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Accelerated Algorithms for a Class of Optimization Problems with Equality and Box Constraints

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Abstract— **Convex optimization with equality and inequality constraints is a ubiquitous problem in several optimization and control problems in large-scale systems. Recently there has been a lot of interest in establishing accelerated convergence of the loss function. A class of high-order tuners was recently proposed in an effort to lead to accelerated convergence for the case when no constraints are present. In this paper, we propose a new high-order tuner that can accommodate the presence of equality constraints. In order to accommodate the underlying box constraints, time-varying gains are introduced in the high-order tuner which leverage convexity and ensure anytime feasibility of the constraints. Numerical examples are provided to support the theoretical derivations.**

I. Introduction

A class of algorithms referred to as High Order Tuners (HT) was proposed for parameter estimation involved in dynamic systems [1]. These iterative algorithms utilize second-order information of the system, enabling faster convergence in discrete time parameter estimation. This property of HT is motivated by Nesterov's algorithm [2], [3]. Additionally, HT has been proven to show stable performance in the presence of time-varying regressors [1] and for adaptive control in the presence of delays and high relative degrees [4], [5]. To further leverage the accelerated convergence of HT in machine learning as well as adaptive control problems, discrete-time version of these algorithms for unconstrained convex optimization was proposed in [6], which demonstrated theoretical guarantees pertaining to convergence and stability.

Accelerated algorithms have been extended to equality-constrained convex optimization problems in [7], [8], [9]. Accelerated HT was extended to a class of constrained convex optimization problems, with a dual objective of deconstructing the loss landscapes of constrained optimization problems and applying HT with theoretical guarantees of stable performance in [10]. Several works in the past [11] have utilized the variable reduction approach to transform a constrained convex optimization problem to an unconstrained optimization problem in a reduced dimension feasible solution space. However, this transformation does not guarantee convexity of the unconstrained optimization problem. In this paper, we present conditions on equality constraints under which the transformed problem is convex.

We first focus on optimization problems with equality constraints and elucidate the conditions under which HT can be deployed with theoretical guarantees pertaining to convergence and stability, using a variable reduction based technique. We then extend these results to optimization with equality constraints and box inequality constraints. Since the guarantees of convexity are only valid within the feasible region, it makes the task of solving hard inequality constraints challenging. To preserve theoretical guarantees pertaining to convergence, it is imperative to constrain the decision variable within a compact set within which the resulting loss function is convex. We show that the same HT proposed in [1], [6] and [10] can be used to guarantee feasibility and convergence to the optimal solution even in the presence of equality and box constraints. We present two numerical example problems in the paper to validate the approach.

The organization of the paper is as follows. Section II describes the broader category of constrained optimization problems and presents theorems pertaining to the performance of High Order Tuner (Algorithm 1) in solving them. In Section III, we present a specific category of constrained optimization problems where the loss function is not convex everywhere, but over known compact sets. We outline a novel formulation based on High Order Tuner with time-varying gains (Algorithm 2) to solve such problems. In Section IV, we look at a scalar example that validates the implementation of Algorithm 2 to a constrained convex optimization problem and demonstrates the accelerated convergence of High Order Tuner. The paper concludes with a summary of main contributions and future directions of research in Section V.

Notation

We employ the following notations throughout the paper. For a vector $x \in \mathbb{R}^n$, with $1 \leq i < j \leq n$, $x_{i:j}$ denotes the subvector with elements from the i-th entry of x to the j-th entry. For a vector-valued function $p : \mathbb{R}^m \to \mathbb{R}^n$, p is convex (respectively, concave) on $\mathbb{R}^m$ implies that scalar function $p_i : \mathbb{R}^m \to \mathbb{R}$ is convex (respectively, concave) on $\mathbb{R}^m$ for all $i \in \{1, \dots, n\}$. We use the shorthand notation PSD to denote a symmetric positive semidefinite matrix. For two sets A and B, $A \times B$ denotes their cartesian product.

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II. Convex Optimization for a Class of Nonconvex Problems

Consider the optimization problem

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & h(x) = 0, \end{aligned} \tag{1}$$

here $x \in \mathbb{R}^n$ is the decision variable, $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^{n-m}$ are continuously differentiable convex functions. Without loss of generality, we assume that problem (1) is not overdetermined, i.e., $m \leq n$. There are two key aspects to solving a constrained optimization problem given by (1); namely, feasibility and optimality. To ensure that the solution is feasible, we first address how to solve equality constraints using variable reduction technique employed in [10].

We define $\mathcal{L}(\cdot)$ as the loss function consisting of the objective function and an optional term penalizing the violation of equality constraints in case it is not feasible to solve the equality constraints:

$$\mathcal{L}(x) = f(x) + \lambda_h \|h(x)\|^2, \lambda_h \geq 0 \tag{2}$$

In this paper, we restrict our attention to problems where $\mathcal{L}$ is convex, which holds for many cases where f and h are convex. To ensure feasibility, we utilize the linear dependence in the feasible solution space introduced by the equality constraints, as illustrated in [11]. x is partitioned into independent and dependent variables; $\theta \in \mathbb{R}^m$ and $z \in \mathbb{R}^{n-m}$ respectively

$$x = [\theta^T \quad z^T]^T, \quad z = p(\theta)$$

Here $p : \mathbb{R}^m \to \mathbb{R}^{n-m}$ is a function that maps the dependence of z on θ, such that $h(x) = 0$. We assume that h is such that given m entries of x, its remaining $(n-m)$ entries can be computed either in closed form or recursively. If $p(\theta)$ can be computed explicitly, we set $\lambda_h = 0$ in (2). Otherwise, λ_h is chosen as a positive real-valued scalar. In other words, we assume that we have knowledge of the function $p : \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that

$$h([\theta^T \quad p(\theta)^T]^T) = 0, \quad \forall \theta \in \mathbb{R}^m$$

Using the function $p(\cdot)$ defined as above, we now define a modified loss function $l : \mathbb{R}^m \to \mathbb{R}$ as

$$l(\theta) = \mathcal{L}([\theta^T \quad p(\theta)^T]^T). \tag{3}$$

The optimization problem in (1) is now reformulated as an unconstrained minimization problem given by

$$\min\ l(\theta), \tag{4}$$

with $\theta \in \mathbb{R}^m$ as the decision variable. We now proceed to delineate conditions under which l is convex in Proposition II.1.

Proposition II.1. *(Convexity of the modified loss function for equality-constrained nonconvex programs): Assume that there exists a convex set $\Omega_n \in \mathbb{R}^n$ such that the functions f and h are convex on Ω_n. Let*

$$\Omega_m = \{\theta \mid \theta = x_{1:m}, x \in \Omega_n\}. \tag{5}$$

If $\mathcal{L}$ is convex and any of the following conditions is satisfied:

(i) h is linear,
(ii) $\nabla \mathcal{L}(x) \geq 0$ for all $x \in \Omega_n$ and p is convex on Ω_m,
(iii) $\nabla \mathcal{L}(x) \leq 0$ for all $x \in \Omega_n$ and p is concave on Ω_m,

then l is convex on Ω_m.

Proof. Readers are referred to [10, Proposition IV.2] for the proof and [12] for details on convexity of composite functions used in the proof. □

Using Implicit Function theorem, we can establish the following sufficient conditions on $h(\cdot)$ for which $p(\cdot)$ is convex or concave. This gives us a readily checkable set of conditions of convexity of $l(\cdot)$ without needing to determine the convexity/concavity of function $p(\cdot)$, which may not always be possible.

Proposition II.2. *(Conditions on h for convexity of p): Following from Proposition II.1, assuming f and h are convex on a given set $\bar{\Omega}_n \subseteq \mathbb{R}^n$, and h_i is twice differentiable $\forall\ i = 1, ..., n-m$, it follows that:*

(i) if $\nabla_p h(x) < 0$ for $x \in \bar{\Omega}_n$ then $p(\theta)$ is convex on Ω_m
(ii) if $\nabla_p h(x) > 0$ for $x \in \bar{\Omega}_n$ then $p(\theta)$ is concave on Ω_m

Additionally, if h is linear in z and convex in θ, then condition (i) follows.

Proof. We prove condition (i), similar arguments can be extended to prove condition (ii). Noting that

$$h([\theta^T \quad p(\theta)^T]^T) = 0, \tag{6}$$

and applying the chain rule and differentiating (6) along the manifold $z = p(\theta)$ twice, for $1 \leq i \leq m$ and $1 \leq j \leq n-m$ we get:

$$\underbrace{\frac{\partial^2 h_i}{\partial \theta^2}}_{I} + \underbrace{\frac{\partial h_i}{\partial z_j}\frac{\partial^2 z_j}{\partial \theta^2}}_{II} + \underbrace{\frac{\partial^2 h_i}{\partial z_j^2}\frac{\partial z_j}{\partial \theta}\left(\frac{\partial z_j}{\partial \theta}\right)^T}_{III} = 0 \tag{7}$$

Since h_i is convex on $\Omega_n\ \forall\ i \in \mathbb{N}$, it follows:

(i) I: $\frac{\partial^2 h_i}{\partial \theta^2}$ is a positive semi-definite matrix
(ii) III: $\frac{\partial z_j}{\partial \theta}\left(\frac{\partial z_j}{\partial \theta}\right)^T$ is a symmetrical dyad, i.e., PSD matrix multiplied by $\nabla^2_{z_j} h_i \geq 0$, hence III is a PSD matrix.

For (7) to be valid, II has to be a negative semi-definite matrix, since it is the negated sum of positive-semi definite matrices. From condition (i) we have $\frac{\partial h_i}{\partial z_j} < 0$ since $z = p(\theta)$. Thus, $\frac{\partial^2 z_j}{\partial \theta^2}$ is PSD. Therefore, $\frac{\partial^2 p_j(\theta)}{\partial \theta^2}$ is PSD, hence $p_j(\theta)$ is convex $\forall \theta \in \Omega_m, \forall j = 1, ..., n-m$. □

Remark 1. Proposition II.2 is one of the main contributions of this work. The explicit availability of function $p(\cdot)$ in closed form is not guaranteed always and p is often estimated iteratively in practice, which is also the rationale for including the penalizing term $\lambda_h \|h(x)\|^2$ in the definition of the loss function $\mathcal{L}(\cdot)$. In the absence of information on p, we can use the chain rule and implicit function theorem to calculate $\nabla_p h$ as demonstrated in [10], [11].

A. Numerical Example

The following example illustrates how the conditions of Proposition II.1 and Proposition II.2 can be verified. We consider the problem in (1) where $x \in \mathbb{R}^2$, $h(x) = {x_1}^2 + (x_2-4)^2 - 1 = 0$ and $f(x) = \log(\sum_{i=1}^{2} e^{x_i})$ are convex functions ($h : \mathbb{R}^2 \to \mathbb{R}$, $m = 1$). With x_1 as the independent variable, we write $x = [x_1 \;\; p(x_1)]^T$, and use the implicit function theorem to determine $p(x_1)$ explicitly. We therefore set $\lambda_h = 0$ while formulating the loss function, and hence $\mathcal{L}$ is the same as f. Now we demonstrate the application of Propostion II.2 to define region Ω_n where $p(\cdot)$ is convex or concave.

Clearly, $\nabla_p h(x) = 2(p(x_1) - 4)$, and it is evident that $\nabla_p h(x) < 0$ for $x_2 < 4$ and $\nabla_p h(x) > 0$ for $x_2 > 4$ *(as $x_2 = p(x_1)$)*. Using Proposition II.2, we conclude that for $x_2 \leq 4$, $p(\cdot)$ is convex and for $x_2 \geq 4$, $p(\cdot)$ is concave. Additionally, to ensure that to $p(x_1)$ evaluates to a real number, we must constrain $-1 \leq x_1 \leq 1$. Thus, following Proposition II.2, we construct sets $\bar{\Omega}_n^1$ and $\bar{\Omega}_n^2$ as:

$$\bar{\Omega}_n^1 = \{x = [x_1 \;\; x_2]^T | -1 \leq x_1 \leq 1, x_2 < 4\} \tag{8}$$

$$\bar{\Omega}_n^2 = \{x = [x_1 \;\; x_2]^T | -1 \leq x_1 \leq 1, x_2 > 4\} \tag{9}$$

Since $\mathcal{L}(x_1, x_2) = \log(e^{x_1} + e^{x_2})$ it is evident that:

$$\nabla_{x_1} \mathcal{L} = \frac{e^{x_1}}{e^{x_1} + e^{x_2}} \tag{10a}$$

$$\nabla_{x_2} \mathcal{L} = \frac{e^{x_2}}{e^{x_1} + e^{x_2}} \tag{10b}$$

Clearly, $\nabla \mathcal{L}(x_1, x_2) > 0$ for all $x \in \mathbb{R}^2$. From Proposition II.1, case (ii), we can see that for $-1 \leq x_1 \leq 1$ and $x_2 \leq 4$, $\nabla \mathcal{L}(x_1, x_2) > 0$ and $p(\cdot)$ is convex. Therefore, $l(x_1)$ is convex in this region. Formally, we define $\Omega_n \equiv \bar{\Omega}_n^1$ as:

$$\Omega_n = \{x = [x_1 \;\; x_2]^T | -1 \leq x_1 \leq 1, x_2 \leq 4\} \tag{11}$$

Within the chosen Ω_n, Proposition-II.1 guarantees that $l(x_1)$ is convex. Consequently, Ω_m is automatically defined as:

$$\Omega_m = \{x_1 \in \mathbb{R} \,|\, -1 \leq x_1 \leq 1\} \tag{12}$$

Indeed, for values of $x_1 \in \Omega_m$ there are two cases for $p(x_1)$ using Implicit Function Theorem [13] given as:

$$p(x_1) = 4 - \sqrt{1 - {x_1}^2} \quad \text{for } x_2 \leq 4 \tag{13}$$

$$p(x_1) = 4 + \sqrt{1 - {x_1}^2} \quad \text{for } x_2 \geq 4 \tag{14}$$

As we seek to expand the set Ω_n where $l(\cdot)$ is convex, we note that due to the simple nature of the problem, it is easy to conclude that $l(\cdot)$ is concave for $x_2 \geq 4$, hence Ω_n for this problem is equivalent to the one in (11).

Hence, using Ω_n as defined in (11), $p(\cdot)$ takes the form mentioned in (13), and by defining $l(\cdot)$ as the one in (3), the optimization problem reduces to the following convex optimization problem which is much simpler to solve:

$$\begin{aligned} &\min \;\; \log(e^{x_1} + e^{4-\sqrt{1-x_1^2}}) \\ &\text{s.t.} \;\; x_1 \in \Omega_m \end{aligned} \tag{15}$$

We now state a HT based Algorithm which guarantees convergence to optimal solution provided the condition $\theta \in \Omega_m$ is satisfied.

B. High Order Tuner

Recently a high-order tuner (HT) was proposed for convex functions and shown to lead to convergence of a loss function. Since this paper builds on this HT, we briefly summarize the underlying HT algorithm presented in [6] in the form of Algorithm 1.

The normalizing signal $\mathcal{N}_k$ is chosen as:

$$\mathcal{N}_k = 1 + H_k, \quad H_k = max\{\lambda : \lambda \in \sigma(\nabla^2 L_k(\theta))\}$$

Here $\sigma(\nabla^2 L_k(\theta))$ denotes the spectrum of Hessian Matrix of the loss function [6].

Algorithm 1 HT Optimizer for equality-constrained nonconvex optimization

1: **Initial conditions** θ_0, ν_0, gains γ, β
2: **for** $k = 1$ to N **do**
3: Compute $\nabla l(\theta)$ and let $\mathcal{N}_k = 1 + H_k$
4: $\nabla \bar{q}_k(\theta_k) = \dfrac{\nabla l(\theta_k)}{\mathcal{N}_k}$
5: $\bar{\theta}_k = \theta_k - \gamma\beta\nabla\bar{q}_k(\theta_k)$
6: $\theta_{k+1} \leftarrow \bar{\theta}_k - \beta(\bar{\theta}_k - \nu_k)$
7: Compute $\nabla l(\theta_{k+1})$ and let
8: $\nabla \bar{q}_k(\theta_{k+1}) = \dfrac{\nabla l(\theta_{k+1})}{\mathcal{N}_k}$
9: $\nu_{k+1} \leftarrow \nu_k - \gamma\nabla\bar{q}_k(\theta_{k+1})$
10: **end for**

In Section II-A we established sufficient conditions for the convexity of the modified loss function in Proposition II.1 which are easily verifiable. We now state Theorem II.3 that shows the convergence of HT in Algorithm 1 for the constrained optimization problem in (1).

Theorem II.3. *If the objective function f and the equality constraint h in* (1) *are convex over a set Ω_n, and $\theta_0 \in \Omega_m$, where Ω_m is defined in* (5)*, and if the sequence of iterates $\{\theta_k\}$ generated by Algorithm 1 satisfy $\{\theta_k\} \in \Omega_m$, then $\lim_{k\to\infty} l(\theta_k) = l(\theta^*)$, where $l(\theta^*) = f([\theta^{*T} \;\; p(\theta^*)^T]^T)$ is the optimal value of* (1) *where γ and β are chosen as $0 < \beta < 1$, $0 < \gamma < \frac{\beta(2-\beta)}{8+\beta}$*

Proof. Readers are referred to [6, Theorem 2] for the proof. □

C. Satisfaction of Box constraints

Theorem II.3 enables us to leverage Algorithm 1, provided that $\theta_k \in \Omega_m \forall k \in \mathbb{N}$, i.e., the parameter remains inside the set over which $l(\cdot)$ is convex. For the numerical example outlined in Section II-A, it is clear that we can utilize Theorem II.3 when $x_1 \in \Omega_1$ where $\Omega_1 = [-1, 1]$. It should however be noted that Theorem II.3 requires that $\theta \in \Omega_m$ for the HT to lead to convergence. In order to constrain the parameter to be within the set Ω_m, we modify (4) into a constrained optimization problem given by:

$$\begin{aligned} &\min\ l(\theta) \\ &\text{s.t. } \ \theta \in \Omega_m. \end{aligned} \tag{16}$$

It must be noted that the set Ω_m in (16) is a subset of the feasible solution-space of (1) If the conditions given by Proposition II.1 are necessary and sufficient, (16) has the same optimizer θ^* as that of (4). While it is tempting to apply a projection procedure for ensuring $\theta \in \Omega_m$, the lack of convexity guarantees of $l(\cdot)$ for all $\theta \in \mathbb{R}^m$ inhibits us from proving the stability of Algorithm 1 as we need $l(\cdot)$ to be convex for all $\overline{\theta}_k, \theta_k, \nu_k \in \mathbb{R}^m$. In the next section, we delineate a general procedure for ensuring that the constraint $\theta \in \Omega_m$ is always satisfied.

III. Constrained Convex Optimization

The starting point of this section is problem (16) where l is convex. Without loss of generality, we assume Ω_m to be a compact set in $\mathbb{R}^m$. For any such Ω_m, it is always possible to find a bounded interval $I = I_1 \times I_2 ... \times I_m \subseteq \Omega_m$ where I_i is a bounded interval in $\mathbb{R}$ defined as $I_i = [\theta^i_{min},\ \theta^i_{max}]$ for all $i = 1, 2, ..., m$.

Using the above arguments, we reformulate the constrained optimization in (16) as:

$$\begin{aligned} &\min\ l(\theta) \\ &\text{s.t. } \ \theta \in I \end{aligned} \tag{17}$$

where θ^* is the solution of problem (17). In Section II, we outlined conditions for which (16) is equivalent to (4), i.e., θ^* is the solution to problem (4) by appropriate selection of sets Ω_n, Ω_m. Note that (16) and (17) are equivalent if $\theta^* \in I$. The conditions under which $\theta^* \in I$ have been summarized in Proposition III.1.

Proposition III.1. *For a given compact set I and convex loss function $l(\theta)$ if there exist θ_1, θ_2 such that $l(\theta_1) = l(\theta_2)$, then $\theta^* \in I$.*

Proof. For a scalar case, i.e., $\Omega_m \subset \mathbb{R}$, Rolle's theorem can be applied to the function l being continuous and differentiable. For a subset $[\theta_1, \theta_2] \subset I$, such that $l(\theta_1) = l(\theta_2)$, by Rolle's Theorem, there exists a $\hat{\theta} \in [\theta_1, \theta_2]$ such that $\nabla l(\hat{\theta}) = 0$. Since l is differentiable and convex, $\nabla l(\hat{\theta}) = 0 \iff \hat{\theta} = \theta^*$. This can be extended to the general case where $m \geq 1, \Omega_m \subset \mathbb{R}^m$, using the vector-version of Rolle's theorem, cf. [14]. □

With the assurance from Proposition III.1 that $\theta^* \in I$, we now revise Algoirithm 1 in the form of Algorithm 2 to guarantee convergence to θ^* while ensuring feasibility that $\theta \in I$. We prove that $\overline{\theta}_k, \nu_k, \theta_k \in I$ for all $k \in \mathbb{N}$ through Proposition III.2 and subsequently provide guarantees of convergence of the iterates $\{\theta_k\}_{k=1}^{\infty}$ generated by Algorithm 2 to θ^* using Theorem III.3.

Algorithm 2 HT Optimizer for localized convex optimization

1: **Initial conditions** θ_0, ν_0, gains γ, β
2: Choose $\theta_0, \nu_0 \in I$
3: **for** $k = 1$ to N **do**
4: Compute $\nabla l(\theta)$ and let $\mathcal{N}_k = 1 + H_k$
5: $\nabla \overline{q}_k(\theta_k) = \dfrac{\nabla l(\theta_k)}{\mathcal{N}_k}$
6: $\overline{\theta}_k = \theta_k - \gamma\beta a_k \nabla \overline{q}_k(\theta_k)$
7: $\theta_{k+1} = \overline{\theta}_k - \beta(\overline{\theta}_k - \nu_k)$
8: Compute $\nabla l(\theta_{k+1})$ and let
9: $\nabla \overline{q}_k(\theta_{k+1}) = \dfrac{\nabla l(\theta_{k+1})}{\mathcal{N}_k}$
10: $\nu_{k+1} \leftarrow \nu_k - \gamma b_{k+1} \nabla \overline{q}_k(\theta_{k+1})$
11: **end for**

Proposition III.2. *Consider Algorithm 2, for a given $k \in \mathbb{N}$, if $\theta_k, \nu_k \in I$, there exist real numbers $a_k > 0$ and $b_{k+1} > 0$ such that $\overline{\theta}_k, \nu_{k+1} \in I$. Consequently, for $0 < \beta \leq 1$, and $\theta_0, \nu_0 \in I$, Algorithm 2 guarantees $\theta_k, \overline{\theta}_k, \nu_k \in I$ for all values of $k \in \mathbb{N}$.*

Proof. We first provide conditions for the selection of a_k.

For a given $\theta_k \in \mathbb{R}^m$ and $i \in \mathbb{N}$, consider $\theta^i_k \in \mathbb{R}$ such that $\theta^i_k \in [\theta^i_{min},\ \theta^i_{max}]$. There are two possible cases:

(i) $\theta^i_k > \theta^{*i} \iff \nabla_i l(\theta_k) > 0$
(ii) $\theta^i_k < \theta^{*i} \iff \nabla_i l(\theta_k) < 0$

For case (ii), using **Step-6** of Algorithm 2 we have,

$$\overline{\theta}^i_k = \theta^i_k + a_k \frac{\gamma\beta |\nabla_i l(\theta_k)|}{\mathcal{N}_k} \tag{18}$$

For $a_k > 0$, $\overline{\theta}^i_k > \theta^i_{min}$ in (18). We need to ensure that $\overline{\theta}^i_k < \theta^i_{max}$ for all i. Therefore, we must ensure:

$$\theta^i_k + a_k \frac{\gamma\beta |\nabla_i l(\theta_k)|}{\mathcal{N}_k} \leq \theta^i_{max} \quad \forall i. \tag{19}$$

Inequality (19) would be true if $a_k \leq \hat{a}_k$, where

$$\hat{a}_k = \min_{i \in \{1,\dots,m\}} \frac{(\theta^i_{max} - \theta^i_k)\mathcal{N}_k}{\gamma\beta |\nabla_i l(\theta_k)|}. \tag{20}$$

Similarly, for case (i), we get the following inequality criteria for a_k to ensure that $\overline{\theta}^i_k > \theta^i_{min}$ for all i:

$$a_k \leq \tilde{a}_k = \min_{i \in \{1,\dots,m\}} \frac{(\theta^i_k - \theta^i_{min})\mathcal{N}_k}{\gamma\beta |\nabla_i l(\theta_k)|} \tag{21}$$

Combining (20) and (21), we have

$$a_k \leq \bar{a}_k = \min\{\hat{a}_k, \tilde{a}_k\} \quad \forall k. \tag{22}$$

We now outline conditions for selection of b_{k+1}, which follows similar approach to selection of a_k, i.e., for given $\nu_k \in I$ we prescribe range of b_{k+1} such that $\nu_{k+1} \in I$. From **Step-10** of Algorithm 2, it could be deduced that for all k, b_{k+1} must satisfy

$$b_{k+1} \leq \bar{b}_{k+1} = \min\{\hat{b}_{k+1}, \tilde{b}_{k+1}\} \tag{23}$$

where

$$\hat{b}_{k+1} = \min_{i\in\{1,\dots,m\}} \frac{(\theta^i_{max}-\nu^i_k)\mathcal{N}_k}{\gamma|\nabla_i l(\theta_{k+1})|}$$
$$\tilde{b}_{k+1} = \min_{i\in\{1,\dots,m\}} \frac{(\nu^i_k-\theta^i_{min})\mathcal{N}_k}{\gamma|\nabla_i l(\theta_{k+1})|}.$$

Note however, that (22), (23) can generate $a_k, b_{k+1} = 0$, which is undesirable. To compensate for that, we introduce an additional rule:

$$a_k = \begin{cases} -\epsilon & \min_{i\in\{1,\dots,m\}}(\theta^i_k - \theta^i_{max}) = 0 \\ \epsilon & \min_{i\in\{1,\dots,m\}}(\theta^i_k - \theta^i_{min}) = 0 \end{cases} \tag{24}$$

Here $0 < \epsilon < 1$ is a very small real number of choice. Similar update rule can be stated for b_{k+1} to avoid the case of $a_k, b_{k+1} = 0$. For a given k, by selecting a_k, b_{k+1} such that (22), (23), (24) are satisfied, we ensure $\overline{\theta}_k, \nu_k \in I$. From **Step-7** of Algorithm 2:

$$\theta_{k+1} = (1-\beta)\overline{\theta}_k + \beta\nu_k \tag{25}$$

Hence, θ_{k+1} is a convex combination of $\overline{\theta}_k$ and ν_k for a given k and $0 < \beta \le 1$. Additionally, set I is compact, hence, if $\overline{\theta}_k, \nu_k \in I$, then $\theta_{k+1} \in I$ for a given $k \in \mathbb{N}$. By choosing $\theta_0, \nu_0 \in I$, by induction it can be shown that Proposition III.2 can be applied iteratively to generate parameters that are bounded within the compact set I. □

Proposition III.2 outlines conditions under which all parameters generated by Algorithm-2 are bounded within set I, where the convexity of loss function $l(\cdot)$ is guaranteed. Theorem III.3 formally establishes the convergence of Algorithm 2 to an optimal solution θ^*.

Theorem III.3. *(Convergence of the HT algorithm constrained to ensure convexity): For a differentiable $\bar{L}_k$-smooth convex loss function $l(.)$, Algorithm 2 with $0 < \beta \le 1$, $0 < \gamma < \frac{\beta(2-\beta)}{8+\beta}$ and a_k, b_{k+1} satisfying $a_k \le \min\{1, \bar{a}_k\}$, $b_{k+1} \le \min\{1, \bar{b}_{k+1}\}$ and* (24)*, where $\bar{a}_{k+1}$ and $\bar{b}_{k+1}$ are defined in* (22) *and* (23) *ensures that $V = \frac{\|\nu-\theta^*\|^2}{\gamma} + \frac{\|\nu-\theta\|^2}{\gamma}$ is a Lyapunov function. Consequently, the sequence of iterates $\{\theta_k\}$ generated by Algorithm 2 satisfy $\{\theta_k\} \in \Omega_m$, and $\lim_{k\to\infty} l(\theta_k) = l(\theta^*)$, where $l(\theta^*) = f([\theta^{*T} \quad p(\theta^*)^T]^T)$ is the optimal value of* (16)*.*

Proof. This proof follows a similar approach to the proof of stability of High Order Tuner for convex optimization, as illustrated in [6, Theorem 2].

Assuming that $\nu_k, \theta_k, \overline{\theta}_k \in I$, function $l(\cdot)$ is convex for all these parameters lying within the set I. Applying convexity and smoothness properties (ref. [6, Section II]) to $l(\cdot)$, the following upper bound is obtained:

$$l(\vartheta_k) - l(\bar{\theta}_k) = l(\vartheta_k) - l(\theta_{k+1}) + l(\theta_{k+1}) - l(\bar{\theta}_k)$$

$$\begin{aligned} &\le \nabla l(\theta_{k+1})^T(\vartheta_k - \theta_{k+1}) + \frac{\bar{L}_k}{2}\|\vartheta_k - \theta_{k+1}\|^2 \\ &\quad + \nabla l(\theta_{k+1})^T(\theta_{k+1} - \bar{\theta}_k) \end{aligned} \tag{26}$$

$$\overset{\text{Alg.2}}{\le} \nabla l(\theta_{k+1})^T(\vartheta_k - \bar{\theta}_k) + \frac{\bar{L}_k}{2}\|\vartheta_k - (1-\beta)\bar{\theta}_k - \beta\vartheta_k)\|^2 \tag{27}$$

$$\begin{aligned} &l(\vartheta_k) - l(\bar{\theta}_k) \\ &\le -\nabla l(\theta_{k+1})^T(\bar{\theta}_k - \vartheta_k) + \frac{\bar{L}_k}{2}(1-\beta)^2\|\bar{\theta}_k - \vartheta_k\|^2. \end{aligned} \tag{28}$$

Similarly, we obtain:

$$\begin{aligned} &l(\bar{\theta}_k) - l(\vartheta_k) \\ &\quad \le \nabla l(\theta_k)^T(\bar{\theta}_k - \vartheta_k) + \frac{a_k^2\bar{L}_k\gamma^2\beta^2}{2\mathcal{N}_k^2}\|\nabla l(\theta_k)\|^2. \end{aligned} \tag{29}$$

Using (28) and (29) we obtain:

$$\begin{aligned} &\nabla l(\theta_{k+1})^T(\bar{\theta}_k - \vartheta_k) \\ &- \frac{\bar{L}_k}{2}(1-\beta)^2\|\bar{\theta}_k - \vartheta_k\|^2 \\ &- \frac{a_k^2\bar{L}_k\gamma^2\beta^2}{2\mathcal{N}_k^2}\|\nabla l(\theta_k)\|^2 \le \nabla l(\theta_k)^T(\bar{\theta}_k - \vartheta_k) \end{aligned} \tag{30}$$

Using Algorithm 2, [6, Theorem 1] and (30), setting $\gamma < \frac{\beta(2-\beta)}{8+\beta}$, $0 < a_k, b_{k+1} \le 1$ and defining $\Delta V_k := V_{k+1} - V_k$, it can be shown that

$$\begin{aligned} \Delta V_k \le \frac{1}{\mathcal{N}_k}\Bigg\{ &-2b_{k+1}(l(\theta_{k+1}) - l(\theta^*)) \\ &-\left(\frac{b_{k+1}}{2\bar{L}_k} - \frac{2\gamma b_{k+1}^2}{\mathcal{N}_k}\right)\|\nabla l(\theta_{k+1})\|^2 \\ &-\left(1 - \frac{\bar{L}_k\gamma\beta a_k}{\mathcal{N}_k}\right)\frac{\gamma\beta^2a_k^2}{\mathcal{N}_k}\|\nabla l(\theta_k)\|^2 \\ &-[\beta - a_k\beta(1-\beta)^2]\bar{L}_k\|\bar{\theta}_k - \nu_k\|^2 \\ &-\left[\frac{\sqrt{b_{k+1}}}{\sqrt{2\bar{L}_k}}\|\nabla l(\theta_{k+1})\| - 2\sqrt{2\bar{L}_k}\|\bar{\theta}_k - \nu_k\|\right]^2 \\ &-4(\sqrt{b_{k+1}} - b_{k+1})\|\bar{\theta}_k - \nu_k\|\|\nabla l(\theta_{k+1})\| \\ &-(8+\beta)\|\bar{\theta}_k - \nu_k\|^2\Bigg\} \le 0 \end{aligned} \tag{31}$$

From (31), it can be seen that:

$$\Delta V_k \le \frac{b_{k+1}}{\mathcal{N}_k}\{-2(l(\theta_{k+1}) - l(\theta^*)\} \le 0 \tag{32}$$

Collecting ΔV_k terms from t_0 to T, and letting $T \to \infty$, it can be seen that $l(\theta_{k+1}) - l(\theta^*) \in \ell_1 \cap \ell_\infty$ and therefore $\lim_{k\to\infty} l(\theta_{k+1}) - l(\theta^*) = 0$. □

IV. Numerical Study

A. An academic example

We consider the same example as before which led to the following constrained convex optimization problem:

$$\begin{aligned} &\min\ \log(e^{x_1} + e^{4-\sqrt{1-x_1^2}}) \\ &\text{s.t.}\ \ -1 \le x_1 \le 1 \end{aligned} \tag{33}$$

The box constraint in (33) is imposed to ensure that the objective function is always real-valued. While we can

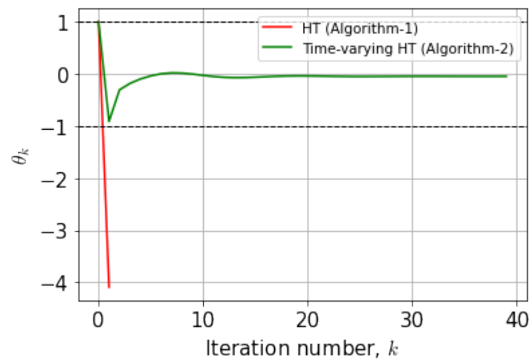

Fig. 1. Convergence of θ using Algorithm 2 for the problem in (33); Algorithm 1 fails to converge

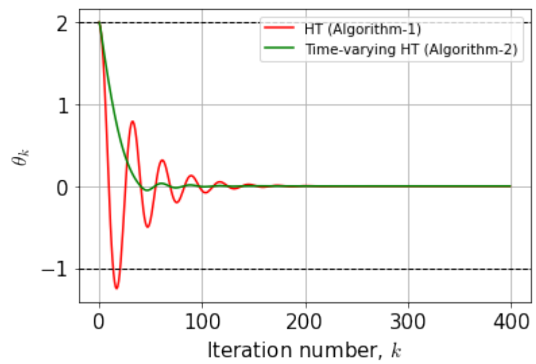

Fig. 2. Constraint satisfaction by Algorithm-2, Algorithm-1 generates iterates that violate the box constraints [-1,2]. Algorithm-2 displays smaller amplitude oscillations compared to Algorithm-1

choose certain step-sizes that ensure that $-1 \leq x_1 \leq 1$ for solving (33), there are no guarantees that such a step-size exists to ensure this constraint. Algorithm 2 solves this problem with suitable choices of γ, β, a_k and b_{k+1} as specified in Theorem III.3. Figure 1 shows the convergence of parameter θ (equivalent to x_1 in (33)) using Algorithm 2 for a chosen value of γ, β. It should be noted in Figure 1 that Algorithm 1 fails to converge. This illustrates the value of Theorem III.3, another contribution of this paper.

B. Provably hard problem of Nesterov

We consider a provably hard problem which corresponds to a strongly convex function (see [6] for details)

$$l(\theta) = \log\left(c_k e^{d_k \theta} + c_k e^{-d_k \theta}\right) + \frac{\mu}{2}\|\theta - \theta_0\|^2. \qquad (34)$$

Here c_k and d_k are positive scalars chosen as $c_k = \frac{1}{2}$ and $d_k = 1$. This function has a unique minimum at $\theta^* = 0$.

In all cases, $\mu = 10^{-4}$, the intial value was chosen to be $\theta_0 = 2$ and the constraints are chosen as $\theta_{min} = -1$ and $\theta_{max} = 2$. It is clear from Figure 2 that with Algorithm 2, the parameters converge to the optimal value while being within $[\theta_{min}, \theta_{max}]$. The speed of convergence is faster than that of Algorithm 1 and Algorithm 2 exhibits lesser oscillations, which is an attractive property.

V. Conclusion and Future Work

In this work, we extend the previously conducted study on using High Order Tuners to solve constrained convex optimization. We propose a new HT that can accommodate the constraints and the non-convexities with guaranteed convergence. We provide academic examples and numerical simulations to validate the theorems presented in the paper pertaining to the accelerated convergence and feasibility guarantees for High Order Tuner. This work establishes a framework to explore potential applications of constrained optimization that would benefit from faster convergence, such as neural network training which can be reformulated in some cases as convex optimization problems [15], and solving Optimal Power Flow (OPF) problems using neural networks as in [16], [17].

References

[1] J. E. Gaudio, A. M. Annaswamy, M. A. Bolender, E. Lavretsky, and T. E. Gibson, "A Class of High Order Tuners for Adaptive Systems," *IEEE Control Systems Letters*, pp. 1–1, 6 2020.
[2] Y. Nesterov, *Lectures on convex optimization*. Springer International Publishing, 2018, vol. 137.
[3] W. Su, S. Boyd, and E. J. Candès, "A differential equation for modeling nesterov's accelerated gradient method: Theory and insights," *Journal of Machine Learning Research*, vol. 17, no. 153, pp. 1–43, 2016.
[4] S. Evesque, A. Annaswamy, S. Niculescu, and A. Dowling, "Adaptive control of a class of time-delay systems," *J. Dyn. Sys., Meas., Control*, vol. 125, no. 2, pp. 186–193, 2003.
[5] A. S. Morse, "High-order parameter tuners for the adaptive control of linear and nonlinear systems," in *Systems, models and feedback: Theory and Applications*. Springer, 1992, pp. 339–364.
[6] J. M. Moreu and A. M. Annaswamy, "A stable high-order tuner for general convex functions," *IEEE Control Systems Letters*, vol. 6, pp. 566–571, 2022.
[7] P. Srivastava and J. Cortés, "Nesterov acceleration for equality-constrained convex optimization via continuously differentiable penalty functions," *IEEE Control Systems Letters*, vol. 5, no. 2, pp. 415–420, 2021.
[8] M. Fazlyab, A. Koppel, V. M. Preciado, and A. Ribeiro, "A variational approach to dual methods for constrained convex optimization," in *2017 American Control Conference (ACC)*, 2017, pp. 5269–5275.
[9] Y. Xu, "Accelerated first-order primal-dual proximal methods for linearly constrained composite convex programming," *SIAM Journal on Optimization*, vol. 27, no. 3, pp. 1459–1484, 2017.
[10] A. Parashar, P. Srivastava, A. M. Annaswamy, B. Dey, and A. Chakraborty, "Accelerated algorithms for a class of optimization problems with constraints," in *2022 IEEE 61st Conference on Decision and Control (CDC)*, 2022, pp. 6960–6965.
[11] P. L. Donti, D. Rolnick, and J. Z. Kolter, "DC3: A learning method for optimization with hard constraints," in *International Conference on Learning Representations*, 2021.
[12] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.
[13] S. G. Krantz and H. R. Parks, *Introduction to the Implicit Function Theorem*. New York, NY: Springer New York, 2013, pp. 1–12.
[14] M. Furi and M. Martelli, "A multidimensional version of rolle's theorem," *The American Mathematical Monthly*, vol. 102, no. 3, pp. 243–249, 1995.
[15] M. Pilanci and T. Ergen, "Neural networks are convex regularizers: Exact polynomial-time convex optimization formulations for two-layer networks," in *Proceedings of the 37th International Conference on Machine Learning*, Jul 2020, pp. 7695–7705.
[16] X. Pan, T. Zhao, M. Chen, and S. Zhang, "DeepOPF: A deep neural network approach for security-constrained DC optimal power flow," *IEEE Transactions on Power Systems*, vol. 36, no. 3, pp. 1725–1735, 2021.
[17] X. Pan, T. Zhao, and M. Chen, "DeepOPF: Deep neural network for dc optimal power flow," 10 2019, pp. 1–6.