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Maximal Dimension of Groups of Symmetries of Homogeneous 2-nondegenerate CR Structures of Hypersurface Type with a 1-dimensional Levi Kernel

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Abstract
We prove that for every $n \geq 3$ the sharp upper bound for the dimension of the symmetry groups of homogeneous, 2-nondegenerate, $(2n+1)$-dimensional CR manifolds of hypersurface type with a 1-dimensional Levi kernel is equal to n^2+7, and simultaneously establish the same result for a more general class of structures characterized by weakening the homogeneity condition. This supports Beloshapka's conjecture stating that hypersurface models with a maximal finite dimensional group of symmetries for a given dimension of the underlying manifold are Levi nondegenerate.

Keywords 2-nondegenerate CR structures · Homogeneous models · Infinitesimal symmetry algebra · Tanaka prolongation · Canonical forms in linear and multilinear algebra

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1 Introduction

A classical problem setting in differential geometry is to find homogeneous structures with the symmetry group of maximal dimension among all geometric structure of a certain class. Homogeneity here means, as usual, that the symmetry group of the structure acts transitively. In Cauchy-Riemann (CR) geometry this problem is classically solved for the class of Levi nondegenerate CR structures of hypersurface

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type of arbitrary dimension ([5, 17]). The present paper solves this problem for 2-nondegenerate CR structures of hypersurface type with a 1-dimensional Levi kernel. This class can be seen as the next one in a hierarchy of nondegeneracies to the class of Levi nondegenerate CR structures of hypersurface type. We furthermore obtain this result for structures that are not necessarily homogeneous, but that rather satisfy a weaker condition we term *admitting a constant reduced modified CR symbol (Definition 3.3 below)*. Previously the answer to this problem was given only in the 5-dimensional case [9, 11, 12], which is the case of the smallest possible dimension in which 2-nondegenrate structures exist. We give the answer for arbitrary dimension (which a priori is odd) greater than 5 extending the previous result of [13] that worked under additional restrictions of regularity of the CR symbol. The definition of the CR symbol and its regularity was introduced in [13] and is discussed in Section 2 below. This result supports Beloshapka's conjecture [9, Conjecture 5.6] stating that the hypersurface models with maximal finite dimensional groups of symmetries for a given dimension of the underlying manifold are Levi nondegenerate.

In more detail, let M be a $(2n+1)$-dimensional CR manifold with CR structure H of hypersurface type, meaning that H is an integrable, totally real, complex rank n distribution contained in the complexified tangent bundle $\mathbb{C}TM$ of M, that is,

$$[H, H] \subset H \quad \text{and} \quad H \cap \overline{H} = 0 \tag{1.1}$$

where the overline in $\overline{H}$ denotes the natural complex conjugation in $\mathbb{C}TM$.

Recall that the *Levi form* of the structure H is a field over M of Hermitian forms defined on fibers of H by the formula

$$\mathcal{L}(X_x, Y_x) := \frac{i}{2}\left[X, \overline{Y}\right]_x \mod H_x \oplus \overline{H}_x \qquad \forall\, X, Y \in \Gamma(H) \text{ and } x \in M. \tag{1.2}$$

Here we are using the notation $\Gamma(E)$ to denote sections of a fiber bundle E. The *kernel* of the Levi form $\mathcal{L}$ is called the *Levi kernel* and will be denoted by K. CR structures with $K = 0$ are called Levi-nondegenerate. It is classically known ([5, 17]) that for Levi-nondegenerate $(2n+1)$-dimensional structures with the Levi form of signature (p, q), where $p + q = n$, the algebra of infinitesimal symmetries of the maximally symmetric model is isomorphic to $\mathfrak{su}(p+1, q+1)$, having dimension $(n+2)^2 - 1$.

In the present paper we assume that the fiber K_x of the Levi kernel is 1-dimensional at every point $x \in M$, that is, K is a rank 1 distribution, and that the following nondegeneracy condition holds: If for $v \in K_x$ and $y \in \overline{H}_x/\overline{K}_x$, we take $V \in \Gamma(K)$ and $Y \in \Gamma(\overline{H})$ such that $V(p) = v$ and $Y(p) \equiv y \mod \overline{K}$, and define a linear map $\mathrm{ad}_v : \overline{H}_x/\overline{K}_x \to H_x/K_x$ by

$$\mathrm{ad}_v(y) := [V, Y]_x \mod K_x \oplus \overline{H}_x, \tag{1.3}$$

and similarly define a linear map $\mathrm{ad}_v : H_x/K_x \to \overline{H}_x/\overline{K}_x$ for $v \in \overline{K}_x$ (or simply take complex conjugates), then there is no nonzero $v \in K_x$ (equivalently, no nonzero $v \in \overline{K}_x$) such that $\mathrm{ad}_v = 0$. A CR structure is called *2-nondegenerate* if this last condition holds. The term *2-nondegeneracy* comes from the more general notion of *k-nondegeneracy*, see, for example, [3, 7, 10], [3, chapter XI] for the generalization of this definition to arbitrary $k \geq 1$ and arbitrary dimension of Levi kernels. The aforementioned CR symbols (defined in Section 2) are basic local invariants of (M, H) equivalent to the local invariants encoded in the Levi form and this family of ad_v operators.

The focus of the present paper is on finding the sharp upper bound for the dimension of the Lie group $\mathrm{Aut}(M, H)$ of symmetries of 2-nondegenerate CR structures (M, H) of hypersurface type with a 1-dimensional Levi kernel admitting a constant reduced modified symbol, which is a property with a rather technical definition given in Section 3 (Definition 3.3). Until we give the exact definition of this property, it will suffice to note that structures admitting constant reduced modified symbols are uniformly 2-nondegenerate and have constant CR symbols. In particular, if (M, H) is homogeneous then it admits a constant reduced modified symbol. As shown in [9, 11, 12] for the lowest dimensional case, that is when $\dim M = 5$, this sharp upper bound is equal to 10, and for the maximally symmetric model the algebra of infinitesimal symmetries is isomorphic to $\mathfrak{so}(3, 2)$. The main result here, see Theorem 2.3 below, gives this sharp upper bound expressed as a function of $\dim M \geq 7$ (equivalently, $n = \frac{1}{2}(\dim M - 1) \geq 3$), namely

$$\dim \mathrm{Aut}(M, H) \leq \frac{1}{4}(\dim M - 1)^2 + 7 = n^2 + 7. \tag{1.4}$$

We also show that symmetries of (M, H) are all determined by their third weighted jet. By the *weighted jet* we mean that the derivatives in various directions are calculated according to the filtration

$$(K \oplus \overline{K}) \cap TM \subset (H \oplus \overline{H}) \cap TM \subset TM$$

of TM so that each derivative in a direction in $(K \oplus \overline{K}) \cap TM$ is assigned weight zero, each derivative in a direction in $\big((H \oplus \overline{H}) \setminus (K \oplus \overline{K})\big) \cap TM$ is assigned weight 1, and each derivative in a direction in $TM \setminus H \oplus \overline{H}$ is assigned weight 2. These results (even without assumption of homogeneity) were previously obtained in [13] for the special class of CR structures whose symbols are known as *regular*, wherein it was shown by example that the upper bound in (1.4) is achieved.

The essential technical bulk of this paper consists of showing that the dimension of $\mathrm{Aut}(M, H)$ for structures with non-regular symbol is strictly less than the right side of (1.4) (in fact it is shown in Theorem 3.7 below that it is strictly less than $(n-1)^2+7$) and that in the non-regular case symmetries of (M, H) are all determined by their first weighted jet. The notion of CR symbols and their regularity is explained in Section 2. Note that, for the considered case $n \geq 3$, the previously treated regular symbols constitute only a finite subset in the space of all CR symbols for each n, which itself depends on continuous parameters.

In the proof of the bound (1.4) we use two results from our previous papers [14] and [15]: the classification of CR symbols [14] and the description of the upper bound for the dimension of symmetry groups in terms of a Tanaka prolongation of the symbol or its reduced version [15]. In the sequel, we calculate these prolongations and their dimensions for each reduced modified symbol corresponding to a non-regular CR symbol. In particular, we show (Theorem 3.7) that the first Tanaka prolongation of each reduced modified symbol corresponding to a non-regular CR symbol is equal to zero and we find the upper bound for the dimension of its Tanaka prolongation. Analogous analysis for regular CR symbols was previously obtained in [13] with the help of the theory of bigraded Tanaka prolongation. The result on the jth-jet determinacy follows from its equivalence to the vanishing of the jth Tanaka prolongation. In Theorem 5.11 for each reduced modified symbol corresponding to a non-regular CR symbol we give more precise upper bound for the dimension of its Tanaka prolongation in terms of the parameters of this non-regular symbol.

Note that at this moment for structures with non-regular CR symbols (and therefore in the general case) we are not able to remove completely the assumption of admitting a constant reduced modified symbol in our results, as this assumption implies that the reduced modified symbols are Lie algebras, and we strongly use the latter fact. So the question of whether or not there exist CR structures from the considered class not admitting a constant reduced modified symbol (Definition 3.3) and with symmetry group of dimension higher than the bound in (1.4) is still open, although the positive answer to this question is highly unlikely.

In the very recent paper [4] it was shown that for $\dim M = 7$, without the homogeneity assumption, the upper bound for the dimension of the group of symmetries of 2-nondegenerate CR structures of hypersurface type with a 1-dimensional Levi kernel is 17. Our sharp bound (1.4) for the homogeneous case is 16 and an example of the structure from the considered class with 17-dimensional symmetry group is unknown. The result of the present paper (communicated in a private correspondence) was in fact used in [4] to reduce the bound from 18, obtained initially by the methods of normal forms, to 17, see Proposition 16 there.

In contrast to the case of $\dim M = 5$, in the case where M is of (odd) dimension greater than or equal to 7, the infinitesimal symmetry algebras of the maximally symmetric homogeneous models are not semisimple. These algebras were calculated in some form in [13, Subsection 5.3]. A more visual description together with a hypersurface realization of these models will feature in future joint work [6].

In the case where $\dim M = 7$, the infinitesimal symmetry algebra of the maximally symmetric models is isomorphic to one of the real forms of the following complex Lie algebra: Let $\mathfrak{s} = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. The complexification of our algebra of interest is isomorphic to the natural semidirect sum of $\mathfrak{s}$ and the 9-dimensional abelian Lie algebra $\mathbb{C}^9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3$ so that the first $\mathfrak{sl}(2, \mathbb{C})$ component in $\mathfrak{s}$ acts irreducibly on the first factor $\mathbb{C}^3$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$, the second component $\mathfrak{sl}(2, \mathbb{C})$ in $\mathfrak{s}$ acts irreducibly on the second factor of $\mathbb{C}^3$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$, and the component $\mathbb{C}$ in $\mathfrak{s}$ acts just by rescaling. The desired real Lie algebra is the natural semidirect sum of the conformal Lorenzian algebra $\mathfrak{co}(3, 1)$ and the 9-dimensional real abelian Lie algebra $\mathbb{R}^9$, where $\mathfrak{co}(3, 1)$ acts irreducibly on $\mathbb{R}^9$. This unique irreducible action is naturally induced from the standard action of $\mathfrak{co}(3, 1)$ on the Minkowski space, if one identifies $\mathbb{R}^9$ with the space of the traceless symmetric bilinear forms on the Minkowski space.

Finally, for completeness, we offer without proof the (local) hypersurface realizations of the maximally symmetric homogeneous models in the considered class (the details will be given in [6]). If, as before, $n = \frac{1}{2}(\dim M - 1)$, and the signature of the form obtained by the reduction of the Levi form at each point x to the space H_x/K_x is equal to (p, q) with $p + q = n - 1$, then in coordinates $(z_1, \dots, z_n, w)$ for $\mathbb{C}^{n+1}$

these are the hypersurfaces are given by the equation

$$\operatorname{Im}\left(w + z_1^2\bar{z}_n\right) = z_1\bar{z}_2 + \bar{z}_1 z_2 + \sum_{i=3}^{n-1} \varepsilon_i z_i \bar{z}_i, \tag{1.5}$$

where $\varepsilon_i \in \{-1, 1\}$ and $\{\varepsilon_i\}_{i=3}^{n-1}$ consists of $p-1$ terms equal to 1 and $q-1$ terms equal to -1 (note that, for $\dim M = 7$, the last sum in the right side of (1.5) disappears).

2 CR Symbols and the Main Results

Our analysis branches depending on properties of the CR structure's local invariants. A basic local invariant of a hypersurface-type CR structure called the *CR symbol* is introduced in [13]. The CR symbol of H (at a point x in M) is a bigraded vector space

$$\mathfrak{g}^0 := \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_{0,-2} \oplus \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,2} \tag{2.1}$$

with involution $\bar{\ }$ whose bigraded components $\mathfrak{g}_{i,j}$ are defined as follows. Ultimately our definitions of $\mathfrak{g}_{i,j}$ will not depend on the point x because going forward we will consider only structures with constant CR symbols, but we still fix x to state the initial definitions. We let ℓ denote the *reduced Levi form*, which is the field of nondegenerate Hermitian forms defined on fibers of the quotient bundle H/K by

$$\ell(X_x + K_x) := \mathcal{L}(X_x).$$

We define the coset spaces

$$\mathfrak{g}_{-2,0} := \mathbb{C}T_xM/H_x, \quad \mathfrak{g}_{-1,-1} := \overline{H}_x/\overline{K}_x, \quad \text{and} \quad \mathfrak{g}_{-1,1} := H_x/K_x.$$

The space

$$\mathfrak{g}_- := \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \tag{2.2}$$

inherits a Heisenberg algebra structure with nontrivial Lie brackets defined in terms of the reduced Levi form by

$$[v, w] := i\ell(v, w) \qquad \forall v \in \mathfrak{g}_{-1,1},\ w \in \mathfrak{g}_{-1,-1}.$$

Note that ℓ formally takes values in $\mathfrak{g}_{-2,0}$. By identifying $\mathfrak{g}_{-2,0}$ and $\mathbb{C}$, we regard ℓ as a $\mathbb{C}$-valued Hermitian form, but, since this identification is not naturally determined by the CR structure, in the sequel we consider the real line $\mathbb{R}\ell$ of $\mathbb{C}$-valued Hermitian forms spanned by ℓ. While the one $\mathbb{C}$-valued form ℓ is not an invariant of the CR structure, the line $\mathbb{R}\ell$ is.

To define $\mathfrak{g}_{0,2}$, we consider special operators associated with vectors in K_x. For a vector v in K_x, define the antilinear operator $A_v : \mathfrak{g}_{-1,1} \to \mathfrak{g}_{-1,1}$ by

$$A_v(x) := \operatorname{ad}_v(\overline{x}). \tag{2.3}$$

The dependence of A_v on v is linear, that is,

$$A_{\lambda v} = \lambda A_v \qquad \forall \lambda \in \mathbb{C},$$

so if the rank of K is equal to 1 then there exists an antilinear operator $\boldsymbol{A}$ such that

$$\{A_v \mid v \in K_x\} = \mathbb{C}\boldsymbol{A}.$$

The fact that H is 2-nondegenerate implies that $\boldsymbol{A} \neq 0$.

The reduced Levi form ℓ naturally extends to define a symplectic form on the space $\mathfrak{g}_{-1} := \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}$ via a standard construction from the study of Heisenberg algebras. Hence $\mathfrak{g}_{-1}$ inherits a symplectic structure from the CR structure with respect to which we obtain the conformal symplectic algebra $\mathfrak{csp}(\mathfrak{g}_{-1})$ defined in the standard way. We define $\mathfrak{g}_{0,2}$ to be the subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ given by the formula

$$\mathfrak{g}_{0,2} := \left\{ \varphi : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1} \,\middle|\, \begin{array}{l} varphi(v) = 0 \quad \forall v \in \mathfrak{g}_{-1,1} \text{ and} \\ \text{there exists } \lambda \in \mathbb{C} \text{ such that} \\ \varphi(v) = \lambda \boldsymbol{A}(\overline{v}) \quad \forall v \in \mathfrak{g}_{-1,-1} \end{array} \right\}.$$

The natural complex conjugation on $\mathbb{C}T_x M$ induces an antilinear involution $v \mapsto \overline{v}$ on $\mathfrak{g}_{-1}$, which in turn induces an antilinear involution on $\mathfrak{csp}(\mathfrak{g}_{-1})$ by the rule

$$\overline{\varphi}(v) := \overline{\varphi(\overline{v})}. \tag{2.4}$$

Using this involution, we define

$$\mathfrak{g}_{0,-2} := \{\varphi \mid \overline{\varphi} \in \mathfrak{g}_{0,2}\}.$$

Lastly, using the standard Lie brackets of $\mathfrak{csp}(\mathfrak{g}_{-1})$ we define

$$\mathfrak{g}_{0,0} := \left\{ v \in \mathfrak{csp}(\mathfrak{g}_{-1}) \,\middle|\, [v, \mathfrak{g}_{0,i}] \subset \mathfrak{g}_{0,i} \quad \forall\, i \in \{-2, 2\} \right\}, \tag{2.5}$$

which completes our definition of the CR symbol $\mathfrak{g}^0$ of H (at the point x). Note that by construction

$$[\mathfrak{g}_{i_1,j_1}, \mathfrak{g}_{i_2,j_2}] \subset \mathfrak{g}_{i_1+i_2,j_1+j_2}, \qquad \forall\, \{(i_1, j_1), (i_2, j_2)\} \neq \{(0, 2), (0, -2)\}. \tag{2.6}$$

Conversely a vector space $\mathfrak{g}^0$ as in (2.1) with $\mathfrak{g}_-$ as in (2.2) being the Heisenberg algebra is called an *abstract CR symbol* for 2-nondegenerate, hypersurface-type CR structures if it satisfies (2.6), $\mathfrak{g}_{0,0}$ is the maximal subalgebra of $\mathfrak{csp}(\mathfrak{g}_-)$ satisfying (2.5), and it is endowed with an antilinear involution $^-$ satisfying (2.4).

Remark 2.1 *The CR symbol $\mathfrak{g}^0$ of a CR structure with a 1-dimensional kernel encodes and is encoded by the pair* $(\mathbb{R}\ell, \mathbb{C}\boldsymbol{A})$.

Note that an abstract CR symbol $\mathfrak{g}^0$ is not necessarily a Lie algebra, as the bigrading conditions in (2.6) are only applied for $\{(i_1, j_1), (i_2, j_2)\} \neq \{(0, 2), (0, -2)\}$, so that $[\mathfrak{g}_{0,-2}, \mathfrak{g}_{0,2}]$ does not necessarily belong to $\mathfrak{g}_{0,0}$ and therefore does not necessarily belong to $\mathfrak{g}^0$. Following the terminology of [13], we say that a CR symbol is *regular* if it is a subalgebra of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_-)$ and *non-regular* otherwise. As shown in [13, Lemma 4.2], the symbol $\mathfrak{g}^0$ of a CR structure with a 1-dimensional kernel corresponding to the pair $(\mathbb{R}\ell, \mathbb{C}\boldsymbol{A})$ is regular if and only if

$$\boldsymbol{A}^3 \in \mathbb{C}\boldsymbol{A}. \tag{2.7}$$

It is shown in [13] that, to any abstract regular CR symbol $\mathfrak{g}^0$, there is a corresponding special homogeneous CR structure fully characterized as the unique structure with the given symbol at every point whose infinitesimal symmetry algebra attains a certain upper bound. This structure is called the *flat CR structure with constant CR symbol* $\mathfrak{g}^0$. As a consequence of [13], see Theorems 3.2, 5.1, 5.3 and the last paragraph of section 5 there, one gets the following theorem.

Theorem 2.2 (Porter and Zelenko [13]) *If (M, H) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel and constant regular symbol, then*

(*1*) *the dimension of the algebra of infinitesimal symmetries of (M, H) is not greater than $\frac{1}{4}(\dim M - 1)^2 + 7$;*
(2) *these symmetries are determined by their third weighted jet;*
(*3*) *the dimension of the algebra of infinitesimal symmetries of (M, H) is equal to $\frac{1}{4}(\dim M - 1)^2 + 7$ if and only if (M, H) has the flat structure (defined in* [13]*) with CR symbol such that the corresponding line of antilinear operators consists of nilpotent ones of rank 1.*

A natural question is whether or not the assumption of regularity of symbol can be removed in the previous theorem. Addressing this question, the main result of the present paper is the following.

Theorem 2.3 *If (M, H) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel admitting a constant reduced modified symbol as in Definition 3.3 (and, in particular, if it is homogeneous), then*

(*1*) *statements (1) and (3) of Theorem 2.2 are valid;*
(2) *if the symbol is non-regular then the (infinitesimal) symmetries of (M, H) are determined by their first weighted jet.*

The proof of this theorem is given in Sections 3 through 6. In Section 3 we give the scheme of the proof of this theorem, based on the constructions and results of our previous paper [15], namely the construction of reduced modified symbols for sufficiently symmetric CR structures and the application of Tanaka prolongation of these reduced modified symbols to obtain an upper bound for the dimension of their infinitesimal symmetry algebras (see Theorem 3.6 below). In this way Theorem 2.3 will be essentially reduced to Theorem 3.7. The latter theorem is proved in Section 6 with the help of the Section 5. In this proof we also use the classification of symbols from our previous paper [14] and the system of matrix equations for the reduced modified symbols derived in [15, section 5]. The latter two topics are briefly reviewed in Section 4 below.

3 Reduced Modified Symbol and the Significance of Its Tanaka Prolongation

Now we will discuss the scheme of the proof of Theorem 2.3, based on the constructions and results of our previous paper [15]. In particular, there we introduced other local invariants of sufficiently symmetric hypersurface-type CR structures encoded in objects called *modified CR symbols* and *reduced modified CR symbols* (see sections 4 and 6 of [15], respectively). Although modified and reduced modified CR symbols are defined in [15], we outline their definitions here for completeness because these objects (especially the latter one) are both nonstandard and fundamental for the present study. Some technical details that are not essential for understanding the principal concepts are omitted here and we refer to [15] for those gaps. Following these definitions, we introduce Theorem 3.7, and describe how Theorem 2.3 essentially follows from Theorem 3.7. The subsequent sections of this paper are dedicated to the proof of Theorem 3.7.

Proceeding, we assume that (M, H) has a constant CR symbol. Let $\mathfrak{g}^0$ be an abstract CR symbol isomorphic to the CR symbol $\mathfrak{g}^0(x)$ of (M, H) at every point x in M. And write $\mathfrak{g}_{i,j}(x)$ to denote the bigraded components of $\mathfrak{g}^0(x)$.

There is a natural way to locally *complexify* M by working in local coordinates and replacing real coordinates with complex ones, and, moreover, the CR structure H, as well as the distributions $\overline{H}$, K, and $\overline{K}$, naturally extend to this complexified manifold (see [15] for full details) yielding a so-called *complexified CR manifold* that we denote by $\mathbb{C}M$ (a detail omitted here is that, since the construction is local, this may only be well defined after replacing M with some neighborhood in M). Note that $\dim_{\mathbb{R}}(\mathbb{C}M) = 2\dim(M)$ and there is a submanifold in $\mathbb{C}M$ that can be naturally identified with M. The distribution $K + \overline{K}$ on $\mathbb{C}M$ is involutive. We let $\mathcal{N}$ be the leaf space of the foliation of $\mathbb{C}M$ generated by $K + \overline{K}$, sometimes called the *Levi leaf space*, and let $\pi : \mathbb{C}M \to \mathcal{N}$ denote the natural projection. That is, points in $\mathcal{N}$ are maximal integral submanifolds of $K + \overline{K}$ in $\mathbb{C}M$.

From the resulting construction, $\mathfrak{g}^0(x)$ remains well defined (in terms of H) for all x in $\mathbb{C}M$. We define the fiber bundle $\mathrm{pr} : P^0 \to \mathbb{C}M$ whose fiber $\mathrm{pr}^{-1}(x)$ over a point x in $\mathbb{C}M$ is comprised of what we call *adapted frames*, that is,

$$\mathrm{pr}^{-1}(x) = \left\{ \varphi : \mathfrak{g}_- \to \mathfrak{g}_-(x) \;\middle|\; \begin{array}{l} \varphi(\mathfrak{g}_{i,j}) = \mathfrak{g}_{i,j}(x) \quad \forall\, (i,j) \in \{(-1,\pm1), (-2,0)\}, \\ \varphi^{-1} \circ \mathfrak{g}_{0,\pm2}(x) \circ \varphi = \mathfrak{g}_{0,\pm2}, \text{ and} \\ \varphi([y_1, y_2]) = [\varphi(y_1), \varphi(y_2)] \quad \forall\, y_1, y_2 \in \mathfrak{g}_- \end{array} \right\}. \tag{3.1}$$

We also consider a second fiber bundle $\pi \circ \mathrm{pr} : P^0 \to \mathcal{N}$, a bundle with total space P^0 and base space $\mathcal{N}$.

For any $\psi \in P^0$ and $\gamma = \pi \circ \mathrm{pr}(\psi)$, the tangent space of the fiber $(P^0)_\gamma = (\pi \circ \mathrm{pr})^{-1}(\gamma)$ of the second bundle at ψ can be identified with a subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ by the map $\theta_0 : T_\psi(P^0)_\gamma \to \mathfrak{csp}(\mathfrak{g}_{-1})$ given by

$$\theta_0\left(\psi'(0)\right) := (\psi(0))^{-1}\psi'(0) \tag{3.2}$$

where $\psi : (-\epsilon, \epsilon) \to (P^0)_\gamma$ denotes an arbitrary curve in $(P^0)_\gamma$ with $\psi(0) = \psi$. The notation θ_0 is used here to match the notation in [15]. Let

$$\mathfrak{g}_0^{\mathrm{mod}}(\psi) := \theta_0(T_\psi(P^0)_\gamma). \tag{3.3}$$

Definition 3.1 *The space* $\mathfrak{g}^{0,\mathrm{mod}}(\psi) := \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{mod}}(\psi)$ *is called the modified CR symbol of the CR structure* H *at the point* $\psi \in P^0$.

Remark 3.2 *Modified CR symbols depend on points in the bundle P^0 rather than points in the original CR manifold. Accordingly, a modified CR symbol is not itself a local invariant of the CR structure from which it arises, but rather, for $x \in M$, the set $\{\mathfrak{g}^{0,\mathrm{mod}}(\psi) \mid \mathrm{pr}(\psi) = x\}$ is a local invariant at x. This invariant encodes more data than is encoded in the corresponding CR symbol.*

We consider the map $\psi \mapsto \varphi_0(\psi)$ sending each point in P^0 to a subspace of $\mathfrak{csp}(\mathfrak{g}_-)$. If, for some subspace $\widetilde{\mathfrak{g}_0} \subset \mathfrak{csp}(\mathfrak{g}_-)$, there is a maximal connected submanifold $\widetilde{P^0}$ of P^0 belonging to the level set

$$\left\{\psi \in P^0 \,\middle|\, \theta_0\left(T_\psi\left(P^0\right)_{\pi\circ\mathrm{pr}(\psi)}\right) = \widetilde{\mathfrak{g}_0}\right\}$$

such that $\mathrm{pr}(\widetilde{P^0}) = \mathbb{C}M$, then we call $\widetilde{P^0}$ a reduction of P^0. After, replacing P^0 and θ_0 with $\widetilde{P^0}$ and the restriction of θ_0 to the vertical tangent vectors of $\pi\circ\mathrm{pr} : \widetilde{P^0} \to \mathcal{N}$, we can repeat this reduction procedure by finding a maximal connected submanifold of P^0 that is in the level set of the new mapping $\psi \mapsto \theta_0\left(T_\psi\left(\widetilde{P^0}\right)_{\pi\circ\mathrm{pr}(\psi)}\right)$ also covering $\mathbb{C}M$ under the projection pr, which we again call a reduction of P^0. In general this reduction procedure can be repeated many times, and eventually terminates in the sense that iterating the reduction procedure again will not yield new reductions. For a reduction $P^{0,\mathrm{red}}$ of P^0 we label the corresponding space

$$\mathfrak{g}_0^{\mathrm{red}}(\psi) := \theta_0\left(T_\psi\left(P^{0,\mathrm{red}}\right)_{\pi\circ\mathrm{pr}(\psi)}\right) \qquad \forall\,\psi \in P^{0,\mathrm{red}}. \tag{3.4}$$

Definition 3.3 *If $P^{0,\mathrm{red}}$ is a reduction of P^0 then the space $\mathfrak{g}^{0,\mathrm{red}}(\psi) := \mathfrak{g}_- \oplus \mathfrak{g}_0^{\mathrm{red}}(\psi)$, with $\mathfrak{g}_0^{\mathrm{red}}(\psi)$ given by (3.4), is called a reduced modified CR symbol of the CR structure H at ψ. We say that H admits a constant reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ if there exists a reduction $P^{0,\mathrm{red}}$ of P^0 together with $\mathfrak{g}_0^{\mathrm{red}}(\psi)$ given by (3.4) such that*

$$\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}^{0,\mathrm{red}}(\psi) \qquad \forall\,\psi \in P^{0,\mathrm{red}}.$$

Lemma 3.4 *If (M, H) is homogeneous then it admits a constant reduced modified symbol, that is, there exists a reduction $P^{0,\mathrm{red}}$ of P^0 such that the map $\psi \mapsto \mathfrak{g}_0^{\mathrm{red}}(\psi)$ given by (3.4) is constant.*

Proof Since (M, H) is homogeneous, so is P^0, and hence each reduction $\widetilde{P^0}$ of P^0 can be taken so that its fibers $\left(\widetilde{P^0}\right)_x := \left\{\psi \in \widetilde{P^0} \mid \pi(\psi) = x\right\}$ have the same image under the mapping $\psi \mapsto \theta_0\left(T_\psi \widetilde{P^0}\right)$. Therefore, if $\psi \mapsto \theta_0\left(T_\psi \widetilde{P^0}\right)$ is not already constant on $\widetilde{P^0}$ then we can repeat the reduction procedure to find a proper submanifold of $\widetilde{P^0}$ that is also a reduction of P^0. Eventually, this iterated procedure ends with a reduction for which either the image of θ_0 applied to its tangent spaces is constant, or its fibers are 0-dimensional. Yet, by homogeneity, the map $\psi \mapsto \theta_0\left(T_\psi P^{0,\mathrm{red}}\right)$ would be constant in the latter case as well. $\square$

For the remainder of this paper, we let $\mathfrak{g}^{0,\mathrm{red}}$ denote a constant reduced modified CR symbol of H. Like the CR symbol of H, $\mathfrak{g}^{0,\mathrm{red}}$ is also a graded subspace of $\mathfrak{g}_- \rtimes \mathfrak{csp}(\mathfrak{g}_{-1})$. It has the decomposition $\mathfrak{g}^{0,\mathrm{red}} = \mathfrak{g}_{-2,0} \oplus \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1} \oplus \mathfrak{g}_0^{\mathrm{red}}$ where the components whose first weight is negative coincide with those of the CR symbol. Here we state some of the properties of $\mathfrak{g}_0^{\mathrm{red}}$. For this we consider weighted components of $\mathfrak{csp}(\mathfrak{g}_{-1})$ defined by

$$(\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,i} = \left\{\varphi \in \mathfrak{csp}(\mathfrak{g}_{-1}) \,\middle|\, \varphi(\mathfrak{g}_{-1,j}) \subset \mathfrak{g}_{-1,i+j}\, \forall j \in \{-1, 1\}\right\}.$$

The space $\mathfrak{g}_0^{\mathrm{red}}$ is a subspace of $\mathfrak{csp}(\mathfrak{g}_{-1})$ with a decomposition

$$\mathfrak{g}_0^{\mathrm{red}} = \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,-}^{\mathrm{red}} \oplus \mathfrak{g}_{0,+}^{\mathrm{red}} \tag{3.5}$$

such that

(1) $\mathfrak{g}_{0,0}^{\mathrm{red}} \subset \mathfrak{g}_{0,0}$;
(2) $\mathfrak{g}_{0,+}^{\mathrm{red}} = \overline{\mathfrak{g}_{0,-}^{\mathrm{red}}}$;
(3) the natural projection of $\mathfrak{csp}(\mathfrak{g}_{-1})$ onto $(\mathfrak{csp}(\mathfrak{g}_{-1}))_{0,2}$ defines an isomorphism between $\mathfrak{g}_{0,+}^{\mathrm{red}}$ and $\mathfrak{g}_{0,2}$;
(4) The subspace $\mathfrak{g}_0^{\mathrm{red}}$ is invariant with respect to the involution on $\mathfrak{csp}(\mathfrak{g}_{-1})$
(5) The subspace $\mathfrak{g}_0^{\mathrm{red}}$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$.

We stress that the decomposition $\mathfrak{g}_0^{\mathrm{red}} = \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,-}^{\mathrm{red}} \oplus \mathfrak{g}_{0,+}^{\mathrm{red}}$ satisfying these properties is not unique, and, furthermore, no such splitting is naturally determined by the CR structure.

Remark 3.5 *The CR symbol of (M, H) is determined by any of its modified CR symbols, which in turn are all determined by any constant reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ that (M, H) admits.*

The underlying theory that we will apply to treat structures with non-regular CR symbols is developed in [15], wherein it is shown that the upper bounds that we wish to compute can be found by computing the universal Tanaka prolongation [18] of $\mathfrak{g}^{0,\mathrm{red}}$, which is defined as follows. Starting with $k = 1$ and setting $\mathfrak{g}_{-2} = \mathfrak{g}_{-2,0}$, we recursively define the vector spaces

$$\mathfrak{g}_k^{\mathrm{red}} := \left\{\varphi \in \bigoplus_{i=-2}^{-1} \mathrm{Hom}(\mathfrak{g}_i, \mathfrak{g}_{i+k}) \,\middle|\, \begin{array}{l} \varphi([v_1, v_2]) = [\varphi(v_1), v_2] + [v_1, \varphi(v_2)] \\ \forall\, v_1, v_2 \in \mathfrak{g}_- \end{array}\right\} \quad \forall k \geq 1, \tag{3.6}$$

The *universal Tanaka prolongation* of $\mathfrak{g}^{0,\mathrm{red}}$ is the vector space

$$\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}}) := \mathfrak{g}_- \oplus \bigoplus_{k\geq 0} \mathfrak{g}_k^{\mathrm{red}}. \tag{3.7}$$

Theorem 3.6 *[follows immediately from* [15, Corollary 2.8 and Theorem 6.2]*] If (M, H) is a 2-nondegenerate CR structure of hypersurface type with a 1-dimensional Levi kernel and constant reduced modified symbol $\mathfrak{g}^{0,\mathrm{red}}$, then the dimension of the algebra of infinitesimal symmetries of (M, H) is not greater than* $\dim \mathfrak{u}\left(\mathfrak{g}^{0,\mathrm{red}}\right)$.

Hence, if we can explicitly calculate $\dim \mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ for non-regular CR symbols, then we can obtain an upper bound for the algebra of infinitesimal symmetries of (M, H). This motivates the following theorem, proved in Section 6.

Theorem 3.7 *If a constant reduced modified CR symbol $\mathfrak{g}^{0,\mathrm{red}}$ corresponds to a non-regular CR symbol then the following statements hold:*

(*1*) *The first Tanaka prolongation $\mathfrak{g}_1^{\mathrm{red}}$ of $\mathfrak{g}^{0,\mathrm{red}}$ vanishes or, equivalently, the universal Tanaka prolongation $\mathfrak{u}(\mathfrak{g}^{0,\mathrm{red}})$ of $\mathfrak{g}^{0,\mathrm{red}}$ is equal to $\mathfrak{g}^{0,\mathrm{red}}$.*

(*2*) $\dim \mathfrak{g}^{0,\mathrm{red}}$ *and therefore the dimension of the algebra of infinitesimal symmetries of a $(2n+1)$-dimensional 2-nondegenerate CR structure of hypersurface type with rank 1 Levi kernel and non-regular CR symbol admitting a constant reduced modified symbol is strictly less than $(n-1)^2+7$.*

(*3*) *For (M, H) as in item (2), the bundle $\mathrm{pr} : \mathfrak{R}(P^0) \to M$, consisting of frames in P^0 that commute with complex conjugation on the CR symbols, is a principal bundle over M whose structure group has the Lie algebra $\mathfrak{g}_{0,0}^{\mathrm{red}}$ and it is equipped with an absolute parallelism invariant under the structure group's action and under the natural induced action of symmetries of (M, H).*

Corollary 3.8 *The dimension of the algebra of infinitesimal symmetries of a homogeneous $(2n+1)$-dimensional 2-nondegenerate CR structure of hypersurface type with rank 1 Levi kernel and non-regular CR symbol is strictly less than $(n-1)^2+7$.*

Theorem 3.7 is proved in Section 6 with the help of preliminary results established in Sections 4 and 5. In Section 4, we introduce a standardized matrix representation of abstract reduced modified symbols, which is necessary for our study because there is no previously developed structure theory for these Lie algebras. In Section 5, we give explicit general formulas for matrix representations of elements in $\mathfrak{g}_{0,0}^{\mathrm{red}}$, and we use these formulas to calculate upper bounds for the dimension of $\mathfrak{g}^{0,\mathrm{red}}$, which are necessary for item (2) of Theorem 3.7. Lastly, in Section 6, we apply the matrix representation formulas derived in Section 5 to prove item (1) of Theorem 3.7 by directly calculating $\mathfrak{g}_1^{\mathrm{red}} = 0$.

Based on the well-known fact [18, Section 6] that an infinitesimal symmetry of a filtered structure is determined by the jth weighted jet, where j is the minimal non-negative integer for which the jth Tanaka prolongation is equal to zero, this theorem immediately implies item (2) of Theorem 2.3. Item (1) of Theorem 2.3 will follow from combining Theorems 3.7 and 3.6. In Theorem 5.11 below, for each reduced modified symbol corresponding to a non-regular CR symbol, we give more precise upper bounds (than the ones in item (2) of Theorem 3.7) for the dimension of its (entire) Tanaka prolongation in terms of the parameters of this non-regular symbol.

Remark 3.9 *To establish Theorem 3.7, we appeal to Theorem 3.6 and the Tanaka-theoretic prolongation procedures developed in [15] which constructs a tower $\mathfrak{R}(P^s) \to \mathfrak{R}(P^{s-1}) \to \cdots \to \mathfrak{R}(P^0) \to M$ of fiber bundles (geometric prolongations) and confers an absolute parallelism onto the largest prolongation $\mathfrak{R}(P^s)$. The familiar reader will notice that item (1) in Theorem 3.7 implies that $\mathfrak{R}(P^0)$ is* *diffeomorphic to the largest prolongation, and may wonder if we can construct a parallelism on P^0 directly without invoking the full prolongation procedure theory. We stress, however, that in general, for a Tanaka structure of depth μ, where μ is the number of negatively graded components, if l is the maximal integer such that the lth algebraic prolongation is not equal to zero, then the parallelism construction requires constructing the $(l+\mu)$th geometric prolongation, and in our setting $\mu = 2$. Contrastingly, the classical prolongation theory for G-structures (whose depth is $\mu = 1$) enjoys greater simplification whenever $\mathfrak{g}_1 = 0$, so that in this case the construction of the parallelism requires the first geometric prolongation only. See* [1, 2, 15, 18, 19] *for detailed exposition of the prolongation procedure.*

4 Matrix Representations of CR and Reduced Modified CR Symbols

Throughout this section, we work with a fixed CR symbol given by the pair $(\mathbb{R}\ell, \mathbb{C}\boldsymbol{A})$, where ℓ is an Hermitian form and $\boldsymbol{A}$ is a self-adjoint antilinear operator on $\mathfrak{g}_{-1,1}$. Let us fix a basis of $\mathfrak{g}_{-1}$. This basis can be fixed such that the pair $(\ell, \boldsymbol{A})$ is represented with respect to it by matrices in a canonical form, which is shown in [14]. We recall one such canonical form below in Theorem 4.1 (there are actually two canonical forms given in [14]).

For $\lambda \in \mathbb{C}$ and a positive integer m, let $J_{\lambda,m}$ denote the $m \times m$ Jordan matrix with a single eigenvalue λ and this eigenvalue has geometric multiplicity 1; let $T_m = J_{0,m}$, and let S_m be the $m \times m$ matrix whose (i, j) entry is 1 if $j + i = m + 1$ and zero otherwise, that is

$$J_{\lambda,m} := \left.\overbrace{\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}}^{m \text{ columns}}\right\} m \text{ rows} \quad \text{and} \quad S_m = \left.\overbrace{\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}}^{m \text{ columns}}\right\} m \text{ rows}.$$

In the sequel, given square matrices $D_1, \ldots D_N$ we will denote by $D_1 \oplus \ldots \oplus D_N$ the block diagonal matrix with diagonal blocks $D_1, \ldots, D_N$ in the order from the top left to the bottom right and all off-diagonal block equal to zero.

For $\lambda \in \mathbb{C}$, we define the $k \times k$ or $2k \times 2k$ matrix $M_{\lambda,k}$ by

$$M_{\lambda,k} := \begin{cases} J_{\lambda,k} & \text{if } \lambda \in \mathbb{R} \\ \begin{pmatrix} 0 & J_{\lambda^2,k} \\ I & 0 \end{pmatrix} & \text{otherwise,} \end{cases}$$

where 0 denotes a matrix of appropriate size with zero in all entries and I denotes the identity matrix. We define corresponding matrices $N_{\lambda,k}$ by

$$N_{\lambda,k} := \begin{cases} S_k & \text{if } \lambda \in \mathbb{R} \\ S_{2k} & \text{otherwise.} \end{cases}$$

For the ℓ-self-adjoint antilinear operator $\boldsymbol{A}$ referred to in the following theorem, let us enumerate the eigenvalues of $\boldsymbol{A}^2$ (counting them with multiplicity) that are contained in the upper-half plane $\{z \in \mathbb{C} \mid \Re(\mathbb{Z}) \geq 0\}$ of $\mathbb{C}$, labeling them as $\lambda_1^2, \ldots, \lambda_\gamma^2$. Furthermore, we take each λ_i to be the principle square root of λ_i^2.

Theorem 4.1 (immediate consequence of the main result in [14]) *Given a nondegenerate Hermitian form ℓ on a vector space V and an ℓ-self-adjoint antilinear operator $\boldsymbol{A}$, there exists a basis of V with respect to which ℓ and $\boldsymbol{A}$ are respectively represented by the matrices H_ℓ and A given by*

$$H_\ell = \bigoplus_{i=1}^{\gamma} \epsilon_i N_{\lambda_i, m_i} \quad \text{and} \quad A = \bigoplus_{i=1}^{\gamma} M_{\lambda_i, m_i}, \tag{4.1}$$

for some sequence $\epsilon_1, \ldots, \epsilon_\gamma$ satisfying $\epsilon_i = \pm 1$ and some sequence of positive integers $m_1, \ldots, m_\gamma$.

Letting H_ℓ and A be matrices representing ℓ and $\boldsymbol{A}$ respectively in some basis of $\mathfrak{g}_{-1}$, we consider the Lie algebras of square matrices α satisfying

$$\alpha A H_\ell^{-1} + A H_\ell^{-1} \alpha^T = \eta A H_\ell^{-1} \quad \text{for some } \eta \in \mathbb{C}$$

and respectively

$$\alpha^T H_\ell \overline{A} + H_\ell \overline{A} \alpha = \eta H_\ell \overline{A} \quad \text{for some } \eta \in \mathbb{C},$$

and define the algebra $\mathscr{A}$ to be their intersection, that is,

$$\mathscr{A} := \left\{ \alpha \;\middle|\; \begin{array}{l} \alpha A H_\ell^{-1} + A H_\ell^{-1} \alpha^T = \eta A H_\ell^{-1} \textit{and} \\ \alpha^T H_\ell \overline{A} + H_\ell \overline{A} \alpha = \eta' H_\ell \overline{A} \text{ for some } \eta, \eta' \in \mathbb{C} \end{array} \right\}. \tag{4.2}$$

Let us fix a splitting of $\mathfrak{g}_0^{\mathrm{red}}$ as given in (3.5). With respect to the basis of $\mathfrak{g}_{-1}$ fixed above, there exists some $(n-1) \times (n-1)$ matrix Ω such that $\mathfrak{g}_{0,+}^{\mathrm{red}}$ and $\mathfrak{g}_{0,-}^{\mathrm{red}}$ have the matrix representations

$$\mathfrak{g}_{0,+}^{\mathrm{red}} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} \Omega & A \\ 0 & -H_\ell^{-1} \Omega^T H_\ell \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{g}_{0,-}^{\mathrm{red}} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} -\overline{H_\ell}^{-1} \Omega^* \overline{H_\ell} & 0 \\ \overline{A} & \overline{\Omega} \end{pmatrix} \right\}. \tag{4.3}$$

In [15], we show that $\mathfrak{g}_0^{\mathrm{red}}$ is a subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$ and establish the following lemma.

Lemma 4.2 *[[15, Proposition 5.4]] There exists a subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ invariant under the transformation $\alpha \mapsto \overline{H_\ell}^{-1} \alpha^* \overline{H_\ell}$ such that*

$$\mathfrak{g}_{0,0}^{\mathrm{red}} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & -H_\ell^{-1} \alpha^T H_\ell \end{pmatrix} + cI \;\middle|\; \alpha \in \mathscr{A}_0, \textit{ and } c \in \mathbb{C} \right\}, \tag{4.4}$$

and there exist coefficients $\{\eta_\alpha\}_{\alpha\in\mathscr{A}_0}\subset\mathbb{C}$ *and* $\mu\in\mathbb{C}$ *such that the system of relations*

$$\left.\begin{array}{ll} i) & \alpha A H_\ell^{-1} + A H_\ell^{-1}\alpha^T = \eta_\alpha A H_\ell^{-1} \\ ii) & [\alpha, \Omega] - \eta_\alpha \Omega \in \mathscr{A}_0 \\ iii) & \Omega^T H_\ell \overline{A} + H_\ell \overline{A}\Omega = \mu H_\ell \overline{A} \\ iv) & \left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}, \Omega\right] + A\overline{A} - \overline{\mu}\Omega - \mu \overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}_0 \end{array}\right\} \tag{4.5}$$

holds for all $\alpha\in\mathscr{A}_0$.

We will need the following basic lemma, whose proof (outlined in the extended version of this text [16]) is a straightforward linear algebra exercise that we leave to the reader.

Lemma 4.3 ([15, Proposition 3.6]) *The following are equivalent.*

(1) $\mathfrak{g}^0$ *is regular.*
(2) $A\overline{A}A$ *is a scalar multiple of* A.

Moreover, if Ω *is in* $\mathscr{A}$ *then* $\mathfrak{g}^0$ *is regular.*

5 Matrix Representations of the Algebra $\mathscr{A}$ A

In this section we give a general formula for elements in the algebra $\mathscr{A}$ defined in (4.2) together with an outline for how the formula can be verified. The complete formula is presented in several parts in Lemmas 5.1, 5.4, and 5.8 and Corollaries 5.3, 5.5, and 5.9. We use this explicit formula to derive upper bounds for the dimension of $\mathscr{A}$ given in Lemma 5.10, which is essential for proving item (2) in Theorem 3.7. These upper bounds also lead to Theorem 5.11, which gives more precise bounds than those in Theorem 3.7. Furthermore, the matrix representation formula presented in this section plays a fundamental role in the proof of item (1) in Theorem 3.7 given in Section 6.

Naturally, it is easier to verify the formula than to derive it, and, since the formula is ancillary to this paper's topic, we omit the analysis used to derive it. To keep this text compact, we omit proofs of several lemmas in this section, and instead provide their full proofs in the extended version of this text [16]. The formula depends on the matrices H_ℓ and A representing the pair $(\ell, \boldsymbol{A})$.

In the sequel we assume that H_ℓ and A are in the canonical form prescribed by Theorem 4.1, namely as given in (4.1). We will also use the notation of Section 4, and, in particular, we let $\lambda_1,\ldots,\lambda_\gamma$, $m_1,\ldots,m_\gamma$, $\epsilon_1,\ldots,\epsilon_\gamma$, M_{λ_i,m_i} and N_{λ_i,m_i} as in Theorem 4.1. Recall that, in particular, this means the real and imaginary parts of each λ_i are both nonnegative.

Define the *bi-orthogonal subalgebra* of $\mathscr{A}$ to be

$$\mathscr{A}^o := \{B\in\mathscr{A} \mid BAH_\ell^{-1} + AH_\ell^{-1}B^T = B^T H_\ell\overline{A} + H_\ell\overline{A}B = 0\},$$

where this name is reflecting the observation that $\mathscr{A}^o$ is analogous to an intersection of two orthogonal algebras. In this section, we first obtain a formula describing the elements in $\mathscr{A}^o$ and then obtain a formula for a subspace $\mathscr{A}^s \subset \mathscr{A}$ complementary to $\mathscr{A}^o$, that is, such that

$$\mathscr{A} = \mathscr{A}^o \oplus \mathscr{A}^s. \tag{5.1}$$

Such a space $\mathscr{A}^s$ is spanned by elements that we call *conformal scaling* elements of $\mathscr{A}$, referring to the observation that these are analogous to non-orthogonal elements in an intersection of two conformally orthogonal algebras.

Let B be an $(n-1)\times(n-1)$ matrix in $\mathscr{A}^o$ and partition B into blocks $\{B_{(i,j)}\}_{i,j=1}^{\gamma}$ where the number of rows in $B_{(i,j)}$ is the same as in the matrix M_{λ_i,m_i} and the number of columns in $B_{(i,j)}$ is the same as in the matrix M_{λ_j,m_j}. Similarly, we partition $H_\ell\overline{A}B$ and BAH_ℓ^{-1} into blocks $\{(H_\ell\overline{A}B)_{(i,j)}\}_{i,j=1}^{\gamma}$ and $\{(BAH_\ell^{-1})_{(i,j)}\}_{i,j=1}^{\gamma}$ whose sizes are the same as in the partition of B.

Let us now derive a relationship between the blocks $B_{(i,j)}$ and $B_{(j,i)}$. To simplify formulas, we assume $\epsilon_i = \epsilon_j$. To treat the more general case where possibly $\epsilon_i \neq \epsilon_j$, one can simply replace N_{λ_i,m_i} (or N_{λ_j,m_j}) with $\epsilon_i N_{\lambda_i,m_i}$ (or $\epsilon_j N_{\lambda_j,m_j}$) in all of the subsequent formulas.

We have

$$\left(BAH_\ell^{-1}\right)_{(i,j)} = B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j} \quad \text{and} \quad (H_\ell\overline{A}B)_{(i,j)} = N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)},$$

so, since $B \in \mathscr{A}$,

$$\left(M_{\lambda_i,m_i}N_{\lambda_i,m_i}\right)^T B_{(j,i)}^T = -B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j}$$

and

$$B_{(j,i)}^T\left(N_{\lambda_j,m_j}\overline{M_{\lambda_j,m_j}}\right)^T = -N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)}.$$

Since A is ℓ-self-adjoint, each matrix $N_{\lambda_k,m_k}\overline{M_{\lambda_k,m_k}}$ and $M_{\lambda_k,m_k}N_{\lambda_k,m_k}$ is symmetric (one can also verify this by directly using the canonical form), and hence

$$M_{\lambda_i,m_i}N_{\lambda_i,m_i}B_{(j,i)}^T = -B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j}, \tag{5.2}$$

and

$$B_{(j,i)}^T N_{\lambda_j,m_j}\overline{M_{\lambda_j,m_j}} = -N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)}. \tag{5.3}$$

Multiplying both sides of (5.3) by $M_{\lambda_j,m_j}N_{\lambda_j,m_j}$ from the right and then applying (5.2) yields

$$\begin{aligned} B_{(j,i)}^T N_{\lambda_j,m_j}\overline{M_{\lambda_j,m_j}}M_{\lambda_j,m_j}N_{\lambda_j,m_j} &= -N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}B_{(i,j)}M_{\lambda_j,m_j}N_{\lambda_j,m_j} \\ &= N_{\lambda_i,m_i}\overline{M_{\lambda_i,m_i}}M_{\lambda_i,m_i}N_{\lambda_i,m_i}B_{(j,i)}^T. \end{aligned} \tag{5.4}$$

Multiplying (5.4) by N_{λ_i,m_i} from the left and by N_{λ_j,m_i} from the right yields

$$\left(N_{\lambda_i,m_i}B_{(i,j)}^T N_{\lambda_j,m_j}\right)\overline{M_{\lambda_j,m_j}}M_{\lambda_j,m_j} = \overline{M_{\lambda_i,m_i}}M_{\lambda_i,m_i}\left(N_{\lambda_i,m_i}B_{(i,j)}^T N_{\lambda_j,m_j}\right). \tag{5.5}$$

Notice that (5.2) is also equivalent to

$$N_{\lambda_i,m_i} M_{\lambda_i,m_i} \left(N_{\lambda_j,m_j} B_{(j,i)} N_{\lambda_i,m_i}\right)^T = -\left(N_{\lambda_i,m_i} B_{(i,j)} N_{\lambda_j,m_j}\right) N_{\lambda_j,m_j} M_{\lambda_j,m_j}. \tag{5.6}$$

Equation (5.5) gives us all restrictions on the general form of $B_{(i,j)}$ that are not coming from the relationship between $B_{(i,j)}$ and other blocks in the matrix B. Equation (5.6), on the other hand, gives us the restrictions on the general form of $B_{(i,j)}$ coming from its relationship with $B_{(j,i)}$. Moreover, if (5.5) and (5.6) are satisfied for i and j then B is in $\mathscr{A}^o$ because (5.2) and (5.3) hold. In other words, our present goal is to solve the system of matrix equations in (5.5) and (5.6), and whenever $(\lambda_i, \lambda_j) \neq (0, 0)$, this exercise is equivalent to first solving the matrix equation

$$X \overline{M_{\lambda_j,m_j}} M_{\lambda_j,m_j} = \overline{M_{\lambda_i,m_i}} M_{\lambda_i,m_i} X, \tag{5.7}$$

and then, for the case where $i = j$, solving the system of equations consisting of (5.7) and

$$N_{\lambda_i,m_i} M_{\lambda_i,m_i} X^T = -X N_{\lambda_i,m_i} M_{\lambda_i,m_i}.$$

The case where $\lambda_i = \lambda_j = 0$ requires special treatment because, in this case, contrary to the case where $(\lambda_i, \lambda_j) \neq (0, 0)$, even if $i \neq j$ solutions for $B_{(i,j)}$ in (5.5) need not satisfy (5.6) for any matrix $B_{(j,i)}$.

Equation (5.7) is of the form analyzed in [8, Chapter 8]. In fact, an explicit solution to (5.7) is given in [8, Chapter 8], but the solution is expressed in terms of a basis with respect to which $\overline{M_{\lambda_i,m_i}} M_{\lambda_i,m_i}$ and $\overline{M_{\lambda_j,m_j}} M_{\lambda_j,m_j}$ have their Jordan normal forms. On the other hand, the transition matrix from the initially considered basis to a basis of the Jordan normal form is block diagonal with the blocks corresponding to the Jordan blocks. Hence, the following lemma can be obtained from the solution in [8, Chapter 8].

Lemma 5.1 *If $\lambda_i \neq \lambda_j$ then $B_{(i,j)} = 0$.*

Given Lemma 5.1, all that remains is to find the general formula for $B_{(i,j)}$ when $\lambda_i = \lambda_j$, which is addressed by the following lemmas.

Lemma 5.2 *Suppose $\lambda_i = \lambda_j$ and $m_i \leq m_j$. The dimension of the space of solutions of (5.7) is equal to*

(1) m_i if $\lambda_i > 0$;
(2) $2m_i$ if $\lambda_i^2 \notin \mathbb{R}$;
(3) $4m_i$ if $\lambda_i^2 < 0$.

Corollary 5.3 *If $m_i \leq m_j$, $\lambda_i = \lambda_j = \lambda$ and $\lambda \neq 0$ then the matrices $B_{(i,j)}$ and $B_{(j,i)}$ are described by one of three formulas, where the correct formula depends wether $\lambda > 0$, $\lambda^2 \notin \mathbb{R}$ or $\lambda^2 < 0$. Letting, as before, T_m denote the $m \times m$ nilpotent*

Jordan block $J_{0,m}$, if $\lambda > 0$ then $B_{(i,j)}$ and $B_{(j,i)}$ respectively equal

$$\begin{array}{l} \quad\overbrace{\hphantom{0 \cdots 0}}^{\substack{m_j - m_i \\ \text{columns}}} \\ \left(\begin{array}{ccc|c} 0 & \cdots & 0 & \\ \vdots & & \vdots & \sum_{k=0}^{m_i-1} a_k T_{m_i}^k \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \end{array}\right) \end{array} \quad \text{and} \quad -\epsilon_i\epsilon_j \left(\begin{array}{ccc} \multicolumn{3}{c}{\sum_{k=0}^{m_i-1} a_k T_{m_i}^k} \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right) \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} \substack{m_j - m_i \\ \text{rows},} \tag{5.8}$$

for some coefficients $\{a_k\}$.

Explicit formulas for the remaining two cases in Lemma 5.3 (i.e., $\lambda^2 \notin \mathbb{R}$ and $\lambda^2 < 0$) are given in the extended version of this text [16], but we omit them here to save space, as they are essential only to additional analysis also appearing in the extended version.

To simplify notation in the following lemma, for an integer q, we let $[q]_2$ denote the residue of q modulo 2, that is, $[q]_2 = 0$ if q is even and $[q]_2 = 1$ if q is odd.

Lemma 5.4 *If $m_i \leq m_j$ and $\lambda_i = \lambda_j = 0$ then*

$$B_{(i,j)} = \begin{array}{l} \quad\overbrace{\hphantom{0 \cdots 0}}^{\substack{m_j - m_i \\ \text{columns}}} \\ \left(\begin{array}{ccc|cccccc} 0 & \cdots & 0 & c_1^1 & c_2^1 & \cdots & & \cdots & c_{m_i}^1 \\ \vdots & & \vdots & 0 & c_1^0 & c_2^0 & \cdots & \cdots & c_{m_i-1}^0 \\ \vdots & & \vdots & 0 & 0 & c_1^1 & c_2^1 & \cdots & c_{m_i-2}^1 \\ \vdots & & \vdots & \vdots & & \ddots & c_1^0 & \cdots & c_{m_i-3}^0 \\ \vdots & & \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & & \cdots & 0 & c_1^{[m_i]_2} \end{array}\right) \end{array}, \tag{5.9}$$

and

$$B_{(j,i)} = -\epsilon_i\epsilon_j \left(\begin{array}{cccccc} c_1^{[m_i+1]_2} & c_2^{[m_i+2]_2} & \cdots & & \cdots & c_{m_i}^{[2m_i]_2} \\ 0 & c_1^{[m_i+2]_2} & c_2^{[m_i+3]_2} & \cdots & \cdots & c_{m_i-1}^{[2m_i]_2} \\ 0 & 0 & c_1^{[m_i+3]_2} & c_2^{[m_i+4]_2} & \cdots & c_{m_i-2}^{[2m_i]_2} \\ \vdots & & \ddots & c_1^{[m_i+4]_2} & \cdots & c_{m_i-3}^{[2m_i]_2} \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & c_1^{[2m_i]_2} \\ \hline 0 \cdots & \cdots & \cdots & \cdots & \cdots 0 & \\ \vdots & & & & \vdots & m_j - m_i \text{ rows} \\ 0 \cdots & \cdots & \cdots & \cdots & \cdots 0 & \end{array}\right) \tag{5.10}$$

for some coefficients $\left\{c_k^1, c_k^0\right\}$.

Corollary 5.5 *For all $i \in \{1, \dots, \gamma\}$,*

$$B_{(i,i)} = \begin{cases} \left(\sum_{k=1}^{\lceil m_i/2 \rceil} a_k T_{m_i}^{m_i-2k+1}\right) I_{\mathrm{alt},m_i} & \text{if } \lambda_i = 0 \\ \begin{pmatrix} 0 & \sum\limits_{k=0}^{m_i-1} a_k T_{m_i}^k \\ \sum\limits_{k=0}^{m_i-1} \left(\sum_{r=0}^{k} a_r\right) T_{m_i}^k & 0 \end{pmatrix} & \text{if } \lambda_i^2 < 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5.11}$$

where $I_{\mathrm{alt},m}$ denotes the $m \times m$ diagonal matrix with a 1 in its odd columns and a -1 in its even columns.

Proof This follows immediately from the formulas in Corollary 5.3 and Lemma 5.4 with $i = j$. $\square$

The previous results provide a general formula for matrices in $\mathscr{A}^o$. We now focus on obtaining a general formula of a subspace $\mathscr{A}^s$ satisfying (5.1).

Lemma 5.6 *Either* $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 1$ *or* $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 2$*, and the latter case occurs if and only if there exists a matrix X in $\mathscr{A}$ satisfying*

$$XAH_\ell^{-1} + AH_\ell^{-1}X^T = 2AH_\ell^{-1} \Leftrightarrow (X - I)^T H_\ell A^{-1} + H_\ell A^{-1}(X - I) = 0,$$
$$X^T H_\ell \overline{A} + H_\ell \overline{A} X = 0.$$

Lemma 5.7 *If $A = M_{m,\lambda}$ and $\lambda \neq 0$ then* $\dim(\mathscr{A}) - \dim(\mathscr{A}^o) = 1$.

With Lemmas 5.6 and 5.7 established one can obtain the general formula for a subspace $\mathscr{A}^s$ of $\mathscr{A}$ satisfying (5.1).

Lemma 5.8 *For a subspace $\mathscr{A}^s$ of $\mathscr{A}$ satisfying (5.1),* $\dim(\mathscr{A}^s) = 2$ *if and only if A is nilpotent. In particular, if*

$$A = J_{0,m_1} \oplus \dots \oplus J_{0,m_\gamma}$$

then, to satisfy (5.1), we can take the subspace $\mathscr{A}^s$ of $\mathscr{A}$ spanned by the identity matrix and the matrix

$$\bigoplus_{i=1}^{\gamma} D_{m_i},$$

where, for an integer m, D_m denotes the $m \times m$ diagonal matrix defined by

$$D_m := \mathrm{Diag}\left(\frac{m}{2}, \frac{m}{2} - 1, \dots, \frac{m}{2} - m + 1\right). \tag{5.12}$$

As a direct consequence of the previous lemmas, since for non-nilpotent A we have $\mathscr{A} = \mathscr{A}^o + \mathbb{C}I$, one gets immediately the following

Corollary 5.9 *If A is not nilpotent then in (4.2) one can take $\eta' = \eta$.*

We conclude this section with one more result.

Lemma 5.10 *If H_ℓ and A are in the canonical form prescribed by Theorem 4.1 and $A \neq 0$ then*

$$\dim(\mathscr{A}) \leq n^2 - 4n + 6. \tag{5.13}$$

Moreover, this bound is attained if and only if $(\ell, \mathbf{A})$ can be represented by the pair (H_ℓ, A) in the canonical form of Theorem 4.1 with

$$A = J_{0,2} \oplus \overbrace{J_{0,1} \oplus \cdots \oplus J_{0,1}}^{n-3 \text{ copies}}. \tag{5.14}$$

Proof Assume that

$$\dim(\mathscr{A}) \geq n^2 - 4n + 6, \tag{5.15}$$

and that (H_ℓ, A) are in the canonical form of Theorem 4.1. We will still use the notation of (4.1), in particular referring to the sequence $(\lambda_1, \ldots, \lambda_\gamma)$.

Suppose that the λ_is are not all the same. Without loss of generality, we can assume that $(\lambda_1, \ldots, \lambda_\gamma)$ is enumerated so that there exists an integer k such that

$$\lambda_1 = \ldots = \lambda_k \quad \text{and} \quad \lambda_j \neq \lambda_1 \qquad \forall j > k. \tag{5.16}$$

Define

$$s = \sum_{i=1}^{k} \left[\text{number of rows in } M_{\lambda_i, m_i}\right]$$

where k is as in (5.16). By Lemma 5.1, for every matrix B in $\dim(\mathscr{A}^o + \text{span}\{I\})$, the upper right $(s) \times (n-1-s)$ block and the lower left $(n-1-s) \times (s)$ block of B is zero. Moreover, since the λ_is are not all zero, there is at least one index i such that $B_{(i,i)}$ has zeros on its main diagonal. Accordingly, if the λ_is are not all the same, then

$$\dim(\mathscr{A}^o) + 1 = \dim(\mathscr{A}^o + \text{span}\{I\}) \leq (n-1)^2 - 2s(n-1-s).$$

Since

$$2n - 4 \leq 2j(n-1-j) \qquad \forall\, 1 \leq j < n-1,$$

it follows that

$$\dim(\mathscr{A}) = \dim(\mathscr{A}^o) + 1 \leq (n-1)^2 - 2s(n-1-s) \leq (n-1)^2 - 2n + 4 = n^2 - 4n + 5,$$

where the identity $\dim(\mathscr{A}) = \dim(\mathscr{A}^o) + 1$ follows from Lemma 5.8 and the assumption that the λ_is are not all the same. Clearly, this contradicts (5.15), so if (5.15) holds then there exists a value $\lambda \in \mathbb{C}$ such that

$$\lambda = \lambda_i \qquad \forall i. \tag{5.17}$$

If (5.17) holds with $\lambda \neq 0$ then Corollaries 5.3 and 5.5 imply that each matrix B in $\mathscr{A}^o$ is fully determined by its entries above the main diagonal, and hence, applying Lemma 5.8,

$$\dim(\mathscr{A}) \leq \frac{(n-1)(n-2)}{2} + 1 < n^2 - 4n + 6, \quad \forall n \geq 2$$

Therefore, if (5.17) holds with $\lambda \neq 0$ then our assumption (5.15) fails.

In other words, — assuming for a moment that (5.15) can be satisfied, which we will prove below by giving an explicit example — if $\dim(\mathscr{A})$ is maximized then we can assume without loss of generality that

$$A = J_{0,m_1} \oplus \cdots \oplus J_{0,m_\gamma} \quad \text{with} \quad m_1 \geq \cdots \geq m_\gamma. \tag{5.18}$$

For B in $\mathscr{A}^o$, let us partition B as is done in Lemma 5.4. By Lemma 5.4, for $i < j$ the $B_{(i,j)}$ and $B_{(j,i)}$ blocks are together determined by $2m_j$ parameters, whereas, by Corollary 5.5, the $B_{(i,i)}$ block is determined by $\lceil \frac{m_i}{2} \rceil$ parameters, where $\lceil \frac{m_i}{2} \rceil$ denotes the ceiling function, i.e., the smallest integer not less than $\frac{m_i}{2}$. Hence, by counting the number of parameters determining B, Lemma 5.4 and Corollary 5.5 imply that if (5.18) holds then

$$\dim(\mathscr{A}^o) = \sum_{k=1}^{\gamma} \left(\left\lceil \frac{m_k}{2} \right\rceil + 2(k-1)m_k \right). \tag{5.19}$$

Let $r \in \{1, \dots, \gamma\}$ be an integer such that

$$m_i = 1 \qquad \forall\, i > r,$$

and to compare with A, let us also consider the matrix

$$A' = J_{0,m_1} \oplus \cdots \oplus J_{0,m_{r-1}} \oplus J_{0,1} \oplus \cdots \oplus J_{0,1}.$$

In other words, A' is obtained from A by replacing the last nonzero block on the diagonal of A with zeros. We will compute the dimension of $\mathscr{A}^o$ corresponding to the case where $A = A'$, but, since are going to compare this to the sum in (5.19), for clarity let $\mathscr{A}'$ denote the algebra that we would otherwise denote by $\mathscr{A}^o$ corresponding to this case where $A = A'$, and let $\mathscr{A}^o$ still denote the algebra referred to in (5.19).

Notice that the kth summand in (5.19) counts the number of parameters determining the blocks $B_{(i,j)}$ of a matrix B in $\mathscr{A}^o$ for which $\max\{i, j\} = k$. If we compare the general formula for a matrix B in $\mathscr{A}^o$ to that of a matrix B' in $\mathscr{A}'$, the only difference appears in the blocks $B_{(i,j)}$ of B for which $\max\{i, j\} = r$, and hence a formula for $\dim(\mathscr{A}')$ should match the formula in (5.19), except that the rth summand will change. Using Lemma 5.4 and Corollary 5.5, it is however straightforward to work out exactly how this rth summand of (5.19).

Specifically, in replacing the formula for B with the formula for B', the $B_{(r,r)}$ block is replaced with the $m_r \times m_r$ matrix having m_r^2 independent parameters, whereas, for all $i < r$, $B_{(i,r)}$ (respectively $B_{(r,i)}$) is replaced with a matrix having m_r independent parameters in its first row (respectively column) and zeros elsewhere. Accordingly,

$$\dim(\mathscr{A}') = \dim(\mathscr{A}^o) - \left(\left\lceil \frac{m_r}{2} \right\rceil + 2(r-1)m_r \right) + m_r^2 + 2(r-1)m_r \geq \dim(\mathscr{A}^o). \tag{5.20}$$

Since equality holds in (5.20) if and only if $m_r = 1$, the dimension of $\mathscr{A}^o$ is maximized with A as in (5.18) if and only if

$$A = J_{0,m_1} \oplus \overbrace{J_{0,1} \oplus \cdots \oplus J_{0,1}}^{n-1-m_1 \text{ copies}}, \tag{5.21}$$

in which case, by (5.19),

$$\dim \mathscr{A}^o = \left\lceil \frac{m_1}{2} \right\rceil + \sum_{k=2}^{n-m_1} (2k-1) = \left\lceil \frac{m_1}{2} \right\rceil + (n-m_1)^2 - 1. \tag{5.22}$$

Since $A \neq 0$, this last sum is maximized with A as in (5.21) if and only if A is as in (5.14), in which case applying (5.22) with $m_1 = 2$ yields (5.13) because, by Lemma 5.8, if A is as in (5.21) then $\dim \mathscr{A} = \dim \mathscr{A}^o + 2$. $\square$

Now, for completeness, given a non-regular CR symbol $\mathfrak{g}^0$ encoded by the pair $(\ell, \boldsymbol{A})$, represented by the pair of matrices (H_ℓ, A) in the canonical basis as in Theorem 4.1 we will give a more precise (i.e., in terms of integers $m_1, \ldots m_\gamma$ and numbers $\lambda_1, \ldots, \lambda_\gamma$) upper bound for the dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate $(2n+1)$-dimensional CR structure of hypersurface type with 1-dimensional Levi kernel admitting a constant reduced modified symbol corresponding to CR symbol $\mathfrak{g}^0$. For this, for every $1 \leq i, j \leq \gamma$, let

$$d(i,j) = \begin{cases} 0, & (\lambda_i \neq \lambda_j) \text{ or } (i = j \text{ and } \lambda_i^2 \text{ is not a nonpositive real number}) \\ \min\{m_i, m_j\} & (i \neq j \text{ and } \lambda_i = \lambda_j > 0) \text{ or } (i = j \text{ and } \lambda_i^2 < 0) \\ 2\min\{m_i, m_j\} & i \neq j, \lambda_i = \lambda_j \text{ and } (\lambda_i^2 \notin \mathbb{R} \text{ or } \lambda_i = 0) \\ 4\min\{m_i, m_j\} & i \neq j, \lambda_i = \lambda_j \text{ and } \lambda_i^2 < 0 \\ \left\lceil \frac{m_i}{2} \right\rceil & i = j \text{ and } \lambda_i = 0 \end{cases}$$

where $\left\lceil \frac{m}{2} \right\rceil$ denotes the ceiling function, i.e., the smallest integer not less than $\frac{m_i}{2}$.

Let

$$d_{\text{total}} := \sum_{i \leq j} d(i,j).$$

Then the following theorem is the direct consequence of item (1) of Theorem 3.7 and Lemmas 5.2, 5.4, Corollary 5.5, and Lemma 5.8:

Theorem 5.11 *Given a non-regular CR symbol $\mathfrak{g}^0$ encoded by the pair $(\ell, \boldsymbol{A})$ in the canonical basis as in Theorem 4.1, the dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate $(2n+1)$-dimensional CR structure of hypersurface type with 1-dimensional Levi kernel admitting a constant reduced modified symbol corresponding to the CR symbol $\mathfrak{g}^0$ is not greater than $d_{\text{total}}+2n+3$ if the operator $\boldsymbol{A}$ is not nilpotent, and it is not greater than $d_{\text{total}}+2n+4$, if the operator $\boldsymbol{A}$ is nilpotent.*

Note that the mentioned Lemmas and Corollaries from Section 5 together with (4.4) imply that $\dim \mathfrak{g}_{0,0}^{\text{red}}$ is either not greater than $d_{\text{total}} + 2$ or $d_{\text{total}} + 3$ depending whether or not A is nilpotent. The estimate for $\mathfrak{u}(\mathfrak{g}^{0,\text{red}}) = \mathfrak{g}^{0,\text{red}}$ in Theorem 5.11 follows from this and the fact that $\dim(\mathfrak{g}_- + \mathfrak{g}_{0,-2} + \mathfrak{g}_{0,2}) = 2n + 1$.

6 Proof of Theorem 3.7

6.1 Preparatory Lemmas and Notations

Let $\sigma : \mathfrak{g}^{0,\mathrm{red}} \to \mathfrak{g}^{0,\mathrm{red}}$ denote the antilinear involution induced by the natural complex conjugation of $\mathbb{C}TM$. We introduce this σ notation to avoid confusion because while working with matrix representations in coordinates we will use the overline notation to denote the standard complex conjugation of coordinates, which is a different involution. Let

$$(e_1, \dots, e_{2n-2})$$

be a basis of $\mathfrak{g}_{-1}$ with respect to which we get the matrix representation of $\mathfrak{g}_0^{\mathrm{red}}$ given by (4.3) and (4.4). Notice in particular that $(e_1, \dots, e_{n-1})$ spans $\mathfrak{g}_{-1,1}$ and

$$\sigma(e_i) = e_{n+i-1} \qquad \forall\, 1 \le i \le n-1.$$

Note that σ extends to an involution defined of $\mathfrak{g}_1^{\mathrm{red}}$ by same formula (see (2.4)) that we used to extend the natural conjugation from $\mathfrak{g}_-$ to be defined on $\mathfrak{csp}(\mathfrak{g}_{-1})$, that is

$$\sigma(\varphi)(v) := \sigma \circ \varphi \circ \sigma(v) \qquad \forall\, v \in \mathfrak{g}^{0,\mathrm{red}},\ \varphi \in \mathfrak{g}_1^{\mathrm{red}} \tag{6.1}$$

defines an involution of $\mathfrak{g}_1^{\mathrm{red}}$.

An element φ in $\mathrm{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) \oplus \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0^{\mathrm{red}})$ belongs to $\mathfrak{g}_1^{\mathrm{red}}$ if and only if

$$\varphi([e_i, e_j]) = (\varphi(e_i))\,(e_j) - \big(\varphi(e_j)\big)\,(e_i) \quad \forall\, i, j \in \{1, \dots, 2n-2\}. \tag{6.2}$$

Note, here $\phi(e_i) \in \mathfrak{g}_0^{\mathrm{red}} \subset \mathfrak{csp}(\mathfrak{g}_{-1})$.

Given any element $v \in \mathfrak{g}_{-1}$ let v_- and v_+ be the canonical projections of v to $\mathfrak{g}_{-1,-1}$ and $\mathfrak{g}_{-1,1}$, respectively, with respect to the splitting $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,-1} \oplus \mathfrak{g}_{-1,1}$.

As a direct consequence of (6.2) and (4.3), if $n \le j \le 2n-2$ and $1 \le i \le n-1$, then

$$\begin{aligned} ((\varphi(e_j))e_i)_+ &\in \mathrm{span}\{\boldsymbol{A}e_{j-n+1}\} - \big(\varphi([e_i, e_j])\big)_+ \subset \mathrm{span}\{\boldsymbol{A}e_{j-n+1}, (\varphi(1))_+\}, \\ ((\varphi(e_i))\,e_j)_- &\in \mathrm{span}\,\{\sigma\,(\boldsymbol{A}e_i)\} - \big(\varphi([e_i, e_j])\big)_- \subset \mathrm{span}\{\boldsymbol{A}e_i, (\varphi(1))_-\} \end{aligned} \tag{6.3}$$

In particular, the upper left $(n-1) \times (n-1)$ block in the matrix $\varphi(e_j)$ and the lower right $(n-1) \times (n-1)$ block in the matrix $\varphi(e_i)$ both have rank at most 2.

Also from (6.2) and the fact that $[e_i, e_j] = 0$ for $n \le i, j \le 2n-2$, we immediately have that

$$\varphi(e_i)e_j = \varphi(e_j)e_i, \quad n \le i, j \le 2n-2. \tag{6.4}$$

Lemma 6.1 *If the antilinear operator $\boldsymbol{A}$ (or, equivalently the matrix A) has rank greater than 1 and $i \ge n$ then $\varphi(e_i) \in \mathfrak{g}_{0,0}^{\mathrm{red}} \oplus \mathfrak{g}_{0,-}^{\mathrm{red}}$, or, equivalently,*

$$\varphi(e_i) = \begin{pmatrix} \alpha_i & 0 \\ c\overline{A} & -H_\ell^{-1}\alpha_i^T H_\ell \end{pmatrix} \qquad \textit{for some } c \in \mathbb{C} \textit{ and } \alpha_i \in \mathscr{A}_0 + \mathbb{C}(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}). \tag{6.5}$$

Proof By (4.3), there exists $c \in \mathbb{C}$ such that for every $n \le j \le 2n-2$

$$\big((\varphi(e_i))\, e_j\big)_+ = c\boldsymbol{A} e_{j-n+1} \quad \text{and} \quad \big((\varphi(e_j))\, e_i\big)_+ \in \operatorname{span}\{\boldsymbol{A} e_{i-n+1}\}.$$

By (6.4), for all $n \le j \le 2n-2$,

$$c\boldsymbol{A} e_{j-n+1} \in \operatorname{span}\{\boldsymbol{A} e_{i-n+1}\}.$$

This implies that $c = 0$, because otherwise rank $\boldsymbol{A} \le 1$, contradicting our assumption. Therefore, $(\varphi(e_i)v)_+ = 0$ for all $v \in \mathfrak{g}_{-1,-1}$, which is equivalent to the statement of the lemma. ☐

Similarly, we have the following Lemma.

Lemma 6.2 *If the antilinear operator $\boldsymbol{A}$ (or, equivalently the matrix A) has rank greater than 1 and $i < n$ then $\varphi(e_i) \in \mathfrak{g}^{\mathrm{red}}_{0,0} \oplus \mathfrak{g}^{\mathrm{red}}_{0,+}$ or, equivalently,*

$$\varphi(e_i) = \begin{pmatrix} \alpha_i & cA \\ 0 & -H_\ell^{-1}\alpha_i^T H_\ell \end{pmatrix} \qquad \text{for some } c \in \mathbb{C} \text{ and } \alpha_i \in \mathscr{A}_0 + \mathbb{C}\Omega. \tag{6.6}$$

Lemma 6.3 *If A has rank greater than 1 and α_i is the matrix defined by (6.5) and (6.6) then, for $i < n$, we have*

$$\left(H_\ell \overline{A}\alpha_i\right)^T + H_\ell \overline{A}\alpha_i = \eta H_\ell \overline{A} \text{ for some } \eta \in \mathbb{C} \tag{6.7}$$

and, for $n \le i$, we have

$$\alpha_i A H_\ell^{-1} + \left(\alpha_i A H_\ell^{-1}\right)^T = \eta A H_\ell^{-1} \text{ for some } \eta \in \mathbb{C}. \tag{6.8}$$

Proof If α_i is as in (6.6) then $\alpha_i \in \mathscr{A} + \mathbb{C}\Omega$, so the definition of $\mathscr{A}$ and item (iii) of (4.5) imply (6.7). If, on the other hand, α_i is as in (6.5) then $\alpha_i \in \mathscr{A} + \mathbb{C}(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell})$, so the definition of $\mathscr{A}$ and item (iii) of (4.5) imply (6.8). ☐

Corollary 6.4 *If the CR symbol is not regular (and hence with* rank $A > 1$*) and the matrix α_i given in (6.5) or (6.6) is zero, then $\varphi(e_i) = 0$.*

Proof Suppose $\alpha_i = 0$. By (4.3), (4.4), and Lemmas 6.1 and 6.2, if $\varphi(e_i) \ne 0$ then either $\Omega \in \mathscr{A}$ or $\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}$. The conditions $\Omega \in \mathscr{A}$ and $\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}$ are, however, equivalent, so either $\varphi(e_i) \ne 0$ or $\Omega \in \mathscr{A}$. If the CR symbol is not regular then, by Lemma 4.3, $\Omega \notin \mathscr{A}$, and hence $\varphi(e_i) = 0$. ☐

Lemma 6.5 *If an element φ in $\mathfrak{g}^{\mathrm{red}}_1$ satisfies $\varphi(1) = 0$ and*

$$\varphi(e_i) = 0 \qquad \forall\, i \ge n \tag{6.9}$$

then $\varphi = 0$.

Proof Since $\varphi(1) = 0$, the left side of (6.2) is zero for all i and j. Accordingly, for any $i \in \{1, \dots, n-1\}$ and $j \in \{n, \dots, 2n-2\}$, (6.2) and (6.9) imply that the j

column of $\varphi(e_i)$ is zero. Hence, for all $i \in \{1, \dots, n-1\}$, the latter $n-1$ columns of $\varphi(e_i)$ are all zero. From this and Lemma 6.2 (and specifically (6.6)), it follows that $H_\ell^{-1}\alpha_i^T H_\ell = 0$. Hence $\alpha_i = 0$, which, again noting (6.6), shows that the initial $n-1$ columns of $\varphi(e_i)$ are also zero. Therefore $\varphi(e_i) = 0$ for any i. □

The *general strategy* of our proof of item (1) of Theorem 3.7 is, for a given arbitrary $\varphi \in \mathfrak{g}_1^{\text{red}}$, first to prove that $\varphi(1) = 0$ and then to prove (6.9).

We will also need the following equations and notation. In the sequel every $(n-1) \times (n-1)$ matrix X will be also regarded as an operator having the matrix representation X with respect to the basis $(e_1, \dots, e_{n-1})$. Let $\{\varphi_i\}_{i=1}^{2n-2} \subset \mathbb{C}$ denote the coefficients satisfying

$$\varphi(1) = \sum_{i=1}^{2n-2} \varphi_i e_i.$$

By (6.5), it follows that

$$\big(\varphi(e_i)e_j\big)_- = -\left(H_\ell^{-1}\alpha_i^T H_\ell\right) e_{j-n+1}, \quad \forall\, n \le i, j \le 2n-2.$$

This together with (6.4) yields

$$\left(H_\ell^{-1}\alpha_i^T H_\ell\right) e_{j-n+1} = \left(H_\ell^{-1}\alpha_j^T H_\ell\right) e_{i-n+1}, \quad \forall\, n \le i, j \le 2n-2. \tag{6.10}$$

Condition (6.10) is crucial in the subsequent analysis, namely in the proof of Lemmas 6.6 and 6.11. Therefore, we need to describe the matrix $H_\ell^{-1}\alpha_j^T H_\ell$, which we begin by first describing the matrix α_j. By (6.5), it follows that, for $n \le j \le 2n-2$ and $1 \le i \le n-1$,

$$\big(\varphi(e_j)e_i\big)_+ = \alpha_j e_i.$$

From this and (6.3), taking into account that the matrix A represents the antilinear operator $\boldsymbol{A}$, we have that there exists the unique tuple $(\kappa_i)_{i=1}^{n-1}$ such that

$$\alpha_j e_i = \kappa_i A e_{j-n+1} - (H_\ell)_{i,j-n+1}\,(\varphi(1))_+ \tag{6.11}$$

for all $1 \le i \le n-1$ and $n \le j \le 2n-2$. The uniqueness of $(\kappa_i)_{i=1}^{n-1}$ follows from the assumption that $A \ne 0$ and that κ_i in (6.11) is independent of j.

6.2 Branching Analysis: the First Two Cases

There are three separate cases determined by properties of $(\ell, \boldsymbol{A})$ that require separate analysis. First is the special case wherein, for some integer m satisfying $2 \le m \le n-1$, we have

$$H_\ell = S_m \oplus H'_\ell \tag{6.12}$$

where H'_ℓ is an arbitrary nondegenerate Hermitian matrix, and

$$A = J_{\lambda,m} \oplus A' \qquad \text{for some } \lambda \ge 0, \tag{6.13}$$

where A' is such that $(\ell, \boldsymbol{A})$ is represented by (H_ℓ, A). Second is the case where we have some integer $1 \leq m \leq n-1$ such that

$$H_\ell = \left(\begin{array}{c|c} S_{2m} & 0 \\ \hline 0 & H'_\ell \end{array}\right), \tag{6.14}$$

where H'_ℓ is an arbitrary nondegenerate Hermitian matrix, and

$$A = \left(\begin{array}{c|c} \overbrace{\begin{array}{c|c} 0 & J_{m,\lambda} \\ \hline I & 0 \end{array}}^{2m \text{ columns}} & 0 \\ \hline 0 & A' \end{array}\right) \Big\} 2m \text{ rows}, \qquad \text{for some } \lambda \in \mathbb{C} \setminus \{x \in \mathbb{R} \,|\, x \geq 0\}, \tag{6.15}$$

where A' is a matrix such that $(\ell, \boldsymbol{A})$ is represented by (H_ℓ, A). Third is the case where (H_ℓ, A) corresponds to a non-regular CR structure and A is diagonal. By Theorem 4.1, we can indeed always take the matrices representing $(\ell, \boldsymbol{A})$ in the form of one of these three special cases.

In the present section, Section 6.2, only the first of these three cases is analyzed, and analysis of the third is deferred to Section 6.3. The additional analysis required for the second case is very similar to that of the first, so we omit it from the present text; full detailed analysis for all three cases is, however, presented in the extended version of this text [16].

Let us assume now that (H_ℓ, A) is in the canonical form of Theorem 4.1, satisfying (6.12) in particular, which implies

$$Ae_1 = \lambda e_1, \quad Ae_i = \lambda e_i + e_{i-1} \qquad \forall\, 2 \leq i \leq m, \tag{6.16}$$

and

$$H_\ell e_i = e_{m+1-i} \qquad \forall\, 1 \leq i \leq m. \tag{6.17}$$

Using (6.11), (6.12), and (6.16) we obtain

$$\alpha_n e_i = \kappa_i \lambda e_1 - \delta_{i,m}(\varphi(1))_+ \quad \forall\, i \in \{1, \ldots, n-1\}, \tag{6.18}$$

and, for $0 < p < m$,

$$\alpha_{n+p} e_i = \kappa_i e_p + \kappa_i \lambda e_{p+1} - \delta_{i,m-p}(\varphi(1))_+ \quad \forall\, i \in \{1, \ldots, n-1\}. \tag{6.19}$$

Now from (6.18), we get

$$\alpha_n^T e_1 = \sum_{j=1}^{n-1} \kappa_j \lambda e_j - \varphi_1 e_m \quad \text{and} \quad \alpha_n^T e_i = -\varphi_i e_m \qquad \forall\, 2 \leq i \leq n-1.$$

Using this together with (6.17) we can get

$$(H_\ell^{-1} \alpha_n^T H_\ell) e_i = -\varphi_{m+1-i} e_1 \qquad \forall\, i \in \{1, \ldots, m-1\}, \tag{6.20}$$

$$(H_\ell^{-1} \alpha_n^T H_\ell) e_m \equiv -\varphi_1 e_1 + \lambda \sum_{j=1}^{m} \kappa_{m+1-j} e_j \quad (\text{mod span}\{e_{m+1}, e_{m+2}, \ldots, e_{n-1}\}), \tag{6.21}$$

and

$$(H_\ell^{-1}\alpha_n^T H_\ell)e_i = -\left(\sum_{j=m+1}^{n-1}(H_\ell)_{j,i}\varphi_j\right)e_1 = -\left(\sum_{j=1}^{n-1-m}(H'_\ell)_{j,i-m}\varphi_{j+m}\right)e_1 \quad \forall i > m, \tag{6.22}$$

where H_ℓ is as in (6.12).

Similarly, for $0 < p < m$, from (6.19) we have

$$\alpha_{n+p}^T e_i = \begin{cases} -\varphi_i e_{m-p}, & i \in \{1,\dots,n-1\}\setminus\{p,p+1\} \\ -\varphi_p e_{m-p} + \sum_{j=1}^{n-1}\kappa_j e_j, & i = p \\ -\varphi_{p+1}e_{m-p} + \lambda\sum_{j=1}^{n-1}\kappa_j e_j & i = p+1, \end{cases}$$

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell)e_i = -\varphi_{m+1-i}e_{p+1} \qquad \forall i \in \{1,\dots,m\}\setminus\{m-p, m-p+1\}, \tag{6.23}$$

$$(H_\ell^{-1}\alpha_{n+p}^T H_\ell)e_{m-p} \equiv -\varphi_{p+1}e_{p+1} + \lambda\sum_{j=1}^{m}\kappa_{m+1-j}e_j \quad (\mathrm{mod}\ \mathrm{span}\{e_{m+1},\dots,e_{n-1}\}), \tag{6.24}$$

and

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right)e_{m-p+1} \equiv -\varphi_p e_{p+1} + \sum_{j=1}^{m}\kappa_{m+1-j}e_j \quad (\mathrm{mod}\ \mathrm{span}\{e_{m+1},\dots,e_{n-1}\}).$$

For $p \geq m$,

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right)e_i \in \mathrm{span}\{e_{m+1},\dots,e_{n-1}\}. \tag{6.25}$$

Lemma 6.6 *Under the assumptions (6.12) and (6.13), if* rank $A > 1$ *then*

$$\varphi(1) = 0. \tag{6.26}$$

Proof We will begin by showing that

$$(\varphi(1))_+ = 0. \tag{6.27}$$

The proof consists of analysis of (6.10) in three cases:

1. (6.10) *for $i = n$ and $j = n + p$ with* $0 \leq p < m - 1$. By (6.20)

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right)e_{p+1} = \varphi_{m-p}e_1 \qquad \forall\ 0 \leq p < m-1, \tag{6.28}$$

and, by (6.23),

$$\left(H_\ell^{-1}\alpha_{n+p}^T H_\ell\right)e_1 = \varphi_m e_{p+1} \qquad \forall\ 0 \leq p < m-1. \tag{6.29}$$

Applying (6.28) and (6.29) to (6.10) with $i = n$ and $j = n + p$ we get

$$\varphi_{m-p}e_1 = \varphi_m e_{p+1} \qquad \forall\ 0 \leq p < m-1.$$

Therefore, using the last equation for $1 \leq p < m-1$ (as for p=0 this equation is a tautology), we get

$$\varphi_2 = \cdots = \varphi_{m-1} = 0,$$

and also that $\varphi_m = 0$ for $m > 2$ (we will give another way to prove the latter identity including the case $m = 2$ in item 3 of the proof below).

2. *(6.10) for $i = n$ and $j = n + p$ with $p \geq m$.* By (6.22) we get that

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = \left(\sum_{j=1}^{n-1-m} \left(H'_\ell\right)_{j,p+1-m} \varphi_{j+m}\right) e_1. \tag{6.30}$$

Using (6.10), from (6.30) and (6.25) it follows that $\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_{p+1} = 0$ or, equivalently,

$$\sum_{j=1}^{n-1-m} \left(H'_\ell\right)_{j,i} \varphi_{j+m} = 0, \quad 1 \leq i \leq n-1-m.$$

Since the matrix H'_ℓ is nonsingular, this yields

$$\varphi_{m+1} = \cdots = \varphi_{n-1} = 0.$$

3. *(6.10) for $i = n$ and $j = n + m - 1$.* If $v = \lambda \sum_{j=1}^{m} \kappa_{m+1-j} e_j$, then, by (6.21),

$$\left(H_\ell^{-1}\alpha_n^T H_\ell\right) e_m \equiv -\varphi_1 e_1 + v \pmod{\operatorname{span}\{e_i\}_{i=m+1}^{n-1}}, \tag{6.31}$$

and, by (6.24),

$$\left(H_\ell^{-1}\alpha_{n+m-1}^T H_\ell\right) e_1 \equiv -\varphi_m e_m + v \pmod{\operatorname{span}\{e_i\}_{i=m+1}^{n-1}}. \tag{6.32}$$

Using (6.10) again and the fact that $m \geq 2$, from (6.31) and (6.32) it follows that $\varphi_1 = 0$ and $\varphi_m = 0$. This completes the proof of (6.27).

Since (6.1) defines an involution of $\mathfrak{g}_1^{\text{red}}$, $\sigma(\varphi)$ also belongs to $\mathfrak{g}_1^{\text{red}}$, so, since φ was an arbitrary element in $\mathfrak{g}_1^{\text{red}}$, the exact same arguments applied above show that $(\sigma(\varphi)(1))_+ = 0$. Since $\sigma(1) = 1$,

$$\sigma\left((\varphi(1))_-\right) = (\sigma \circ \varphi(1))_+ = (\sigma(\varphi)(1))_+ = 0,$$

and hence $(\varphi(1))_- = 0$, which, together with (6.27) implies (6.26). □

Lemma 6.7 *Under the assumptions (6.12) and (6.13), if* $\operatorname{rank} A > 2$ *then* $(\kappa_1, \ldots, \kappa_n)A = 0$.

Proof Consider now the equation in (6.8) with $i = n$. The matrix on the right side of (6.8) is either zero or it has rank equal to $\operatorname{rank} A$, which is at least 3 under this lemma's hypothesis. On the other hand, applying (6.16), (6.17), (6.18) and Lemma 6.6, we get

$$\left(\alpha_n A H_\ell^{-1}\right) e_i \in \operatorname{span}\{e_1\} \qquad \forall\, i \in \{1, \ldots, m-1\}, \tag{6.33}$$

and, applying (6.19) additionally, if $\lambda = 0$ then

$$\left(\alpha_{n+1} A H_\ell^{-1}\right) e_i \in \operatorname{span}\{e_1\} \qquad \forall\, i \in \{1, \ldots, m-1\}. \tag{6.34}$$

Hence, by (6.33),

$$\operatorname{rank}\left(\alpha_n A H_\ell^{-1}\right) \leq 1 \tag{6.35}$$

and $\operatorname{rank}\left(\alpha_n A H_\ell^{-1} + \left(\alpha_n A H_\ell^{-1}\right)^T\right) \leq 2$ because $\alpha_n A H_\ell^{-1}$ has at most one nonzero row. Similarly, if $\lambda = 0$ then (6.34) implies

$$\operatorname{rank}\left(\alpha_{n+1} A H_\ell^{-1}\right) \leq 1 \tag{6.36}$$

and $\operatorname{rank}\left(\alpha_{n+1} A H_\ell^{-1} + \left(\alpha_{n+1} A H_\ell^{-1}\right)^T\right) \leq 2$. Since the matrix on the left side of (6.8) has rank at most 2 whenever $i = n$ or $(\lambda, i) = (0, n+1)$, the matrix on the right side of (6.8) is zero whenever $i = n$ or $(\lambda, i) = (0, n+1)$. Thus by (6.8) the matrix $\alpha_n A H_\ell^{-1}$ is skew symmetric, and the matrix $\alpha_{n+1} A H_\ell^{-1}$ is skew symmetric whenever $\lambda = 0$. This together with (6.35) implies that

$$\alpha_n A H_\ell^{-1} = 0, \tag{6.37}$$

whereas applying (6.36) yields

$$\alpha_{n+1} A H_\ell^{-1} = 0, \tag{6.38}$$

whenever $\lambda = 0$. By (6.37) and (6.18) for $\lambda \neq 0$, or by (6.38) and (6.19) for $\lambda = 0$, we get that the vector $(\kappa_1, \ldots, \kappa_n) A H_\ell^{-1} = 0$, which completes this proof. $\square$

In the next four lemmas, 6.8-6.11, we prove item (1) of Theorem 3.7 in four special cases that together cover all non-regular CR symbols not treated in the final section, 6.3.

Lemma 6.8 *Under the assumptions (6.12) and (6.13), if* $\operatorname{rank} A > 2$ *and* $(\lambda, m) \notin \{(0, 2), (0, 3)\}$ *then* $\mathfrak{g}_1^{\text{red}} = 0$.

Proof Let $\varphi \in \mathfrak{g}_1^{\text{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (6.11). It will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$. Indeed, first plugging this condition and the conclusion (6.26) of Lemma 6.6 into relation (6.11) we obtain that $\alpha_j = 0$ for all $n \leq j \leq 2n-2$. This and Corollary 6.4 imply (6.9). Thus, the conclusion of the present lemma will follow from (6.26) and Lemma 6.5.

Notice that since $(\kappa_1, \ldots, \kappa_n) A = 0$, we have that $\kappa_i = 0$ for $1 \leq i \leq m$ if $\lambda \neq 0$, and $\kappa_i = 0$ for $1 \leq i \leq m-1$ if $\lambda = 0$. In particular, as $m \geq 2$ we have $\kappa_1 = \kappa_2 = 0$ always, and, since it is assumed that $m > 3$ when $\lambda = 0$, if $\lambda = 0$ then $\kappa_3 = 0$ as well.

To produce a contradiction, assume that there exists an index r such that $\kappa_r \neq 0$ and let r be the minimal such index. By (6.18),

$$\alpha_n e_i = \delta_{i,r} \kappa_i \lambda e_1 \quad \forall\, i \leq r, \tag{6.39}$$

and, by (6.19), for $0 < p < m$,

$$\alpha_{n+p} e_i = \delta_{i,r} (\kappa_i e_p + \kappa_i \lambda e_{p+1}) \quad \forall\, i \leq r. \tag{6.40}$$

Note that, by Lemma 6.1, $\operatorname{span}\{\alpha_n, \alpha_{n+1}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1} \Omega^* \overline{H}_\ell)$. Since $\mathscr{A}$ is a subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1} \Omega^* \overline{H}_\ell)$ of codimension at most

1, the subspaces span$\{\alpha_n, \alpha_{n+1}\}$ and $\mathscr{A}$ have a nontrivial intersection. That is, there exist $b_1, b_2 \in \mathbb{C}$ such that $(b_1, b_2) \neq (0, 0)$ and

$$b_1\alpha_n + b_2\alpha_{n+1} \in \mathscr{A}. \tag{6.41}$$

By (6.39) and (6.40) again the first $r - 1$ columns of the matrix $b_1\alpha_n + b_2\alpha_n$ vanish and

$$(b_1\alpha_n + b_2\alpha_{n+1})e_r = \kappa_r\Big((\lambda b_1 + b_2)e_1 + \lambda b_2 e_2\Big) \tag{6.42}$$

By applying formulas from Section 5, we can derive a contradiction from the assumption $\lambda \neq 0$ as follows. Let $b_1\alpha_n + b_2\alpha_{n+1}$ be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of A (referring to the block diagonal partition of A given in (4.1)).

By (6.41), if $\lambda > 0$ then each (i, j) block of $b_1\alpha_n + b_2\alpha_{n+1}$ is either characterized by Lemma 5.1 or Corollary 5.5 and identically zero or it is characterized by Corollary 5.3 and more specifically characterized by (5.8). In particular, if the $(1, j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is nonzero (and therefore characterized by (5.8)) and contains part of the r column of $b_1\alpha_n + b_2\alpha_{n+1}$, then (5.8) implies that the $(j, 1)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ is nonzero and contained in the first $r - 1$ columns of $b_1\alpha_n + b_2\alpha_{n+1}$, which contradicts our definition of r. Accordingly, if $\lambda > 0$ then the $(1, j)$ block of $b_1\alpha_n + b_2\alpha_{n+1}$ containing part of the r column of $b_1\alpha_n + b_2\alpha_{n+1}$ is identically zero, which implies $\lambda b_1 + b_2 = 0$ and $\lambda b_2 = 0$ by (6.42). So, if $\lambda > 0$, then we obtain the contradiction $(b_1, b_2) = (0, 0)$.

On the other hand, if $\lambda = 0$ then, by Lemma 6.1, span$\{\alpha_{n+2}, \alpha_{n+3}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H}_\ell^{-1}\Omega^*\overline{H}_\ell)$. Similarly to the previous case, $\mathscr{A}$ and span$\{\alpha_{n+2}, \alpha_{n+3}\}$ have a nontrivial intersection, that is, there exist $b_1, b_2 \in \mathbb{C}$ such that $(b_1, b_2) \neq (0, 0)$ and

$$b_1\alpha_{n+2} + b_2\alpha_{n+3} \in \mathscr{A}. \tag{6.43}$$

Note that we are now redefining b_1 and b_2 because the previous definition is no longer needed, and that the b_is in (6.43) are not related to the b_is in (6.41). By (6.39) and (6.40) the first $r - 1$ columns of the matrix $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ vanish and

$$(b_1\alpha_{n+2} + b_2\alpha_{n+3})e_r = \kappa_r\Big(b_1e_2 + b_2e_3\Big). \tag{6.44}$$

By applying formulas from Section 5 again, we can derive a contradiction now from the assumption $\lambda = 0$. For this, let $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ in (6.43) be partitioned as a block matrix whose diagonal blocks have the same size as the diagonal blocks of A. By (6.43), if $\lambda = 0$ then each (i, j) block of $b_1\alpha_n + b_2\alpha_{n+1}$ is either characterized by Lemma 5.1 and identically zero or it is characterized by Lemmas 5.4 and 5.8 and Corollary 5.5 and more specifically characterized by (5.9), (5.10), (5.11), and (5.12). In particular, if $\lambda = 0$ and the $(1, j)$ block of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ contains part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$, and, furthermore, we assume that the $(1, j)$ block is not identically zero, then this $(1, j)$ block is either characterized by (5.11) and (5.12) or by (5.9) and (5.10).

Considering the first possibility where the $(1, j)$ block containing part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ is characterized by (5.11) and (5.12) (i.e., $j = 1$), by (6.44), the first m entries of $b_1e_2 + b_2e_3$ form the r column of the $(1, 1)$ block of

$b_1\alpha_{n+2}+b_2\alpha_{n+3}$. Since we are assuming that this $(1, 1)$ block is a linear combination of matrices (5.11) and (5.12) with the latter being a diagonal matrix, noting that $r > 3$, it follows that the first entry in the $r-1$ column of this $(1, 1)$ block is $-b_1$ and the second entry in the $r-1$ column of this $(1, 1)$ block is $-b_2$. Yet the $r-1$ column of the $(1, 1)$ block of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ is zero by the definition of r, so we have obtained the contradiction that $(b_1, b_2) = (0, 0)$.

Considering the remaining possibility, which is where the $(1, j)$ block containing part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ is characterized by (5.9) or (5.10), if this $(1, j)$ block is nonzero then (5.9) and (5.10) imply that the $(j, 1)$ block is nonzero and contained in the first $r-1$ columns of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$, which contradicts the definition of r.

Hence, the $(1, j)$ block containing part of the r column of $b_1\alpha_{n+2} + b_2\alpha_{n+3}$ must be identically zero because all other possibilities yield contradictions, and yet, by (6.44), setting this $(1, j)$ block equal to zero again implies the contradiction $(b_1, b_2) = (0, 0)$. Therefore, there is no index r such that $\kappa_r \neq 0$. □

Lemma 6.9 *Under the assumptions (6.12) and (6.13), if there is a basis with respect to which* $\boldsymbol{A}$ *is represented by the matrix*

$$A = J_{0,3} \oplus J_{1,c} \oplus A'' \quad \textit{for some } c > 0 \tag{6.45}$$

or

$$A = J_{0,2} \oplus J_{1,c} \oplus J_{1,c'} \oplus A'' \quad \textit{for some } c, c' > 0. \tag{6.46}$$

then $\mathfrak{g}_1^{\mathrm{red}} = 0$.

Proof Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (6.11). By the same arguments as in the beginning of the proof of Lemma 6.8 , it will suffice to show that $\kappa_i = 0$ for every $1 \le i \le n-1$. Note that, by Lemma 6.6, in the considered cases $\varphi(1) = 0$. It is more convenient to work with matrices

$$\widetilde{A} = J_{c,1} \oplus J_{0,3} \oplus A'' \tag{6.47}$$

or

$$\widetilde{A} = J_{c,1} \oplus J_{c',1} \oplus J_{0,2} \oplus A'' \tag{6.48}$$

instead of A in (6.45) and (6.46), respectively. This can be done by an obvious permutation of the basis. Also, in the considered cases the rank assumptions of Lemma 6.7 with A replaced by $\widetilde{A}$ holds. Therefore, using (6.11) with A replaced by $\widetilde{A}$ we get

$$\kappa_1 = \kappa_2 = \kappa_3 = 0. \tag{6.49}$$

Note that if we would not replace A by $\widetilde{A}$ we could conclude that $\kappa_1 = \kappa_2 = \kappa_4 = 0$ in the case of (6.45) and that $\kappa_1 = \kappa_3 = \kappa_4 = 0$ in the case of (6.46), so that is why we make this permutation of the blocks.

Assume for a proof by contradiction that there exists r such that $\kappa_r \neq 0$ and moreover that this is the minimal such index, that is, $\kappa_i = 0$ for all $i < r$. By (6.49), $r > 3$. From (6.11) with A replaced by $\widetilde{A}$ it follows that in both cases the first $r-1$

columns of the matrices α_i with $n \le i \le n+3$ vanish,

$$\alpha_n e_r = \kappa_r c e_1, \quad \text{and} \quad \alpha_{n+3} e_r = \kappa_r e_3. \tag{6.50}$$

Further,

$$\alpha_{n+2} e_r = \kappa_r e_2 \tag{6.51}$$

if $\widetilde{A}$ satisfies (6.47), and

$$\alpha_{n+1} e_r = \kappa_r c' e_2 \tag{6.52}$$

if $\widetilde{A}$ satisfies (6.48). Note that, by Lemma 6.1, each α_i in these equations belongs to $\mathscr{A} + \mathbb{C}\left(\overline{H_\ell}^{-1}\Omega^* \overline{H_\ell}\right)$.

Hence, using similar arguments as in the proof of Lemma 6.8 we get that the 3-dimensional subspace $\operatorname{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}$ in the first case and $\operatorname{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}$ in the second case has at least a two dimensional intersection with $\mathscr{A}$. Notice further that in either case, the rth columns of matrices in these intersections must have a two-dimensional span because the natural map from the space $\operatorname{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}$ (or $\operatorname{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}$) to $\mathbb{C}^{n-1}$ sending a matrix to its r column in this space is injective.

Let us now first assume that $\widetilde{A}$ satisfies (6.47). Let $B^{(1)}$ and $B^{(2)}$ be matrices belonging to the intersection of $\operatorname{span}\{\alpha_n, \alpha_{n+2}, \alpha_{n+3}\}$ and $\mathscr{A}$ such that the r column of $B^{(1)}$ is linearly independent from the r column of $B^{(2)}$. For an $(n-1)\times(n-1)$ matrix B, let $(B_{(i,j)})$ be a partition of B into a block matrix whose diagonal blocks have the same size as the diagonal blocks of A. Let j be the index such that $B_{(1,j)}$ contains part of the r column of B. By Lemma 5.1, since $c \neq 0$ there exists $i \in \{1, 2\}$ such that $B_{(i,j)} = 0$ for all $B \in \mathscr{A}$, because otherwise Lemma 5.1 implies that the $(1, 1)$ and $(2, 2)$ blocks of $A\overline{A}$ have the same eigenvalues. In particular, at most one of the $(1, j)$ and $(2, j)$ blocks of any linear combination of $B^{(1)}$ and $B^{(2)}$ is nonzero. It follows that, for each $k \in \{1, 2\}$, $B^{(k)}_{(1,j)} = 0$ and $B^{(k)}_{(2,j)} \neq 0$ because otherwise the r column of each $B^{(k)}$ belongs to $\operatorname{span}\{e_1\}$, which contradicts our choice of $B^{(1)}$ and $B^{(2)}$. Moreover, by (6.50) and (6.51), the first nonzero column of each block $B^{(k)}_{(2,j)}$ has zero in all but its first two entries.

Each $B^{(k)}_{(2,j)}$ is either characterized by Lemma 5.1 and is identically zero or characterized by Lemma 5.4 and Corollary 5.5 and more specifically characterized by (5.9), (5.10), or (5.11) (with $\lambda_i = 0$). If $B^{(k)}_{(2,j)}$ is characterized by (5.11) then $j = 2$ and, by (5.11), the second entry of the first nonzero column of $B^{(k)}_{(2,2)}$ is zero. If, on the other hand, $B^{(k)}_{(2,j)}$ is characterized by (5.9) (or (5.10)) and the second entry of the first nonzero column of $B^{(k)}_{(2,j)}$ is nonzero, then, by (5.10) (or respectively (5.9)), the $B^{(k)}_{(j,2)}$ block of $B^{(k)}$ is nonzero and contained in the first $r-1$ columns of $B^{(k)}$, which contradicts our choice of r. Therefore if $B^{(k)}_{(2,j)}$ is nonzero then the second entry of the first nonzero column of $B^{(k)}_{(2,j)}$ is zero. Yet this contradicts our choice of $B^{(1)}$ and $B^{(2)}$ because it means that the only nonzero entry in the r column of $B^{(1)}$ and $B^{(2)}$ is the second entry.

Let us now address the remaining case, that is, assume that $\widetilde{A}$ satisfies (6.48). Again, let j be the index such that $B_{(1,j)}$ contains part of the r column a given $(n-1)\times(n-1)$ matrix B. Let $B^{(1)}$ and $B^{(2)}$ be matrices belonging to the intersection of $\operatorname{span}\{\alpha_n, \alpha_{n+1}, \alpha_{n+3}\}$ and $\mathscr{A}$ such that the r column of $B^{(1)}$ is linearly independent from the r column of $B^{(2)}$. From this independence condition and the fact that nonzero entries of these respective rth columns of $B^{(1)}$ and of $B^{(2)}$ appear within their first three entries (the latter is a consequence of (6.50) and (6.52)), it follows that there exists a matrix B in $\operatorname{span}\{B^{(1)}, B^{(2)}\}$ such that there exists $i \in \{1, 2\}$ with $B_{(i,j)} \neq 0$ (because otherwise, the third entry is the only nonzero entry of rth columns of $B^{(1)}$ and $B^{(2)}$, which contradicts the independence of these columns). Since $r > 3$ it follows that $j > 2$. Thus, it follows from Lemma 5.1 and Corollary 5.3 that this nonzero $B_{(i,j)}$ with $i \in \{1, 2\}$ is characterized by (5.8). Yet (5.8) implies that the $B_{(j,i)}$ is a nonzero block contained in the first $r-1$ rows of B, which contradicts our choice of r. □

Lemma 6.10 *Under the assumptions (6.12) and (6.13), if*

$$A = \overbrace{J_{0,2} \oplus \cdots \oplus J_{0,2}}^{k \text{ copies}} \oplus J_{c,1} \oplus J_{0,1} \oplus \cdots \oplus J_{0,1},$$

for some integer k and some $c > 0$ then $\mathfrak{g}_1^{\mathrm{red}} = 0$.

Proof Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (6.11). By the same arguments as in the beginning of the proof of Lemma 6.8 , it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$. We work with (H_ℓ, A) in the canonical form of Theorem 4.1, so H_ℓ is as in (4.1), that is

$$H_\ell = \epsilon_1 N_{0,2} \oplus \cdots \oplus \epsilon_k N_{0,2} \oplus \epsilon_{k+1} N_{c,1} \oplus \cdots \oplus \epsilon_\gamma N_{0,1}$$

for some coefficients $\epsilon_i = \pm 1$.

For a matrix B in $\mathscr{A}$, let $(B_{(i,j)})$ be a partition of B into a block matrix whose diagonal blocks have the same size as the diagonal blocks of A. By Lemma 5.4 and Corollary 5.5, we have

$$B_{(i,j)} = \epsilon_i \begin{pmatrix} b & c \\ 0 & d \end{pmatrix} \quad \text{and} \quad B_{(j,i)} = -\epsilon_j \begin{pmatrix} b & e \\ 0 & d \end{pmatrix} \qquad \forall\, i, j \leq k$$

and

$$B_{(i,j)} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad \text{and} \quad B_{(j,i)} = \begin{pmatrix} 0 & b \end{pmatrix} \qquad \forall\, i \leq k < j$$

for some $b, c, d, e \in \mathbb{C}$ that depend on (i, j). By Corollary 5.5 and Lemma 5.8,

$$B_{1,1} = B_{2,2} = \cdots = B_{2k+1,2k+1}, \tag{6.53}$$

where here $B_{i,j}$ denotes the (i, j) entry of B rather than the (i, j) block $B_{(i,j)}$. By Lemma 5.1 and Corollary 5.5,

$$B_{(i,k+1)} = 0 \quad \text{and} \quad B_{(k+1,i)} = 0 \qquad \forall\, i \neq k. \tag{6.54}$$

Since, by Lemma 6.7, $(\kappa_1, \ldots, \kappa_{n-1})A = 0$, we have

$$\kappa_i = 0 \qquad \text{whenever } i \text{ is odd and } i \leq 2k+1. \tag{6.55}$$

From (6.11) and Lemma 6.6 it follows that, for $0 \leq p \leq n-1$, the i column of the matrix α_{n+p} is equal to κ_i times the $p+1$ column of A. In particular, the (i, j) entry of α_{n+2k} is

$$(\alpha_{n+2k})_{i,j} = \kappa_j c \delta_{i,2k+1}. \tag{6.56}$$

Since, by Lemma 6.1, each α_{n+p} belongs to $\mathscr{A}_0 + \mathbb{C}\left(\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}\right)$ and α_{n+2k} does not belong to $\mathscr{A}_0 \setminus \{0\}$, which can be seen by contrasting (6.54) and (6.56), it follows that

$$\text{either} \quad \alpha_{n+2k} = 0 \quad \text{or} \quad \overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}_0 + \operatorname{span}_{\mathbb{C}}\{\alpha_{n+2k}\}.$$

But $\alpha_{n+2k} = 0$ if and only if $\kappa_1 = \cdots = \kappa_{n-1} = 0$, which is equivalent to what we want to show, so let us proceed assuming

$$\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} \in \mathscr{A}_0 + \operatorname{span}_{\mathbb{C}}\{\alpha_{n+2k}\}$$

in order to produce a contradiction. Accordingly, let $\Omega_0 \in \mathscr{A}_0$ and $s \in \mathbb{C}$ be such that

$$\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell} = \overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell} + s\alpha_{n+2k}, \tag{6.57}$$

or, equivalently,

$$\Omega = \Omega_0 + \overline{s}\,\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}. \tag{6.58}$$

Here we will apply another result from Section 5, namely Corollary 5.9, which states that for $B \in \mathscr{A}$, since A is not nilpotent, if $\left(H_\ell\overline{A}B\right)^T + H_\ell\overline{A}B = \mu H_\ell\overline{A}$ then $BAH_\ell^{-1} + AH_\ell^{-1}B^T = \mu AH_\ell^{-1}$. Noting that, by (6.55) and (6.56), $\overline{AH_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell} = 0$, item (iii) in (4.5) and (6.58) imply that

$$\left(H_\ell\overline{A}\Omega_0\right)^T + H_\ell\overline{A}\Omega_0 = \mu H_\ell\overline{A}, \tag{6.59}$$

and hence Corollary 5.9 implies that

$$\eta_{\Omega_0} = \mu,$$

where this notation η_{Ω_0} refers to the coefficient with that label in items (i) and (ii) or (4.5).

Since the matrix equation $\left(H_\ell\overline{A}X\right)^T + H_\ell\overline{A}X = \mu H_\ell\overline{A}$ is equivalent to

$$\left(\overline{H_\ell}^{-1}X^*\overline{H_\ell}\right)AH_\ell^{-1} + AH_\ell^{-1}\left(\overline{H_\ell}^{-1}X^*\overline{H_\ell}\right)^T = \overline{\mu}AH_\ell^{-1},$$

(6.59) implies

$$\eta_{\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell}} = \overline{\mu}. \tag{6.60}$$

By (6.60), items (i) and (ii) in (4.5) imply

$$\left[\Omega, \overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell}\right] + \overline{\mu}\Omega \in \mathscr{A}_0, \tag{6.61}$$

and applying the transformation $X \mapsto \overline{H_\ell}^{-1}X^*\overline{H_\ell}$ to the matrix in (6.60) yields

$$\left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}, \Omega_0\right] - \mu\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell} \in \mathscr{A}_0. \tag{6.62}$$

Now we analyze item (iv) of (4.5). Using (6.57), (6.58), and lastly (6.61), we have

$$\begin{aligned}\left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell},\Omega\right] &= \left[\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell},\Omega\right]+\left[s\alpha_{n+2k},\Omega_0\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]\\ &\equiv \overline{\mu}\Omega+\left[s\alpha_{n+2k},\Omega_0\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right] \pmod{\mathscr{A}_0}.\end{aligned}$$

Substituting the last equation into item (iv) of (4.5) we get, after the obvious cancellation, that

$$\left[s\alpha_{n+2k},\Omega_0\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]+A\overline{A}-\mu\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell}\in\mathscr{A}_0. \tag{6.63}$$

Similarly, (6.57), (6.58), and then (6.62) yields

$$\begin{aligned}\left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell},\Omega\right] &= \left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell},\Omega_0\right]+\left[\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell},\overline{s}\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]\\ &\equiv \mu\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell}+\left[\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell},\overline{s}\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right],\end{aligned}$$

where the equivalence is modulo $\mathscr{A}_0$. Substituting the last equation into item (iv) of (4.5) we get

$$\left[\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell},\overline{s}\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]+A\overline{A}-\overline{\mu}\Omega\in\mathscr{A}_0. \tag{6.64}$$

On the other hand, again from (6.57) , (6.58), and using that $\left[\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell},\Omega_0\right]\in\mathscr{A}_0$, we can write

$$\left[\overline{H_\ell}^{-1}\Omega^*\overline{H_\ell},\Omega\right]\equiv\left[s\alpha_{n+2k},\Omega_0\right]+\left[\overline{H_\ell}^{-1}\Omega_0^*\overline{H_\ell},\overline{s}\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right],$$

where here again the equivalence is modulo $\mathscr{A}_0$. By subtracting the matrix in item (iv) of (4.5) from the sum of the matrices in (6.63) and (6.64) and using the last relation, we get

$$A\overline{A}+|s|^2\left[\alpha_{n+2k},\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right]\in\mathscr{A}_0,$$

or, equivalently,

$$\left(A\overline{A}+|s|^2\alpha_{n+2k}\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\right)-|s|^2\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\alpha_{n+2k}\in\mathscr{A}_0. \tag{6.65}$$

Notice that the first two terms in (6.65), grouped together by parentheses, are matrices whose only potentially nonzero entry is the $(2k+1, 2k+1)$ entry, whereas the other term has the same value in the first $2k+1$ entries of its main diagonal. By (6.53), each matrix in $\mathscr{A}_0$ also has the same values in the first $2k+1$ entries of its main diagonal. Moreover, the $(2k+1, 2k+1)$ entry of $A\overline{A}$ is nonzero. Therefore, by (6.65),

$$A\overline{A}=-|s|^2\alpha_{n+2k}\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}. \tag{6.66}$$

Defining

$$\alpha:=|s|^2\overline{H_\ell}^{-1}\alpha_{n+2k}^*\overline{H_\ell}\alpha_{n+2k},$$

(6.65) and (6.66) imply that α is in $\mathscr{A}_0$.

It is straightforward to check that, with this definition for α, $\eta_\alpha = 0$ in the notation of item (i) of (4.5) (by calculating, for example, the (1, 1) entries of the terms in item (i)), and hence items (i) and (ii) of (4.5) yield $[\Omega, \alpha] \in \mathscr{A}_0$. Or, equivalently, by (6.58), noting that $[\Omega_0, \alpha] \in \mathscr{A}_0$,

$$\overline{s}\left[\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}, \alpha\right] \in \mathscr{A}_0. \tag{6.67}$$

Notice that $\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\alpha = 0$ because $\left(\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\right)^2 = 0$, and hence (6.67) implies

$$\overline{s}|s|^2\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\left(\alpha_{n+2k}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\right) \in \mathscr{A}_0. \tag{6.68}$$

Applying (6.66), we get

$$\begin{aligned}
-\frac{\overline{s}|s|^2}{|c|^2}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\left(\alpha_{n+2k}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\right) &= \frac{\overline{s}}{|c|^2}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell}\left(A\overline{A}\right) \\
&= \overline{s}\overline{H_\ell}^{-1}\alpha^*_{n+2k}\overline{H_\ell},
\end{aligned} \tag{6.69}$$

where this last equality follows easily from (6.56).

By (6.58), (6.68), and (6.69), we get that Ω is in $\mathscr{A}_0$, but this contradicts Lemma 4.3. Therefore, the assumption that $\alpha_{n+2k} \neq 0$ must be false, which in turn implies that $a_1 = \cdots = a_{n-1} = 0$, completing this proof. □

Establishing the following lemma requires analysis of the second special case (i.e., wherein (6.14) and (6.15) hold) analogous to the analysis of the first case carried out above. Although we omit the analysis for the second case, it is fully presented in the extended version of this text [16].

Lemma 6.11 *Under the assumptions (6.14) and (6.15), if A corresponds to a non-regular CR structure then* $\mathfrak{g}_1^{\mathrm{red}} = 0$.

6.3 Branching Analysis: the Final Case:

In this subsection, 6.3, we consider the special case where (H_ℓ, A) corresponds to a non-regular CR structure and A is diagonal. Working in the normal form of Theorem 4.1, H_ℓ is diagonal too. Since A corresponds to a non-regular CR structure, the matrix $A\overline{A}$ has at least two distinct nonzero eigenvalues, so we can assume without loss of generality that there are numbers $\lambda_1, \ldots, \lambda_{n-1}, \in \mathbb{C}$ and $\epsilon_1, \ldots, \epsilon_{n-1} \in \{1, -1\}$ such that $|\lambda_1| \neq |\lambda_2|$, $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and

$$A = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{n-1}) \quad \text{and} \quad H_\ell = \mathrm{diag}\,(\epsilon_1, \ldots, \epsilon_{n-1}).$$

Accordingly, by (6.11),

$$\alpha_{n+p}e_i = \kappa_i\lambda_{p+1}e_{p+1} - \delta_{i,p+1}\varphi(1) \qquad \forall\, 0 \leq p < n, \tag{6.70}$$

$$\alpha_n A H_\ell^{-1} e_i = \lambda_i \varepsilon_i \kappa_i \left(\lambda_1 e_1 - \delta_{i,1}\varphi(1)\right), \tag{6.71}$$

$$H^{-1}\alpha_{n+p}^T He_1 = \pm\varphi_1 e_{p+1} \qquad \forall 0 \leq p < n, \tag{6.72}$$

and

$$H^{-1}\alpha_n^T He_{p+1} = \pm\varphi_{p+1} e_1 \qquad \forall 0 < p < n. \tag{6.73}$$

By (6.10), we can equate $H^{-1}\alpha_n^T He_{p+1}$, and hence (6.72) and (6.73) yields

$$\varphi_1 = \varphi_2 = \cdots = \varphi_{n-1} = 0. \tag{6.74}$$

Formula in (6.71) now simplifies giving that $\alpha_n AH_\ell^{-1}$ is a matrix with at most 1 nonzero row, and hence the left side of (6.8) (when setting $i = n$) cannot be a diagonal matrix of rank greater than one. Yet the right side of (6.8) is a diagonal matrix that is either zero or of rank greater than 1, so the right side of (6.8) must be zero for the equation to hold. Since the left side of (6.8) is zero, (6.71) and (6.74) imply that

$$\lambda_1\kappa_1 = \lambda_2\kappa_2 = \cdots = \lambda_{n-1}\kappa_{n-1} = 0$$

because $\lambda_1 \neq 0$. In particular,

$$\kappa_1 = \kappa_2 = 0 \tag{6.75}$$

because λ_1 and λ_2 are both nonzero.

Lemma 6.12 *If (H_ℓ, A) corresponds to a non-regular CR structure and A is diagonal then $\mathfrak{g}_1^{\mathrm{red}} = 0$.*

Proof Let $\varphi \in \mathfrak{g}_1^{\mathrm{red}}$ and let $(\kappa_i)_{i=1}^{n-1}$ be as in (6.11). Recall that $(\varphi(1))_+ = 0$ implies $\varphi(1) = 0$, by the same argument applied at the end of the proof of Lemma 6.6, and hence $\varphi(1) = 0$ by (6.74). Accordingly, by the same arguments as in the beginning of the proof of Lemma 6.8, it will suffice to show that $\kappa_i = 0$ for every $1 \leq i \leq n-1$.

Assume that there exists r such that $\kappa_r \neq 0$ and r is the minimal index with this property. By (6.75) we have that $r > 2$. Noting (6.70), by Lemma 6.1, $\kappa_r \neq 0$ implies $\mathrm{span}\{\alpha_n, \alpha_{n+1}\}$ is a 2-dimensional subspace in $\mathscr{A} + \mathbb{C}(\overline{H_\ell}^{-1}\Omega^*\overline{H}_\ell)$. Accordingly, $\kappa_r \neq 0$ yields that $\mathrm{span}\{\alpha_n, \alpha_{n+1}\}$ and $\mathscr{A}$ have at least a 1-dimensional intersection. By (6.75) and(6.70), nonzero entries in the matrices in $\mathrm{span}\{\alpha_n, \alpha_{n+1}\}$ can only appear in their first two rows and moreover they do not appear in their first two columns. Yet, in the Section 5, we describe the matrices in $\mathscr{A}$ explicitly. In particular, given that H_ℓ and A are diagonal, the description of $\mathscr{A}$ in Section 5 implies that every matrix in $\mathscr{A}$ with nonzero entries in its first two rows also has nonzero entries in its first two columns, which implies that $\mathrm{span}\{\alpha_n, \alpha_{n+1}\}$ and $\mathscr{A}$ have a trivial intersection, a clear contradiction. □

By combining the results of Lemmas 6.8, 6.9, 6.10, 6.11, and 6.12, we finish the proof of item (1) of Theorem 3.7, because these lemmas account for all non-regular symbols. To prove item (2) of Theorem 3.7 note that by (4.4) and Lemma 3.7, for the reduced modified CR symbol corresponding to a non-regular symbol,

$$\dim \mathfrak{g}_{0,0}^{\mathrm{red}} = \dim \mathscr{A} + 1 < n^2 - 4n + 7.$$

Therefore, from item (1) of the theorem under consideration and the fact that $\dim \mathfrak{g}_0^{\text{red}} = \dim \mathfrak{g}_{0,0}^{\text{red}} + 2$ and $\dim \mathfrak{g}_- = 2n - 1$, it follows that

$$\dim \mathfrak{u}\left(\mathfrak{g}^{0,\text{red}}\right) = \dim \mathfrak{g}^{0,\text{red}} < (2n-1) + (n^2 - 4n + 7) + 2 = (n-1)^2 + 7,$$

which together with Theorem 3.6 completes the proof of item (2) of Theorem 3.7. Item (3) of Theorem 3.7 follows from item (1) of Theorem 3.7 and the parallelism construction referred to in [15, Theorem 6.2].

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Declarations

Conflict of Interest The authors declare no competing interests.

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