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ABSTRACT

This paper presents the derivation of a general wave dispersion relation for warm magnetized plasma under the two-fluid formalism. The discussion is quite general except for the assumption of low frequency and slow phase speed, for which the displacement current is negligible, under the implicit assumption that the plasma is sufficiently dense to satisfy the condition $\omega_{pe} > \omega_{ce}$, where ω_{pe} and ω_{ce} denote the plasma oscillation frequency and electron gyro frequency, respectively. The present discussion does not invoke charge neutrality associated with the fluctuations although it is implicitly satisfied. The resulting dispersion relation that includes the fluid thermal effects shows that there are three eigen modes, which include those corresponding to ideal MHD, namely, fast, slow, and kinetic Alfvén waves, as well as higher-frequency modes including the ion and electron cyclotron and lower-hybrid resonances. The fluid effects in the ideal MHD wave branches are influenced by the finite Larmor radius scales, and when the wave number in the cross field direction is comparable to these values, the fluid effects become significant. It is found that the Larmor radius should be interpreted in the sense as ion-acoustic gyro-radius instead of ion thermal gyro radius only. That is, it is found that the electrons also contribute to the non-ideal effect associated with the kinetic Alfvén wave. A comprehensive explanation of the polarization of each mode is also presented. The present findings indicate that the polarity may change its sign only for the kinetic Alfvén mode branch and that such a transition is based on the propagation angle. When such a change does take place, it is found that the kinetic Alfvén wave transits to an ion-acoustic mode. For each branch, it is also found that the electric field along the ambient magnetic field is purely transverse.

I. INTRODUCTION

Despite more than five decades of research, issues that relate to the heating and acceleration of charged particles in space are not fully resolved. Alfvén waves are believed to be the one of the candidates for particle acceleration and heating.[1–5] In this regard, Hasegawa and Chen[6] suggested that linear Alfvén wave with ion kinetic effects may explain the particle acceleration along the auroral field lines. Later Goertz and Boswell[7] found that in typical space environment above the auroral ionosphere, it is the electron inertial effect associated with the Alfvén wave, which can be more important than the kinetic correction that arises from finite ion pressure. Thus, kinetic effects on the Alfvén wave are manifested in two limits: one is the pressure-induced "kinetic" Alfvén wave,[6] while the other is the electron inertia-induced "inertial" Alfvén wave. Hui and Seyler[8] demonstrated by means of numerical simulation that Alfvén waves in the inertial scale with small perpendicular (with respect to magnetic field) wavelength may lead to an electron acceleration up to several keV energy, which is typical of the auroral electrons. Alfvén waves with small perpendicular wavelength are observed in sounding rocket experiment[9,10] and in Freja satellite mission.[11,12] Alfvén waves are also found and discussed in the context of various environments of the space and solar system such as in the planetary magnetospheres,[13–18] cometary jets,[19] solar radio bursts,[20] solar flares,[21] and in the context of solar coronal heating.[22]

On account of the significance of parallel electric field in the particle energization, understanding the polarization of the Alfvén wave as it relates to the parallel **E** field (with respect to external magnetic field) is important.[23] In the literature, a number of discussions can be found that attempt to address the various properties of kinetic Alfvén waves (KAWs) within the framework of two-fluid plasma model. For example, Wu[24] and Chen and Wu[25] discussed the two-fluid theory of kinetic Alfvén waves and showed that both ions and electrons affect the pressure-driven as well as the inertia-driven corrections to the standard Alfvén waves. Their works, however, made simplifying assumptions by ignoring terms of order m_e/m_i (where m_e and m_i are electron and proton masses, respectively) in order to reduce the mathematical complexity. Historically, discussions of low-frequency waves that include the thermal and inertia corrections within the warm two-fluid theoretical framework go back to the pioneering work of Stringer,[26] who provided a comprehensive treatment of low-frequency waves within the warm two-fluid theory, but the treatise is restricted to low β (the ratio of thermal to magnetic pressure). Formisano and Kennel[27] subsequently extended the discussion to high β regime. Along such a line of research, Bellan[28] made further contribution by formulating the two-fluid warm plasma wave analysis in an alternative way. In the intervening years, Hollweg[29] also made significant contributions to the two-fluid plasma wave theory for low-frequency waves. In this regard, one of the present authors also considered the property of low-frequency waves from the perspective of the Hall-magnetohydrodynamics[30]—see also, Ref. 31, for a similar discussion.

Of course, all the above-cited works pertain to the macroscopic description that lacks the wave–particle interactions, which requires kinetic prescription.[15,16,31–33] However, if one is to limit the discussion within the framework of warm two-fluid theory despite the fact that such a theory lacks wave damping effects, then a completely general theoretical approach for the analysis of waves in a warm magnetized two-fluid plasma can be found in the monumental work by Goedbloed *et al.*[34] By a brute force method, the authors of the above-referenced monograph managed to obtain a completely general dispersion equation for warm two-fluid electron–proton plasma waves, which includes not only the low-frequency but also high-frequency waves, given as a sixth-order polynomial equation in ω^2, the square of the angular frequency (or 12th-order equation in ω). Their complicated polynomial dispersion equation is specified by 19 coefficients that are functions of $\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}$, $\omega_{pi} = (4\pi n_0 e^2/m_i)^{1/2}$, $\omega_{ce} = eB_0/(m_e c)$, $\omega_{ci} = eB_0/(m_i c)$, $v_{the} = (\gamma_e T_e/m_e)^{1/2}$, and $v_{thi} = (\gamma_i T_i/m_i)^{1/2}$, where these are the electron and ion (proton) plasma oscillation frequencies, the electron and ion (proton) cyclotron frequencies, electron and ion (proton) fluid thermal speeds, in that order. Here, e, m_e, m_i, c, T_e, T_i, n_0, and B_0 stand for the unit charge, electron and proton masses, speed of light *in vacuo*, electron and ion fluid temperatures (in the unit of energy), ambient density, and the ambient magnetic field intensity, respectively. The quantities γ_e and γ_i denote the ratios of specific heat, which are related to polytropic indices. Note that a similar effort has also been carried out by Kakuwa.[35] The general dispersion relation found in Ref. 34 is continually being analyzed to this day.[36–38]

Despite the availability of such a general result, because of the fact that the result is so general, there have been attempts to re-derive, as it were, a reduced form of warm two-fluid plasma dispersion relations restricted to the low-frequency modes, satisfying the condition $ck/\omega > 1$. To reduce the general formalism to the low-frequency regime, all one has to do is to ignore the displacement current in the Maxwell's equation. Of course, one must exercise caution when implementing such an approximation. The slow-mode dispersion relation that results from the procedure of ignoring the displacement current is accurate only if the plasma is characterized by the condition that the frequency ratio ω_{pe}/ω_{ce} is sufficiently higher than unity. The subsequent analysis is implicitly applicable under such a condition. As a matter of fact, the above-cited references[24,25] appear to represent just such an effort—although curiously, these works do not reference the completely general treatise mentioned above.[34] However, as noted, Refs. 24 and 25 made a simplifying assumption of ignoring terms of order m_e/m_i. A formalism that is directly relevant to the present paper is the paper by Zhao *et al.*,[39] who (re) derived the warm two-fluid plasma dispersion relation in full generality, except that they ignored the displacement current at the outset, thus restricting the discussion to the low-frequency, or more accurately, the slow mode regime characterized by the sub-luminal condition, $ck > \omega$. A similar discussion can also be found in Ref. 40. In the derivation of Refs. 39 and 40, however, the authors made an assumption of quasi-neutrality as it relates to the perturbed electron and ion density fluctuations. We find, however, that such an assumption is superfluous and that it is possible to formulate the entire problem along the line of Refs. 34 and 35, except for the fact that the simplifying assumption of ignoring the displacement current is made.

The aim of this paper is, first, to revisit the work by Zhao *et al.*[39] and Zhao[40] and, second, to understand the polarization characteristics of the low-frequency modes including the kinetic Alfvén wave. The organization of the present paper is as follows: In Sec. II, we discuss the dispersion relation for warm two-fluid plasma under the assumption of low frequency, or slow ($ck > \omega$) mode, and we also discuss the polarization of the electric fields within the formalism of two-fluid equation. In Sec. III, we restrict the discussion to the low-frequency modes analogous to the MHD modes, namely, fast, kinetic Alfvén, and slow wave and discuss the corresponding branches in ideal MHD limit. In Sec. IV, polarization properties associated with different branches of the three modes are discussed in detail. In Sec. V, a brief discussion on relation between low frequency, displacement current, and charge neutrality is given and transverse properties of the parallel electric field is shown. Finally, in Sec. VI, we briefly summarize the present paper.

II. FULL TWO-FLUID DISPERSION RELATION FOR SLOW WAVES

The present analysis starts from the two-fluid plasma equations, which are given by

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \tag{1a}$$

$$m_\alpha \left(\frac{\partial \mathbf{v}_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha \right) = q_\alpha \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{\nabla P_\alpha}{n_\alpha}, \tag{1b}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \left(\frac{P_\alpha}{n^{\gamma_\alpha}} \right) = 0, \tag{1c}$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha n_\alpha \mathbf{v}_\alpha, \tag{1d}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{1e}$$

where $\alpha = e, i$ denotes electrons and ions, respectively, with the corresponding charge densities specified by $q_e = -e$, $q_i = e$, respectively, and $\gamma_\alpha = (N+2)/N$, where N is degree of the freedom. Other quantities are standard, n_α, $\mathbf{v}_\alpha$, P_α denoting the fluid density, velocity, and pressure, respectively, while $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic field vectors, respectively. We separate all physical quantities into average and perturbed quantities, distinguished by subscripts 0 and 1, respectively. In what follows, we assume that the zeroth-order electron and ion pressures are specified by $P_{\alpha 0} = n_{\alpha 0} k_B T_\alpha$, where k_B is the Boltzmann constant. Here, $n_{\alpha 0}$ and T_α denote the average fluid density and temperature, respectively. We assume zero average fluid velocity, $\mathbf{v}_{\alpha 0} = 0$, and zero net average electric field, $\mathbf{E}_0 = 0$. The average magnetic field is assumed to be constant, $\mathbf{B}_0 = \text{const}$. Linearized two-fluid equations are given as follows:

$$\frac{\partial n_{\alpha 1}}{\partial t} + \nabla \cdot (n_{\alpha 0} \mathbf{v}_{\alpha 1}) = 0, \tag{2a}$$

$$m_\alpha \frac{\partial \mathbf{v}_{\alpha 1}}{\partial t} = q_\alpha \left(\mathbf{E}_1 + \frac{1}{c} \mathbf{v}_{\alpha 1} \times \mathbf{B}_0 \right) - \frac{\nabla P_{\alpha 1}}{n_{\alpha 0}}, \tag{2b}$$

$$P_{\alpha 1} = \gamma_\alpha P_{\alpha 0} n_{\alpha 0}^{-1} n_{\alpha 1} = \gamma_\alpha k_B T_\alpha n_{\alpha 1}, \tag{2c}$$

$$\nabla \times \mathbf{B}_1 - \frac{1}{c}\frac{\partial \mathbf{E}_1}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha n_{\alpha 0} \mathbf{v}_{\alpha 1}, \tag{2d}$$

$$\nabla \times \mathbf{E}_1 = -\frac{1}{c}\frac{\partial \mathbf{B}_1}{\partial t}. \tag{2e}$$

Upon carrying out the spectral transformation of the above linearized warm two-fluid equations under the assumption that the wave vector $\mathbf{k}$ lies in yz plane, $\mathbf{k} = k_y \hat{\mathbf{y}} + k_z \hat{\mathbf{z}}$ and that the ambient magnetic field is directed along z axis, $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$, it is possible to derive the following dispersion relation, which can be found in many standard works—see, e.g., Refs. 34, 35, 39, and 41

$$\mathbf{a} \cdot \boldsymbol{E} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = 0, \tag{3}$$

where

$$a_{11} = -\frac{c^2 k^2}{\omega^2} - \sum_\alpha \frac{(\omega^2 - k^2 v_{th\alpha}^2)\omega_{p\alpha}^2}{D_\alpha}, \tag{4a}$$

$$a_{12} = -a_{21} = -\sum_\alpha \frac{i\omega_{c\alpha}}{\omega} \frac{\omega_{p\alpha}^2 (\omega^2 - k_z^2 v_{th\alpha}^2)}{D_\alpha}, \tag{4b}$$

$$a_{13} = -a_{31} = -k_y k_z \sum_\alpha \frac{i\omega_{c\alpha}}{\omega} \frac{\omega_{p\alpha}^2 v_{th\alpha}^2}{D_\alpha}, \tag{4c}$$

$$a_{22} = -\frac{c^2 k_z^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2 (\omega^2 - k_z^2 v_{th\alpha}^2)}{D_\alpha}, \tag{4d}$$

$$a_{23} = a_{32} = k_y k_z \left(\frac{c^2}{\omega^2} - \sum_\alpha \frac{v_{th\alpha}^2 \omega_{p\alpha}^2}{D_\alpha} \right), \tag{4e}$$

$$a_{33} = -\frac{c^2 k_y^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2 \left(\omega^2 - \omega_{c\alpha}^2 - k_y^2 v_{th\alpha}^2 \right)}{D_\alpha}, \tag{4f}$$

$$D_\alpha = \omega^4 - \omega^2 (k^2 v_{th\alpha}^2 + \omega_{c\alpha}^2) + \omega_{c\alpha}^2 k_z^2 v_{th\alpha}^2, \tag{4g}$$

and where we have ignored the displacement current. Here, various quantities are defined by $\omega_{p\alpha}^2 = 4\pi n_0 q_\alpha^2 / m_\alpha$, $\omega_{c\alpha} = |q_\alpha| B_0 / (m_\alpha c)$, $v_{th\alpha} = \gamma_\alpha T_\alpha / m_\alpha$. As with the standard works, e.g., Refs. 34, 35, and 39, we have carried out the matrix determinant calculation, following the dictum "[t]he human qualities that are required to carry out these [$\cdots$] calculations are, in small proportion, insight, and in large proportion, stamina" (quote from Stix,[41] p. 266, which is also quoted by Goedbloed *et al.*,[34] p. 91), but in our case, we have also taken advantage of the help provided by the symbolic mathematics package, `Mathematica`. The important point is that we have not invoked the assumption of charge-neutrality associated with the density perturbation, unlike Ref. 39. In spite of this, however, as it will be shown in Sec. V, our formalism effectively reduces to an implicit charge neutrality. This may be the reason, as it will be shown subsequently, why the present formalism and that of Ref. 39, which are slightly different, virtually lead to indistinguishable numerical results. But before we discuss such a final outcome, let us return to the basic formalism first. The laborious process of the matrix determinant calculation leads to the final result, which can be summarized as follows:

$$0 = D_6 \omega^6 + D_4 \omega^4 + D_2 \omega^2 + D_0, \tag{5}$$

where

$$D_6 = -\left(\omega_{pe}^2 + \omega_{pi}^2\right)\left(c^2 k^2 + \omega_{pe}^2 + \omega_{pi}^2\right)^2, \tag{6a}$$

$$\begin{aligned} D_4 = {} & c^4 k^2 \Big[\omega_{pi}^2 (k^4 v_{the}^2 + k^2 \omega_{ce}^2 + k_z^2 \omega_{ci}^2) \\ & + \omega_{pe}^2 (k^4 v_{thi}^2 + k^2 \omega_{ci}^2 + k_z^2 \omega_{ce}^2) \Big] + \left(\omega_{pe}^2 + \omega_{pi}^2\right) \\ & \times \left[k^2 \left(\omega_{pe}^2 + \omega_{pi}^2\right)\left(v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2\right) + \left(\omega_{ci}\omega_{pe}^2 + \omega_{ce}\omega_{pi}^2\right)^2 \right] \\ & + 2c^2 \left(\omega_{pe}^2 + \omega_{pi}^2\right) \Big[\left(k_y^4 v_{thi}^2 + k_z^4 v_{thi}^2 + k_z^2 \omega_{ci}^2\right) \omega_{pe}^2 \\ & + \left(k_y^4 v_{the}^2 + k_z^4 v_{the}^2 + k_z^2 \omega_{ce}^2\right) \omega_{pi}^2 \Big] \\ & + c^2 k_y^2 \Big[2\omega_{ci}^2 \omega_{pe}^4 + 2\omega_{ce}^2 \omega_{pi}^4 + (\omega_{ce} + \omega_{ci})^2 \omega_{pe}^2 \omega_{pi}^2 \\ & + 4k_z^2 \left(\omega_{pe}^2 + \omega_{pi}^2\right)\left(v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2\right) \Big], \end{aligned} \tag{6b}$$

$$\begin{aligned} D_2 = {} & -k_z^2 \left(v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2\right) \\ & \times \left[(\omega_{ci}\omega_{pe}^2 + \omega_{ce}\omega_{pi}^2)^2 + 2c^2 k^2 \left(\omega_{ci}^2 \omega_{pe}^2 + \omega_{ce}^2 \omega_{pi}^2\right) \right] \\ & - c^4 k^2 k_z^2 \Big[\omega_{ce}^2 \omega_{ci}^2 \left(\omega_{pe}^2 + \omega_{pi}^2\right) \\ & + k^2 (\omega_{ce}^2 + \omega_{ci}^2)\left(v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2\right) \Big], \end{aligned} \tag{6c}$$

$$D_0 = c^4 k_z^4 k^2 \omega_{ce}^2 \omega_{ci}^2 \left(v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2\right). \tag{6d}$$

With appropriate definitions, we make the following substitutions:

$$\begin{gathered} \omega \to \Omega k_z V_A, \quad \omega_{pe}^2 + \omega_{pi}^2 \to \omega_{pe}^2 (1+Q), \quad c \to \omega_{pe} \lambda_e, \\ v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2 \to (1+Q) V_A^2 \beta \omega_{pe}^2, \\ \omega_{ce} \to -\frac{V_A}{\sqrt{Q}\lambda_e}, \quad \omega_{ci} \to \frac{\sqrt{Q} V_A}{\lambda_e}, \quad \omega_{pi}^2 \to Q\omega_{pe}^2, \\ v_{thi}^2 \to (1+Q) V_A^2 \beta - Q v_{the}^2, \quad \beta \lambda_e^2 \to Q \rho_L, \quad \lambda_i \to \frac{\lambda_e}{\sqrt{Q}}. \end{gathered} \tag{7}$$

Obviously, $Q = m_e/m_i$, and $\lambda_e = c/\omega_{pe}$ and $\lambda_i = c/\omega_{pi}$ enjoy the interpretation of being the electron and ion (proton) skin depths, respectively. The Alfvén speed V_A is defined via $V_A^2 = B_0/(4\pi n_0 m_i)$, and the total plasma beta includes the contribution from the ion-acoustic speed. The quantity β and Alfvén speed are, thus, related via $\beta = c_s^2/V_A^2$, where $c_s = \sqrt{(\gamma_i T_i + \gamma_e T_e)/(m_i + m_e)}$ is the ion-acoustic speed. Substituting Eq. (7) to Eqs. (6a)–(6d), we obtain

$$D_6 = -\Omega^6 \omega_{pe}^6 k_z^6 V_A^6 (1+Q)\left(1+Q+k^2\lambda_e^2\right)^2, \tag{8a}$$

$$D_4 = \Omega^4 \omega_{pe}^6 k_z^4 V_A^6 (1+Q)\left\{\left[k_z^2 + k^2 + k^2\beta\left(1+Q+k^2\lambda_e^2\right)\right] \times \left(1+Q+k^2\lambda_e^2\right) + k_z^2(1-Q)^2\frac{k^2\lambda_e^2}{Q}\right\}, \tag{8b}$$

$$D_2 = -\Omega^2 \omega_{pe}^6 k_z^4 V_A^6 k^2 (1+Q)\left[1 + 2(1+Q)\beta + k^2(1+Q^2)\rho_L^2\right], \tag{8c}$$

$$D_0 = \omega_{pe}^6 k_z^4 V_A^6 k^2 (1+Q)\beta. \tag{8d}$$

One can see that each term in Eqs. (8a)–(8d) contains a common factor, $\omega_{pe}^6 k_z^4 V_A^6 (1+Q)$, which when eliminated, leads to

$$\begin{aligned} 0 &= k_z^2 A^2 \Omega^6 - \left[\left(k_z^2 + k^2 + k^2\beta A\right)A + k_z^2(1-Q)^2 k^2\lambda_i^2\right]\Omega^4 \\ &\quad + k^2[B + 2(1+Q)\beta]\Omega^2 - k^2\beta, \\ &\quad A = 1+Q+k^2\lambda_e^2, \quad B = 1 + k^2(1+Q^2)\rho_L^2. \end{aligned} \tag{9}$$

Here, $\rho_L = c_s/\omega_{ci}$ is the ion-acoustic gyro-radius. In Eq. (9), the definition of beta is different from the conventional one in that it is now given as

$$\beta = \frac{\gamma_i T_i + \gamma_e T_e}{(m_i + m_e)V_A^2} = \frac{1}{2(1+Q)}(\beta_e + \beta_i), \tag{10}$$

where $\beta_e = 2\gamma_e T_e/m_i V_A^2$ and $\beta_i = 2\gamma_i T_i/m_i V_A^2$. It is worth noting that the overall dispersion relation (9) does not depend on individual, but only on β as defined in Eq. (10) which we call it the "ion-acoustic" beta. As we will discuss subsequently, although the dispersion relation only depends on the ion-acoustic beta, we find that the wave polarization depends on individual betas, β_e and β_i.

This is a good place to discuss a similar dispersion relation, which is also derived by Zhao *et al.*[39] and further analyzed by Zhao.[40] Our dispersion relation (9) is alternatively written in terms of normalized wave frequency $\varpi = \omega/\omega_{ce}$ and wave number $\kappa = ck/\omega_{pe}$ as

$$\begin{aligned} 0 &= (1+Q)(1+Q+\kappa^2)^2\varpi^6 \\ &\quad -\kappa^2\left\{Q(1+Q)^2(1+\cos^2\theta) + \kappa^2(1+Q)\right. \\ &\quad \times \left.\left[Q + (1-Q+Q^2)\cos^2\theta\right] + Q(1+Q+\kappa^2)^2\beta\right\}\varpi^4 \\ &\quad + Q\kappa^4\cos^2\theta\left[Q(1+Q) + 2Q(1+Q)\beta + (1+Q^2)\kappa^2\beta\right]\varpi^2 \\ &\quad - Q^3\kappa^6\beta\cos^4\theta, \end{aligned} \tag{11}$$

where $\cos\theta = k_z/k$. Also, to be consistent with Zhao's result, we used old definition $\beta = (\beta_i + \beta_e)/2$. Now this can be compared with the counterpart derived by Zhao *et al.*,[39]

$$\begin{aligned} 0 &= (1+Q)(1+Q+\kappa^2)^2\varpi^6 \\ &\quad -\kappa^2\left\{Q(1+Q)^2(1+\cos^2\theta) + \kappa^2(1+Q)\right. \\ &\quad \times \left.\left[Q + (1-Q+Q^2)\cos^2\theta\right] + Q(1+Q+\kappa^2)^2\beta\right\}\varpi^4 \\ &\quad + Q\kappa^4\cos^2\theta\left[Q(1+Q) + 2Q(1+Q)\beta + \overbrace{Q}(1+Q^2)\kappa^2\beta\right]\varpi^2 \\ &\quad - Q^3\kappa^6\beta\cos^4\theta, \end{aligned} \tag{12}$$

where we have indicated the difference, by an over-brace. As one may appreciate, the discrepancy involves a small term related to the quantity $Q = m_e/m_i$. As such, in actual numerical solutions, the difference hardly matters. Nevertheless, we believe that our representation is more accurate—at least, it is alternative to the expression found in the paper by Zhao *et al.*[39] Note that both expressions become identical if we set $\beta = 0$ (cold plasma limit).

Before moving on, we consider the numerical solutions of Eq. (11), which is virtually indistinguishable from the same obtained on the basis of Eq. (12), for various values of β. We first consider the case of $\beta = 0$, in which case both Eqs. (11) and (12) reduce to a quadratic equation in ϖ^2. Figure 1, thus, plots the cold-plasma dispersion relation. The top surface depicts the fast-magnetosonic mode for low frequency, which smoothly turns into the whistler mode as the frequency increases, until ω approaches the electron cyclotron frequency, $\omega \to \omega_{ce}$, at which point, the mode approaches the electron–cyclotron resonance frequency. On the other hand, as the frequency increases along perpendicular k_y direction, the mode approaches the lower-hybrid resonance frequency, $\omega_{LH} = [(\omega_{ce}\omega_{ci})^{-1} + \omega_{pi}^{-2}]^{-1/2}$. The lower branch depicts the shear Alfvén wave, which approaches the ion (proton) cyclotron frequency as ω increases. Along k_y direction, the Alfvén mode simply decreases in its frequency until it reaches zero (note that we have used the logarithmic scales in the plot).

Once we include a finite value of β, the dispersion characteristics becomes significantly modified as a third branch appears among the solution. To show this, we next consider $\beta = 0.01$, which is showcased

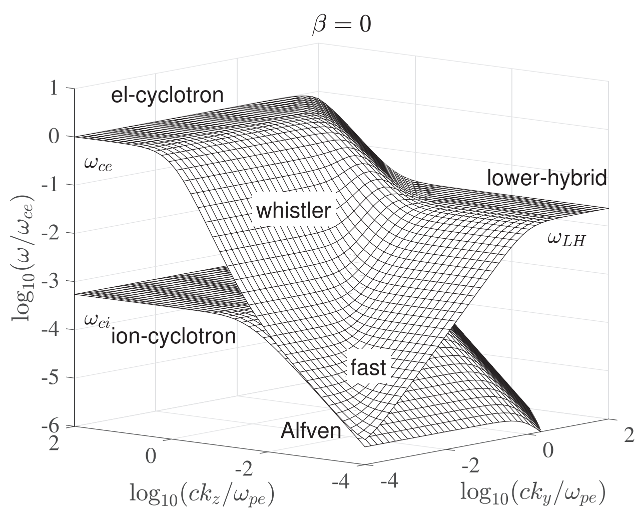

FIG. 1. The dispersion surfaces for the sub-luminal ($ck > \omega$) cold ($\beta = 0$) magnetized plasma. Various mode designations that are commonly discussed in the literature are indicated.

in Fig. 2. We reiterate that the numerical solutions based on either Eq. (11) or Eq. (12) remain indistinguishable for a wide range of β, including up to $\beta \propto \mathcal{O}(10)$. The most obvious difference is along k_z direction, where the ion-acoustic mode branch with the slope $\sim c_s$ is now present. Upon a careful examination, one can see that the Alfvén-ion cyclotron mode now attains the characteristics of kinetic Alfvén mode along k_y direction. As noted in the Introduction, the kinetic Alfvén wave (KAW) is influenced by two aspects, one relates to the thermal correction, $(\omega/k_z V_A)^2 = 1 + k_y^2\rho_L^2$, which leads to the increase in the wave frequency as k_y increases, and the other comes from the finite electron inertia effect, $(\omega/k_z V_A)^2 = 1/(1 + k_y^2\lambda_e^2)$, which leads to a decrease in wave frequency for large k_y. The kinetic Alfvén correction as indicated by "KAW" in Fig. 2 shows first an increase and subsequent decrease in ω as k_y increases for a fixed value of k_z, thus demonstrating that the mode surface correctly features the expected behavior associated with KAW, that is, $(\omega/k_z V_A)^2 = (1 + k_y^2\rho_L^2)/(1 + k_y^2\lambda_e^2)$. The fast-magnetosonic-whistler-lower hybrid branch, on the other hand, undergoes an interesting mode-switchover behavior along k_y direction. It is seen that the quasi-perpendicular ion-acoustic branch interferes with the whistler-lower hybrid branch such that the mode no longer becomes a resonance mode at ω_{LH}, but instead, its frequency increases as k_y increases. It is interesting to note that along k_z direction, aside from the presence of ion-acoustic mode, the proton and electron cyclotron modes largely retain their cold-plasma characteristics. From this, it seems that the finite-β effect is important for quasi-perpendicular propagation directions.

We next consider the case of increased beta of $\beta = 1$, and the result is shown in Fig. 3. For such a high beta value, we recognize that the fluid description is incomplete as the two-fluid theory does not have the wave–particle resonance and the associated cyclotron/Landau damping, but as far as the real frequency is concerned, the fluid description still provides useful information. One of the most striking features for the present high-beta situation relates to the behavior of kinetic Alfvén wave (KAW). Upon a careful visual examination, it can be gleaned that the KAW mode for high k_y range smoothly transitions to the ion-acoustic mode surface rather than the shear-Alfvén wave branch. It is also noteworthy that the quasi-perpendicular range of what was whistler-lower hybrid mode surface no longer bears any signature of lower-hybrid resonance, but rather, the dispersion surface around the region that formerly was associated with ω_{LH} is now completely dominated by the ion-acoustic mode surface. In fact, the top two surfaces are a mixture of whistler mode and ion-acoustic mode surfaces, which undergo mode interchange. These subtle mode structures are best seen if one chooses a single propagation angle and plots the various dispersion curves, as was done by Goedbloed *et al.*,[34] Kakuwa,[35] Chen and Wu,[25] Zhao *et al.*,[39] Zhao,[40] etc., but on the other hand, the present way of plotting the dispersion surfaces also provides a broader view of the overall mode configuration.

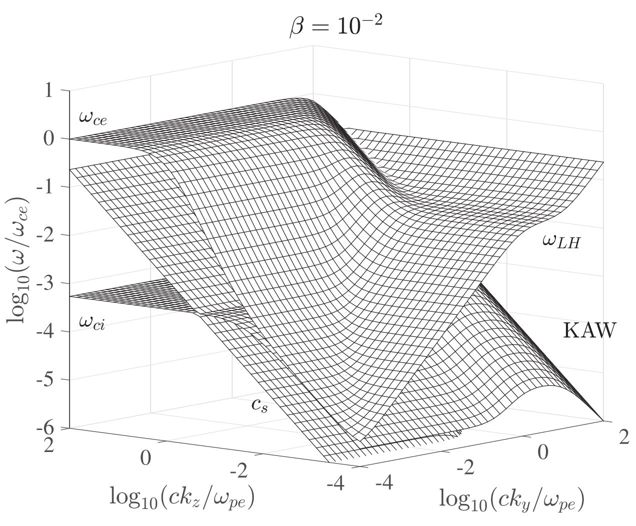

FIG. 2. The dispersion surfaces for finite beta ($\beta = 0.01$) plasma. In addition to the two previous mode surfaces, an additional mode is now present, which corresponds to the ion-acoustic mode. The distinguishing feature as compared with the cold case is the manifestation of kinetic Alfvén mode characteristics (KAW) along increasing k_y axis, and the fact that the whistler-lower hybrid branch is no longer resonant at ω_{LH}, but the mode frequency steadily increases as k_y increases.

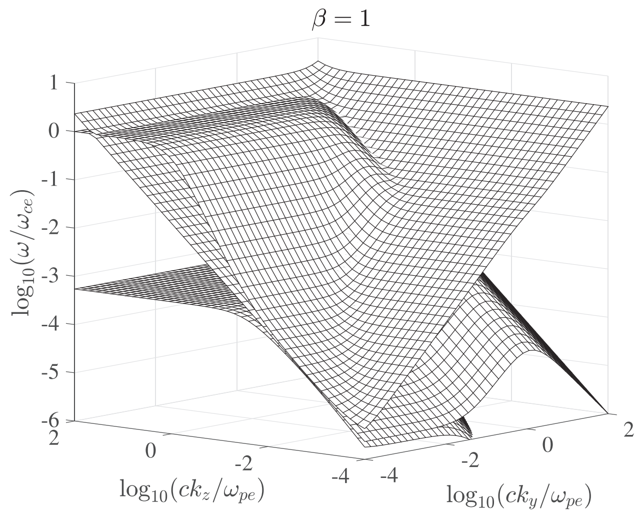

FIG. 3. The dispersion surfaces for an increased beta of $\beta = 1$. In this case, the quasi-perpendicular kinetic Alfvén mode smoothly transits to ion-acoustic mode as the mode becomes more quasi parallel. Also, noteworthy is the fact that any signature associated with the lower-hybrid resonance is now completely wiped out. Note also that the whistler and ion-acoustic mode surfaces are intermingled.

III. RELATION BETWEEN IDEAL MHD AND TWO-FLUID PICTURE

In this section, we will investigate dispersion relation given by Eq. (9) in detail, but our focus henceforth will be on the structure of the dispersion relations as analogues and generalization of the ideal MHD modes. The dispersion relation (9) is given by the sixth-order polynomial equation in Ω has powers up to 6, or quartic equation in Ω^2, given by the (Cardan) form[25,28]

$$\begin{aligned} 0 &= \left(\Omega^2 - \Omega_0^2\right)^3 + p\left(\Omega^2 - \Omega_0^2\right) + q, \\ \Omega_0^2 &= a/3, \quad p = b - 3\Omega_0^4, \quad q = -2\Omega_0^6 + b\Omega_0^2 - c, \\ a &= \frac{(k_z^2 + k^2 + k^2\beta A)A + k_z^2(1-Q)^2 k^2\lambda_i^2}{k_z^2 A^2}, \\ b &= \frac{k^2[B + 2(1+Q)\beta]}{k_z^2 A^2}, \quad c = \frac{k^2\beta}{k_z^2 A^2} \end{aligned} \tag{13}$$

with the general solution (three independent solutions are designated by F, S, and A),

$$\Omega_F^2 = \Omega_0^2 + S + T, \tag{14a}$$

$$\Omega_S^2 = \Omega_0^2 - \frac{S+T}{2} + \frac{i\sqrt{3}}{2}(S-T), \tag{14b}$$

$$\Omega_A^2 = \Omega_0^2 - \frac{S+T}{2} - \frac{i\sqrt{3}}{2}(S-T), \tag{14c}$$

where $S = \left(R + \sqrt{W^3+R^2}\right)^{1/3}$, $T = \left(R - \sqrt{W^3+R^2}\right)^{1/3}$, $R = -q/2$, and $W = p/3$. Equation (14a) is the warm two-fluid generalization of MHD fast-mode dispersion relation, Eq. (14b) is the same for the slow mode, and Eq. (14c) is for the kinetic Alfvén wave. Taking the limit $Q \to 0$ and $\lambda_i \to 0$, it can be shown that these expressions reduce to the ideal MHD dispersion relation. It can be shown that the discriminant $\Delta = W^3 + R^2$ is always negative such that the three solutions are all real and there is no imaginary part associated with the solutions.

To see how the non-ideal effects in the warm two-fluid formalism modify the ideal MHD waves, we consider the following limits:

$$k_z \ll k, \quad Q \sim \beta \ll 1. \tag{15}$$

Under these limits, we have $\Omega_0^2 \approx k^2/(2k_z^2 A)$, $\sqrt{W^3+R^2} \approx iBk^4/(6\sqrt{3}A^3k_z^4)$, $S+T \approx 2k^2/(3k_z^2 A) - B/A$, and $S-T \approx (B/A)(i/\sqrt{3})$. Thus, we have $\Omega_F^2 = (k^2V_A^2)/(Ak_z^2)$, $\Omega_S^2 = (c_s^2k_z^2)/(k_z^2V_A^2)$, and $\Omega_A^2 = B/A$, or

$$\omega_F^2 \approx \frac{k_\perp^2 V_A^2}{1 + k_\perp^2 \lambda_e^2}, \quad \omega_S^2 \approx c_s^2 k_z^2, \quad \omega_A^2 = k_z^2 V_A^2 \frac{1 + k^2\rho_L^2}{1 + k^2\lambda_e^2}. \tag{16}$$

Thus, we recover the well-known results, except that the kinetic Alfvén wave dispersion relation includes the "finite Larmor radius" corrections which accounts for both ion and electrons by way of ρ_L.

IV. PROPERTIES OF WAVE POLARIZATION

The polarization characteristics of the eigenmodes can be discussed on the basis of Eq. (3). If we choose to represent E_{1z} in terms of E_{1x}, then, we have

$$E_{1z} = \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{22}a_{33} - a_{23}^2} E_{1x}. \tag{17}$$

Making use of explicit definitions given by Eqs. (4a)–(4g), one may express the numerator as

$$a_{13}a_{22} - a_{12}a_{23} = \frac{ik_yk_z}{\omega}\left(\frac{\omega^2\omega_{pe}^2\omega_{pi}^2\left(v_{the}^2 - v_{thi}^2\right)(\omega_{ce} - \omega_{ci})}{D_eD_i} + c^2\sum_\alpha \frac{\omega_{c\alpha}\omega_{p\alpha}^2}{D_\alpha}\right). \tag{18}$$

After some lengthy algebra, Eq. (18) is rewritten as

$$\begin{aligned} a_{13}a_{22} - a_{12}a_{23} &= \frac{ik_yk_z}{\omega}\frac{1}{D_eD_i}\left(\tilde{G}_4\Omega^4 + \tilde{G}_2\Omega^2 + \tilde{G}_0\right), \\ \tilde{G}_4 &= -\frac{1+Q}{\lambda_e\sqrt{Q}}k_z^2\omega_{pe}^4V_A^5(1-Q)k_z^2\lambda_e^2, \\ \tilde{G}_2 &= -\frac{1+Q}{\lambda_e\sqrt{Q}}k_z^2\omega_{pe}^4V_A^5\left[-(Q\beta - \beta_e)(1+Q) + k^2(\beta_e - \beta)\lambda_e^2\right], \\ \tilde{G}_0 &= -\frac{1+Q}{\lambda_e\sqrt{Q}}k_z^2\omega_{pe}^4V_A^5(Q\beta - \beta_e), \end{aligned} \tag{19}$$

where we have made substitution $v_{the}^2 \to \beta_e V_A^2/Q$.

For the denominator, we have

$$\begin{aligned} a_{22}a_{33} - a_{23}^2 = &-k_y^2k_z^2\left(-\frac{c^2}{\omega^2} + \sum_\alpha \frac{v_{th\alpha}^2\omega_{p\alpha}^2}{D_\alpha}\right)^2 \\ &+ \left(-\frac{c^2k_z^2}{\omega^2} + \sum_\beta \frac{\left(k_z^2v_{th\beta}^2 - \omega^2\right)\omega_{p\beta}^2}{D_\beta}\right) \\ &\times \left(-\frac{c^2k_y^2}{\omega^2} + \sum_\gamma \frac{\left(k_y^2v_{th\gamma}^2 + \omega_{c\gamma}^2 - \omega^2\right)\omega_{p\gamma}^2}{D_\gamma}\right). \end{aligned} \tag{20}$$

After some lengthy algebra, which we have double checked with the aid of symbolic math software `Mathematica`, we arrive at

$$\begin{aligned} a_{22}a_{33} - a_{23}^2 = \frac{1}{\omega^2D_eD_i}\Big\{&\omega^6(\omega_{pe}^2 + \omega_{pi}^2)(c^2k^2 + \omega_{pe}^2 + \omega_{pi}^2) \\ &+\omega^4\Big[-\left(\omega_{pe}^2 + \omega_{pi}^2\right)\left(k^2v_{thi}^2\omega_{pe}^2 + k^2v_{the}^2\omega_{pi}^2\right. \\ &\left.+ \omega_{ci}^2\omega_{pe}^2 + \omega_{ce}^2\omega_{pi}^2\right) - c^2\omega_{pe}^2\left(k^4v_{thi} + k^2\omega_{ci}^2 + k_z^2\omega_{ce}^2\right) \\ &-c^2\omega_{pi}^2\left(k^4v_{the}^2 + k^2\omega_{ce}^2 + k_z^2\omega_{ci}^2\right)\Big] \\ &+\omega^2\Big[\left(k_z^2v_{thi}^2\omega_{pe}^2 + k_z^2v_{the}^2\omega_{pi}^2\right)\left(\omega_{ci}^2\omega_{pe}^2 + \omega_{ce}^2\omega_{pi}^2\right) \\ &+ c^2k_z^2\omega_{ce}^2\omega_{ci}^2\left(\omega_{pe}^2 + \omega_{pi}^2\right) + c^2k_z^2k^2\left(\omega_{ce}^2 + \omega_{ci}^2\right) \\ &\times\left(v_{thi}^2\omega_{pe}^2 + v_{the}^2\omega_{pi}^2\right)\Big] - c^2k_z^4\omega_{ce}^2\omega_{ci}^2 \\ &\times\left(v_{thi}^2\omega_{pe}^2 + v_{the}^2\omega_{pi}^2\right)\Big\}. \end{aligned} \tag{21}$$

We now change the variable as given by Eq. (7). Then, Eq (21) can be simplified as

$$\begin{aligned} a_{22}a_{33} - a_{23}^2 = \frac{k_z^4\omega_{pe}^4V_A^6(1+Q)}{\omega^2Q\lambda_e^2D_eD_i}\Big(&k_z^2Q\lambda_e^2\left(1 + Q + k^2\lambda_e^2\right)\Omega^6 \\ &-\Omega^4\Big\{Q(1+Q) + \lambda_e^2\Big[k_z^2 + k_y^2Q + k_z^2Q^2 \\ &+k^2(1+Q)Q\beta\Big] + Q\beta k^4\lambda_e^4\Big\} \\ &+\Omega^2\left[Q(1+\beta+Q\beta) + k^2\left(1+Q^2\right)\beta\lambda_e^2\right] - Q\beta\Big). \end{aligned} \tag{22}$$

At this point, we simplify Eq. (22) by means of the dispersion relation (9), with the end result

$$a_{22}a_{33} - a_{23}^2 = \frac{k_z^4 \omega_{pe}^4 V_A^6 (1+Q)^2}{\omega^2 \lambda_e^2 D_e D_i k^2} \Omega^2 (A\Omega^2 - 1)(k^2\beta - k_z^2\Omega^2). \tag{23}$$

Thus, finally writing down the entire term,

$$\begin{aligned} \frac{E_{1z}}{E_{1x}} &= \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{22}a_{33} - a_{23}^2} \\ &= \frac{ik_y k_z}{\omega} \frac{1}{D_e D_i} \frac{\omega^2 \lambda_e^2 D_e D_i k^2}{k_z^4 \omega_{pe}^4 V_A^6 (1+Q)^2} \left(-\frac{1+Q}{\lambda_e \sqrt{Q}} k_z^2 \omega_{pe}^4 V_A^5 \right) \\ &\times \frac{(Q\beta - \beta_e)[1 - (1+Q)\Omega^2] + k^2\lambda_e^2(\beta_e - \beta)\Omega^2 + k_z^2(1-Q)\lambda_e^2\Omega^4}{\Omega^2(A\Omega^2 - 1)(k^2\beta - k_z^2\Omega^2)}, \end{aligned} \tag{24}$$

which upon canceling out some terms further simplifies

$$\begin{aligned} E_{1z} &= -\frac{iE_{1x} k_y k^2 \lambda_i C_E}{(1+Q)\Omega(k^2\beta - k_z^2\Omega^2)(-1 + A\Omega^2)}, \\ C_E &= (Q\beta - \beta_e)[1 - (1+Q)\Omega^2] \\ &\quad + k^2\lambda_e^2(\beta_e - \beta)\Omega^2 + k_z^2(1-Q)\lambda_e^2\Omega^4. \end{aligned} \tag{25}$$

This is the wave polarization of E_{1z} expressed in terms of E_{1x}.

We may likewise express E_{1y} in terms of E_{1x},

$$E_{1y} = \frac{a_{12}a_{33} - a_{13}a_{23}}{a_{22}a_{33} - a_{23}^2} E_{1x}. \tag{26}$$

As the denominator $a_{22}a_{33} - a_{23}^2$ is already know, the remaining task is to compute the numerator,

$$\begin{aligned} a_{12}a_{33} - a_{13}a_{23} &= \left(\sum_\alpha \frac{\omega_{c\alpha}}{\omega} \frac{\omega_{p\alpha}^2 (\omega^2 - k_z^2 v_{th\alpha}^2)}{D_\alpha} \right) \\ &\times \left(\frac{c^2 k_y^2}{\omega^2} + \sum_\beta \frac{\omega_{p\beta}^2 \left(\omega^2 - \omega_{c\beta}^2 - k_y^2 v_{th\beta}^2 \right)}{D_\beta} \right) \\ &- k_y^2 k_z^2 \left(\sum_\alpha \frac{i\omega_{c\alpha}}{\omega} \frac{\omega_{p\alpha}^2 v_{th\alpha}^2}{D_\alpha} \right) \left(\frac{c^2}{\omega^2} - \sum_\beta \frac{v_{th\beta}^2 \omega_{p\beta}^2}{D_\beta} \right). \end{aligned} \tag{27}$$

Again, after some lengthy algebra, it can be shown that Eq. (27) is expressed as follows:

$$\begin{aligned} a_{12}a_{33} - a_{13}a_{23} = \frac{i}{\omega D_e D_i} \Big\{ &\omega^4 \left(c^2 k_y^2 + \omega_{pe}^2 + \omega_{pi}^2 \right) \\ &\times \left(\omega_{ce}\omega_{pe}^2 + \omega_{ci}\omega_{pi}^2 \right) + \omega^2 \Big[-\omega_{pe}^4 \omega_{ce} (k^2 v_{thi}^2 + \omega_{ci}^2) \\ &- \omega_{pi}^4 \omega_{ci} (k^2 v_{the}^2 + \omega_{ce}^2) - \omega_{pe}^2 \omega_{pi}^2 \big(\omega_{ce}^2 \omega_{ci} + \omega_{ci}^2 \omega_{ce} \\ &+ k_y^2 v_{thi}^2 \omega_{ce} + k_y^2 v_{the}^2 \omega_{ci} + k_z^2 v_{the}^2 \omega_{ce} + k_z^2 v_{thi}^2 \omega_{ci} \big) \Big] \\ &+ k_z^2 \omega_{ce} \omega_{ci} \Big[\left(v_{thi}^2 \omega_{pe}^2 + v_{the}^2 \omega_{pi}^2 \right) \left(\omega_{ci}\omega_{pe}^2 + \omega_{ce}\omega_{pi}^2 \right) \\ &+ c^2 k_y^2 \left(\omega_{ci} v_{thi}^2 \omega pe^2 + \omega_{ce} v_{the}^2 \omega_{pi}^2 \right) \Big] \Big\}. \end{aligned} \tag{28}$$

Again making use of the convention defined in Eq. (7), we obtain

$$\begin{aligned} a_{12}a_{33} - a_{13}a_{23} &= \frac{i}{\omega D_e D_i} \left(\tilde{K}_4 \Omega^4 + \tilde{K}_2 \Omega^2 + \tilde{K}_0 \right), \\ \tilde{K}_4 &= -\frac{k_z^2 (1+Q) V_A^5 \omega_{pe}^4}{\sqrt{Q}\lambda_e} k_z^2 (1-Q) \left(1 + Q + k_y^2 \lambda_e^2 \right), \\ \tilde{K}_2 &= \frac{k_z^2 (1+Q) V_A^5 \omega_{pe}^4}{\sqrt{Q}\lambda_e} \Big[k_z^2 (1-Q)\beta + k_y^4 (\beta - \beta_e)\lambda_e^2 \\ &\quad + k_y^2 (\beta - \beta_e)(1 + Q + k_z^2 \lambda_e^2) \Big], \\ \tilde{K}_0 &= -\frac{k_z^2 (1+Q) V_A^5 \omega_{pe}^4}{\sqrt{Q}\lambda_e} k_y^2 (Q\beta - \beta_e). \end{aligned} \tag{29}$$

Making further connection with C_E as defined in Eq. (25), we can write Eq. (29) as

$$\begin{aligned} a_{12}a_{33} - a_{13}a_{23} &= -\frac{i}{\omega D_e D_i} \frac{k_z^2 V_A^5 \omega_{pe}^4 (1+Q)}{\sqrt{Q}\lambda_e^3} \\ &\times \left[(A - k_z^2 \lambda_e^2) C_E - (1 - A\Omega^2)(1+Q)(Q\beta - \beta_e) \right]. \end{aligned} \tag{30}$$

With Eq. (23) and eliminating the common factor, we finally arrive at

$$E_{1y} = -\frac{iE_{1x} k^2 \lambda_i \left[(A - k_z^2 \lambda_e^2) C_E - (1 - A\Omega^2)(1+Q)(Q\beta - \beta_e) \right]}{k_z (1+Q)\lambda_e^2 \Omega (k^2\beta - k_z^2\Omega^2)(-1 + A\Omega^2)}. \tag{31}$$

This can be used to depict the wave polarization E_{1y}/E_{1x}. There are several ways to arrive at this conclusion, but we find that the present treatise is the most straightforward way. In Sec. V, we will demonstrate the importance of such expressions. Note that the factor i, which appears on the right-hand side of E_{1y}, indicates that electric field is elliptically polarized in the plane perpendicular to the magnetic field.

For the case of magnetic field polarization, starting from Faraday's low, the ratio of perturbed magnetic field along and across the external field is given by

$$\frac{B_{1z}}{B_{1x}} = \frac{k_y E_{1x}}{k_z E_{1y} - k_y E_{1z}} = \frac{k_y}{k_z E_{1y}/E_{1x} - k_y E_{1z}/E_{1x}}. \tag{32}$$

Making use of

$$\begin{aligned} \frac{k_z E_{1y} - k_y E_{1z}}{E_{1x}} &= -ik^2\lambda_i \frac{(A - k_z^2\lambda_e^2)C_E - (1 - A\Omega^2)(1+Q)(Q\beta - \beta_e) - k_y^2 \lambda_e^2 C_E}{(1+Q)\lambda_e^2 \Omega (k^2\beta - k_z^2\Omega^2)(-1 + A\Omega^2)} \\ &= ik^2\lambda_i \frac{(1-Q)\Omega}{-1 + A\Omega^2}, \end{aligned} \tag{33}$$

we may express the compressional magnetic field as

$$B_{1z} = \frac{ik_y (A\Omega^2 - 1) B_{1x}}{k^2 (1-Q)\lambda_i \Omega}. \tag{34}$$

Despite the complicated intermediate calculations, the final solution is amazingly simple. It is interesting to note that when the wave dispersion is characterized by the kinetic Alfvén wave, then, we know from

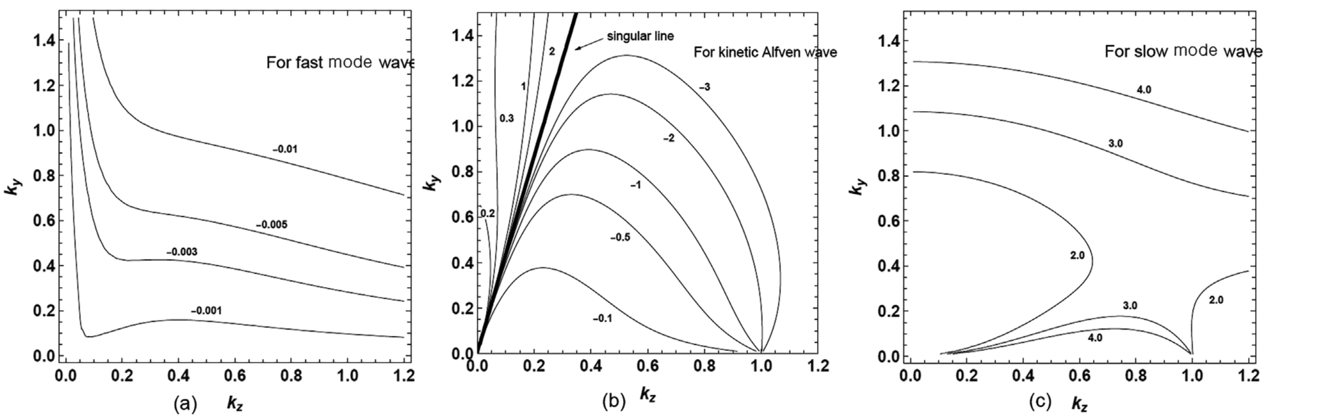

FIG. 4. Contour plot of E_{1z}/E_{1x} for (a) fast mode wave, (b) kinetic Alfvén wave, and (c) slow mode wave with $\beta = 0.05$, $Q = 1/1800$, $R = \beta_e/\beta = 0.1$, $\lambda_e = 0.1$, and $\Omega = \omega/k_z V_A$. Bold line is ion acoustic dispersion relation.

Sec. III that $\Omega^2 = B/A$. Thus, for the case of kinetic Alfvén wave, the compressional component of magnetic field is given by

$$B_{1z} = \frac{ic^2 k_y \rho_L c_s B_{1x}}{V_A}\sqrt{\frac{A}{B}} \tag{35}$$

with $Q \ll 1$. This result is same as given by Hollweg[29] if we set $A = B$—see Eq. (11) in Ref. 29. However, when finite Larmor radius effect is included, then, for small $k\rho_L$, we have

$$B_{1z} \approx \frac{ic^2 k_y \rho_L \sqrt{\beta} B_{1x}}{1 + k^2\rho_L^2/2}. \tag{36}$$

The factor 1/2 is different from Eq. (28) of Ref. 29. This is because Hollweg[29] included the finite Larmor radius effect starting from the most simple MHD relations, and such a finite Larmor radius effect has been included through diamagnetic current by intuition. Thus for the case of kinetic Alfvén wave, compressional component of magnetic field is proportional to $k_y\rho_L$, the perpendicular structure of the wave, and the square root of plasma beta. So, compressional component of kinetic Alfvén wave is important when β becomes larger and the two-potential approximation introduced by Hasegawa and Uberoi[32] and Goertz[42] is not applicable.

In Fig. 4, we show the polarity E_{1z}/E_{1x} for different wave branches with $\beta = 0.05$, $Q = 1/1800$, $R = \beta_e/\beta = 0.1$, $\lambda_e = 0.1$, and $\Omega = \omega/k_z V_A$. For the case of the kinetic Alfvén branch, there is a polarization inversion across a singular line which is given as $k^2\beta - k_z^2\Omega^2 = 0$. Such a line provides a critical propagation angle above which it has positive polarization and below it has negative polarization. For our choice of parameters, it is about 78°. Also, such a singular line indicates that kinetic Alfvén waves with such a propagation angle will have linear polarization with $E_{1x} = 0$. For the case of fast and slow mode branches, we do not have the change of polarization inversion as shown in Figs. 4(a) and 4(c).

In Fig. 5, we display the polarity of E_{1y}/E_{1x}. Again just as with the E_{1z} case, the polarity inversion and a singular line occurs only in the kinetic Alfvén wave branch such as in Fig. 5(b), which is plotted with the same parameter values. It is interesting to note that besides the singular line, we have another line of polarity inversion for certain

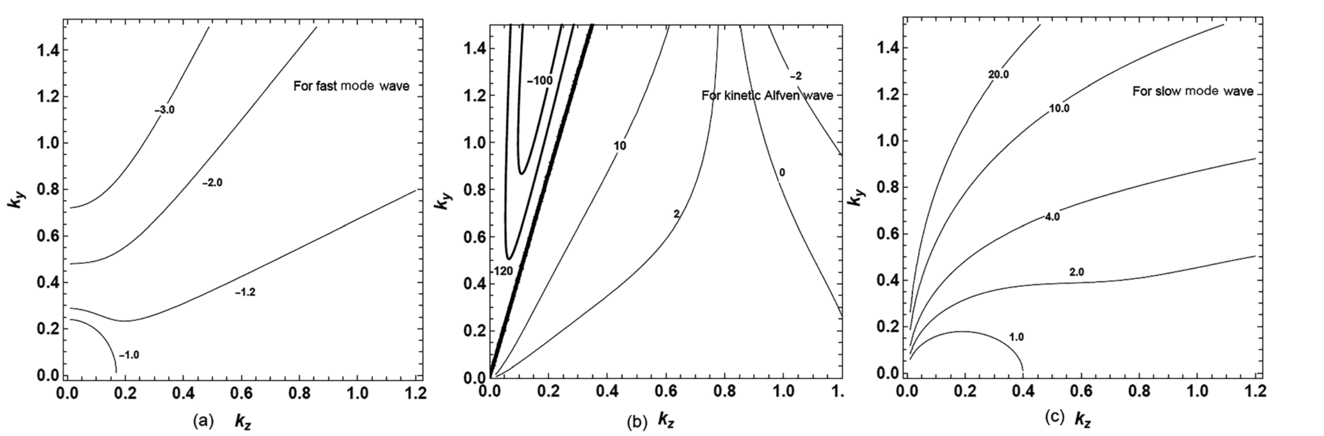

FIG. 5. Contour plot of E_{1y}/E_{1x} for (a) fast mode wave, (b) kinetic Alfvén wave, and (c) slow mode wave with $\beta = 0.05$, $Q = 1/1800$, $R = \beta_e/\beta = 0.1$, $\lambda_e = 0.1$, and $\Omega = \omega/k_z V_A$. Bold line is ion acoustic dispersion relation.

values of wave numbers. Along such lines, we have $E_{1y} = 0$ which again indicates linear polarization. However, for such cases, these lines do not correspond to a certain angle, but given instead as a curve that relates the wave number in x and y directions.

V. CHARGE NEUTRALITY AND PROPERTIES OF THE ELECTRIC FIELD

In this session, we will first review the connection between charge neutrality and displacement current as mentioned in Sec. I, and second, from the polarization of the electric field, we will show how the charge separation should be induced, or in other words, how the electro-static component of the electric field should be generated. Throughout the present paper, we have not explicitly made an assumption on the charge neutrality condition at the outset, but it turns out that the assumption of slow wave, namely, $c^2k^2 > \omega^2$, is equivalent to an implicit charge neutrality condition. To see this, recall the Fourier transformed Ampère's law, which along perpendicular direction is given by

$$\mathbf{E}_\perp\left(1 - \frac{\omega^2}{c^2k^2}\right) = -\frac{4\pi e\omega i}{ck^2}(n_i\mathbf{v}_{i\perp} - n_e\mathbf{v}_{e\perp}), \tag{37}$$

where $\perp$ refers to perpendicular direction with respect to $\mathbf{k}$. Neglecting the displacement current is equivalent to taking the assumption of $\omega^2 \ll c^2k^2$, which can be also interpreted as low frequency modes—that is, modes whose frequency is sufficiently lower than the radiation modes, under the implicit assumption that the plasma frequency is much higher than the electron gyro frequency, $\omega_{pe} \gg \omega_{ce}$. Along the wave propagation direction, together with continuity equation, we have

$$0 = \frac{\omega}{c}\mathbf{k}\cdot\mathbf{E} - \frac{4\pi\omega\rho_c}{c}, \tag{38}$$

where $\rho_c = \sum_\alpha q_\alpha n_\alpha$ is the overall charge density. Obviously, this is Poisson's equation. The first term is the displacement current term, and neglecting this term is equivalent to assuming $\rho_c = 0$. So, in our reduced model, perturbed charge density is automatically zero. In other words, charge neutrality is implicitly presumed in the model although we did not explicitly impose such a condition when we derived the slow-wave dispersion relation.

As shown above, the charge neutrality is implicitly assumed although not explicitly invoked. However, the electrostatic component of the electric field is not zero, that is, $\mathbf{k}\cdot\mathbf{E} \neq 0$. This is reminiscent of the ion acoustic mode in unmagnetized plasma where different mobility of electrons and ions induce an electrostatic field, but the smallness of wave number $\mathbf{k}$ guarantees the "charge neutrality" of the plasma. In our model, we may extract electromagnetic E_{1ym} and electrostatic components E_{1ys} from $E_{1y} = E_{1ys} + E_{1ym}$ as

$$E_{1ys} = -\frac{iE_{1x}k^2\lambda_i\left[AC_E - (1-A\Omega^2)(1+Q)(Q\beta-\beta_e)\right]}{k_z(1+Q)\lambda_e^2\Omega(k^2\beta - k_z^2\Omega^2)(-1+A\Omega^2)}, \tag{39a}$$

$$E_{1ym} = \frac{iE_{1x}k^2k_z\lambda_iC_E}{(1+Q)\Omega(k^2\beta - k_z^2\Omega^2)(-1+A\Omega^2)}. \tag{39b}$$

Making use of Eq. (25) and from the properties of the electromagnetic field, we get

$$\begin{aligned}\mathbf{k}\cdot\mathbf{E}_{EM} &= k_zE_{1z} + k_yE_{1ym}\\ &= -\frac{iE_{1x}k_zk_yk^2\lambda_iC_E}{(1+Q)\Omega(k^2\beta - k_z^2\Omega^2)(-1+A\Omega^2)}\\ &\quad + \frac{iE_{1x}k^2k_yk_z\lambda_iC_E}{(1+Q)\Omega(k^2\beta - k_z^2\Omega^2)(-1+A\Omega^2)} = 0.\end{aligned} \tag{40}$$

Here, we note that the electric field along the magnetic field, that is E_{1z}, exactly cancels out E_{1ym}, so the Fourier transformed Poisson's equation gives

$$i\mathbf{k}\cdot\mathbf{E} = ik_yE_{1y} + ik_zE_{1z} = ik_yE_{1ys} = 4\pi\rho_c. \tag{41}$$

By virtue of the exact cancelation of E_{1z} *á la* Eq. (40), the charge separation can only be induced through E_{1ys}, that is the electric field "perpendicular" to the magnetic field, but not along the wave propagation direction.

The reason why perpendicular electric field induces charge separation is that it stems from the mobility of electrons and ions. For the case of magnetized plasma, the distinct particle motions along and cross the magnetic field imply that the mobility along the field lines is orders of magnitude faster than that across the field. For low-frequency reduced model, assuming zero pressure, the perpendicular electric field is related to the perpendicular current as

$$\mathbf{J}_{perp} \sim \frac{c^2\rho_M}{B^2}\frac{d\mathbf{E}_{perp}}{dt}, \tag{42}$$

where ρ_M is center of mass density and subscript *perp* means perpendicular to the magnetic field. It is clear that this is polarization current of the plasma and the equation can be understood as the polarization current across the magnetic field has induced an electrostatic field.

In short, when the wave in plasma is associated with perturbed electric field $\mathbf{E}$, the electrons and ions both undergo $\mathbf{E}\times\mathbf{B}$ motion, which has the same velocity perturbation. However, to a higher order, difference of the finite Larmor radius for electrons and ions leads to the polarization current perpendicular to the magnetic field. It is this polarization current that induces the electric field perpendicular to the magnetic field. Since the wave propagates oblique to the magnetic field (that is, $k_y \neq 0$), there should be electric field component along the wave vector $\mathbf{k}$, and such an electric field can possibly contribute to the separation of charge, although it will be small just as in the case of the ion acoustic wave in unmagnetized plasmas. Thus, instead of the electric field, as noted by Hasegawa and Chen,[6] parallel component of the magnetic field is the key for the wave energy absorption, and such a field can provide the magnetic field components along the ambient magnetic field for the transit time damping.

VI. CONCLUSIONS

In this paper, we have (re)derived a general dispersion relation for low-frequency waves by making use of warm two-fluid theory. As noted in the Introduction, completely general dispersion relations for warm magnetized electron–proton plasma have been derived on the basis of the two-fluid theory,[34,35] but it was also noted that a separate derivation of the reduced form of warm two-fluid plasma dispersion relations restricted to the low-frequency modes (or more accurately, slow or sub-luminal modes satisfying the condition $ck > \omega$—which can be derived by ignoring the displacement current) is also useful. In this regard, Zhao *et al.*[39] carried out just such an analysis, but under a simplifying assumption of quasi-neutrality as it relates to the perturbed electron and ion density fluctuations. The purpose of the present paper was primarily to revisit Ref. 39 by without making an explicit assumption of quasi-neutrality at the outset although it is shown later that the present formalism is implicitly equivalent to the charge-neutral situation. We have, thus, re-derived the desired dispersion relation (11) and have compared our result with that of Ref. 39—Eq. (11) vs (12). We have also found that while the two equations differ slightly, the numerical solutions were virtually indistinguishable (the comparative results are not shown). A close connection between our approach and the formalism that invokes the charge neutrality[39] is shown, and it is verified that even if we have not invoked the charge neutrality at the outset, it is implied in Ampère's Law, thus showing that the present formalism and Ref. 39 share a conceptual similarity, and this may explain the virtually identical numerical results between the two formalisms. Despite this, we believe that our result is more formally correct. We have also presented a detailed analysis of the warm magnetized plasma dispersion surfaces in Figs. 1–3. In the second part of the present paper, we have focused on the low-frequency MHD-like modes and their polarization characteristics. Based upon the present analysis, we have reached the following new conclusions:

The overall dispersion relation depends on the total beta of the plasma that is sum of electron and ion betas. Such a finite beta effect naturally enters the dispersion relation in the form of finite Larmor radius effect. Although it is generally believed that finite Larmor radius effect is due to the large ion Larmor radius combined with high perpendicular wave number, we found that it is actually the ion-acoustic Larmor radius effect such that even without the ion thermal gyromotion, $T_i = 0$, the finite electron temperature, T_e, can also have a non-ideal contribution to the dispersion relation. That is, generally speaking, kinetic contributions to Alfvén wave should include both electron and ion thermal effects. All these findings are consistent with previous findings by various authors.[34,35,39,40,43–45]

We have found that the electric field along the ambient magnetic field is always transverse, implying that the parallel electric field does not involve charge separation. Owing to the oblique nature of the wave propagation, we found that the electrostatic components, which could possibly lead to the charge separation, is the polarization current. The polarization current relates to the perpendicular motion of the electrons and ions as shown in Eq. (42). The reason why the perpendicular motion dominates the charge separation is that electron and ion mobility across the magnetic field is orders of magnitude smaller when compared to the parallel mobility.

Finally, we also have found that only the kinetic Alfvén wave branch can change the direction of electric field polarization as shown in Figs. 4 and 5. Here, the line of such a polarization inversion is characterized by $\Omega^2 k_z^2 = k^2\beta$. That is, this is none other than the ion-acoustic mode dispersion relation, $\omega^2 = c_s^2 k^2$. This indicates that, in some specific directions, the transverse field E_x becomes zero and kinetic Alfvén wave loses its characteristics and, thus, becomes ion acoustic mode. This feature is consistent with the dispersion surface properties for high β case discussed in Fig. 3. Such a sudden change of the wave characteristics is not surprising given that we have a very similar case in the literature, that is X mode—a full electromagnetic wave that propagates exactly perpendicular to the magnetic field.[41]

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Cheong R. Choi: Investigation (equal). **Minho Woo:** Investigation (supporting). **Kwangsun Ryu:** Funding acquisition (equal). **Dae-Young Lee:** Investigation (supporting). **Peter H. Yoon:** Conceptualization (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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