



Relationships Between Polyhedral Convex Sets and Generalized Polyhedral Convex Sets

Nguyen Ngoc Luan¹ · Nguyen Mau Nam² · Nguyen Nang Thieu³ ·
Nguyen Dong Yen³

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Abstract

In this paper, we study some relationships between polyhedral convex sets and generalized polyhedral convex sets. In particular, we clarify by a counterexample that the necessary and sufficient conditions for the separation of a convex set and a polyhedral convex set obtained by Ng et al. (Nonlinear Anal. 55:845–858, 2003; Theorem 3.1) are no longer valid when considering generalized polyhedral convex sets instead of polyhedral convex sets. We also introduce and study the notions of generalized polyhedral multifunctions and optimal value functions generated by generalized polyhedral convex multifunctions along with their generalized differentiation calculus rules.

Keywords Convex polyhedron · Generalized convex polyhedron · Separation · Normal cone · Coderivative · Subdifferential · Optimal value function

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✉ Nguyen Mau Nam
mnn3@pdx.edu

Nguyen Ngoc Luan
luannn@hnue.edu.vn

Nguyen Nang Thieu
nnthieu@math.ac.vn

Nguyen Dong Yen
ndyen@math.ac.vn

¹ Department of Mathematics and Informatics, Hanoi National University of Education, 136 Xuan Thuy, Hanoi, Vietnam

² Fariborz Maseeh Department of Mathematics and Statistics, Portland State University, Portland, OR 97207, USA

³ Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam

1 Introduction

Polyhedral convex sets and related concepts have been studied broadly in the framework of convex analysis in both finite dimensions and infinite dimensions. Among many notable results involving polyhedral convex sets, necessary and sufficient conditions for a separation of a convex set and a polyhedral convex set play a crucial role in developing generalized differentiation involving polyhedral convex sets with applications to optimization. An important result was established by Ng and Song (see [10, Theorem 3.86], [12, Theorem 3.1]) providing necessary and sufficient conditions for separating a convex set and a polyhedral convex set in locally convex topological vector spaces. This is a generalization of a well-known result obtained by Rockafellar in finite dimensions; see [15, Theorem 20.2]. The result by Ng and Song was then used in [4] to obtain comprehensive generalized differentiation calculus for nonsmooth functions and multifunctions in locally convex topological vector spaces.

Given the crucial role of polyhedral convex sets in convex analysis and applications, a concept called *generalized polyhedral convex sets* has been introduced and studied in infinite dimensions; see [1, 8, 20] and the references therein. In a series of recent papers, Luan, Yao, and Yen have established new results on generalized polyhedral convex sets and generalized polyhedral convex functions in locally convex topological vector spaces and have provided many applications in convex optimization, linear vector optimization, piecewise linear vector optimization, etc. These developments also shed new light on many known results involving polyhedral convex sets. In particular, we refer the reader to the paper by Luan et al. [7] in which several constructions such as sum of sets, sum of functions, directional derivative, infimal convolution, normal cone, subdifferential, conjugate function involving polyhedral convex sets and generalized polyhedral convex sets were thoroughly investigated.

The notion of polyhedral convex set was employed to define polyhedral convexity of extended-real-valued functions by requiring the epigraphs of such functions to be polyhedral convex. Similarly, a multifunction is said to be polyhedral convex if its graph is a polyhedral convex set in the product space. Polyhedral convex functions and multifunctions have been extensively investigated in the literature in both theoretical aspects and applications to optimization; see [10, 13–15, 17] and the references therein. These notions inspire us to define a new notion called the *generalized polyhedral convex multifunction* by requiring that the graph of the function in question is a generalized polyhedral convex set.

The remarkable role of polyhedral convex sets and generalized polyhedral convex sets raises an important question asking for the clarification whether a certain result which holds for polyhedral convex sets also holds for generalized polyhedral convex sets or not. One of the main goals of this paper is to answer the question. In particular, we provide a counterexample showing that the aforementioned separation result by Ng and Song for polyhedral convex sets is no longer true for generalized polyhedral convex sets in general, and thus this counterexample somehow discourages the possibility for full generalizations of the results in [4] to the case of generalized polyhedral convex sets. The second main goal of this paper is to study polyhedral convex multifunctions and generalized polyhedral convex multifunctions and derive new calculus rules of generalized differentiation in the case where one multifunction involved is

polyhedral convex, while the other multifunction is generalized polyhedral convex. We explore the notion of *coderivative* introduced by Mordukhovich and show that in the afore-mentioned setting coderivative calculus rules are satisfied without requiring any qualification condition, which was needed in the general convex and nonconvex cases from [4, 9, 10, 16]. We also study generalized differentiation of optimal value functions generated by polyhedral convex and generalized polyhedral convex multifunctions. When developing these generalized differentiation calculus rules, we recover a number of known results obtained by Luan et al. [7] for polyhedral convex and generalized polyhedral convex functions.

This paper is organized as follows. In Sect. 2, we study some relationships between polyhedral convex sets and generalized polyhedral convex sets in locally convex Hausdorff topological vector spaces. Section 3 is devoted to generalized differentiation for polyhedral convex multifunctions and generalized polyhedral convex multifunctions. Generalized differentiation of optimal value functions generated by polyhedral convex multifunctions and generalized polyhedral convex set-valued mappings is investigated in Sect. 4. The final section gives some concluding remarks.

2 Relationships Between Polyhedral Convex Sets and Generalized Polyhedral Convex Sets

The main goal of this section is to study some relationships between polyhedral convex sets and generalized polyhedral convex sets. We first establish a necessary and sufficient condition for a generalized polyhedral convex set to be a polyhedral convex set. Then we show by a counterexample that the necessary and sufficient conditions for the separation of a convex set and a polyhedral convex set obtained by Ng and Song in [12, Theorem 3.1] are no longer valid when considering generalized polyhedral convex sets instead of polyhedral convex sets.

Let X be a locally convex Hausdorff topological vector space over the reals with its topological dual denoted by X^* . For simplicity of presentation, we assume that all spaces under consideration are locally convex Hausdorff topological vector spaces. The cone generated by a nonempty subset Ω of X (resp., the closure of Ω) is denoted by $\text{cone}(\Omega)$ (resp., $\overline{\Omega}$). Thus, $\text{cone}(\Omega) = \{tx \mid t \geq 0, x \in \Omega\}$. In the sequel, let $\mathbb{R} := (-\infty, \infty]$.

Definition 2.1 (a) A subset P of X is said to be a *polyhedral convex set* or a *convex polyhedron* if there exist $x_1^*, \dots, x_m^* \in X^*$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ such that

$$P = \{x \in X \mid \langle x_i^*, x \rangle \leq \alpha_i \text{ for all } i = 1, \dots, m\}.$$

(b) A subset Q of X is said to be a *generalized polyhedral convex set* or a *generalized convex polyhedron* if there exist $x_1^*, \dots, x_m^* \in X^*$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, and a closed affine subspace M of X such that

$$Q = \{x \in X \mid x \in M, \langle x_i^*, x \rangle \leq \alpha_i \text{ for all } i = 1, \dots, m\}. \quad (1)$$

It follows from the definitions that a generalized polyhedral convex set can be represented as the intersection of a polyhedral convex set and a closed affine subspace.

The proposition presented below establishes a condition that is both necessary and sufficient for the polyhedral convexity of a generalized polyhedral convex set of the form (1). This condition is based on the dimensionality of the closed affine subspace M involved.

Proposition 2.1 *Let Q be a nonempty generalized polyhedral convex set given by*

$$Q = \{x \in X \mid x \in M, \langle x_i^*, x \rangle \leq \alpha_i \text{ for all } i = 1, \dots, m\},$$

where $x_1^*, \dots, x_m^* \in X^*$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, and M is a closed affine subspace. Then Q is a polyhedral convex set if and only if $\text{codim } M < \infty$.

Proof $\Leftarrow$: Let $L := M - M$ and observe that L is the unique closed linear subspace parallel to M . Since M has finite codimension, $\dim X/L = k$ for some positive integer k . Consider the quotient mapping $\Phi: X \rightarrow X/L$. Choose $a \in M$ and get $[a] \in X/L$ with $\Phi(M) = [a]$. Suppose that $[b_1], \dots, [b_k]$ form a basis for X/L . Then we have the representation

$$[a] = \sum_{i=1}^k \beta_i [b_i],$$

where $\beta_1, \dots, \beta_k \in \mathbb{R}$. For each $i \in \{1, \dots, k\}$, we consider the linear mapping $g_i: X/L \rightarrow \mathbb{R}$ with

$$g_i\left(\sum_{i=1}^k \mu_i [b_i]\right) = \mu_i.$$

By [18, Lemma 1.20 and Theorem 1.21] (note that the same results and proofs are valid for linear mappings defined on $\mathbb{R}^n$), g_i is continuous for all $i \in \{1, \dots, k\}$. Thus, the function $u_i^* := g_i \circ \Phi$ is linear and continuous for all $i \in \{1, \dots, k\}$. Hence, $u_i^* \in X^*$ and $u_i^* \neq 0$ for all $i \in \{1, \dots, k\}$. We will now prove that

$$M = \{x \in X \mid u_i^*(x) = \beta_i, \forall i \in \{1, \dots, k\}\}. \quad (2)$$

For all $x \in M$, we have $\Phi(x) = \Phi(a) = [a] = \sum_{i=1}^k \beta_i [b_i]$. Then, $u_i^*(x) = g_i(\Phi(x)) = \beta_i$ for all $i \in \{1, \dots, k\}$. Therefore,

$$M \subset \{x \in X \mid u_i^*(x) = \beta_i, \forall i \in \{1, \dots, k\}\}.$$

To prove the reverse inclusion, taking any x in the right-hand side of (2), one has $u_i^*(x) = \beta_i$ for all $i \in \{1, \dots, k\}$. Thus, $g_i(\Phi(x)) = \beta_i$ for all $i \in \{1, \dots, k\}$, which means that

$$\Phi(x) = \sum_{i=1}^k \beta_i [b_i] = \Phi(a).$$

Thus, $\Phi(x - a) = [0]$. Hence, $x - a \in L$. This implies that $x \in M$. We have thus proved that the equality (2) is valid. Now, Q can be represented as

$$Q = \{x \in X \mid \langle u_i^*, x \rangle = \beta_i \text{ for } i = 1, \dots, k, \langle x_i^*, x \rangle \leq \alpha_i \text{ for } i = 1, \dots, m\}.$$

Therefore, the set Q is a polyhedral convex set.

$\Rightarrow$: Suppose that Q is a polyhedral convex set. Then there exist $z_1^*, \dots, z_p^* \in X^*$ and $\gamma_1, \dots, \gamma_p \in \mathbb{R}$ such that

$$Q = \{x \in X \mid \langle z_i^*, x \rangle \leq \gamma_i \text{ for all } i = 1, \dots, p\}.$$

Choosing $x_0 \in Q$ gives $x_0 \in M$ and $\langle z_i^*, x_0 \rangle \leq \gamma_i$ for all $i = 1, \dots, p$. Let $c_i := \gamma_i - \langle z_i^*, x_0 \rangle$ for $i = 1, \dots, p$. Then $c_i \geq 0$ for all $i = 1, \dots, p$ and we have

$$Q - x_0 = \{y \in X \mid \langle z_i^*, y \rangle \leq c_i\} \subset L := M - x_0.$$

Let $L_1 := \bigcap_{i=1}^p \ker z_i^*$. Then $L_1 \subset Q - x_0 \subset L$. It follows that

$$\text{codim } L \leq \text{codim } L_1 < \infty,$$

which completes the proof. $\square$

The next corollary is a direct consequence of Proposition 2.1.

Corollary 2.1 *A closed linear subspace M of X is a polyhedral convex set if and only if M has finite codimension.*

Next, we present an example of a generalized polyhedral convex set that is not a polyhedral convex set.

Example 2.1 Let $X = \ell_2$ and let

$$Q = \{x = (x_n) \mid x_{2k} = 0 \text{ for all } k \in \mathbb{N}\}.$$

Then Q is a generalized polyhedral convex set. In fact, it is a closed linear subspace of X . Since $\text{codim } Q = \infty$, the set Q is not a polyhedral convex set by Proposition 2.1.

Definition 2.2 Let Ω_1 and Ω_2 be a nonempty convex sets in X . We say that Ω_1 and Ω_2 can be separated by a closed hyperplane that does not contain Ω_2 if there exist $x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, x \rangle \leq \alpha \leq \langle x^*, y \rangle \text{ whenever } x \in \Omega_1, y \in \Omega_2,$$

and there exists $\hat{y} \in \Omega_2$ such that $\alpha < \langle x^*, \hat{y} \rangle$.

In the setting of Definition 2.2, define $\mathcal{H} = \{x \in X^* \mid \langle x^*, x \rangle = \alpha\}$,

$$\mathcal{H}_+ = \{x \in X^* \mid \langle x^*, x \rangle \geq \alpha\}, \quad \mathcal{H}_- = \{x \in X^* \mid \langle x^*, x \rangle \leq \alpha\}.$$

Since x^* is obviously nonzero, $\mathcal{H}$ is a closed hyperplane. We have

$$\Omega_1 \subset \mathcal{H}_-, \quad \Omega_2 \subset \mathcal{H}_+, \quad \Omega_2 \not\subset \mathcal{H}.$$

Definition 2.3 (See [10, Definition 2.168]) Let Ω be a convex subset of X . The *quasi-relative interior* of Ω is the set

$$\text{qri}(\Omega) := \{x \in \Omega \mid \overline{\text{cone}}(\Omega - x) \text{ is a linear subspace of } X\}.$$

In [7] and the references therein, several important results for polyhedral convex sets have been generalized for generalized polyhedral convex sets. We present below a number of important results which hold for polyhedral convex sets but do not hold for generalized polyhedral convex sets. The first one is a convex separation theorem involving a polyhedral convex set and a convex set in X ; see [10, Theorem 3.86] and [12, Theorem 3.1].

Theorem 2.1 *Let P be a nonempty polyhedral convex set and let Ω be a nonempty convex set in X . Suppose that $\text{qri}(\Omega) \neq \emptyset$. Then P and Ω can be separated by a closed hyperplane that does not contain Ω if and only if $P \cap \text{qri}(\Omega) = \emptyset$.*

This result plays a crucial role in developing generalized differentiation for nonsmooth functions and multifunctions in the case where some functions and multifunctions involved are generated by polyhedral convex sets.

The following example will show that the conclusion of Theorem 2.1 may not hold true when instead of P one takes a subspace M of infinite codimension.

Example 2.2 Let $\Omega_0 = \{x \in \ell_2 \mid x \text{ has finitely many nonzero coordinates}\}$,

$$y = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right), \quad z = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right),$$

and $M = \{\mu z \mid \mu \in \mathbb{R}\}$. Clearly, $y \in \ell_2$ and $z \in \ell_2$. Put $\Omega = y + \Omega_0$. Then, Ω is an affine subset of ℓ_2 . Hence, Ω is convex and $\text{qri}(\Omega) = \text{iri}(\Omega) = \Omega$. We have $M \cap \Omega = \emptyset$. Indeed, if $M \cap \Omega \neq \emptyset$, then for some $u \in \Omega_0$, we have $y + u \in M$. Thus, there is $\mu \in \mathbb{R}$ and $\bar{k} \in \mathbb{N}$ such that $\frac{1}{2^k} = \frac{\mu}{k+1}$ for all $k \geq \bar{k}$, which is a contradiction. As $\Omega = \text{qri}(\Omega)$, it follows that $M \cap \text{qri}(\Omega) = \emptyset$.

Next, we will prove that there is no closed hyperplane which does not contain Ω and separates M and Ω . Suppose on the contrary that there exists a closed hyperplane $\mathcal{H} \subset X$ such that $\mathcal{H}$ does not contain Ω and separates M and Ω . Then there is $x^* \in \ell_2 \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\mathcal{H} = \{x \in \ell_2 \mid \langle x^*, x \rangle = \alpha\}$. Since $\mathcal{H}$ separates Ω and M and $\mathcal{H}$ does not contain Ω , we have

$$\sup_{x \in M} \langle x^*, x \rangle \leq \alpha \leq \inf_{x \in \Omega} \langle x^*, x \rangle \quad (3)$$

and there exists $w \in \Omega$ such that $\alpha < \langle x^*, w \rangle$. As $\langle x^*, x \rangle \leq \alpha$ for all $x \in M$, one has $\langle x^*, \mu z \rangle \leq \alpha$ for all $\mu \in \mathbb{R}$. Thus, $\langle x^*, z \rangle = 0$. Then, the relation (3) is equivalent to

$$0 \leq \alpha \leq \inf_{x \in \Omega} \langle x^*, x \rangle.$$

So, we have

$$0 \leq \alpha \leq \langle x^*, y \rangle + \langle x^*, u \rangle$$

for all $u \in \Omega_0$. This implies $-\langle x^*, y \rangle \leq \langle x^*, u \rangle$ for all $u \in \Omega_0$. Since Ω_0 is dense in ℓ_2 , the latter property yields $-\langle x^*, y \rangle \leq \langle x^*, v \rangle$ for all $v \in \ell_2$. This means that $x^* = 0$, which contradicts the choice of x^* . Hence, there is no closed hyperplane not containing Ω which separates M and Ω .

Remark 2.1 In Example 2.2, the set Ω is not closed. However, the assertion of Theorem 2.1 may still be false when a subspace M of infinite codimension plays the role of P and Ω is a closed set in X . Indeed, by [7, Remark 2.12], there exists a locally convex topological vector space X with two closed linear subspaces L and M such that $\bar{L} + \bar{M} = X$ but $L + M \neq X$. Taking any $a \in X \setminus (L + M)$ and setting $\Omega = a + L$, one sees that Ω is a closed affine set. Hence, $\text{iri}(\Omega) = \text{qri}(\Omega) = \Omega$. First, we will show that $M \cap \Omega = \emptyset$. Suppose on the contrary that there is some $u \in M \cap \Omega$. Then, $u \in M$ and $u = a + v$ for some $v \in L$. Therefore, $a = u - v \in L + M$, which is a contradiction. Thus, $M \cap \Omega = \emptyset$. Next, we will prove that M and Ω cannot be separated by any hyperplane. Suppose on the contrary that there exist a nonzero linear functional $x^* \in X^*$ and a real number α such that

$$\sup_{x \in M} \langle x^*, x \rangle \leq \alpha \leq \inf_{y \in \Omega} \langle x^*, y \rangle. \quad (4)$$

If there is some $\bar{x} \in M$ such that $\langle x^*, \bar{x} \rangle = \beta \neq 0$, then by taking $t = (\alpha + 1)/\beta$, we have $t\bar{x} \in M$ and $\langle x^*, t\bar{x} \rangle = \alpha + 1 > \alpha$, which contradicts the fact that $\sup_{x \in M} \langle x^*, x \rangle \leq \alpha$. Hence $\langle x^*, x \rangle = 0$ for all $x \in M$. Thus, (4) yields

$$0 \leq \langle x^*, a \rangle + \inf_{z \in L} \langle x^*, z \rangle.$$

Since $\langle x^*, a \rangle$ is fixed, the latter implies that $\langle x^*, z \rangle = 0$ for all $z \in L$. Therefore, we have shown that $\langle x^*, x + z \rangle = 0$ for all $x \in M$ and $z \in L$. Recalling that $\bar{L} + \bar{M} = X$, we can infer that $\langle x^*, u \rangle = 0$ for all $u \in X$. This contradicts the choice of x^* . We have thus proved that $M \cap \text{qri}(\Omega) = \emptyset$ and that M and Ω cannot be separated by any hyperplane.

3 Generalized Differentiation for Polyhedral Convex Multifunctions and Generalized Polyhedral Convex Multifunctions

In this section, we study generalized differentiation for polyhedral convex and generalized polyhedral convex multifunctions in locally convex topological vector spaces. The main concept under consideration is the *coderivative* defined by Mordukhovich (see [9]) for multifunctions. The counterexample in Example 2.2 shows that analogues of the generalized differentiation results involving polyhedral convex sets in [10] may not hold for generalized polyhedral convex sets. Herein, we establish new positive results showing that qualification conditions needed for coderivative calculus (see, e.g., [4, 9, 10, 16]) are no longer required in our setting. Our developments shed new light on the theory of polyhedral convexity and generalized polyhedral convexity from the existing literature including [1, 7, 8, 10, 13–15, 17].

Given a nonempty convex set Ω in X , recall that the *normal cone* to the set Ω at $\bar{x} \in \Omega$ is given by

$$N(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega\}.$$

Lemma 3.1 *Let P be a polyhedral convex set and let M be a closed affine subspace in X . Then*

$$N(\bar{x}; P \cap M) = N(\bar{x}; P) + N(\bar{x}; M) = N(\bar{x}; P) + L^\perp \text{ for all } \bar{x} \in P \cap M,$$

where L is the linear subspace parallel to M .

Proof Fix any $\bar{x} \in P \cap M$. Then $\bar{x} \in P \cap \text{qri}(M) = P \cap M$, so $P \cap \text{qri}(M) \neq \emptyset$. It is follows from [10, Theorem 3.87] that

$$N(\bar{x}; P \cap M) = N(\bar{x}; P) + N(\bar{x}; M).$$

Since $N(\bar{x}; M) = L^\perp$, this completes the proof. $\square$

Lemma 3.2 *Let P_1 and P_2 be two convex polyhedra in X . Then*

$$N(\bar{x}; P_1 \cap P_2) = N(\bar{x}; P_1) + N(\bar{x}; P_2) \text{ for all } \bar{x} \in P_1 \cap P_2.$$

Proof If $P = \{x \in X \mid \langle x_i^*, x \rangle \leq \alpha_i \text{ for all } i = 1, \dots, m\}$, then

$$N(\bar{x}; P) = \text{cone}\{x_i^* \mid i \in I(\bar{x})\},$$

where $I(\bar{x}) := \{i \mid i = 1, \dots, m, \langle x_i^*, \bar{x} \rangle = \alpha_i\}$. Clearly, the desired formula follows from this observation. $\square$

Next, we present a new proof for the important result obtained by Luan et al. [7, Theorem 4.10].

Theorem 3.1 Let P be a polyhedral convex set and let Q be a generalized polyhedral convex set. Then

$$N(\bar{x}; P \cap Q) = N(\bar{x}; P) + N(\bar{x}; Q) \quad \text{for all } \bar{x} \in P \cap Q.$$

Proof Fix a point $\bar{x} \in P \cap Q$. Since Q is a generalized polyhedral convex set, we have the representation $Q = P_1 \cap M$, where P_1 is a polyhedral convex set and M is a closed affine subspace. Then by Lemmas 3.1 and 3.2 we have

$$\begin{aligned} N(\bar{x}; P \cap Q) &= N(\bar{x}; (P \cap P_1) \cap M) = N(\bar{x}; P \cap P_1) + N(\bar{x}; M) \\ &= N(\bar{x}; P) + N(\bar{x}; P_1) + N(\bar{x}; M) = N(\bar{x}; P) + N(\bar{x}; P_1 \cap M) \\ &= N(\bar{x}; P) + N(\bar{x}; Q). \end{aligned}$$

This completes the proof. $\square$

In what follows, let X and Y be locally convex Hausdorff topological vector spaces over the reals. For a multifunction $F: X \rightrightarrows Y$, one defines the *graph* and the *effective domain* of F , respectively, by

$$\text{gph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

$$\text{and } \text{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}.$$

Definition 3.1 Let $F: X \rightrightarrows Y$ be a multifunction.

- (a) F is said to be *convex* if $\text{gph}(F)$ is a convex set in $X \times Y$.
- (b) F is said to be *polyhedral convex* if $\text{gph}(F)$ is a polyhedral convex set in $X \times Y$.
- (c) F is said to be *generalized polyhedral convex* if $\text{gph}(F)$ is a generalized polyhedral convex set in $X \times Y$.

It follows from definitions that the following implications hold for a multifunction $F: X \rightrightarrows Y$:

$$\boxed{\text{polyhedral convexity}} \implies \boxed{\text{generalized polyhedral convexity}} \implies \boxed{\text{convexity}}$$

Thus, we can use tools of convex analysis to study generalized differential properties of generalized polyhedral convex multifunctions and polyhedral convex multifunctions.

Definition 3.2 Let $F: X \rightrightarrows Y$ be a convex multifunction and let $(\bar{x}, \bar{y}) \in \text{gph}(F)$. The multifunction $D^*F(\bar{x}, \bar{y}): Y^* \rightrightarrows X^*$ with the values

$$D^*F(\bar{x}, \bar{y})(v^*) := \{u^* \in X^* \mid (u^*, -v^*) \in N((\bar{x}, \bar{y}); \text{gph}(F))\}, \quad v^* \in Y^*, \quad (5)$$

is called the *coderivative* of F at $(\bar{x}, \bar{y})$.

Example 3.1 Given a subset Θ of X , define $\Delta_\Theta: X \rightrightarrows Y$ by

$$\Delta_\Theta(x) := \begin{cases} 0 & \text{if } x \in \Theta, \\ \emptyset & \text{if } x \notin \Theta. \end{cases}$$

Then $\text{gph}(\Delta_\Theta) = \Theta \times \{0\}$. Suppose that Θ is a convex set and $\bar{x} \in \Theta$. We have $N((\bar{x}, 0); \text{gph}(\Delta_\Theta)) = N(\bar{x}; \Theta) \times Y$ and hence

$$D^* \Delta_\Theta(\bar{x}, 0)(v^*) = N(\bar{x}; \Theta) \quad \text{for all } v^* \in Y^*. \quad (6)$$

For any convex multifunctions $F_1, F_2: X \rightrightarrows Y$, we can show that their sum, which is defined by setting $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ for all $x \in X$, is a convex multifunction with $\text{dom}(F_1 + F_2) = \text{dom}(F_1) \cap \text{dom}(F_2)$. Our first calculus result concerns representing the coderivative of $F_1 + F_2$ at a given point $(\bar{x}, \bar{y}) \in \text{gph}(F_1 + F_2)$. To formulate this result, consider the set

$$S(\bar{x}, \bar{y}) := \{(\bar{y}_1, \bar{y}_2) \in Y \times Y \mid \bar{y} = \bar{y}_1 + \bar{y}_2, \bar{y}_i \in F_i(\bar{x}), i = 1, 2\}. \quad (7)$$

The following theorem gives us the coderivative sum rule for polyhedral convex multifunctions and generalized polyhedral convex multifunctions.

Theorem 3.2 *Let $F_1: X \rightrightarrows Y$ be a polyhedral convex multifunction and let $F_2: X \rightrightarrows Y$ be a generalized polyhedral convex multifunction. Then the equality*

$$D^*(F_1 + F_2)(\bar{x}, \bar{y})(v^*) = D^*F_1(\bar{x}, \bar{y}_1)(v^*) + D^*F_2(\bar{x}, \bar{y}_2)(v^*)$$

holds for every $v^ \in Y^*$ whenever $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$, where S is defined in (7).*

Proof Let $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ and $v^* \in Y^*$ be given arbitrarily. Fix any

$$u^* \in D^*(F_1 + F_2)(\bar{x}, \bar{y})(v^*). \quad (8)$$

Then the inclusion $(u^*, -v^*) \in N((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2))$ is valid. Consider the convex sets

$$\begin{aligned} \Omega_1 &:= \{(x, y_1, y_2) \in X \times Y \times Y \mid y_1 \in F_1(x)\}, \\ \Omega_2 &:= \{(x, y_1, y_2) \in X \times Y \times Y \mid y_2 \in F_2(x)\} \end{aligned}$$

and deduce from the normal cone definition that

$$(u^*, -v^*, -v^*) \in N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2). \quad (9)$$

Observe that $\Omega_1 = (\text{gph}F_1) \times Y$ and thus it is a polyhedral convex set in $X \times Y \times Y$ by the assumption made on F_1 . Similarly, Ω_2 is a generalized polyhedral convex sets

in $X \times Y \times Y$ by the condition imposed on F_2 . Then we can employ Theorem 3.1 and get

$$(u^*, -v^*, -v^*) \in N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2) = N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1) + N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2).$$

Therefore, we obtain the relationships

$$(u^*, -v^*, -v^*) = (u_1^*, -v^*, 0) + (u_2^*, 0, -v^*),$$

where $(u_i^*, -v^*) \in N((\bar{x}, \bar{y}_i); \text{gph} F_i)$ for $i = 1, 2$. This implies by the coderivative definition that

$$u^* = u_1^* + u_2^* \in D^*F_1(\bar{x}, \bar{y}_1)(v^*) + D^*F_2(\bar{x}, \bar{y}_2)(v^*).$$

So, we have proved that

$$D^*(F_1 + F_2)(\bar{x}, \bar{y})(v^*) \subset D^*F_1(\bar{x}, \bar{y}_1)(v^*) + D^*F_2(\bar{x}, \bar{y}_2)(v^*).$$

To prove the reverse inclusion, take any $u_1^* \in D^*F_1(\bar{x}, \bar{y}_1)(v^*)$ and $u_2^* \in D^*F_2(\bar{x}, \bar{y}_2)(v^*)$. Then

$$\langle u_i^*, -v^* \rangle, (x - \bar{x}, y_i - \bar{y}_i) \rangle \leq 0$$

for every $x \in X$ and $y_i \in F_i(x)$ with $i = 1, 2$. It follows that

$$\langle u_1^*, -v^*, 0 \rangle, (x - \bar{x}, y_1 - \bar{y}_1, y_2 - \bar{y}_2) \rangle \leq 0$$

and

$$\langle u_1^*, 0, -v^* \rangle, (x - \bar{x}, y_1 - \bar{y}_1, y_2 - \bar{y}_2) \rangle \leq 0.$$

Adding these inequalities side-by-side yields

$$\langle u^*, -v^*, -v^* \rangle, (x - \bar{x}, y_1 - \bar{y}_1, y_2 - \bar{y}_2) \rangle \leq 0$$

for every $x \in X$ and $y_i \in F_i(x)$ with $i = 1, 2$, where $u^* := u_1^* + u_2^*$. Hence one gets the inclusion (9), which clearly implies that

$$(u^*, -v^*) \in N((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2)).$$

Hence (8) is valid, and thus we have verified the claimed sum rule. $\square$

Let $f: X \rightarrow \overline{\mathbb{R}} = (-\infty, \infty]$ be an extended-real-valued function. Recall that the *epigraph* and the *effective domain* of f are given, respectively, by

$$\text{epi}(f) := \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$$

and $\text{dom}(f) := \{x \in X \mid f(x) < \infty\}$.

Definition 3.3 Let $f: X \rightarrow \overline{\mathbb{R}}$ be an extended-real-valued function.

- (a) f is said to be *convex* if $\text{epi}(f)$ is a convex set in $X \times \mathbb{R}$.
- (b) f is said to be *polyhedral convex* if $\text{epi}(f)$ is a polyhedral convex set in $X \times \mathbb{R}$.
- (c) f is said to be *generalized polyhedral convex* if $\text{epi}(f)$ is a generalized polyhedral convex set in $X \times \mathbb{R}$.

If f is convex, then the *subdifferential* $\partial f(\bar{x})$ of f at $\bar{x} \in \text{dom}(f)$ is defined by setting

$$\begin{aligned}\partial f(\bar{x}) &= \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \text{ for all } x \in X\} \\ &= \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})); \text{epi}(f))\}.\end{aligned}$$

Theorem 3.2 allows us to obtain the next subdifferential sum rule for polyhedral convex functions and generalized polyhedral convex functions.

Corollary 3.1 Let $f_1, f_2: X \rightarrow \overline{\mathbb{R}}$ be two extended-real-valued functions. Suppose that f_1 is a polyhedral convex function and f_2 is a generalized polyhedral convex function. Then

$$\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}) \text{ for every } \bar{x} \in \text{dom}(f_1) \cap \text{dom}(f_2).$$

Proof Fix any $\bar{x} \in \text{dom}(f_1) \cap \text{dom}(f_2)$. Let $F_i(x) := [f_i(x), \infty)$ for all $x \in X$ and get $\text{gph}(F_i) = \text{epi}(f_i)$ for $i = 1, 2$. Thus, F_1 is a polyhedral convex multifunction and F_2 is a generalized polyhedral convex multifunction. In addition,

$$D^*F_i(\bar{x}, f_i(\bar{x}))(1) = \partial f_i(\bar{x}) \text{ for } i = 1, 2.$$

Let $\bar{y} := f_1(\bar{x}) + f_2(\bar{x})$. Then $S(\bar{x}, \bar{y}) = \{(f_1(\bar{x}), f_2(\bar{x}))\}$, where $S(\bar{x}, \bar{y})$ is defined in (7). Applying Theorem 3.2 gives

$$\begin{aligned}\partial(f_1 + f_2)(\bar{x}) &= D^*(F_1 + F_2)(\bar{x}, \bar{y})(1) \\ &= D^*F_1(\bar{x}, f_1(\bar{x}))(1) + D^*F_2(\bar{x}, f_2(\bar{x}))(1) \\ &= \partial f_1(\bar{x}) + \partial f_2(\bar{x}).\end{aligned}$$

This completes the proof. $\square$

Consider the composition of two mappings $F: X \rightrightarrows Y$ and $G: Y \rightrightarrows Z$. It follows from the definition that $G \circ F$ is convex provided that both F and G have this property. Given $\bar{z} \in (G \circ F)(\bar{x})$, we consider the set

$$M(\bar{x}, \bar{z}) := F(\bar{x}) \cap G^{-1}(\bar{z}). \quad (10)$$

The following theorem establishes the coderivative chain rule for multifunctions.

Theorem 3.3 Let $F: X \rightrightarrows Y$ and $G: Y \rightrightarrows Z$ be multifunctions. Suppose that F is a polyhedral multifunction and G is a generalized polyhedral multifunction or vice versa. Then for any $(\bar{x}, \bar{z}) \in \text{gph}(G \circ F)$ and $w^* \in Z^*$ we have the coderivative chain rule

$$D^*(G \circ F)(\bar{x}, \bar{z})(w^*) = D^*F(\bar{x}, \bar{y}) \circ D^*G(\bar{y}, \bar{z})(w^*) \quad (11)$$

whenever $\bar{y} \in M(\bar{x}, \bar{z})$.

Proof Picking $u^* \in D^*(G \circ F)(\bar{x}, \bar{z})(w^*)$ and $\bar{y} \in M(\bar{x}, \bar{z})$ gives us the inclusion $(u^*, -w^*) \in N((\bar{x}, \bar{z}); \text{gph}(G \circ F))$, which means that

$$\langle u^*, x - \bar{x} \rangle - \langle w^*, z - \bar{z} \rangle \leq 0 \quad \text{for all } (x, z) \in \text{gph}(G \circ F). \quad (12)$$

Define two convex subsets of $X \times Y \times Z$ by

$$\Omega_1 := (\text{gph}(F)) \times Z \quad \text{and} \quad \Omega_2 := X \times (\text{gph}G).$$

We can directly deduce from (12) and the definitions that

$$(u^*, 0, -w^*) \in N((\bar{x}, \bar{y}, \bar{z}); \Omega_1 \cap \Omega_2). \quad (13)$$

Applying Theorem 3.1 together with the conditions made on F_1 and F_2 tells us that

$$(u^*, 0, -w^*) \in N((\bar{x}, \bar{y}, \bar{z}); \Omega_1 \cap \Omega_2) = N((\bar{x}, \bar{y}, \bar{z}); \Omega_1) + N((\bar{x}, \bar{y}, \bar{z}); \Omega_2) \quad (14)$$

and thus there exists a vector $v^* \in Y^*$ such that we have the representation

$$(u^*, 0, -w^*) = (u^*, -v^*, 0) + (0, v^*, -w^*) \quad (15)$$

with $(u^*, -v^*) \in N((\bar{x}, \bar{y}); \text{gph}(F))$ and $(v^*, -w^*) \in N((\bar{y}, \bar{z}); \text{gph}G)$. This shows by the coderivative definition in (5) that

$$u^* \in D^*F(\bar{x}, \bar{y})(v^*) \quad \text{and} \quad v^* \in D^*G(\bar{y}, \bar{z})(w^*), \quad (16)$$

and so we get the inclusion “ $\subset$ ” in (11). The reverse inclusion can be proved as follows. Given any $u^* \in D^*F(\bar{x}, \bar{y}) \circ D^*G(\bar{y}, \bar{z})(w^*)$, one can find some $v^* \in Y^*$ such that (16) holds. Then (15) is fulfilled and, moreover, one has $(u^*, -v^*, 0) \in N((\bar{x}, \bar{y}, \bar{z}); \Omega_1)$ and $(0, v^*, -w^*) \in N((\bar{x}, \bar{y}, \bar{z}); \Omega_2)$. Since the inclusion

$$N((\bar{x}, \bar{y}, \bar{z}); \Omega_1) + N((\bar{x}, \bar{y}, \bar{z}); \Omega_2) \subset N((\bar{x}, \bar{y}, \bar{z}); \Omega_1 \cap \Omega_2)$$

is valid whenever F and G are merely convex multifunctions, one gets (13), which implies (12). Hence, $u^* \in D^*(G \circ F)(\bar{x}, \bar{z})(w^*)$. The proof is complete. $\square$

The next rule for computing subdifferentials of the composition of a polyhedral convex function and an affine mapping is a corollary of the preceding theorem.

Corollary 3.2 *Let $B: X \rightarrow Y$ be an affine mapping given by*

$$B(x) := A(x) + b \quad \text{for } x \in X,$$

where $A: X \rightarrow Y$ is a continuous linear mapping and $b \in Y$. If $f: Y \rightarrow \overline{\mathbb{R}}$ is a polyhedral convex function, then

$$\partial(f \circ B)(\bar{x}) = A^*(\partial f(\bar{y})) \quad \text{for all } \bar{x} \in \text{dom}(f \circ B), \quad (17)$$

where $\bar{y} := B(\bar{x})$.

Proof Let $F(x) := \{B(x)\}$ for $x \in X$ and let $G(y) := [f(y), \infty)$ for $y \in Y$. Then

$$(G \circ F)(x) = [(f \circ B)(x), \infty) \quad \text{for all } x \in X. \quad (18)$$

By our assumptions, $F: X \rightrightarrows Y$ is a generalized polyhedral multifunction and $G: Y \rightrightarrows \mathbb{R}$ is a polyhedral multifunction. Applying the coderivative chain rule from Theorem 3.3, we can get the desired result. Indeed, take any $\bar{x} \in \text{dom}(f \circ B)$ and put $\bar{y} = B(\bar{x})$, $\bar{z} = f(\bar{y})$. Then $(\bar{x}, \bar{z}) \in \text{gph}(G \circ F)$ and $\bar{y} \in M(\bar{x}, \bar{z}) = F(\bar{x}) \cap G^{-1}(\bar{z})$. Therefore, by (11) we have

$$\begin{aligned} D^*(G \circ F)(\bar{x}, \bar{z})(-1) &= D^*F(\bar{x}, \bar{y}) \circ D^*G(\bar{y}, \bar{z})(-1) \\ &= D^*F(\bar{x}, \bar{y})(\partial f(\bar{y})) \\ &= A^*(\partial f(\bar{y})), \end{aligned}$$

which together with (18) implies the equality in (17). $\square$

Let $F: X \rightrightarrows Y$ be a multifunction and let $\Theta \subset Y$ be a given set. The *preimage* or *inverse image* of Θ under the mapping F is defined by

$$F^{-1}(\Theta) = \{x \in X \mid F(x) \cap \Theta \neq \emptyset\}.$$

The theorem below gives us a formula to compute the normal cone to $F^{-1}(\Theta)$ at a point of interest via the normal cone to Θ and the coderivative of F at certain points.

Theorem 3.4 *Let $F: X \rightrightarrows Y$ be a multifunction and let $\Theta \subset Y$. Suppose that F is a polyhedral convex multifunction and Θ is a generalized polyhedral convex set, or F is a generalized polyhedral convex multifunction and Θ is a polyhedral convex set. Then for any $\bar{x} \in F^{-1}(\Theta)$ and $\bar{y} \in F(\bar{x}) \cap \Theta$ we have the representation*

$$N(\bar{x}; F^{-1}(\Theta)) = D^*F(\bar{x}, \bar{y})(N(\bar{y}; \Theta)). \quad (19)$$

Proof Similarly as in Example 3.1, let us consider the indicator mappings $\Delta_\Theta: Y \rightrightarrows Y$ and $\Delta_{F^{-1}(\Theta)}: X \rightrightarrows Y$. Note that

$$\Delta_{F^{-1}(\Theta)}(x) = (\Delta_\Theta \circ F)(x) \quad \text{for all } x \in X. \quad (20)$$

Then the representation (19) can be obtained by using Example 3.1 and Theorem 3.3 with $G := \Delta_\Theta$. Indeed, given any $\bar{x} \in F^{-1}(\Theta)$ and $\bar{y} \in F(\bar{x}) \cap \Theta$, we set $\bar{z} = 0 \in Y$. It can be easily verified that $(\bar{x}, \bar{z}) \in \text{gph}(G \circ F)$ and $\bar{y} \in M(\bar{x}, \bar{z})$, where the last set is defined by (10). By our assumptions, F is a polyhedral convex multifunction and G is a generalized polyhedral convex multifunction or vice versa. So, fixing any $w^* \in Y^*$, by Theorem 3.3 and formula (6) we can infer that

$$\begin{aligned} D^*(G \circ F)(\bar{x}, \bar{z})(w^*) &= D^*F(\bar{x}, \bar{y}) \circ D^*G(\bar{y}, \bar{z})(w^*) \\ &= D^*F(\bar{x}, \bar{y})(N(\bar{y}; \Theta)). \end{aligned}$$

Since the relation $D^*(G \circ F)(\bar{x}, \bar{z})(w^*) = D^*(\Delta_{F^{-1}(\Theta)})(\bar{x}, \bar{z})(w^*)$ is valid by (20), this together with (6) establishes (19). $\square$

Next, consider a function $f: X \rightarrow \overline{\mathbb{R}}$ and define the *sublevel sets*

$$\mathcal{L}_\gamma := \{x \in X \mid f(x) \leq \gamma\}, \quad \gamma \in \mathbb{R}.$$

Our goal is to establish a formula for the normal cone to the sublevel sets associated with a generalized polyhedral convex function. To continue, for $\bar{x} \in \text{dom}(f)$ we use the following notation

$$\lambda \odot \partial f(\bar{x}) := \begin{cases} \lambda \partial f(\bar{x}) & \text{if } \lambda > 0, \\ \partial^\infty f(\bar{x}) & \text{if } \lambda = 0. \end{cases}$$

Here $\partial^\infty f(\bar{x})$ denotes the *singular subdifferential* of f at $\bar{x}$ defined by

$$\partial^\infty f(\bar{x}) = \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, f(\bar{x})); \text{epi}(f))\}.$$

It follows from the definition that for the epigraphical mapping $E_f: X \rightarrow \mathbb{R}$ given by $E_f(x) := [f(x), \infty)$ for all $x \in X$ we have

$$D^*E_f(\bar{x}, f(\bar{x}))(\lambda) = \lambda \odot \partial f(\bar{x}) \quad \text{for all } \bar{x} \in \text{dom}(f), \lambda \geq 0. \quad (21)$$

Corollary 3.3 *Let $f: X \rightarrow \overline{\mathbb{R}}$ be a generalized polyhedral convex function with $\bar{x} \in \text{dom}(f)$ and $f(\bar{x}) = \gamma$. Then*

$$N(\bar{x}; \mathcal{L}_\gamma) = \bigcup_{\lambda \geq 0} \lambda \odot \partial f(\bar{x}).$$

Proof Let $\Theta := (-\infty, \gamma]$ and $F(x) := E_f(x)$ for $x \in X$. Then one has $\mathcal{L}_\gamma = F^{-1}(\Theta)$. Since $\text{gph}(F) = \text{epi}(f)$, we see that F is a generalized convex polyhedral multifunction. In addition, Θ is a polyhedral convex set. Observe also that

$$N(f(\bar{x}); \Theta) = N(\gamma; (-\infty, \gamma]) = [0, \infty).$$

Therefore, by Theorem 3.4 and (21) we have

$$\begin{aligned} N(\bar{x}; \mathcal{L}_\gamma) &= N(\bar{x}; F^{-1}(\Theta)) = D^*F(\bar{x}, f(\bar{x}))(N(\bar{y}; \Theta)) \\ &= \bigcup_{\lambda \geq 0} D^*F(\bar{x}, f(\bar{x}))(\lambda) \\ &= \bigcup_{\lambda \geq 0} \lambda \odot \partial f(\bar{x}), \end{aligned}$$

which completes the proof of the corollary. $\square$

In a more general setting, consider m functions $f_i: X \rightarrow \overline{\mathbb{R}}$ together with $\gamma_i \in \mathbb{R}$ for $i = 1, \dots, m$. Let $\gamma := (\gamma_1, \dots, \gamma_m)$ and define

$$\mathcal{L}_\gamma := \{x \in X \mid f_i(x) \leq \gamma_i \text{ for all } i = 1, \dots, m\}.$$

Let $I := \{1, \dots, m\}$. Given $\bar{x} \in \bigcap_{i \in I} \text{dom}(f_i)$, define

$$I(\bar{x}) := \{i \in I \mid f_i(\bar{x}) = \gamma_i\}.$$

The next theorem extends the result in Corollary 3.3 to the case where the functions involved are continuous.

Theorem 3.5 *Let $f_i: X \rightarrow \overline{\mathbb{R}}$ for $i = 1, \dots, m$, and let $\gamma := (\gamma_1, \dots, \gamma_m) \in \mathbb{R}^m$, $m \geq 2$. Suppose that among f_i , $i = 1, \dots, m$, there are at least $m - 1$ polyhedral convex functions and the remaining one is generalized polyhedral convex. Then for any $\bar{x} \in \mathcal{L}_\gamma$ we have*

$$\begin{aligned} N(\bar{x}; \mathcal{L}_\gamma) &= \left\{ \sum_{i \in I(\bar{x})} \lambda_i \odot \partial f(\bar{x}) \mid \lambda_i \geq 0 \text{ for all } i \in I(\bar{x}) \right\} \\ &\quad + \sum_{i \notin I(\bar{x})} N(\bar{x}; \text{dom}(f_i)). \end{aligned} \quad (22)$$

Proof Consider the multifunction $F: X \rightrightarrows \mathbb{R}^m$ given by

$$F(x) := [f_1(x), \infty) \times \cdots \times [f_m(x), \infty), \quad x \in X.$$

It can be shown that F is a generalized polyhedral convex multifunction. Consider the set $\Theta := (-\infty, \gamma_1] \times \cdots \times (-\infty, \gamma_m]$ and observe that Θ is a polyhedral convex set in $\mathbb{R}^m$. Clearly, $\mathcal{L}_\gamma = F^{-1}(\Theta)$.

Define the following subsets of $X \times \mathbb{R}^m$:

$$\begin{aligned} \Omega_1 &:= \{(x, \lambda_1, \dots, \lambda_m) \mid \lambda_1 \geq f_1(x)\} = \text{epi}(f_1) \times \mathbb{R}^{m-1}, \\ \Omega_2 &:= \{(x, \lambda_1, \dots, \lambda_m) \mid \lambda_2 \geq f_2(x)\}, \\ &\dots \\ \Omega_m &:= \{(x, \lambda_1, \dots, \lambda_m) \mid \lambda_m \geq f_m(x)\}. \end{aligned}$$

By our assumptions, among these sets there are at least $m - 1$ polyhedral convex sets and the remaining one is a generalized polyhedral convex. Note also that $\text{gph}(F) = \bigcap_{i \in I} \Omega_i$.

By induction, from the last equality and Theorem 3.1 we have

$$N\left((\bar{x}, f_1(\bar{x}), \dots, f_m(\bar{x})); \text{gph}(F)\right) = \sum_{i \in I} N\left((\bar{x}, f_1(\bar{x}), \dots, f_m(\bar{x})); \Omega_i\right).$$

Thus, by the special construction of Ω_i , $i \in I$, one has

$$(x^*, -\lambda_1, \dots, -\lambda_m) \in N((\bar{x}, f_1(\bar{x}), \dots, f_m(\bar{x})); \text{gph}(F))$$

if and only if there exists x_i^* for $i = 1, \dots, m$ such that

$$(x_i^*, -\lambda_i) \in N((\bar{x}, f_i(\bar{x})); \text{epi}(f_i))$$

for each $i \in I$ and $x^* = \sum_{i \in I} x_i^*$. It follows that

$$D^*F(\bar{x}, f_1(\bar{x}), \dots, f_m(\bar{x}))(\lambda_1, \dots, \lambda_m) = \sum_{i \in I} \lambda_i \odot \partial f_i(\bar{x}), \quad (23)$$

provided that $\lambda_i \geq 0$ for all $i \in I$. We also see that

$$N((f_1(\bar{x}), \dots, f_m(\bar{x})); \Theta) = \{(\lambda_1, \dots, \lambda_m) \mid \lambda_i \geq 0 \forall i \in I, \lambda_i = 0 \text{ if } i \notin I(\bar{x})\}.$$

By Theorem 3.4 we have

$$\begin{aligned} N(\bar{x}; \mathcal{L}_\gamma) &= N(\bar{x}; F^{-1}(\Theta)) \\ &= D^*F(\bar{x}, f_1(\bar{x}), \dots, f_m(\bar{x}))(N((f_1(\bar{x}), \dots, f_m(\bar{x})); \Theta)). \end{aligned}$$

Therefore, taking into account that $\partial^\infty f_i(\bar{x}) = N(\bar{x}; \text{dom}(f_i))$ for every $i \in I$, we can obtain (22) from (23). $\square$

The following corollary provides a simplified version of (22) in the case where f_i is continuous at $\bar{x}$ for every $i \in I$.

Corollary 3.4 *Under the assumptions of Theorem 3.5, assume in addition that all the functions f_i for $i \in I$ are continuous at $\bar{x} \in \mathcal{L}_\gamma$. Then we have*

$$N(\bar{x}; \mathcal{L}_\gamma) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i \partial f_i(\bar{x}) \mid \lambda_i \geq 0 \text{ for all } i \in I(\bar{x}) \right\}. \quad (24)$$

Proof For each $i \in I$, since f_i is continuous at $\bar{x}$, we have $\bar{x} \in \text{int}(\text{dom}(f_i))$. Thus,

$$\partial^\infty f_i(\bar{x}) = N(\bar{x}; \text{dom}(f_i)) = \{0\}.$$

Moreover, as $\partial f_i(\bar{x}) \neq \emptyset$ by the continuity of f_i , we see that

$$\lambda_i \odot \partial f_i(\bar{x}) = \lambda_i \partial f_i(\bar{x})$$

whenever $\lambda_i \geq 0$. Therefore, the equality (24) follows directly from (22). $\square$

4 Generalized Differentiation for Optimal Value Functions

In this section we consider the *optimal value/marginal* function given by

$$\mu(x) := \inf \{ \varphi(x, y) \mid y \in F(x) \} \quad (25)$$

for all $x \in X$, where $F: X \rightrightarrows Y$ is a multifunction and $\varphi: X \times Y \rightarrow \overline{\mathbb{R}}$ is an extended-real-valued function. For simplicity of the presentation, we assume that $\mu(x) > -\infty$ for all $x \in X$.

Theorem 4.1 *Let μ be an optimal value function of the form (25). Suppose that φ is a polyhedral convex function and F is a generalized polyhedral convex multifunction, or φ is a generalized polyhedral convex function and F is a polyhedral convex multifunction. For any $\bar{x} \in \text{dom}(\mu)$, consider the solution set $S(\bar{x}) := \{ \bar{y} \in F(\bar{x}) \mid \mu(\bar{x}) = \varphi(\bar{x}, \bar{y}) \}$. If $S(\bar{x})$ is nonempty, then for any $\bar{y} \in S(\bar{x})$ we have*

$$\partial \mu(\bar{x}) = \bigcup_{(u,v) \in \partial \varphi(\bar{x}, \bar{y})} [u + D^* F(\bar{x}, \bar{y})(v)]. \quad (26)$$

Proof The inclusion “ $\supset$ ” in (26) holds when φ is merely a convex function and F is merely a convex multifunction. Indeed, given w^* from the right-hand side of (26), we can find $(u, v) \in \partial \varphi(\bar{x}, \bar{y})$ such that $w^* - u \in D^* F(\bar{x}, \bar{y})(v)$. Hence, $(w^* - u, -v) \in N((\bar{x}, \bar{y}); \text{gph}(F))$. It follows that

$$\langle (w^* - u, -v), (x - \bar{x}, y - \bar{y}) \rangle \leq \forall (x, y) \in \text{gph}(F).$$

Then we have

$$\langle w^*, x - \bar{x} \rangle \leq \langle (u, v), (x, y) - (\bar{x}, \bar{y}) \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y})$$

for any $x \in X$ and $y \in F(x)$. Taking the infimum of the right-hand side of the inequality $\langle w^*, x - \bar{x} \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y})$ and using the condition $\bar{y} \in S(\bar{x})$ yield

$$\langle w^*, x - \bar{x} \rangle \leq \mu(x) - \mu(\bar{x}) \quad \forall x \in X.$$

This means that $w^* \in \partial \mu(\bar{x})$.

Let us verify the inclusion “ $\subset$ ” in (26) under the assumptions that φ is a polyhedral convex function and F is a generalized polyhedral convex multifunction. Pick an element $\bar{y} \in S(\bar{x})$ and let $w^* \in \partial \mu(\bar{x})$ be given arbitrarily. For any $x \in X$, we have

$$\begin{aligned}\langle w^*, x - \bar{x} \rangle &\leq \mu(x) - \mu(\bar{x}) = \mu(x) - \varphi(\bar{x}, \bar{y}) \\ &\leq \varphi(x, y) - \varphi(\bar{x}, \bar{y})\end{aligned}$$

for all $y \in F(x)$. This implies that, whenever $(x, y) \in X \times Y$, the next inequality holds:

$$\langle w^*, x - \bar{x} \rangle + \langle 0, y - \bar{y} \rangle \leq [\varphi(x, y) + \delta((x, y); \text{gph}(F))] - [\varphi(\bar{x}, \bar{y}) + \delta((\bar{x}, \bar{y}); \text{gph}(F))].$$

Hence, considering the function $f(x, y) := \varphi(x, y) + \delta((x, y); \text{gph}(F))$ for $(x, y) \in X \times Y$, we have $(w^*, 0) \in \partial f(\bar{x}, \bar{y})$. Letting $h(x, y) := \delta((x, y); \text{gph}(F))$ for $(x, y) \in X \times Y$, we see that

$$\text{epi}(h) = \text{gph}(F) \times [0, \infty)$$

is a generalized polyhedral convex set. Since φ is a polyhedral convex function, by the subdifferential sum rule in Theorem 3.1 one has

$$(w^*, 0) \in \partial f(\bar{x}, \bar{y}) = \partial \varphi(\bar{x}, \bar{y}) + \partial h(\bar{x}, \bar{y}) = \partial \varphi(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \text{gph}(F)).$$

This shows that $(w^*, 0) = (u_1^*, v_1^*) + (u_2^*, v_2^*)$ for some $(u_1^*, v_1^*) \in \partial \varphi(\bar{x}, \bar{y})$ and

$$(u_2^*, v_2^*) \in N((\bar{x}, \bar{y}); \text{gph}(F)).$$

It follows that $v_2^* = -v_1^*$; hence $(u_2^*, -v_1^*) \in N((\bar{x}, \bar{y}); \text{gph}(F))$. Thus, we get $u_2^* \in D^*F(\bar{x}, \bar{y})(v_1^*)$ and therefore

$$w^* = u_1^* + u_2^* \in u_1^* + D^*F(\bar{x}, \bar{y})(v_1^*).$$

So, the inclusion “ $\subset$ ” in (26) is valid.

The verification of the inclusion “ $\supset$ ” in (26) under the assumptions that φ is a generalized polyhedral convex function and F is a polyhedral convex multifunction can be done in the same way. Namely, in the above notations, it suffices to note that $\text{epi}(h)$ is a polyhedral convex set. As φ is a generalized polyhedral convex function, we can apply the subdifferential sum rule in Theorem 3.1 to get the desired result.

The proof of the theorem is completed. $\square$

From Theorem 4.1 we get the following useful *chain rule* for convex compositions.

Corollary 4.1 *Let $f: X \rightarrow \mathbb{R}$ be a real-valued convex function and $\phi: \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be a nondecreasing convex function. Take $\bar{x} \in X$ and let $\bar{y} = f(\bar{x}) \in \text{dom}(\phi)$. If f is a generalized polyhedral convex function and ϕ is a polyhedral convex function or vice versa, then*

$$\partial(\phi \circ f)(\bar{x}) = \bigcup_{\lambda \in \partial \phi(\bar{y})} \lambda \odot \partial f(\bar{x}). \quad (27)$$

If we assume in addition that f is continuous at $\bar{x}$, then

$$\partial(\phi \circ f)(\bar{x}) = \bigcup_{\lambda \in \partial\phi(\bar{y})} \lambda \partial f(\bar{x}). \quad (28)$$

Proof Clearly, the composition $\phi \circ f$ is a convex function. Let $\varphi(x, y) := \phi(y)$ for $(x, y) \in X \times \mathbb{R}$ and $F(x) := [f(x), \infty)$ for $x \in X$. Since φ is nondecreasing, one has

$$(\phi \circ f)(x) = \inf_{y \in F(x)} \phi(y) = \inf_{y \in F(x)} \varphi(x, y).$$

Hence we can let $\phi \circ f$ play the role of the optimal value function μ in Theorem 4.1. As f is a generalized polyhedral convex function and ϕ is a polyhedral convex function or vice versa, by Theorem 4.1 we have

$$\partial(\phi \circ f)(\bar{x}) = \bigcup_{\lambda \in \partial\phi(\bar{y})} D^*F(\bar{x}, \bar{y})(\lambda).$$

Taking again into account that ϕ is nondecreasing yields $\lambda \geq 0$ for every $\lambda \in \partial\phi(\bar{y})$. It follows that

$$\partial(\phi \circ f)(\bar{x}) = \bigcup_{\lambda \in \partial\phi(\bar{y})} D^*F(\bar{x}, \bar{y})(\lambda) = \bigcup_{\lambda \in \partial\phi(\bar{y})} \lambda \odot \partial f(\bar{x}),$$

which implies (27).

The simplified version (28) under the continuity of f at $\bar{x}$ follows from the observation in the proof of Corollary 3.4. $\square$

5 Concluding Remarks

In the first main part of the paper, we provided a full answer to an open question raised by Mordukhovich and Nam in [10, Exercise 3.118]. Our answer clarifies that the necessary and sufficient conditions for the separation of a convex set and a polyhedral convex set obtained by Ng and Song in [12, Theorem 3.1] are no longer valid when considering generalized polyhedral convex sets instead of polyhedral convex sets.

In the second main part of the paper, we developed new calculus rules of generalized differentiation for polyhedral convex and generalized polyhedral multifunctions with specifications for nonsmooth functions. In this setting, the qualification conditions via the notion of *quasi-relative interior* introduced by Borwein and Lewis [3] are no longer needed as shown by the related results in [10] for operations on convex multifunction and polyhedral convex multifunctions.

In our future work, we plan to explore the role of relative interior and generalized relative interiors (see [2, 5, 6, 11, 19]) including the quasi-relative interior in developing convex generalized differentiation which aims to unify convex analysis in finite and infinite dimensions.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.

References

1. Bonnans, J.F.: Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, New York (2000)
2. Borwein, J.M., Goebel, R.: Notions of relative interior in Banach spaces. *J. Math. Sci.* **115**, 2542–2553 (2003)
3. Borwein, J.M., Lewis, A.S.: Partially finite convex programming, Part I: quasi-relative interiors and duality theory. *Math. Program.* **57**, 15–48 (1992)
4. Cuong, D.V., Mordukhovich, B.S., Nam, N.M., Sandine, G.: Generalized differentiation and duality in infinite dimensions under polyhedral convexity. *Set-Valued Var. Anal.* **30**, 1503–1526 (2022)
5. Hiriart-Urruty, J.B., Lemaréchal, C.: Convex Analysis and Minimization Algorithms, I: Fundamentals. Springer, Berlin (1993)
6. Khan, A.A., Tammer, C., Zălinescu, C.: Set-Valued Optimization. Springer, New York (2015)
7. Luan, N.N., Yao, J.-C., Yen, N.D.: On some generalized polyhedral convex constructions. *Numer. Funct. Anal. Optim.* **39**, 537–570 (2018)
8. Luan, N.N., Yen, N.D.: A representation of generalized convex polyhedra and applications. *Optimization* **69**, 471–492 (2020)
9. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications. Springer, Berlin (2006)
10. Mordukhovich, B.S., Nam, N.M.: Convex Analysis and Beyond, Vol. I: Basic Theory. Springer, Cham, Switzerland (2022)
11. Mordukhovich, B.S., Nam, N.M.: An Easy Path to Convex Analysis and Applications, 2nd edn. Springer, Cham (2023)
12. Ng, K.F., Song, W.: Fenchel duality in finite-dimensional setting and its applications. *Nonlinear Anal.* **55**, 845–858 (2003)
13. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. *Math. Program. Study* **14**, 206–214 (1981)
14. Robinson, S.M.: Solution continuity in monotone affine variational inequalities. *SIAM J. Optim.* **18**, 1046–1060 (2007)
15. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)
16. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Berlin (1998)
17. Rockafellar, R.T.: Convex algebra and duality in dynamic models of production. In: Łoś, J., Łoś, M.W. (eds.) Mathematical Models of Economics, pp. 351–378. North-Holland, Amsterdam (1974)
18. Rudin, W.: Functional Analysis, 2nd edn. McGraw-Hill, New York (1991)
19. Zălinescu, C.: On the use of the quasi-relative interior in optimization. *Optimization* **64**, 1795–1823 (2015)
20. Zheng, X.Y.: Pareto solutions of polyhedral-valued vector optimization problems in Banach spaces. *Set-Valued Anal.* **17**, 389–408 (2009)

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