

HARMONIC FUNCTIONS ON FINITELY CONNECTED TORI*

CHIU-YEN KAO[†], BRAXTON OSTING[‡], AND ÉDOUARD OUDET[§]

Abstract. In this paper, we prove a logarithmic conjugation theorem on finitely connected tori. The theorem states that a harmonic function can be written as the real part of a function whose derivative is analytic and a finite sum of terms involving the logarithm of the modulus of a modified Weierstrass sigma function. We implement the method using arbitrary precision and use the result to find approximate solutions to the Laplace problem and the Steklov eigenvalue problem. Using a posteriori estimation, we show that the solution of the Laplace problem on a torus with a few holes has error less than 10^{-100} using a few hundred degrees of freedom and the Steklov eigenvalues have similar error.

Key words. harmonic function, Laplace equation, finitely connected torus, doubly periodic domain, elliptic function, Weierstrass elliptic function

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1. Introduction. Harmonic functions satisfying the Laplace equation, $\Delta u = 0$, arise in many physical applications, including potential flow in fluid dynamics, the stationary solution of heat conduction, and electrostatics in the absence of charges, to name just a few. Efficient and robust numerical approaches to solving the Laplace equation on a general domain with different boundary conditions are crucial for understanding the aforementioned applications. In this paper, *we are particularly interested in solving the Laplace equation on finitely connected tori*, which serves as a model problem for the study of heat or electrical conduction in the exterior of a periodic lattice of inclusions with prescribed boundary conditions or for fluid flow through a doubly periodic array of obstacles.

Harmonic functions. It is well known that every harmonic function u on a simply connected domain $\Omega \subset \mathbb{C}$ can be written as the real part of an analytic function, $f(z)$, where

$$(1.1) \quad u(z) = \Re f(z).$$

For finitely connected domains, the analogous result is known as the *logarithmic conjugation theorem* [2, 21]. Let $\Omega \subset \mathbb{C}$ be a finitely connected region, which means that $\mathbb{C} \setminus \Omega$ has only finitely many bounded connected components, $\{K_j\}_{j \in [b]}$, with

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[†]Department of Mathematical Sciences, Claremont McKenna College, Claremont, CA 91711 USA (Chiu-Yen.Kao@ClaremontMcKenna.edu).

[‡]Department of Mathematics, University of Utah, Salt Lake City, UT 84112 USA (osting@math.utah.edu).

[§]LJK, Université Grenoble Alpes, 38041 Grenoble Cedex 9, France (Edouard.Oudet@imag.fr).

$b \in \mathbb{N} \setminus \{0\}$. For each $j \in [b]$, let a_j be a point in K_j . If u is a harmonic function on Ω , then there exist an analytic function f on Ω and real numbers c_j , $j \in [b]$, such that

$$(1.2) \quad u(z) = \Re f(z) + \sum_{j \in [b]} c_j \log |z - a_j|, \quad z \in \Omega.$$

Our main result is to extend the logarithmic conjugation theorem to finitely connected tori. We consider a torus $\mathbb{T}_\omega = \mathbb{C}/L_\omega$, where $L_\omega = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$ is a lattice and $\omega = (\omega_1, \omega_2) \in \mathbb{C}^2$ are half-periods, assumed not to be colinear. Let

$$(1.3) \quad \Omega = \mathbb{T}_\omega \setminus \bigcup_{j \in [b]} K_j$$

denote the finitely connected torus after removing $b \in \mathbb{N} \setminus \{0\}$ disjoint, connected compact sets $\{K_j\}_{j \in [b]}$ with smooth boundary. We also introduce the parallelogram (fundamental domain)

$$(1.4) \quad \mathcal{P} = \{2\omega_1 x + 2\omega_2 y \in \mathbb{C} : (x, y) \in [0, 1]^2\} \setminus \bigcup_{j \in [b]} K_j.$$

Note that Ω is obtained from $\mathcal{P}$ after identification of opposite sides. Recall that a meromorphic, doubly periodic function is called an *elliptic* function. Let

$$(1.5) \quad \hat{\sigma}(z, \omega) = e^{-\frac{1}{2}\gamma_2 z^2 - \frac{1}{2}\pi|z|^2/A} \sigma(z, \omega)$$

denote the modified Weierstrass sigma function [13], where $\sigma(z, \omega)$ is the Weierstrass sigma function, $\gamma_2 = \gamma_2(\omega) \in \mathbb{C}$ is a lattice invariant, and $A = \text{area}(\mathbb{T}_\omega)$. We further discuss $\hat{\sigma}(z, \omega)$ in section 2, but for now just note that it is a nonholomorphic function with a pole of order 2 at $z = 0$ such that $|\hat{\sigma}(z, \omega)|$ is doubly periodic.

THEOREM 1.1. *Let Ω and $\mathcal{P}$ be defined as in (1.3) and (1.4). For each $j \in [b]$, let a_j be a point in K_j . If u is a harmonic function on Ω (equivalently, harmonic and doubly periodic on $\mathcal{P}$), then there exist an analytic function $\hat{f}$ on $\mathcal{P}$ and real numbers c_j , $j \in [b]$, satisfying $\sum_{j \in [b]} c_j = 0$, such that $\hat{f}'$ is elliptic and*

$$(1.6) \quad u(z) = \Re \hat{f}(z) + \sum_{j \in [b]} c_j \log |\hat{\sigma}(z - a_j, \omega)|, \quad z \in \Omega.$$

If there is only one connected boundary component (i.e., $b = 1$), then $c_1 = 0$ and $u(z) = \Re \hat{f}(z)$.

A proof of Theorem 1.1 is given in section 3. We comment that the result in Theorem 1.1 differs from the logarithmic conjugation theorem for finitely connected domains in several important ways. First, the modified Weierstrass sigma function, $\log |\hat{\sigma}(z, \omega)|$, plays the role of $\log |z|$. Second, and perhaps surprisingly, while the derivative $\hat{f}'$ is elliptic, the function $\hat{f}$ cannot always be taken to be elliptic.

Computing harmonic functions on finitely connected tori. There are a variety of methods for computing harmonic functions on finitely connected tori, including integral equation methods with multipole acceleration [3] and the finite element method [12]. In our approach, we are inspired by the work in [21] to use Theorem 1.1 to represent doubly periodic harmonic functions using a series solution. Let $\wp(z) = \wp(z, \omega)$ denote the Weierstrass elliptic function, let $\wp^{(k)}(z, \omega)$ denote the k th derivative, and let $\hat{\zeta}(z) = \hat{\zeta}(z, \omega)$ denote the “modified” Weierstrass zeta function that is doubly periodic; these will be defined in section 2.

THEOREM 1.2. Let Ω be a finitely connected torus as in (1.3). For each $j \in [b]$, let a_j be a point in K_j . If u is a harmonic function on Ω , then there exist a constant $C \in \mathbb{R}$ and real coefficients $(a_{j,k})$, $(b_{j,k})$, and (c_j) such that

$$(1.7) \quad u(z) = C + \sum_{j \in [b]} \left[a_{j,-1} \Re \hat{\zeta}(z - a_j) + b_{j,-1} \Im \hat{\zeta}(z - a_j) + \sum_{k \geq 0} a_{j,k} \Re \wp^{(k)}(z - a_j) + \sum_{k \geq 0} b_{j,k} \Im \wp^{(k)}(z - a_j) + c_j \log |\hat{\sigma}(z - a_j)| \right],$$

where $\sum_{j \in [b]} c_j = 0$.

A proof of Theorem 1.2 is given in section 3. We have chosen to represent the elliptic function f' using a sum of Weierstrass functions. Similar representations have been used to find doubly periodic solutions in several applications, including doubly periodic stress distributions in perforated plates [16], solitary wave solutions to a nonlinear wave equation [8] and a nonlinear Schrödinger equation [9], lowest-Landau-level wavefunctions on the torus [13], and simulation of oil recovery [1]. Other representations for elliptic functions are possible, including as $R(\wp) + \wp' S(\wp)$ for some rational functions R and S [7].

In section 4, we use the series representation (1.7) to solve the Laplace problem

$$(1.8a) \quad \Delta u = 0 \quad \text{in } \Omega,$$

$$(1.8b) \quad u = f \quad \text{on } \partial\Omega = \cup_{j \in [b]} \partial K_j,$$

where $f \in L^2(\partial\Omega)$ is given, and the Steklov eigenvalue problem

$$(1.9a) \quad \Delta u = 0 \quad \text{in } \Omega,$$

$$(1.9b) \quad \partial_n u = \sigma u \quad \text{on } \partial\Omega = \cup_{j \in [b]} \partial K_j.$$

As in [21], the series solution (1.7) are not convergent series. The coefficients depend on the truncation of the sum (in k). For the Laplace problem, by the maximum principle, the accuracy of the solution can be computed by looking at the error on the boundary, $\sup_{x \in \partial\Omega} |u(x) - f(x)|$. For the Steklov problem, we bound the error in the eigenvalues using an a posteriori estimate [6, 10]. We implement the proposed numerical method in Julia using arbitrary precision and use the result to find approximate solutions to the Laplace problem and the Steklov eigenvalue problem. For a few circular holes, the solution of the Laplace problem has error less than 10^{-100} using a few hundred degrees of freedom and the Steklov eigenvalues have similar error. We show the solution to the Laplace problem with 25 disks removed in Figure 1. The spectral accuracy is also demonstrated for nonconvex holes in Figure 4.

We conclude in section 5 with a discussion.

2. Weierstrass elliptic functions. Here we recall some background material on Weierstrass elliptic functions and establish notation used in the paper. Excellent references include [7, 13, 20].

We consider the lattice

$$L_\omega = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z},$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{C}^2$ are half-periods, assumed not to be colinear. A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is said to be *doubly periodic* if it satisfies

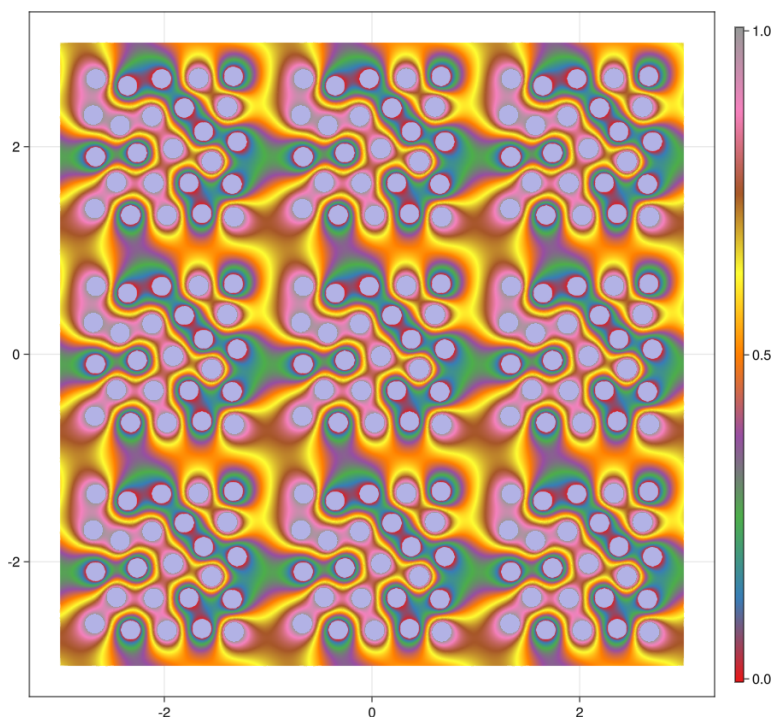


FIG. 1. Approximate solution to the Laplace equation on a square torus with 25 disks removed. Dirichlet boundary conditions equal to 0 or 1 are imposed on the boundary of each disk. The computational domain is $[-1, 1]^2$, and nine copies are displayed to emphasize periodicity. See subsection 4.2 for more details.

$$\begin{aligned} f(z + 2\omega_1) &= f(z), \\ f(z + 2\omega_2) &= f(z) \end{aligned}$$

for all $z \in \mathbb{C}$. A function is said to be *elliptic* if it is meromorphic and doubly periodic. An example of an elliptic function is the Weierstrass elliptic function

$$\wp(z, \omega) := \frac{1}{z^2} + \sum_{\ell \in L_\omega \setminus \{0\}} \left(\frac{1}{(z - \ell)^2} - \frac{1}{(\ell)^2} \right).$$

The subtraction of the last term ensures the convergence of the series. Furthermore, the derivative of a Weierstrass elliptic function is an odd function satisfying the differential equation

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3,$$

where $g_2 := \sum_{\ell \neq 0} 60 \frac{1}{(\ell)^4}$ and $g_3 := \sum_{\ell \neq 0} 140 \frac{1}{(\ell)^6}$. This differential equation can be used to compute higher-order derivatives of $\wp$. We obtain

$$\wp^{(2)}(z) = 6\wp^2(z) - \frac{g_2}{2}$$

and

$$\wp^{(n+2)}(z) = 6 \sum_{k=0}^n \binom{n}{k} \wp^{(n-k)}(z) \wp^{(k)}(z), \quad n \geq 1.$$

The Weierstrass zeta function is defined by

$$(2.1) \quad \zeta(z) = \frac{1}{z} + \sum_{\ell \neq 0} \left\{ \frac{1}{z-\ell} + \frac{1}{\ell} + \frac{z}{\ell^2} \right\} = \frac{1}{z} + \sum_{\ell \neq 0} \frac{z^3}{\ell^2(z^2 - \ell^2)}$$

and satisfies

$$\frac{d\zeta}{dz} = -\wp(z).$$

It has a Laurent expansion near $z = 0$:

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} \gamma_{2k} z^{2k-1},$$

where $\gamma_{2k} = \sum_{\ell \neq 0} \frac{1}{\ell^{2k}}$, $k \geq 2$. In contrast to $\wp(z)$, the function $\zeta(z)$ does not possess the doubly periodic property. Instead, it satisfies the quasi-periodic condition:

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1,$$

$$\zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2,$$

where $\eta_1 = \zeta(\omega_1)$ and $\eta_2 = \zeta(\omega_2)$. The values η_1 , η_2 , ω_1 , ω_2 are not independent but are related by the Legendre identity

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}.$$

The ζ function can be modified so that it is periodic:

$$\hat{\zeta}(z) = \zeta(z) - \gamma_2 z - \frac{\pi}{A} z^*,$$

where A is the area of the fundamental cell of the lattice and γ_2 is given by an Eisenstein summation and satisfies $\zeta(\omega_i) \equiv \eta_i = \gamma_2\omega_i + \frac{\pi\omega_i^*}{A}$, $i = 1, 2$ [13]. Note that since $\hat{\zeta}$ depends on z^* , it is no longer meromorphic.

Finally, the Weierstrass sigma function is defined by

$$\sigma(z, \omega) = \lim_{\varepsilon \rightarrow 0} \varepsilon \exp \left(\int_{\varepsilon}^z \zeta(w, \omega) dw \right),$$

which is an odd, nondoubly periodic, holomorphic function with simple zeros at the lattice points. It satisfies

$$(2.2) \quad \zeta(z, \omega) = \frac{\sigma'(z, \omega)}{\sigma(z, \omega)}.$$

As for the zeta function, the sigma function can be modified as in (1.5). When defined this way, its modulus has the lattice periodicity [13].

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. The first part of the proof closely follows the proof of Axler for the logarithmic conjugation theorem [2]. Define $h: \Omega \rightarrow \mathbb{C}$ by

$$h(z) := u_x(z) - u_y(z).$$

The Cauchy–Riemann equations can be used to check that h is analytic on Ω . For each $j \in [b]$, let Γ_j be a closed curve in Ω that circles K_j once and no other K_k , $k \neq j$. Define

$$c_j := \frac{1}{2\pi i} \oint_{\Gamma_j} h(w) dw.$$

We see that $\Im c_j = -\frac{1}{2\pi} \Re \oint_{\Gamma_j} h(w) dw = -\frac{1}{2\pi} \Re \oint_{\Gamma_j} u_x(w)dx + u_y(w)dy = 0$, so c_j is a real number for each $j \in [b]$. Since u is doubly periodic, so is h , and by the Cauchy integral theorem, we have that

$$(3.1) \quad \sum_{j \in [b]} c_j = 0.$$

We consider h to be a function on $\mathcal{P}$, which we still denote by h . Fix a point $z_0 \in \mathcal{P}$, and define $f: \mathcal{P} \rightarrow \mathbb{C}$ by

$$f(z) := \int_{z_0}^z h(w) - \sum_{j \in [b]} c_j \zeta(w - a_j, \omega) dw,$$

where the integral is taken over any path in $\mathcal{P}$ from z_0 to z and ζ is the Weierstrass zeta function as in (2.1). To show that f is well-defined, we check that the above integral is independent of the path from z_0 to z . Take two paths from z_0 to z , and reverse the direction of transversal in one to form a closed curve. Thus, we need only show that

$$\frac{1}{2\pi i} \oint_{\gamma} h(z) dz = \frac{1}{2\pi i} \sum_{j \in [b]} c_j \oint_{\gamma} \zeta(w - a_j, \omega) dw$$

for any closed curve $\gamma \subset \Omega$. By the Cauchy integral theorem and the definition of c_j , the left-hand side is given by $\sum_{j \in [b]} c_j I_j(\gamma)$, where $I_j(\gamma)$ denotes the winding number of γ about K_j . Using the Laurent expansion for $\zeta(z, \omega)$, which has a single pole of order one, by the Cauchy integral theorem, the right-hand side is also seen to be equal to $\sum_{j \in [b]} c_j I_j(\gamma)$, as desired. The function $f(z)$ is analytic on $\mathcal{P}$, and we compute the derivative

$$(3.2) \quad f'(z) = h(z) - \sum_{j \in [b]} c_j \zeta(z - a_j, \omega).$$

Now define

$$(3.3) \quad q(z) := \Re f(z) + \sum_{j \in [b]} c_j \log |\sigma(z - a_j, \omega)|.$$

We claim that $u_x(z) = q_x(z)$ and $u_y(z) = q_y(z)$, so that, after adding a constant to f , we obtain $u(z) = q(z)$, $z \in \mathcal{P}$. Using (2.2), we compute

$$q_x(z) = \Re f'(z) + \sum_{j \in [b]} c_j \Re \zeta(z - a_j, \omega) = \Re h(z) = u_x$$

and

$$q_y(z) = \Re(\imath f'(z)) + \sum_{j \in [b]} c_j \Re(\imath \zeta(z - a_j, \omega)) = \Re(\imath h(z)) = u_y.$$

We have established that $u(z) = q(z)$ up to a constant, $z \in \mathcal{P}$, and it remains to show that we can rewrite $q(z)$ in (3.3) so that the two terms on the right-hand side are each doubly periodic, so that they can be thought of as functions on Ω . In (3.3), the second term on the right-hand side is not doubly periodic since σ is not doubly periodic. By (1.5), this term can be rewritten as

$$\sum_{j \in [b]} c_j \log |\sigma(z - a_j)| = \sum_{j \in [b]} c_j \log |\hat{\sigma}(z - a_j)| + \Re g(z),$$

where

$$\begin{aligned} g(z) &= \frac{1}{2} \sum_{j \in [b]} c_j (\gamma_2(z - a_j)^2 + \pi |z - a_j|^2 / A) \\ &= \alpha z + \beta z^* + \gamma, \end{aligned}$$

where α , β , and γ are constants and we have used (3.1) to drop the quadratic terms.

From (3.2), f' is doubly periodic since h is doubly periodic and $\sum_{j \in [b]} c_j = 0$. There exist α_1, α_2 such that for all admissible z

$$\begin{cases} f(z + 2\omega_1) = f(z) + \alpha_1, \\ f(z + 2\omega_2) = f(z) + \alpha_2. \end{cases}$$

Let us introduce (μ_1, μ_2) , the unique solution of

$$\begin{cases} \omega_1 \mu_1 + \omega_1^* \mu_2 = -\alpha_1/2, \\ \omega_2 \mu_1 + \omega_2^* \mu_2 = -\alpha_2/2. \end{cases}$$

Notice that the previous system is nonsingular since the determinant is proportional to the area of the fundamental domain, which is nonzero. Moreover, a straightforward computation shows that

$$f(z) + \mu_1 z + \mu_2 z^*$$

is a doubly periodic function. Thus, for a suitable μ , $\hat{f}(z) = f(z) + \mu z$ is an analytic function and $\Re \hat{f}$ is also doubly periodic. Note that $\Im \hat{f}(z)$ is not necessarily doubly periodic and that $\hat{f}'$ is elliptic.

Summarizing our results, we have established that there exist $\hat{f}$ analytic with doubly periodic real part and $(\nu, \xi) \in \mathbb{C}^2$ such that

$$u = \Re \hat{f} + \sum_{j \in [b]} c_j \log |\hat{\sigma}(z - a_j)| + \Re(\nu z + \xi z^*).$$

Observing that both the left-hand side and the two first terms of the right-hand side are doubly periodic, we obtain $\nu = \xi = 0$, which concludes the proof. $\square$

Proof of Theorem 1.2. Let $\hat{f}'$ be the elliptic function from Theorem 1.1 associated with the harmonic function u . Using a representation of elliptic functions (see, e.g., [22, p. 450] or [20, p. 23]), we may write

$$\hat{f}'(z) = \tau + \sum_{j \in [b]} \left(\alpha_j \zeta(z - a_j) + \sum_{k \geq 0} \beta_{j,k} \wp^{(k)}(z - a_j) \right),$$

where $\tau \in \mathbb{C}$, $\alpha_j \in \mathbb{C}$, and $\beta_{j,k} \in \mathbb{C}$ are constants. Consequently, there exists a constant $\rho \in \mathbb{C}$ such that

$$\hat{f}(z) = \tau z + \rho + \sum_{j \in [b]} \left(\alpha_j \log \sigma(z - a_j) + \beta_{j,0} \zeta(z - a_j) + \sum_{k \geq 1} \beta_{j,k} \wp^{(k-1)}(z - a_j) \right).$$

Introducing the periodic modifications $\hat{\zeta}$ and $\log |\hat{\sigma}|$ of ζ and $\log |\sigma|$ functions, respectively, we obtain that there exist real coefficients $C, a_{j,k}, b_{j,k}$ such that

$$u(z) = C + \sum_{j \in [b]} \left[a_{j,-1} \Re \hat{\zeta}(z - a_j) + b_{j,-1} \Im \hat{\zeta}(z - a_j) + \sum_{k \geq 0} a_{j,k} \Re \wp^{(k)}(z - a_j) + \sum_{k \geq 0} b_{j,k} \Im \wp^{(k)}(z - a_j) + c_j \log |\hat{\sigma}(z - a_j)| \right] + g(x, y)$$

for some affine function g . By periodicity of all other terms, the function g also has to be doubly periodic, so it must be identically equal to zero. Finally, $\sum_{j \in [b]} c_j = 0$ is deduced from the harmonicity of all the terms, except the $\log |\hat{\sigma}|$ terms, which have a constant Laplacian. $\square$

4. Computational method and experiments. Here we develop a computational method based on a series solution of the form (1.7) to solve the Laplace problem (1.8) and the Steklov eigenvalue problem (1.9).

4.1. Computational method. Let Ω be a finitely connected torus as in (1.3). For simplicity, we will take each region K_j , $j \in [b]$, to be a closed disk, $K_j = \overline{B}(a_j, r_j)$, that is centered at the point a_j and has radius r_j . The centers and radii are chosen such that $K_i \cap K_j = \emptyset$ for $i \neq j$. Based on Theorem 1.2, we consider a series solution of the form (1.7), where the sums on k are truncated at $k = k_{\max}$. We collect the (real) coefficients in the series solution into a vector $v = [C, (a_{j,k}), (b_{j,k}), (c_j)] \in \mathbb{R}^m$, where $m = 1 + 2b(k_{\max} + 2) + (b - 1)$. For each coefficient, v_i , we let ϕ_i , $i \in [m]$, denote the corresponding basis function (e.g., the real part of a Weierstrass $\wp$ function), so that

$$(4.1) \quad u(z) = \sum_{i \in [m]} v_i \phi_i(z).$$

On each boundary component ∂K_j , we uniformly sample points with respect to arclength and denote the collection of all sampled points in the union of the boundary components by $(p_\ell)_{\ell \in [S]}$. In the experiments below, we report the value of m and take $S = 3m$. Define the matrices $A, B \in \mathbb{R}^{S \times m}$ by

$$A_{\ell,i} = \frac{\partial \phi_i}{\partial n}(p_\ell), \\ B_{\ell,i} = \phi_i(p_\ell).$$

Details about the computation of the normal derivatives of basis functions are given in Appendix A.

4.2. Laplace problem. We solve the Laplace problem (1.8) with boundary data $f(x)$, $x \in \partial\Omega$, as follows. Define the vector $b \in \mathbb{R}^S$ by $b_\ell = f(p_\ell)$. The least-squares solution is found by solving the normal equations

$$(4.2) \quad B^t B v = B^t b.$$

The solution v then is used with the expansion in (4.1) as an approximate solution of (1.8). By the maximum principle, the accuracy of the solution can be computed by looking at the error on the boundary, $\sup_{x \in \partial\Omega} |u(x) - f(x)|$.

We implement the numerical method in Julia using arbitrary precision provided by the packages *GenericLinearAlgebra.jl* and *ArbNumerics.jl* (a wrapper of the *Arb* C library). All computational experiments were performed with a precision of 2^{10} bits, which corresponds to a machine epsilon approximately equal to 10^{-300} .

We first consider a finitely connected square torus with half-periods $(\omega_1, \omega_2) = (1, i)$. The complement is taken to be $b = 1$ disks with $K_1 = B(a_1, r)$ with $a_1 = 0$ and $r = 0.4$. We take $f(\theta) = \sin(5\theta)$, where θ is the polar angle centered at a_1 . The resulting solution is plotted in the top left panel of Figure 2. Using the maximum principle to bound the $L^\infty(\Omega)$ error of the solution, we estimate $\|u(x) - f(x)\|_{L^\infty(\partial\Omega)}$ in the bottom left panel of Figure 2 for increasing numbers of degrees of freedom, m . This estimate is based on the maximum value obtained at the sampled points, after doubling the number of sampled points. Spectral convergence is observed. With $k_{\max} = 150$ ($m = 305$ degrees of freedom), the solution has error less 10^{-100} corresponding to at least 100 digits of accuracy.

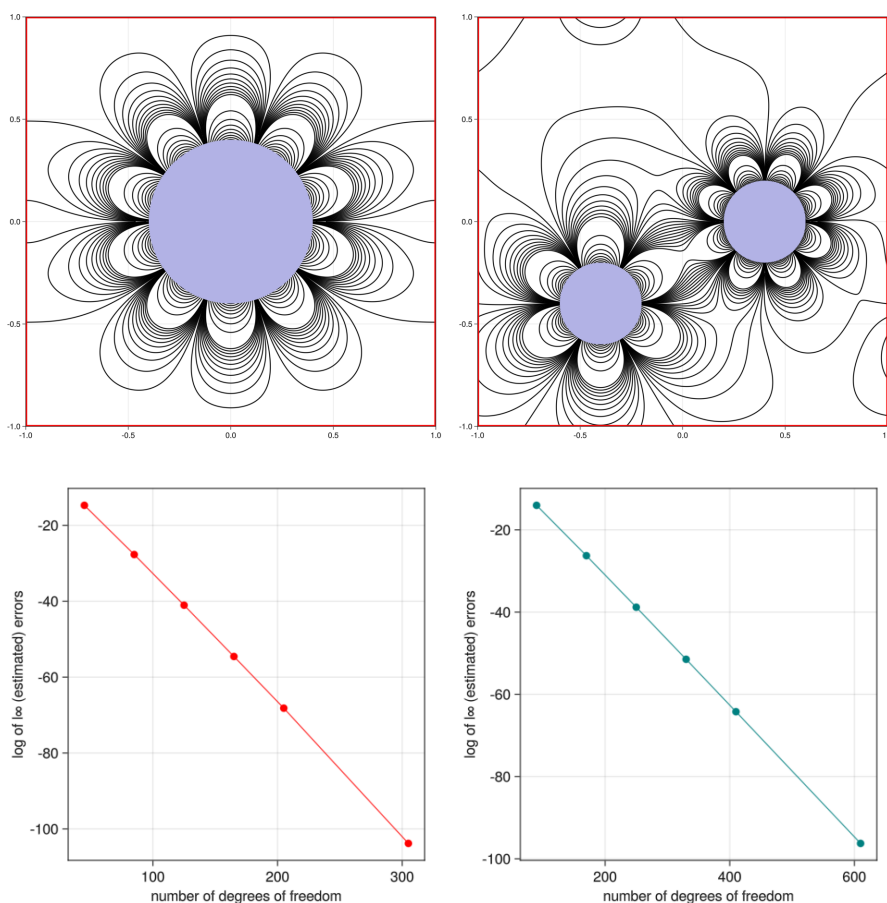


FIG. 2. (Upper panels) Approximate solution to the Laplace problem in a square torus with one and two circular holes. (Lower panels) Spectral convergence is observed for each of the two geometries. See subsection 4.2.

Next, we again consider a finitely connected square torus with half-periods $(\omega_1, \omega_2) = (1, i)$. The complement is taken to be $b = 2$ disks with $K_i = B(a_i, r_i)$, $i = 1, 2$, with $a_1 = 0.4$, $a_2 = -0.4 - 0.4i$, and $r_1 = r_2 = 0.2$. On each boundary, we set $f(\theta) = \sin(4\theta)$ for circle $i = 1$ and $f(\theta) = \sin(3\theta)$ for circle $i = 2$. The resulting solution is plotted in the top right panel of Figure 2. In the bottom right panel of Figure 2, we can see that the solutions have error similar to that in the previous example, albeit using more degrees of freedom.

Next, we consider a finitely connected equilateral torus with half-periods $(\omega_1, \omega_2) = (1, \frac{1}{2} + \frac{\sqrt{3}}{2}i)$. The complement is taken to be the same sets as above with one and two circular holes. We plot the results in Figure 3. The solutions have error similar to those in the previous examples.

In Figure 4, we provide an approximate solution to the Laplace problem for two nonconvex holes in a square torus. The polar parametrizations of the boundaries of the two holes K_1 and K_2 are given by $A_i + \rho(\theta + \theta_i)(\cos(\theta), \sin(\theta))$, where

$$\rho(\theta) = \frac{3}{10} + \frac{1}{10} \cos(3\theta),$$

$A_1 = (0.4, 0.4)$, $A_2 = -A_1$, $\theta_1 = 0$, and $\theta_2 = \frac{\pi}{3}$. We impose the Dirichlet condition 0 on the first boundary component and 1 on the second. The sampled points are obtained using the previous parametrization together with a uniform sampling of the angles. As previously, in the right panel of Figure 4, we observe exponential convergence but notice that the obtained accuracy is significantly lower than in previous cases with the same number of degrees of freedom.

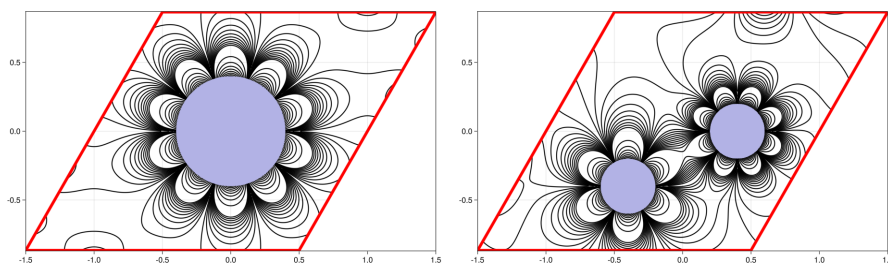


FIG. 3. Approximate solution to the Laplace problem in an equilateral torus with one and two circular holes. See subsection 4.2.

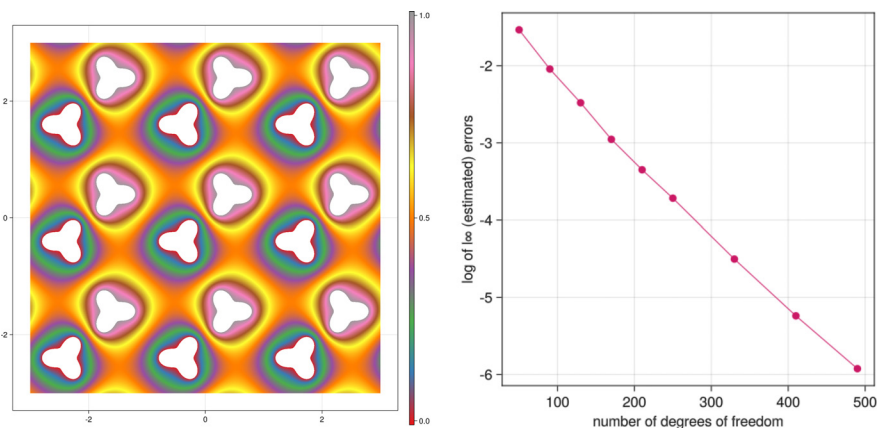


FIG. 4. Approximate solution to the Laplace problem in a square torus with two nonconvex holes.

Finally, we consider the Laplace equation on a finitely connected square torus with 25 disks removed. Dirichlet boundary conditions equal to 0 or 1 are imposed on the boundary of each disk. The results are plotted in Figure 1. The solution has error less than 10^{-16} .

4.3. Steklov eigenvalue problem. To solve the Steklov eigenvalue problem (1.9), we consider a generalized eigenvalue problem

$$(4.3) \quad A v = \sigma B v.$$

We can approximate solutions to this eigenvalue problem by multiplying both sides by B^t and considering the nonsymmetric generalized eigenvalue problem, $B^t A v = \sigma B^t B v$. For $k_{\max} \leq 50$, this formulation leads to exponentially converging eigenvalue approximations, as expected. As has been observed by several authors [5, 4, 10], this formulation with a larger number of degrees of freedom may produce ill-conditioned matrices. To illustrate this, in Figure 5, for the example considered above with two nonconvex holes (see Figure 4), we plot the condition number of $B^t B$ and $B^t A$ as the number of degrees of freedom varies. To overcome this difficulty and avoid spurious modes, we followed the SVD approach described in [5]: for a (small) set of randomly sampled interior points $(q_r)_{r \in [R]}$, we consider the evaluation matrix $C \in \mathbb{R}^{R \times m}$:

$$(4.4) \quad C_{r,i} = \phi_i(q_r).$$

In all our experiments, we set $R = 50$. We define $s(\sigma)$ to be the smallest (always nonnegative) eigenvalue of the generalized eigenvalue problem

$$(4.5) \quad D(\sigma)x(\sigma) = s(\sigma)C^t C x(\sigma),$$

where $D(\sigma) = (A - \sigma B)^t(A - \sigma B)$. From a computational point of view, $s(\sigma)$ can be efficiently evaluated using a standard power method or an orthogonal subspace approach if the multiplicity is suspected to be greater than one. Local minimizers of $s(\sigma)$ provide stable approximations of Steklov eigenvalues. To numerically identify these local extrema, we used the golden section algorithm.

To bound the error in the eigenvalues, we use the following a posteriori estimate for Steklov eigenvalues in [6], which extends previous estimates for Laplace–Dirichlet eigenvalues [10, 18].

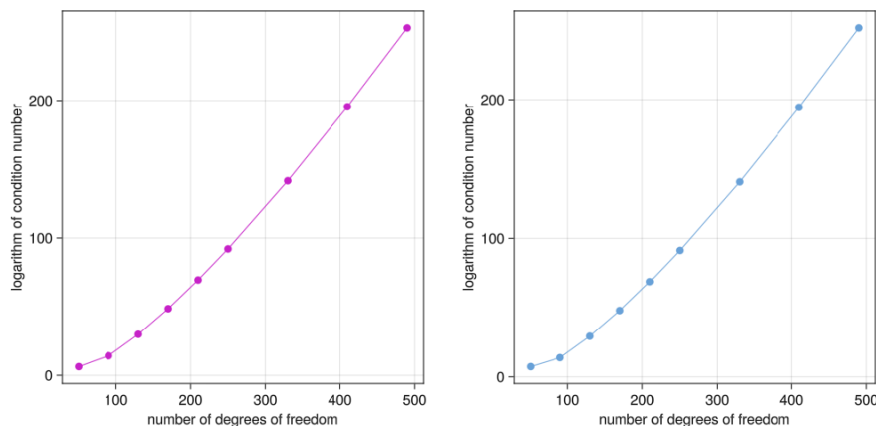


FIG. 5. Exponential growth of condition numbers of matrices $B^t B$ (left) and $B^t A$ (right) with respect to the number of degrees of freedom.

PROPOSITION 4.1 (see [6]). *Consider Ω a bounded open regular domain, and suppose that u_ε solve the following approximate eigenvalue problem:*

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega, \\ \partial_n u_\varepsilon &= \sigma_\varepsilon u_\varepsilon + f_\varepsilon && \text{on } \partial\Omega. \end{aligned}$$

Then if $\|f_\varepsilon\|_{L^2(\partial\Omega)}$ is small, there exist a constant C , depending only on Ω , and a Steklov eigenvalue σ_k satisfying

$$\frac{|\sigma_\varepsilon - \sigma_k|}{\sigma_k} \leq C \|f_\varepsilon\|_{L^2(\partial\Omega)}.$$

We study three geometrical configurations: tori which are the complement of $K_1 = B(a_1, r)$ with $a_1 = 0$ and $r = 0.4$, the complement of $K_i = B(a_i, r_i)$, $i = 1, 2$, with $a_1 = 0.2$, $a_2 = -0.2 + 0.2i$, and $r_1 = r_2 = 0.1$, and the complement of $K_i = B(a_i, r_i)$, $i = 1, 2, 3$, with $a_1 = 0.3$, $a_2 = 0.3i$, $a_3 = -0.3 - 0.3i$ and $r_1 = r_2 = 0.1$, $r_3 = 0.05$. We approximated Steklov eigenfunctions of the square torus with half-periods $(\omega_1, \omega_2) = (1, i)$ (see Figures 6 to 8) and of the equilateral torus with half-periods $(\omega_1, \omega_2) = (1, \frac{1}{2} + \frac{\sqrt{3}}{2}i)$ in these three configurations (see Figures 9 to 11). The first eigenvalue is zero, which corresponds to a constant eigenfunction. In these figures, Steklov eigenfunctions of indices 2 to 7 are plotted. The Steklov eigenfunctions, as expected, are oscillatory near the boundary and decay exponentially away from the boundary. We used Proposition 4.1 to estimate the approximation error of the Steklov eigenvalues. We approximated the L^2 boundary term by a (periodic) trapezoidal quadrature formula after doubling the number of sampled points. Convergence plots for Steklov eigenvalues on a square domain with one, two, and three punctured circular holes are given in Figure 12. As expected, spectral convergence is also observed in these situations. The same convergence rate has also been obtained studying the equilateral case.

Finally, in Appendix B, we report in Tables 1 to 6 our approximation of the first six nontrivial eigenvalues of the square and equilateral tori with $b = 1, 2$, and 3 circular

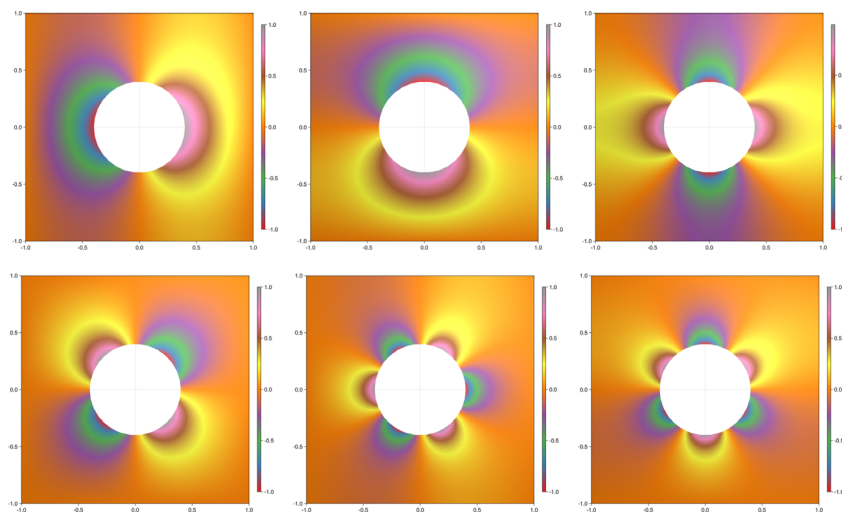


FIG. 6. Approximate Steklov eigenfunctions of indices 2 to 7 on a punctured square torus with one hole. See subsection 4.3.

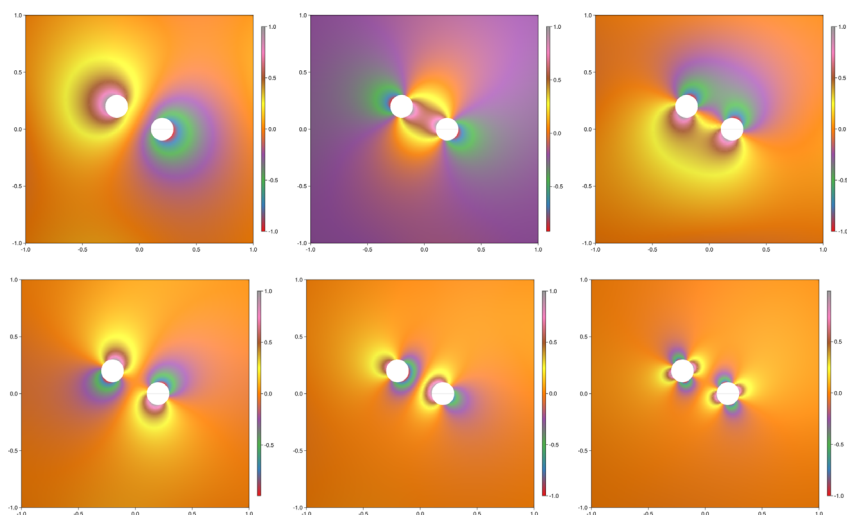


FIG. 7. Approximate Steklov eigenfunctions of indices 2 to 7 on a punctured square torus with two circular holes. See subsection 4.3.

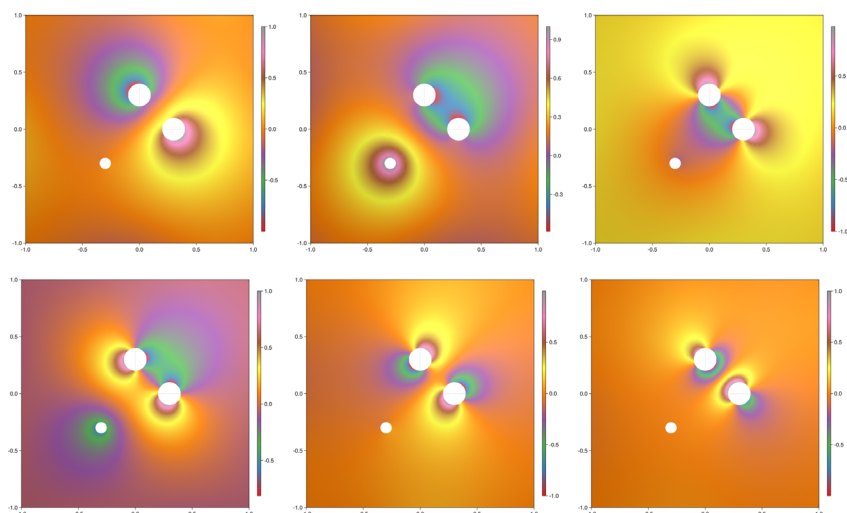


FIG. 8. Approximate Steklov eigenfunctions of indices 2 to 7 on a punctured square torus with three circular holes. See subsection 4.3.

holes. We believe that the reported 50 digits are correct in each case. As indicated in Theorem 1.1, when there is only one connected boundary component ($b = 1$), the eigenfunctions do not involve the logarithmic term and are oscillatory along the boundary, as shown in Figures 6 and 9. In general, eigenfunctions corresponding to larger Steklov eigenvalues are more oscillatory near the boundary. Note that the tori parameters, ω , effect the multiplicity of the eigenvalues. On a square torus with one circular hole, $\sigma_2 = \sigma_3$ and $\sigma_6 = \sigma_7$, while, on an equilateral torus with one circular hole, $\sigma_2 = \sigma_3$ and $\sigma_4 = \sigma_5$. Since the domains with two or three circular holes do not possess symmetry, we observe that all the obtained eigenvalues are simple.

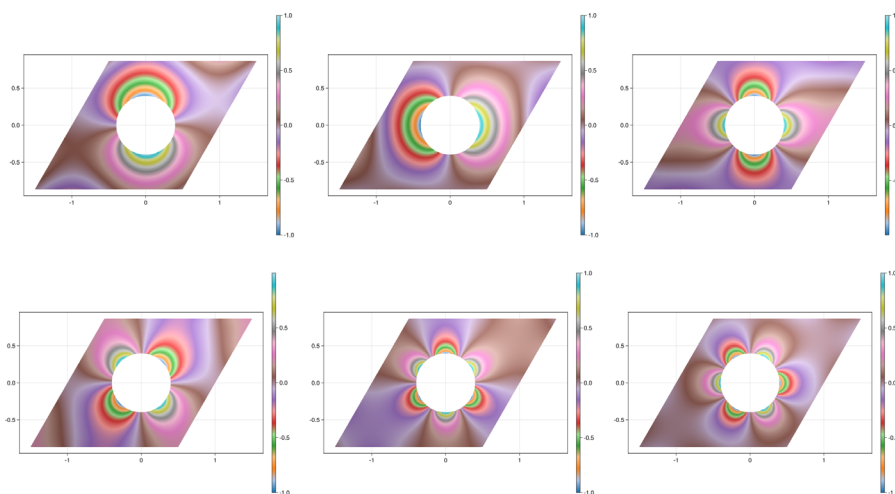


FIG. 9. *Approximate Steklov eigenfunctions of indices 2 to 7 on a punctured equilateral torus with one hole. See subsection 4.3.*

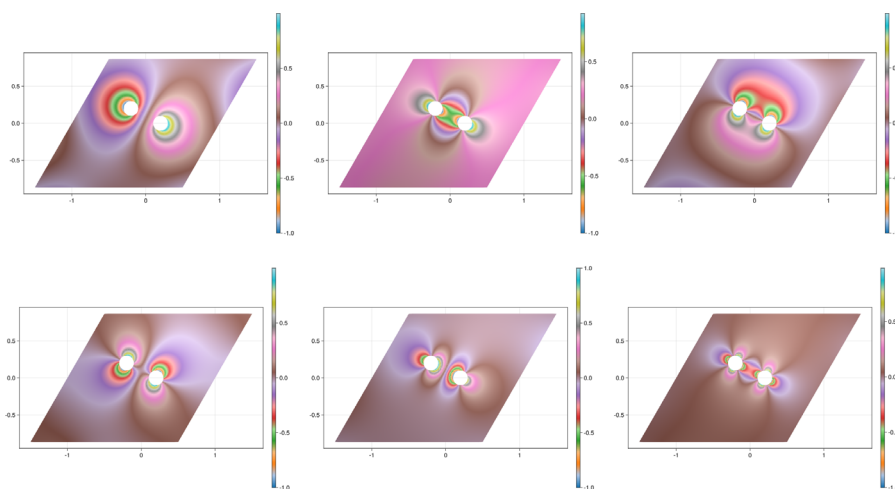


FIG. 10. *Approximate Steklov eigenfunctions of indices 2 to 7 on a punctured equilateral torus with two circular holes. See subsection 4.3.*

5. Discussion. In this paper, we established Theorem 1.1, a logarithmic conjugation theorem on finitely connected tori. We used the theorem to find a series solution representation of harmonic functions on finitely connected tori; see Theorem 1.2. Implementing the numerical method in Julia using arbitrary precision, we approximate solutions to the Laplace problem (1.8) and the Steklov eigenvalue problem (1.9); see section 4. Using a posteriori estimation, we show that the approximate solution of the Laplace problem has error less than 10^{-100} using a few hundred degrees of freedom and the Steklov eigenvalues have similar error.

There are several future directions for this work. The fundamental solution of Laplacian on flat tori can be expressed as a logarithmic function involving the first Jacobi theta function [15, 17]; we think it would be interesting to develop integral equation methods to approximate harmonic functions on finitely connected tori in the

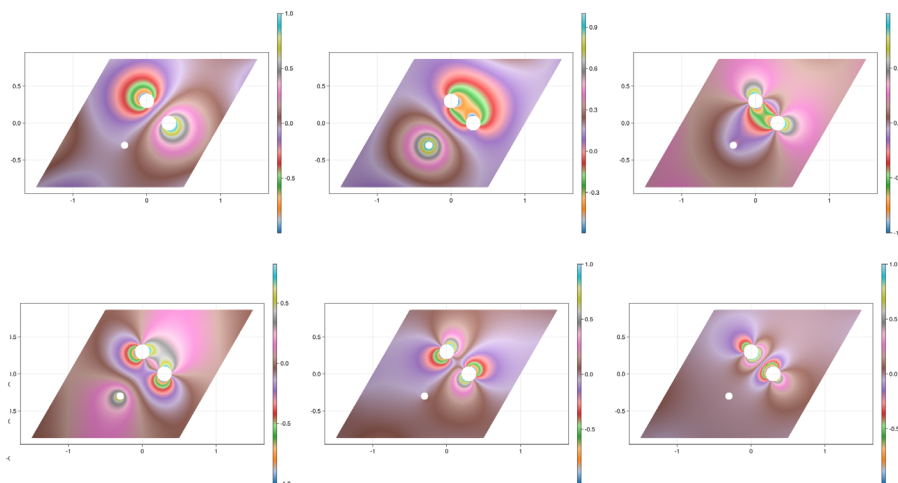


FIG. 11. Approximate Steklov eigenfunctions of indices 2 to 7 on a punctured equilateral torus with three circular holes. See subsection 4.3.

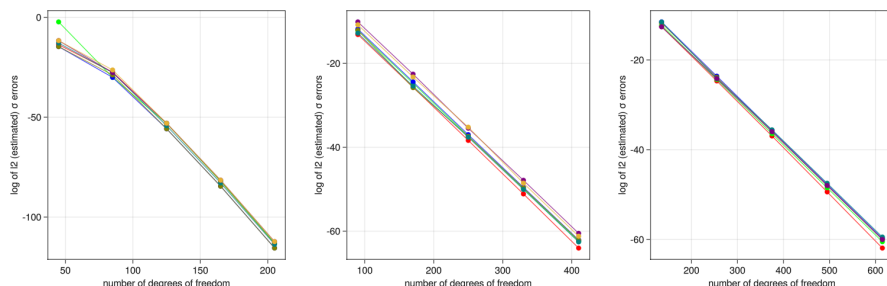


FIG. 12. Convergence plots for Steklov eigenvalues on a square domain with one, two, and three punctured holes. Each line corresponds to one of the first seven eigenvalues. See subsection 4.3.

spirit of [3]. We have focused on the case where the domain complement, $\cup_{j \in [b]} K_j$, has a smooth boundary. We think it would be interesting to extend the methods in [11] to improve the order of convergence for nonsmooth boundaries. Finally, we think it would be interesting to apply the developed numerical methods to the numerical problem of computing extremal Steklov eigenvalue problems for finitely connected flat tori [14, 19].

Appendix A. Computing normal derivatives. In this appendix, we provide some details for computing normal derivatives of functions of a complex variable. Denote $f(z) = u(x, y) + v(x, y)$, with $z = x + iy$. Since f is analytic, we have $u_x = v_y$ and $u_y = -v_x$. Furthermore, $f_x = f'(z)$ and $f_y = if'(z)$. Thus, with $n = n_1 + in_2$, we have

$$\begin{aligned} u_n &= n_1 u_x + n_2 u_y = n_1 u_x - n_2 v_x = \Re[(n_1 + in_2)(u_x + v_x)] = \Re(n f'(z)), \\ v_n &= n_1 v_x + n_2 v_y = -n_1 u_y + n_2 v_y = -\Im(n_1 + in_2)(u_y + v_y) = \Im(n f'(z)). \end{aligned}$$

For example, $f(z) = z^k$,

$$\begin{aligned} (\Re(z^k))_n &= k \Re(n z^{k-1}), \\ (\Im(z^k))_n &= k \Im(n z^{k-1}). \end{aligned}$$

If $f(z) = \wp^{(k)}(z - a_j)$,

$$\begin{aligned}\left(\Re\left(\wp^{(k)}(z - a_j)\right)\right)_n &= \Re(n\wp^{(k+1)}(z - a_j)), \\ \left(\Im\left(\wp^{(k)}(z - a_j)\right)\right)_n &= \Im(n\wp^{(k+1)}(z - a_j)).\end{aligned}$$

If $f(z) = \hat{\zeta}(z - a_j)$,

$$\begin{aligned}\left(\Re\left(\hat{\zeta}(z - a_j)\right)\right)_n &= \left(\Re(\zeta(z - a_j))\right)_n - \left(\Re\left(\gamma_2(z - a_j) + \frac{\pi}{A}(z - a_j)^*\right)\right)_n \\ &= -\Re(n\wp(z - a_j)) - \left(\left(\Re(\gamma_2) + \frac{\pi}{A}\right)n_1 - \Im(\gamma_2)n_2\right), \\ \left(\Im\left(\hat{\zeta}(z - a_j)\right)\right)_n &= \left(\Im(\zeta(z - a_j))\right)_n - \left(\Im\left(\gamma_2(z - a_j) + \frac{\pi}{A}(z - a_j)^*\right)\right)_n \\ &= -\Im(n\wp(z - a_j)) - \left(\Im(\gamma_2)n_1 + \left(\Re(\gamma_2) - \frac{\pi}{A}\right)n_2\right),\end{aligned}$$

If $f(z) = \log|\hat{\sigma}(z - a_j)|$,

$$\begin{aligned}(\log|\hat{\sigma}(z - a_j)|)_n &= n_1(\log|\hat{\sigma}(z - a_j)|)_x + n_2(\log|\hat{\sigma}(z - a_j)|)_y \\ &= n_1\left(\Re\left(-\frac{1}{2}\gamma_2 z^2 - \frac{1}{2}\frac{\pi}{A}|z|^2\right)\right)_x + n_2\left(\Re\left(-\frac{1}{2}\gamma_2 z^2 - \frac{1}{2}\frac{\pi}{A}|z|^2\right)\right)_y \\ &\quad + n_1(\log|\sigma(z - a_j)|)_x + n_2(\log|\sigma(z - a_j)|)_y \\ &= n_1\left(\left(-\Re(\gamma_2) - \frac{\pi}{A}\right)x + \Im(\gamma_2)y\right) + n_2\left(\Im(\gamma_2)x + \left(\Re(\gamma_2) - \frac{\pi}{A}\right)y\right) \\ &\quad + n_1(\Re(\zeta(z - a_j))) + n_2(-\Im(\zeta(z - a_j))).\end{aligned}$$

Appendix B. Numerical values of computed Steklov eigenvalues. Values of computed Steklov eigenvalues are given in Tables 1 to 6; see subsection 4.3 for details.

TABLE 1
Steklov eigenvalues of a square torus with one circular hole.

σ_2	3.21737540790552735473880286001400036767774798208487
σ_3	3.21737540790552735473880286001400036767774798208487
σ_4	4.85099530552467697892257589130439715581461931719259
σ_5	5.15358084940676223549771471754234765157435969419525
σ_6	7.50305008416767542642635086056165243882709526430554
σ_7	7.50305008416767542642635086056165243882709526430554

TABLE 2
Steklov eigenvalues of a square torus with two circular holes.

σ_2	6.45837308842285506198400983365912091999317179119988
σ_3	9.04038374077713587651429965380130970292955686420981
σ_4	9.32931391918711635803886895114746515357566095450257
σ_5	11.02561512617948586321622981756262835104523220458063
σ_6	12.69568331719729045908212485186369130848598658103989
σ_7	19.72884655790348748027382339516572459572547368362522

TABLE 3

Steklov eigenvalues of a square torus with three circular holes.

σ_2	6.54721983775026738598476089606442586801693676638247
σ_3	6.79298688602543949226783518103096724408533776232952
σ_4	9.02715360305747386008778464587475727275979230551042
σ_5	9.75911376018587533254687022367601130464658416864329
σ_6	11.11563661826511047191549742109769883301063010901993
σ_7	13.08067309361125105561475152956318658177620553096385

TABLE 4

Steklov eigenvalues of an equilateral torus with one hole.

σ_2	3.34865594380260534169550288243470971962587318064277
σ_3	3.34865594380260534169550288243470971962587318064277
σ_4	4.99978881548382813234141616969113198885117552416465
σ_5	4.99978881548382813234141616969113198885117552416465
σ_6	7.44392530690947308002824485738760008901145380307620
σ_7	7.55649710043624518482844840631875099119732734059433

TABLE 5

Steklov eigenvalues of an equilateral torus with two circular holes.

σ_2	6.53794803818597918794030349125758145344842633243163
σ_3	9.03760803330365503342990995931942991592541389841134
σ_4	9.37148419781059159007737134528684902568383756667729
σ_5	11.02904931936017784776119216982004095594847249520813
σ_6	12.70222698966325001285792418382443547595163064198654
σ_7	19.72940718569248148461882657324755544321541234433839

TABLE 6

Steklov eigenvalues of an equilateral torus with three circular holes.

σ_2	6.63530737085667505246439432756580077469480498850424
σ_3	6.94424494471680808970061612948991192950478141474806
σ_4	9.02318420302479178183722837786227147321263458092783
σ_5	9.69311259795549433304394048074564041975314036318590
σ_6	11.14183481942696624006786357768349746325365435641531
σ_7	13.09270988086125229485281063758500867332548835687565

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