_{\star\diamond}}$ to $\widetilde{\mathbf{U}}^\imath$. Applying $\pi_{\bar{\varsigma}_{\star\diamond}}$ to (9.14)–(9.15), we have, for any $x \in \mathbf{U}^\imath_{\varsigma_{\star\diamond}}$,

$$\begin{aligned} \mathbf{T}'_{i,+1;\bar{\varsigma}_{\star\diamond}}(x)\, T'_{\mathbf{r}_i,+1}(\mathrm{Y}^{-1}_{i,\bar{\varsigma}_{\star\diamond}}) &= T'_{\mathbf{r}_i,+1}(\mathrm{Y}^{-1}_{i,\bar{\varsigma}_{\star\diamond}})\, T'_{\mathbf{r}_i,+1}(x), \\ \mathbf{T}''_{i,-1;\bar{\varsigma}_{\star\diamond}}(x)\mathrm{Y}_{i,\bar{\varsigma}_{\star\diamond}} &= \mathrm{Y}_{i,\bar{\varsigma}_{\star\diamond}} T''_{\mathbf{r}_i,-1}(x). \end{aligned} \tag{10.6}$$

Since ς is a balanced parameter, by the proof of Proposition 2.7, $\phi_\varsigma \phi^{-1}_{\bar{\varsigma}_{\star\diamond}}$ is the restriction of $\Phi_{\bar{\varsigma}^{-1}_{\star\diamond}\varsigma} = \Phi_{\varsigma_{\star\diamond}\varsigma}$. Define

$$\mathscr{T}''_{i,-1;\varsigma} := \Phi_{\varsigma_{\star\diamond}\varsigma} T''_{i,-1} \Phi^{-1}_{\varsigma_{\star\diamond}\varsigma}, \qquad \mathscr{T}'_{i,+1;\varsigma} := \Phi_{\varsigma_{\star\diamond}\varsigma} T'_{i,+1} \Phi^{-1}_{\varsigma_{\star\diamond}\varsigma}. \tag{10.7}$$

Applying $\phi_\varsigma \phi^{-1}_{\bar{\varsigma}_{\star\diamond}}$ to (10.6), we have established the following.

Proposition 10.2. *Let ς be a balanced parameter. The automorphisms $\mathbf{T}'_{i,+1;\varsigma}$ and $\mathbf{T}''_{i,-1;\varsigma}$ on $\mathbf{U}^\imath_\varsigma$ satisfy the following intertwining relations, for all $x \in \mathbf{U}^\imath_\varsigma$:*

$$\begin{aligned} \mathbf{T}'_{i,+1}(x)\ \mathscr{T}'_{\mathbf{r}_i,+1;\varsigma}(\mathrm{Y}^{-1}_{i,\varsigma}) &= \mathscr{T}'_{\mathbf{r}_i,+1;\varsigma}(\mathrm{Y}^{-1}_{i,\varsigma})\ \mathscr{T}'_{\mathbf{r}_i,+1;\varsigma}(x), \\ \mathbf{T}''_{i,-1}(x)\mathrm{Y}_{i,\varsigma} &= \mathrm{Y}_{i,\varsigma}\mathscr{T}''_{\mathbf{r}_i,-1;\varsigma}(x). \end{aligned}$$

10.2 | Compatible actions of $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on U-modules

Denote by $E_i^{(n)}, F_i^{(n)}$ the divided powers $\frac{E_i^n}{[n]_i!}, \frac{F_i^n}{[n]_i!}$ in $\mathbf{U}$, for $n \in \mathbb{N}$.

Let $\mathcal{F}$ be the category of finite-dimensional $\mathbf{U}$-modules of type $\mathbf{1}$. By definition, $M \in \mathcal{F}$ has a weight space decomposition (with respect a fixed $i \in \mathbb{I}$)

$$M = \bigoplus_{n\in\mathbb{Z}} M_n, \qquad M_n = \{v \in M | K_i v = q_i^n v\}.$$

Following [28], we define linear operators $T'_{i,e}, T''_{i,e}, e = \pm 1$ on M by

$$T'_{i,e}(v) = \sum_{\substack{a,b,c\geqslant 0;\\ a-b+c=m}} (-1)^b q_i^{e(b-ac)} F_i^{(a)} E_i^{(b)} F_i^{(c)} v, \qquad v \in M_m, \tag{10.8}$$

$$T''_{i,e}(v) = \sum_{\substack{a,b,c\geqslant 0;\\ -a+b-c=m}} (-1)^b q_i^{e(b-ac)} E_i^{(a)} F_i^{(b)} E_i^{(c)} v, \qquad v \in M_m. \tag{10.9}$$

Proposition 10.3 [28, 39.4.3]. *Let $M \in \mathcal{F}$. Then, for any $u \in \mathbf{U}, v \in M, e = \pm 1$, we have*

$$T'_{i,e}(uv) = T'_{i,e}(u)T'_{i,e}(v), \qquad T''_{i,e}(uv) = T''_{i,e}(u)T''_{i,e}(v). \tag{10.10}$$

Recall $\mathscr{T}'_{i,e;\varsigma}, \mathscr{T}''_{i,e;\varsigma}$ are merely rescalings of $T'_{i,e}, T''_{i,e}$ defined in (10.1) and (10.7). Applying exactly the same rescalings to the operators on modules (10.8)–(10.9), we obtain operators $\mathscr{T}'_{i,e;\varsigma}, \mathscr{T}''_{i,e;\varsigma}$ on M that satisfy

$$\mathscr{T}'_{i,e;\varsigma}(uv) = \mathscr{T}'_{i,e;\varsigma}(u)\mathscr{T}'_{i,e;\varsigma}(v), \qquad \mathscr{T}''_{i,e;\varsigma}(uv) = \mathscr{T}''_{i,e;\varsigma}(u)\mathscr{T}''_{i,e;\varsigma}(v). \tag{10.11}$$

for any $u \in \mathbf{U}, v \in M$.

We regard the $\mathbf{U}$-module M as a $\mathbf{U}^\imath$-module by restriction.

Definition 10.4. Define linear operators $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on M, for $i \in \mathbb{I}_\circ$ and $e = \pm 1$, by

$$\begin{aligned}
\mathbf{T}'_{i,-1}(v) &:= \mathrm{Y}_{i,\varsigma}\mathscr{T}'_{\mathbf{r}_i,-1;\varsigma}(v),\\
\mathbf{T}''_{i,+1}(v) &:= \mathscr{T}''_{\mathbf{r}_i,+1;\varsigma}(\mathrm{Y}_{i,\varsigma}^{-1})\mathscr{T}''_{\mathbf{r}_i,+1;\varsigma}(v),\\
\mathbf{T}'_{i,+1}(v) &:= \mathscr{T}'_{\mathbf{r}_i,+1;\varsigma}(\mathrm{Y}_{i,\varsigma}^{-1})\mathscr{T}'_{\mathbf{r}_i,+1;\varsigma}(v),\\
\mathbf{T}''_{i,-1}(v) &:= \mathrm{Y}_{i,\varsigma}\mathscr{T}''_{\mathbf{r}_i,-1;\varsigma}(v),
\end{aligned} \tag{10.12}$$

for any $v \in M$.

(In these notations, we have suppressed the dependence on ς on these operators.)

The automorphisms $\mathbf{T}'_{i,e}, \mathbf{T}''_{i,e}$ on M in (10.12) are compatible with the corresponding automorphisms on $\mathbf{U}^\imath_\varsigma$.

Theorem 10.5. *Let $M \in \mathcal{F}$, $i \in \mathbb{I}_\circ$ and $e = \pm 1$. Then we have*

$$\mathbf{T}'_{i,e}(xv) = \mathbf{T}'_{i,e}(x)\mathbf{T}'_{i,e}(v), \qquad \mathbf{T}''_{i,e}(xv) = \mathbf{T}''_{i,e}(x)\mathbf{T}''_{i,e}(v), \tag{10.13}$$

for any $x \in \mathbf{U}^\imath_\varsigma$, $v \in M$.

Proof. We prove the identity for $\mathbf{T}'_{i,-1}$; the proofs for the remaining ones are similar. In the proof, we omit the subindex ς for $\mathrm{Y}_{i,\varsigma}$ and $\mathscr{T}'_{\mathbf{r}_i,-1;\varsigma}$ as there is no confusion.

Since $\mathscr{T}'_{\mathbf{r}_i,-1}(xv) = \mathscr{T}'_{\mathbf{r}_i,-1}(x)\mathscr{T}'_{\mathbf{r}_i,-1}(v)$, we have

$$\mathrm{Y}_i\mathscr{T}'_{\mathbf{r}_i,-1}(xv) = \Big(\mathrm{Y}_i\mathscr{T}'_{\mathbf{r}_i,-1}(x)\mathrm{Y}_i^{-1}\Big)\mathrm{Y}_i\mathscr{T}'_{\mathbf{r}_i,-1}(v). \tag{10.14}$$

By Proposition 10.1, we have $\mathrm{Y}_i\mathscr{T}_{\mathbf{r}_i,-1}(x)\mathrm{Y}_i^{-1} = \mathbf{T}'_{i,-1}(x)$. Hence, using the definition (10.12), the identity (10.14) implies that $\mathbf{T}'_{i,-1}(xv) = \mathbf{T}'_{i,-1}(x)\mathbf{T}'_{i,-1}(v)$ as desired. □

10.3 | Relative braid relations on U-modules

Let m_{ij} denotes the order of $\mathbf{r}_i\mathbf{r}_j$ in W°.

Theorem 10.6. *Let $M \in \mathcal{F}$. Then the relative braid relations hold for the linear operators $\mathbf{T}'_{i,e}$ (and, respectively, $\mathbf{T}''_{i,e}$) on M; that is, for any $i \neq j \in \mathbb{I}_{\circ,\tau}$ and for any $v \in M$, we have*

$$\underbrace{\mathbf{T}'_{i,e}\mathbf{T}'_{j,e}\mathbf{T}'_{i,e}\cdots}_{m_{ij}}(v) = \underbrace{\mathbf{T}'_{j,e}\mathbf{T}'_{i,e}\mathbf{T}'_{j,e}\cdots}_{m_{ij}}(v), \tag{10.15}$$

$$\underbrace{\mathbf{T}''_{i,e}\mathbf{T}''_{j,e}\mathbf{T}''_{i,e}\cdots}_{m_{ij}}(v) = \underbrace{\mathbf{T}''_{j,e}\mathbf{T}''_{i,e}\mathbf{T}''_{j,e}\cdots}_{m_{ij}}(v). \tag{10.16}$$

TABLE 3 Rank 2 formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ $(i \neq j \in \mathbb{I}_{\circ,\tau})$.

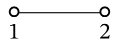
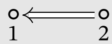
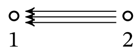
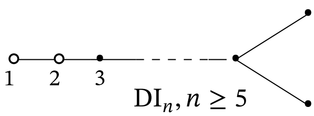
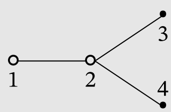
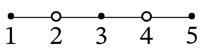
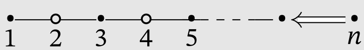
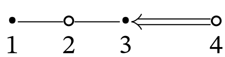
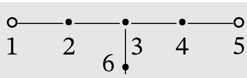
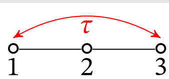
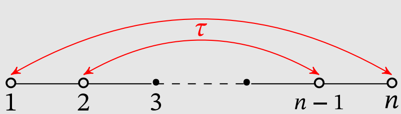
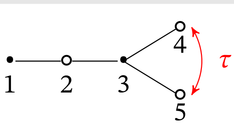
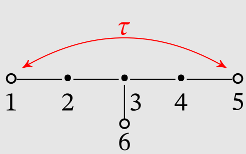
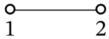
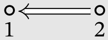
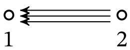
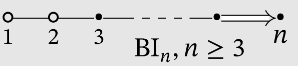
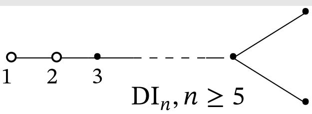
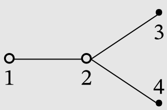
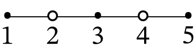
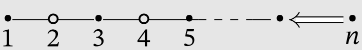
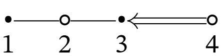
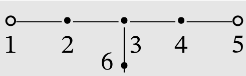
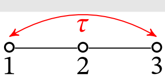
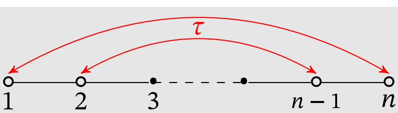
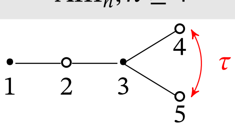
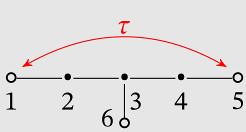
Rank 2 Satake diagrams		Formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$
AI_2		$\widetilde{\mathbf{T}}'_{1,-1}(B_2) = [B_1, B_2]_q$
CI_2		$\widetilde{\mathbf{T}}'_{1,-1}(B_2) = \frac{1}{[2]_{q_1}}\big[B_1, [B_1, B_2]_{q_1^2}\big] - q_1^2 B_2 \mathcal{K}_1$
G_2		$\widetilde{\mathbf{T}}'_{1,-1}(B_2) = \frac{1}{[3]!}\Big[B_1, \big[B_1, [B_1, B_2]_{q^3}\big]_q\Big]_{q^{-1}}$ $-\frac{1}{[3]!}\big(q(1+[3])[B_1, B_2]_{q^3} + q^3[3][B_1, B_2]_{q^{-1}}\big)\widetilde{k}_1$
$\mathrm{BI}_n, n \geq 3$		$\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, B_1]_{q_2}\big]_{q_2} - q_2 B_1 \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)$
$\mathrm{DI}_n, n \geq 5$		$\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, B_1]_q\big]_q - q B_1 \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)$
DIII_4		$\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathscr{T}}_{w_\bullet}(B_2), [B_2, B_1]_q\big]_q - q B_1 \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)$
AII_5		$\widetilde{\mathbf{T}}'_{4,-1}(B_2) = [\widetilde{\mathscr{T}}_3(B_4), B_2]_q$
$\mathrm{CII}_n, n \geq 5$		$\widetilde{\mathbf{T}}'_{4,-1}(B_2) = \big[[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4), \widetilde{\mathscr{T}}_3(B_4)]_{q_2}, B_2\big]_{q_2}$ $-q_2 \widetilde{\mathscr{T}}_3^{-2}(B_2)\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4)$
CII_4		$\widetilde{\mathbf{T}}'_{4,-1}(B_2) = \big[[B_4, F_3]_{q_4}, B_2\big]_{q_3}$ $\widetilde{\mathbf{T}}'_{2,-1}(B_4) = \big[\widetilde{\mathscr{T}}_3(B_2), [\widetilde{\mathscr{T}}_3(B_2), B_4]_{q_3^2}\big]$ $-(q_3 - q_3^{-1})[F_3, B_4]_{q_3^2} E_1 \widetilde{\mathscr{T}}_3(\mathcal{K}_2) K_1'^{-1}$
EIV		$\widetilde{\mathbf{T}}'_{1,-1}(B_5) = \big[\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_3\widetilde{\mathscr{T}}_2(B_1), B_5\big]_q$
AIII_3		$\widetilde{\mathbf{T}}'_{1,-1}(B_2) = \big[B_3, [B_1, B_2]_q\big]_q - q B_2 \mathcal{K}_3$
$\mathrm{AIII}_n, n \geq 4$		$\widetilde{\mathbf{T}}'_{1,-1}(B_2) = [B_1, B_2]_q$ $\widetilde{\mathbf{T}}'_{2,-1}(B_1) = \big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}), [B_2, B_1]_q\big]_q - B_1 \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1})$
DIII_5		$\widetilde{\mathbf{T}}'_{2,-1}(B_4) = [\widetilde{\mathscr{T}}_3(B_2), B_4]_q$ $\widetilde{\mathbf{T}}'_{4,-1}(B_2) = \big[B_4, [\widetilde{\mathscr{T}}_3(B_5), B_2]_q\big]_q - \widetilde{\mathscr{T}}_3^{-2}(B_2)\mathcal{K}_4$
EIII		$\widetilde{\mathbf{T}}'_{6,-1}(B_1) = [\widetilde{\mathscr{T}}_{23}(B_6), B_1]_q$ $\widetilde{\mathbf{T}}'_{1,-1}(B_6) = \big[\widetilde{\mathscr{T}}_4(B_5), [\widetilde{\mathscr{T}}_{32}(B_1), B_6]_q\big]_q$ $-\widetilde{\mathscr{T}}_{32323}^{-1}(B_6)\widetilde{\mathscr{T}}_4(\mathcal{K}_5)$

TABLE 4 Rank 2 formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$ $(i \neq j \in \mathbb{I}_{\circ,\tau})$.

Rank 2 Satake diagrams		**Formulas for $\widetilde{\mathbf{T}}''_{i,+1}(B_j)$**
AI_2		$\widetilde{\mathbf{T}}''_{1,+1}(B_2) = [B_2, B_1]_q$
CI_2		$\widetilde{\mathbf{T}}''_{1,+1}(B_2) = \frac{1}{[2]_{q_1}}\big[[B_2, B_1]_{q_1^2}, B_1\big] - q_1^2 B_2 \mathcal{K}_1$
G_2		$\widetilde{\mathbf{T}}''_{1,+1}(B_2) = \frac{1}{[3]_1!}\Big[\big[[B_2, B_1]_{q_1^3}, B_1\big]_{q_1}, B_1\Big]_{q_1^{-1}}$ $-\frac{1}{[3]_1!}\big(q_1(1+[3]_1)[B_2, B_1]_{q_1^3} + q_1^3[3]_1[B_2, B_1]_{q_1^{-1}}\big)\widetilde{k}_1$
$\mathrm{BI}_n, n \geq 3$		$\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_{q_2}, \mathscr{T}_{w_\bullet}^{-1}(B_2)\big]_{q_2} - q_2 B_1 \mathcal{K}_2$
$\mathrm{DI}_n, n \geq 5$		$\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_q, \mathscr{T}_{w_\bullet}^{-1}(B_2)\big]_q - q B_1 \mathcal{K}_2$
DIII_4		$\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_q, \mathscr{T}_{w_\bullet}^{-1}(B_2)\big]_q - q B_1 \mathcal{K}_2$
AII_5		$\widetilde{\mathbf{T}}''_{4,+1}(B_2) = [B_2, \mathscr{T}_3^{-1}(B_4)]_q$
$\mathrm{CII}_n, n \geq 5$		$\widetilde{\mathbf{T}}''_{4,+1}(B_2) = \big[B_2, [\mathscr{T}_3^{-1}(B_4), \mathscr{T}_{5\cdots n\cdots 5}^{-1}(B_4)]_{q_2}\big]_{q_2}$ $-q_2^2 \mathscr{T}_3^2(B_2)\mathscr{T}_3(\mathcal{K}_4)$
CII_4		$\widetilde{\mathbf{T}}''_{4,+1}(B_2) = \big[B_2, [F_3, B_4]_{q_4}\big]_{q_3}$ $\widetilde{\mathbf{T}}''_{2,+1}(B_4) = \big[[B_4, \mathscr{T}_3^{-1}(B_2)]_{q_3^2}, \mathscr{T}_3^{-1}(B_2)\big]$ $-(q_3 - q_3^{-1})[B_4, F_3]_{q_3^2} E_1 \mathcal{K}_2 K_1'^{-1}$
EIV		$\widetilde{\mathbf{T}}''_{1,+1}(B_5) = \big[B_5, \mathscr{T}_4^{-1}\mathscr{T}_3^{-1}\mathscr{T}_2^{-1}(B_1)\big]_q$
AIII_3		$\widetilde{\mathbf{T}}''_{1,+1}(B_2) = \big[[B_2, B_1]_q, B_3\big]_q - q B_2 \mathcal{K}_1$
$\mathrm{AIII}_n, n \geq 4$		$\widetilde{\mathbf{T}}''_{1,+1}(B_2) = [B_2, B_1]_q$ $\widetilde{\mathbf{T}}''_{2,+1}(B_1) = \big[[B_1, B_2]_q, \mathscr{T}_{w_\bullet}^{-1}(B_{n-1})\big]_q - \mathcal{K}_2 B_1$
DIII_5		$\widetilde{\mathbf{T}}''_{2,+1}(B_4) = [B_4, \mathscr{T}_3^{-1}(B_2)]_q$ $\widetilde{\mathbf{T}}''_{4,+1}(B_2) = \big[[B_2, \mathscr{T}_3^{-1}(B_5)]_q, B_4\big]_q - q \mathscr{T}_3^2(B_2)\mathscr{T}_3(\mathcal{K}_5)$
EIII		$\widetilde{\mathbf{T}}''_{6,+1}(B_1) = [B_1, \mathscr{T}_2^{-1}\mathscr{T}_3^{-1}(B_6)]_q$ $\widetilde{\mathbf{T}}''_{1,+1}(B_6) = \big[[B_6, \mathscr{T}_3^{-1}\mathscr{T}_2^{-1}(B_1)]_q, \mathscr{T}_4^{-1}(B_5)\big]_q$ $-q \mathscr{T}_{32323}(B_6)\mathscr{T}_{s_4 w_\bullet}(\mathcal{K}_1)$

Proof. We prove the first identity for $e = -1$; the proofs for the remaining ones are similar and skipped.

Set $m = m_{ij}$. We keep the notations $\boldsymbol{w}_\circ, \boldsymbol{w}'_\circ, \boldsymbol{w}_k, \boldsymbol{w}'_k$ for $1 \leqslant k \leqslant m$ from the proof of Theorem 9.1. We shall write $\mathscr{T}_{\mathbf{r}_i}^{-1}$ for $\mathscr{T}'_{\mathbf{r}_i,-1,\varsigma}$ and omit the subindex ς for $\mathrm{Y}_{i,\varsigma}$ in the proof, since there is no confusion.

By definition (10.12), for any $v \in M$, we have

$$\underbrace{\mathbf{T}'_{i,-1}\mathbf{T}'_{j,-1}\mathbf{T}'_{i,-1}\cdots}_{m}(v) = (\mathrm{Y}_i \mathscr{T}_{\boldsymbol{w}_1}^{-1}\mathrm{Y}_j \mathscr{T}_{\boldsymbol{w}_2}^{-1}\mathrm{Y}_i\cdots)\underbrace{\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{\mathbf{r}_j}^{-1}\mathscr{T}_{\mathbf{r}_i}^{-1}\cdots}_{m}(v). \tag{10.17}$$

By taking a central reduction to (9.7), the first factor on RHS (10.17) is $\sigma(\mathrm{Y}_{\boldsymbol{w}_\circ})$. Hence, we have

$$\underbrace{\mathbf{T}'_{i,-1}\mathbf{T}'_{j,-1}\mathbf{T}'_{i,-1}\cdots}_{m}(v) = \sigma(\mathrm{Y}_{\boldsymbol{w}_\circ})\underbrace{\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{\mathbf{r}_j}^{-1}\mathscr{T}_{\mathbf{r}_i}^{-1}\cdots}_{m}(v). \tag{10.18}$$

Similarly, by switching i, j in (10.18), we obtain

$$\underbrace{\mathbf{T}'_{j,-1}\mathbf{T}'_{i,-1}\mathbf{T}'_{j,-1}\cdots}_{m}(v) = \sigma(\mathrm{Y}_{\boldsymbol{w}'_\circ})\underbrace{\mathscr{T}_{\mathbf{r}_j}^{-1}\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{\mathbf{r}_j}^{-1}\cdots}_{m}(v). \tag{10.19}$$

Applying a central reduction to Theorem 8.1, we have $\mathrm{Y}_{\boldsymbol{w}_\circ} = \mathrm{Y}_{\boldsymbol{w}'_\circ}$. Since $\mathscr{T}_i$ are defined by rescaling $T''_{i,+1}$ in (10.1), they satisfy the braid relations. Hence, we have

$$\underbrace{\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{\mathbf{r}_j}^{-1}\mathscr{T}_{\mathbf{r}_i}^{-1}\cdots}_{m}(v) = \underbrace{\mathscr{T}_{\mathbf{r}_j}^{-1}\mathscr{T}_{\mathbf{r}_i}^{-1}\mathscr{T}_{\mathbf{r}_j}^{-1}\cdots}_{m}(v). \tag{10.20}$$

Combining (10.18)–(10.20), we have proved the first identity for $e = -1$. □

APPENDIX: PROOFS OF PROPOSITION 5.11 AND TABLE 3

In this appendix, we shall provide constructive proofs for Proposition 5.11 and verify the rank 2 formulas for $\widetilde{\mathbf{T}}'_{i,-1}(B_j)$ in Table 3. The proofs are based on type-by-type computations in $\widetilde{\mathbf{U}}$ for each rank 2 Satake diagram. Along the way, we will also specify a reduced expression for $\mathbf{r}_i$ in W.

A.1 | Some preparatory lemmas

Denote the t-commutator

$$[C, D]_t = CD - tDC,$$

for various q-powers t. Let $(\mathbb{I} = \mathbb{I}_\bullet \cup \mathbb{I}_\circ, \tau)$ be an arbitrary Satake diagram. Recall that $B_i = F_i + \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau i})K'_i$ and $B^\sigma_i = F_i + K_i\widetilde{\mathscr{T}}_{w_\bullet}^{-1}(E_{\tau i})$.

Lemma A.1. *Suppose that $i, j \in \mathbb{I}_\circ$ such that $j \notin \{i, \tau i\}$. Then we have*

$$[B^\sigma_i, F_j]_{q^{-(\alpha_i,\alpha_j)}} = [F_i, F_j]_{q^{-(\alpha_i,\alpha_j)}}, \tag{A.1}$$

$$[B_i, \widetilde{\mathscr{T}}_{w_\bullet}(\widetilde{E}_{\tau j})K'_j]_{q^{-(\alpha_i,\alpha_j)}} = q^{-(\alpha_i, w_\bullet(\alpha_{\tau j}))}\widetilde{\mathscr{T}}_{w_\bullet}\Big([E_{\tau i}, E_{\tau j}]_{q^{-(\alpha_i,\alpha_j)}}\Big)K'_i K'_j. \tag{A.2}$$

Proof. Follows by a simple computation and using the identity $[E_k, F_j] = 0$, for $k \neq j$. □

Introduce the following operator (see Lemma 4.4 for some of the notations)

$$\ni := \widetilde{\mathscr{T}}_{w_0} \widetilde{\mathscr{T}}_{w_\bullet} \widehat{\tau}_0 \widehat{\tau}. \tag{A.3}$$

We shall formulate several basic properties for $\ni$ below. A systematic use of $\ni$ throughout the Appendices will allow us to reduce the proofs of many challenging identities to easier ones.

Lemma A.2. *We have*

$$\ni(B_i^\sigma) = -q^{-(\alpha_i,\alpha_i)} B_i \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{\tau i}^{-1}), \tag{A.4}$$

$$\ni(F_j) = -q_j^{-2} \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j}) K'_j \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{\tau j}^{-1}). \tag{A.5}$$

Proof. We rewrite the identity (4.19) as follows:

$$\begin{aligned} B_i \widetilde{\mathscr{T}}_{\mathbf{r}_i}(\mathcal{K}_{\tau_{\bullet,i}\tau i}) &= -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})} \widetilde{\mathscr{T}}_{w_\bullet} \widetilde{\mathscr{T}}_{w_{\bullet,i}}(B^\sigma_{\tau_{\bullet,i}\tau i}) \\ &= -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})} \widetilde{\mathscr{T}}_{w_\bullet} \widetilde{\mathscr{T}}_{w_0}(B^\sigma_{\tau_0 \tau i}) = -q^{-(\alpha_i, w_\bullet \alpha_{\tau i})} \ni(B_i^\sigma). \end{aligned} \tag{A.6}$$

Since $\widetilde{\mathscr{T}}_{\mathbf{r}_i}(\mathcal{K}_{\tau_{\bullet,i}\tau i}) = \widetilde{\mathscr{T}}_{w_\bullet} \widetilde{\mathscr{T}}_{w_{\bullet,i}}(\mathcal{K}_{\tau_{\bullet,i}\tau i}) = \varsigma_{i,\diamond}^2 \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{\tau i}^{-1})$, the formula (A.4) follows from (A.6).

By Lemma 4.4, we have $\ni(F_j) = -K^{-1}_{w_\bullet(\tau j)} \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j}) = -q_j^{-2} \widetilde{\mathscr{T}}_{w_\bullet}(E_{\tau j}) K'_j \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{\tau j}^{-1})$. This proves (A.5). □

Lemma A.3. *The operator $\ni$ commutes with $\widetilde{\mathscr{T}}_{\mathbf{r}_i}, \widetilde{\mathscr{T}}_j$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$.*

Proof. Since $w_0 s_k = s_{\tau_0 k} w_0$, for $k \in \mathbb{I}$, we have $\widetilde{\mathscr{T}}_{w_0} \widetilde{\mathscr{T}}_k^{-1} = \widetilde{\mathscr{T}}_{w_0 s_k} = \widetilde{\mathscr{T}}_{s_{\tau_0 k} w_0} = \widetilde{\mathscr{T}}_{\tau_0 k}^{-1} \widetilde{\mathscr{T}}_{w_0}$. Hence, $\widetilde{\mathscr{T}}_{w_0} \widetilde{\mathscr{T}}_k = \widetilde{\mathscr{T}}_{\tau_0 k} \widetilde{\mathscr{T}}_{w_0}$ for any $k \in \mathbb{I}$. Therefore, $\widetilde{\mathscr{T}}_{w_0} \widehat{\tau}_0$ commutes with $\widetilde{\mathscr{T}}_k$ ($k \in \mathbb{I}$) and thus commutes with $\widetilde{\mathscr{T}}_{\mathbf{r}_i}, \widetilde{\mathscr{T}}_j$, for $i \in \mathbb{I}_\circ, j \in \mathbb{I}_\bullet$.

Similarly, one can show that $\widetilde{\mathscr{T}}_{w_\bullet} \widehat{\tau}$ commutes with $\widetilde{\mathscr{T}}_j$, for $j \in \mathbb{I}_\bullet$. Hence, by definition (A.3), the operator $\ni$ commutes with $\widetilde{\mathscr{T}}_j$ for $j \in \mathbb{I}_\bullet$.

On the other hand, by definition (2.14), $\widetilde{\mathscr{T}}_{\mathbf{r}_i}$, for $i \in \mathbb{I}_\circ$, commutes with both $\widetilde{\mathscr{T}}_{w_\bullet}$ and $\widehat{\tau}$. Hence, $\ni$ also commutes with $\widetilde{\mathscr{T}}_{\mathbf{r}_i}$. □

A.2 | Split types of rank 2

Consider rank 2 split Satake diagrams ($\mathbb{I} = \mathbb{I}_\circ = \{i, j\}$, Id). In this case, we have $\mathbf{r}_i = s_i$, $B_i^\sigma = F_i + K_i E_i$.

A.2.1 | $c_{ij} = -1$

In this case, in line with the first line of Table 3, Proposition 5.11 is reformulated and proved as follows.

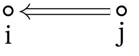
Lemma A.4. *We have*

$$\widetilde{\mathscr{T}}_i^{-1}(F_j) = [B_i^\sigma, F_j]_{q_i}, \qquad \widetilde{\mathscr{T}}_i^{-1}(E_jK_j') = [B_i, E_jK_j']_{q_i}. \tag{A.7}$$

Proof. Follows immediately by Lemma A.1 and the definition of $\widetilde{\mathscr{T}}_i$. □

A.2.2 | $c_{ij} = -2$

In this case, the rank 2 Satake diagram is given by



and in line with Table 3, Proposition 5.11 can be reformulated and proved as follows.

Lemma A.5. *We have*

$$\widetilde{\mathscr{T}}_i^{-1}(F_j) = \frac{1}{[2]_i}\Big[B_i^\sigma, [B_i^\sigma, F_j]_{q_i^2}\Big] - q_i^2 F_j K_i K_i', \tag{A.8}$$

$$\widetilde{\mathscr{T}}_i^{-1}(E_jK_j') = \frac{1}{[2]_i}\Big[B_i, [B_i, E_jK_j']_{q_i^2}\Big] - q_i^2 E_jK_j'K_iK_i'. \tag{A.9}$$

Proof. We prove the formula (A.8). By Lemma A.1, we have $[B_i^\sigma, F_j]_{q_i^2} = [F_i, F_j]_{q_i^2}$. By Proposition 4.2, we have $\widetilde{\mathscr{T}}_i^{-1}(F_j) = \frac{1}{[2]_i}[F_i, [F_i, F_j]_{q_i^2}]$. Now we compute the first term on RHS (A.8) using Lemma A.1 as follows:

$$\begin{aligned}
\Big[B_i^\sigma, [B_i^\sigma, F_j]_{q_i^2}\Big] &= \Big[B_i^\sigma, [F_i, F_j]_{q_i^2}\Big] \\
&= \Big[F_i, [F_i, F_j]_{q_i^2}\Big] + \Big[K_iE_i, [F_i, F_j]_{q_i^2}\Big] \\
&= [2]_i \widetilde{\mathscr{T}}_i^{-1}(F_j) + K_i\Big[\frac{K_i - K_i'}{q_i - q_i^{-1}}, F_j\Big]_{q_i^2} \\
&= [2]_i \widetilde{\mathscr{T}}_i^{-1}(F_j) + q_i^2 [2]_i F_j K_i K_i'.
\end{aligned}$$

Hence, the formula (A.8) holds.

We next prove the formula (A.9). In this case, we read (A.3) as $\backepsilon = \widetilde{\mathscr{T}}_{w_0}$, and note that $\mathcal{K}_i = \widetilde{k}_i$. By Lemma A.3, $\backepsilon$ it commutes with $\widetilde{\mathscr{T}}_i^{-1}$. Applying this operator $\backepsilon$ to the formula (A.8) and then using (A.4)–(A.5), we obtain

$$\widetilde{\mathscr{T}}_i^{-1}(E_jK_j')\widetilde{\mathscr{T}}_i^{-1}(\widetilde{k}_j^{-1}) = \frac{q_i^{-4}}{[2]_i}\Big[B_i\widetilde{k}_i^{-1}, [B_i\widetilde{k}_i^{-1}, E_jK_j'\widetilde{k}_j^{-1}]_{q_i^2}\Big] - q_i^2 E_jK_j'\widetilde{k}_j^{-1}\widetilde{\mathscr{T}}_{w_0}(\widetilde{k}_i). \tag{A.10}$$

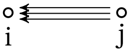
Recall that our symmetries $\widetilde{\mathscr{T}}_j$ are defined in §4.1 by normalizing a variant of Lusztig's symmetries $\widetilde{\mathscr{T}}''_{j,+1}$. In this case, we have $\widetilde{\mathscr{T}}_{w_0}(\widetilde{k}_i) = q_i^{-4}\widetilde{k}_i^{-1}$ and $\widetilde{\mathscr{T}}_i^{-1}(\widetilde{k}_j^{-1}) = q_i^{-4}\widetilde{k}_j^{-1}\widetilde{k}_i^{-2}$. Hence, since $\widetilde{k}_i, \widetilde{k}_j$ are central, (A.10) is simplified as the following formula:

$$\widetilde{\mathscr{T}}_i^{-1}(E_jK_j')\widetilde{k}_j^{-1}\widetilde{k}_i^{-2} = \left(\frac{1}{[2]_i}\Big[B_i, [B_i, E_jK_j']_{q_i^2}\Big] - q_i^2 E_jK_j'K_iK_i'\right)\widetilde{k}_j^{-1}\widetilde{k}_i^{-2}, \tag{A.11}$$

which clearly implies the formula (A.9). □

A.2.3 | $c_{ij} = -3$

Consider the Satake diagram of split type G_2



In this case, we have $q_i = q$ and $q_j = q^3$.

Lemma A.6. *We have*

$$\big[K_iE_i, [F_i, F_j]_{q^3}\big]_q = q^3[3]F_jK_iK_i', \tag{A.12}$$

$$\Big[K_iE_i, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}} = q(1+[3])[B_i^\sigma, F_j]_{q^3}K_iK_i'. \tag{A.13}$$

Proof. The first identity (A.12) is derived as follows:

$$\text{LHS}(A.12) = K_i\big[E_i, [F_i, F_j]_{q^3}\big] = K_i[\frac{K_i - K_i^{-1}}{q-q^{-1}}, F_j]_{q^3} = q^3[3]K_iK_i'F_j = \text{RHS}(A.12).$$

We next compute

$$\begin{aligned}
\text{LHS}(A.13) &= K_i\Big[E_i, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big] \\
&= K_i\left[\frac{K_i - K_i^{-1}}{q-q^{-1}}, [F_i, F_j]_{q^3}\right]_q + K_i\left[F_i, [\frac{K_i - K_i'}{q-q^{-1}}, F_j]_{q^3}\right]_q \\
&= qK_iK_i'[F_i, F_j]_{q^3} + q^3[3]K_i\big[F_i, K_i'F_j\big]_q \\
&= (q + q[3])[F_i, F_j]_{q^3}K_iK_i' \\
&= (q + q[3])[B_i^\sigma, F_j]_{q^3}K_iK_i',
\end{aligned}$$

where the last equality follows from Lemma A.1. This proves (A.13). □

In line with Table 3, Proposition 5.11 can be reformulated and proved as follows.

Lemma A.7. *We have*

$$\begin{aligned}
\widetilde{\mathscr{T}}_i^{-1}(F_j) = &\frac{1}{[3]!}\Big[B_i^\sigma, \big[B_i^\sigma, [B_i^\sigma, F_j]_{q^3}\big]_q\Big]_{q^{-1}} \\
&- \frac{1}{[3]!}\big(q(1+[3])[B_i^\sigma, F_j]_{q^3} + q^3[3][B_i^\sigma, F_j]_{q^{-1}}\big)\widetilde{k}_i.
\end{aligned} \tag{A.14}$$

Proof. By Proposition 4.2, $\widetilde{\mathscr{T}}_i^{-1}(F_j) = \frac{1}{[3]!}[F_i, [F_i, [F_i, F_j]_{q^3}]_q]_{q^{-1}}$. By Lemma A.1, we have $[B_i^\sigma, F_j]_{q^3} = [F_i, F_j]_{q^3}$. Then we have

$$\Big[B_i^\sigma, \big[B_i^\sigma, [B_i^\sigma, F_j]_{q^3}\big]_q\Big]_{q^{-1}} = \Big[B_i^\sigma, \big[F_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}} + \Big[B_i^\sigma, \big[K_iE_i, [F_i, F_j]_{q^3}\big]_q\Big]_{q^{-1}}. \tag{A.15}$$

Using Lemma A.6, we rewrite RHS (A.15) as

$$\Big[F_i,\Big[F_i,[F_i,F_j]_{q^3}\Big]_q\Big]_{q^{-1}}+\Big[K_iE_i,\Big[F_i,[F_i,F_j]_{q^3}\Big]_q\Big]_{q^{-1}}+q^3[3][B_i^\sigma,F_j]_{q^{-1}}K_iK_i'$$
$$=[3]!\widetilde{\mathscr{T}}_i^{-1}(F_j)+q(1+[3])[B_i^\sigma,F_j]_{q^3}K_iK_i'+q^3[3][B_i^\sigma,F_j]_{q^{-1}}K_iK_i'. \tag{A.16}$$

Now the desired formula (A.14) follows from (A.15)–(A.16). □

Lemma A.8. *We have*

$$\widetilde{\mathscr{T}}_i^{-1}(E_jK_j')=\frac{1}{[3]!}\Big[B_i,\Big[B_i,[B_i,E_jK_j']_{q^3}\Big]_q\Big]_{q^{-1}}$$
$$-\frac{1}{[3]!}\Big(q(1+[3])[B_i,E_jK_j']_{q^3}-q^3[3][B_i,E_jK_j']_{q^{-1}}\Big)\widetilde{k}_i. \tag{A.17}$$

Proof. In this case, $\mathcal{K}_i=\widetilde{k}_i$ and $\mathcal{K}_j=\widetilde{k}_j$ are central. By (A.4)–(A.5), we have

$$\backepsilon(F_j)=-q_j^{-2}E_jK_j'\widetilde{k}_j^{-1},\qquad \backepsilon(B_i^\sigma)=-q^{-2}B_i\widetilde{k}_i^{-1}. \tag{A.18}$$

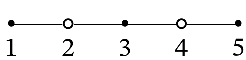
Recall from Lemma A.3 that $\backepsilon$ commutes with $\widetilde{\mathscr{T}}_i$. Applying $\backepsilon$ to (A.14) and then using (A.18), we have

$$\widetilde{\mathscr{T}}_i^{-1}(E_jK_j')\widetilde{\mathscr{T}}_i^{-1}(\widetilde{k}_j^{-1})$$
$$=-q^{-6}\frac{1}{[3]!}\Big[B_i,\Big[B_i,[B_i,E_jK_j']_{q^3}\Big]_q\Big]_{q^{-1}}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3} \tag{A.19}$$
$$+q^{-2}\frac{1}{[3]!}\Big(q(1+[3])[B_i^\sigma,E_jK_j']_{q^3}-q^3[3][B_i^\sigma,E_jK_j']_{q^{-1}}\Big)\backepsilon(\widetilde{k}_i)\widetilde{k}_j^{-1}\widetilde{k}_i^{-1}.$$

Since $s_i(\alpha_j)=\alpha_j+3\alpha_i$, by Proposition 4.2, we have $\widetilde{\mathscr{T}}_i^{-1}(\widetilde{k}_j^{-1})=-q^{-6}\widetilde{k}_j^{-1}\widetilde{k}_i^{-3}$. Note also that $\backepsilon(\widetilde{k}_i)=q^{-4}\widetilde{k}_i^{-1}$. Hence, (A.19) implies the desired formula (A.17). □

A.3 | Type AII

Consider the rank 2 Satake diagram of type AII_5



$\mathbf{r}_4=s_4s_3s_5s_4.$

In this case, Proposition 5.11 is reformulated and proved as follows.

Lemma A.9. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2)=[\widetilde{\mathscr{T}}_3(B_4^\sigma),F_2]_q,$$
$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big)=[\widetilde{\mathscr{T}}_3(B_4),\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2']_q.$$

Proof. The first formula follows by $\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1} = \widetilde{\mathscr{T}}_{4354}^{-1}$, Proposition 4.2, and the formula (A.1).

We prove the second formula. By (A.4)–(A.5), we have

$$\ni(F_2) = -q^{-2}\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}), \qquad \ni(B_4^\sigma) = -q^{-2}B_4\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1}). \tag{A.20}$$

Recall from Lemma A.3 that the operator $\ni$ in (A.3) commutes with $\widetilde{\mathscr{T}}_3, \widetilde{\mathscr{T}}_{\mathbf{r}_4}$. Applying the operator $\ni$ to both sides of the first formula and then using (A.20), we have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1}) = -q^{-2}[\widetilde{\mathscr{T}}_3(B_4)\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1}), \widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})]_q. \tag{A.21}$$

For a weight reason, we have

$$\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2' = q\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1}),$$

$$\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3(B_4) = q\widetilde{\mathscr{T}}_3(B_4)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).$$

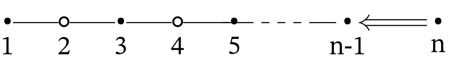
Using these two identities, we simplify (A.21) as

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1}) = -q^{-1}[\widetilde{\mathscr{T}}_3(B_4), \widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2']_q\widetilde{\mathscr{T}}_5(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \tag{A.22}$$

Finally, by Proposition 4.2, we have $\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2) = -q\widetilde{\mathscr{T}}_5(\mathcal{K}_4)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)$. Hence, (A.22) implies the second desired formula. □

A.4 | Type CII$_n$, $n \geqslant 5$

Consider the rank 2 Satake diagram of type CII$_n$, for $n \geqslant 5$:



$$\varsigma_{2,\diamond} = -q_2^{-1}, \qquad \varsigma_{4,\diamond} = -q_4^{-1/2}$$

$$\mathbf{r}_4 = s_{4\cdots n\cdots 4}s_3s_{4\cdots n\cdots 4}.$$

Note that $q_2 = q_4 = q$. The notation $4 \cdots n \cdots 4$ (with the local minima/maxima indicated) denotes a sequence $4\ 5 \cdots n-1\ n\ n-1 \cdots 5\ 4$, and we denote $s_{4\cdots n\cdots 4} = s_4 \cdots s_n \cdots s_4$.

In this case, Proposition 5.11 is reformulated and proved as Lemmas A.10–A.11 below.

Lemma A.10. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) = \Big[[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4^\sigma), \widetilde{\mathscr{T}}_3(B_4^\sigma)]_q, F_2\Big]_q - q\widetilde{\mathscr{T}}_3^{-2}(F_2)\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4). \tag{A.23}$$

Proof. Since $s_{5\cdots n\cdots 5}s_4s_{5\cdots n\cdots 5}(\alpha_4) = \alpha_4$, we have $\widetilde{\mathscr{T}}_4^{-1}\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}^{-1}(F_4) = \widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(F_4)$. Then

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) = \widetilde{\mathscr{T}}_{4\cdots n\cdots 4}^{-1}[F_3, F_2]_q = \Big[\widetilde{\mathscr{T}}_4^{-1}\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}^{-1}([F_4, F_3]_q), F_2\Big]_q$$

$$= \Big[\Big[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(F_4), [F_4, F_3]_q\Big]_q, F_2\Big]_q.$$

On the other hand, we compute RHS (A.23) as follows. First, note that

$$[K_4\widetilde{\mathscr{T}}_{w_\bullet}^{-1}(E_4),F_3]_q = q^{-1}\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}^{-1}(E_4)K_3K_4,$$

and hence,

$$\Big[[K_4\widetilde{\mathscr{T}}_{w_\bullet}(E_4),F_3]_q,F_2\Big]_q = [\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}^{-1}(E_4),F_2]K_3K_4 = 0.$$

Thus, we have

$$\begin{aligned}
&\Big[[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4^\sigma),\widetilde{\mathscr{T}}_3(B_4^\sigma)]_q,F_2\Big]_q\\
&=\Big[\Big[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4^\sigma),[B_4^\sigma,F_3]_q\Big]_q,F_2\Big]_q\\
&=\Big[\Big[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4^\sigma),[F_4,F_3]_q\Big]_q,F_2\Big]_q\\
&=\Big[\Big[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(F_4),[F_4,F_3]_q\Big]_q,F_2\Big]_q+\Big[\Big[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4)\widetilde{\mathscr{T}}_3^{-1}(E_4),[F_4,F_3]_q\Big]_q,F_2\Big]_q\\
&=\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2)+q\Big[\Big[\widetilde{\mathscr{T}}_3^{-1}(E_4),[F_4,F_3]_q\Big],F_2\Big]_q\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4)\\
&=\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2)+q\Big[[\widetilde{\mathscr{T}}_3^{-1}(F_3),F_3],F_2\Big]_{q^2}\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4)\\
&=\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2)+q\widetilde{\mathscr{T}}_3^{-2}(F_2)\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4),
\end{aligned}$$

as desired. This proves the formula (A.23). □

Lemma A.11. *We have*

$$\begin{aligned}
\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2') = {}&\Big[[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4),\widetilde{\mathscr{T}}_3(B_4)]_q,\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big]_q\\
&-q\widetilde{\mathscr{T}}_3^{-2}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big)\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4). \qquad (A.24)
\end{aligned}$$

Proof. By Lemma A.3, the operator ∍ in (A.3) commutes with $\widetilde{\mathscr{T}}_3, \widetilde{\mathscr{T}}_{5\cdots n\cdots 5}, \widetilde{\mathscr{T}}_{\mathbf{r}_4}$. Applying ∍ to (A.23) and then using (A.4)–(A.5), we obtain

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2')\widetilde{\mathscr{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1})\\
&=q^{-4}\Big[[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4)\widetilde{\mathscr{T}}_3(\mathcal{K}_4^{-1}),\widetilde{\mathscr{T}}_3(B_4)\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})]_q,\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\Big]_q\\
&-q\widetilde{\mathscr{T}}_3^{-2}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\backepsilon(\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4)). \qquad (A.25)
\end{aligned}$$

Recalling ck_i from (3.23), we have

$$\mathcal{K}_4 B_4 = q^{-3} B_4 \mathcal{K}_4,$$

$$\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4)\widetilde{\mathscr{T}}_3(B_4) = \widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4)\widetilde{\mathscr{T}}_3(B_4)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2),$$

$$\backepsilon(\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4)) = q^{-1}\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4).$$

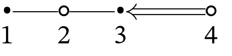
Using these formulas, we simplify (A.25) as

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2')\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2^{-1})\\
&= q^{-1}\Big[[\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(B_4), \widetilde{\mathscr{T}}_3(B_4)]_q, \widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big]_q \widetilde{\mathscr{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\\
&- \widetilde{\mathscr{T}}_3^{-2}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big)\widetilde{\mathscr{T}}_3(K_4')\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(K_4)\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_3(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}). \qquad \text{(A.26)}
\end{aligned}$$

Finally, by (3.23), we have $\widetilde{\mathscr{T}}_{w_{\bullet,4}}(\mathcal{K}_2) = q\widetilde{\mathscr{T}}_3(\mathcal{K}_4)\widetilde{\mathscr{T}}_{5\cdots n\cdots 5}(\mathcal{K}_4)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2)$. Therefore, the formula (A.24) follows from (A.26). □

A.5 | Type CII$_4$

Consider the rank 2 Satake diagram of type CII$_4$:



$$\mathbf{r}_4 = s_4 s_3 s_4, \qquad \mathbf{r}_2 = s_2 s_1 s_3 s_2.$$

In this case, Proposition 5.11 is reformulated and proved as Lemmas A.12–A.13 below.

Lemma A.12. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) = \Big[[B_4^\sigma, F_3]_{q_4}, F_2\Big]_{q_3}, \qquad \text{(A.27)}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_4) = \Big[\widetilde{\mathscr{T}}_3(B_2^\sigma), [\widetilde{\mathscr{T}}_3(B_2^\sigma), F_4]_{q_3^2}\Big] - (q_3 - q_3^{-1})[F_3, F_4]_{q_3^2} E_1 K_2 K_2' K_3. \qquad \text{(A.28)}$$

Proof. The first formula (A.27) follows by a direct computation.

We prove (A.28). We have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_4) = \Big[\widetilde{\mathscr{T}}_2^{-1}(F_3), [\widetilde{\mathscr{T}}_2^{-1}(F_3), F_4]_{q_3^2}\Big] = \Big[\widetilde{\mathscr{T}}_3(F_2), [\widetilde{\mathscr{T}}_3(F_2), F_4]_{q_3^2}\Big].$$

Hence, recalling that $B_2^\sigma = F_2 + K_2\widetilde{\mathscr{T}}_{13}^{-1}(E_2)$, we have

$$\begin{aligned}
\Big[\widetilde{\mathscr{T}}_3(B_2^\sigma), [\widetilde{\mathscr{T}}_3(B_2^\sigma), F_4]_{q_3^2}\Big] &= \Big[\widetilde{\mathscr{T}}_3(B_2^\sigma), [\widetilde{\mathscr{T}}_3(F_2), F_4]_{q_3^2}\Big]\\
&= \Big[\widetilde{\mathscr{T}}_3(F_2), [\widetilde{\mathscr{T}}_3(F_2), F_4]_{q_3^2}\Big] + \Big[K_2K_3\widetilde{\mathscr{T}}_1^{-1}(E_2), [\widetilde{\mathscr{T}}_3(F_2), F_4]_{q_3^2}\Big]\\
&= \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_4) + \Big[[\widetilde{\mathscr{T}}_1^{-1}(E_2), \widetilde{\mathscr{T}}_3(F_2)], F_4\Big]_{q_3^2} K_2K_3\\
&= \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_4) + (q_3 - q_3^{-1})[F_3, F_4]_{q_3^2} E_1 K_2 K_2' K_3.
\end{aligned}$$

Thus, (A.28) is proved. □

Lemma A.13. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big) = \Big[[B_4, F_3]_{q_4}, \widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\Big]_{q_3}, \tag{A.29}$$

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4'\Big) &= \Big[\widetilde{\mathscr{T}}_3(B_2), [\widetilde{\mathscr{T}}_3(B_2), \widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}\Big] \\ &\quad -(q_3 - q_3^{-1})[F_3, \widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}E_1K_2K_2'K_3.\end{aligned} \tag{A.30}$$

Proof. We shall prove the formula (A.30) only, and skip a similar proof for (A.29).

By Lemma A.3, the operator $\backepsilon$ defined in (A.3) commutes with $\widetilde{\mathscr{T}}_3, \widetilde{\mathscr{T}}_{\mathbf{r}_2}$. Applying $\backepsilon$ to the identity (A.28) and then using (A.4)–(A.5), we have

$$\begin{aligned}&\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\mathcal{K}_4^{-1}) \\ &= q_2^{-4}\Big[\widetilde{\mathscr{T}}_3(B_2)\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1}), [\widetilde{\mathscr{T}}_3(B_2)\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1}), \widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})]_{q_3^2}\Big] \\ &- (q_3 - q_3^{-1})q_3^{-4}[F_3K_3K_3'^{-1}, \widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})]_{q_3^2}E_1K_1^{-1}K_1'\backepsilon(K_2K_2'K_3).\end{aligned} \tag{A.31}$$

For a weight reason, we have

$$\begin{aligned}\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(E_4) &= q_3^2\widetilde{\mathscr{T}}_{w_\bullet}(E_4)\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1}),\\ \widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_3(B_2) &= q_3^2\widetilde{\mathscr{T}}_3(B_2)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1}),\\ \widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3(B_2)\widetilde{\mathscr{T}}_{w_\bullet}(E_4) &= \widetilde{\mathscr{T}}_3(B_2)\widetilde{\mathscr{T}}_{w_\bullet}(E_4)\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1}),\\ \widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_3(B_2) &= \widetilde{\mathscr{T}}_3(B_2)\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1}).\end{aligned}$$

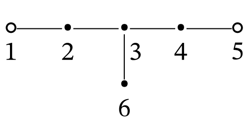
We also have $\backepsilon(K_2K_2'K_3) = q_2^{-2}\widetilde{\mathscr{T}}_{w_\bullet}(K_2K_2')^{-1}K_3$. Hence, (A.31) is simplified as

$$\begin{aligned}&\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\mathcal{K}_4^{-1}) \\ &\quad = q_2^{-2}\Big[\widetilde{\mathscr{T}}_3(B_2), [\widetilde{\mathscr{T}}_3(B_2), \widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}\Big]\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})^2 \\ &\quad - (q_3 - q_3^{-1})q_2^{-2}[F_3, \widetilde{\mathscr{T}}_{w_\bullet}(E_4)K_4']_{q_3^2}E_1K_2K_2'K_3\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})^2.\end{aligned} \tag{A.32}$$

By the definition of $\mathcal{K}_i$ in (3.23), we have $\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\mathcal{K}_4^{-1}) = q_2^{-2}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_4^{-1})\widetilde{\mathscr{T}}_1(\mathcal{K}_2^{-1})^2$. Thus, (A.32) implies the desired formula (A.30). □

A.6 | Type EIV

Consider the rank 2 Satake diagram of type EIV:



$$\mathbf{r}_1 = s_1s_2s_3s_4s_6s_3s_2s_1.$$

In this case, Proposition 5.11 is reformulated and proved as Lemma A.14 below.

Lemma A.14.

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_5) = \Big[\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_3\widetilde{\mathscr{T}}_2(B_1^{\sigma}), F_5\Big]_q, \tag{A.33}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_5'\Big) = \Big[\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_3\widetilde{\mathscr{T}}_2(B_1), \widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_5'\Big]_q. \tag{A.34}$$

Proof. We prove the formula (A.33). Indeed, we have

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_5) = \widetilde{\mathscr{T}}_1^{-1}\widetilde{\mathscr{T}}_2^{-1}\widetilde{\mathscr{T}}_3^{-1}[F_4, F_5]_q &= [\widetilde{\mathscr{T}}_1^{-1}\widetilde{\mathscr{T}}_2^{-1}\widetilde{\mathscr{T}}_3^{-1}(F_4), F_5]_q \\ &= [\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_3\widetilde{\mathscr{T}}_2(F_1), F_5]_q = [\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_3\widetilde{\mathscr{T}}_2(B_1^{\sigma}), F_5]_q.\end{aligned}$$

We next prove the formula (A.34). Recall from Lemma A.3 that $\widetilde{\mathscr{T}}_j$, for $j \in \mathbb{I}_\bullet$, commutes with $\ni$ in (A.3). Applying $\ni$ to the formula (A.33) and then using (A.4)–(A.5), we have

$$\begin{aligned}&\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_5'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,1}}(\mathcal{K}_5^{-1}) \\ &= -q^{-2}\Big[\widetilde{\mathscr{T}}_{432}(B_1)\widetilde{\mathscr{T}}_{632}(\mathcal{K}_1^{-1}), \widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_5'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1})\Big]_q.\end{aligned} \tag{A.35}$$

By a weight consideration, we have

$$\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1})\widetilde{\mathscr{T}}_{432}(B_1) = q\widetilde{\mathscr{T}}_{432}(B_1)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1}),$$

$$\widetilde{\mathscr{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(E_5) = q\widetilde{\mathscr{T}}_{w_\bullet}(E_5)\widetilde{\mathscr{T}}_{632}(\mathcal{K}_1^{-1}).$$

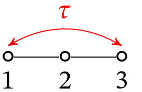
Hence, using these two identities, (A.35) is simplified as

$$\begin{aligned}&\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_5'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,1}}(\mathcal{K}_5^{-1}) \\ &= -q^{-1}\Big[\widetilde{\mathscr{T}}_{432}(B_1), \widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_5'\Big]_q\widetilde{\mathscr{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1}).\end{aligned} \tag{A.36}$$

Finally, by the definition (3.23) of $\mathcal{K}_i$, $\widetilde{\mathscr{T}}_{w_{\bullet,1}}(\mathcal{K}_5^{-1}) = -q^{-1}\widetilde{\mathscr{T}}_{632}(\mathcal{K}_1^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1})$. Then, (A.36) implies the desired formula (A.34). □

A.7 | Type AIII_3

Consider the rank 2 Satake diagram of type AIII_3:



$$\varsigma_{1,\diamond} = \varsigma_{3,\diamond} = -q^{-1}, \qquad \varsigma_{2,\diamond} = -q^{-2}$$
$$\mathbf{r}_1 = s_1s_3, \qquad \mathbf{r}_2 = s_2.$$

In this case, Proposition 5.11 is reformulated and proved as the following lemma.

Lemma A.15. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_2) = \big[B_3^\sigma, [B_1^\sigma, F_2]_q\big]_q - qF_2K_3K_1', \tag{A.37}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(E_2K_2') = \big[B_3, [B_1, E_2K_2']_q\big]_q - qE_2K_2'K_3K_1'. \tag{A.38}$$

Proof. By Lemma A.1, we have $[B_1^\sigma, F_2]_q = [F_1, F_2]_q$. Then, the first term on the RHS of (A.37) is computed as follows:

$$\begin{aligned}
\big[B_3^\sigma, [B_1^\sigma, F_2]_q\big]_q &= \big[K_3E_1, [F_1, F_2]_q\big]_q + \big[F_3, [F_1, F_2]_q\big]_q \\
&= q[[E_1, F_1], F_2]_q K_3 + \big[F_3, [F_1, F_2]_q\big]_q \\
&= q[\frac{K_1 - K_1'}{q - q^{-1}}, F_2]_q K_3 + \big[F_3, [F_1, F_2]_q\big]_q \\
&= qF_2K_3K_1' + \big[F_3, [F_1, F_2]_q\big]_q \\
&= \widetilde{\mathscr{T}}_{13}^{-1}(F_2) + qF_2K_3K_1'.
\end{aligned}$$

This proves the formula (A.37).

We next prove (A.38). In this case, $\tau_0 = \tau \neq \mathrm{Id}$, $\tau_{\bullet,1} = \mathrm{Id}$, and we simplify $\ni$ in (A.3) as $\ni = \widetilde{\mathscr{T}}_{w_0}$. We also have $\mathcal{K}_i = \widetilde{k}_i$ for $i = 1, 2, 3$. Applying the operator $\ni = \widetilde{\mathscr{T}}_{w_0}$ to the identity (A.37) and then using (A.4)–(A.5), we have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(E_2K_2')\widetilde{\mathscr{T}}_{\mathbf{r}_1}(\widetilde{k}_2^{-1}) = q^{-4}\big[B_3\widetilde{k}_1^{-1}, [B_1\widetilde{k}_3^{-1}, E_2K_2'\widetilde{k}_2^{-1}]_q\big]_q - qE_2K_2'\widetilde{k}_2^{-1}\ni(K_3K_1'). \tag{A.39}$$

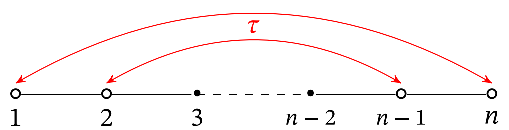
We have $\ni(K_3K_1') = q^{-2}\widetilde{k}_1^{-1}\widetilde{k}_3^{-1}K_3K_1'$. Note also that $\widetilde{k}_2$ is central and $\widetilde{k}_3, \widetilde{k}_1$ commute with E_2. Hence, (A.39) can be rewritten as

$$\begin{aligned}
\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(E_2K_2')\widetilde{\mathscr{T}}_{\mathbf{r}_1}(\widetilde{k}_2^{-1}) = q^{-2}&\big[B_3, [B_1, E_2K_2']_q\big]_q\widetilde{k}_1^{-1}\widetilde{k}_3^{-1}\widetilde{k}_2^{-1} \\
&- q^{-1}E_2K_2'\widetilde{k}_2^{-1}\widetilde{k}_1^{-1}\widetilde{k}_3^{-1}K_3K_1'.
\end{aligned} \tag{A.40}$$

Finally, since $\mathbf{r}_1(\alpha_2) = \alpha_2 + \alpha_1 + \alpha_3$, we have $\widetilde{\mathscr{T}}_{\mathbf{r}_1}(\widetilde{k}_2^{-1}) = q^{-2}\widetilde{k}_2^{-1}\widetilde{k}_1^{-1}\widetilde{k}_3^{-1}$. Therefore, the desired formula (A.38) follows from (A.40). □

A.8 | Type AIII$_n$, $n \geqslant 4$

Consider the rank 2 Satake diagram of type AIII$_n$, $n \geqslant 4$:



$$\varsigma_{1,\diamond} = \varsigma_{n,\diamond} = -q^{-1}, \qquad \varsigma_{2,\diamond} = \varsigma_{n-1,\diamond} = -q^{-1/2},$$
$$\mathbf{r}_1 = s_1s_n, \qquad \mathbf{r}_2 = s_2\cdots s_{n-1}\cdots s_2.$$

We first have a simple observation.

Lemma A.16. *For any* $3 \leqslant s \leqslant n-2$, $\widetilde{\mathscr{T}}_{2\cdots n-2}(F_{n-1})$ *is fixed by* $\widetilde{\mathscr{T}}_s$.

Proof. Recall from Proposition 4.2 that $\widetilde{\mathscr{T}}_s$ satisfies the braid relation. Then we have

$$\begin{aligned}\widetilde{\mathscr{T}}_s\widetilde{\mathscr{T}}_{2\cdots n-2}(F_{n-1}) &= \widetilde{\mathscr{T}}_{2\cdots s-2}\widetilde{\mathscr{T}}_s\widetilde{\mathscr{T}}_{s-1}\widetilde{\mathscr{T}}_s\widetilde{\mathscr{T}}_{s+1\cdots n-2}(F_{n-1})\\ &= \widetilde{\mathscr{T}}_{2\cdots s-2}\widetilde{\mathscr{T}}_{s-1}\widetilde{\mathscr{T}}_s\widetilde{\mathscr{T}}_{s-1}\widetilde{\mathscr{T}}_{s+1\cdots n-2}(F_{n-1})\\ &= \widetilde{\mathscr{T}}_{2\cdots s-2}\widetilde{\mathscr{T}}_{s-1}\widetilde{\mathscr{T}}_s\widetilde{\mathscr{T}}_{s+1\cdots n-2}\widetilde{\mathscr{T}}_{s-1}(F_{n-1})\\ &= \widetilde{\mathscr{T}}_{2\cdots s-2}\widetilde{\mathscr{T}}_{s-1}\widetilde{\mathscr{T}}_s\widetilde{\mathscr{T}}_{s+1\cdots n-2}(F_{n-1}) = \widetilde{\mathscr{T}}_{2\cdots n-2}(F_{n-1}).\end{aligned}$$

Hence, $\widetilde{\mathscr{T}}_{2\cdots n-2}(F_{n-1})$ is fixed by $\widetilde{\mathscr{T}}_s$ for $3 \leqslant s \leqslant n-2$. □

In this case, Proposition 5.11 is reformulated and proved as Lemmas A.17–A.18 below.

Lemma A.17. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_2) = [B_1^\sigma, F_2]_q, \tag{A.41}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) = \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}^\sigma), [B_2^\sigma, F_1]_q\Big]_q - F_1K_2'K_{w_\bullet(\alpha_{n-1})}. \tag{A.42}$$

Proof. The formula (A.41) follows from Lemma A.1.

We prove (A.42). By a direct computation, we have

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) &= \Big[\widetilde{\mathscr{T}}_{2\cdots n-1\cdots 3}^{-1}(F_2), [F_2, F_1]_q\Big]_q\\ &= \Big[\widetilde{\mathscr{T}}_{2\cdots n-2}^{-1}\widetilde{\mathscr{T}}_{2\cdots n-2}(F_{n-1}), [F_2, F_1]_q\Big]_q\\ &= \Big[\widetilde{\mathscr{T}}_{3\cdots n-2}(F_{n-1}), [F_2, F_1]_q\Big]_q\\ &= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(F_{n-1}), [F_2, F_1]_q\Big]_q,\end{aligned}$$

where the last equality follows by applying Lemma A.16 and noting that $w_\bullet(\alpha_{n-1}) = s_{3\cdots n-2}(\alpha_{n-1})$. Recalling that $B_{n-1}^\sigma = F_{n-1} + K_{n-1}\widetilde{\mathscr{T}}_{w_\bullet}^{-1}(E_2)$, we compute the RHS of (A.42) as follows:

$$\begin{aligned}\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}^\sigma), [B_2^\sigma, F_1]_q\Big]_q &= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}^\sigma), [F_2, F_1]_q\Big]_q\\ &= \Big[\widetilde{\mathscr{T}}_{w_\bullet}(F_{n-1}), [F_2, F_1]_q\Big]_q + \Big[\widetilde{\mathscr{T}}_{w_\bullet}(K_{n-1})E_2, [F_2, F_1]_q\Big]_q\\ &= \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) + \big[E_2, [F_2, F_1]_q\big]\widetilde{\mathscr{T}}_{w_\bullet}(K_{n-1})\\ &= \widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_1) + F_1K_2'K_{w_\bullet(\alpha_{n-1})}.\end{aligned}$$

This proves the formula (A.42). □

Lemma A.18. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_{n-1})K_2'\Big) = [B_1, \widetilde{\mathscr{T}}_{w_\bullet}(E_{n-1})K_2']_q, \tag{A.43}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_nK_1') = \Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}), [B_2, E_nK_1']_q\Big]_q - E_nK_1'K_2'K_{w_\bullet(\alpha_{n-1})}. \tag{A.44}$$

Proof. Note that $\widetilde{\mathscr{T}}_{\mathbf{r}_1} = \widetilde{\mathscr{T}}_1\widetilde{\mathscr{T}}_n$ commutes with $\widetilde{\mathscr{T}}_{w_\bullet}$. Hence, we have

$$\begin{aligned}
\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_{n-1})K_2'\Big) &= \varsigma_{1,\diamond}^{-1/2}\widetilde{\mathscr{T}}_{w_\bullet}\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(E_{n-1})K_2'K_1' \\
&= \varsigma_{1,\diamond}^{-1}\widetilde{\mathscr{T}}_{w_\bullet}\big([E_{n-1}E_n]_{q^{-1}}\big)K_2'K_1' \\
&= \widetilde{\mathscr{T}}_{w_\bullet}[E_n, E_{n-1}]_qK_2'K_1' \\
&= [B_1, \widetilde{\mathscr{T}}_{w_\bullet}(E_{n-1})K_2']_q,
\end{aligned}$$

where the last step follows from Lemma A.1. Hence, we have proved (A.43).

We next prove (A.44). In this case, $\tau_0 = \tau$, $\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_n) = \mathcal{K}_n = \widetilde{k}_n$, and we simplify $\ni$ in (A.3) as $\ni = \widetilde{\mathscr{T}}_{w_\bullet}\widetilde{\mathscr{T}}_{w_0}$. Applying $\ni$ to (A.42) and then using (A.4)–(A.5), we have

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_nK_1')\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\widetilde{k}_n^{-1}) \\
&\quad = q^{-4}\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1})\mathcal{K}_2^{-1}, [B_2\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}), E_nK_1'\widetilde{k}_n^{-1}]_q\Big]_q \\
&\qquad - E_nK_1'\widetilde{k}_n^{-1}\ni(K_2'K_{w_\bullet(\alpha_{n-1})}).
\end{aligned} \tag{A.45}$$

For a weight reason, we have

$$\begin{aligned}
\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})E_n &= qE_n\widetilde{\mathscr{T}}_{w_\bullet}(\widetilde{k}_{n-1}^{-1}), \\
\widetilde{k}_n^{-1}B_2 &= qB_2\widetilde{k}_n^{-1}, \\
\mathcal{K}_2^{-1}B_2E_n &= q^2B_2E_n\mathcal{K}_2^{-1}, \\
\widetilde{k}_n^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}) &= q^2\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1})\widetilde{k}_n^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1}).
\end{aligned}$$

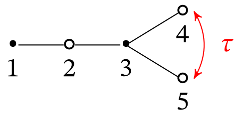
In addition, by (3.23), we have $\ni(K_2'K_{w_\bullet\alpha_{n-1}}) = q^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}K_2'K_{w_\bullet\alpha_{n-1}}$. Using these formulas, we rewrite (A.45) as

$$\begin{aligned}
\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(E_nK_1')\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\widetilde{k}_n^{-1}) &= q^{-1}\Big[\widetilde{\mathscr{T}}_{w_\bullet}(B_{n-1}), [B_2, E_nK_1']_q\Big]_q\widetilde{k}_n^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1} \\
&\quad - q^{-1}E_nK_1'\widetilde{k}_n^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}K_2'K_{w_\bullet\alpha_{n-1}}.
\end{aligned} \tag{A.46}$$

Finally, we have $\widetilde{\mathscr{T}}_{w_{\bullet,2}}(\widetilde{k}_n^{-1}) = q^{-1}\widetilde{k}_n^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_{n-1}^{-1})\mathcal{K}_2^{-1}$. Then the formula (A.44) follows from (A.46). □

A.9 | Type DIII$_5$

Consider the rank 2 Satake diagram of type DIII$_5$:



$$\varsigma_{2,\diamond} = -q^{-1}, \qquad \varsigma_{4,\diamond} = \varsigma_{5,\diamond} = -q^{-1/2},$$
$$\mathbf{r}_2 = s_2 s_1 s_3 s_2, \qquad \mathbf{r}_4 = s_4 s_5 s_3 s_4 s_5.$$

In this case, Proposition 5.11 is reformulated and proved as Lemmas A.19–A.20 below.

Lemma A.19. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(F_4) = [\widetilde{\mathscr{T}}_3(B_2^\sigma), F_4]_q, \tag{A.47}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) = \Big[B_4^\sigma, [\widetilde{\mathscr{T}}_3(B_5^\sigma), F_2]_q\Big]_q - \widetilde{\mathscr{T}}_3^{-2}(F_2) K_4 K_5' K_3'. \tag{A.48}$$

Proof. The proof for (A.47) is similar to that of Lemma A.9, and thus omitted.

We prove (A.48). By a direct computation, we have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) = \Big[\big[F_4, [F_5, F_3]_q\big]_q, F_2\Big]_q = \Big[F_4, [\widetilde{\mathscr{T}}_3(F_5), F_2]_q\Big]_q.$$

Note that $B_5^\sigma = F_5 + K_5 \widetilde{\mathscr{T}}_3^{-1}(E_4)$. Since $[\widetilde{\mathscr{T}}_3(K_5)E_4, F_2]_q = q[E_4, F_2]K_3K_5 = 0$, we have $[\widetilde{\mathscr{T}}_3(B_5^\sigma), F_2]_q = [\widetilde{\mathscr{T}}_3(F_5), F_2]_q$. We now compute the first term of RHS (A.48) as

$$\begin{aligned}
\Big[B_4^\sigma, [\widetilde{\mathscr{T}}_3(B_5^\sigma), F_2]_q\Big]_q &= \Big[B_4^\sigma, [\widetilde{\mathscr{T}}_3(F_5), F_2]_q\Big]_q \\
&= \Big[F_4, [\widetilde{\mathscr{T}}_3(F_5), F_2]_q\Big]_q + \Big[K_4 \widetilde{\mathscr{T}}_3^{-1}(E_4), [\widetilde{\mathscr{T}}_3(F_5), F_2]_q\Big]_q \\
&= \widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) + K_4 \Big[\widetilde{\mathscr{T}}_3^{-1}(E_4), [\widetilde{\mathscr{T}}_3(F_5), F_2]_q\Big] \\
&= \widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) - q^{-1}\big[[E_3, F_3]_{q^2}, F_2\big]_q K_4 K_5' \\
&= \widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(F_2) + \widetilde{\mathscr{T}}_3^{-2}(F_2) K_4 K_5' K_3'.
\end{aligned}$$

This proves (A.48). □

Lemma A.20. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_2}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}(E_5) K_4') = [\widetilde{\mathscr{T}}_3(B_2), \widetilde{\mathscr{T}}_{w_\bullet}(E_5) K_4']_q, \tag{A.49}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}(\widetilde{\mathscr{T}}_{w_\bullet}(E_2) K_2') = \Big[B_4, [\widetilde{\mathscr{T}}_3(B_5), \widetilde{\mathscr{T}}_{w_\bullet}(E_2) K_2']_q\Big]_q - \widetilde{\mathscr{T}}_3^{-2}(\widetilde{\mathscr{T}}_{w_\bullet}(E_2) K_2') K_4 K_5' K_3'. \tag{A.50}$$

Proof. We prove (A.50). The proof for (A.49) is easier and hence omitted.

By Lemma A.3, the operator $\backepsilon$ defined in (A.3) commutes with $\widetilde{\mathscr{T}}_3, \widetilde{\mathscr{T}}_{\mathbf{r}_4}$. Applying $\backepsilon$ to (A.48) and using (A.4)–(A.5), we have

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}\big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathscr{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) \\
&\quad = q^{-4}\Big[B_4\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1}), [\widetilde{\mathscr{T}}_3(B_5)\mathcal{K}_4^{-1}, \widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})]_q\Big]_q \\
&\quad - \widetilde{\mathscr{T}}_3^{-2}\big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\backepsilon(K_4K_5'K_3').
\end{aligned} \tag{A.51}$$

For a weight reason, we have

$$\mathcal{K}_4^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(E_2) = q\widetilde{\mathscr{T}}_{w_\bullet}(E_2)\mathcal{K}_4^{-1},$$
$$\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3(B_5) = q\widetilde{\mathscr{T}}_3(B_5)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}),$$
$$\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1})\widetilde{\mathscr{T}}_3(B_5)\widetilde{\mathscr{T}}_{w_\bullet}(E_2) = q^2\widetilde{\mathscr{T}}_3(B_5)\widetilde{\mathscr{T}}_{w_\bullet}(E_2)\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1}),$$
$$\mathcal{K}_4^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})B_4 = q^2B_4\mathcal{K}_4^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1}).$$

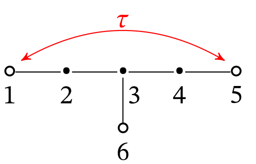
We also have $\backepsilon(K_4K_5'K_3') = q^{-1}\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}K_4K_5'K_3'$. Hence, (A.51) is written as

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_4}^{-1}\big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\big)\widetilde{\mathscr{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) \\
&\quad = q^{-1}\Big[B_4, [\widetilde{\mathscr{T}}_3(B_5), \widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2']_q\Big]_q\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1} \\
&\quad - q^{-1}\widetilde{\mathscr{T}}_3^{-2}\big(\widetilde{\mathscr{T}}_{w_\bullet}(E_2)K_2'\big)K_4K_5'K_3'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}.
\end{aligned} \tag{A.52}$$

Finally, by definition of $\mathcal{K}_i$ (3.23), we have $\widetilde{\mathscr{T}}_{w_\bullet,4}(\mathcal{K}_2^{-1}) = q^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_2^{-1})\widetilde{\mathscr{T}}_3(\mathcal{K}_5^{-1})\mathcal{K}_4^{-1}$. Thus, (A.52) implies (A.50). □

A.10 | Type EIII

Consider the rank 2 Satake diagram of type EIII:



$$\varsigma_{1,\diamond} = \varsigma_{5,\diamond} = -q^{-1/2}, \qquad \varsigma_{6,\diamond} = -q^{-1}$$
$$\mathbf{r}_1 = s_1 \cdots s_5 \cdots s_1, \qquad \mathbf{r}_6 = s_6s_3s_2s_4s_3s_6$$
$$w_\bullet = s_3s_2s_4s_3s_2s_4 = s_2s_4s_3s_2s_4s_3.$$

In this case, Proposition 5.11 is reformulated and proved as Lemmas A.21–A.22 below.

Lemma A.21. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_6}^{-1}(F_1) = [\widetilde{\mathscr{T}}_{23}(B_6^\sigma), F_1]_q, \tag{A.53}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_6) = \Big[\widetilde{\mathscr{T}}_4(B_5^\sigma), [\widetilde{\mathscr{T}}_{32}(B_1^\sigma), F_6]_q\Big]_q - \widetilde{\mathscr{T}}_{32323}^{-1}(F_6)K_1'K_2'K_3'K_4K_5. \tag{A.54}$$

Proof. We have

$$\widetilde{\mathscr{T}}_{\mathbf{r}_6}^{-1}(F_1) = \widetilde{\mathscr{T}}_{632}^{-1}(F_1) = [\widetilde{\mathscr{T}}_{63}^{-1}(F_2), F_1]_q = [\widetilde{\mathscr{T}}_{23}(F_6), F_1]_q = [\widetilde{\mathscr{T}}_{23}(B_6^\sigma), F_1]_q.$$

Hence, (A.53) follows.

We next prove (A.54). We have

$$\begin{aligned}\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_6) &= \widetilde{\mathscr{T}}_{1\dots5\dots3}^{-1}(F_6) = \widetilde{\mathscr{T}}_{123}^{-1}[\widetilde{\mathscr{T}}_{454}^{-1}(F_3), F_6]_q = \widetilde{\mathscr{T}}_{123}^{-1}[\widetilde{\mathscr{T}}_{34}(F_5), F_6]_q \\ &= \Big[\widetilde{\mathscr{T}}_4(F_5), [\widetilde{\mathscr{T}}_{12}^{-1}(F_3), F_6]_q\Big]_q = \Big[\widetilde{\mathscr{T}}_4(F_5), [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q\Big]_q. \end{aligned} \tag{A.55}$$

Recall that $B_1^\sigma = F_1 + K_1\widetilde{\mathscr{T}}_{w_\bullet}^{-1}(E_5)$. Hence,

$$\begin{aligned}[\widetilde{\mathscr{T}}_{32}(B_1^\sigma), F_6]_q &= [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q + [K_{123}\widetilde{\mathscr{T}}_3\widetilde{\mathscr{T}}_{434}^{-1}(E_5), F_6]_q \\ &= [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q + K_{123}[\widetilde{\mathscr{T}}_4^{-1}(E_5), F_6] = [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q. \end{aligned} \tag{A.56}$$

On the other hand, we have

$$\begin{aligned}\widetilde{\mathscr{T}}_{32323}^{-1}(F_6) &= \widetilde{\mathscr{T}}_{323}^{-1}[\widetilde{\mathscr{T}}_2^{-1}(F_3), F_6]_q = \widetilde{\mathscr{T}}_{323}^{-1}[\widetilde{\mathscr{T}}_3(F_2), F_6]_q \\ &= -[\widetilde{\mathscr{T}}_3^{-1}(E_2K_2'^{-1}), \widetilde{\mathscr{T}}_{232}^{-1}(F_6)]_q \\ &= -q^{-1}[\widetilde{\mathscr{T}}_3^{-1}(E_2), \widetilde{\mathscr{T}}_{23}^{-1}(F_6)]_{q^2}K_2'^{-1}K_3'^{-1} \\ &= -q^{-1}\Big[\widetilde{\mathscr{T}}_3^{-1}(E_2), [\widetilde{\mathscr{T}}_2^{-1}(F_3), F_6]_q\Big]_{q^2}K_2'^{-1}K_3'^{-1} \\ &= -q^{-1}\Big[[\widetilde{\mathscr{T}}_3^{-1}(E_2), \widetilde{\mathscr{T}}_2^{-1}(F_3)]_{q^2}, F_6\Big]_q K_2'^{-1}K_3'^{-1}. \end{aligned} \tag{A.57}$$

We now rewrite RHS (A.54) as follows:

$$\begin{aligned}&\Big[\widetilde{\mathscr{T}}_4(B_5^\sigma), [\widetilde{\mathscr{T}}_{32}(B_1^\sigma), F_6]_q\Big]_q \\ &\overset{(A.56)}{=} \Big[\widetilde{\mathscr{T}}_4(B_5^\sigma), [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q\Big]_q \\ &= \Big[\widetilde{\mathscr{T}}_4(F_5), [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q\Big]_q + \Big[K_4K_5\widetilde{\mathscr{T}}_{232}^{-1}(E_1), [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q\Big]_q \\ &\overset{(A.55)}{=} \widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_6) + K_4K_5\Big[\widetilde{\mathscr{T}}_{32}^{-1}(E_1), [\widetilde{\mathscr{T}}_{32}(F_1), F_6]_q\Big] \\ &= \widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_6) - q^{-1}\Big[[\widetilde{\mathscr{T}}_3^{-1}(E_2), \widetilde{\mathscr{T}}_3(F_2)]_{q^2}, F_6\Big]_q K_1'K_4K_5 \\ &\overset{(A.57)}{=} \widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}(F_6) + \widetilde{\mathscr{T}}_{32323}^{-1}(F_6)K_1'K_2'K_3'K_4K_5. \end{aligned}$$

Therefore, the formula (A.54) follows. □

Lemma A.22. *We have*

$$\widetilde{\mathscr{T}}_{\mathbf{r}_6}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_1'\Big) = [\widetilde{\mathscr{T}}_{23}(B_6), \widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_1']_q, \tag{A.58}$$

$$\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\Big) = \Big[\widetilde{\mathscr{T}}_4(B_5), [\widetilde{\mathscr{T}}_{32}(B_1), \widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6']_q\Big]_q - \widetilde{\mathscr{T}}_{32323}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\Big)K_1'K_2'K_3'K_4K_5. \tag{A.59}$$

Proof. Recall from Lemma A.3 that the operator $\ni$ defined in (A.3) commutes with each of the automorphisms $\widetilde{\mathscr{T}}_4, \widetilde{\mathscr{T}}_{32}, \widetilde{\mathscr{T}}_{23}, \widetilde{\mathscr{T}}_{\mathbf{r}_1}, \widetilde{\mathscr{T}}_{\mathbf{r}_6}$.

We first prove the formula (A.58). Applying $\ni$ to (A.53) and then using (A.4)–(A.5), we obtain

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_6}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_1'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,6}}(\mathcal{K}_5^{-1})\\
&= -q^{-2}[\widetilde{\mathscr{T}}_{23}(B_6)\widetilde{\mathscr{T}}_{432}(\mathcal{K}_6^{-1}), \widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_1'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1})]_q\\
&= -q^{-1}[\widetilde{\mathscr{T}}_{23}(B_6), \widetilde{\mathscr{T}}_{w_\bullet}(E_5)K_1']_q\widetilde{\mathscr{T}}_{432}(\mathcal{K}_6^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1}),
\end{aligned} \tag{A.60}$$

where the last equality follows by a weight consideration. On the other hand, we have $\widetilde{\mathscr{T}}_{w_{\bullet,6}}(\mathcal{K}_5^{-1}) = -q^{-1}\widetilde{\mathscr{T}}_{432}(\mathcal{K}_6^{-1})\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1})$. Thus, the formula (A.58) follows from (A.60).

We next prove the formula (A.59). Applying $\ni$ in the identity (A.3) to (A.54) and using (A.4)–(A.5), we obtain

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})\\
&= q^{-4}\Big[\widetilde{\mathscr{T}}_4(B_5)\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_1^{-1}), [\widetilde{\mathscr{T}}_{32}(B_1)\widetilde{\mathscr{T}}_{32}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1}), \widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1})]_q\Big]_q\\
&- \widetilde{\mathscr{T}}_{32323}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\Big)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\ni(K_1'K_2'K_3'K_4K_5).
\end{aligned} \tag{A.61}$$

Note that $\widetilde{\mathscr{T}}_4\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_1^{-1}) = \widetilde{\mathscr{T}}_{32}(K_1^{-1})\widetilde{\mathscr{T}}_4(K_5')^{-1}$ and $\widetilde{\mathscr{T}}_{32}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_5^{-1}) = \widetilde{\mathscr{T}}_4(K_5^{-1})\widetilde{\mathscr{T}}_{32}(K_1')^{-1}$. We also note that $K_1'K_2'K_3'K_4K_5 = \widetilde{\mathscr{T}}_{32}(K_1')\widetilde{\mathscr{T}}_4(K_5)$ and then $\ni(K_1'K_2'K_3'K_4K_5) = q^{-1}\widetilde{\mathscr{T}}_4(K_5')^{-1}\widetilde{\mathscr{T}}_{32}(K_1^{-1})$. Hence, (A.61) can be rewritten as

$$\begin{aligned}
&\widetilde{\mathscr{T}}_{\mathbf{r}_1}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\Big)\widetilde{\mathscr{T}}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})\\
&= q^{-4}\Big[\widetilde{\mathscr{T}}_4(B_5)\widetilde{\mathscr{T}}_{32}(K_1^{-1})\widetilde{\mathscr{T}}_4(K_5')^{-1}, [\widetilde{\mathscr{T}}_{32}(B_1K_1'^{-1})\widetilde{\mathscr{T}}_4(K_5^{-1}), \widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1})]_q\Big]_q\\
&- q^{-1}\widetilde{\mathscr{T}}_{32323}^{-1}\Big(\widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6'\Big)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathscr{T}}_4(K_5')^{-1}\widetilde{\mathscr{T}}_{32}(K_1^{-1}).
\end{aligned} \tag{A.62}$$

For a weight reason, we have

$$\widetilde{\mathscr{T}}_4(K_5^{-1})\widetilde{\mathscr{T}}_{32}(K_1')^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(E_6) = q\widetilde{\mathscr{T}}_{w_\bullet}(E_6)\widetilde{\mathscr{T}}_4(K_5^{-1})\widetilde{\mathscr{T}}_{32}(K_1')^{-1},$$

$$\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathscr{T}}_{32}(B_1) = q\widetilde{\mathscr{T}}_{32}(B_1)\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1}),$$

$$\widetilde{\mathscr{T}}_{32}(K_1)\widetilde{\mathscr{T}}_4(K_5')[\widetilde{\mathscr{T}}_{32}(B_1), \widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6']_q = q^{-2}[\widetilde{\mathscr{T}}_{32}(B_1), \widetilde{\mathscr{T}}_{w_\bullet}(E_6)K_6']_q\widetilde{\mathscr{T}}_{32}(K_1)\widetilde{\mathscr{T}}_4(K_5'),$$

$$\widetilde{\mathscr{T}}_4(K_5^{-1})\widetilde{\mathscr{T}}_{32}(K_1')^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1})\widetilde{\mathscr{T}}_4(B_5) = q^2\widetilde{\mathscr{T}}_4(B_5)\widetilde{\mathscr{T}}_4(K_5^{-1})\widetilde{\mathscr{T}}_{32}(K_1')^{-1}\widetilde{\mathscr{T}}_{w_\bullet}(\mathcal{K}_6^{-1}).$$

Using the above four identities, we rewrite (A.62) as

$$\begin{aligned}
&\mathscr{T}_{\mathbf{r}_1}^{-1}\Big(\mathscr{T}_{w_\bullet}(E_6)K_6'\Big)\mathscr{T}_{w_{\bullet,1}}(\mathcal{K}_6^{-1})\\
&= q^{-1}\Big[\mathscr{T}_4(B_5), [\mathscr{T}_{32}(B_1), \mathscr{T}_{w_\bullet}(E_6)K_6']_q\Big]_q \mathscr{T}_{w_\bullet}(\mathcal{K}_6^{-1})\mathscr{T}_{32}(K_1K_1')^{-1}\mathscr{T}_4(K_5K_5')^{-1}\\
&- q^{-1}\mathscr{T}_{32323}^{-1}\Big(\mathscr{T}_{w_\bullet}(E_6)K_6'\Big)K_1'K_2'K_3'K_4K_5\mathscr{T}_{w_\bullet}(\mathcal{K}_6^{-1})\mathscr{T}_{32}(K_1K_1')^{-1}\mathscr{T}_4(K_5K_5')^{-1}. \qquad \text{(A.63)}
\end{aligned}$$

Moreover, we have $\mathscr{T}_{w_{\bullet,1}}(\mathcal{K}_6^{-1}) = q^{-1}\mathscr{T}_{w_\bullet}(\mathcal{K}_6^{-1})\mathscr{T}_{32}(K_1K_1')^{-1}\mathscr{T}_4(K_5K_5')^{-1}$. Thus, (A.63) implies the desired formula (A.59). □

ACKNOWLEDGMENTS

We thank Stefan Kolb and Ming Lu for helpful comments and suggestions. We thank an anonymous referee for a careful reading and helpful comments. WW is partially supported by the NSF grant DMS-2001351. WZ is supported by a GSAS fellowship at University of Virginia and WW's NSF Graduate Research Assistantship.

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The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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