ON WRONSKIANS AND qq-SYSTEMS

ANTON M. ZEITLIN

Abstract. We discuss the qq-systems, the functional form of the Bethe ansatz equations for the twisted Gaudin model from a new geometric point of view. We use a concept of G-Wronskians, which are certain meromorphic sections of principal G-bundles on the projective line. In this context, the qq-system, similar to its difference analog, is realized as the relation between generalized minors of the G-Wronskian. We explain the link between G-Wronskians and twisted G-oper connections, which are the traditional source for the qq-systems.

Contents

1. Introduction

The impact of quantum integrable models on modern mathematics is enormous. The important examples of this kind are the so-called spin chain models [B1, KBI, R]. While many of the algebraic structures observed there found themselves in pure mathematics, in particular in the modern theory of quantum groups, the original method of solution of these models, known as algebraic Bethe ansatz [TF, R] remained popular mainly in the framework of mathematical physics. The centerpiece of this method, which on the folklore level is a method of diagonalization of a mutually commuting set of operators (transfer-matrices) in finite-dimensional vector space [KBI, R] is the resulting algebraic equations, known as *Bethe equations* [B2, KBI] which at first sight have no particular mathematical meaning. In recent years a lot of activity has been devoted to finding the geometric context in which these equations appear naturally. In a particular case of XXX/XXZ spin chains [TF, OW], the integrable models, based on Yangians $(Y_\hbar(\mathfrak{g}))$/quantum affine algebras $(U_\hbar(\mathfrak{g}))$, the Bethe equations emerge as the relations for the quantum equivariant cohomology/K-theory of a certain variety [PSZ, KPSZ, AO, O] as conjectured in the theoretical physics context [NS1, NS2]. At the same time, these relations may straightforwardly be written in terms of a system of difference equations, known as QQ-systems. Incidentally for $U_\hbar(\mathfrak{g})$ the QQ-systems emerge as the relations in the extended Grothendieck ring of finite-dimensional representations of quantum affine algebras [BLZ, HJ, FH1, FH2].

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Much earlier, another geometric realization due to B. Feigin, E. Frenkel and their collaborators [FFR1, F1, F2, FFTL, FFR2] was achieved for the semiclassical version of the aforementioned integrable models, the so-called Gaudin model. It turned out the Bethe equations, in this case, describe certain principal ${}^LG$-bundle connections (group ${}^LG$ corresponds to the Langlands dual ${}^L\mathfrak{g}$) on the projective line, called opers, with a prescribed singularity structure. This geometrization of Gaudin Bethe equations was a part of a far bigger story: this correspondence is an example of a geometric Langlands correspondence.

A naturally arising question is whether there exists a deformation of this example if a similar correspondence holds for Bethe equations of XXX/XXZ models, and what is a proper generalization of the principal bundle connection. In [KSZ, FKSZ] we introduced the deformed version of the connection, which we called $({}^LG, \hbar)$-opers for Bethe equations of XXZ type [1]. While [FKSZ] we treated $({}^LG, \hbar)$-opers on Lie-theoretic level, in [KSZ, KZ2] for $(SL(N), \hbar)$-opers we exploited a different approach, which used interpretations of QQ-systems as minors in the deformed and twisted version of the Wronskian matrix. At the time, it seemed like a construction specific for defining the representation of $SL(N)$. Still, later, it was observed in [KZ1] that a QQ-system emerges from a new object, $({}^LG, \hbar)$-Wronskian: a meromorphic section of a principal ${}^LG$-bundle satisfying a certain difference equation. We established the explicit correspondence between $({}^LG, \hbar)$-opers and $({}^LG, \hbar)$-Wronskians in [KZ1], so that the QQ-system emerges as relations between generalized minors [FZ1, FZ2] of $(G, \hbar)$-Wronskian.

In [BSZ] we described the classical limit of the QQ-system: we called it qq-system, which is a system of differential equations representing the original Gaudin model/oper context. On the level of equations one obtains QQ-system from qq-system via proper application of $\hbar \to 1$ limit and rescaling. In this note we explain how to obtain the differential G-Wronskian and generalized minor interpretation of the qq-system.

The exposition is as follows. First, Section 2 reviews the concept of generalized minors following Fomin and Zelevinsky. Then, in Section 3, we introduce the idea of G-Wronskian for simply connected simple group G and its relation to the qq-system. Next, in Section 4, we discuss the class of G-opers, which correspond to the qq-system, following [BSZ]. Finally, in Section 5, we establish the relation between two objects: G-opers and G-Wronskians and discuss the differences between G-Wronskians and their deformed analogs.

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2. Generalized minors

2.1. **Group-theoretic data.** Let G be a connected, simply connected, simple algebraic group of rank r over $\mathbb{C}$. We fix a Borel subgroup B_- with unipotent radical $N_- = [B_-, B_-]$ and a maximal torus $H \subset B_-$. Let B_+ be the opposite Borel subgroup containing H. Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of positive simple roots for the pair $H \subset B_+$. Let $\{\check{\alpha}_1, \dots, \check{\alpha}_r\}$ be the corresponding coroots; the elements of the Cartan matrix of the Lie algebra $\mathfrak{g}$ of G are given by $a_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle$.

The Lie algebra $\mathfrak{g}$ has Chevalley generators $\{e_i, f_i, \check{\alpha}_i\}_{i=1,\dots,r}$, so that $\mathfrak{b}_- = \mathrm{Lie}(B_-)$ is generated by the f_i's and the $\check{\alpha}_i$'s and $\mathfrak{b}_+ = \mathrm{Lie}(B_+)$ is generated by the e_i's and the $\check{\alpha}_i$'s, while $\mathfrak{h} = \mathfrak{b}_\pm/[\mathfrak{b}_\pm, \mathfrak{b}_\pm]$ is generated by $\check{\alpha}_i$'s. Let $\{\omega_i\}_{i=1,\dots,r}$ and $\{\check{\omega}_i\}_{i=1,\dots,r}$ be the

[1] In the non-simply laced case of $\mathfrak{g}$, the situation is more involved, see recent paper [FHR].

fundamental weights and coweights correspondingly, defined by $\langle \omega_i, \check{\alpha}_j\rangle = \langle \check{\omega}_i, \alpha_j\rangle \delta_{ij}$. The element $\mathrm{ad}_{\check{\rho}}$, where $\check{\rho} = \sum_{i=1}^r \check{\omega}_i$ defines a principal gradation on $\mathfrak{b}_+ = \oplus_{i\geq 0}\mathfrak{b}_{+,i}$.

Let $W_G = N(H)/H$ be the Weyl group of G. Let $w_i \in W$, $(i = 1, \dots, r)$ denote the simple reflection corresponding to α_i. We also denote by w_0 be the longest element of W, so that $B_+ = w_0(B_-)$. We also fix representatives $s_i \in N(H)$ of w_i, and in general will denote $\tilde{w}$ the representative of w in $N(H)$.

2.2. **Generalized Minors and their properties.** Consider the big cell $G_0 \in G$ in Bruhat decomposition: $G_0 = N_- H N_+$. Any element $g \in G_0$ can be represented as follows:

$$g = n_-\ h\ n_+. \tag{2.1}$$

for some $n_+ \in N_+$, $h \in H$, and $n_- \in N_-$. Let V_i be the irreducible representation of G with highest weight ω_i and highest weight vector ν_i which is the eigenvector for any element $h \in H$, i.e. $h\nu_i = [h]^{\omega_i}\nu_i$, where $[h]^{\omega_i} \in \mathbb{C}^\times$ denotes an eigenvalue of h.

We formulate the following definition.

Definition 2.1. [FZ1] Regular functions $\{\Delta^{\omega_i}\}_{i=1,\dots,r}$ on G, whose values on a dense set G_0 are given by

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \dots, r \tag{2.2}$$

will be referred to as *principal minors* of a group element g.

In case of $G = SL(r+1)$, these functions coincide with the principal minors of the standard matrix realization of $SL(r+1)$. We define other generalized minors using the action of the lifts of the Weyl group elements on the right and the left and then applying principal minors to the result. In other words, we have the following definition.

Definition 2.2. [FZ1] For $u, v \in W_G$, we define a regular function $\Delta_{u\cdot\omega_i, v\cdot\omega_i}$ on G by setting

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g\ \tilde{v}), \tag{2.3}$$

where $\tilde{u}, \tilde{v}$ are lifts of Weyl group elements u, v to G.

Notice that in this notation $\Delta_{\omega_i,\omega_i}(g) = \Delta^{\omega_i}(g)$. Consider the orbit $\mathcal{O}_{W_G} = W_G \cdot \mathbb{C}\nu_i$. Then we have the following Proposition.

Proposition 2.3. *Action of the group element on the highest weight vector $\nu_i \in V_i$ is as follows:*

$$g \cdot \nu_i = \sum_{w\in W} \Delta_{w\cdot\omega_i,\omega_i}(g)\tilde{w}\cdot\nu_i + \dots, \tag{2.4}$$

where dots stand for the vectors, which do not belong to the orbit $\mathcal{O}_W$.

The set of generalized minors $\{\Delta_{w\cdot\omega_i,\omega_i}\}_{w\in W; i=1,\dots,r}$ creates a set of coordinates on G/B_+, known as *generalized Plücker coordinates.* In particular, the set of zeroes of each of $\Delta_{w\cdot\omega_i,\omega_i}$ is a uniquely and unambiguously defined hypersurface in G/B. This feature is important for characterizing Schubert cells as quasi-projective subvarieties of a generalized flag variety, see [FZ2] for details.

We started this section from the explicit definition of the principal minors by means of Gaussian decomposition. The following proposition (see Corollary 2.5 in [FZ1]) provides a necessary and sufficient condition of its existence for a given group element.

Proposition 2.4. *An element $g \in G$ admits the Gaussian decomposition if and only if $\Delta^{\omega_i}(g) \neq 0$ for any $i = 1, \dots, r$.*

Finally, we introduce the fundamental relation ([FZ1], Theorem 1.17) between generalized minors, which will be crucial in the following.

Proposition 2.5. *Let, $u, v \in W$, such that for $i \in \{1, \dots, r\}$, $\ell(uw_i) = \ell(u) + 1$, $\ell(vw_i) = \ell(v) + 1$. Then*

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}(g)\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i}(g) - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}(g)\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i}(g) = \prod_{j\neq i}\Big[\Delta_{u\cdot\omega_j, v\cdot\omega_j}(g)\Big]^{-a_{ji}}. \tag{2.5}$$

3. G-Wronskians

3.1. **Differential equations and G-Wronskian.** Consider the irreducible representation V_i of G with highest weight ω_i. It comes equipped with a 2-dimensional subspace $W_i = L_i^+ \oplus L_i^-$, $L_i^+ = \mathbb{C}\nu_i$, $L_i^- = \mathbb{C}f_i\nu_i$, which is invariant under the action of B_+.

Suppose we have a principal G-bundle $\mathcal{F}_G$ and its B_+-reduction $\mathcal{F}_{B_+}$ and thus an H-reduction $\mathcal{F}_H$, where $H = B_+/[B_+, B_+]$ as well. Therefore for each $i = 1, \dots, r$, the vector bundle

$$\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} V_i$$

associated to V_i contains an H-line subbundles

$$\mathcal{L}_i^+ = \mathcal{F}_H \underset{H}{\times} L_i^+, \quad \mathcal{L}_i^- = \mathcal{F}_H \underset{H}{\times} L_i^-$$

associated to $L_i^\pm \subset V_i$.

Definition 3.1. G-Wronskian on $\mathbb{P}^1$ is the quadruple $(\mathcal{F}_G, \mathcal{F}_{B_+}, \mathcal{G}, \nabla^Z)$, where $\mathcal{G}$ is a meromorphic section of a principle bundle $\mathcal{F}_G$, $\mathcal{F}_{B_+}$ is a reduction of $\mathcal{F}_G$ to B_+, ∇^Z is an H-connection, so that $H = B_+/[B_+, B_+]$, satisfying the following condition. There exist a Zariski open dense subset $U \subset \mathbb{P}^1$ together with the trivialization $\imath_{B_+}$ of $\mathcal{F}_{B_+}$ so that $i_{\frac{d}{dz}}\nabla^Z = \nabla^Z_z = \partial_z - Z$, where $Z \in \mathfrak{h} = \mathfrak{b}_+/[\mathfrak{b}_+, \mathfrak{b}_+]$, so that for certain sections $\{v_i^\pm\}_{i=1,\dots,r}$ of line bundles $\{\mathcal{L}_i^\pm\}_{i=1,\dots,r}$ on U we have $\mathcal{G}$ as an element of $G(z)$ satisfy the following conditions:

$$\nabla^Z_z(\mathcal{G}\cdot v_i^+) = \mathcal{G}\cdot v_i^-, \quad i = 1, \dots, r \tag{3.1}$$

In local terms we have the following. Representing the corresponding section $\mathcal{G} = \mathcal{G}(z) \in G(z)$, we have:

$$(\partial_z - Z)\mathcal{G}(z)\nu_i = \mathcal{G}(z)p_{-1}^{\phi}(z)\nu_i, \tag{3.2}$$

where $p_{-1}^{\phi}(z) = \sum_{i=1}^r \phi_i(z)f_i$, and $\phi_i(z) \in \mathbb{C}(z)$. We are interested in the case when $\{\phi_i(z)\}_{i=1,\dots,r}$ are polynomials.

Definition 3.2. G-Wronskian has regular singularities if $p_{-1}^{\phi}(z) = p_{-1}^{\Lambda}(z) = \sum_{i=1}^r \Lambda_i(z)f_i$, where $\Lambda_i(z) \in \mathbb{C}[z]$.

We also are interested to impose some non-degeneracy conditions on $\mathcal{G}(z)$, which has to do with their generalized minor structure.

Definition 3.3. We say that G-Wronskian with regular singularities is *nondegenerate* if $\Delta_{w\cdot\omega_i,\omega_i}(\mathcal{G}(z))$ are nonzero polynomials for all $w \in W$ and $i = 1, \dots, r$.

Now let us look at the equations, satisfied by non-degenerate genralized G-Wronskians:

$$(\partial_z - Z)\mathscr{G}(z)\nu_i = \mathscr{G}(z)p^{\Lambda}_{-1}(z)\nu_i \tag{3.3}$$

Restricting this equation to $W_i \subset V_i$, we have the following:

$$\begin{aligned}&(\partial_z - \langle Z,\omega_i\rangle)\Delta_{\omega_i,\omega_i}(\mathscr{G}(z)) = \Lambda_i(z)\Delta_{\omega_i,w_i\cdot\omega_i}(\mathscr{G}(z))\\ &(\partial_z - \langle Z,\omega_i-\alpha_i\rangle)\Delta_{w_i\cdot\omega_i,\omega_i}(\mathscr{G}(z)) = \Lambda_i(z)\Delta_{w_i\cdot\omega_i,w_i\cdot\omega_i}(\mathscr{G}(z))\end{aligned} \tag{3.4}$$

Let us make use of the relation (2.5), when $u,v=1$, applying it to $\mathscr{G}(z)$:

$$\Delta_{\omega_i,\omega_i}(\mathscr{G}(z))\Delta_{w_i\cdot\omega_i,w_i\cdot\omega_i}(\mathscr{G}(z)) - \Delta_{w_i\cdot\omega_i,\omega_i}(\mathscr{G}(z))\Delta_{\omega_i,w_i\cdot\omega_i}(\mathscr{G}(z)) = \prod_{j\neq i}\Big[\Delta_{\omega_j,\omega_j}(\mathscr{G}(z))\Big]^{-a_{ji}}. \tag{3.5}$$

With the help of (3.4) we obtain:

$$\begin{aligned}&\Lambda_i^{-1}(z)\Delta_{\omega_i,\omega_i}(\mathscr{G}(z))(\partial_z - \langle Z,\omega_i-\alpha\rangle)\Delta_{w_i\omega_i,\omega_i}(\mathscr{G}(z)) -\\ &\Lambda_i^{-1}(z)\Delta_{w_i\cdot\omega_i,\omega_i}(\mathscr{G}(z))(\partial_z - \langle Z,\omega_i\rangle)\Delta_{\omega_i,\omega_i}(\mathscr{G}(z)) = \prod_{j\neq i}\Delta^{-a_{ji}}_{\omega_j,\omega_j}(\mathscr{G}(z)).\end{aligned} \tag{3.6}$$

Denoting $\Delta_{\omega_i,\omega_i}(\mathscr{G}(z)) = q^i_+(z)$, $\Delta_{w_i\cdot\omega_i,\omega_i}(\mathscr{G}(z)) = q^i_-(z)$, then simplifying, and collecting such equations for all fundamental weights, we arrive to the following system:

$$\begin{aligned}&q^i_+(z)\partial_z q^i_-(z) - q^i_-(z)\partial_z q^i_+(z) + \langle Z,\alpha_i\rangle q^i_+(z)q^i_-(z) =\\ &\Lambda_i(z)\prod_{j\neq i}\Big[q^j_+(z)\Big]^{-a_{ji}}, \quad i=1,\dots,r.\end{aligned} \tag{3.7}$$

Applying $\tilde{w}\in G$ which is a lift of the element of $w\in W$ to (3.3), we have:

$$(\partial_z - Z^w)\mathscr{G}^w(z)\nu_i = \mathscr{G}^w(z)p^{\Lambda}_{-1}(z)\nu_i\ , \tag{3.8}$$

where $Z^w \in \tilde{w}Z\tilde{w}^{-1}$ and $\mathscr{G}^w(z) = \tilde{w}\,\mathscr{G}(z)$. Let us denote $\Delta_{w^{-1}\cdot\omega_i,\omega_i}(\mathscr{G}(z)) = q^{i,w}_+(z)$. Then, if $\ell(w^{-1}s_i) = \ell(w^{-1})+1$, we have $\Delta_{w^{-1}\cdot\omega_i,\omega_i}(\mathscr{G}(z)) = q^{i,w}_+(z)$, so that $\Delta_{s_i\cdot\omega_i,\omega_i}(\mathscr{G}(z)) = q^i_-(z)$ and the following relations hold:

$$\begin{aligned}&q^{i,w}_+(z)\partial_z q^{i,w}_-(z) - q^{i,w}_-(z)\partial_z q^{i,w}_+(z) + \langle Z^w,\alpha_i\rangle q^{i,w}_+(z)q^{i,w}_-(z) =\\ &\Lambda_i(z)\prod_{j\neq i}\Big[q^{j,w}_+(z)\Big]^{-a_{ji}}, \quad i=1,\dots,r; \quad w\in W.\end{aligned} \tag{3.9}$$

Definition 3.4. The collection of equations (3.7) on polynomials $\{q^i_\pm\}_{i=1,\dots,r}(z)$ is called the *qq-system*. The collection of equations (3.9) is called *the full qq-system*.

Using this definition we can formulate the following Theorem.

Theorem 3.5. *i) The element $\mathscr{G}(z)\in G(z)$, which defines the nondegenerate G-Wronskian has a Gaussian decomposition $\mathscr{G}(z) = \mathscr{B}_-(z)\mathscr{N}_+(z)$.*
ii) Generalized minors, which determine $\mathscr{B}_-(z)$ are the solutions of the qq-system, according to the formula $q^{i,w}_+(z) = \Delta_{w^{-1}\cdot\omega_i,\omega_i}(\mathscr{G}(z))$.
iii) Representing $\mathscr{N}_+(z) = e^{n_+(z)}$, the restriction $n_+(z)|_{\mathfrak{b}_{+,1}(z)}$ is determined by the solution of the qq-system.
iv) Given a solution $\mathscr{G}(z)$ of (3.8), we obtain that $\mathscr{G}(z)U_+(z)$, where $U_+(z)\in[N_+,N_+](z)$, i.e. $U_+(z) = e^{u_+(z)}$, so that $u_+(z)\in\oplus_{k\geq 2}\mathfrak{n}_k$.

Proof. To prove i), it is enough to refer to nondegeneracy condition of $\mathscr{G}(z)$ and use 2.4. We derived ii) above. Part iii) follows from the fact that all components of $n_+(z)|_{\mathfrak{b}_{+,1}(z)}$ which is a linear combination of e_i's appear in the right hand side of (3.3), since they act on $f_i\nu_i$. For the same reason iv) is true since $e^{u_+(z)}|_{W_i} = \mathrm{id}_{W_i}$, if $u_+(z) \in \oplus_{k\geq 2}\mathfrak{n}_k$. □

Part iv) of the above Theorem motivates the following definition.

Definition 3.6. Two G-Wronskians $\mathscr{G}_1(z), \mathscr{G}_2(z)$ are called *equivalent* if $\mathscr{G}_1(z) = \mathscr{G}_2(z)U_+(z)$, where $U_+(z) \in [N_+, N_+](z)$.

Theorem (3.5) implies that there is a map from equivalence classes of nondegenerate G-Wronskians to the solution of the full qq-systems, where $\{q_+^{w,i}(z)\}_{w\in W, i=1,\dots,r}$ are nonzero. As one could guess, we would like an inverse map. We will construct it in the last section (we also refer to the last section for the explicit $SL(2)$ example). An important tool for that is the concept of Z-twisted Miura G-oper with regular singularities, which we review in the next section.

4. Z-twisted G-opers on $\mathbb{P}^1$ and the qq-systems

4.1. **Z-twisted G-opers on $\mathbb{P}^1$.** We now define meromorphic G-oper conenctions, or, simply, G-opers, on $\mathbb{P}^1$. Let us consider the pair $(\mathcal{F}_G, \nabla)$ of a principal G-bundle on $\mathbb{P}^1$ and a connection, which is automatically flat. Let $\mathcal{F}_{B_+}$ be a reduction of $\mathcal{F}_G$ to the Borel subgroup B_+. If ∇' is any connection which preserves $\mathcal{F}_{B_+}$, then $\nabla - \nabla'$ induces a well-defined one-form on $\mathbb{P}^1$ with values in the associated bundle $(\mathfrak{g}/\mathfrak{b}_+)_{\mathcal{F}_{B_+}}$. We denote this 1-form by $\nabla/\mathcal{F}_{B_-}$.

Following [BD] (see also [F3]) we will define a G-oper as a G-connection $(\mathcal{F}_G, \nabla)$ together with a reduction $\mathcal{F}_{B_+}$, such that this reduction is not preserved by the connection but instead satisfies a certain condition on the 1-form $\nabla/\mathcal{F}_{B_+}$.

Let $\mathbf{O} \in [\mathfrak{n}_+, \mathfrak{n}_+]^\perp/\mathfrak{b}_+ \in \mathfrak{g}/\mathfrak{b}_+$ be the open B_+-orbit consisting of vectors stabilized by N_+ and such that all of the simple root components with respect to the adjoint action of B_+/N_+, are non-zero, where the orthogonal complement is taken with respect to the Killing form.

Definition 4.1. A meromorphic G-*oper* on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_+})$, where pair $(\mathcal{F}_G, \Delta)$ is a principal G-bundle on $\mathbb{P}^1$ with a meromorphic connection and $\mathcal{F}_{B_+}$ is a reduction of $\mathcal{F}_G$ to B_+ satisfying the following condition: there exists a Zariski open dense subset $U \subset \mathbb{P}^1$ together with a trivialization of $\mathcal{F}_{B_+}$ such that the restriction of the 1-form $\nabla/\mathcal{F}_{B_+}$ to U, written as an element of $\mathfrak{g}/\mathfrak{b}_+(z)$, belongs to $\mathbf{O}(z)$.

Using the trivialization $\imath_{B_-}$, the G-oper can be written as a differential operator:

$$\nabla = \partial_z + \sum_{i=1}^{r} \phi_i(z) f_i + b(z) \tag{4.1}$$

where $\phi_i(z) \in \mathbb{C}(z)$ and $b(z) \in \mathfrak{b}_+(z)$ are regular on U and moreover $\phi_i(z)$ has no zeros in U.

One can impose the following restrictions on ϕ_i:

Definition 4.2. We say that a meromorphic G-oper has regular singularities if $\phi_i(z) = \Lambda_i(z)$, where $\Lambda_i(z) \in \mathbb{C}[z]$ for all $i = 1, \dots, r$.

4.2. **Z-twisted Miura G-opers with regular singularities.**

Definition 4.3. A *Miura G-oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, where $(\mathcal{F}_G, \nabla, \mathcal{F}_{B_+})$ is a meromorphic G-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_-}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_- that is preserved by the connection ∇.

Let us discuss the relative position of the two reductions over any $x \in \mathbb{P}^1$. This relative position will be an element of the Weyl group. To define this, first note that the fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at x is a G-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+ respectively. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $gB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with hB_+ for some $g, h \in G$. The quotient $g^{-1}h$ specifies an element of the double coset space $B_-\backslash G/B_+$. The Bruhat decomposition gives a bijection between this spaces and the Weyl group, so we obtain a well-defined element of G. We say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have *generic relative position* at $x \in \mathbb{P}^1$ if the relative position is the identity element of W. More concretely, this mean that the quotient $g^{-1}h$ belongs to the open dense Bruhat cell $B_-B_+ \subset G$. It turns out the following theorem holds.

Theorem 4.4. *i) For any Miura G-oper on $\mathbb{P}^1$, there exists an open dense subset $V \subset \mathbb{P}^1$ such that the reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in generic relative position for all $x \in V$.*
ii) For any Miura G-oper with regular singularities on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper connection has the form

$$\nabla = \partial_z - \sum_{i=1}^{r} g_i(z)\check{\alpha}_i + \sum_{i=1}^{r} \Lambda_i(z) f_i, \tag{4.2}$$

where $g_i(z), \phi_i(z) \in \mathbb{C}(z)$.

Let us impose a strong condition, which picks up a subset of opers we are interested in, namely the Miura G-opers, which are gauge equivalent to a constant connection.

Definition 4.5. A *Z-twisted Miura G-oper* on $\mathbb{P}^1$ is a Miura G-oper that is equivalent to the constant element $Z \in \mathfrak{b}_- \subset \mathfrak{g}(z)$ under the gauge action of $G(z)$.

Here we immediately can decompose the twist Z into

$$Z = Z^H + Z^{N_-}, \quad Z^H = \sum_{i=1}^{r} \zeta_i \check{\alpha}_i \ , \tag{4.3}$$

breaking it into Cartan and nilpotent part.

For untwisted opers, there is a full flag variety G/B_- of associated Miura opers. If $Z = Z^H$, this space is discrete and is one-to-one correspondence with Weyl group W. In, general, in the twisted case, we must introduce certain closed subvarieties of the flag manifold of the form $(G/B_-)_Z = \{gB_- \mid g^{-1}Zg \in \mathfrak{b}_-\}$, known as *Springer fibers* (see, for example, Chapter 3 of [CG]). For $SL(n)$ (or $GL(n)$), a Springer fiber may be viewed as the space of complete flags in $\mathbb{C}^n$ preserved by a fixed endomorphism.

Proposition 4.6. [BSZ] *The map from Miura Z-twisted opers to Z-twisted opers is a fiber bundle with fiber $(G/B_-)_Z$.*

Taking the quotient of $\mathcal{F}_{B_-}$ by $N_- = [B_-, B_-]$, we obtain an H-bundle $\mathcal{F}_{B_-}/N_-$ endowed with an H-connection, which we will refer to as *associated Cartan connection*: $\nabla^H =$

$\partial_z + A^H(z)$, so that

$$A^H(z) = \sum_{i=1}^{r} g_i(z)\check{\alpha}_i. \tag{4.4}$$

For Z-twisted Miura G-opers, we immediately obtain that

$$g_i(z) = \zeta_i - y_i(z)^{-1}\partial_z y_i(z), \tag{4.5}$$

where $y_i(z) \in \mathbb{C}(z)$. We refer to Section 5 for the explicit $SL(2)$ example.

4.3. **Nondegenerate Z-twisted Miura G-opers, qq-systems and Bäcklund transformations.** Now we will impose nondegeneracy conditions on $y_i(z)$, which will lead us to the relation between Miura G-opers and the qq-systems. We will formulate it in the algebraic manner and refer to [BSZ] for more geometric formulation.

Definition 4.7. A Z-twisted Miura G-oper is called nondegenerate, if:
i) it has the form (4.2) with $g_i(z)$ satisfying (4.5), where:

(1) $y_i(z)$ are polynomials with no multiple zeros;
(2) if $a_{ik} \neq 0$, then the roots of $\Lambda_k(z)$ are distinct from the the zeros and poles of $y_i(z)$; and
(3) if $i \neq j$ and there exists k for which $a_{ik}, a_{jk} \neq 0$, then the zeros and poles of $y_i(z)$ and $y_j(z)$ are distinct from each other.

A related definition, which imposes a similar type of conditions on the solution of the qq-system is as follows:

Definition 4.8. A polynomial solution $\{q^i_+(z), q^i_-(z)\}_{i=1,\dots,r}$ of (3.7) is called *nondegenerate* if each $q^i_+(z)$ is relatively prime to $q^i_-(z)$, and the $q^i_+(z)$'s satisfy the conditions in Definition 4.7.

Then the following statement is true.

Theorem 4.9. [BSZ] *There is one-to-one correspondence between nondegenerate Z-twisted Miura G-opers and nondegenerate solutions of the qq-system, which allows polynomial solutions to the full qq-system, so that*

$$y_i(z) = q^i_+(z), \quad i = 1, \dots, r. \tag{4.6}$$

Moreover, any Z-twisted Miura G-oper is Z^H-twisted.

One can write explicit algebraic equations on zeroes of $q^i_+(z) = \prod_{\ell=1}^{\deg(q^i_+(z))}(z - w^i_\ell)$ polynomials (without the loss of generality we can assume $q^i_+(z)$ to be monic). These algebraic equations are known as *Bethe equations* for Z-twisted ${}^L\mathfrak{g}$-Gaudin model [FFTL], [FFR2]:

$$\langle \alpha_i, Z^H \rangle + \sum_{j=1}^{N} \frac{\langle \alpha_i, \check{\lambda}_j \rangle}{w^i_\ell - z_j} - \sum_{(j,s)\neq(i,\ell)} \frac{a_{ji}}{w^i_\ell - w^j_s} = 0, \tag{4.7}$$
$$i = 1, \dots, r, \quad \ell = 1, \dots, \deg(q^i_+(z)).$$

The correspondence between qq-systems and Bethe ansatz equations is summarized in the following Theorem (see [MV2, MV3], [BSZ].)

Theorem 4.10. *i) If Z^H is regular, there is a bijection between the solutions of the Bethe ansatz equations (4.7) and the nondegenerate polynomial solutions of the qq-system (3.7).*
ii) If $\langle \alpha_l, Z^H \rangle = 0$, for $l = i_1, \dots, i_k$ and is nonzero otherwise, then $\{q^i_+(z)\}_{i=1,\dots,r}$ and $\{q^i_-(z)\}_{i\neq i_1,\dots,i_k}$, are uniquely determined by the Bethe ansatz equations, but each $\{q^{i_j}_-(z)\}$ for $j = 1, \dots k$ is only determined up to an arbitrary transformation $q^{i_j}_-(z) \longrightarrow q^{i_j}_-(z) + c_j q^{i_j}_+(z)$, where $c_j \in \mathbb{C}$.

From now on we will drop the superscript H over Z and consider Z to be an element of Cartan subalgebra. We also remark, that in the case of Gaudin model, one puts the restrictions on degrees of $\{q^i_+(z)\}_{i=1,\dots,r}$, so that they determine a certain weight:

$$\check{\Lambda} = \sum_{i=1}^{N} \check{\lambda}_i - \sum_i \deg(q^i_+(z))\check{\alpha}_i \tag{4.8}$$

in the representation of ${}^L\mathfrak{g}$, and the degrees of $\{q^{i,w}_+\}_{w\in W,\ i=1,\dots,r}$ in the full qq-system determine $w\cdot\check{\Lambda} = \sum_{i=1}^N \check{\lambda}_i - \sum_i \deg(q^{w,i}_+(z))\check{\alpha}_i$. This means that with this restriction on the degrees, the qq-system can be extended to the full qq-system, allowing polynomial solutions. From now on we will assume that qq-systems allow such extension to the full qq-system.

The Miura G-oper connection operator, which correspond to the qq-system can be explicitly written as follows:

$$\nabla = \partial_z - Z + \sum_{i=1}^{r} \partial_z \log(q^i_+(z))\check{\alpha}_i + \sum_{i=1}^{r} \Lambda_i(z) f_i\ . \tag{4.9}$$

Let us explain what role the full qq-system will play in this context. To do that, we use the following Proposition to introduced *Bäcklund transformations*, aligned with the action of the generators of the Weyl group [2].

Proposition 4.11. [BSZ] *Let $\{q^j_+, q^j_-\}_{j=1,\dots,r}$ be a polynomial solution of the qq-system (3.7), and let ∇ be the connection in the form (4.9). Let $\nabla^{(i)}$ be the connection obtained from ∇ via the gauge transformation by $e^{\mu_i(z)e_i}$, where*

$$\mu_i(z) = \Lambda_i(z)^{-1}\left[\partial_z \log\left(\frac{q^i_-(z)}{q^i_+(z)}\right) + \langle \alpha_i, Z^H\rangle\right]. \tag{4.10}$$

Then $\nabla^{(i)}$ is obtained by making the following substitutions in (4.9):

$$\begin{aligned} &q^j_+(z) \mapsto q^j_+(z), && j \neq i, \\ &q^i_+(z) \mapsto q^i_-(z), && Z \mapsto s_i(Z^H) = Z^H - \langle \alpha_i, Z^H \rangle\ \check{\alpha}_i\ . \end{aligned} \tag{4.11}$$

Thus the sequence of Bäcklund transformations produces Z^w-twisted Miura G-oper, where

$$Z^w = \tilde{w} Z \tilde{w}^{-1}, \quad w \in W, \tag{4.12}$$

corresponding to a given G-oper, each of which is constructed from $q^{i,w}_\pm(z)$.

[2] We note here that the degenerate version of the qq-system (Z=0), as well as degenerate version of the Proposition below were introduced by Mukhin and Varchenko [MV1].

5. From Z-twisted Miura G-opers to G-Wronskians

We see that there is a crtain relation between nondegenerate G-Wronskians and nondegenerate G-opers. Let us make this correspondence explicit. The Z-twisted condition implies that that there exist an element $\mathscr{B}_-(z)$, such that

$$\nabla = \mathcal{B}_-^{-1}(z)(\partial_z - Z)\mathcal{B}_-(z), \tag{5.1}$$

where

$$\mathcal{B}_-(z) = \prod_{i=1}^{r} e^{\frac{q^i_-(z)}{q^i_+(z)}f_i} \prod_{i=1}^{r}\Big[q^i_+(z)\Big]^{\check{\alpha}_i}\dots, \tag{5.2}$$

where dots stand for the terms from $[N_-, N_-](z)$. Let us use the following upper-triangular transformation, choosing a certain order in the product below:

$$\mathcal{N}_+(z) = \prod_{i=1}^{r} e^{\frac{(\partial_z \log(q^i_+(z)) - \zeta_i)e_i}{\Lambda_i(z)}} \tag{5.3}$$

such that

$$\mathcal{N}_+^{-1}(z)\nabla\mathcal{N}_+(z) = \partial_z + p^{\Lambda}_{-1} + n_+(z) \tag{5.4}$$

where $n_+(z) \in \mathfrak{n}_+(z)$. Then applying this operator to the highest weight vector ν in some representation V the product $\mathcal{G}(z) = \mathcal{B}_-(z)\mathcal{N}_+(z)$ satisfies the equation

$$\mathcal{G}(z)^{-1}(\partial_z - Z)\mathcal{G}(z)\nu = p^{\Lambda}_{-1}\nu, \tag{5.5}$$

which coincides with equations 3.3 when ν varies over highest weights corresponding to fundamental representations. Turns out this correspondence works in a different direction as well, namely we want to show the following.

Theorem 5.1. *There is a one-to-one correspondence between nondegnerate Z-twisted Miura G-opers, and equivalence classes of G-Wronskians, corresponding to the solution of the nondegenerate full qq-system. The element $\mathcal{G}(z) = \mathcal{B}_-(z)\mathcal{N}_+(z) \in G(z)$, where $\mathscr{B}_-(z)$, $\mathscr{N}_+(z)$ are defined in (5.2), (5.3) correspondingly, is a representative in the class of G-Wronskians corresponding to a given solution of the full qq-system.*

Proof. Given $\mathscr{G}(z) = \mathscr{B}_-(z)\mathscr{N}_+(z) \in G(z)$ corresponding to a G-Wronskian, we have the following equations:

$$\mathscr{N}_+(z)^{-1}\mathscr{B}_-(z)^{-1}(\partial_z - Z)\mathscr{B}_-(z)\mathscr{N}_+(z)\nu_i = p^{\Lambda}_{-1}\nu_i, \quad ,i = 1,\dots,r. \tag{5.6}$$

This means $\mathscr{N}_+(z)^{-1}A(z)\mathscr{N}_+(z)\nu_i = p^{\Lambda}_{-1}\nu_i$, where

$$\partial_z + A(z) = \mathscr{B}_-(z)^{-1}(\partial_z - Z)\mathscr{B}_-(z), \tag{5.7}$$

so that $A(z) \in \mathfrak{b}_-(z)$. Then $A(z) \in p^{\Lambda}_{-1} + \mathfrak{h}_+(z)$, namely a Miura oper connection. That, however, implies that $A(z)$ defines a Miura oper connection. The diagonal part of $\mathscr{B}_-(z)$ is given by principal minors, which are exactly the $q^i_+(z)$, thus reproducing the diagonal part of Z-twisted Miura-Plucker oper. □

At the same time, the nondegeneracy conditions on qq-systems is an open condition, which was only needed for Miura G-oper connection itself and the corresponding Z-twisted condition, which is not the case for G-Wronskians, which only care about the operator (4.9), which satisfies Z-twisted condition as long as $\{q^i_+(z)\}_{i=1,\dots,r}$ satisfy the qq-system. Therefore we have the following theorem.

Theorem 5.2. *There is a one-to-one correspondence between solutions of the full qq-system, such that $q_+^{i,w}(z) \neq 0$ and equivalence classes of nondegenerate G-Wronskians.*

Example: $SL(2)$-opers and $SL(2)$-Wronskians. The Z-twisted Miura $SL(2)$-oper connection is given by the following matrix operator in defining representation of $\mathfrak{sl}(2)$:

$$\partial_z + \begin{pmatrix} \partial_z \log(q_+) - \zeta & \Lambda(z) \\ 0 & \zeta - \partial_z \log(q_+) \end{pmatrix} \tag{5.8}$$

where $\Lambda(z)$ is a polynomial, defining regular singularities $q_+(z)$ is a polynomial, satisfying qq-system:

$$q_+(z)\partial_z q_-(z) - q_-(z)\partial_z q_+(z) + 2\zeta q_+(z) q_-(z) = \Lambda(z) \tag{5.9}$$

The matrices corresponding to $\mathcal{B}_+(z)$ and $\mathcal{N}_+(z)$ are represented by the following matrices:

$$\mathcal{B}_-(z) = \begin{pmatrix} q_+(z) & 0 \\ q_-(z) & q_+(z)^{-1} \end{pmatrix} \quad \mathcal{N}_+(z) = \begin{pmatrix} 1 & \Lambda(z)^{-1}(\partial_z \log[q_+(z)] - \zeta) \\ 0 & 1 \end{pmatrix}, \tag{5.10}$$

so that their product give a version of a Wronskian (one needs to use the qq-system (5.9) to obtain the answer):

$$\mathcal{G}(z) = \mathcal{B}_-(z)\mathcal{N}_+(z) = \begin{pmatrix} q_+(z) & \Lambda^{-1}(\partial_z - \zeta) q_+(z) \\ q_-(z) & \Lambda^{-1}(\partial_z + \zeta) q_-(z) \end{pmatrix}. \tag{5.11}$$

Comparison to $(G,\hbar)$-oper case. In [KZ1] we considered a similar construction related to $(G,\hbar)$-opers. The $(G,\hbar)$-Wronskian equations, which is the q-difference version of (3.3) is as follows:

$$Z^{-1}\mathscr{G}(\hbar z)\nu_i = \mathscr{G}(z) s_\Lambda^{-1}(z)\nu_i, \quad i = 1, \dots r, \tag{5.12}$$

where $\mathscr{G}(z) \in G(z)$, $Z \in H$, $s_\Lambda(z) = \prod_{i=1}^r \Lambda_i^{\check{\alpha}_i}(z) s_i$ is a lift of a chosen Coxeter element to $G(z)$ and $\Lambda_i(z)$ are polynomials, corresponding to regular singularities. As in the differential case, one can define a class of those: namely, if $\mathscr{G}(z)$ is a solution of (5.12), then so is $\mathscr{G}(z)n_+(z)$, as long as $s_\Lambda n_+(z) s_\Lambda^{-1} \in N_+(z)$. Choosing Coxeter element in such a way that $s_\Lambda^{h/2} = w_0^\Lambda$, where h is the Coxeter number and w_0^Λ is a lift of the Weyl reflection, corresponding to the longest root to $G(z)$, we can iterate the equations 5.12:

$$\begin{aligned} &Z^{-k}\mathscr{G}(\hbar^k z)\nu_i = \mathscr{G}(z) s_\Lambda^{-1}(z) s_\Lambda^{-1}(\hbar z) \dots s_\Lambda^{-1}(\hbar^{k-1} z)\nu_i, \\ &i = 1, \dots r, \quad k = 1, \dots h/2. \end{aligned} \tag{5.13}$$

The solution of this kind picks up a unique element in the equivalence class of $(G,\hbar)$-Wronskians, moreover, we were able to write down a universal formula for it. In the $SL(N)$ case that gives a suitably twisted $(G,\hbar)$-Wronskian matrix. Unfortunately it is non-obvious whether if is possible to write a similar universal formula in differential case: the blunt approach using differential analogue of equations (5.13) do not seem to work beyond defining representation of $SL(N)$.

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Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
Email: zeitlin@lsu.edu,
http://math.lsu.edu/∼zeitlin