# $q$-OPERS, $QQ$-SYSTEMS, AND BETHE ANSATZ

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Abstract. We introduce the notions of $(G,q)$-opers and Miura $(G,q)$-opers, where $G$ is a simply connected simple complex Lie group, and prove some general results about their structure. We then establish a one-to-one correspondence between the set of $(G,q)$-opers of a certain kind and the set of nondegenerate solutions of a system of Bethe Ansatz equations. This may be viewed as a $q$DE/IM correspondence between the spectra of a quantum integrable model (IM) and classical geometric objects ($q$-differential equations). If $\mathfrak{g}$ is simply laced, the Bethe Ansatz equations we obtain coincide with the equations that appear in the quantum integrable model of XXZ-type associated to the quantum affine algebra $U_q\widehat{\mathfrak{g}}$. However, if $\mathfrak{g}$ is non-simply laced, then these equations correspond to a different integrable model, associated to $U_q{}^L\widehat{\mathfrak{g}}$ where ${}^L\widehat{\mathfrak{g}}$ is the Langlands dual (twisted) affine algebra. A key element in this $q$DE/IM correspondence is the $QQ$-system that has appeared previously in the study of the ODE/IM correspondence and the Grothendieck ring of the category $\mathcal{O}$ of the relevant quantum affine algebra.

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## 1. Introduction

In his celebrated 1931 paper [B1], Hans Bethe proposed a method of diagonalization of the Hamiltonian of the XXX spin chain model that was introduced three years earlier by Werner Heisenberg. The elegance and simplicity of his method, dubbed the Bethe Ansatz, has dazzled several generations of physicists and mathematicians. Richard Feynman wrote [F]: "I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better." In this paper, we make another attempt to understand all this better.

1.1. **Gaudin model.** To explain our main idea, let us discuss a close relative of the XXX spin chain: the quantum Gaudin model corresponding to a simple Lie algebra $\mathfrak{g}$. The space of states of this model is a representation of the corresponding (polynomial) loop algebra

$\mathfrak{g}[t,t^{-1}]$. Suppose for simplicity that it is the tensor product $\otimes_{i=1}^{N} V_{\lambda_i}(z_i)$ of finite-dimensional evaluation representations with highest weights $\lambda_i$ and distinct evaluations parameters $z_i$. In this case, it was shown in [FFR, F3] that the spectrum of the quantum Gaudin Hamiltonians can be encoded by objects of an entirely different nature, namely, certain ordinary differential operators on $\mathbb{P}^1$ called *opers*. (This concept was introduced by Beilinson and Drinfeld [BD2] and goes back to the work of Drinfeld and Sokolov on generalized KdV systems [DS]). The opers that encode the spectrum have regular singularities at the points $z_1, \ldots, z_N$ and $\infty$, and trivial monodromy.

Remarkably, the opers that encode the spectra of the Gaudin model associated to $\mathfrak{g}$ are naturally associated to the *Langlands dual* Lie algebra ${}^L\mathfrak{g}$ rather than $\mathfrak{g}$ itself. This is no accident; as explained in [F1], this correspondence may be viewed as a special case of the construction [BD1] of the geometric Langlands correspondence.

As a bonus, we obtain (at least, in a generic situation) explicit formulas for the eigenvectors and eigenvalues of the Gaudin Hamiltonians [FFR], which are analogous to Bethe's original formulas. An interesting aspect is that the ${}^L\mathfrak{g}$-opers corresponding to these eigenvalues can be expressed in terms of the Miura transformation well-known in the theory of generalized KdV equations [FFR, F3]. This means that we may alternatively encode the eigenvalues by so-called Miura opers. The same is true if we replace finite-dimensional representations $V_{\lambda_i}$ by other highest weight $\mathfrak{g}$-modules. (For more general $\mathfrak{g}$-modules, the spectral problem becomes more complicated; a strategy for describing the relevant ${}^L\mathfrak{g}$-opers and the corresponding Bethe Ansatz equations using the data of 4D gauge theories was proposed in [NW, NRS].)

Accordingly, we obtain a link between two worlds that at first glance seem to be far apart: the quantum world of integrable models and the classical world of geometry and differential equations. The mystery is partially resolved when we realize that this link can be deformed in such a way that both sides correspond to quantum field theories. Namely, the spectral problem of the Gaudin model deforms to the KZ equations on conformal blocks in a WZW model associated to $\mathfrak{g}$ (see [RV]), whereas opers deform to conformal blocks of the $\mathcal{W}$-algebra corresponding to ${}^L\mathfrak{g}$.

Thus, the original Gaudin/opers correspondence can be recovered in a special limit of a more familiar duality of QFTs. In this limit, the symmetry algebra on one side remains quantum but develops a large commutative subalgebra (in fact, it can be obtained from the center of $\widetilde{U}(\widehat{\mathfrak{g}})$ at the critical level [FFR, F3] so that the quadratic Hamiltonians correspond to the limit of the rescaled stress tensor given by the Segal-Sugawara formula). In contrast, the limit of the symmetry algebra on the other side is purely classical (opers and related structures); thus, we get a commutative algebra with a Poisson structure. This Poisson structure is a hint of the existence of a one-parameter deformation, as is the appearance of the Miura transformation (which is known to preserve Poisson structures) in the formula for the eigenvalues of quantum Hamiltonians, when they are expressed as opers.

1.2. **Quantum KdV systems.** The above construction has various generalizations. One possibility, discussed in [Z], is to replace the Lie algebra $\mathfrak{g}$ by a Lie superalgebra. Another possibility is to replace the finite-dimensional simple Lie algebra $\mathfrak{g}$ by the corresponding affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. These affine Gaudin models were introduced in [FF], where it was argued that the spectra of the corresponding quantum Hamiltonians (in this case, there are both local and non-local Hamiltonians) should be encoded by affine analogues of ${}^L\mathfrak{g}$-opers, called ${}^L\widehat{\mathfrak{g}}$-opers. Here, ${}^L\widehat{\mathfrak{g}}$ is the affine algebra that is Langlands dual to $\widehat{\mathfrak{g}}$.

In particular, ${}^L\widehat{\mathfrak{g}}$ is a twisted affine Kac-Moody algebra if $\widehat{\mathfrak{g}}$ is an untwisted affine Lie algebra associated to a non-simply laced $\mathfrak{g}$, so this duality is more subtle than in the finite-dimensional case.

It was shown in [FF] that with particular choices of the available parameters, the affine Gaudin model associated to $\widehat{\mathfrak{g}}$ can be viewed as a quantization of the classical $\widehat{\mathfrak{g}}$-KdV system defined in [DS]. The conjecture of [FF] (see also [FH2]) states that the spectra of the corresponding quantum $\widehat{\mathfrak{g}}$-Hamiltonians can be encoded by ${}^L\widehat{\mathfrak{g}}$-opers on $\mathbb{P}^1$ with particular analytic properties. In the case of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ (so that we are dealing with the KdV system proper), these opers can be written as second order ordinary differential operators with spectral parameter on $\mathbb{P}^1$. The conjecture of [FF] then becomes the conjecture made earlier by Bazhanov, Lukyanov, and Zamolodchikov [BLZ], which was in fact a motivation for [FF]. (A similar conjecture was also made in the case of $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_3$ in [BHK].)

For general $\mathfrak{g}$, ${}^L\widehat{\mathfrak{g}}$-opers are Lie algebra-valued ordinary differential operators on $\mathbb{P}^1$ (or a finite cover, if $\mathfrak{g}$ is non-simply laced [FH2]) with spectral parameter. Therefore the conjecture of [FF, FH2] fits the general paradigm of the *ODE/IM correspondence* (see e.g. [DDT]), i.e. a correspondence between spectra of quantum Hamiltonians in a quantum integrable model (IM) and ordinary differential operators (ODE).

In [MRV1, MRV2], Masoero, Raimondo, and Valeri made an important discovery. They analyzed the ${}^L\widehat{\mathfrak{g}}$-affine opers that according to [FF] should encode the eigenvalues on the ground eigenstates of the $\widehat{\mathfrak{g}}$-KdV system. They were able to assign to each of them a collection of functions $\{Q_i(z), \widetilde{Q}_i(z)\}_{i=1,\dots,r}$ solving a novel system of equations (dubbed the $Q\widetilde{Q}$-system in [FH2]) generalizing the quantum Wronskian relation found in [BLZ]. Furthermore, they showed that in a generic situation, these equations imply that the roots of the functions $Q_i(z)$ satisfy a version of the system of Bethe Ansatz equations proposed earlier in [OW, RW, R]. These results were generalized in [MR] to the affine opers corresponding to the excited eigenstates of the $\widehat{\mathfrak{g}}$-KdV system for simply laced $\mathfrak{g}$. Thus, we obtain that the Bethe Ansatz equations describe objects on the ODE side of the ODE/IM correspondence. (This is analogous to the interpretation of the Bethe Ansatz equations of the $\mathfrak{g}$-Gaudin model in terms of the corresponding ${}^L\mathfrak{g}$-opers, see Section 1.1.)

To see that the same Bethe Ansatz equations also appear on the IM side, we need to interpret the $Q\widetilde{Q}$-system directly in terms of the quantum Hamiltonians. This was done in [FH2], where it was shown that the $Q\widetilde{Q}$-system arises from universal relations between the quantum $\widehat{\mathfrak{g}}$-KdV Hamiltonians corresponding to certain specific transfer-matrices of $U_q\widehat{\mathfrak{g}}$ (or rather its Borel subalgebra).

Thus, we obtain an analogue of the Gaudin/oper correspondence of Section 1.1: the ODE/IM correspondence between the spectra of the quantum Hamiltonians of the quantum KdV system and classical geometric objects, namely, affine opers. The two sides of the ODE/IM correspondence share the $Q\widetilde{Q}$-system and the Bethe Ansatz equations (see [FH2] and Section 6.2 for more details).

1.3. **Quantum spin chains.** The goal of the present paper is to express in the same spirit the spectra of the quantum integrable models of the type considered by Bethe in his original work [B1].

In fact, it is known that the XXX spin chain naturally corresponds to the Yangian of $\mathfrak{sl}_2$, and this model can be generalized so that the symmetry algebra is the Yangian of an arbitrary simple Lie algebra $\mathfrak{g}$. In this paper, we focus on the trigonometric versions of these models. In the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$, this is the Heisenberg XXZ model, in which the symmetry algebra is the quantum affine algebra $U_q\widehat{\mathfrak{sl}}_2$, where $q$ corresponds to the parameter $\Delta$ of the XXZ model by the formula $\Delta = (q + q^{-1})/2$. (The XXX model can be obtained from the XXZ model, if we take the limit $q \to 1$ in a particular fashion.) This model can be generalized to an arbitrary $\mathfrak{g}$, so that the symmetry algebra is the quantum affine algebra $U_q\widehat{\mathfrak{g}}$. We will refer to these models as $U_q\widehat{\mathfrak{g}}$ *XXZ-type models*.

Analogues of the Bethe Ansatz equations for these models were proposed in the 1980's [OW,RW,R]. In the 1990's, Reshetikhin and one of the authors [FR2] explained how to relate them to the spectra of the transfer-matrices associated to finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$, which are quantum Hamiltonians of this model. This derivation was based on a conjectural formula (generalizing Baxter's $TQ$-relation in the case of $\mathfrak{g} = \mathfrak{sl}_2$) for the eigenvalues of these transfer-matrices in terms of the $q$-characters of the corresponding representations of $U_q\widehat{\mathfrak{g}}$. This conjecture was proved by Hernandez and one of the authors in [FH1].

Thus, as in the previous two types of quantum integrable systems, we obtain a description of the spectra of quantum Hamiltonians in terms of solutions of a system of Bethe Ansatz equations.

The question then is: *How to construct the geometric objects encoding the solutions of these Bethe Ansatz equations?* If we answer this question, we obtain another instance of the duality discussed above.

The first step in this direction was made by Mukhin and Varchenko [MV2] in the closely related case of the XXX-type models corresponding to Yangians rather than quantum affine algebras. To each solution of the corresponding system of Bethe Ansatz equations, they assigned a difference operator that they called a "discrete Miura oper." However, since they did not give an independent definition of such objects, this does not give a duality correspondence of the kind discussed above.

In [KSZ], three of the authors proposed such a definition in the case of the $U_q\widehat{\mathfrak{sl}}_n$ XXZ-type model. Namely, they introduced $(\mathrm{SL}(n), q)$-opers (as scalar $q$-difference operators of order $n$) and related them to nondegenerate solutions of the corresponding Bethe Ansatz equations.

In this paper, we elucidate the results of [KSZ] and generalize them to other $U_q\widehat{\mathfrak{g}}$ XXZ-type models.

1.4. **$q$-opers and Miura $q$-opers.** We start in Section 2 with an intrinsic, coordinate-independent definition of the geometric objects dual to the spectra of the $U_q\widehat{\mathfrak{g}}$ XXZ-type models. These are the *$q$-opers* and *Miura $q$-opers* associated to an arbitrary simple simply-connected complex Lie group $G$. (Our definition can be generalized in a straightforward fashion to an arbitrary reductive $G$.) They are analogues of ordinary opers and Miura opers.

Unlike opers (or more general connections on principle bundles), which can be defined over an arbitrary complex algebraic curve $X$, the definition of a $q$-connection (with $q$ not a root of unity) requires that $X$ carry an automorphism of infinite order. In this paper, we focus on the case $X = \mathbb{P}^1$, with the automorphism being $z \mapsto qz$. However, a similar definition can also be given in the additive case (a translation of the affine line, which can be extended to $\mathbb{P}^1$) or the elliptic case, where $X$ is an elliptic curve and we choose a translation by a generic element of the abelian group of its points. Most of our results can be generalized to these two cases.

Having introduced the notion of a $q$-connection on a principal $G$-bundle on the projective line $\mathbb{P}^1$, we define a $(G, q)$-oper as a triple consisting of a principal $G$-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$

together with a reduction $\mathcal{F}_{B_-}$ of $\mathcal{F}_G$ to a Borel subgroup $B_-$ and a $q$-connection satisfying a certain condition with respect to $\mathcal{F}_{B_-}$, which is analogous to the oper condition. In defining the $q$-oper condition, we follow the definition of $q$-Drinfeld-Sokolov reduction given in [FRSTS, SS], which involves the Bruhat cell of a Coxeter element of the Weyl group of $G$.

We then define a Miura $(G, q)$-oper by analogy with the differential case (see [F2, F3]) as a $(G, q)$-oper with an additional structure: a reduction $\mathcal{F}_{B_+}$ of $\mathcal{F}_G$ to an opposite Borel subgroup $B_+$ that is preserved by the oper $q$-connection. We prove a fundamental property of the two reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$: they are necessarily in generic relative position on a dense Zariski open subset of $\mathbb{P}^1$. This is analogous to the situation in the differential case [F2, F3].

1.5. **$q$DE/IM correspondence.** Consider now the $U_q\widehat{\mathfrak{g}}$ XXZ-type model, where $\mathfrak{g}$ is a simple Lie algebra. Assume that the space of states on which the quantum Hamiltonians (the transfer-matrices of $U_q\widehat{\mathfrak{g}}$) act is finite-dimensional. In the case of the Gaudin model, we saw in Section 1.1 that the opers encoding the spectra in the finite-dimensional case have trivial monodromy, i.e. are gauge equivalent to the trivial connection. The XXZ-type models that we consider in this paper are slightly more general, in that we include a non-trivial twist of the boundary conditions, which is represented by a regular semisimple element $Z$ of the Lie group $G$. (In the case of the Gaudin model, this corresponds to allowing opers on $\mathbb{P}^1$ with an irregular singularity at $\infty$ [FFTL]; a similar twist can also be included in XXX-type models.) Because of this twist, we impose the condition that our $q$-opers are $q$-gauge equivalent to the constant $q$-connection equal to $Z$. We call such $q$-opers, and the corresponding Miura $q$-opers, *$Z$-twisted*.

We further introduce certain intermediate objects, which we call *$Z$-twisted Miura-Plücker $q$-opers* on $\mathbb{P}^1$. The main theorem of this paper establishes a one-to-one correspondence between the set of these objects satisfying an explicit nondegeneracy condition and the set of nondegenerate solutions of a system of Bethe Ansatz equations.

This is proved by constructing a bijection between each of the above two sets and the set of nondegenerate solutions of a system of algebraic equations. Remarkably, for simply laced $\mathfrak{g}$, this system is a version of the $Q\widetilde{Q}$-system discussed in Section 1.2 in the context of the quantum $\widehat{\mathfrak{g}}$-KdV systems. Here, we refer to it as the *$QQ$-system*. Following the results of [FH1, FH2], we expect that this system of equations is satisfied by a certain family of transfer-matrices of the $U_q\widehat{\mathfrak{g}}$ XXZ-type model (see Section 6.2 for more details). Accordingly, we can link the spectra of these transfer-matrices and $q$-opers.

Thus, for simply laced $\mathfrak{g}$ we fulfill our goal and obtain a dual description of the spectra of quantum Hamiltonians of the $U_q\widehat{\mathfrak{g}}$ XXZ-type model in terms of geometric objects, namely, $Z$-twisted Miura-Plücker $(G, q)$-opers on $\mathbb{P}^1$, where $Z$ corresponds to the twist in the boundary condition of the XXZ-type model.

By analogy with the ODE/IM correspondence discussed in Section 1.2, we call it the *$q$DE/IM correspondence*.

If $\mathfrak{g}$ is non-simply laced, this $q$DE/IM correspondence becomes more subtle. In fact, the $QQ$-system and the system of Bethe Ansatz equations we obtain in this case are different from the systems that arise from the $U_q\widehat{\mathfrak{g}}$ XXZ-type model. (This was already noted in [MV2] in the case of XXX-type models.) In light of the results and conjectures of [FR1, FH2], we expect that they arise from a novel quantum integrable model associated to the *twisted* quantum affine algebra $U_q{}^L\widehat{\mathfrak{g}}$, where ${}^L\widehat{\mathfrak{g}}$ is the affine Kac-Moody algebra Langlands dual to $\widehat{\mathfrak{g}}$. This will be explained in [FHR].

In summary, we expect that *the qDE/IM correspondence links Z-twisted Miura-Plücker $(G, q)$-opers on $\mathbb{P}^1$ and the spectra of quantum Hamiltonians in a quantum integrable model associated to $U_q{}^L\widehat{\mathfrak{g}}$.*

1.6. **Quantum $q$-Langlands Correspondence.** In the case of the Gaudin model, both sides of the duality between spectra and opers can be deformed to QFTs, as we discussed in Section 1.1. It turns out that such a deformation also exists in the case of the $q$DE/IM correspondence. It was proposed in [AFO] under the name "quantum $q$-Langlands correspondence." Specifically, it was shown in [AFO] that for simply laced $\mathfrak{g}$, one can identify solutions of the $q$KZ equations, which can be viewed as deformed conformal blocks for the quantum affine algebra, and deformed conformal blocks for the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$. (The notation is a bit misleading because what we previously denoted by $q$ is now the $t$ of $\mathcal{W}_{q,t}(\mathfrak{g})$, and the $q$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ is the product of $t$ and the parameter of the quantum affine algebra on the other side; thus, we obtain an XXZ-type model in the limit in which this $q$ goes to 1 and so we should consider $t$-opers rather than $q$-opers.)

As argued in [AFO], this quantum duality has its origins in the duality of the little string theory with defects on an ADE-type surface. The defects wrap compact and noncompact cycles of the internal Calabi-Yau manifold, thereby producing screening and vertex operators respectively of the deformed conformal blocks of $\mathcal{W}_{q,t}(\mathfrak{g})$. On the other hand, little string theory, in the limit where it becomes a conformal $(0, 2)$ theory in six dimensions, localizes on the defects, yielding quiver gauge theories whose Higgs branches are Nakajima quiver varieties. Using powerful methods of enumerative equivariant K-theory [O], the authors of [AFO] were able to express these deformed conformal blocks in terms of solutions of the $q$KZ equations.

The quantum $q$-Langlands correspondence for the QFTs associated to non-simply laced Lie algebras is more subtle. In [AFO], it is studied by introducing an $H$-twist to the little string theory which, together with the corresponding outer automorphism of the ADE Dynkin diagram, acts on the complex line supporting the defects. (A similar construction was considered in [DHKM] and [KP].)

In the critical level limit, solutions of the $q$KZ equations corresponding to $U_q\widehat{\mathfrak{g}}$ give rise to eigenvectors of the $U_q\widehat{\mathfrak{g}}$ XXZ-type model. The results of the present paper suggest that for non-simply laced $\mathfrak{g}$, the limits of the correlation functions on the other side should give rise *not* to $q$-opers associated to ${}^LG$, as one might expect by analogy with the duality in the case of the Gaudin model, but rather to some twisted $q$-opers associated to a twisted affine Kac-Moody group. These can probably be defined similarly to the definition of twisted opers in [FG] and twisted affine opers in [FH2].

The appearance of the Langlands dual affine Lie algebra in this duality can also be seen from the Poisson structure on the space of $(G, q)$-opers which was defined in [FRSTS, SS] using $q$-Drinfeld-Sokolov reduction. (This is, of course, a hint at the existence of the quantum duality which is analogous to the classical duality involving the Gaudin models, see Section 1.1.) The results and conjectures of [FR1] (see Conjecture 3, Section 6.3, and Appendix B) show that if $G$ is non-simply laced, then this Poisson algebra (with $q$ replaced by $t^{r^\vee}$, $r^\vee$ being the lacing number of $\mathfrak{g}$) is isomorphic to the limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ as $q \to 1$, which is in turn related to the center of $U_t\widehat{\mathfrak{g}}^\vee$ at the critical level and to the algebra of transfer-matrices of $U_t\widehat{\mathfrak{g}}^\vee$, where $\widehat{\mathfrak{g}}^\vee = {}^L(\widehat{{}^L\mathfrak{g}})$. On the other hand, Conjecture 4 of [FR1] states that the limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ as $q \to e^{\pi i/r^\vee}$, is related to the center of $U_t{}^L\widehat{\mathfrak{g}}$ at the critical level and to the algebra of transfer-matrices of $U_t{}^L\widehat{\mathfrak{g}}$. Both of these limits and their potential connections to the duality of [AFO] in the non-simply laced case certainly deserve investigation. This will be further discussed in [FHR].

1.7. **A brane construction.** Finally, we want to mention another approach to the $q$DE/IM correspondence. In the case of the Gaudin model discussed in Section 1.1, it was shown in [NW, GW] that the spectra of the Gaudin Hamiltonians can be identified with the intersection of the brane of opers and another brane in the corresponding Hitchin moduli space of Higgs bundles. Recent work of Elliott and Pestun [EP] suggests that there is a $q$-deformation of this construction, in which one should consider a multiplicative version of Higgs bundles and a brane of $q$-opers. In the case of finite-dimensional representations of quantum affine algebras, this idea is supported by the fact, originally observed in [FR1, FR2] (see also Section 8 below), which is discussed in [EP], that the $q$-character homomorphism and the corresponding generalized Baxter $TQ$-relations can be interpreted in terms of a $q$-deformation of the Miura transformation.

It is possible that this construction can also be generalized to infinite-dimensional representations of quantum affine algebras along the lines of [NRS], with the brane of opers replaced by a brane of $q$-opers, which can now be rigorously defined using the results of the present paper.

1.8. **Plan of the Paper.** The paper is organized as follows. In Section 2, we give the definition of meromorphic $q$-opers and Miura $q$-opers. (We consider the case of the curve $\mathbb{P}^1$ but the same definition can be given for any curve equipped with an automorphism of infinite order.) We prove Theorem 2.3 about the relative position of the two Borel reductions of a $q$-Miura oper. In Section 3, we define $Z$-twisted $(G, q)$-opers with regular singularities on $\mathbb{P}^1$ as well as the corresponding Miura $(G, q)$-opers and Cartan $q$-connections. In Section 4, we define Miura-Plücker $q$-opers and introduce a nondegeneracy condition for them.

In Section 5, we consider in detail the case $G = \mathrm{SL}(2)$ and show, elucidating the results of [KSZ], that the nondegenerate $Z$-twisted Miura $q$-opers (which are the same as Miura-Plücker $q$-opers in this case) are in bijection with nondegenerate solutions of the $QQ$-system and Bethe Ansatz equations (Theorems 5.1 and 5.2). In Section 6, we generalize these results to an arbitrary simply connected simple complex Lie group $G$ (Theorems 6.1 and 6.4).

In Section 7, we define Bäcklund-type transformations on Miura-Plücker $q$-opers, following the construction of similar transformations in the Yangian case given in [MV2]. We use these transformation to give a sufficient condition for a Miura-Plücker $q$-oper to be a Miura $q$-oper. Finally, in Section 8, we discuss the relation between $(G, q)$-opers and the $q$-Drinfeld-Sokolov reduction defined in [FRSTS, SS]. We construct a canonical system of coordinates on the space of $(G, q)$-opers with regular singularities. Following [FR1, FR2], we conjecture that if $G$ is simply laced, then the formulas expressing these canonical coordinates in terms of the polynomials $Q^i_+(z)$ coincide with the generalized Baxter $TQ$-relations established in [FH1].

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## 2. $(G, q)$-opers with regular singularities

2.1. **Group-theoretic data.** Let $G$ be a connected, simply connected, simple algebraic group of rank $r$ over $\mathbb{C}$. We fix a Borel subgroup $B_-$ with unipotent radical $N_- = [B_-, B_-]$ and a maximal torus $H \subset B_-$. Let $B_+$ be the opposite Borel subgroup containing $H$. Let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of positive simple roots for the pair $H \subset B_+$. Let $\{\check{\alpha}_1, \ldots, \check{\alpha}_r\}$ be the corresponding coroots; the elements of the Cartan matrix of the Lie algebra $\mathfrak{g}$ of $G$ are given by $a_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle$. The Lie algebra $\mathfrak{g}$ has Chevalley generators $\{e_i, f_i, \check{\alpha}_i\}_{i=1,\ldots,r}$, so that $\mathfrak{b}_- = \mathrm{Lie}(B_-)$ is generated by the $f_i$'s and the $\check{\alpha}_i$'s and $\mathfrak{b}_+ = \mathrm{Lie}(B_+)$ is generated by the $e_i$'s and the $\check{\alpha}_i$'s. Let $\omega_1, \ldots \omega_r$ be the fundamental weights, defined by $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$.

Let $W_G = N(H)/H$ be the Weyl group of $G$. Let $w_i \in W$, $(i = 1, \ldots, r)$ denote the simple reflection corresponding to $\alpha_i$. We also denote by $w_0$ be the longest element of $W$, so that $B_+ = w_0(B_-)$. Recall that a Coxeter element of $W$ is a product of all simple reflections in a particular order. It is known that the set of all Coxeter elements forms a single conjugacy class in $W_G$. We will fix once and for all (unless otherwise specified) a particular ordering $(\alpha_{i_1}, \ldots, \alpha_{i_r})$ of the simple roots. Let $c = w_{i_1} \ldots w_{i_r}$ be the Coxeter element associated to this ordering. In what follows (unless otherwise specified), all products over $i \in \{1, \ldots, r\}$ will be taken in this order; thus, for example, we write $c = \prod_i w_i$. We also fix representatives $s_i \in N(H)$ of $w_i$. In particular, $s = \prod_i s_i$ will be a representative of $c$ in $N(H)$.

Although we have defined the Coxeter element $c$ using $H$ and $B_-$, it is in fact the case that the Bruhat cell $BcB$ makes sense for any Borel subgroup $B$. Indeed, let $(\Phi, \Delta)$ be the root system associated to $G$, where $\Delta$ is the set of simple roots as above and $\Phi$ is the set of all roots. These data give a realization of the Weyl group of $G$ as a Coxeter group, i.e., a pair $(W_G, S)$, where $S$ is the set of Coxeter generators $w_i$ of $W_G$ associated to elements of $\Delta$. Now, given any Borel subgroup $B$, set $\mathfrak{b} = \mathrm{Lie}(B)$. Then the dual of the vector space $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ comes equipped with a set of roots and simple roots, and this pair is canonically isomorphic to the root system $(\Phi, \Delta)$ [CG, §3.1.22]. The definition of the sets of roots and simple roots on this space involves a choice of maximal torus $T \subset B$, but these sets turn out to be independent of the choice. Accordingly, the group $N(T)/T$ together with the set of its Coxeter generators corresponding to these simple roots is isomorphic to $(W_G, S)$ as a Coxeter group. Under this isomorphism, $w \in W_G$ corresponds to an element of $N(T)/T$ by the following rule: we write $w$ as a word in the Coxeter generators of $W_G$ corresponding to elements of $S$ and then replace each Coxeter generator in it by the corresponding Coxeter generator of $N(T)/T$. Accordingly, the Bruhat cell $BwB$ is well-defined for any $w \in W_G$.

2.2. **Meromorphic $q$-opers.** The definitions given below can be given for an arbitrary algebraic curve equipped with an automorphism of infinite order. For the sake of definitiveness, we will focus here on the case of the curve $\mathbb{P}^1$ and its automorphism $M_q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ sending $z \mapsto qz$, where $q \in \mathbb{C}^\times$ is *not* a root of unity.

Given a principal $G$-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$ (in the Zariski topology), let $\mathcal{F}_G^q$ denote its pullback under the map $M_q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ sending $z \mapsto qz$. A meromorphic $(G, q)$-*connection* on a principal $G$-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ is a section $A$ of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$, where $U$ is a Zariski open dense subset of $\mathbb{P}^1$. We can always choose $U$ so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to $U$ is isomorphic to the trivial $G$-bundle. Choosing such an isomorphism, i.e. a trivialization of $\mathcal{F}_G|_U$, we also obtain a trivialization of $\mathcal{F}_G|_{M_q^{-1}(U)}$. Using these trivializations, the restriction of $A$ to the Zariski open dense subset $U \cap M_q^{-1}(U)$ can be written as a section of the trivial $G$-bundle on $U \cap M_q^{-1}(U)$, and hence as an element $A(z)$ of $G(z)$.[1] Changing the trivialization of $\mathcal{F}_G|_U$ via $g(z) \in G(z)$ changes $A(z)$ by the following *$q$-gauge transformation*:

$$A(z) \mapsto g(qz)A(z)g(z)^{-1}. \tag{2.1}$$

This shows that the set of equivalence classes of pairs $(\mathcal{F}_G, A)$ as above is in bijection with the quotient of $G(z)$ by the $q$-gauge transformations (2.1).

Following [FRSTS, SS], we define a $(G, q)$-oper as a $(G, q)$-connection on a $G$-bundle on $\mathbb{P}^1$ equipped with a reduction to the Borel subgroup $B_-$ that is not preserved by the $(G, q)$-connection but instead satisfies a special "transversality condition" which is defined in terms of the Bruhat cell associated to the Coxeter element $c$. Here is the precise definition.

**Definition 2.1.** A meromorphic $(G, q)$-*oper* (or simply a $q$-*oper*) on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$, where $A$ is a meromorphic $(G, q)$-connection on a $G$-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ and $\mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to $B_-$ satisfying the following condition: there exists a Zariski open dense subset $U \subset \mathbb{P}^1$ together with a trivialization $\imath_{B_-}$ of $\mathcal{F}_{B_-}$ such that the restriction of the connection $A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q$ to $U \cap M_q^{-1}(U)$, written as an element of $G(z)$ using the trivializations of $\mathcal{F}_G$ and $\mathcal{F}_G^q$ on $U \cap M_q^{-1}(U)$ induced by $\imath_{B_-}$, takes values in the Bruhat cell $B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$.

Note that this property does not depend on the choice of trivialization $\imath_{B_-}$.

Since $G$ is assumed to be simply connected, any $q$-oper connection $A$ can be written (using a particular trivialization $\imath_{B_-}$) in the form

$$A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i}\, s_i) n(z), \tag{2.2}$$

where $\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$ are such that their zeros and poles are outside the subset $U \cap M_q^{-1}(U)$ of $\mathbb{P}^1$.

We remark that the choice of a particular Coxeter element $c$ in this definition can be viewed as a choice of a particular gauge, at least for orderings differing by a cyclic permutation. Indeed, we will see below in Proposition 4.10 that the spaces of $q$-opers we consider for such a pair of Coxeter elements are isomorphic under a specific $q$-gauge transformation.

2.3. **Miura $q$-opers.** We will also need a $q$-difference version of the notion of differential Miura opers introduced in [F2, F3]. These are $q$-opers together with an additional datum: a reduction of the underlying $G$-bundle to the Borel subgroup $B_+$ (opposite to $B_-$) that is preserved by the oper $q$-connection.

**Definition 2.2.** A *Miura* $(G, q)$-*oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic $(G, q)$-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_+}$ is a reduction of the $G$-bundle $\mathcal{F}_G$ to $B_+$ that is preserved by the $q$-connection $A$.

Forgetting $\mathcal{F}_{B_+}$, we associate a $(G, q)$-oper to a given Miura $(G, q)$-oper. We will refer to it as the $(G, q)$-oper underlying the Miura $(G, q)$-oper.

The following result is an analogue of a statement about differential Miura opers proved in [F2, F3].

Suppose we are given a principal $G$-bundle $\mathcal{F}_G$ on any smooth complex manifold $X$ equipped with reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ to $B_-$ and $B_+$ respectively. We then assign to

[1]Throughout the paper, if $K$ is a complex algebraic group, we set $K(z) = K(\mathbb{C}(z))$.

any point $x \in X$ an element of the Weyl group $W_G$. To see this, first note that the fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at $x$ is a $G$-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to $B_-$ and $B_+$ respectively. Choose any trivialization of $\mathcal{F}_{G,x}$, i.e. an isomorphism of $G$-torsors $\mathcal{F}_{G,x} \simeq G$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with $bB_+$. Then, $a^{-1}b$ is a well-defined element of the double quotient $B_-\backslash G/B_+$, which is in bijection with $W_G$. Hence, we obtain a well-defined element of $W_G$.

We will say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have *generic relative position* at $x \in X$ if the element of $W_G$ assigned to them at $x$ is equal to 1. This means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_-B_+ \subset G$.

**Theorem 2.3.** *For any Miura $(G,q)$-oper on $\mathbb{P}^1$, there exists an open dense subset $V \subset \mathbb{P}^1$ such that the reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in generic relative position for all $x \in V$.*

*Proof.* Let $U$ be a Zariski open dense subset on $\mathbb{P}^1$ as in Definition 2.1. Choosing a trivialization $\imath_{B_-}$ of $\mathcal{F}_G$ on $U \cap M_q^{-1}(U)$, we can write the $q$-connection $A$ in the form (2.2). On the other hand, using the $B_+$-reduction $\mathcal{F}_{B_+}$, we can choose another trivialization of $\mathcal{F}_G$ on $U \cap M_q^{-1}(U)$ such that the $q$-connection $A$ acquires the form $\widetilde{A}(z) \in B_+(z)$. Hence there exists $g(z) \in G(z)$ such that

$$g(zq)n'(z)\prod_i(\phi_i(z)^{\check{\alpha}_i}\, s_i)n(z)g(z)^{-1} = \widetilde{A}(z) \in B_+(z). \tag{2.3}$$

Recall the Bruhat decomposition (see [B2][Theorem 21.15]):

$$G(z) = \bigsqcup_{w\in W_G} B_+(z)wN_-(z). \tag{2.4}$$

The statement of the proposition is equivalent to the statement that

$$g(z) \in B_+(z)N_-(z)$$

(corresponding to $w = 1$), or equivalently, that $g(z) \notin\ B_+(z)wN_-(z)$ for $w \neq 1$.

Suppose that this is not the case. Then $g(z) = b_+(z)wn_-(z)$ for some $b_+(z) \in B_+(z), n_-(z) \in N_-(z)$, and $w \neq 1$. Setting $\widetilde{n'}(z) = n_-(zq)n'(z)$ and $\widetilde{n}(z) = n(z)n_-(z)^{-1}$, we can rewrite (2.3) as

$$\widetilde{n'}(z)\prod_i(\phi_i(z)^{\check{\alpha}_i}\, s_i)\widetilde{n}(z) \in wB_+(z)w^{-1}. \tag{2.5}$$

Now, the Borel subgroup decomposes as

$$wB_+w^{-1} = H(N_- \cap wN_+w^{-1})(N_+ \cap wN_+w^{-1})$$

because $wN_+w^{-1} = (N_- \cap wN_+w^{-1})(N_+ \cap wN_+w^{-1}))$. Hence, denoting the element (2.5) by $A$, we can write

$$A = hu_-u_+, \qquad h \in H, \quad u_- \in N_- \cap wN_+w^{-1}, \quad u_+ \in N_+ \cap wN_+w^{-1}.$$

It follows that $u_+ \in B_-cB_- \cap N_+ \cap wN_+w^{-1}$. In particular, $u_+ \in N_+ \cap wN^+w^{-1}$, which is the product of the one-dimensional unipotent subgroups $X_\alpha$, where $\alpha$ runs over the set of positive roots for which $w(\alpha)$ is positive.

On the other hand, according to Theorem 2.5, every element of $N_- \prod_i \phi_i(z)^{\check{\alpha}_i} s_i N_- \cap B_+$ can be written in the form

$$\prod_i g_i(z)^{\check{\alpha}_i} e^{\frac{\phi_i(z)t_i}{g_i(z)}e_i}, \qquad g_i(z) \in \mathbb{C}(z)^\times, t_i \in \mathbb{C}^\times.$$

Therefore, $u_+ = h'(z)\prod_i e^{a_i(z)e_i}$, where $h \in H$ and $a_i(z) \neq 0$ for all $i = 1,\dots,r$. Such an element $u_+$ can belong to $wN_+w^{-1}$ only if $w^{-1}$ maps all positive simple roots to positive roots, i.e. if it preserves the set of positive roots. But this can only happen for $w = 1$. This completes the proof. □

**Corollary 2.4.** *For any Miura $(G,q)$-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying $G$-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper $q$-connection has the form*

$$A(z) \in N_-(z)\prod_i((\phi_i(z)^{\check{\alpha}_i}s_i)N_-(z) \,\cap\, B_+(z). \tag{2.6}$$

*Proof.* In the course of the proof of Theorem 2.3, we showed that we can choose a trivialization of $\mathcal{F}_G$ so that the oper $q$-connection has the form

$$\widetilde{A}(z) = g(zq)n'(z)\prod_i(\phi_i(z)^{\check{\alpha}_i}\, s_i)n(z)g(z)^{-1},$$

where $n(z), n'(z) \in N_-(z)$ and $g(z) = b_+(z)n_-(z)$, with $b_+(z) \in B_+(z), n_-(z) \in N_-(z)$. Therefore, changing the trivialization by $b_+(z)$, we obtain the $q$-connection

$$A(z) = b_+(zq)^{-1}\widetilde{A}(z)b_+(z) \in N_-(z)\prod_i((\phi_i(z)^{\check{\alpha}_i}s_i)N_-(z) \,\cap\, B_+(z).$$

□

2.4. **Explicit representatives.** Our proof of Theorem 2.3 relies on the following general result, which might be of independent interest.

For a field $F$, consider the group $G(F)$ and the corresponding subgroups $N_-(F)$, $B_+(F)$, and $H(F)$. As before, we denote by $s_i$ a lifting of $w_i \in W_G$ to $G(F)$.

**Theorem 2.5.** *Let $F$ be any field, and fix $\lambda_i \in F^\times, i = 1,\dots,r$. Then every element of the set $N_-\prod_i \lambda_i^{\check{\alpha}_i}s_iN_- \,\cap\, B_+$ can be written in the form*

$$\prod_i g_i^{\check{\alpha}_i}e^{\frac{\lambda_i t_i}{g_i}e_i}, \qquad g_i \in F^\times, \tag{2.7}$$

*where each $t_i \in F^\times$ is determined by the lifting $s_i$.*

We start with the following

**Lemma 2.6.** *Every element of $\lambda_i^{\check{\alpha}_i}s_iN_-$ may be written in either of the following two forms:*

$$n_-\lambda_i^{\check{\alpha}_i}s_i \qquad or \qquad n_-g^{\check{\alpha}_i}e^{\frac{\lambda_i t_i}{g}e_i}$$

*for some $n_- \in N_-(F)$, $g \in F^\times$, and with each $t_i \in F^\times$ determined by the lifting $s_i$.*

*Proof.* First, note that $\lambda_i^{\check{\alpha}_i}s_ie^{af_i}$ with $a \neq 0$ is of the form $n_-g^{\check{\alpha}_i}e^{\frac{\lambda_i t_i}{g}e_i}$, where $n_- \in N_-$ and $g = a\lambda_i t_i$. This follows from the equality of $2\times 2$ matrices

$$\begin{pmatrix}\lambda_i & 0\\ 0 & \lambda_i^{-1}\end{pmatrix}\begin{pmatrix}0 & t_i\\ -t_i^{-1} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ n_- & 1\end{pmatrix}\begin{pmatrix}a\lambda_i t_i & \lambda_i t_i\\ 0 & (a\lambda_i t_i)^{-1}\end{pmatrix},$$

where $n_- = -\frac{1}{at_i^2\lambda_i^2}$.

An arbitrary element $u$ of $N_-$ can be expressed as a product

$$u = \prod_k e^{a_kf_k}\prod_{s<r}e^{a_{s,r}[f_s,f_r]}\dots,$$

where the ellipses stand for exponentials of higher commutators and we assume a particular order in the first two products. Notice that if $a_i \neq 0$, then

$$\lambda_i^{\check{\alpha}_i} s_i \prod_k e^{a_k f_k} = n_-^1 g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i},$$

for some $n_-^1 \in N_-$ and where $g_i = a_i t_i \lambda_i$. On the other hand, if $a_i = 0$, then

$$\lambda_i^{\check{\alpha}_i} s_i \prod_k e^{a_k f_k} = n_-^2 \lambda_i^{\check{\alpha}_i} s_i$$

for some $n_-^2 \in N_-$ since $s_i(f_k)$, $k \neq i$, belong to the Lie algebra of $N_-$. Therefore, we have

$$\lambda_i^{\check{\alpha}_i} s_i n_- = \begin{cases} n_-^1 g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i} \prod_{s<r} e^{a_{s,r}[f_s,f_r]} \dots & \text{if } a_i \neq 0, \\ n_-^2 \lambda_i^{\check{\alpha}_i} s_i \prod_{s<r} e^{a_{s,r}[f_s,f_r]} \dots & \text{if } a_i = 0. \end{cases} \tag{2.8}$$

Denote one of the products of commutators and higher commutators in (2.8) by $X$. Clearly, $X \in \exp([\mathfrak{n}_-, \mathfrak{n}_-])$, and therefore $s_i(X)$ and $e^{b_i e_i} X e^{-b_i e_i}$ belong to $\exp(\mathfrak{n}_-)$. This allows us to move the elements $g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i}$, $\lambda_i^{\check{\alpha}_i} s_i$ to the right of the products in (2.8) at the expense of multiplying $n_-^1, n_-^2$ by additional elements from $N_-$. This completes the proof of the lemma. □

The following proposition is proved by repeated applications of this lemma. Suppose that $s = s_{i_1} \dots s_{i_r}$. Below, the product over $i$ means the ordered product corresponding to this decomposition.

**Proposition 2.7.** *For every element $A$ of $N_- \prod_i \lambda_i^{\check{\alpha}_i} s_i N_-$, there exists a particular ordered subset $J = \{j_1, ..., j_k\} \subset \{i_1, \dots, i_s\}$ such that $A$ can be written as $n_- \prod_i \epsilon_i$, where $n_- \in N_-$ and $\epsilon_i = g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i}$ for $i \in J$ and $\lambda_i^{\check{\alpha}_i} s_i$ for $i \notin J$, with $g_i, t_i \in \mathbb{C}^\times$.*

We are now ready to prove Theorem 2.5.

*Proof of Theorem 2.5.* In the notation of Proposition 2.7, $n_- \prod_i \epsilon_i$ belongs to $N_- w B_+$, where $w = s_{j_1} \dots s_{j_k}$. Such an element can only belong to $B_+$ if $w = 1$, i.e. if the subset $J = \{j_1, \dots, j_k\}$ is empty. This proves the theorem. □

2.5. **Connection to Fomin-Zelevinsky factorization.** Theorem 2.5 is closely related to a statement about the Fomin-Zelevinsky factorization of a particular double Bruhat cell (defined over $\mathbb{C}$) [FZ]. Here, we will only recall this factorization for double Bruhat cells of the form $G^w = B_+ \cap B_- w B_-$, where $w \in W$. If $w = w_{i_1} \dots w_{i_k}$ is a reduced expression, Fomin and Zelevinsky have shown that the map

$$\begin{aligned} H \times \mathbb{C}^k &\to G \\ (a, u_1, \dots, u_k) &\mapsto a e^{u_1 e_{i_1}} \dots e^{u_k e_{i_k}} \end{aligned} \tag{2.9}$$

induces an isomorphism of algebraic varieties $\psi^w : H \times (\mathbb{C}^\times)^k \cong G_0^w$, where $G_0^w$ is a certain Zariski-open and dense subset of $G^w$ [FZ]. In particular, $G^w$ is irreducible, and the dimension of $G^w$ is $r + \ell(w)$.

If one fixes a lifting $\widetilde{w}$ of $w$, one can also consider the subvariety $C^{\widetilde{w}} = B_+ \cap N_- \widetilde{w} N_-$. It is easy to see that $G^w = H C^{\widetilde{w}}$, so $C^{\widetilde{w}}$ is irreducible and has dimension $\ell(w)$. Moreover, $C_0^{\widetilde{w}} = C^{\widetilde{w}} \cap G_0^w$ is a Zariski-open and dense subset of $C^{\widetilde{w}}$, and any element of $C_0^{\widetilde{w}}$ can be expressed uniquely in terms of the Fomin-Zelevinsky map (2.9).

We now specialize to the case of the Coxeter element $c$. The factorization in Theorem 2.5 yields an explicit version of the Fomin-Zelevinsky factorization for $C_0^{\widetilde{c}}$, where $\widetilde{c} = \prod \lambda_i^{\check{\alpha}_i} s_i$. Theorem 2.5 thus implies that $C_0^{\widetilde{c}} = C^{\widetilde{c}}$ and $G_0^c = G^c$, i.e., in this case, the Fomin-Zelevinsky map (2.9) gives a factorization for the *entire* double Bruhat cell. In fact, the same argument applies to show $G_0^w = G^w$ for any $w$ whose reduced decompositions do not involve repeated simple reflections. We remark that this statement is apparently known to specialists and may also be proved using cluster algebra techniques.[2]

2.6. **$q$-opers and Miura $q$-opers with regular singularities.** Let $\{\Lambda_i(z)\}_{i=1,\dots,r}$ be a collection of nonconstant polynomials.

**Definition 2.8.** A $(G, q)$-*oper with regular singularities determined by* $\{\Lambda_i(z)\}_{i=1,\dots,r}$ is a $q$-oper on $\mathbb{P}^1$ whose $q$-connection (2.2) may be written in the form

$$A(z) = n'(z) \prod_i (\Lambda_i(z)^{\check{\alpha}_i} \, s_i) n(z), \qquad n(z), n'(z) \in N_-(z). \tag{2.10}$$

**Definition 2.9.** *A Miura $(G, q)$-oper with regular singularities determined by polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ is a Miura $(G, q)$-oper such that the underlying $q$-oper has regular singularities determined by $\{\Lambda_i(z)\}_{i=1,\dots,r}$.*

According to Corollary 2.4, we can write the $q$-connection underlying such a Miura $(G, q)$-oper in the form

$$A(z) \in N_-(z) \prod_i ((\Lambda_i(z)^{\check{\alpha}_i} s_i) N_-(z) \ \cap \ B_+(z).$$

Recall Theorem 2.5. Observe that we can choose liftings $s_i$ of the simple reflections $w_i \in W_G$ in such a way that $t_i = 1$ for all $i = 1, \dots, r$. From now on, we will only consider such liftings.

The following theorem follows from Theorem 2.5 in the case $F = \mathbb{C}(z)$ and gives an explicit parametrization of generic elements of the above intersection.

**Theorem 2.10.** *Every element of $N_-(z) \prod_i (\Lambda_i(z))^{\check{\alpha}_i} s_i) N_-(z) \ \cap \ B_+$ may be written in the form*

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} \, e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \qquad g_i(z) \in \mathbb{C}(z)^\times. \tag{2.11}$$

**Corollary 2.11.** *For every Miura $(G, q)$-oper with regular singularities determined by the polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$, the underlying $q$-connection can be written in the form* (2.11).

## 3. $Z$-twisted $q$-opers and Miura $q$-opers

Next, we consider a class of (Miura) $q$-opers that are gauge equivalent to a constant element of $G$ (as $(G, q)$-connections). Let $Z$ be an element of the maximal torus $H$. Since $G$ is simply connected, we can write

$$Z = \prod_{i=1}^r \zeta_i^{\check{\alpha}_i}, \qquad \zeta_i \in \mathbb{C}^\times. \tag{3.1}$$

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[2]We thank Greg Muller for a discussion of these matters.

**Definition 3.1.** A *$Z$-twisted $(G,q)$-oper* on $\mathbb{P}^1$ is a $(G,q)$-oper that is equivalent to the constant element $Z \in H \subset H(z)$ under the $q$-gauge action of $G(z)$, i.e. if $A(z)$ is the meromorphic oper $q$-connection (with respect to a particular trivialization of the underlying bundle), there exists $g(z) \in G(z)$ such that

$$A(z) = g(qz)Zg(z)^{-1}. \tag{3.2}$$

**Definition 3.2.** A *$Z$-twisted Miura $(G,q)$-oper* is a Miura $(G,q)$-oper on $\mathbb{P}^1$ that is equivalent to the constant element $Z \in H \subset H(z)$ under the $q$-gauge action of $B_+(z)$, i.e.

$$A(z) = v(qz)Zv(z)^{-1}, \qquad v(z) \in B_+(z). \tag{3.3}$$

3.1. **From $Z$-twisted $q$-opers to Miura $q$-opers.** It follows from Definition 3.1 that any $Z$-twisted $(G,q)$-oper is also $Z'$-twisted for any $Z'$ in the $W_G$-orbit of $Z$. However, if we endow it with the structure of a $Z$-twisted Miura $(G,q)$-oper (by adding a $B_+$-reduction $\mathcal{F}_{B_+}$ preserved by the oper $q$-connection), then we fix a specific element in this $W_G$-orbit.

Indeed, suppose that $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a $Z$-twisted $(G,q)$-oper. Choose a trivialization of $\mathcal{F}_G$ on a Zariski open dense subset $U$ of $\mathbb{P}^1$ with respect to which $A$ is equal to $Z$. Then a choice of an $A$-invariant $B_+$-reduction $\mathcal{F}_{B_+}$ of $\mathcal{F}_G$ on a Zariski open dense subset $V \subset U$ is the same as a choice of a $Z$-invariant $B_+$-reduction of the fiber $\mathcal{F}_{G,v}$ of $\mathcal{F}_G$ at any point $v \in V$. Our trivialization of $\mathcal{F}_G|_U$ identifies $\mathcal{F}_{G,v}$ with $G$, and hence a $B_+$-reduction of $\mathcal{F}_{G,v}$ with a right coset $gB_+$ of $G$. The $Z$-invariance of this $B_+$-reduction means that $gB_+$, viewed as a point of the flag variety $G/B_+$, is a fixed point of $Z$. This is equivalent to $Z \in gB_+g^{-1}$ or $g^{-1}Zg \in B_+$.

Adding the $B_+$-reduction $\mathcal{F}_{B_+}$ corresponding to a coset $gB_+$ satisfying this property to our $(G,q)$-oper $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$, we endow it with the structure of a Miura $(G,q)$-oper. A choice of trivialization of $\mathcal{F}_{B_+}$ is equivalent to a choice of an identification of the coset $gB_+$ with $B_+$, which is the same as a choice of an element of this coset (this element corresponds to $1 \in B_+$ under the given isomorphism $B_+ \simeq gB_+$). Without loss of generality, we denote this element also by $g$. Then, with respect to the corresponding trivialization of the $(G,q)$-oper bundle $\mathcal{F}_G$, the $q$-connection becomes equal to $g^{-1}Zg \in B_+$. However, note that because we can multiply $g$ on the right by any element of $B_+$, we still have the freedom to conjugate $g^{-1}Zg$ by an element of $B_+$, and there is a unique element in the $B_+$-conjugacy class of $g^{-1}Zg$ of the form $w^{-1}Zw$, where $w \in W_G$. Denote this element by $Z'$. We now conclude that the Miura $(G,q)$-oper obtained by endowing our $(G,q)$-oper with the $B_+$-reduction $\mathcal{F}_{B_+}$ corresponding to $gB_+$ is $Z'$-twisted.

As a result, we also construct a map $\mu_Z$ from $(G/B_+)^Z = \{f \in G/B_+ \mid Z \cdot f = f\}$ to $W_G \cdot Z$, sending $gB_+$ with $g^{-1}Zg \in B_+$ to the unique element $Z'$ of $W_G \cdot Z$ that is $B_+$-conjugate to $g^{-1}Zg$. According to the above construction, the set of points of the fiber $\mu_Z^{-1}(Z')$ of $\mu_Z$ over a specific $Z' \in W_G \cdot Z$ is in bijection with the set of $A$-invariant $B_+$-reductions $\mathcal{F}_{B_+}$ on our $(G,q)$-oper such that the corresponding Miura $(G,q)$-oper is $Z'$-twisted.

Thus, we have proved the following result.

**Proposition 3.3.** *Let $Z \in H$. For any $Z$-twisted $(G,q)$-oper $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ and any choice of $B_+$-reduction $\mathcal{F}_{B_+}$ of $\mathcal{F}_G$ preserved by the oper $q$-connection $A$, the resulting Miura $(G,q)$-oper is $Z'$-twisted for a particular $Z' \in W_G \cdot Z$.*

*Moreover, the set of $A$-invariant $B_+$-reductions $\mathcal{F}_{B_+}$ on the $(G,q)$-oper $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ making it into a $Z'$-twisted Miura $(G,q)$-oper is in bijection with the set of points of $\mu_Z^{-1}(Z')$.*

Consider two extreme examples of $Z$. If $Z = 1$, we always obtain $Z' = 1$, so the set of $B_+$-reductions is the set of points of the flag manifold $G/B_+$. On the other hand, if $Z \in H$ is regular, then the map $\mu_Z$ is a bijection. Thus, for each $Z' \in W_G \cdot Z$, there is a unique $B_+$-reduction on the $(G, q)$-oper $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ making it into a $Z'$-twisted Miura $(G, q)$-oper. We will focus on the regular case in the rest of this paper.

We also note that the first statement of Proposition 3.3 has a more concrete reformulation:

**Corollary 3.4.** *Let $Z \in H$. For any $Z$-twisted $(G, q)$-oper whose $q$-connection $A(z)$ has the form (3.2) and any $Z' \in W_G \cdot Z$, $A(z)$ can be written in the form*

$$A(z) = n(qz)v(qz)\, Z'\, v(z)^{-1}n(z)^{-1}, \qquad n(z) \in N_-(z), \quad v(z) \in B_+(z). \tag{3.4}$$

*Proof.* We give an independent proof. Let $Z' = w^{-1}Zw, w \in W_G$. Formula (3.2) implies

$$(g(qz)s)^{-1}A(z)(g(z)s) = Z', \tag{3.5}$$

where $s$ is a lifting of $w$ to $N(H)$. Since $Z' \in B_+ \subset B_+(z)$, Theorem 2.3 implies that

$$g(z)s = n(z)v(z), \qquad n(z) \in N_-(z), \quad v(z) \in B_+(z).$$

Substituting this back into formula (3.5), we obtain formula (3.4). □

3.2. **The associated Cartan $q$-connection.** From now on, let $Z$ be a regular element of the maximal torus $H \subset G$.

Consider a Miura $(G, q)$-oper with regular singularities determined by polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$. By Corollary 2.11, the underlying $(G, q)$-connection can be written in the form (2.11). Since it preserves the $B_+$-bundle $\mathcal{F}_{B_+}$ that is part of the data of this Miura $(G, q)$-oper (see Definition 2.2), it may be viewed as a meromorphic $(B_+, q)$-connection on $\mathbb{P}^1$. Taking the quotient of $\mathcal{F}_{B_+}$ by $N_+ = [B_+, B_+]$ and using the fact that $B/N_+ \simeq H$, we obtain an $H$-bundle $\mathcal{F}_{B_+}/N_+$ endowed with an $(H, q)$-connection, which we denote by $A^H(z)$. According to formula (2.11), it is given by the formula

$$A^H(z) = \prod_i g_i(z)^{\check{\alpha}_i}. \tag{3.6}$$

We call $A^H(z)$ the *associated Cartan $q$-connection* of the Miura $q$-oper $A(z)$.

Now, if our Miura $q$-oper is $Z$-twisted (see Definition 3.2), then we also have $A(z) = v(qz)Zv(z)^{-1}$, where $v(z) \in B_+(z)$. Since $v(z)$ can be written as

$$v(z) = \prod_i y_i(z)^{\check{\alpha}_i} n(z), \qquad n(z) \in N_+(z), \quad y_i(z) \in \mathbb{C}(z)^\times, \tag{3.7}$$

the Cartan $q$-connection $A^H(z)$ has the form

$$A^H(z) = \prod_i y_i(qz)^{\check{\alpha}_i}\ Z\ \prod_i y_i(z)^{-\check{\alpha}_i} \tag{3.8}$$

and hence we will refer to $A^H(z)$ as a *$Z$-twisted Cartan $q$-connection*. This formula shows that $A^H(z)$ is completely determined by $Z$ and the rational functions $y_i(z)$. Indeed, comparing this equation with (3.6) gives

$$g_i(z) = \zeta_i \frac{y_i(qz)}{y_i(z)}\,. \tag{3.9}$$

It is also the case that $A^H(z)$ determines the $y_i(z)$'s uniquely up to scalar. Indeed, if

$$\prod_i \widetilde{y}_i(qz)^{\check{\alpha}_i} Z \prod_i \widetilde{y}_i(z)^{-\check{\alpha}_i} = Z$$

as well, then

$$\prod_i \left(\frac{\widetilde{y}_i(qz)}{y_i(qz)}\right)^{\check{\alpha}_i} Z \prod_i \left(\frac{\widetilde{y}_i(z)}{y_i(z)}\right)^{-\check{\alpha}_i} = Z.$$

The commutativity of $H(z)$ immediately implies that the rational functions $h_i(z) = \frac{\widetilde{y}_i(z)}{y_i(z)}$ satisfy $h_i(qz) = h_i(z)$. Since $q$ is not a root of unity, we find that $h_i(z) \in \mathbb{C}^\times$.

## 4. Nondegenerate Miura-Plücker $q$-opers

Our main goal is to link Miura $q$-opers to solutions of a certain system of equations called the $QQ$-system, which is in turn related to the system of Bethe Ansatz equations. We do this in two steps: first, we introduce the notion of $Z$-twisted Miura-Plücker $(G, q)$-opers; these are Miura $q$-opers satisfying a slightly weaker condition than the $Z$-twisted Miura $q$-opers discussed in the previous section. Second, we impose two *nondegeneracy* conditions on these $Z$-twisted Miura-Plücker $(G, q)$-opers.

We start in Section 4.1 with an outline of the Plücker description of $B_+$-bundles. When $G$ has rank greater than 1, we use this to associate to a Miura $(G, q)$-oper a collection of Miura $(\mathrm{GL}(2), q)$-opers indexed by the fundamental weights of $G$. This will motivate the definition of $Z$-twisted Miura-Plücker $(G, q)$-opers in Section 4.2. We will then define two nondegeneracy conditions for these objects: the first one in Section 4.3, and the second one in Sections 4.4 (for $G = \mathrm{SL}(2)$) and 4.5 (for general $G$).

### 4.1. The associated Miura $(\mathrm{GL}(2), q)$-opers.

In this section, we associate to a Miura $(G, q)$-oper with regular singularities a collection of Miura $(\mathrm{GL}(2), q)$-opers indexed by the fundamental weights. This is done using the Plücker description of $B_+$-bundles which we learned from V. Drinfeld.

Recall that $\omega_i$ denotes the $i$th fundamental weight of $G$. Let $V_i$ be the irreducible representation of $G$ with highest weight $\omega_i$ with respect to $B_+$. It comes equipped with a line $L_i \subset V_i$ (of highest weight vectors) stable under the action of $B_+$. Likewise, there is a $B_+$-stable line $L_\lambda$ in the irreducible representations $V_\lambda$ of $G$ for every dominant integral highest weight $\lambda$. These lines satisfy the following generalized *Plücker relations*: for any two dominant integral weights $\lambda$ and $\mu$, $L_{\lambda+\mu} \subset V_{\lambda+\mu}$ is the image of $L_\lambda \otimes L_\mu \subset V_\lambda \otimes V_\mu$ under the canonical projection $V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}$. Conversely, given a collection of lines $L_\lambda \subset V_\lambda$ for all $\lambda$ satisfying the Plücker relations, there is a Borel subgroup $B \subset G$ such that $L_\lambda$ is stabilized by $B$ for all $\lambda$. A choice of $B$ is equivalent to a choice of $B_+$-torsor in $G$. Hence, we can identify the datum of a $B_+$-reduction $\mathcal{F}_{B_+}$ of a $G$-bundle $\mathcal{F}_G$ with the "linear algebra" data of line subbundles $\mathcal{L}_i$ of the associated vector bundles $\mathcal{V}_\lambda = \mathcal{F} \underset{G}{\times} V_\lambda$ satisfying the Plücker relations. If we have a connection or a $q$-connection on $\mathcal{F}_G$ that preserves $\mathcal{F}_{B_+}$ (or has another relation with $\mathcal{F}_{B_+}$, such as the oper or $q$-oper condition), we can use this formalism to express the properties of $A(z)$ in terms of these "linear algebra" data.

In our discussion here, we will not make full use of this formalism. What we need is the following simple fact. Let $\nu_{\omega_i}$ be a generator of the line $L_i \subset V_i$. It is a vector of weight $\omega_i$ with respect to our maximal torus $H \subset B_+$. The subspace of $V_i$ of weight $\omega_i - \alpha_i$ is one-dimensional and is spanned by $f_i \cdot \nu_{\omega_i}$. Therefore, the two-dimensional subspace $W_i$ of $V_i$ spanned by the weight vectors $\nu_{\omega_i}$ and $f_i \cdot \nu_{\omega_i}$ is a $B_+$-invariant subspace of $V_i$.

Now, let $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura $(G,q)$-oper with regular singularities determined by polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ (see Definition 2.9). Recall that $\mathcal{F}_{B_+}$ is a $B_+$-reduction of a $G$-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ preserved by the $(G,q)$-connection $A$. Therefore for each $i = 1,\dots,r$, the vector bundle

$$\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} V_i$$

associated to $V_i$ contains a rank two subbundle

$$\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle

$$\mathcal{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$$

associated to $L_i \subset W_i$.

Denote by $\phi_i(A)$ the $q$-connection on the vector bundle $\mathcal{V}_i$ (or equivalently, a $(\mathrm{GL}(V_i),q)$-connection) corresponding to the above Miura $q$-oper connection $A$. Since $A$ preserves $\mathcal{F}_{B_+}$ (see Definition 2.2), we see that $\phi_i(A)$ preserves the subbundles $\mathcal{L}_i$ and $\mathcal{W}_i$ of $\mathcal{V}_i$. Denote by $A_i$ the corresponding $q$-connection on the rank 2 bundle $\mathcal{W}_i$.

Let us trivialize $\mathcal{F}_{B_+}$ on a Zariski open subset of $\mathbb{P}^1$ so that $A(z)$ has the form (2.11) with respect to this trivialization (see Corollary 2.11). This trivializes the bundles $\mathcal{V}_i$, $\mathcal{W}_i$, and $\mathcal{L}_i$, so that the $q$-connection $A_i(z)$ becomes a $2\times 2$ matrix whose entries are in $\mathbb{C}(z)$.

Direct computation using formula (2.11) yields the following result.

**Lemma 4.1.** *We have*

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z)\prod_{j>i} g_j(z)^{-a_{ji}} \\ & \\ 0 & g_i^{-1}(z)\prod_{j\neq i} g_j(z)^{-a_{ji}} \end{pmatrix}, \tag{4.1}$$

*where we use the ordering of the simple roots determined by the Coxeter element $c$.*

Using the trivialization of $\mathcal{W}_i$ in which $A_i(z)$ has the form (4.1), we represent $\mathcal{W}_i$ as the direct sum of two line subbundles. The first is $\mathcal{L}_i$, generated by the basis vector $\begin{pmatrix}1\\0\end{pmatrix}$. The second, which we denote by $\widetilde{\mathcal{L}}_i$, is generated by the basis vector $\begin{pmatrix}0\\1\end{pmatrix}$. The subbundle $\mathcal{L}_i$ is $A_i$-invariant, whereas the subbundle $\widetilde{\mathcal{L}}_i$ and $A_i$ satisfy the following $(\mathrm{GL}(2),q)$-oper condition.

**Definition 4.2.** A $(\mathrm{GL}(2),q)$-*oper* on $\mathbb{P}^1$ is a triple $(\mathcal{W}, A, \widetilde{\mathcal{L}})$, where $\mathcal{W}$ is a rank 2 bundle on $\mathbb{P}^1$, $A : \mathcal{W} \longrightarrow \mathcal{W}^q$ is a meromorphic $q$-connection on $\mathcal{W}$, and $\widetilde{\mathcal{L}}$ is a line subbundle of $\mathcal{W}$ such that the induced map $\bar{A} : \widetilde{\mathcal{L}} \longrightarrow (\mathcal{W}/\widetilde{\mathcal{L}})^q$ is an isomorphism on a Zariski open dense subset of $\mathbb{P}^1$.

A Miura $(\mathrm{GL}(2),q)$-*oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{W}, A, \widetilde{\mathcal{L}}, \mathcal{L})$, where $(\mathcal{W}, A, \widetilde{\mathcal{L}})$ is a $(\mathrm{GL}(2),q)$-oper and $\mathcal{L}$ is an $A$-invariant line subbundle of $\mathcal{W}$.

Using this definition, one obtains an alternative definition of (Miura) $(\mathrm{SL}(2),q)$-opers: these are the (Miura) $(\mathrm{GL}(2),q)$-opers defined by the above triples (resp. quadruples) satisfying the additional property that there exists an isomorphism between $\mathrm{Det}\,\mathcal{W}$ and the trivial line bundle on a Zariski open dense subset of $\mathbb{P}^1$ and under this isomorphism $\det(A)$ is equal to the identity.

Our quadruple $(\mathcal{W}_i, A, \widetilde{\mathcal{L}}_i, \mathcal{L}_i)$ is clearly a Miura $(\mathrm{GL}(2), q)$-oper. It is not clear whether it is an $(\mathrm{SL}(2), q)$-oper because $\det A_i(z)$ is not necessarily equal to 1 in the above trivialization of $\mathcal{W}_i$.

We now make a further assumption that our initial Miura $(G, q)$-oper $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ with regular singularities is such that the associated Cartan connection $A^H(z)$ has the form (3.8):

$$A^H(z) = \prod_i y_i(qz)^{\check{\alpha}_i}\ Z \prod_i y_i(z)^{-\check{\alpha}_i}, \qquad y_i(z) \in \mathbb{C}(z). \tag{4.2}$$

We claim that if this condition is satisfied, then there exists another trivialization of $\mathcal{W}_i$ in which the $q$-connection $A_i$ has a constant determinant (albeit not equal to 1, in general). In other words, we can apply to $A_i(z)$ given by formula (4.1) a $q$-gauge transformation so that the resulting $q$-connection has constant determinant. This provides a particularly convenient gauge for $A_i$.

To see this, let us apply to $A_i(z)$ the $q$-gauge transformation by the diagonal matrix

$$u(z) = \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j \neq i} y_j(z)^{a_{ji}} \end{pmatrix}.$$

This gives

$$\widetilde{A}_i(z) = u(qz) A_i(z) u^{-1}(z) = \begin{pmatrix} \zeta_i \frac{y_i(qz)}{y_i(z)} & \rho_i(z) \\ 0 & \prod_{j \neq i} \zeta_j^{-a_{ji}} \zeta_i^{-1} \frac{y_i(z)}{y_i(qz)} \end{pmatrix}, \tag{4.3}$$

where

$$\rho_i(z) = \Lambda_i(z) \prod_{j>i} (\zeta_j y_j(qz))^{-a_{ji}} \prod_{j<i} y_j(z)^{-a_{ji}}. \tag{4.4}$$

Since $a_{ij} \leq 0$ for $i \neq j$, $\rho_i(z)$ is a polynomial if all $y_j(z), j = 1, \ldots, r$, are polynomials.

Let $G_i \cong \mathrm{SL}(2)$ be the subgroup of $G$ corresponding to the $\mathfrak{sl}(2)$-triple spanned by $\{e_i, f_i, \check{\alpha}_i\}$. Note that the group $G_i$ preserves $W_i$. Consider the Miura $(G_i, q)$-oper $(\mathcal{W}_i, \mathcal{A}_i, \widetilde{\mathcal{L}}_i, \mathcal{L}_i)$ with $\widetilde{\mathcal{L}}_i = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $\mathcal{L}_i = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, and

$$\mathcal{A}_i(z) = g_i^{\check{\alpha}_i}(z)\, e^{\frac{\rho_i(z)}{g_i(z)} e_i} = \begin{pmatrix} \zeta_i \frac{y_i(qz)}{y_i(z)} & \rho_i(z) \\ 0 & \zeta_i^{-1} \frac{y_i(z)}{y_i(qz)} \end{pmatrix}. \tag{4.5}$$

We can now express our $q$-connection $\widetilde{A}_i(z)$ given by formula (4.3) as follows:

$$\widetilde{A}_i(z) = \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j \neq i} \zeta_j^{-a_{ji}} \end{pmatrix} \mathcal{A}_i(z) \tag{4.6}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \prod_{j \neq i} \zeta_j^{-a_{ji}} \end{pmatrix} g_i^{\check{\alpha}_i}(z)\, e^{\frac{\rho_i(z)}{g_i(z)} e_i}. \tag{4.7}$$

This shows that $\det(\widetilde{A}_i(z))$ is constant; namely, it is equal to $\prod_{j \neq i} \zeta_j^{-a_{ji}}$.

Thus, under the assumption (4.2), our Miura $(G, q)$-oper $A(z)$ gives rise to a collection of meromorphic Miura $(\mathrm{SL}(2), q)$-opers $\mathcal{A}_i(z)$ for $i = 1, \ldots, r$. It should be noted that it has regular singularities in the sense of Definition 2.8 if and only if $\rho_i(z)$ is a polynomial.

For example, this holds for all $i$ if all $y_j(z), j = 1, \ldots, r$, are polynomials, an observation we will use below.

4.2. **$Z$-twisted Miura-Plücker $q$-opers.** Recall Definition 3.2 of $Z$-twisted Miura $(G,q)$-opers, where $Z$ is a regular semisimple element of the maximal torus $H$. These are Miura $(G,q)$-opers whose underlying $q$-connection can be written in the form (3.3):

$$A(z) = v(qz)Zv(z)^{-1}, \qquad v(z) \in B_+(z). \tag{4.8}$$

We will now relax this condition using the Miura $(\mathrm{GL}(2),q)$-opers $A_i(z)$ (or equivalently, the Miura $(\mathrm{SL}(2),q)$-opers $\mathcal{A}_i(z)$) associated to $A(z)$. Instead, we will require that there exists an element $v(z)$ from $B_+(z)$ such that $A_i(z)$ satisfies formula (4.8) with $v(z)$ replaced by $v_i(z) = v(z)|_{W_i} \in \mathrm{GL}(2)$ and $Z$ replaced by $Z|_{W_i}$ for all $i = 1, \ldots, r$.

**Definition 4.3.** A *$Z$-twisted Miura-Plücker* $(G,q)$*-oper* is a meromorphic Miura $(G,q)$-oper on $\mathbb{P}^1$ with underlying $q$-connection $A(z)$ satisfying the following condition: there exists $v(z) \in B_+(z)$ such that for all $i = 1, \ldots, r$, the Miura $(\mathrm{GL}(2),q)$-opers $A_i(z)$ associated to $A(z)$ by formula (4.1) can be written in the form

$$A_i(z) = v(zq)Zv(z)^{-1}|_{W_i} = v_i(zq)Z_i v_i(z)^{-1}, \tag{4.9}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$.

The difference between $Z$-twisted Miura $(G,q)$-opers and $Z$-twisted Miura-Plücker $(G,q)$-opers may be explained as follows: the former is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ as in Definition 2.2 such that there exists a trivialization of $\mathcal{F}_{B_+}$ with respect to which the $q$-connection $A$ is a constant element of $G(z)$ equal to our element $Z \in H$. For the latter, we only ask that there exists a trivialization of $\mathcal{F}_{B_+}$ with respect to which the $q$-connections $A_i(z)$ are constant elements of $\mathrm{GL}(2)(z)$ equal to $Z_i$ for all $i = 1, \ldots, r$.

Thus, every $Z$-twisted Miura $(G,q)$-oper is automatically a $Z$-twisted Miura-Plücker $(G,q)$-oper, but the converse is not necessarily true if $G \neq SL(2)$.

Note, however, that it follows from the above definition that the $(H,q)$-connection $A^H(z)$ associated to a $Z$-twisted Miura-Plücker $(G,q)$-oper can be written in the same form (4.2) as the $(H,q)$-connection associated to a $Z$-twisted Miura $(G,q)$-oper.

4.3. **$H$-nondegeneracy condition.** We now introduce two nondegeneracy conditions for $Z$-twisted Miura-Plücker $q$-opers. The first of them, called the $H$-nondegeneracy condition, is applicable to arbitrary Miura $q$-opers with regular singularities. Recall from Corollary 2.11 that the underlying $q$-connection can be represented in the form (2.11).

In what follows, we will say that $v, w \in \mathbb{C}^\times$ are *$q$-distinct* if $q^{\mathbb{Z}}v \cap q^{\mathbb{Z}}w = \varnothing$.

**Definition 4.4.** A Miura $(G,q)$-oper $A(z)$ of the form (2.11) is called *$H$-nondegenerate* if the corresponding $(H,q)$-connection $A^H(z)$ can be written in the form (3.8), where for all $i,j,k$ with $i \neq j$ and $a_{ik} \neq 0, a_{jk} \neq 0$, the zeros and poles of $y_i(z)$ and $y_j(z)$ are $q$-distinct from each other and from the zeros of $\Lambda_k(z)$.

4.4. **Nondegenerate $Z$-twisted Miura** $(\mathrm{SL}(2),q)$**-opers.** Next, we define the second nondegeneracy condition. This condition applies to $Z$-twisted Miura-Plücker $(G,q)$-opers. In this subsection, we give the definition for $G = \mathrm{SL}(2)$. (Note that $Z$-twisted Miura-Plücker $(\mathrm{SL}(2),q)$-opers are the same as $Z$-twisted Miura $(\mathrm{SL}(2),q)$-opers.) In the next subsection, we will give it in the case of an arbitrary simply connected simple complex Lie group $G$.

Consider a Miura $(\mathrm{SL}(2),q)$-oper given by formula (2.11), which reads in this case:

$$A(z) = g(z)^{\check{\alpha}} \ \exp\left(\frac{\Lambda(z)}{g(z)}e\right) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}.$$

According to formula (4.2), the corresponding Cartan $q$-connection $A^H(z)$ is equal to

$$A^H(z) = g(z)^{\check{\alpha}} = y(qz)^{\check{\alpha}} Z y(z)^{-\check{\alpha}} = \begin{pmatrix} y(qz)y(z)^{-1} & 0 \\ 0 & y(qz)^{-1}y(z) \end{pmatrix},$$

where $y(z)$ is a rational function. Let us assume that $A(z)$ is $H$-nondegenerate (see Definition 4.4). This means that the zeros of $\Lambda(z)$ are $q$-distinct from the zeros and poles of $y(z)$.

If we apply a $q$-gauge transformation by an element of $h(z)^{\check{\alpha}} \in H[z]$ to $A(z)$, we obtain a new $q$-oper connection

$$\widetilde{A}(z) = \widetilde{g}(z)^{\check{\alpha}} \ \exp\left(\frac{\widetilde{\Lambda}(z)}{\widetilde{g}(z)}e\right), \tag{4.10}$$

where

$$\widetilde{g}(z) = g(z)h(zq)h(z)^{-1}, \qquad \widetilde{\Lambda}(z) = \Lambda(z)h(zq)h(z). \tag{4.11}$$

It also has regular singularities, but for a different polynomial $\widetilde{\Lambda}(z)$, and $\widetilde{A}(z)$ may no longer be $H$-nondegenerate. However, it turns out there is an essentially unique gauge transformation from $H[z]$ for which the resulting $\widetilde{A}(z)$ is $H$-nondegenerate and $\widetilde{y}(z)$ is a polynomial. This choice allows us to fix the polynomial $\Lambda(z)$ determining the regular singularities of our $(\mathrm{SL}(2),q)$-oper.

**Lemma 4.5.** (1) *There is an $H$-nondegenerate $(\mathrm{SL}(2),q)$-oper $\widetilde{A}(z)$ in the $H[z]$-gauge class of $A(z)$, say with $\widetilde{A}^H(z) = \widetilde{g}(z)^{\check{\alpha}}$, for which the rational function $\widetilde{y}(z)$ is a polynomial. This oper is unique up to a scalar $a \in \mathbb{C}^\times$ that leaves $\widetilde{g}(z)$ unchanged, but multiplies $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ by $a$ and $a^2$ respectively.*

(2) *This $(\mathrm{SL}(2),q)$-oper $\widetilde{A}(z)$ may also be characterized by the property that $\widetilde{\Lambda}(z)$ has maximal degree subject to the constraint that it is $H$-nondegenerate.*

*Proof.* Write $y(z) = \frac{P_1(z)}{P_2(z)}$, where $P_1, P_2$ are relatively prime polynomials. For a nonzero polynomial $h(z) \in \mathbb{C}(z)^\times$, the gauge transformation of $A(z)$ by $h(z)^{\check{\alpha}}$ is given by formulas (4.10) and (4.11). Thus, in order for $\widetilde{y}(z) = h(z)\frac{P_1(z)}{P_2(z)}$ to be a polynomial, we need $h(z)$ to be divisible by $P_2(z)$. If, however, $\deg(h/P_2) > 0$, then $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ would have a zero in common, so $\widetilde{A}(z)$ would not by $H$-nondegenerate. Hence, we must have $h(z) = aP_2(z)$ for some $a \in \mathbb{C}^\times$. Thus, $h(z)$ is uniquely defined by multiplication by $a$, which leave $\widetilde{g}(z)$ unchanged, but multiplies $\widetilde{y}(z)$ and $\widetilde{\Lambda}(z)$ by $a$ and $a^2$ respectively.

For the second statement, note that if $h(z)$ is a polynomial for which the zeros of $h(z)h(qz)\Lambda(z)$ are $q$-distinct from the zeros and poles of $h(z)\frac{P_1(z)}{P_2(z)}$, we must have $h|P_2$. If $h(z)$ is not an associate of $P_2(z)$, we have $\deg(h) < \deg(P_2)$, so $\deg(h(z)h(qz)\Lambda(z)) < \deg(\widetilde{\Lambda})$. □

This motivates the following definition.

**Definition 4.6.** A $Z$-twisted Miura $(\mathrm{SL}(2),q)$-oper is called *nondegenerate* if it is $H$-nondegenerate and the rational function $y(z)$ appearing in formula (3.8) is a polynomial.

4.5. **Nondegenerate Miura-Plücker $(G, q)$-opers.** We now turn to the general case. Recall Definition 4.3 of $Z$-twisted Miura-Plücker $(G, q)$-opers. Further recall that to every Miura $(G, q)$-oper $A(z)$, we have associated a Miura $(\mathrm{SL}(2), q)$-oper $\mathcal{A}_i(z), i = 1, \ldots, r$, given by formula (4.5). (It can be obtained from the Miura $(\mathrm{GL}(2), q)$-oper $A_i(z) = A(z)|_{W_i}$ using formulas (4.3) and (4.6)). It follows from the definition that if $A(z)$ is $Z$-twisted with $Z$ given by (3.1), then $\mathcal{A}_i(z)$ is $\check{\alpha}_i(\zeta_i)$-twisted.

**Definition 4.7.** Suppose that the rank of $G$ is greater than 1. A $Z$-twisted Miura-Plücker $(G, q)$-oper $A(z)$ is called *nondegenerate* if it is $H$-nondegenerate and each $\check{\alpha}_i(\zeta_i)$-twisted Miura $(\mathrm{SL}(2), q)$-oper $\mathcal{A}_i(z)$ is nondegenerate.

It turns out that this simply means that in addition to $A(z)$ being $H$-nondegenerate, each $y_i(z)$ from formula (3.8) is a polynomial satisfying a mild condition on its roots.

**Proposition 4.8.** *Suppose that the rank of $G$ is greater than 1, and let $A(z)$ be a $Z$-twisted Miura-Plücker $(G, q)$-oper. The following statements are equivalent:*

(1) *$A(z)$ is nondegenerate.*
(2) *$A(z)$ is $H$-nondegenerate, and each $\mathcal{A}_i(z)$ has regular singularities, i.e. $\rho_i(z)$ given by formula (4.4) is in $\mathbb{C}[z]$.*
(3) *Each $y_i(z)$ from formula (3.8) may be chosen to be a monic polynomial, and for all $i, j, k$ with $i \neq j$ and $a_{ik} \neq 0, a_{jk} \neq 0$, the zeros of $y_i(z)$ and $y_j(z)$ are $q$-distinct from each other and from the zeros of $\Lambda_k(z)$.*

*Proof.* To prove that (2) implies (3), we need only show that if each $\rho_i(z)$ given by formula (4.4) is in $\mathbb{C}[z]$, then the $y_i(z)$'s are polynomials. Suppose $y_i(z)$ is not a polynomial, and choose $j \neq i$ such that $a_{ij} \neq 0$. Then $-a_{ij} > 0$ and so the denominator of $y_i(z)$ or $y_i(qz)$ appears in the denominator of $\rho_j(z)$. Moreover, since the poles of $y_i(z)$ are $q$-distinct from the zeros of $\Lambda_j(z)$ and the other $y_k(z)$'s, the poles of $y_i(z)$ or $y_i(qz)$ would give rise to poles of $\rho_j(z)$. But then $\mathcal{A}_j(z)$ would not have regular singularities.

Next, assume (3). Then $A(z)$ is $H$-nondegenerate by Definition 4.4. Since all the $y_i(z)$'s are polynomials, the same is true for the $\rho_i(z)$'s. (Here, we are using the fact that the off-diagonal elements of the Cartan matrix, $a_{ij}$ with $i \neq j$, are less than or equal to 0.) Since $\rho_i(z)$ is a product of polynomials whose roots are $q$-distinct from the roots of $y_i(z)$, we see that the Cartan $q$-connection associated to $\mathcal{A}_i(z)$ is nondegenerate.

Finally, (2) is a trivial consequence of (1). ☐

If we apply a $q$-gauge transformation by an element $h(z) \in H[z]$ to $A(z)$, we get a new $Z$-twisted Miura-Plücker $(G, q)$-oper. However, the following proposition shows that it is only nondegenerate if $h(z) \in H$. As a consequence, the $\Lambda_k$'s of a nondegenerate $q$-oper are determined up to scalar multiples. If we further impose the condition that each $y_i(z)$ is a *monic* polynomial, then $h(z) = 1$, and this fixes the $\Lambda_k$'s.

**Proposition 4.9.** *If $A(z)$ is a nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-oper and $h(z) \in H[z]$, then $h(qz)A(z)h(z)^{-1}$ is nondegenerate if and only if $h(z)$ is a constant element of $H$.*

*Proof.* Write $h(z) = \prod h_i(z)^{\check{\alpha}_i}$. Gauge transformation of $A(z)$ by $h(z)$ induces a gauge transformation of $A_i(z)$ by $h_i(z)$. Since $A_i(z)$ is nondegenerate, Lemma 4.5 implies that the new Miura $(\mathrm{SL}(2), q)$-oper is nondegenerate if and only $h_i \in \mathbb{C}^\times$. ☐

4.6. **Dependence on the Coxeter element.** We end this section with a preliminary result on the dependence of our results on the specific Coxeter element fixed in the definition of $q$-opers. We will see later in Section 7.4 that the $QQ$-systems obtained from different choices of Coxeter element are equivalent. Here, we show that if two Coxeter elements $c$ and $c'$ are related by a cyclic permutation of their simple reflection factors, then the corresponding spaces of $(G,q)$-opers with regular singularities are isomorphic via a map defined in terms of $B_+(z)$-gauge transformations. Moreover, this map preserves nondegeneracy.

**Proposition 4.10.** *Let $c$ and $c'$ be two Coxeter elements that differ by a cyclic permutation of their simple reflection factors. Then, there is an isomorphism between the spaces of $Z$-twisted Miura $(G,q)$-opers with regular singularities defined in terms of $c$ and $c'$ of the form $A(z)\mapsto f_A(qz)A(z)f_A(z)^{-1}$, where $f_A\in B_+(z)$. This isomorphism takes nondegenerate opers to nondegenerate opers.*

*Proof.* Without loss of generality, we may assume that $c=w_{i_1}\dots w_{i_r}$ and $c'=w_{i_2}\dots w_{i_r}w_{i_1}$. Given

$$A(z)=\prod_{j=1}^{r} g_{i_j}(z)^{\check{\alpha}_{i_j}}\; e^{\frac{\Lambda_{i_j}(z)}{g_{i_j}(z)}e_{i_j}},$$

set

$$f_A(qz)=\left(g_{i_1}(z)^{\check{\alpha}_{i_1}}e^{\frac{\Lambda_{i_1}(z)}{g_{i_1}(z)}e_{i_1}}\right)^{-1}.$$

The effect of gauge transformation by $f_A(z)$ is to move the $q^{-1}$-shift of the $i_1$ component of $A$ to the end of the product, thereby giving the order corresponding to $c'$. The new $y_i$'s and $\Lambda_i$'s are the same except for the $q^{-1}$-shift of $y_{i_1}$ and $\Lambda_{i_1}$, so it is obvious that the new $q$-oper also has regular singularities and is nondegenerate if the original $q$-oper was. It is also clear that this map is an isomorphism. □

## 5. $(SL(2),q)$-opers and the Bethe Ansatz equations

Our goal is to establish a bijection between the set of nondegenerate $Z$-twisted Miura-Plücker $(G,q)$-opers and the set of nondegenerate solutions of a system of Bethe Ansatz equations. In this section, we show this for $G=\mathrm{SL}(2)$, which corresponds to the XXZ model. This was already shown in [KSZ], in which a slightly different definition of $(\mathrm{SL}(2),q)$-opers was used. Below, we explain the connection to the formalism used in [KSZ].

5.1. **From non-degenerate $(\mathrm{SL}(2),q)$-opers to the $QQ$-system.** Suppose we have a $Z$-twisted nondegenerate Miura (equivalently, a Miura-Plücker) $(\mathrm{SL}(2),q)$-oper. As explained in Section 4.4, the underlying $q$-connection may be written in the form

$$A(z)=\begin{pmatrix} g(z) & \Lambda(z)\\ 0 & g(z)^{-1}\end{pmatrix},$$

and furthermore, there exists $v(z)\in B_+(z)$ such that

$$A(z)=v(zq)Zv(z)^{-1},\qquad Z=\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}. \tag{5.1}$$

Write

$$v(z)=\begin{pmatrix} y(z) & 0\\ 0 & y(z)^{-1}\end{pmatrix}\begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)}\\ 0 & 1\end{pmatrix}=\begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)}\\ 0 & y(z)^{-1}\end{pmatrix}, \tag{5.2}$$

where $Q_+(z)$ and $Q_-(z)$ are relatively prime polynomials such that $Q_+(z)$ is a monic polynomial. Formula (5.1) then yields

$$g(z) = \zeta_i y(zq) y(z)^{-1}$$

and

$$\Lambda(z) = y(z)y(zq)\left(\zeta \frac{Q_-(z)}{Q_+(z)} - \zeta^{-1}\frac{Q_-(zq)}{Q_+(zq)}\right). \tag{5.3}$$

Nondegeneracy (see Definition 4.6) means that $\Lambda(z)$ and $y(z)$ are polynomials whose roots are $q$-distinct from each other. This can only be satisfied if $y(z)$ equals a scalar multiple of $Q_+(z)$. Since we have the freedom to rescale $y(z)$, without loss of generality we can and will assume that $y(z) = Q_+(z)$. Equation (5.3) then becomes

$$\zeta Q_-(z)Q_+(zq) - \zeta^{-1}Q_-(zq)Q_+(z) = \Lambda(z). \tag{5.4}$$

We call equation (5.4) the *$QQ$-system* associated to SL(2). (See the last paragraph of Section 5.2 and Section 6.2 for a discussion of the origins of this system in the XXZ model.) Here, $\Lambda(z)$ is fixed: it is the polynomial used in the definition of a Miura $(\mathrm{SL}(2), q)$-opers which contains the information about their regular singularities. Thus, the $QQ$-system is an equation on two polynomials $Q_+(z), Q_-(z)$.

Let us call a solution $\{Q_+(z), Q_-(z)\}$ of (5.4) *nondegenerate* if $Q_+(z)$ is a monic polynomial whose roots are $q$-distinct from the roots of the polynomial $\Lambda(z)$. No conditions are imposed on $Q_-(z)$, but note that the nondegeneracy condition and formula (5.4) imply that $Q_+(z)$ and $Q_-(z)$ are *relatively prime*. The above discussion is summarized in the following statement.

**Theorem 5.1.** *There is a one-to-one correspondence between the set of nondegenerate $Z$-twisted $(\mathrm{SL}(2), q)$-opers with regular singularity determined by a polynomial $\Lambda(z)$ and the set of nondegenerate solutions of the $QQ$-system* (5.4)*.*

5.2. **From the $QQ$-system to the Bethe Ansatz equations.** Under our assumption that $Q_+(z)$ is a monic polynomial, we can write

$$Q_+(z) = \prod_{k=1}^{m}(z - w_k).$$

Evaluating (5.4) at $q^{-1}z$, we get

$$\Lambda(q^{-1}z) = \zeta Q_-(q^{-1}z)Q_+(z) - \zeta^{-1}Q_-(z)Q_+(q^{-1}z).$$

If we divide (5.4) by this equation and evaluate at the roots $w_k$ of $Q_+(z)$, we obtain the following equations:

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1, \dots, m. \tag{5.5}$$

These equations are equivalent to the Bethe Ansatz equations of the XXZ model, i.e., the quantum spin chain associated to $U_q\widehat{\mathfrak{sl}}_2$. To express them in a more familiar form, suppose that $\Lambda(z)$ is a monic polynomial all of whose roots are non-zero and simple. Recalling that we do not require the roots of $\Lambda(z)$ to be mutually $q$-distinct, we write $\Lambda(z)$ explicitly as

$$\Lambda(z) = \prod_{p=1}^{L}\prod_{j_p=0}^{r_p-1}(z - q^{-j_p}z_p), \tag{5.6}$$

where the $z_p$'s are mutually $q$-distinct and non-zero. Setting $r = \sum_{p=1}^{L} t_p$, the equations (5.5) become

$$q^r \prod_{p=1}^{L} \frac{w_k - q^{1-r_p} z_p}{w_k - q z_p} = -\zeta^2 q^m \prod_{j=1}^{m} \frac{q w_k - w_j}{w_k - q w_j}, \qquad k = 1, \dots, m. \tag{5.7}$$

This is a more familiar form of the Bethe Ansatz equations in the XXZ model (see e.g. [FH1], Section 5.6).

Let us call a solution $Q_+(z)$ of the system of Bethe Ansatz equations (5.5) *nondegenerate* if $Q_+(z)$ is a monic polynomial whose roots are $q$-distinct from the roots of $\Lambda(z)$. It is clear that if $\{Q_+(z), Q_-(z)\}$ is a nondegenerate solution of (5.4), then $Q_+(z)$ is a nondegenerate solution of (5.5), and vice versa. The above calculation, combined with Theorem 5.1, proves the following result.

**Theorem 5.2.** *There is a one-to-one correspondence between the set of nondegenerate $Z$-twisted* $(\mathrm{SL}(2), q)$*-opers with regular singularity determined by a polynomial $\Lambda(z)$ and the set of nondegenerate solutions of the Bethe Ansatz equations* (5.5)*.*

It is known that the Bethe Ansatz equations (5.5) parametrize the spectra of the quantum transfer-matrices in the XXZ model corresponding to $U_{q'}\widehat{\mathfrak{sl}}_2$, where $q' = q^{-2}$, with the space of states being the tensor product of finite-dimensional representations of $U_{q'}\widehat{\mathfrak{sl}}_2$ (see e.g. [FH1]). The polynomial $\Lambda(z)$ is the product of the Drinfeld polynomials of these representations, up to multiplicative shifts by powers of $q$. Furthermore, we expect that the $QQ$-system (5.4) can be derived from the $Q\widetilde{Q}$-relation in the Grothendieck ring of the category $\mathcal{O}$ of $U_{q'}\widehat{\mathfrak{sl}}_2$ proved in [FH2].

5.3. **An approach using the $q$-Wronskian.** In [KSZ], the equations (5.4) and (5.5) were derived in a slightly different way, and analogous results were also obtained for $G = \mathrm{SL}(n)$. We now make an explicit connection between this approach and the approach of the preceding section.

Recall Definition 4.2 of $(\mathrm{GL}(2), q)$-opers. Adding the condition that the underlying rank two vector bundle $\mathcal{W}$ can be identified with the trivial line bundle so that $\det(A) = 1$, we obtain the definition of Miura $(\mathrm{SL}(2), q)$-opers. The oper condition is now expressed as the existence of a line subbundle $\widetilde{\mathcal{L}} \subset \mathcal{W}$ for which $\bar{A} : \widetilde{\mathcal{L}} \longrightarrow \mathcal{W}/\widetilde{\mathcal{L}}$ is an isomorphism on a open dense subset of $\mathbb{P}^1$. Choose any trivialization of $\mathcal{W}$ on an open dense subset $U$, and let $s(z)$ be a section of $\mathcal{W}$ on this subset that generates the line subbundle $\widetilde{\mathcal{L}}$. The $q$-connection $A(z)$ then satisfies the condition

$$s(qz) \wedge A(z)s(z) \neq 0$$

on a Zariski open dense subset $V$ of $U$. This is the definition of a general meromorphic $(\mathrm{SL}(2), q)$-oper.

From this perspective, $(\mathrm{SL}(2), q)$-opers with regular singularities are defined in [KSZ] as follows.

**Definition 5.3.** An $(\mathrm{SL}(2), q)$*-oper with regular singularities determined by* $\Lambda(z)$ is a meromorphic $(\mathrm{SL}(2), q)$-oper $(\mathcal{E}, A, \widetilde{\mathcal{L}})$ such that $s(qz) \wedge A(z)s(z) = \Lambda(z)$.

This definition is equivalent to Definition 2.8.

Consider a diagonal matrix $Z = \mathrm{diag}(\zeta, \zeta^{-1})$ with $\zeta \neq \pm 1$. Recall that an $(\mathrm{SL}(2), q)$-oper $(\mathcal{E}, A, \widetilde{\mathcal{L}})$ is a *$Z$-twisted $q$-oper* if $A$ is gauge equivalent to $Z$. (We remark that in [KSZ], a

$Z$-twisted $q$-oper was actually defined to be one that is gauge equivalent to $Z^{-1}$. This does not matter at the level of $q$-opers, but does matter if we consider the corresponding Miura $q$-opers.)

Now, suppose that $(\mathcal{E}, A, \mathcal{L})$ is a $Z$-twisted $(\mathrm{SL}(2), q)$-oper with regular singularities determined by a monic polynomial $\Lambda(z)$. (Note that we can assume that $\Lambda(z)$ is monic after multiplying the section $s$ by a nonzero constant.) Choose a trivialization of $\mathcal{E}$ with respect to which the $q$-connection matrix is $Z$. Since $\widetilde{\mathcal{L}}$ can be trivialized on $\mathbb{P}^1 \setminus \infty$, it is generated by a section

$$s(z) = \begin{pmatrix} Q_-(z) \\ Q_+(z) \end{pmatrix}, \tag{5.8}$$

where $Q_+(z)$ and $Q_-(z)$ are relatively prime polynomials and $Q_+(z)$ is monic. Furthermore, the polynomials $Q_+(z), Q_-(z)$ satisfying these conditions are uniquely determined by $\widetilde{\mathcal{L}}$.

Regular singularity of the $q$-oper then becomes an explicit equation for the *$q$-Wronskian* of $Q_-(z)$ and $Q_+(z)$:

$$\zeta Q_-(z) Q_+(qz) - \zeta^{-1} Q_-(qz) Q_+(z) = \Lambda(z). \tag{5.9}$$

This is just the $QQ$-system (5.4).

Note that in [KSZ] it was assumed that $\Lambda(z)$ has the form (5.6), i.e. that the polynomial $\Lambda(z)$ only has simple roots, and 0 is not a root. However, the same derivation of (5.9) works for any monic polynomial $\Lambda(z)$.

Next, we explain the link between the section $s(z)$ and the $q$-oper (5.1). Recall that a Miura $(\mathrm{SL}(2), q)$-oper in the sense of Definition 4.2 is a quadruple $(\mathcal{W}, A, \widetilde{\mathcal{L}}, \mathcal{L})$, where $\widetilde{\mathcal{L}}$ is a line subbundle satisfying the $q$-oper condition with respect to $A$ and $\mathcal{L}$ is an $A$-invariant line subbundle.

In the particular trivialization of $\mathcal{W}$ that we are now considering, the $q$-connection $A$ is equal to $Z$, $\widetilde{\mathcal{L}}$ is generated by the section (5.8), and $\mathcal{L}$ is generated by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. To bring it to the form (5.1), we need to change the trivialization of $\mathcal{W}$ in such a way that $\mathcal{L}$ is still generated by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $\widetilde{\mathcal{L}}$ is generated by$\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. With respect to this new trivialization, the oper $q$-connection becomes equal to the $q$-gauge transformation of $Z$ by the corresponding element $U(z) \in G(z)$. The above conditions means that $U(z)$ should preserve $\mathcal{L}$, i.e. it should be in $B_+(z)$ and should satisfy

$$U(z)s(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

There is a unique such $U(z)$, namely,

$$U(z) = \begin{pmatrix} Q_+(z) & -Q_-(z) \\ 0 & Q_+^{-1}(z) \end{pmatrix}. \tag{5.10}$$

Applying the $q$-gauge transformation by $U(z)$ to $Z$, we obtain a formula for the oper $q$-connection $A(z)$ in the new trivialization of $\mathcal{W}$:

$$A(z) = \begin{pmatrix} Q_+(qz) & -Q_-(qz) \\ 0 & Q_+^{-1}(qz) \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} Q_+(z)^{-1} & Q_-(z) \\ 0 & Q_+(z) \end{pmatrix} \tag{5.11}$$

$$= \begin{pmatrix} \zeta Q_+(qz) Q_+^{-1}(z) & \Lambda(z) \\ 0 & \zeta^{-1} Q_+^{-1}(qz) Q_+(z) \end{pmatrix}, \tag{5.12}$$

where $\Lambda(z)$ is the $q$-Wronskian (5.9).

Thus, we have arrived at a nondegenerate $Z$-twisted Miura $(\mathrm{SL}(2), q)$-oper in the sense of Section 4: $A(z) = g^{\check{\alpha}}(z) e^{\frac{\Lambda(z)}{g(z)} e}$, where $g(z) = \zeta Q_+(zq) Q_+(z)^{-1}$.

## 6. Miura-Plücker $q$-opers, $QQ$-system, and Bethe Ansatz equations

In this section, we generalize the results of the previous section to an arbitrary simply connected simple complex Lie group $G$. We establish a one-to-one correspondence between the set of nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-opers and the set of nondegenerate solutions of a system of Bethe Ansatz equations associated to $G$. A key element of the construction is an intermediate object between these two sets: the set of nondegenerate solutions of the so-called $QQ$-system.

6.1. **Miura $(G, q)$-opers and the $QQ$-system.** First, we construct a one-to-one correspondence between the set of nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-opers and the set of nondegenerate solutions of the $QQ$-system.

Recall that we have chosen a set of non-zero polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$, which we will assume from now on to be monic, and a set of non-zero complex numbers $\{\zeta_i\}_{i=1,\ldots,r}$ that correspond to a regular element $Z$ of the maximal torus $H \subset G$ by formula (3.1). In this section, these data are assumed to be fixed. (In the next section, we will also consider elements $w(Z)$ of the orbit of $Z$ under the action of the Weyl group $W_G$ of $G$ and the corresponding $\zeta_i$'s.)

From now on, we will assume that our element $Z = \prod_i \zeta_i^{\check{\alpha}_i} \in H$ satisfies the following property:

$$\prod_{i=1}^{r} \zeta_i^{a_{ij}} \notin q^{\mathbb{Z}}, \qquad \forall j = 1, \ldots, r\,. \tag{6.1}$$

Since $\prod_{i=1}^{r} \zeta_i^{a_{ij}} \neq 1$ is a special case of (6.1), this implies that $Z$ is *regular semisimple.*

Introduce the following system of equations:

$$\widetilde{\xi}_i Q^i_-(z) Q^i_+(qz) - \xi_i Q^i_-(qz) Q^i_+(z) = \Lambda_i(z) \prod_{j>i} \Big[ Q^j_+(qz) \Big]^{-a_{ji}} \prod_{j<i} \Big[ Q^j_+(z) \Big]^{-a_{ji}}, \qquad i = 1, \ldots, r, \tag{6.2}$$

where

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}} \tag{6.3}$$

and we use the ordering of simple roots from the definition of $(G, q)$-opers.

We call this the *$QQ$-system* associated to $G$ and a collection of polynomials $\Lambda_i(z)$, $i = 1, \ldots, r$.

A polynomial solution $\{Q^i_+(z), Q^i_-(z)\}_{i=1,\ldots,r}$ of (6.2) is called *nondegenerate* if it has the following properties: condition (6.1) holds for the $\zeta_i$'s; for all $i, j, k$ with $i \neq j$ and $a_{ik}, a_{jk} \neq 0$, the zeros of $Q^j_+(z)$ and $Q^j_-(z)$ are $q$-distinct from each other and from the zeros of $\Lambda_k(z)$; and the polynomials $Q^i_+(z)$ are monic.

Recall Definition 4.3 of nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-opers.

**Theorem 6.1.** *There is a one-to-one correspondence between the set of nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-opers and the set of nondegenerate polynomial solutions of the $QQ$-system (6.2).*

*Proof.* Let $A(z)$ be a nondegenerate $Z$-twisted Miura-Plücker $(G,q)$-oper. According to Corollary 2.11, it can be written in the form (2.11):

$$A(z)=\prod_i g_i(z)^{\check{\alpha}_i}\, e^{\frac{\Lambda_i(z)}{g_i(z)}e_i}, \qquad g_i(z)\in\mathbb{C}(z)^\times, \tag{6.4}$$

and there exists $v(z)\in B_+(z)$ such that for all $i=1,\dots,r$, the Miura $(\mathrm{GL}(2),q)$-opers $A_i(z)$ associated to $A(z)$ by formula (4.1) can be written in the form (4.9):

$$A_i(z)=v_i(zq)Z_iv_i(z)^{-1}, \qquad i=1,\dots,r, \tag{6.5}$$

where $v_i(z)=v(z)|_{W_i}$ and $Z_i=Z|_{W_i}$.

The element $v(z)$ can be expressed in the form

$$v(z)=\prod_{i=1}^r y_i(z)^{\check{\alpha}_i}\prod_{i=1}^r e^{-\frac{Q^i_-(z)}{Q^i_+(z)}e_i}\dots, \tag{6.6}$$

where the dots stand for the exponentials of higher commutator terms in $\mathfrak{n}_+=\mathrm{Lie}\,N_+$ (these terms will not matter in the computations below) and $Q^i_+(z),Q^i_-(z)$ are relatively prime polynomials with $Q^i_+(z)$ monic for each $i=1,\dots,r$. Formula (6.5) shows that, without loss of generality, we can and will assume that each $y_i(z)$ is a monic polynomial.

Acting on the two-dimensional subspace $W_i$ introduced in Section 4.1, $v(z)$ has the form

$$v(z)|_{W^i}=\begin{pmatrix} y_i(z) & 0\\ 0 & y_i^{-1}(z)\prod_{j\neq i}y_j^{-a_{ji}}(z)\end{pmatrix}\begin{pmatrix}1 & -\frac{Q^i_-(z)}{Q^i_+(z)}\\ 0 & 1\end{pmatrix} \tag{6.7}$$

while $Z$ has the form

$$Z|_{W_i}=\begin{pmatrix}\zeta_i & 0\\ 0 & \zeta_i^{-1}\prod_{j\neq i}\zeta_j^{-a_{ji}}\end{pmatrix}. \tag{6.8}$$

We now apply (4.1) and (6.5)to relate the $y_i(z)$'s and $Q^i_\pm(z)$'s. First, comparing the diagonal entries on both sides of (6.5) gives formula (3.9):

$$g_i(z)=\zeta_i\frac{y_i(qz)}{y_i(z)}. \tag{6.9}$$

Second, by comparing the upper triangular entries on both sides of (6.5), we obtain

$$\Lambda_i(z)\prod_{j>i}g_j(z)^{-a_{ji}} = y_i(z)y_i(qz)\prod_{j\neq i}y_j(z)^{a_{ji}}\left[\zeta_i\frac{Q^i_-(z)}{Q^i_+(z)}-\zeta_i^{-1}\prod_{j\neq i}\zeta_j^{-a_{ji}}\frac{Q^i_-(qz)}{Q^i_+(qz)}\right]. \tag{6.10}$$

Since $\Lambda_i(z)$ and $y_i(z)$ are monic polynomials, the nondegeneracy conditions can only be satisfied if

$$y_i(z)=Q^i_+(z), \qquad i=1,\dots,r. \tag{6.11}$$

Substituting (6.11) into (6.10), we see that the polynomials $Q^i_+(z),Q^i_-(z)$, $i=1,\dots,r$, satisfy the system of equations (6.2). Thus, we obtain a map from the set of nondegenerate Miura $(G,q)$-opers to the set of nondegenerate solutions of (6.2).

To show that this map is a bijection, we construct its inverse. Suppose that we are given a nondegenerate solution $\{Q^i_+(z),Q^i_-(z)\}_{i=1,\dots,r}$ of the system (6.2). The nondegeneracy condition implies that the polynomials $Q^i_+(z)$ and $Q^i_-(z)$ are relatively prime. We then define $A(z)$ by formula (6.4), where we set

$$g_i(z) = \zeta_i \frac{Q^i_+(qz)}{Q^i_+(z)},$$

i.e.

$$A(z) = \prod_j \left[\zeta_j \frac{Q^j_+(qz)}{Q^j_+(z)}\right]^{\check{\alpha}_j} e^{\frac{\Lambda_j(z)Q^j_+(z)}{\zeta_j Q^j_+(qz)} e_i} \tag{6.12}$$

$$= \prod_j \left[\zeta_j Q^j_+(qz)\right]^{\check{\alpha}_j} e^{\frac{\Lambda_j(z)}{\zeta_j Q^j_+(qz) Q^j_+(z)} e_j} \left[Q^j_+(z)\right]^{-\check{\alpha}_j}. \tag{6.13}$$

We also set

$$v(z) = \prod_{i=1}^r y_i(z)^{\check{\alpha}_i} \prod_{i=1}^r e^{-\frac{Q^j_-(z)}{Q^j_+(z)} e_i}. \tag{6.14}$$

Equations (6.5) are satisfied for all $i = 1, \ldots, r$. Using Proposition 4.8, we check that the nondegeneracy conditions on $A(z)$ are satisfied. Therefore, $A(z)$ defines a nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-oper. This completes the proof. □

*Remark* 6.2. The system (6.2) depends on our choice of ordering of the simple roots of $G$. In Section 7.4 we will show that the systems corresponding to different orderings are equivalent. □

6.2. **Prior work on the $QQ$-system.** The system (6.2) has an interesting history. As far as we know, for $G = \mathrm{SL}(2)$ the corresponding equation (5.4) with $\Lambda(z) = 1$ was first written by Bazhanov, Lukyanov, and Zamolodchikov [BLZ] in their study of the quantum KdV system. It was then generalized to the case $G = \mathrm{SL}(3)$ (also with $\Lambda_i(z) = 1$) in [BHK]. However, in both of these works, the conditions imposed on $Q^i_\pm(z)$ are different from those considered here; they are not polynomials, but rather entire functions in $z$ with a particular asymptotic behavior as $z \to \infty$.

For a general simply laced $G$, the system (6.2) with $\Lambda_i(z) = 1$ is equivalent to a system that, as far as we know, was first proposed by Masoero, Raimondo, and Valeri in [MRV1], in their study of (differential) affine opers introduced in [FF]. (For $G = \mathrm{SL}(n)$, a Yangian version of this system is closely related to the system introduced in [BFL$^+$]; see Remark 3.4 of [FH2].) The goal of [MRV1] was to generalize the results of [BLZ] in light of the conjecture of [FF] (see also [FH2]) linking the spectra of quantum $\widehat{\mathfrak{g}}$-KdV system and affine ${}^L\widehat{\mathfrak{g}}$-opers on $\mathbb{P}^1$ of a special kind. Here, ${}^L\widehat{\mathfrak{g}}$ is the affine Kac-Moody algebra that is Langlands dual to $\widehat{\mathfrak{g}}$, i.e., its Cartan matrix is the transpose of that of $\widehat{\mathfrak{g}}$. If $\mathfrak{g}$ is simply laced, then ${}^L\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}$. The authors of [MRV1] considered the simplest of the $\widehat{\mathfrak{g}}$-opers proposed in [FF], those corresponding to the ground states of the quantum $\widehat{\mathfrak{g}}$-KdV system, and associated to each of them a solution of a system equivalent to (6.2) with $\Lambda_i(z) = 1$. (This was subsequently generalized in [MR] by Masoero and Raimondo to the $\widehat{\mathfrak{g}}$-opers conjectured in [FF] to correspond to the excited states of the quantum $\widehat{\mathfrak{g}}$-KdV system.) However, the meaning of this system from the point of view of quantum integrable systems remained unclear.

The meaning was revealed in [FH2], where it was shown that the system of [MRV1] is a system of relations in the Grothendieck ring $K_0(\mathcal{O})$ of the category $\mathcal{O}$ associated to $U_q\widehat{\mathfrak{g}}$, which was introduced in [HJ]. (Actually, it is $U_{q'}\widehat{\mathfrak{g}}$, where $q' = q^{-2}$, but we will ignore this here.) Recall that $\mathcal{O}$ is a category of representations of the Borel subalgebra $U_q\widehat{\mathfrak{b}}_+$ of $U_q\widehat{\mathfrak{g}}$ (with respect to the Drinfeld-Jimbo realization) which decompose into a direct sum of finite-dimensional weight spaces with respect to the finite-dimensional Cartan subalgebra. There are two sets of representations from this category corresponding to $Q^i_+(z)$ and $Q^i_-(z), i = 1, \ldots, r$, whose classes in the Grothendieck ring $K_0(\mathcal{O})$ were proved in [FH2] to satisfy the relations of a system equivalent to the $QQ$-system (with $\Lambda_i(z) = 1$). The polynomials $Q^+_i(z), i = 1, \ldots, r$, correspond to the classes of the so-called prefundamental representations of $U_q\widehat{\mathfrak{b}}_+$, whereas $Q^-_i(z), i = 1, \ldots, r$, correspond to the classes of another less familiar set of representations of $U_q\widehat{\mathfrak{b}}_+$ which were introduced in [FH2].

Note that in [FH2], $Q^i_+(z)$ was denoted by $Q_i(z)$ and $Q^i_-(z)$ by $\widetilde{Q}_i(z)$, and the system (6.2) was called the $Q\widetilde{Q}$-system. Here, we call the system (6.2) the *$QQ$-system*.

According to the results of [FH2], for every quantum integrable system in which the commutative algebra $K_0(\mathcal{O})$ maps to the algebra of quantum Hamiltonians, we obtain the system (6.2) for each common set of eigenvalues of the Hamiltonians corresponding to $Q^i_+(z), Q^i_-(z), i = 1, \ldots, r$. Examples of such integrable systems include the $U_q\widehat{\mathfrak{g}}$ XXZ-type model (the XXZ model corresponds to $U_q\widehat{\mathfrak{sl}}_2$, as discussed in the Introduction) and the quantum $\widehat{\mathfrak{g}}$-KdV system. In both cases, the Hamiltonians corresponding to the $Q^i_+(z)$'s and $Q^i_-(z)$'s can be expressed as the transfer-matrices associated to the above representations of $U_q\widehat{\mathfrak{b}}_+$ from the category $\mathcal{O}$.

The difference between the two types of systems is reflected in the difference between the analytic properties of the $Q^i_+(z)$ and $Q^i_-(z)$. Namely, for $U_q\widehat{\mathfrak{g}}$ XXZ-type quantum models with the space of states $V$ being the tensor product of irreducible finite-dimensional representations, these should be polynomials up to a universal factor. (This has been proved for $Q^i_+(z)$ in [FH1], and we expect the same to be true for $Q^i_-(z)$.) These factors should naturally combine to form the polynomials $\Lambda_i(z)$ appearing in the $QQ$-system (6.2), which should be equal to products of the Drinfeld polynomials of the finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$ appearing as factors in $V$ (up to multiplicative shifts by powers of $q$). On the other hand, in quantum KdV systems, the functions $Q^i_+(z)$ and $Q^i_-(z)$ are expected to be entire functions on the complex plane [BLZ, BHK, MRV1, MR].

In the case of $\mathfrak{g} = \mathfrak{sl}_2$, the polynomial $Q^1_+(z)$ corresponds to the eigenvalues of the so-called Baxter operator in the XXZ-type quantum spin chain. Together with the transfer-matrix of the two-dimensional evaluation representation of $U_q\widehat{\mathfrak{sl}}_2$, it obeys the celebrated Baxter $TQ$-relation. This relation was generalized in [FH1] from $U_q\widehat{\mathfrak{sl}}_2$ to $U_q\widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is an arbitrary simple Lie algebra $\mathfrak{g}$, thereby proving a conjecture of [FR2].

*Remark* 6.3. For simply laced $\mathfrak{g}$, the form of the $QQ$-system (6.2) differs slightly from that of [MRV1, FH2]. One can relate the two by making small notational adjustments. For example, in the case of $\mathfrak{g} = \mathfrak{sl}_n$ and the standard ordering of the simple roots, we obtain

$$\Lambda_i(z)Q^{i+1}_+(qz)Q^{i-1}_+(z) = \widetilde{\xi}_i Q^i_-(z)Q^i_+(qz) - \xi_i Q^i_-(qz)Q^i_+(z), \tag{6.15}$$

which is equivalent to

$$\Lambda_i(q^{-1/2}z)Q^{i+1}_+(q^{1/2}z)Q^{i-1}_+(q^{-1/2}z) = \widetilde{\xi}_i Q^i_-(q^{-1/2}z)Q^i_+(q^{1/2}z) - \xi_i Q^i_-(q^{1/2}z)Q^i_+(q^{-1/2}z).$$

Upon making the substitution $Q^i_\pm(z) = \mathbf{Q}^i_\pm(q^{\frac{N-i}{2}}z)$ and $\mathbf{\Lambda}_i(z) = \Lambda_i(q^{\frac{N-i-1}{2}}z)$, we obtain a more symmetric form of the system which was considered in [KSZ]:

$$\mathbf{\Lambda}_i(z)\mathbf{Q}^{i+1}_+(z)\mathbf{Q}^{i-1}_+(z) = \frac{\zeta_i}{\zeta_{i+1}}\mathbf{Q}^i_-(q^{-1/2}z)\mathbf{Q}^i_+(q^{1/2}z) - \frac{\zeta_{i+1}}{\zeta_i}\mathbf{Q}^i_-(q^{1/2}z)\mathbf{Q}^i_+(q^{-1/2}z).$$

If we set $\mathbf{\Lambda}_i(z) = 1$, the latter is equivalent to the system from [MRV1, FH2] corresponding to $U_{q'}\widehat{\mathfrak{sl}}_n$ with $q' = q^{-2}$.

Now, suppose that $\mathfrak{g}$ is non-simply laced. In this case, the system (6.2) is *different* from the $Q\widetilde{Q}$-system of [MRV2] and [FH2] corresponding to $U_q\widehat{\mathfrak{g}}$. Instead, it can be obtained by "folding" the $Q\widetilde{Q}$-system corresponding to $U_q\widehat{\mathfrak{g}}'$, where $\mathfrak{g}'$ is the simply laced Lie algebra with an automorphism $\sigma$ such that $(\mathfrak{g}')^\sigma = \mathfrak{g}$. This will be discussed in [FHR].

6.3. **$QQ$-system and Bethe Ansatz equations.** As we will see, the $QQ$-system (6.2) gives rise to a system of equations only involving the $Q^i_+(z)$'s. Let $\{w^k_i\}_{k=1,\ldots,m_i}$ be the set of roots of the polynomial $Q^i_+(w)$. We call the system of equations

$$\frac{Q^i_+(qw^k_i)}{Q^i_+(q^{-1}w^k_i)}\prod_j \zeta_j^{a_{ji}} = -\frac{\Lambda_i(w^k_i)\prod_{j>i}\Big[Q^j_+(qw^k_i)\Big]^{-a_{ji}}\prod_{j<i}\Big[Q^j_+(w^k_i)\Big]^{-a_{ji}}}{\Lambda_i(q^{-1}w^k_i)\prod_{j>i}\Big[Q^j_+(w^k_i)\Big]^{-a_{ji}}\prod_{j<i}\Big[Q^j_+(q^{-1}w^k_i)\Big]^{-a_{ji}}} \tag{6.16}$$

for $i = 1,\ldots,r$, $k = 1,\ldots,m_i$ the *Bethe Ansatz equations* for the group $G$ and the set $\{\Lambda_i(z)\}_{i=1,\ldots,r}$.

For simply laced $G$, this system is equivalent to the system of Bethe Ansatz equations that appears in the $U_q\widehat{\mathfrak{g}}$ XXZ-type model [OW, RW, R]. However, for non-simply laced $G$, we obtain a different system of Bethe Ansatz equations, which, as far as we know, has not yet been studied in the literature on quantum integrable systems. (As we mentioned in the Introduction, an additive version of this system appeared earlier in [MV2].) As will be explained in [FHR], these Bethe Ansatz equations correspond to a novel quantum integrable model in which the spaces of states are representations of the twisted quantum affine Kac-Moody algebra $U_q{}^L\widehat{\mathfrak{g}}$, where ${}^L\widehat{\mathfrak{g}}$ is the Langlands dual Lie algebra of $\widehat{\mathfrak{g}}$.

Recall the nondegeneracy condition for the solutions of the $QQ$-system. We apply the same notion to the solutions of (6.16).

**Theorem 6.4.** *There is a bijection between the sets of nondegenerate polynomial solutions of the $QQ$-system (6.2) and the Bethe Ansatz equations (6.16).*

*Proof.* Let $\{Q^i_+(z), Q^i_-(z)\}_{i=1,\ldots,r}$ be a nondegenerate solution of the $QQ$-system (6.2). Set

$$\phi_i(z) = \frac{Q^i_-(z)}{Q^i_+(z)} \tag{6.17}$$

and

$$f_i(z) = \Lambda_i(z)\prod_{j>i}\Big[Q^j_+(qz)\Big]^{-a_{ji}}\prod_{j<i}\Big[Q^j_+(z)\Big]^{-a_{ji}}. \tag{6.18}$$

Then, the $i$th equation of the $QQ$-system may be rewritten as

$$\widetilde{\xi}_i\phi_i(z) - \xi_i\phi_i(qz) = \frac{f_i(z)}{Q^i_+(z)Q^i_+(qz)}. \tag{6.19}$$

The nondegeneracy condition implies that we have the following partial fraction decompositions in which all the denominators are pairwise relatively prime:

$$\frac{f_i(z)}{Q^i_+(z)Q^i_+(qz)} = h_i(z) + \sum_{k=1}^{m_i} \frac{b_k}{z - w^i_k} + \sum_{k=1}^{m_i} \frac{c_k}{qz - w^i_k}, \tag{6.20}$$

$$\phi_i(z) = \widetilde{\phi}_i(z) + \sum_{k=1}^{m_i} \frac{d_k}{z - w^i_k}. \tag{6.21}$$

Here, $h_i(z)$ and $\widetilde{\phi}_i(z)$ are polynomials and $\{w^i_k\}_{k=1,\dots,m_i}$ is the set of roots of the polynomial $Q^i_+(z)$.

The residues at $z = w^i_k$ (resp. $z = w^i_k q^{-1}$) on the two sides of (6.19) must coincide. Therefore,

$$d_k = \frac{b_k}{\xi_i}, \qquad \text{resp.} \quad d_k = -\frac{c_k}{\widetilde{\xi}_i} \tag{6.22}$$

for all $k = 1, \dots, m_i$. Thus, we obtain

$$\frac{b_k}{\xi_i} + \frac{c_k}{\widetilde{\xi}_i} = 0,$$

or equivalently,

$$\mathrm{Res}_{z=w^i_k}\left[\frac{f_i(z)}{\xi_i Q^i_+(z)Q^i_+(qz)}\right] + \mathrm{Res}_{z=w^i_k}\left[\frac{f_i(q^{-1}z)}{\widetilde{\xi}_i Q^i_+(q^{-1}z)Q^i_+(z)}\right] = 0 \tag{6.23}$$

which is just equation (6.16) for $i, k$. Thus, we have a nondegenerate solution $\{Q^i_+(z)\}_{i=1,\dots,r}$ of the system (6.16).

Next, we define the inverse map. Suppose that we have a nondegenerate solution $\{Q^i_+(z)\}_{i=1,\dots,r}$ of (6.16). We need to construct polynomials $\{Q^i_-(z)\}_{i=1,\dots,r}$ that together with the polynomials $\{Q^i_+(z)\}_{i=1,\dots,r}$ solve the $QQ$-system (6.2). To do this, we will construct a rational function $\phi_i(z)$ that has the same set of poles as the set of roots of the polynomial $Q^i_+(z)$ and define $Q^i_-(z)$ by the formula

$$Q^i_-(z) = \phi_i(z) Q^i_+(z)$$

(compare with (6.17)).

We will define the rational function $\phi_i(z)$ via the partial fraction decomposition (6.21), where we use (6.22) to define the residues $d_k$. (Note that the $d_k$'s are completely determined by the $Q^i_+(z)$'s.) It remains to find the polynomial part of $\phi_i(z)$,

$$\widetilde{\phi}_i(z) = \sum_{m \geq 0} r_m z^m.$$

Let

$$h_i(z) = \sum_{m \geq 0} s_m z^m$$

be the polynomial appearing in equation (6.20). Note that $h_i(z)$, and hence $\{s_m\}_{m \geq 0}$, are completely determined by the $Q^i_+(z)$'s.

Now, observe that (6.19), which is the $i$th equation of the $QQ$-system, is satisfied if and only if the following equations on the $\{r_m\}_{m \geq 0}$ are satisfied:

$$r_m(\widetilde{\xi}_i q^m - \xi_i) = s_m, \qquad m \geq 0. \tag{6.24}$$

This follows from our assumption on $Z$ in (6.1) because $\widetilde{\xi}_i/\xi_i = \prod_{j=1}^r \zeta_j^{a_{ji}}$. Therefore, each of the equations (6.24) has a unique solution.

It then follows that there exist unique polynomials $\{Q^i_-(z)\}_{i=1,\dots,r}$ that together with $\{Q^i_+(z)\}_{i=1,\dots,r}$ satisfy the $QQ$-system (6.2). Furthermore, by construction, it follows that this solution of the $QQ$-system is nondegenerate. □

## 7. Bäcklund-type transformations

Theorems 6.1 and 6.4 establish a bijection between the set of nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-opers and the sets of polynomial nondegenerate solutions of the $QQ$-system and the Bethe Ansatz equations (6.16).

Now, the set of nondegenerate $Z$-twisted Miura-Plücker $(G, q)$-opers includes as a subset those $Z$-twisted Miura-Plücker $(G, q)$-opers which are actually $Z$-twisted Miura $(G, q)$-opers. Recall the difference between the two: a $Z$-twisted Miura $(G, q)$-oper is one whose $q$-connection can be represented in the form (4.8):

$$A(z) = v(qz)Zv(z)^{-1}, \qquad v(z) \in B_+(z), \tag{7.1}$$

whereas a $Z$-twisted Miura-Plücker $(G, q)$-oper is one for which only the associated $(\mathrm{GL}(2), q)$-opers $A_i(z)$ have this property (compare with (6.5)). When we constructed the inverse map in the proof of Theorem 6.1, we defined an element $v(z)$ of $B_+(z)$ by formula (6.14). This $v(z)$ satisfies the equations (6.5), so we do get a $Z$-twisted Miura-Plücker $(G, q)$-oper, but it is not clear whether this $v(z)$ can be extended to an element of $B_+(z)$ satisfying formula (7.1). More precisely, equations (6.5) uniquely fix the image $\overline{v}(z)$ of $v(z)$ in the quotient $B_+/[N_+, N_+]$, and the question is whether we can lift this $\overline{v}(z)$ to an element $v(z) \in B_+(z)$ such that equation (7.1) is satisfied.

In this section, we will give a sufficient condition for this to hold (see Theorem 7.10 and Remark 7.11). It is based on transformations described in the next subsection for generating new solutions of the $QQ$-system from an existing one. (There is one such transformation for each simple root of $G$.). We call them Bäcklund-type transformations.

Here, we follow an idea of Mukhin and Varchenko [MV1, MV2], who introduced similar procedures for the solutions of the Bethe Ansatz equations arising from the XXX-type models associated to Yangians. However, in contrast to their setting, we have a non-trivial twist represented by a regular semisimple element $Z$ of the Cartan subalgebra. As a result, our transformations generically give rise to solutions labeled by elements of the Weyl group of $G$, rather than by points of the flag manifold of $G$ as in [MV1, MV2].

### 7.1. Definition of Bäcklund-type transformations.

Consider a $Z$-twisted Miura-Plücker $(G, q)$-oper given by formula (6.12). We now define a transformation associated to the $i$th simple reflection from the Weyl group $W_G$ on the set of such Miura $q$-opers.

**Proposition 7.1.** *Consider the $q$-gauge transformation of the $q$-connection $A$ given by formula (6.12):*

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \text{where} \quad \mu_i(z) = \frac{\prod_{j\neq i} \left[Q^j_+(z)\right]^{-a_{ji}}}{Q^i_+(z)Q^i_-(z)}. \tag{7.2}$$

*Then $A^{(i)}(z)$ can be obtained from $A(z)$ by substituting in formula (6.12) (or (6.13))*

$$Q_+^j(z) \mapsto Q_+^j(z), \qquad j \neq i, \tag{7.3}$$

$$Q_+^i(z) \mapsto Q_-^i(z), \qquad Z \mapsto s_i(Z)\,. \tag{7.4}$$

In the proof of Theorem 7.1, we will use the following lemma, which is proved by a direct computation. (The results of the lemma have appeared previously in [MV2]).

**Lemma 7.2.** *The following relations hold for any $u, v \in \mathbb{C}$:*

$$\begin{aligned} u^{\check\alpha_i} e^{v e_j} &= \exp\left(u^{a_{ji}} v\, e_i\right) u^{\check\alpha_i} \\ u^{\check\alpha_i} e^{v f_j} &= \exp\left(u^{-a_{ji}} v\, f_i\right) u^{\check\alpha_i} \\ e^{u e_i}\ e^{v f_i} &= \exp\left(\frac{v}{1+uv} f_i\right)\ (1+uv)^{\check\alpha_i}\ \exp\left(\frac{u}{1+uv} e_i\right). \end{aligned}$$

*Proof of Proposition 7.1.* Using the first identity from Lemma 7.2, we can move all factors $Q_+^j(qz)$ in formula (6.12) to the left and all factors $Q_+^j(z)$ to the right. The resulting expression is

$$A(z) = \prod_k \left[Q_+^k(qz)\right]^{\check\alpha_k} \left[\prod_i \zeta_i^{\check\alpha_i} e^{\widetilde\Lambda_i(z) e_i}\right] \prod_l \left[Q_+^l(z)\right]^{-\check\alpha_l}, \tag{7.5}$$

where

$$\widetilde\Lambda_i(z) = \frac{\prod\limits_{j<i} \left[Q_+^j(z)\right]^{-a_{ji}} \prod\limits_{j>i} \left[Q_+^j(qz)\right]^{-a_{ji}} \Lambda_i(z)}{\zeta_i Q_+^i(qz) Q_+^i(z)}. \tag{7.6}$$

Let

$$\widetilde\mu_i(z) = \mu_i(z) \prod_j \left[Q_+^j(z)\right]^{a_{ji}}. \tag{7.7}$$

Then, applying the second identity from Lemma 7.2 to (7.5), we obtain

$$\widetilde A^{(i)} = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i} = \dots \zeta_i^{\check\alpha_i} e^{w\, f_i} \cdot e^{u\, e_i} \cdot e^{v\, f_i} \dots \tag{7.8}$$

where

$$w = \zeta_i^2 \prod_{j<i} \zeta_j^{a_{ji}} \widetilde\mu_i(qz)\,, \qquad u = \widetilde\Lambda_i(z)\,, \qquad v = -\prod_{j>i} \zeta_j^{a_{ji}} \widetilde\mu_i(z)\,, \tag{7.9}$$

and the ellipses stand for all other terms including the elements of the maximal torus and the exponentials of $e_j$ with $j \neq i$. We now use the third identity from Lemma 7.2 to reshuffle the middle and the last exponent in (7.8):

$$\widetilde A^{(i)} = \dots \zeta_i^{\check\alpha_i}\ \exp\left(\left(w + \frac{v}{1+uv}\right) f_i\right)\ (1+uv)^{\check\alpha_i}\ \exp\left(\frac{u}{1+uv} e_i\right) \dots. \tag{7.10}$$

In order to prove the proposition, we first need to show that

$$w + \frac{v}{1+uv} = 0, \tag{7.11}$$

or, in other words,

$$\zeta_i^2 \prod_{j<i} \zeta_j^{a_{ji}} \widetilde{\mu}_i(qz) - \frac{\prod_{j>i} \zeta_j^{a_{ji}} \widetilde{\mu}_i(z)}{1 - \widetilde{\Lambda}_i(z) \prod_{j>i} \zeta_j^{a_{ji}} \widetilde{\mu}_i(z)} = 0. \tag{7.12}$$

Let us demonstrate that this is indeed the case. The above equation is equivalent to

$$\prod_{j>i} \zeta_j^{-a_{ji}} \widetilde{\mu}_i^{-1}(z) - \zeta_i^{-2} \prod_{j<i} \zeta_j^{-a_{ji}} \widetilde{\mu}_i^{-1}(qz) = \widetilde{\Lambda}_i(z)\,. \tag{7.13}$$

Substituting $\widetilde{\Lambda}_i(z)$ from (7.6) into this equation gives

$$\begin{aligned} \zeta_i \prod_{j>i} \zeta_j^{-a_{ji}}\, \widetilde{\mu}_i^{-1}(z) Q_+^i(qz) Q_+^i(z) - \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}\, \widetilde{\mu}_i^{-1}(qz) Q_+^i(qz) Q_+^i(z) \\ = \prod_{j<i} \Big[ Q_+^j(z) \Big]^{-a_{ji}} \prod_{j>i} \Big[ Q_+^j(qz) \Big]^{-a_{ji}} \Lambda_i(z)\,. \end{aligned} \tag{7.14}$$

Keeping in mind from (7.2) and (7.7) that $\widetilde{\mu}_i(z) = \frac{Q_+^i(z)}{Q_-^i(z)}$, we recover the $QQ$-system equations:

$$\begin{aligned} \zeta_i \prod_{j>i} \zeta_j^{-a_{ji}}\, Q_+^i(qz) Q_-^i(z) - \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}\, Q_-^i(qz) Q_+^i(z) \\ = \prod_{j<i} \Big[ Q_+^j(z) \Big]^{-a_{ji}} \prod_{j>i} \Big[ Q_+^j(qz) \Big]^{-a_{ji}} \Lambda_i(z)\,. \end{aligned} \tag{7.15}$$

We now check that the two remaining terms in (7.10) are indeed as prescribed by (7.4). Note that

$$\begin{aligned} 1 + uv &= 1 - \widetilde{\Lambda}_i(z) \prod_{j>i} \zeta_j^{a_{ji}} \widetilde{\mu}_i(z) \\ &= 1 - \frac{Q_+^i(z)}{Q_-^i(z)} \prod_{j>i} \zeta_j^{a_{ji}} \frac{\prod_{j<i} \Big[ Q_+^j(z) \Big]^{-a_{ji}} \prod_{j>i} \Big[ Q_+^j(qz) \Big]^{-a_{ji}} \Lambda_i(z)}{\zeta_i Q_+^i(qz) Q_+^i(z)}\,. \end{aligned} \tag{7.16}$$

Thus, the $QQ$-equations imply that

$$1 + uv = \zeta_i^{-2} \prod_{j\neq i} \zeta_j^{-a_{ji}} \frac{Q_-^i(qz) Q_+^i(z)}{Q_+^i(qz) Q_-^i(z)}\,. \tag{7.17}$$

The corresponding contribution to the q-connection is

$$(1+uv)^{\check{\alpha}_i} = s_i(Z) Z^{-1} \left[ \frac{Q_-^i(qz)}{Q_+^i(qz)} \right]^{\check{\alpha}_i} \left[ \frac{Q_+^i(z)}{Q_-^i(z)} \right]^{\check{\alpha}_i}\,. \tag{7.18}$$

Finally, we need to evaluate the last term in (7.10). Computing, we obtain

$$\frac{u}{1+uv} = \frac{\dfrac{\prod_{j<i}\left[Q_+^j(z)\right]^{-a_{ji}}\prod_{j>i}\left[Q_+^j(qz)\right]^{-a_{ji}}\Lambda_i(z)}{\zeta_i Q_+^i(qz)Q_+^i(z)}}{\zeta_i^{-2}\prod_{j\neq i}\zeta_j^{-a_{ji}}\dfrac{Q_-^i(qz)Q_+^i(z)}{Q_+^i(qz)Q_-^i(z)}}$$
$$= \frac{\prod_{j<i}\left[Q_+^j(z)\right]^{-a_{ji}}\prod_{j>i}\left[Q_+^j(qz)\right]^{-a_{ji}}\Lambda_i(z)}{\zeta_i^{-1}\prod_{j\neq i}\zeta_j^{-a_{ji}}Q_-^i(qz)Q_-^i(z)}\left[\frac{Q_-^i(z)}{Q_+^i(z)}\right]^2.$$

Thus, from (7.10), we have

$$\widetilde{A}^{(i)} = \dots\zeta_i^{-\check{\alpha}_i-\langle\check{\alpha}_j,\alpha_i\rangle\check{\alpha}_i}\left[\frac{Q_-^i(qz)}{Q_+^i(qz)}\right]^{\check{\alpha}_i} e^{\widetilde{\Lambda}_i^-(z)e_i}\left[\frac{Q_+^i(z)}{Q_-^i(z)}\right]^{\check{\alpha}_i}\dots, \tag{7.19}$$

where

$$\widetilde{\Lambda}_i^-(z) = \frac{\prod_{j<i}\left[Q_+^j(z)\right]^{-a_{ji}}\prod_{j>i}\left[Q_+^j(qz)\right]^{-a_{ji}}\Lambda_i(z)}{\zeta_i^{-1}\prod\limits_{j\neq i}\zeta_j^{-a_{ji}}Q_-^i(qz)Q_-^i(z)}.$$

Moving the elements of the maximal torus corresponding to the ratios of the $Q$'s to the left and right side of (7.8), we obtain the statement of Proposition 7.1. □

7.2. **Nondegeneracy.** Suppose that $A(z)$ is a nondegenerate $Z$-twisted Miura-Plücker $(G,q)$-oper. In Proposition 7.1, we apply to it the $q$-gauge transformation by $e^{\mu_i(z)f_i}$, which belongs to $N_-$ and not to $B_+$. Hence, it is not clear whether the $q$-connection $A^{(i)}(z)$ constructed in Proposition 7.1 corresponds to a $Z$-twisted Miura-Plücker $(G,q)$-oper, let alone a nondegenerate one.

Seen from the perspective of Theorem 6.1, there exist unique polynomials $\{Q_-^j(z)\}_{j=1,\dots,r}$ that together with $\{Q_+^j(z)\}_{j=1,\dots,r}$ give rise to a nondegenerate solution of the $QQ$-system (6.2) corresponding to the element $Z\in H$. However, the nondegeneracy condition for the latter solution does *not* include any information on the roots of the polynomials $\{Q_-^j(z)\}_{j=1,\dots,r}$, other than the fact that the roots of $Q_-^j(z)$ are distinct from the roots of $Q_+^j(z)$ for all $j=1,\dots,r$. (We know this from the construction of $Q_-^j(z)$ given in the proof of Theorem 6.4).

By construction, $A^{(i)}(z)$ is an $s_i(Z)$-twisted Miura-Plücker $(G,q)$-oper, which corresponds to the polynomials $\{\widetilde{Q}_+^j(z)\}_{j=1,\dots,r}$, where $\widetilde{Q}_j^+(z)=Q_j^+(z)$ for $j\neq i$ and $\widetilde{Q}_i^+(z)=Q_i^-(z)$. The conditions for it to be nondegenerate are spelled out in the following lemma.

**Lemma 7.3.** *Suppose that the roots of the polynomial $Q_-^i(z)$ constructed in the proof of Theorem 6.4 are $q$-distinct from the roots of $\Lambda_k(z)$ for $a_{ik}\neq 0$ and from the roots of $Q_+^j(z)$ for $j\neq i$ and $a_{jk}\neq 0$. Then, the data*

$$\{\widetilde{Q}_+^j\}_{j=1,\dots,r} = \{Q_+^1,\dots,Q_+^{i-1},Q_-^i,Q_+^{i+1}\dots,Q_+^r\}; \tag{7.20}$$
$$\{\widetilde{\zeta}_j\}_{j=1,\dots,r} = \{\zeta_1,\dots,\zeta_{i-1},\zeta_i^{-1}\prod_{j\neq i}\zeta_j^{-a_{ji}},\dots,\zeta_r\}$$

*give rise to a nondegenerate solution of the Bethe Ansatz equations (6.16) corresponding to $s_i(Z)\in H$. Furthermore, there exist polynomials $\{\widetilde{Q}_-^j\}_{j=1,\dots,r}$ that together with*

$\{\widetilde{Q}^j_+\}_{j=1,\dots,r}$ *give rise to a nondegenerate solution of the QQ-system* (6.2) *corresponding to* $s_i(Z)$.

*Proof.* Let us rewrite the $QQ$-system as follows:

$$\Lambda_i(z)\prod_j\left[Q^j_+(q^{b_{ji}}z)\right]^{-a_{ji}}=\widetilde{\xi}_iQ^i_-(z)Q^i_+(qz)-\xi_iQ^i_-(qz)Q^i_+(z), \tag{7.21}$$

where $b_{ji}=1$ if $j>i$ and $b_{ji}=0$ otherwise. Note that for $i\neq j$,

$$b_{ji}+b_{ij}=1\,. \tag{7.22}$$

Dividing both sides by $Q^i_+(z)Q^i_-(z)$ gives

$$\widetilde{\xi}_i\frac{Q^i_+(qz)}{Q^i_+(z)}-\xi_i\frac{Q^i_-(qz)}{Q^i_-(z)}=\frac{\Pi_i(z)}{Q^i_+(z)Q^i_-(z)}\,, \tag{7.23}$$

where

$$\Pi_i(z)=\Lambda_i(z)\prod_{j\neq i}\left[Q^j_+(q^{b_{ij}}z)\right]^{-a_{ji}}.$$

Evaluating this equation at each $q^{-b_{im}}w^k_m$, where $m\neq i$ and the $w^k_m$'s are the roots of the polynomial $Q^m_+(z)$, we obtain

$$\Big(\prod_j\zeta_j^{a_{ji}}\Big)\frac{Q^i_+(q^{1-b_{im}}w^k_m)}{Q^i_+(q^{-b_{im}}w^k_m)}=\frac{Q^i_-(q^{1-b_{im}}w^k_m)}{Q^i_-(q^{-b_{im}}w^k_m)}\,. \tag{7.24}$$

Comparing the above formula with the Bethe Ansatz equations (6.16) and using (7.22), we see that the data (7.20) satisfy the Bethe Ansatz equations (6.16). The existence of $\{\widetilde{Q}^j_+\}_{j=1,\dots,r}$ then follows from Theorem 6.4. $\square$

Thus, if the conditions of Lemma 7.3 are satisfied, we can associate to every nondegenerate $Z$-twisted Miura-Plücker $q$-oper a nondegenerate $s_i(Z)$-twisted Miura-Plücker oper via $A(z)\mapsto A^{(i)}(z)$. We call this procedure a *Bäcklund-type transformation* associated to the $i$th simple reflection of the Weyl group $W_G$. We now generalize this transformation to other Weyl group elements.

**Definition 7.4.** Let $w=s_{i_1}\dots s_{i_k}$ be a reduced decomposition of an element $w$ of the Weyl group. A solution of the $QQ$-system (6.2) is called $(i_1\dots i_k)$-generic if by consecutively applying the procedure described in Lemma 7.3 with $i=i_k,\dots,i_1$, we obtain a sequence of nondegenerate solutions of the $QQ$-systems corresponding to the elements $w_j(Z)\in H$, where $w_k=s_{i_{k-j+1}}\dots s_{i_k}$ with $j=1,\dots,k$.

*Remark* 7.5. Note that in this definition, we only assume the existence of a sequence of transformations as described in Lemma 7.3 for a particular reduced decomposition of $w$. We do not assume that such a sequence exists for other reduced decompositions of $w$.

Now, we derive an important consequence of this property for $(G,q)$-opers.

**Proposition 7.6.** *Let* $w=s_{i_1}\dots s_{i_k}$ *be a reduced decomposition. Then, for each* $(i_1\dots i_k)$*-generic solution of the QQ-system* (6.2), *there exists an element* $b_-(z)\in B(z)$ *of the form*

$$b_-(z)=e^{c_{i_1}(z)f_{i_1}}e^{c_{i_2}(z)f_{i_2}}\dots e^{c_{i_k}(z)f_{i_k}}h(z),$$

*where* $c_{i_j}(z)$ *are non-zero rational functions and* $h(z)\in H(z)$, *such that*

$$b_-(qz)w(Z)v=A(z)b_-(z)v. \tag{7.25}$$

*Here, $A(z)$ is given by equation (6.12) and $v$ is a highest weight vector in any irreducible finite-dimensional representation of $G$.*

*Proof.* The idea is to construct $b_-(z)$ as a composition of the elements of $B_-(z)$ appearing in the $q$-gauge transformations from the Proposition 7.1 corresponding to different simple reflections in the reduced decomposition of $w$.

If $w = s_i$, then we have the single $q$-gauge transformation from formula (7.2):

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \text{where} \quad \mu_i = \frac{\prod_{j\neq i} \left[Q^j_+(z)\right]^{-a_{ji}}}{Q^i_+(z)Q^i_-(z)}. \tag{7.26}$$

According to Proposition 7.1, if we apply $A^{(i)}$ to a highest weight vector $v$, we obtain

$$A^{(i)}v = \prod_j \left[\frac{\widetilde{Q}^j_+(qz)}{\widetilde{Q}^j_+(z)}\right]^{\check{\alpha}_j} s_i(Z)\ v, \tag{7.27}$$

where the polynomials $\widetilde{Q}^j_+(z)$ are defined by formula (7.20). After multiplying both sides by

$$e^{-\mu_i(qz)f_i} \prod_j \left[\widetilde{Q}^j_+(z)\right]^{\langle\check{\alpha}_j,\lambda\rangle},$$

where $\lambda$ is the weight of $v$, we obtain:

$$A(z)e^{-\mu_i(z)f_i} \prod_j \left[\widetilde{Q}^j_+(z)\right]^{\check{\alpha}_j} v = e^{-\mu_i(qz)f_i} \prod_j \left[\widetilde{Q}^j_+(qz)\right]^{\check{\alpha}_j} s_i(Z)v. \tag{7.28}$$

Thus, $b_-(z) = e^{-\mu_i(z)f_i} \prod_j \left[\widetilde{Q}^j_+(z)\right]^{\check{\alpha}_j}$ is the sought-after element of $B_-(z)$.

For $w = s_{i_1} \dots s_{i_k}$, we successively apply the $q$-gauge transformations (7.26) with $i = i_k, i_{k-1}, \dots, i_1$. Let $\mu_{i_k}(z), \mu_{i_{k-1}}(z), \dots, \mu_{i_1}(z)$ be the corresponding functions as in (7.2), and let $\{\overline{Q}^j_+(z)\}$ be the set of polynomials obtained from $\{Q^j_+(z)\}$ by successively applying formula (7.20) with $i = i_k, i_{k-1}, \dots, i_1$. It then follows that

$$b_-(z) = e^{-\mu_{i_1}(z)f_{i_1}} e^{-\mu_{i_2}(z)f_{i_2}} \dots e^{-\mu_{i_k}(z)f_{i_k}} \prod_j \left[\overline{Q}^j_+(z)\right]^{\check{\alpha}_j} \tag{7.29}$$

satisfies (7.25). □

We also formulate an additional result regarding elements in Bruhat cells. Recall the following well-known fact about the product of Bruhat cells (see e.g. [H, Lemma 29.3.A]):

**Lemma 7.7.** *If $u, v \in W_G$ satisfy $\ell(u) + \ell(v) = \ell(uv)$, then $B_-uB_-vB_- = B_-uvB_-$.*

The following result is a direct consequence of Lemma 7.7.

**Proposition 7.8.** *If $w \in W$ has a reduced decomposition $w = s_{i_1}s_{i_2}\dots s_{i_k}$, then*

$$e^{a_{i_1}e_{i_1}} e^{a_{i_2}e_{i_2}} \dots e^{a_{i_k}e_{i_k}} \in B_-wB_-, \quad e^{a_{i_1}f_{i_1}} e^{a_{i_2}f_{i_2}} \dots e^{a_{i_k}f_{i_k}} \in B_+wN_+$$

*if $a_{i_j} \neq 0$ for all $j$.*

**Definition 7.9.** A $Z$-twisted Miura-Plücker $(G, q)$-oper is called $(i_1 \dots i_k)$-generic if it corresponds to an $(i_1 \dots i_k)$-generic solution of the $QQ$-system via the bijection in Theorem 6.1.

7.3. **From Miura-Plücker to Miura $q$-opers.** We shall now describe a sufficient condition for a $Z$-twisted Miura-Plücker $(G,q)$-opers to be a Miura $(G,q)$-oper.

Let $w_0 = s_{i_1} \dots s_{i_\ell}$ be a reduced decomposition of the longest element of the Weyl group. In what follows, we refer to an $(i_1,\dots,i_\ell)$-generic object as $w_0$-*generic*.

**Theorem 7.10.** *Every $w_0$-generic $Z$-twisted Miura-Plücker $(G,q)$-oper is a nondegenerate $Z$-twisted Miura $(G,q)$-oper.*

*Proof.* Let

$$A(z) = \prod_j \left[\zeta_j \frac{Q^j_+(qz)}{Q^j_+(z)}\right]^{\check{\alpha}_j} e^{\frac{\Lambda_j(z)e_j}{g_j(z)}}$$

be the $w_0$-generic $Z$-twisted Miura-Plücker $(G,q)$-oper coming from a $w_0$-generic solution $\{Q^j_+\}$ of the $QQ$-system. By Proposition 7.6, there exists an element $b_-(z) \in B_-(z)$ such that

$$b_-(qz)w_0(Z)v = A(z)b_-(z)v,$$

where $v$ is any highest weight vector in a finite-dimensional irreducible representation of $G$; moreover,

$$b_-(z) = e^{c_{i_1}f_{i_1}} e^{c_{i_2}f_{i_2}} \dots e^{c_{i_k}f_{i_\ell}} h(z)$$

with $c_{i_j}(z) \in \mathbb{C}(z)^\times$ and $h(z) \in H(z)$.

By Proposition 7.8,

$$b_-(z) = b_+(z)w_0n_+(z),$$

where $b_+(z) \in B_+(z)$ and $n_+(z) \in N_+(z)$. Therefore, we have

$$b_+(qz)Zw_0v = A(z)b_+(z)w_0v,$$

so if we set

$$U(z) = Z^{-1}b_+^{-1}(qz)A(z)b_+(z) \in B_+(z), \tag{7.30}$$

then

$$w_0v = U(z)w_0v$$

for any irreducible finite-dimensional representation of $G$ with highest weight vector $v$. Thus, $U(z)$ is an element of $B_+(z)$ which fixes the lowest weight vector $w_0v$ of any irreducible finite-dimensional representation of $G$. This means that $U(z) = 1$. Equation (7.30) then implies that $A(z)$ satisfies

$$A(z) = b_+(qz)Zb_+(z)^{-1} \tag{7.31}$$

for some $b_+(z) \in B_+(z)$. Thus, we have proved that every $w_0$-generic $Z$-twisted Miura-Plücker $(G,q)$-oper is a nondegenerate $Z$-twisted Miura $(G,q)$-oper. Equivalently, every $w_0$-generic solution of the $QQ$-system corresponds to a nondegenerate $Z$-twisted Miura $(G,q)$-oper. □

*Remark* 7.11. Given a regular semisimple element $Z \in H$ and a collection of polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ as above, consider the following three sets of objects on $\mathbb{P}^1$:

- $q\text{-MPOp}_G^Z$, the set of nondegenerate $Z$-twisted Miura-Plücker $(G,q)$-opers;
- $q\text{-MPOp}_G^{Z,w_0}$, the set of $w_0$-generic $Z$-twisted Miura-Plücker $(G,q)$-opers;
- $q\text{-MOp}_G^Z$, the set of nondegenerate $Z$-twisted Miura $(G,q)$-opers.

According to the discussion at the beginning of this section and Theorem 7.10, we have the inclusions

$$q\text{-MPOp}_G^{Z,w_0} \subset q\text{-MOp}_G^Z \subset q\text{-MPOp}_G^Z\,. \tag{7.32}$$

It would be interesting to find out under what conditions either (or both) of these inclusions is an equality. □

7.4. **Equivalence of the $QQ$-Systems.** The $QQ$-system (6.2) depends on the choice of the Coxeter element $c$ that we used in the definition of $(G,q)$-opers; the ordering of the simple reflections in it is the ordering of the simple roots in (6.2). In this subsection, we show that $QQ$-systems corresponding to different Coxeter elements (or equivalently, different orderings of the simple roots) are equivalent. First, we explain what we mean by equivalence.

We say that two $QQ$-systems are *gauge equivalent* if they are related to each other via the transformations

$$\begin{aligned} &Q^i_\pm(x) \mapsto Q^i_\pm(D^{\{i\}}x)\,, \qquad && \Lambda^i(x) \mapsto \Lambda^i(D^{\{i\}}x)\,, \\ &\xi_i \mapsto \alpha^{(i)}\xi_i\,, && \widetilde{\xi}_i \mapsto \alpha^{(i)}\widetilde{\xi}_i\,, \end{aligned} \tag{7.33}$$

where the $D^{\{i\}}$'s and $\alpha^{(i)}$'s are non-zero complex numbers.

The following theorem is a multiplicative version of Theorem 2.5 and Corollary 2.6 in [MV2].

**Theorem 7.12.** *The $QQ$-system associated to the Coxeter element $c$, parameters $\{\zeta_i\}$ and polynomials $\{\Lambda_i(z)\}$ is gauge equivalent to the $QQ$-system associated to a Coxeter element $c'$ with the same parameters $\{\zeta_i\}$ and polynomials $\{\Lambda_i(q^{d^{\{i\}}}z)\}$ for some $d^{\{i\}} \in \mathbb{Z}$.*

*Proof.* Recall from (7.21) that the $QQ$-system defined using the Coxeter element $c = w_{i_1}\dots w_{i_r}$ may be written in terms of integers $\{b_{ij}\}_{i\neq j}$, where $b_{ij}$ equals 1 (resp. 0) if $w_j$ comes before (resp. after) $w_i$ in this order. To prove the theorem, it suffices to show that this system is gauge equivalent to one defined in terms of other parameters $\hat{b}_{ij}$ whose definition only involves the Dynkin diagram.

Since $G$ is simple, the underlying graph of the Dynkin diagram is a tree whose vertices are labeled by the simple reflections $\alpha_i$. We let $\delta_i$ denote the distance from $\alpha_i$ to $\alpha_1$.

We now define $\hat{b}_{ij}$ via

$$\hat{b}_{ij} = \begin{cases} 1 & \text{if } \delta_i < \delta_j, \text{ or } \delta_i = \delta_j \text{ and } i<j; \\ 0 & \text{otherwise.} \end{cases} \tag{7.34}$$

Let $\widetilde{\hat{\xi}}_i = \prod_j \zeta_j^{a_{ji}}$ and $\hat{\xi}_i = 1$. Our original $QQ$-system is gauge equivalent to the system described by the parameters $\hat{b}_{ij}, \widetilde{\hat{\xi}}_i, \hat{\xi}_i$. Indeed, one can set $\alpha^{(i)} = \xi_i$ and

$$D^{\{i\}} = q^{b_{i_1i_2}+b_{i_2i_3}+\dots+b_{i_{k-1}i_k}-(k-1)}. \tag{7.35}$$

□

## 8. $(G,q)$-OPERS AND BAXTER RELATIONS

The space of $\mathfrak{g}$-opers on the punctured disc may be identified with the phase space of the Drinfeld-Sokolov reduction [DS]. Likewise, the space of $(G,q)$-opers on the punctured disc can be described in terms of the $q$-difference version of the Drinfeld-Sokolov reduction, which was defined in [FRSTS, SS]. For example, just as $\mathfrak{sl}_n$-opers can be represented as $n$th order differential operators, $(\mathrm{SL}(n),q)$-opers can be represented as $n$th order $q$-difference operators. In this section, we apply results of [FRSTS,SS] to construct a system of canonical coordinates for $q$-opers on $\mathbb{P}^1$ with regular singularities for all simply connected simple complex Lie groups $G$ other than $E_6$. We conjecture that if $G$ is simply laced, then the relations between these canonical coordinates and the polynomials $Q^i_+(z)$ introduced above are equivalent to the generalized Baxter $TQ$-relations established in [FH1].

8.1. **Presentations of unipotent subgroups.** Let $\widetilde{s}_i \in N(H)$ be a lifting of the simple reflection $w_i \in W_G$ (not necessarily equal to the liftings $s_i$ used before). Given $w \in W_G$, let $\widetilde{w} = \widetilde{s}_{i_1} \dots \widetilde{s}_{i_k}$ of $w$ ne the ocorresponding lifting Suppose that $w \in W_G$ has a reduced decomposition $w = w_{i_1} \dots w_{i_k}$ where the $w_{i_j}$'s are distinct, and consider the lifting $\widetilde{w} = \widetilde{s}_{i_1} \dots \widetilde{s}_{i_k}$ of $w$.

Let $X_\alpha$ denote the root subgroup corresponding to $\alpha$. Consider the subgroup

$$N_-^w = N_- \cap \widetilde{w} N_+ \widetilde{w}^{-1}.$$

It is well-known that $N_-^w$ is a subgroup of $N_-$ of dimension $\ell(w)$ [H, Section 28], which does not depend on the choice of the lifting $\widetilde{w}$. Moreover,

$$N_-^w = \prod_{\alpha<0, w\cdot\alpha>0} X_\alpha,$$

where the product can be taken in any order. In the case of a Coxeter element, or more generally, for any Weyl group element with a reduced decomposition consisting of distinct simple reflections, this group can be described more explicitly.

**Proposition 8.1.** *Let $w$ be a Weyl group element whose reduced decompositions consist of distinct simple reflections. Then, we have the presentation*

$$N_-^w = \widetilde{s}_{i_1} X_{\alpha_{i_1}} \dots \widetilde{s}_{i_r} X_{\alpha_{i_k}} \widetilde{w}^{-1}. \tag{8.1}$$

*Proof.* Consider the roots $\beta_j = w_{i_1} \dots w_{i_j}(\alpha_{i_j})$ for $j = 1, \dots, k = \ell(w)$. We will show that these are precisely the negative roots $\alpha$ such that $w^{-1}(\alpha)$ is positive. First, observe that each $\beta_j$ is negative. Indeed, $\beta_j = w_{i_1} \dots w_{i_{j-1}}(-\alpha_{i_j}) = -\alpha_{i_j} + \gamma_j$, where $\gamma_j$ is a linear combination of the $\alpha_{i_s}$'s for $1 \le s \le j-1$. Since the $\alpha_{i_s}$'s are distinct, the coefficient of $\alpha_{i_j}$ is negative, so $\beta_j$ is negative. Next, we show that the $\beta_j$'s are distinct. Suppose $\beta_j = \beta_s$ with $s > j$. Then, we have $\alpha_{i_j} = w_{i_{j+1}} \dots w_{i_s}(\alpha_{i_s})$. However, the root on the right is negative by the argument above, contradicting the fact that $\alpha_{i_j}$ is positive. Finally, $w^{-1}\beta_j = w_{i_k} \dots w_{i_{j+1}}(\alpha_{i_j})$, which is positive.

We conclude that $N_-^w = X_{\beta_1} \dots X_{\beta_k}$. The result now follows from the fact that

$$\begin{aligned} X_{\beta_1} \dots X_{\beta_k} &= \widetilde{s}_{i_1} X_{\alpha_{i_1}} \widetilde{s}_{i_1}^{-1} \widetilde{s}_{i_1} \widetilde{s}_{i_2} X_{\alpha_{i_1}} \widetilde{s}_{i_1}^{-1} \widetilde{s}_{i_2}^{-1} \dots \widetilde{w} X_{\alpha_{i_k}} \widetilde{w}^{-1} \\ &= \widetilde{s}_{i_1} X_{\alpha_{i_1}} \dots \widetilde{s}_{i_r} X_{\alpha_{i_k}} \widetilde{w}^{-1}. \end{aligned}$$

□

8.2. **Canonical coordinates on $q$-opers.** It follows from Definition 2.8 that a $(G,q)$-oper with regular singularities determined by the nonconstant polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ is an equivalence class of $q$-connections of the form

$$A(z) = n'(z) \prod_i (\Lambda_i(z)^{\check{\alpha}_i}\, s_i) n(z), \qquad n(z), n'(z) \in N_-(z),$$

under the action of $q$-gauge transformations by elements $N_-(z)$. We now describe a set of canonical representatives for these equivalence classes, using Theorem 3.1 from [SS]. Since this theorem was proved in [SS] for all simple Lie groups except $E_6$, in the rest of this section we will assume that $G \neq E_6$.

**Theorem 8.2.** *For every $A(z) \in n'(z)\prod_i(\Lambda_i(z)^{\check{\alpha}_i}\, s_i)n(z)$, there exists a unique $u'(z) \in N_-(z)$ such that*

$$u'(qz)A(z){u'}^{-1}(z) \in N_-^s(z)\prod_i(\Lambda_i(z)^{\check{\alpha}_i}\, s_i).$$

*Moreover, there exist unique $T_i(z) \in \mathbb{C}(z)$ such that*

$$u'(qz)A(z){u'}^{-1}(z) = \prod_i\Big[\Lambda_i(z)^{\check{\alpha}_i} s_i e^{-T_i(z)e_i}\Big]. \tag{8.2}$$

*As usual, the order in the product is determined by the Coxeter element $c$.*

*Proof.* Note that $\widetilde{s}_i = \Lambda_i(z)^{\check{\alpha}_i}\, s_i$ is a lifting of $w_i$ and $\widetilde{s} = \prod_i(\Lambda_i(z)^{\check{\alpha}_i}\, s_i)$ is the corresponding lifting of $c$. Let $\widetilde{N}_-^c = N_- \cap cN_-c^{-1}$. It is a standard fact that $N_- = N_-^c\widetilde{N}_-^c$ [H, Section 28]. If we write $n'(z) = u(z)v(z)$ with $u(z) \in N_-^c$ and $v(z) \in \widetilde{N}_-^c$, then

$$A(z) = u(z)\,\widetilde{s}\,\widetilde{s}^{-1}\,v(z)\,\widetilde{s}\,n(z) = u(z)\,\widetilde{s}\,\widetilde{n}(z),$$

for some $\widetilde{n}(z) \in N_-(z)$. Theorem 3.1 of [SS] now applies to show that there is a unique $u'(z)$ such that

$$u'(qz)A(z){u'}^{-1}(z) \in N_-^c(z)\widetilde{s}.$$

It then follows from Proposition 8.1 that

$$u'(qz)A(z){u'}^{-1}(z) = \prod_i \widetilde{s}_i x_{\alpha_i},$$

where $x_{\alpha_i} \in X_{\alpha_i}(z)$. Since $x_{\alpha_i} = e^{-T_i(z)e_i}$ for a unique $T_i(z) \in \mathbb{C}(z)$, we obtain the desired factorization. □

*Remark* 8.3. Note that the same proof applies if we consider general meromorphic $(G,q)$-opers instead of those with regular singularities, so that the polynomials $\Lambda_i(z)$ are replaced by arbitrary non-zero rational functions $\phi_i(z)$ as in (2.2).

Thus, we obtain a system of canonical representatives (8.2) for $(G,q)$-opers with regular singularities determined by $\{\Lambda_i(z)\}_{i=1,\dots,r}$ and hence, a system of canonical coordinates $(T_1(z),\dots,T_r(z))$ on the space of these $(G,q)$-opers.

### 8.3. Generalized Baxter relations.

Now, suppose that our $(G,q)$-oper has the structure of a nondegenerate $Z$-twisted Miura-Plücker $(G,q)$-oper. According to Theorem 6.4, it is then uniquely determined by the polynomials $\{Q_+^i(z)\}_{i=1,\dots,r}$ giving a nondegenerate solution of the Bethe Ansatz equations (6.16). Therefore, we can express the coordinates $T_j(z)$ of this $(G,q)$-oper in terms of $\{Q_+^i(z)\}_{i=1,\dots,r}$ and $\{\Lambda_i(z)\}_{i=1,\dots,r}$.

Consider the case $G = \mathrm{SL}_2$. Choose a canonical representative

$$\widetilde{A}(z) = \begin{pmatrix} 0 & \Lambda(z) \\ -\Lambda(z)^{-1} & \Lambda(z)T(z)\end{pmatrix}$$

for a $q$-gauge equivalence class as in (8.2). Suppose that the $q$-connection $A(z)$ given by formula (5.12) is in this equivalence class. Then, there exists $u(z) \in \mathbb{C}(z)$ such that

$$\begin{pmatrix} 1 & 0 \\ u(zq) & 1 \end{pmatrix} \begin{pmatrix} 0 & \Lambda(z) \\ -\Lambda(z)^{-1} & \Lambda(z)T(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u(z) & 1 \end{pmatrix} = \begin{pmatrix} \zeta Q_+(qz)Q_+^{-1}(z) & \Lambda(z) \\ 0 & \zeta^{-1}Q_+^{-1}(qz)Q_+(z) \end{pmatrix}. \tag{8.3}$$

Indeed, $u(z)$ is uniquely determined by the equation $u(z) = -\zeta\frac{Q_+(zq)}{Q_+(z)}\Lambda(z)^{-1}$. Substituting this into equation (8.3) gives

$$T(z) = \zeta\frac{1}{\Lambda(zq)}\frac{Q_+(zq^2)}{Q_+(zq)} + \zeta^{-1}\frac{1}{\Lambda(z)}\frac{Q_+(z)}{Q_+(zq)}. \tag{8.4}$$

This equation, which we have now obtained on the $q$DE side of the $q$DE/IM correspondence also makes perfect sense on the IM side; it is a version of the celebrated *Baxter $TQ$-relation* for the XXZ model. It relates the eigenvalues of the Baxter $Q$-operator and the eigenvalues of the transfer-matrix $T(z)$ of the two-dimensional evaluation representation of $U_{q'}\widehat{\mathfrak{sl}}_2$, suitably normalized, acting on a finite-dimensional representation $V$ of $U_{q'}\widehat{\mathfrak{sl}}_2$ (here $q' = q^{-2}$); see Example 5.13(i) of [FH1]. Note that up to multiplicative shifts by powers of $q$, $\Lambda(z)$ is the product of Drinfeld polynomials of the irreducible factors of $V$. Moreover, the Bethe Ansatz equations (5.5) follow from formula (8.4) if we assume that $T(z)$ does not have poles at the points $z = w_k q$, where $\{w_k\}$ are the zeros of $Q_+(z)$.

The fact that the relation between the canonical coordinates for $q$-opers and $Q_+(z)$ is equivalent to the Baxter $TQ$-relation was known before. Indeed, it was shown in [FRSTS] that the $q$-analogue of the Miura transformation arising from the $q$-Drinfeld-Sokolov reduction of SL(2) coincides with the formula for the $q$-character of the fundamental representation of $U_{q'}\widehat{\mathfrak{sl}}_2$. On the other hand, Baxter's $TQ$-relation can also be interpreted as this $q$-character, in which we substitute the ratio of the shifts of the Baxter polynomial just as in formula (8.4), see [FH1].

In fact, an analogue of the last statement for an arbitrary quantum affine algebra was conjectured in [FR2] and proved in [FH1]. On the other hand, according to the results and conjectures of [FR1, FR2], the link between the $q$-Miura transformation arising from $q$-Drinfeld-Sokolov reduction and the $q$-character homomorphism is expected to hold for all simply laced groups $G$.

We thus formulate the following conjecture. Recall (see [FH1]) that the generalized Baxter $TQ$-relation expresses the eigenvalues of the transfer-matrix of the $j$th fundamental representation of $U_{q'}\widehat{\mathfrak{g}}$ (suitably normalized) on the space of states determined by the Drinfeld polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$ in terms of the generalized Baxter polynomials $\{Q^i_+(z)\}_{i=1,\ldots,r}$ (up to multiplicative shifts by powers of $q$). (See [FH1] for the definition of the generalized Baxter polynomials.)

**Conjecture 8.4.** *Suppose that $G$ is simply laced. For any nondegenerate $Z$-twisted Miura-Plücker $(G,q)$-oper, the formula expressing its canonical coordinate $T_j(z)$ in terms of the corresponding polynomials $\{Q^i_+(z)\}_{i=1,\ldots,r}$ and $\{\Lambda_i(z)\}_{i=1,\ldots,r}$ coincides with the generalized Baxter $TQ$-relation.*

If $G$ is non-simply laced, the picture becomes more complicated. (See Conjecture 3, Section 6.3 and Appendix B in [FR1].) We hope to return to it in the future.

## References


[AFO] M. Aganagic, E. Frenkel, and A. Okounkov, *Quantum q-Langlands Correspondence*, Trans. Moscow Math. Soc. **79** (2018), 1–83, 1701.03146.

[B1] H. Bethe, *Zur Theorie der Metalle I. Eigenwerte und Eigenfunktionen der linearen Atomkette*, Z. Phys. **71** (1931), 205–226.

[B2] A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics (1991).

[BD1] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's Integrable System and Hecke Eigensheaves*, http://math.uchicago.edu/%7edrinfeld/langlands/QuantizationHitchin.pdf.

[BD2] A. Beilinson and V. Drinfeld, *Opers* (2005), math/0501398.

[BFL$^+$] V. Bazhanov, R. Frassek, T. Lukowski, C. Meneghelli, and M. Staudacher, *Baxter Q-Operators and Representations of Yangians*, Nucl. Phys. **B850** (2011), 148–174, 1010.3699.

[BHK] V. Bazhanov, A. Hibberd, and S. Khoroshkin, *Integrable Structure of W(3) Conformal Field Theory, Quantum Boussinesq Theory and Boundary Affine Toda Theory*, Nucl. Phys. **B622** (2002), 475–547, hep-th/0105177.

[BLZ] V. Bazhanov, S. Lukyanov, and A. Zamolodchikov, *Higher-level eigenvalues of Q-operators and Schrodinger equation*, Advances in Theoretical and Mathematical Physics **7** (2003), no. 4, 711–725, hep-th/0307108.

[CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997.

[DDT] P. Dorey, C. Dunning, and R. Tateo, *The ODE/IM Correspondence*, J.Phys. **A40** (2007), R205, hep-th/0703066.

[DHKM] A. Dey, A. Hanany, P. Koroteev, and N. Mekareeya, *On Three-Dimensional Quiver Gauge Theories of Type B*, JHEP **09** (2017), 067, 1612.00810.

[DS] V. Drinfeld and V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, Journal of Soviet Mathematics **30:2** (1985), 1975–2036.

[EP] C. Elliott and V. Pestun, *Multiplicative Hitchin Systems and Supersymmetric Gauge Theory* (2018), 1812.05516.

[F] R. Feynman, Asia-Pacific Physics News **3** (1988June/July), no. 22.

[FF] B. Feigin and E. Frenkel, *Quantization of soliton systems and Langlands duality*, in Exploration of New Structures and Natural Constructions in Mathematical Physics, pp. 185–274, Adv. Stud. Pure Math. **61**, Math. Soc. Japan, Tokyo (2011), 0705.2486.

[F1] E. Frenkel, *Affine algebras, Langlands duality and Bethe ansatz*, Proceedings of the International Congress of Mathematical Physics, Paris, 1994, ed. D. Iagolnitzer, pp. 606–642, International Press. (1995), q-alg/9506003.

[F2] E. Frenkel, *Opers on the projective line, flag manifolds and Bethe Ansatz*, Mosc. Math. J. **4** (2003), 655–705, math/0308269.

[F3] E. Frenkel, *Gaudin Model and Opers*, Infinite Dimensional Algebras and Quantum Integrable Systems, eds. P. Kulish, e.a., Progress in Math. **237** (2004), 1–60, math/0407524.

[FFR] B. Feigin, E. Frenkel, and N. Reshetikhin, *Gaudin Model, Bethe Ansatz and Critical Level*, Commun. Math. Phys. **166** (1994), 27–62, hep-th/9402022.

[FFTL] B. Feigin, E. Frenkel, and V. Toledano Laredo, *Gaudin models with irregular singularities*, Adv. Math. **223** (2010), 873–948, math/0612798.

[FG] E. Frenkel and B. Gross, *A Rigid Irregular Connection on the Projective Line*, Ann. of Math. (2) **170** (2009), no. 3, 1469–1512, 0901.2163.

[FH1] E. Frenkel and D. Hernandez, *Baxter's relations and spectra of quantum integrable models*, Duke Math. J. **164** (2015), no. 12, 2407–2460, 1308.3444.

[FH2] E. Frenkel and D. Hernandez, *Spectra of quantum KdV Hamiltonians, Langlands duality, and affine opers*, Commun. Math. Phys. **362** (2018), 361–414, 1606.05301.

[FHR] E. Frenkel, D. Hernandez, and N. Reshetikhin, *Folded quantum integrable models and deformed $\mathcal{W}$-algebras*, Lett. Math. Phys., to appear, 2110.14600.

[FR1] E. Frenkel and N. Reshetikhin, *Deformations of W-algebras associated to simple Lie algebras*, Commun. Math. Phys. **197** (1998), 1–32, q-alg/9708006.

[FR2] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras*, Contemp. Math., 248, Amer. Math. Soc. (1999), 163–205, math/9810055.

[FRSTS] E. Frenkel, N. Reshetikhin, and M. Semenov-Tian-Shansky, *Drinfeld–Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras I: The case of Virasoro algebra*, Commun. Math. Phys. **192** (1998), 605–629, q-alg/9704011.

[FZ] S. Fomin and A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380, math/9802056.

[GW] D. Gaiotto and E. Witten, *Knot invariants from four-dimensional gauge theory*, Adv. Theor. Math. Phys. **16** (2012), no. 3, 935–1086, 1106.4789.

[H] J. Humphreys, *Linear algebraic groups*, Springer, 1975.

[HJ] D. Hernandez and M. Jimbo, *Asymptotic Representations and Drinfeld Rational Fractions*, Compositio Mathematica **148** (2012), no. 5, 1593–1623, 1104.1891.

[KP] T. Kimura and V. Pestun, *Fractional quiver W-algebras*, Lett. Math. Phys. **108** (2018), no. 11, 2425–2451, 1705.04410.

[KSZ] P. Koroteev, D. Sage, and A. Zeitlin, *(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality*, Commun. Math. Phys. **381** (2021), 641–672, 1811.09937.

[MR] D. Masoero and A. Raimondo, *Opers for higher states of quantum KdV models* (2018), 1812.00228.

[MRV1] D. Masoero, A. Raimondo, and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case*, Commun. Math. Phys. **344** (2016), no. 3, 719–750, 1501.07421.

[MRV2] D. Masoero, A. Raimondo, and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie algebra–Valued Connections II: The Non Simply–Laced Case*, Communications in Mathematical Physics **349** (2017), no. 3, 1063–1105, 1511.00895.

[MV1] E. Mukhin and A. Varchenko, *Populations of solutions of the XXX Bethe equations associated to Kac-Moody algebras* (2002), math/0212092.

[MV2] E. Mukhin and A. Varchenko, *Discrete Miura Opers and Solutions of the Bethe Ansatz Equations*, Commun. Math. Phys. **256** (2005), 565–588, math/0401137.

[NRS] N. Nekrasov, A. Rosly, and S. Shatashvili, *Darboux coordinates, Yang-Yang functional, and gauge theory*, Nucl. Phys. Proc. Suppl. **216** (2011), 69–93, 1103.3919.

[NW] N. Nekrasov and E. Witten, *The Omega Deformation, Branes, Integrability, and Liouville Theory*, JHEP **1009** (2010), 092, 1002.0888.

[O] A. Okounkov, *Lectures on K-theoretic computations in enumerative geometry* (2015December), 1512.07363.

[OW] E. Ogievetsky and P. Wiegmann, *Factorized S-matrix and the Bethe ansatz for Simple Lie Groups*, Physics Letters B **168** (1986), no. 4, 360 –366.

[R] N. Reshetikhin, *The spectrum of the transfer matrices connected with Kac-Moody algebras*, Lett. Math. Phys. **14** (1987), 235–246.

[RV] N. Reshetikhin and A. Varchenko, *Quasiclassical asymptotics of solutions to the KZ equations* (1994), hep-th/9402126.

[RW] N. Reshetikhin and P. Weigmann, *Towards the classification of completely integrable quantum field theories (the Bethe-Ansatz associated with Dynkin diagrams and their automorphisms)*, Physics Letters B **189** (1987), no. 1, 125 –131.

[SS] M. Semenov-Tian-Shansky and A. Sevostyanov, *Drinfeld-Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras II. The general semisimple case*, Communications in Mathematical Physics **192** (1998), 631–647, q-alg/9702016.

[Z] A. Zeitlin, *Superopers on Supercurves*, Lett. Math. Phys. **105** (2015), no. 2, 149–167, 1311.5997.

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