The Astrophysical Journal Letters, 960:L15 (6pp), 2024 January 10 https://doi.org/10.3847/2041-8213/ad17cc
© 2024. The Author(s). Published by the American Astronomical Society.
OPEN ACCESS

A Fokker–Planck Framework for Studying the Variability of the Magnetic Field Direction in the Alfvénic Streams of the Solar Wind

N. H. Bian[1] and Gang Li[2]
[1] CSPAR, University of Alabama in Huntsville, Huntsville, AL, 35899, USA; nbian@hotmail.com
[2] Department of Space Science and CSPAR, University of Alabama in Huntsville, Huntsville, AL, 35899, USA; gangli.uah@gmail.com
Received 2023 November 18; revised 2023 December 18; accepted 2023 December 20; published 2024 January 8

Abstract

Turbulent rotations of the magnetic field vector are observed in the Alfvénic streams of the solar wind where the magnetic field strength remains close to a constant. They can lead to reversals of the radial magnetic field component or switchbacks. It is not ruled out from the data that the rotations are divisible into the sum of small random angular deflections. In this work, we develop tools aimed at the analysis of the one-point statistical properties of the directional fluctuations of the magnetic field vector in the solar wind. The angular fluctuations are modeled by a drift-diffusion process which admits the exponential distribution as steady-state solution. Realizations of the stochastic process are obtained by solving the corresponding Langevin equation. It is shown that the cumulative effects of consecutive small-angle deflections can yield frequent reversals of the magnetic field vector even when the concentration parameter of the directional data is large. The majority of the rotations are associated with nearly transverse magnetic field fluctuations in this case.

Unified Astronomy Thesaurus concepts: Interplanetary turbulence (830); Interplanetary magnetic fields (824)

1. Introduction

The interplanetary space is filled with solar wind plasma, which expands radially outward, carrying the Sun's magnetic fields. In Parker's (1958) model, the solar wind fluid particles are accelerated within the high-temperature corona. They attain a near-terminal velocity at a source surface where they cease to be significantly accelerated. The solar wind plasma is highly conducting, and hence, the magnetic field lines connect the same fluid particles continuously ejected outward from the rotating Sun. It follows that the magnetic field lines coincide with the solar wind streamlines, in a steady state. This yields the Archimedean spiral structure of the interplanetary magnetic field lines. Measurements of the solar wind magnetic fields confirm Parker's predictions "on average." The average magnetic field direction is close to the spiral angle predicted by Parker, but there are fluctuations about the time average that have been studied over a wide range of averaging timescales. The long-term fluctuations, over a solar rotation or a cycle, contain the broadest amount of information on the angular variability, including the magnetic sectors and their boundaries, but they inherently mix different solar wind conditions (Ness & Wilcox 1966, 1967; Forsyth et al. 1996; Borovsky 2010; Xu et al. 2015; Chang et al. 2022). With the launch of the Parker Solar Probe mission, the statistical properties of the angular deflections of the magnetic field with respect to the theoretical Parker spiral direction can now be studied within the young solar wind streams in the vicinity of the Sun (Dudok de Wit et al. 2020; Fargette et al. 2021, 2022).

As pointed out originally by Jokipii & Parker (1969), the solar wind is turbulent (Coleman 1968), and hence, the Parker spirals are stochastic (Bian & Li 2021, 2022a). On the basis of Leighton's (1964) model of diffusive magnetic flux transfer at the Sun, Jokipii & Parker (1969) developed a boundary-driven model for the angular diffusion of the Parker spirals in the heliosphere, providing a plausible explanation for the angular spread of solar cosmic rays (Meyer et al. 1956) inferred by the Pioneer missions (Fan et al. 1968). Since then, the angular dispersion of solar energetic particles has remained an active subject of investigation, in both longitude and latitude. The observations by Cohen et al. (2017) show that the longitudinal widths of solar energetic particles are largely independent of the charge-to-mass ratio, suggesting that the angular dispersion of the solar energetic particles is mainly determined by the angular dispersion of the magnetic field lines as anticipated by Jokipii & Parker (1969).

From the mean magnetic field direction $\boldsymbol{b}_0$ at the spacecraft, a convenient orthogonal coordinate system for expressing the three components of the fluctuating magnetic field vector is formed by $\boldsymbol{u}_1 = \boldsymbol{b}_0 \times \boldsymbol{u}_r$, $\boldsymbol{u}_2 = \boldsymbol{b}_0 \times (\boldsymbol{b}_0 \times \boldsymbol{u}_r)$ and $\boldsymbol{u}_3 = \boldsymbol{b}_0$, where the unit vector $\boldsymbol{u}_r$ is the direction toward the spacecraft's location from the center of the Sun. The Mariner's observations of Belcher & Davis (1971) already revealed a 5: 4: 1 anisotropy of the magnetic field fluctuations. The nature of the anecdotal $\langle B_1^2\rangle/\langle B_2^2\rangle \sim 1.2$ variance anisotropy within the plane perpendicular to the mean field is not fully understood. The more substantial $(\langle B_1^2\rangle + \langle B_2^2\rangle)/\langle B_3^2\rangle \sim 10$ variance anisotropy can be explained in terms of the predominantly transverse polarization of Alfvénic fluctuations in the turbulent solar wind. Since, in the Alfvénic streams of the solar wind, the magnetic field fluctuations are overall spherically polarized (Belcher & Davis 1971; Barnes 1981), this explanation requires that the bulk of the fluctuations are composed of small-angle rotations of the magnetic field vector around the Parker field direction. Small angular deflections are indeed the most frequent in the Alfvénic streams.

Nonetheless, the bulk of the Alfvénic stream fluctuations coexist with scarce but large amplitude deflections. Some of them can even produce radial switchbacks and spatial folds in the magnetic field. The directional fluctuations of the magnetic

Original content from this work may be used under the terms of the Creative Commons Attribution 4.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

field around the Parker direction are correlated with fluctuations in the radial component of the magnetic field and with fluctuations in the radial component of the velocity field, as expected for Alfvénic fluctuations. Reversals of the radial magnetic field component are often associated with the atypical sunward streaming of the Strahl (Kahler & Lin 1994; Owens et al. 2017), a suprathermal beam of electrons that is formed at the Sun (Feldman et al. 1978; Pilipp et al. 1987). Therefore, it transpires that the role played by the turbulent magnetic field deflections is not limited to the transport of the solar cosmic rays but also has a substantial impact on the transport of the Strahl electrons. The possibility of observing radial switchbacks heavily depends on the angle between the Parker field and the radial direction and on the magnitude of the turbulent fluctuations. As summarized by Borovsky (2016), switchbacks have been observed in the coronal hole plasma by a fleet of space exploration missions, at various distances from the Sun and heliographic latitudes. It includes Helios, Ulysses, Wind, and Ace spacecraft. Magnetic switchbacks became an important subject of investigations after the launch of the Parker Solar Probe (Bale et al. 2019, 2021; Kasper et al. 2019; Dudok de Wit et al. 2020; Macneil et al. 2020; Mozer et al. 2020; Fargette et al. 2021, 2022).

In this work we develop a general framework, based on the Fokker–Planck equation for studying the statistical properties of the angular deflections of the magnetic field with respect to the Parker direction in the fast streams of the solar wind.

2. Directional Statistics of the Magnetic Field Vector

The magnetic field vector $\boldsymbol{B}(t)$ measured by a magnetometer on board the spacecraft in the solar wind can be decomposed into a time-averaged and a fluctuating component according to $\boldsymbol{B}(t) = \langle \boldsymbol{B} \rangle + \delta\boldsymbol{B}(t)$. The solar wind plasma is magnetized, and the time-averaged component provides the privileged direction from which the magnetic field vector deflects at the spacecraft location. Under Taylor's frozen-in turbulence hypothesis, the time evolution of the magnetic field vector can be related to the spatial variations of the magnetic field via the solar wind speed. The recorded time series are interpreted as spatial cuts into the solar wind plasma at the time of the measurement. In this interpretation, the magnetic field vector deflects as a function of the spatial coordinate in the direction of the solar wind velocity.

From the time series of the magnetic field vector at the spacecraft, a hodograph can be constructed. The hodograph is the curve followed by the tip of a time-varying vector with the origin of the vector being fixed. The magnetic field strength $B(t)$ remains nearly a constant within the Alfvénic streams of the solar wind. There, the magnetic field vector predominantly rotates as a function of time while preserving its length: its hodograph lies on a sphere. Hodographs of the solar wind magnetic field vector, including animated ones, can be found in the review by Bruno & Carbone (2013). Moreover, hodographs resulting from Alfvén simple waves (Barnes & Hollweg 1974) can be found in the work of Webb et al. (2010). In the specific model examples discussed by Webb et al. (2010), the hodograph closes on itself after a not-too-large number of turning points. Closed loops in the hodographs are unlikely to be the rule in the turbulent conditions of the solar wind. Noticeable exceptions are provided by magnetic switchbacks (Laker et al. 2022) and also by the coherent pulse-like Alfvénic events (Riley et al. 1996; Gosling et al. 2011), which appear to be arc polarized.

Let us denote the time-dependent direction of the magnetic vector by $\boldsymbol{b}(t) = \boldsymbol{B}(t)/B(t)$. While it requires two angles to draw its hodograph, we will here focus on the statistical properties of the "pitch-angle" cosine, defined as

$$\mu(t) = \boldsymbol{b}(t) \cdot \boldsymbol{b}_0 = \cos\theta(t), \tag{1}$$

where $\boldsymbol{b}_0$ is the mean field direction. We are here more concerned by the evolution of the polar angle, chosen to vary from 0 to π, than by the evolution of the azimuthal angle $\phi(t)$. In Parker's model, the direction $\boldsymbol{b}_0$ is tangential to Parker's spirals and, from a Lagrangian viewpoint, it coincides with the direction of the motion of the solar wind fluid particles in the frame corotating with the Sun.

In this work, we are primarily interested in the modeling of the turbulent rotations of the magnetic field vector by directional stochastic processes. Directional statistics is a subdiscipline of statistics that deals with the directions of vectors and their rotations. Stochastic processes are mathematical models for the behavior of multivariate random variables or random vectors. Therefore, directional stochastic processes belong to the subclass of stochastic processes aimed at modeling the fluctuations in the direction of random vectors and their rotations. Perhaps the most fundamental directional stochastic process is given by the rotational Brownian motion $\boldsymbol{b}(t)$, which is the solution of the Langevin equation

$$\frac{d\boldsymbol{b}(t)}{dt} = \sqrt{\nu}\boldsymbol{b}(t) \times \boldsymbol{\zeta}(t), \tag{2}$$

where $\boldsymbol{\zeta}(t)$ is the unit Gaussian white noise vector and ν is the angular diffusivity. The spherical Brownian motion describes continuous random rotations of a vector. An important solar physics application of the spherical Brownian motion is provided by Leighton's (1964) model for the turbulent dispersion of the magnetic footpoints on the photosphere, which reads $d\boldsymbol{r}(t)/dt = \boldsymbol{r}(t) \times [\boldsymbol{\Omega} + \sqrt{\nu}\boldsymbol{\zeta}(t)]$, where $\boldsymbol{\Omega}$ is the rotation vector of the Sun. The boundary-driven stochastic Parker spirals of Jokipii & Parker (1969) straightforwardly derives from the Leighton's model (1964) by using $r = r_0 + V_{\rm sw}t$ describing the trajectories of the solar wind fluid particles continuously emitted from the footpoints at the spherical source surface of radius r_0. It follows that the angular diffusivity of the boundary-driven stochastic Parker spirals, measuring the amount of their angular spread per unit of the radial distance from the Sun, is given by $\nu/V_{\rm sw}$. We note that the solar wind fluid particle trajectories are straight lines in the stochastic Parker spiral model of Jokipii & Parker (1969), and hence, the basic structure of the solar wind is left intact with respect to Parker's (1958) model. Therefore, this boundary-driven model does not account for the effects of in situ solar wind velocity field fluctuations. We also note that the spherical Brownian motion is nowhere differentiable, as is the Brownian motion. Therefore, it is *not* possible to assign a direction, nor a length, to the stochastic Parker spirals in their original form. A smooth version of Leighton's (1964) model was considered in Li & Bian (2023), where the diffusion process is replaced by an Ornstein–Uhlenbeck process (Chandrasekhar 1943). Based on the analysis of the Lagrangian properties of Alfvénic turbulence, Bian & Li (2022b) established a refined model for the local structure of the stochastic Parker spirals, which is consistent with the $k_{\parallel}^{-2}$ spectrum (Horbury et al. 2008; Podesta 2009; Wicks et al. 2010) of the magnetic field fluctuations derived from in situ measurements in the solar wind. The spectral index equal to -2 is predicted by theories elucidating the importance of the critical balance condition in Alfvénic turbulence (Goldreich & Sridhar 1995; Boldyrev 2006; Schekochihin et al. 2009). Bian & Li (2022b) also pointed out that the magnetic field line diffusivity, and hence the angular diffusivity, can directly be inferred from the observations by extrapolating to $k_{\|} = 0$ the measured spectral energy distribution of the magnetic field fluctuations.

Directional data have the property that they cannot have Gaussian statistics. However, the normal distribution can be used to fit directional data when their concentration is large. A large angular concentration is the hallmark of the small-angle approximation applied to directional stochastic processes. A fundamental probability distribution function, in directional statistics, is the von Mises–Fisher distribution (Watson 1982). It is given by

$$P(\boldsymbol{b}; \boldsymbol{b}_0, \kappa) = \frac{\kappa}{4\pi \sinh \kappa} \exp[\kappa \boldsymbol{b} \cdot \boldsymbol{b}_0], \quad (3)$$

where $\boldsymbol{b}_0$ is the mean direction and κ is the concentration parameter of the directional distribution. When the concentration parameter κ is zero, the density is constant. It corresponds to the situation where the directional data cover the sphere in a uniform manner. In terms of the polar angle θ, the uniform distribution is $P(\theta) = \sin\theta$, and the Fisher distribution is $P(\theta, \kappa) = (\kappa/2\sinh\kappa)\sin\theta \exp[\kappa\cos\theta]$. We note that the Fischer distribution corresponds to the Gibbs–Boltzmann distribution for the orientation of a statistical ensemble of noninteracting atoms in a paramagnetic material in the presence of an applied magnetic field. In his classical theory of paramagnetism, Langevin assigns the Hamiltonian $H(\theta) = -\boldsymbol{M} \cdot \boldsymbol{B}$ to the atom, where $\boldsymbol{M}$ is the magnetic moment. It yields the Gibbs distribution $f(\theta) = Z^{-1}\exp[MB\cos\theta/k_B T]$ in the equilibrium state with $Z = \int e^{MB\cos\theta/k_B T}\sin(\theta)d\theta$ the partition function. Coincidentally, the Fisher distribution is also the steady-state angular distribution of the velocity vector of a solar cosmic ray undergoing pitch-angle scattering while interacting with the fluctuating components of the solar wind magnetic field. Scattering of particles due to magnetic field fluctuations is elastic. Conservation of the first adiabatic invariant focuses the pitch angle of the velocity vector in the mean field direction $\boldsymbol{b}_0$. Similarly, the steady-state pitch-angle distribution of the fluctuating magnetic vector in the Alfvénic streams of the solar wind is well described by the exponential distribution (Bavassano & Mariani 1983; Borovsky 2016; Dudok de Wit et al. 2020)

$$P(\mu) = \frac{\kappa}{2\sinh\kappa}\exp[\kappa\mu], \quad (4)$$

over the domain $[-1, 1]$. Here, $\mu = 1$ corresponds to the Parker field direction. The mean pitch-angle cosine of this directional distribution is related to the concentration parameter by the Langevin function

$$\langle\mu\rangle = \coth\kappa - \frac{1}{\kappa}, \quad (5)$$

showing that when the concentration parameter κ is large, $\langle\mu\rangle \to 1$, and on the contrary, when $\kappa \to 0$, then $\langle\mu\rangle \to \kappa/3$. We observe that when the concentration parameter is large, it is legitimate to make the small-angle approximation, which yields the Gaussian function $P(\theta, \kappa)/\theta \propto \exp[-\kappa\theta^2/2]$. The Gaussian distribution has often formed the basis of the fit to the observed angular distributions of the magnetic field vector in the solar wind. The accuracy of the Gaussian fit to the angular data, which ought to deviate in distribution from the normal, relies on the accuracy of the small-angle approximation $\sin\theta \sim \theta$, $\cos\theta \sim 1 - \theta^2/2$. It corresponds to the strong focusing limit in the context of the transport of solar energetic particles (Bian & Emslie 2019, 2020).

Let us now recall the principles of the drift-diffusive modeling of stochastic processes (Risken 1989), here pertinent to the directional variability of a vector. The following brief presentation is intended to apply to the one-point statistical properties of the angular deflections of the magnetic field vector in the solar wind. From the measurements of the fluctuating magnetic field vector, the time-ordered sequence of pitch-angle cosine μ_0, μ_1..., μ_n, corresponding to $t_0 < t_1 < ... < t_n$, can be formed. The joint probability distribution function of these angular deflections is denoted by $P(\mu_0, \mu_1,...., \mu_n)$. This joint probability can be conceived as a path probability in the history of the angular variability of the magnetic vector. For a Markov process, the conditional probability distribution function $P(\mu_n|\mu_{n-1}, \ldots, \mu_0)$ depends only on the last angular deflection: $P(\mu_n|\mu_{n-1}, \ldots, \mu_0) = P(\mu_n|\mu_{n-1})$. The continuous-time probability density distribution $P(\mu, t) = \langle\delta(\mu(t) - \mu)\rangle$ is a statistical average over the ensemble of realizations $\mu(t)$, conditional to $\mu = \mu_0$ at time $t = t_0$. For a Markov process, it obeys the integro-differential equation

$$\frac{\partial P(\mu, t)}{\partial t} = \int [W(\mu|\mu')P(\mu', t) - W(\mu'|\mu)P(\mu, t)]d\mu', \quad (6)$$

which is known as the master equation. The master equation states that the time variation of the probability density to observe the direction μ with accuracy $d\mu$ at time t results from the difference between the sum of all the angular deflections from μ' to μ, occurring with probability $W(\mu|\mu')$ per unit time and the sum of all the angular deflections from μ to μ' occurring with probability $W(\mu'|\mu)$ per unit time. The transition probabilities $W(\mu'|\mu)$ and $W(\mu|\mu')$ are angular deflection probabilities. They generally depend on μ and on time. A particular example of a master equation is the nonlocal diffusion equation $\partial P(\mu, t)/\partial t = \int W(\mu' - \mu)P(\mu', t)d\mu' - \lambda_\mu P(\mu, t)$, where the deflection probabilities only depend on the deflection amplitude $\Delta\mu = \mu - \mu'$ and where the frequency λ_μ is defined by $\lambda_\mu = \int W(\Delta\mu)d\Delta\mu$. Nonlocal diffusion processes involve a convolution kernel, which can be expressed as a differential operator via a Taylor expansion. The Taylor expansion of a nonlocal diffusion equation, with a time-independent Gaussian step-size probability, is the starting point of Einstein's theory of the Brownian motion. Writing $W(\mu|\mu') = W(\mu', \Delta\mu)$, the master equation is Taylor-expanded into

$$\frac{\partial P(\mu, t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial\mu^n}[a_n(\mu, t)P(\mu, t)],$$
$$a_n(\mu, t) = \int \Delta\mu^n W(\mu, \Delta\mu) d(\Delta\mu). \quad (7)$$

Equation (7) constitutes the Kramers–Moyal expansion of the master equation and Pawula's theorem states that if a_3 is zero,

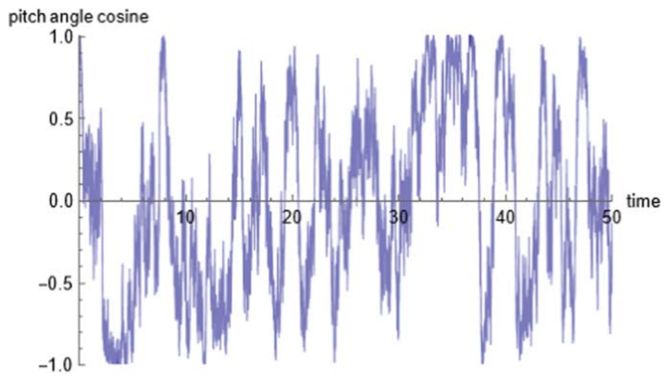

Figure 1. Modeled time series of the pitch-angle cosine $\mu(t) = \cos\theta(t)$ of the magnetic field vector, solution of Equation (13) for $\kappa = 0$. The concentration parameter is zero. It corresponds to Barnes's (1981) spherical diffusion model. Time is normalized by the inverse angular diffusivity of the magnetic field vector.

then all higher-order moments are zero; and when the fourth-order moment a_4 is nonzero, then all the even moments of order higher than four are also nonzero (Risken 1989). Therefore, the evolution of the magnetic vector direction can be classified into three main categories depending on whether (1) only a_1 is nonzero, and hence the dynamics is deterministic without fluctuations; (2) a_3 is zero, and hence, the Kramers–Moyal expansion of the master equation truncates at a_2 and the Markov process is a drift-diffusion process described by the Fokker–Planck equation; and (3) a_4 is nonzero, and hence the high degree of nonlocality involved in a single-step angular deflection is such that the Fokker–Planck description does not apply. We observe that the degree of nonlocality is necessarily bounded by π for directional stochastic processes.

Let us assume that the time evolution of the magnetic field vector direction, measured at the spacecraft location, is a random Markov process. Moreover, let us assume that a_4 is much smaller than a_2. Therefore, cases (1) and (3) above can be ruled out. It leaves us with the Fokker–Planck or Kolmogorov forward equation

$$\frac{\partial P(\mu, t)}{\partial t} + \frac{\partial}{\partial \mu}[a_1(\mu)P(\mu, t)] = \frac{1}{2}\frac{\partial^2}{\partial \mu^2}[a_2(\mu)P(\mu, t)], \quad (8)$$

describing the stochastic variability of the magnetic vector direction measured at one point in the solar wind. We took care of spelling the main assumptions involved in a Fokker–Planck model. It is important to note that the probability $P(\mu, t)$ is generally *not* Gaussian because the drift and the diffusion coefficients are nonlinear functions of μ. The only linear Gaussian stochastic models are the Wiener and the Ornstein–Uhlenbeck processes, which are inapplicable in the context of angular statistics, except at small angles. Directional stochastic processes are necessarily non-Gaussian and nonlinear. The steady-state solution of Equation (8) is given by $P(\mu) = [C/a_2(\mu)]\exp\int^{\mu}[2a_1(\mu)/a_2(\mu)]d\mu$. Therefore, the drift and the diffusion coefficients are connected via the steady-state distribution $P(\mu)$. It follows that the knowledge of the steady-state distribution and of the diffusion coefficient $a_2(\mu)$ uniquely constrains the form of the drift coefficient $a_1(\mu)$, and vice versa. Obtaining the steady-state distribution $P(\mu)$ from the directional data is tantamount to the histogram of the time series $\mu(t)$, under the stationary assumption. More challenging is the inverse problem of deriving both the drift and the diffusion coefficients, in the Fokker–Planck equation, from the measured time series. The Fokker–Planck Equation (8) is equivalent to the stochastic differential equation

$$\frac{d\mu(t)}{dt} = a_1(\mu) + \sqrt{a_2(\mu)}\,\zeta(t), \quad (9)$$

where $\zeta(t)$ is the unit Gaussian white noise. The coefficient $a_1(\mu)$ is called the drift insofar as it determines the average evolution via $d\langle\mu(t)\rangle/dt = \langle a_1(\mu)\rangle$. This equation for the first moment generally couples to those for the higher-order moments, except in the special case where the drift coefficient is a linear function of μ. When $a_1(\mu) = \nu\mu$, the average evolution determines the autocorrelation function $C_\mu(t) = \langle\mu(0)\mu(t)\rangle$, which ought to decay exponentially at a rate given by the angular diffusivity. A Fourier transform yields the Lorentzian spectrum. At frequencies $\omega \gg \nu$, the spectral energy density of the process is the power law $E_\mu(\omega) \propto \omega^{-2}$, which coincides with the power spectrum of the Brownian motion. We note that using the Alfvén wave dispersion relation $\omega = V_A k_\parallel$, it follows that $E_\mu(k_\parallel) \propto k_\parallel^{-2}$.

The modeling of the time evolution of the solar wind magnetic field vector direction by a Fokker–Planck equation was originally advocated by Barnes (1981). The stochastic framework adopted by Barnes (1981) received some credence from the Helios measurements analyzed by Bavassano & Mariani (1983). We first briefly review his approach and extend it the next. Barnes's (1981) model is the rotational Brownian motion for the magnetic field vector direction given by Equation (2). In this model, the probability density distribution of the pitch-angle cosine evolves according to

$$\frac{\partial P(\mu, t)}{\partial t} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial P}{\partial \mu}\right), \qquad D_{\mu\mu} = \frac{1}{2}\nu(1 - \mu^2), \quad (10)$$

where the angular diffusivity ν is assumed constant. The diffusion form of Equation (10) allows to immediately infer the steady-state pitch-angle distribution, which is the uniform distribution in this model. Therefore, all the solutions of Equation (10) relax over time to the distribution function given by Equation (4) with the concentration parameter $\kappa = 0$. Equation (10) is written in the form of a diffusion equation, not yet in the form of a Fokker–Planck equation. Nevertheless, Barnes's (1981) diffusion model can be written in the form of a Fokker–Planck Equation (8) with the drift and the diffusion coefficients given by $a_1(\mu) = \partial D_{\mu\mu}/\partial\mu = -\nu\mu$, $a_2(\mu)/2 = D_{\mu\mu} = \nu(1 - \mu^2)/2$. Barnes's (1981) model is the restriction to the polar angle θ of the spherical diffusion equation. The spherical diffusion equation is itself the restriction to the sphere of the standard diffusion equation. Equation (10) can be solved by the separation of the variables yielding the particular solution $P(\mu, t) = P_\alpha(\mu)\exp\left[-\frac{1}{2}\alpha(1 + \alpha)\nu t\right]$, where $P_\alpha(\mu)$ is the Legendre function of the first kind of order α. The admissible values of α are dictated by the boundary conditions. Diffusion over the whole sphere imposes that α is an integer. It follows that the solution of the spherical diffusion equation can be expressed as an infinite sum of Legendre polynomials. A realization of $\mu(t)$ in Barnes's (1981) spherical diffusion model is plotted in Figure 1. Barnes (1981) also discusses the case of a reflecting boundary on the sphere. A realization of $\mu(t)$ with such a reflecting boundary condition at $\mu = 0$ would look like the one plotted in Figure 1 but constrained to vary between 1 and 0. In any case, all the directional processes investigated by Barnes (1981) have a uniform steady-state distribution either on the whole sphere or over any solid angle subtended by the sphere at its center, as expected.

Let us add a drift term in Barnes's (1981) diffusion model given by Equation (10) and require that the steady-state probability density distribution is given by Equation (4). There is only one functional form of the drift coefficient that is consistent with Equation (4). It yields the drift-diffusion equation

$$\frac{\partial P(\mu, t)}{\partial t} + \frac{\partial}{\partial \mu}\left[\frac{1}{2}\kappa\nu(1-\mu^2)P\right] = \frac{\partial}{\partial \mu}\left[\frac{1}{2}\nu(1-\mu^2)\frac{\partial P}{\partial \mu}\right]. \quad (11)$$

Equation (11) is the *unique* drift-diffusion model describing the turbulent deflections of the magnetic vector direction that yields the exponential steady-state distribution, provided the angular diffusivity is a constant. It follows from Equation (11) that the drift and the diffusion coefficients in the Fokker–Planck Equation (8) are now given by

$$a_1(\mu) = \frac{1}{2}\kappa\nu(1-\mu^2) - \nu\mu, \qquad a_2(\mu) = \nu(1-\mu^2). \quad (12)$$

This Fokker–Planck equation is equivalent to the stochastic differential equation for the pitch-angle cosine $\mu(\tilde{t})$ given by

$$\frac{d\mu(\tilde{t})}{d\tilde{t}} = \frac{1}{2}\kappa(1-\mu^2) - \mu + \sqrt{(1-\mu^2)}\,\zeta(\tilde{t}), \qquad \tilde{t} = \nu t, \quad (13)$$

where time is measured in units of the inverse of the angular diffusivity.

A realization of $\mu(t)$ for $\kappa = 0$ was plotted in Figure 1. A realization of $\mu(t)$ for $\kappa = 3$ is plotted in Figure 2. As it can be seen from Figure 2, the small-angle deflections around $\mu = 1$ constitute the bulk of the fluctuations of the magnetic field vector. This property of the model is consistent with the observations of a substantial variance anisotropy, first revealed by Belcher & Davis (1971) from Mariner's measurements of the magnetic field in the Alfvénic streams of the solar wind. The small-angle deflections correspond to turbulent magnetic field fluctuations that are near transverse to the mean field direction. We note that there is a broader class of drift-diffusion models yielding the exponential steady-state distribution, including the case where the angular diffusivity ν in Equation (11) is a function of μ. In this case, taking the small-angle θ limit, the μ-dependent angular diffusivity $\nu(\mu)$ can be approximated by the first, the constant term, in its Taylor expansion. It also yields Equation (11).

3. Discussions and Conclusion

In this work, we established the basis of a Fokker–Planck framework for studying the variability of the magnetic field direction in the Alfvénic streams of the solar wind. The time series of the pitch-angle cosine $\mu(t) = \boldsymbol{b}(t) \cdot \boldsymbol{b}_0$ of the magnetic field vector $\boldsymbol{B}(t)$ recorded by spacecraft in the turbulent solar wind are modeled by the Langevin equation

$$\mu(t+\tau) = \mu(t) + a_1(\mu)\tau + \sqrt{a_2(\mu)\tau}\,\zeta, \quad (14)$$

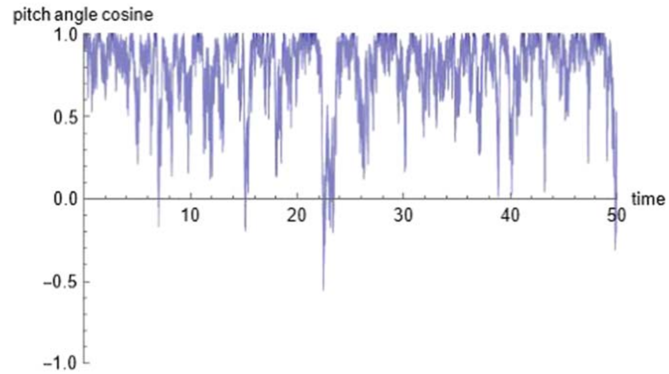

Figure 2. Modeled time series of the pitch-angle cosine $\mu(t) = \cos\theta(t)$ for the concentration parameter $\kappa = 3$. The value $\mu = 1$ corresponds to Parker's prediction.

where ζ is a unit Gaussian random number. The drift $a_1(\mu)$ and the diffusion $a_2(\mu)$ coefficients are related to the two first moments of the increments $\mu(t+\tau) - \mu(t)$, conditional to $\mu(t) = \mu$ at time t. They derive from $\mu(t)$ via the conditional average

$$a_n(\mu) = \lim_{\tau\to 0}\frac{1}{\mathrm{n}!\tau}\langle[\mu(\mathrm{t}+\tau) - \mu(\mathrm{t})]^n | \mu(\mathrm{t}) = \mu\rangle, \quad (15)$$

where $n = 1, 2$. In the Alfvénic streams of the solar wind, the steady-state probability density $P(\mu)$ is often observed to be close to the exponential distribution $P(\mu) = (\kappa/2\sinh\kappa)\exp[\kappa\mu]$, where the concentration parameter κ is the unique parameter controlling its shape. We note that while Dudok de Wit et al. (2020) did not attempt to fit the measured directional distribution by the exponential distribution, it appears from their Figure 4 that the logarithm of the distribution is close to linear. The slope of the linear function in a log-linear plot is precisely the concentration parameter κ of the directional distribution. In the small-angle approximation $\mu \sim 1 - \theta^2/2$, and hence, the exponential in μ becomes Gaussian in θ, yielding the standard fit to the observed angular distributions of the magnetic field vector in the solar wind. Guided by this observation, we modeled the time series $\mu(t)$ recorded at the spacecraft by the Langevin Equation (14) where the drift and the diffusion coefficients are given by Equation (12). The main assumptions are that the directional variability is stochastic and Markov. The present model is the natural extension of Barnes's (1981) spherical diffusion model which is recovered for $\kappa = 0$. Barnes's (1981) model received some credence from the Helios measurements analyzed by Bavassano & Mariani (1983). Bavassano & Mariani (1983) also considered the statistical properties of the relative rotation or the angle between two consecutively measured magnetic vectors, for a fixed time lag. The evolution with the time lag was extensively studied more recently by Matteini et al. (2018). Bavassano & Mariani (1983) observe that the frequency distribution of μ does not relax to the uniform distribution with $\kappa = 0$, and hence, $\langle\mu\rangle$ does not converge to zero at large times, as anticipated from the existence of a mean field: the Parker field. Nevertheless, Bavassano & Mariani (1983) use the Helios data to show that the first-order moment relaxes to nearly the same constant value $\langle\mu\rangle \sim 0.7$, on a timescale of about 10 minutes at 0.3 au and on a larger timescale of about 15 minutes near the Earth. From examination of the Langevin function that relates $<\mu>$ and κ, we can deduce the value of the concentration parameter $\kappa \simeq 3$ in these measurements. A realization of the directional drift-diffusion process is plotted in Figure 2 for $\kappa = 3$. It is clear from Figure 2 that, because of the exponential form of the steady-state directional distribution, there are many occurrences of reversals even when the concentration parameter is large. In this case, the bulk of the rotations are resulting from nearly transverse magnetic field fluctuations. Following Barnes (1981), we adopted here a forward-modeling approach. However, the drift and the diffusion coefficients can be extracted from the solar wind data themselves by evaluating the two first, $n = 1$, 2 conditional moments of $\mu(t+\tau) - \mu(t)$ with $\mu(t) = \mu$ in the domain $[-1, -1]$. As pointed out by Friedrich et al. (2011), the Fokker–Planck equation provides a solid framework for the analysis of complex time series. The validity of the Langevin equation can be tested from the solar wind data and the drift and the diffusion coefficients obtained by solving an inverse problem. The Fokker–Planck approach that deals here with the properties of one-point Eulerian directional data can be extended to two-point Eulerian directional data, for instance, the angle $\Delta\theta(\tau)$ between two consecutive magnetic field vectors $\boldsymbol{B}(t)$ and $\boldsymbol{B}(t+\tau)$, as a function of the time lag τ. Matteini et al. (2018) have shown that at the largest time lags corresponding to the $1/f$ range, the probability density distribution of $\cos\Delta\theta$ does not become a constant, corresponding to directional fluctuations uniformly covering the whole sphere but instead saturates to a scale-independent exponential distribution with a concentration parameter of the order of unity. We note that a Fokker–Planck approach has been used to investigate the self similarity of the probability distribution function of certain solar wind plasma quantities (Hnat et al. 2003), which also pertains to the intermittency of the solar wind MHD turbulence, as we have investigated here.

Acknowledgments

This work is supported in part by NASA grants 80NSSC19K0075, 80NSSC21K1327, and 80NSSC22K0268, and NSF ANSWERS grant 2149771 at UAH. Support by ISSI and ISSI-BJ through the international team 469 is also acknowledged.

ORCID iDs

N. H. Bian https://orcid.org/0000-0003-0142-8669
Gang Li https://orcid.org/0000-0003-4695-8866

References

Bale, S. D., Badman, S. T., Bonnell, J. W., et al. 2019, Natur, 576, 237
Bale, S. D., Horbury, T. S., Velli, M., et al. 2021, ApJ, 923, 174
Barnes, A. 1981, JGR, 86, 7498
Barnes, A., & Hollweg, J. V. 1974, JGR, 79, 2302
Bavassano, B., & Mariani, F. 1983, in NASA Conf. Publ. 228, Solar Wind 5 Conf. Proc. (Washington, DC: NASA), 99
Belcher, J. W., & Davis, L. J. 1971, JGR, 76, 3534
Bian, N. H., & Emslie, A. G. 2019, ApJ, 880, 11
Bian, N. H., & Emslie, A. G. 2020, ApJ, 897, 34
Bian, N. H., & Li, G. 2021, ApJ, 908, 45
Bian, N. H., & Li, G. 2022a, ApJ, 924, 120
Bian, N. H., & Li, G. 2022b, ApJ, 941, 58
Boldyrev, S. 2006, PhRvL, 96, 115002
Borovsky, J. E. 2010, JGRA, 115, A09101
Borovsky, J. E. 2016, JGRA, 121, 5055
Bruno, R., & Carbone, V. 2013, LRSP, 10, 2
Chandrasekhar, S. 1943, RvMP, 15, 1
Chang, Q., Xu, X., Wang, X., et al. 2022, ApJ, 931, 105
Cohen, C. M. S., Mason, G. M., & Mewaldt, R. A. 2017, ApJ, 843, 132
Coleman, Paul J., Jr. 1968, ApJ, 153, 371
Dudok de Wit, T., Krasnoselskikh, V. V., Bale, S. D., et al. 2020, ApJS, 246, 39
Fan, C. Y., Pick, M., Pyle, R., Simpson, J. A., & Smith, D. R. 1968, JGR, 73, 1555
Fargette, N., Lavraud, B., Rouillard, A. P., et al. 2021, ApJ, 919, 96
Fargette, N., Lavraud, B., Rouillard, A. P., et al. 2022, A&A, 663, A109
Feldman, W. C., Asbridge, J. R., Bame, S. J., Gosling, J. T., & Lemons, D. S. 1978, JGR, 83, 5285
Forsyth, R. J., Balogh, A., Smith, E. J., Erdös, G., & McComas, D. J. 1996, JGR, 101, 395
Friedrich, R., Peinke, J., Sahimi, M., & Reza Rahimi Tabar, M. 2011, PhR, 506, 87
Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 763
Gosling, J. T., Tian, H., & Phan, T. D. 2011, ApJL, 737, L35
Hnat, B., Chapman, S. C., & Rowlands, G. 2003, PhRvE, 67, 056404
Horbury, T. S., Forman, M., & Oughton, S. 2008, PhRvL, 101, 175005
Jokipii, J. R., & Parker, E. N. 1969, ApJ, 155, 777
Kahler, S., & Lin, R. P. 1994, GeoRL, 21, 1575
Kasper, J. C., Bale, S. D., Belcher, J. W., et al. 2019, Natur, 576, 228
Laker, R., Horbury, T. S., Matteini, L., et al. 2022, MNRAS, 517, 1001
Leighton, R. B. 1964, ApJ, 140, 1547
Li, G., & Bian, N. H. 2023, ApJ, 945, 150
Macneil, A. R., Owens, M. J., Wicks, R. T., et al. 2020, MNRAS, 494, 3642
Matteini, L., Stansby, D., Horbury, T. S., & Chen, C. H. K. 2018, ApJL, 869, L32
Meyer, P., Parker, E. N., & Simpson, J. A. 1956, PhRv, 104, 768
Mozer, F. S., Agapitov, O. V., Bale, S. D., et al. 2020, ApJS, 246, 68
Ness, N. F., & Wilcox, J. M. 1966, ApJ, 143, 23
Ness, N. F., & Wilcox, J. M. 1967, SoPh, 2, 351
Owens, M. J., Lockwood, M., Riley, P., & Linker, J. 2017, JGRA, 122, 980
Parker, E. N. 1958, ApJ, 128, 664
Pilipp, W. G., Miggenrieder, H., Montgomery, M. D., et al. 1987, JGR, 92, 1075
Podesta, J. J. 2009, ApJ, 698, 986
Riley, P., Sonett, C. P., Tsurutani, B. T., et al. 1996, JGR, 101, 19987
Risken, H. 1989, The Fokker–Planck Equation. Methods of Solution and Applications (2nd ed.; Berlin: Springer)
Schekochihin, A. A., Cowley, S. C., Dorland, W., et al. 2009, ApJS, 182, 310
Watson, G. S. 1982, J. Appl. Probab., 19, 265
Webb, G. M., Hu, Q., Dasgupta, B., Roberts, D. A., & Zank, G. P. 2010, ApJ, 725, 2128
Wicks, R. T., Horbury, T. S., Chen, C. H. K., & Schekochihin, A. A. 2010, MNRAS, 407, L31
Xu, F., Li, G., Zhao, L., et al. 2015, ApJ, 801, 58