



Which Schubert Varieties are Hessenberg Varieties?

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Abstract

After proving that every Schubert variety in the full flag variety of a complex reductive group G is a general Hessenberg variety, we show that not all such Schubert varieties are adjoint Hessenberg varieties. In fact, in types A and C, we provide pattern avoidance criteria implying that the proportion of Schubert varieties that are adjoint Hessenberg varieties approaches zero as the rank of G increases. We show also that in type A, some Schubert varieties are not isomorphic to any adjoint Hessenberg variety.

1 Introduction

Hessenberg varieties have been studied by applied mathematicians, combinatorialists, geometers, representation theorists, and topologists. See [2] for a survey of some recent developments. Our goal is to understand better the structure of these varieties, and in particular what restrictions on such structure exist. To this end, we address herein a question raised by Tymoczko:

Is every Schubert variety in a full flag variety a Hessenberg variety?

Schubert varieties comprise a well-studied collection of subvarieties of the flag variety and naturally embody the interplay between combinatorics and geometry. For example, their singularities can be characterized in terms of pattern avoidance (see, e.g., [3]). Because the flag variety has an underlying combinatorial structure, one can ask how other subvarieties fit into this framework. As we will discuss below, conditions known to be satisfied by the most closely studied Hessenberg varieties are also satisfied by

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Schubert varieties. So, Tymoczko's question is pertinent. There are several possible interpretations of the question. Moreover, there are several ways to define a Hessenberg variety, giving rise to progressively more general classes of varieties. The answer to Tymoczko's question depends on the chosen definition, as we shall see.

Let G be a connected, reductive complex algebraic group. Fix a Borel subgroup $B \leq G$ and a maximal torus $T \leq B$. Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$ be, respectively, the Lie algebras of G , B , and T . We write Ad for the adjoint representation of G on $\mathfrak{g}$.

Let $N = N_G(T)$ be the normalizer of T in G and let $W = N/T$ be the associated Weyl group. The flag variety $\mathcal{B} := G/B$ is the union of Schubert cells C_w , over all $w \in W$. For each w , the Schubert variety X_w is the closure of C_w in $\mathcal{B}$.

Our first definition of a Hessenberg variety is as follows.

Definition 1.1 Given $x \in \mathfrak{g}$ and a subspace H of $\mathfrak{g}$ such that $[\mathfrak{b}, H] \subseteq H$, the **adjoint Hessenberg variety** $\mathcal{B}(x, H)$ consists of those $gB \in \mathcal{B}$ such that $Ad(g^{-1})(x) \in H$.

Adjoint Hessenberg varieties were defined and studied by De Mari et al. in [10], after being defined and studied for $G = GL_n(\mathbb{C})$ only by De Mari and Shayman in [11]. In both [11] and [10], it is assumed that $\mathfrak{b} \subseteq H$ and that x is a generic (regular semisimple) element of $\mathfrak{g}$. We make neither assumption here, as otherwise the topology of $\mathcal{B}(x, H)$ is restricted considerably. Indeed, it follows from the results in [10, 11] that if $\mathfrak{b} \subseteq H$ and x is regular semisimple, then the Euler characteristic $\chi(\mathcal{B}(x, H))$ is equal to $|W|$. In particular, the only Schubert variety in $\mathcal{B}$ that is a Hessenberg variety under these assumptions is $\mathcal{B}$ itself.

A larger class of varieties than that given in Definition 1.1 is defined in [15] by Goresky, Kottwitz and MacPherson, who allow an arbitrary representation of G , rather than restricting to the adjoint representation.

Definition 1.2 Let $\psi : G \rightarrow GL(V)$ be a (finite-dimensional, rational) representation. Given $x \in V$ and a B -invariant subspace H of V , the **Hessenberg variety** $\mathcal{B}(x, H)$ consists of those $gB \in \mathcal{B}$ such that $\psi(g^{-1})x \in H$.

We will always use the modifier “adjoint” when referring to the Hessenberg varieties described in Definition 1.1 and sometimes use the modifier “general” when discussing the Hessenberg varieties described in Definition 1.2. We will use repeatedly, and without reference, the fact that if H is a subspace of $\mathfrak{g}$, then $[\mathfrak{b}, H] \subseteq H$ if and only if H is $Ad(B)$ -invariant (see for example [19, Definition 8.1.22]). So, every adjoint Hessenberg variety is a general Hessenberg variety. The subspace H appearing in either definition is called a **Hessenberg space**.

With definitions in hand, we turn to possible interpretations of Tymoczko's question, and list three precise questions.

Question 1.3 (the equality problem). *Is it true that for every G , B , and Schubert variety $X_w \subseteq \mathcal{B} = G/B$, there is a Hessenberg variety $\mathcal{B}(x, H) \subseteq \mathcal{B}$ such that $X_w = \mathcal{B}(x, H)$?*

Question 1.4 (the isomorphism problem). *Is it true that for every G , B and Schubert variety $X_w \subseteq \mathcal{B} = G/B$, there is a Hessenberg variety $\mathcal{B}(x, H) \subseteq \mathcal{B}$ such that X_w is isomorphic with $\mathcal{B}(x, H)$?*

Question 1.5 (the general isomorphism problem). *Is it true that for every G, B and Schubert variety $X_w \subseteq G/B$, there exist a complex reductive group G^* with Borel subgroup B^* and a Hessenberg variety $\mathcal{B}^*(x^*, H^*) \subseteq G^*/B^*$ such that X_w is isomorphic with $\mathcal{B}^*(x^*, H^*)$?*

The answer to Question 1.3 (and therefore to Questions 1.4 and 1.5) is “Yes” for general Hessenberg varieties, as our first main result shows. (Relevant terminology will be discussed in Sects. 2 and 3.)

Theorem 1.6 (See Theorem 3.6 below). *Let λ be a strictly dominant weight for G , with associated highest weight representation $\psi : G \rightarrow GL(V(\lambda))$ and highest weight vector v_λ . Given $w \in W$, let $\dot{w}$ be a representative of w in N , and let $H_{w^{-1}(\lambda)}$ be the Demazure module generated by $\dot{w}^{-1}(v_\lambda)$. Then, $X_w = \mathcal{B}(v_\lambda, H_{w^{-1}(\lambda)})$.*

Theorem 1.6 follows directly from a result of Bernšteĭn, Gel’fand, and Gel’fand in [6].

The situation is more interesting when we consider adjoint Hessenberg varieties. We obtain negative results of various types. Before describing these, we discuss why the consideration of Schubert varieties is appropriate in the study of adjoint Hessenberg varieties.

We know of no restrictions on the structure of general Hessenberg varieties, and it is reasonable to wonder what such restrictions might exist. In addition to Theorem 1.6, one can point to Section 9 of [22], in which various interesting curves and surfaces are shown to be general Hessenberg varieties, as evidence that such restrictions are not so easy to come by. A certain class of general Hessenberg varieties is examined by Chen et al. [9] in their study of Springer Theory for symmetric spaces.

Adjoint Hessenberg varieties are a different matter. Given $x \in \mathfrak{g}$, we consider the usual Jordan decomposition $x = x_s + x_n$ with x_s semisimple and x_n nilpotent. If x_n is regular in a Levi subalgebra of $\mathfrak{g}$, then $\mathcal{B}(x, H)$ admits an affine paving for every Hessenberg space $H \subseteq \mathfrak{g}$. This is proved under the assumption $\mathfrak{b} \subseteq H$ by Tymoczko in type A (see [28]) and by the second author for arbitrary G (see [23]), although the assumption is not necessary for the relevant arguments in either paper. When H is a nilpotent subspace, Fresse proved $\mathcal{B}(x, H)$ is paved by affines for all x when G is a classical group in [14] and Xue has extended these results to groups of type G_2 , F_4 , and E_6 in [29]. So, in some sense, $\mathcal{B}(x, H)$ is paved by affines for “most” x, H , and indeed for all x, H in type A. This restricts considerably the structure of adjoint Hessenberg varieties. One can ask if there are any obvious additional restrictions. As Schubert varieties admit affine pavings, Tymoczko’s question is a good starting point in the search for such constraints.

We turn now to our results. First, if the root system Φ for G has an irreducible component not of type A_1 or A_2 , then there are some $y \in W$ such that the Schubert variety $X_y \subseteq \mathcal{B}$ is not equal to any adjoint Hessenberg variety in $\mathcal{B}$.

Theorem 1.7 (See Theorem 3.7 below). *If some nonabelian simple ideal of $\mathfrak{g}$ is isomorphic with neither $\mathfrak{sl}_2(\mathbb{C})$ nor $\mathfrak{sl}_3(\mathbb{C})$, then there is some $w \in W$ such that no adjoint Hessenberg variety in $\mathcal{B}$ is equal to X_w . In particular, assume that the root system Φ for G is irreducible and not of type A_1 or A_2 . Let θ be the highest root in Φ and let W_θ*

be the stabilizer of θ in W . Let w_0 be the longest element of W . If y is a nonidentity element of W_θ , then no adjoint Hessenberg variety in $\mathcal{B}$ is equal to X_{yw_0} .

We observe in the case where Φ is irreducible, there is some function f such that $[W : W_\theta] \geq f(\dim_{\mathbb{C}} \mathfrak{h})$ and $\lim_{n \rightarrow \infty} f(n) = \infty$. So, Theorem 1.7 applies to an eventually negligible portion of the Schubert varieties in $\mathcal{B}$. When Φ is of type A or type C, we can do much better.

Theorem 1.8 (See Theorem 4.4 below). *Assume $G = GL_n(\mathbb{C})$ or $G = SL_n(\mathbb{C})$ and let B be the Borel subgroup of G consisting of upper triangular matrices. Fix w in the Weyl group $W = S_n$ of G . If there exist $x \in \mathfrak{g}$ and a Hessenberg space $H \subseteq \mathfrak{g}$ such that $\mathcal{B}(x, H) = X_w$, then w avoids the pattern [4231].*

It follows from Theorem 1.8 and the Marcus–Tardos Theorem (see [21]) that the number of Schubert varieties in $SL_n(\mathbb{C})/B$ that are (equal to) adjoint Hessenberg varieties grows at most exponentially in n . So, in type A, the portion of Schubert varieties that are equal to adjoint Hessenberg varieties is eventually negligible. A more precise enumerative result appears in Section 4, along with a proof of the theorem. Given Theorem 1.8, one might hope that the set of Type A Schubert varieties that are (equal to) adjoint Hessenberg varieties is characterized by pattern avoidance. This is not the case. We show in Section 4 that $X_{[14235]}$ is an adjoint Hessenberg variety in $SL_5(\mathbb{C})/B_5(\mathbb{C})$, but $X_{[1423]}$ is not an adjoint Hessenberg variety in $SL_4(\mathbb{C})/B_4(\mathbb{C})$.

We also obtain a negative answer to the isomorphism question for adjoint Hessenberg varieties in type A, although our result applies to far fewer Schubert varieties than Theorem 1.8.

Theorem 1.9 (See Theorem 5.1 below) *Suppose $n \geq 6$. Assume that $G = GL_n(\mathbb{C})$ or $G = SL_n(\mathbb{C})$ and B is the subgroup of G consisting of upper triangular matrices. Let w_0 be the longest element of the Weyl group $W = S_n$, and for $i \in [n-1]$, let $s_i \in W$ be the transposition $(i, i+1)$. If $3 \leq i \leq n-3$, then there do not exist $x \in \mathfrak{g}$ and Hessenberg space $H \subseteq \mathfrak{g}$ such that $\mathcal{B}(x, H)$ is isomorphic with $X_{s_i w_0}$.*

We prove in Section 5 that under the conditions given in Theorem 1.9, there is no irreducible adjoint Hessenberg variety $\mathcal{B}(x, H)$ with the same Betti numbers as $X_{s_i w_0}$. In fact, we show that unless $(n, i) \in \{(8, 3), (8, 5)\}$, no such $\mathcal{B}(x, H)$ has the same Euler characteristic as $X_{s_i w_0}$. We use the following result, reminiscent of the restriction $\chi(\mathcal{B}(x, H)) = |W|$ obtained by De Mari et al. in [10] under the assumption that $\mathfrak{b} \subseteq H$ and x is regular semisimple.

Proposition 1.10 (See Proposition 5.3 below). *Let G and B be as in Theorem 1.9. If the adjoint Hessenberg variety $\mathcal{B}(x, H)$ is irreducible and of codimension one in $\mathcal{B}$, then $\chi(\mathcal{B}(x, H))$ is divisible by $(n-2)!$.*

We do not know the answer to the general isomorphism problem for adjoint Hessenberg varieties, and pose the following question. Observe that $[653421] \in S_6$ is the “first” Schubert variety to which Theorem 1.9 applies.

Question 1.11 *Is there an adjoint Hessenberg variety (in a flag variety for an arbitrary reductive group) that is isomorphic to the Schubert variety $X_{[653421]}$ in the flag variety of type A_5 ?*

One might also consider Question 1.11 for the Schubert variety $X_{[4231]}$ in the flag variety of type A_3 . It follows from either Theorem 1.7 or Theorem 1.8 that $X_{[4231]}$ is not equal to any adjoint Hessenberg variety, and $X_{[4231]}$ is the “first” type A Schubert variety to which these theorems apply. However, there is an adjoint Hessenberg variety in $SL_4(\mathbb{C})/B$ with the same Betti numbers as $X_{[4231]}$ (see Example 5.2 below).

We have a type C version of Theorem 1.8. Consider the embedding $\phi : Sp_{2n}(\mathbb{C}) \rightarrow SL_{2n}(\mathbb{C})$ whose image stabilizes the alternating form $\langle \cdot, \cdot \rangle$ given by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i < j = 2n + 1 - i, \\ -1 & i > j = 2n + 1 - i \\ 0 & \text{otherwise.} \end{cases}$$

Here $e_1, \dots, e_{2n}$ denotes the standard basis of $\mathbb{C}^{2n}$. One obtains from this the embedding ϕ^* of the type C Weyl group into S_{2n} whose image consists of those permutations w satisfying $w_{2n+1-i} = 2n + 1 - w_i$ for all i .

Theorem 1.12 (See Theorem 6.10 below). *Let $G = Sp_{2n}(\mathbb{C})$ and let $B \leq G$ be the Borel subgroup whose image under ϕ consists of upper triangular matrices. Fix $w \in W$. If there exist $x \in \mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ and a subspace H of $\mathfrak{g}$ such that $[b, H] \subseteq H$ and $B(x, H) = X_w$, then $\phi^*(w)$ avoids the pattern [4231].*

Our proof of Theorem 1.12 utilizes an interesting fact relating adjoint Hessenberg varieties in types A and C. Let E be the linear transformation on $\mathbb{C}^{2n}$ such that $\langle v, w \rangle = v^{\text{tr}} E w$ for all $v, w \in \mathbb{C}^{2n}$, and define the automorphism σ of $G_A := SL_{2n}(\mathbb{C})$ by

$$\sigma(g) = E(g^{\text{tr}})^{-1} E^{-1}.$$

Then $G_C := \phi(Sp_{2n}(\mathbb{C}))$ is the group of σ -fixed points in G_A . Moreover, if B_A is the group of upper triangular matrices in G_A , then σ fixes B_A setwise, and $B_C := (B_A)^\sigma$ is the image under ϕ of the Borel subgroup of $Sp_{2n}(\mathbb{C})$ described in Theorem 1.12. We observe that the action of σ on G_A induces an automorphism of the variety $\mathcal{B}_A := G_A/B_A$. The map ϕ' from $\mathcal{B}_C := G_C/B_C$ to $\mathcal{B}_A := G_A/B_A$ sending gB_C to $\phi(g)B_A$ is a well-defined embedding, and $(\mathcal{B}_A)^\sigma = \phi'(\mathcal{B}_C)$ (see for example [20, Proposition 6.1.1.1]).

Theorem 1.13 (See Theorem 6.3 below) *If $\mathcal{V}_C$ is an adjoint Hessenberg variety in the type C flag variety $\mathcal{B}_C$, then there is some adjoint Hessenberg variety $\mathcal{V}_A$ in the type A flag variety $\mathcal{B}_A$ such that $\phi'(\mathcal{V}_C) = (\mathcal{V}_A)^\sigma$.*

The paper follows the outline presented above. Section 2 is devoted to background, notation, and terminology. Our study of general Hessenberg varieties and a proof of Theorem 1.6 can be found in Section 3, in which we prove also that the answer to the equality question (Question 1.3) is “No” for adjoint Hessenberg varieties in all Lie types. We then focus our attention on adjoint Hessenberg varieties in the type A flag variety in Sections 4 and 5, proving Theorems 1.8 and 1.9. Finally, our study of adjoint Hessenberg varieties in the type C flag variety is undertaken in Section 6, in which we prove Theorems 1.12 and 1.13.

2 Definitions, Notation, and Preliminary Results

We review here various known results about algebraic groups, Weyl groups, flag varieties, and Hessenberg varieties. A reader familiar with basic facts about these objects can skip this section and refer back when necessary. Facts stated without reference or argument can be found in at least one of [4, 7, 12, 18].

As in the introduction, G is a reductive algebraic group with Borel subgroup B and maximal torus $T \leq B$. The Lie algebras of G , B , T are denoted, respectively, by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$. The Weyl group of G is $W := N_G(T)/T$ where $N_G(T)$ is the normalizer in G of T . We fix a representative $\dot{w} \in N_G(T)$ for each Weyl group element $w \in W$.

2.1 The Root System and the Bruhat Order

The Lie algebra $\mathfrak{g}$ admits a Cartan decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Phi} \mathfrak{g}_{\gamma}.$$

Here $\Phi \subseteq \mathfrak{h}^*$ is the root system for $\mathfrak{g}$, and each root space $\mathfrak{g}_{\gamma}$ is a 1-dimensional subspace of $\mathfrak{g}$ satisfying $[h, x] = \gamma(h)x$ whenever $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_{\gamma}$. We can choose a set of simple roots $\Delta \subseteq \Phi$ such that each $\gamma \in \Phi$ is either a non-negative linear combination of elements of Δ or a non-positive such combination. This gives a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$. We may (and do) choose Δ so that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Phi^+} \mathfrak{g}_{\gamma}.$$

There is a partial order on Φ given by

$$\beta \leq \gamma \text{ whenever } \gamma - \beta \text{ is a nonnegative linear combination of positive roots.} \quad (2.1)$$

Each finite, irreducible root system Φ contains a unique maximal element with respect to this order called the **highest root** of Φ and denoted herein by $\theta \in \Phi^+$.

If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{h}$ the Cartan subalgebra of diagonal matrices, we write $\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\}$ with positive roots $\Phi^+ = \{\epsilon_i - \epsilon_j \in \Phi : i < j\}$. Here $\epsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$ denotes projection to the i -th diagonal entry. We assume furthermore that $\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\}$. The highest root of Φ in this case is $\theta = \epsilon_1 - \epsilon_n$.

The restriction of the adjoint action of G on $\mathfrak{g}$ to $N_G(T)$ preserves the Cartan decomposition and factors through T . Thus we get an action of W on $\mathfrak{h}^*$ that restricts to an action on Φ . For each $\alpha \in \Delta$, there is the simple reflection $s_{\alpha} \in W$, which acts on $\mathfrak{h}^*$ as the reflection through the hyperplane orthogonal to α . In the type A case if $\alpha = \epsilon_i - \epsilon_{i+1}$ we write $s_{\alpha} = s_i$ for the corresponding simple transposition exchanging i and $i+1$ in $W = S_n$. The set S of simple reflections generates W , and the length $\ell(w)$ of $w \in W$ is the shortest length of a list of elements of S (called a reduced word) whose product is w .

The **Bruhat order** $\leq_{\text{Br}}$ is the partial order on W defined by $v \leq_{\text{Br}} w$ if some reduced word for v is a subword of some reduced word for w . We write w_0 for the unique maximal element of W in the Bruhat order.

2.2 The Flag Variety and Schubert Varieties

The group G acts on the flag variety $\mathcal{B} := G/B$ by translation. Each B -orbit in this action contains exactly one coset $\dot{w}B$ ($w \in W$), and so

$$\mathcal{B} = \bigsqcup_{w \in W} B\dot{w}B/B.$$

We write C_w for $B\dot{w}B/B$, called a **Schubert cell**. Each Schubert cell is isomorphic to the affine space $\mathbb{C}^{\ell(w)}$. The closure of C_w in G/B is the **Schubert variety** $X_w := \overline{C_w}$. We have that

$$X_w = \bigsqcup_{v \leq_{\text{Br}} w} C_v.$$

2.3 Adjoint Hessenberg Varieties

Let $H \subseteq \mathfrak{g}$ be a Hessenberg space, that is, a subspace such that $[\mathfrak{b}, H] \subseteq H$, and fix $x \in \mathfrak{g}$. Equivalently, $H \subseteq \mathfrak{g}$ is B -invariant with respect to the Adjoint action (see for example [19, Definition 8.1.22]). Given a root $\gamma \in \Phi$, let π_γ denote the projection of $\mathfrak{g}$ to the root space $\mathfrak{g}_\gamma$. As $\dim_{\mathbb{C}} \mathfrak{g}_\gamma = 1$ and the T -module $\mathfrak{h} \oplus \bigoplus_{\beta \in \Phi \setminus \{\gamma\}} \mathfrak{g}_\beta$ has no quotient isomorphic to $\mathfrak{g}_\gamma$, it follows from basic facts about direct sums that, for any Hessenberg space H in $\mathfrak{g}$,

$$\text{if } \pi_\gamma(H) \neq 0 \text{ then } \mathfrak{g}_\gamma \subseteq H. \quad (2.2)$$

We will use (2.2) and its consequences repeatedly below, frequently without reference.

As in the introduction, we define the adjoint Hessenberg variety corresponding to x and Hessenberg space H by

$$\mathcal{B}(x, H) := \{gB \in \mathcal{B} : g^{-1} \cdot x \in H\}.$$

Here $g \cdot x := \text{Ad}(g)(x)$.

2.4 Type A: The Tableau Criterion, Pattern Avoidance, and Flags in $\mathbb{C}^n$

Now, we consider the case $G = SL_n(\mathbb{C})$ or $G = GL_n(\mathbb{C})$. We record here various results that will be used below. In this case, W is isomorphic to the symmetric group S_n . We write elements of S_n in one-line notation,

$$w = [w_1 w_2 \cdots w_n]$$

where $w_i = w(i)$ in the natural action of w on $[n] := \{1, 2, \dots, n\}$.

Given $w \in S_n$ and $1 \leq j \leq k \leq n$, we write $I_{j,k}(w)$ for the j^{th} smallest element of $\{w_i : 1 \leq i \leq k\}$. So, for example, if $w = [52341] \in S_5$, then $I_{2,4}(w) = 3$, as 3 is the second smallest element of $\{5, 2, 3, 4\}$. The following characterization of the Bruhat order can be found in [4].

Theorem 2.1 (Tableau Criterion). *Let $v, w \in S_n$. Then $v \leq_{\text{Br}} w$ if and only if $I_{j,k}(v) \leq I_{j,k}(w)$ for all $1 \leq j \leq k \leq n$.*

Given $v \in S_m$ and $w \in S_n$ with $m \leq n$, we say that w **contains the pattern** v if there exist $1 \leq i_1 < i_2 < \dots < i_m \leq n$ such that, for all $j, k \in [m]$, $w_{i_j} < w_{i_k}$ if and only if $v_j < v_k$. So, for example, $[631524] \in S_6$ contains the pattern $[4231] \in S_4$, realized by the subsequence 6352. We say w **avoids** v if w does not contain the pattern v . For example, $[631524]$ avoids $[4321]$. We write $S_n(v)$ for the set of all $w \in S_n$ avoiding v . Marcus and Tardos proved in [21] that (as conjectured independently by Stanley and Wilf) for every fixed v , $|S_n(v)|$ grows exponentially with n . It follows immediately that

$$\lim_{n \rightarrow \infty} \frac{|S_n(v)|}{|S_n|} = 0. \quad (2.3)$$

In this type A setting, we take B to be the Borel subgroup consisting of upper triangular matrices in G . Write $e_1, e_2, \dots, e_n$ for the standard basis of $\mathbb{C}^n$. We set $F_k := \mathbb{C}\{e_j : j \in [k]\}$, and observe that B is the stabilizer in G of the flag

$$\mathcal{F}_\bullet := (0 = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n).$$

As G acts transitively on the set

$$\text{Flag}(n) := \{\mathcal{V}_\bullet = (0 = V_0 \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n) : \dim_{\mathbb{C}} V_i = i\}$$

of all full flags in $\mathbb{C}^n$, we obtain a bijection $\mathcal{B} = G/B \rightarrow \text{Flag}(n)$ defined by

$$gB \mapsto g\mathcal{F}_\bullet := (0 = gF_0 \subset gF_1 \subset \dots \subset gF_n = \mathbb{C}^n). \quad (2.4)$$

Let us assume (temporarily) that a Hessenberg space H contains the Borel algebra $\mathfrak{b}$ of upper triangular matrices in $\mathfrak{g}$. In this context, we define the **Hessenberg vector** $h = h(H) := (h_1, \dots, h_n)$ by setting h_j to be the largest integer $i > j$ such that the elementary matrix E_{ij} is contained in H , if such i exists, and $h_j = j$ otherwise. We observe that the sequence h is weakly increasing (as H is $\text{Ad}(B)$ -invariant) and satisfies $h_j \geq j$ for all j (as $\mathfrak{b} \subseteq H$). Moreover, the Hessenberg space H is determined by, and uniquely determines, the Hessenberg vector $h(H)$. A direct computation shows that under the restriction of the bijection from (2.4) to $\mathcal{B}(x, H)$ we have

$$\mathcal{B}(x, H) \simeq \{\mathcal{V}_\bullet \in \text{Flag}(n) : xV_i \subseteq V_{h_i} \text{ for all } i \in [n] \text{ where } h = h(H)\}. \quad (2.5)$$

We will make use this identification below whenever it is convenient to do so.

We will also consider the image of a Schubert variety $X_w = \overline{B\dot{w}B/B}$ under this identification. To this end, for $w \in S_n$ and $p, q \in [n]$, we set

$$r_{p,q}(w) := |\{i \in [p] : w_i \in [q]\}|.$$

Then, the correspondence between cosets and flags maps X_w to the set of all $\mathcal{V}_\bullet$ satisfying

$$\dim_{\mathbb{C}}(V_p \cap F_q) \geq r_{p,q}(w)$$

for all $p, q \in [n]$, see [13, §10.5].

2.5 Representations of Reductive Groups

Returning to the setting of an arbitrary reductive algebraic group G , let $\psi : G \rightarrow GL(V)$ be a (rational, finite-dimensional) representation. Then the differential $d\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a Lie algebra homomorphism. Both $\psi(T)$ and $d\psi(\mathfrak{h})$ are diagonalizable and thus there exist (finitely many) $\lambda \in \mathfrak{h}^*$ such that

- $V = \bigoplus_{\lambda} V_{\lambda}$ with $V_{\lambda} \neq 0$, where
- $d\psi(h)(v) = \lambda(h)v$ for all $v \in V_{\lambda}$ and $h \in \mathfrak{h}$, and
- $\psi(t)$ acts as a scalar transformation on V_{λ} for all $t \in T$ and all λ .

Each such $\lambda \in \mathfrak{h}^*$ is a **weight** and the subspace V_{λ} is called a **weight space** of V . We write Λ for the set of all weights of all representations of G , which forms a lattice in $\mathfrak{h}^*$. The set of weights of the adjoint representation $Ad : G \rightarrow GL(\mathfrak{g})$ is $\Phi \sqcup \{0\}$, hence $\Phi \subseteq \Lambda$.

If ψ is irreducible, then there is a unique 1-dimensional $\psi(B)$ -invariant subspace of V , which is a weight space. The associated weight λ is called the **highest weight** of ψ , and we write $V(\lambda)$ for V . Any nonzero $v_{\lambda} \in V(\lambda)_{\lambda}$ is a **highest weight vector** in $V(\lambda)$.

A weight $\lambda \in \Lambda$ is **dominant** if λ is the highest weight for some irreducible representation. The action of W on $\mathfrak{h}^*$ induces an action of W on Λ . We observe that $\dim_{\mathbb{C}} V_{\lambda} = \dim_{\mathbb{C}} V_{w(\lambda)}$ for every weight λ for V and every $w \in W$. A dominant weight λ is **strictly dominant** if the stabilizer of λ in W is trivial. The partial order $\leq$ on Φ defined above in (2.1) extends to a partial order on Λ . We write $\rho \leq \lambda$ if $\lambda - \rho$ is a nonnegative linear combination of positive roots.

We remark that every representation ψ' of $\mathfrak{g}$ is of the form $\psi' = d\psi$ for some representation ψ of G , and that such a ψ' is irreducible if and only if ψ is irreducible. Thus, we may also use the terminology defined above when referring to representations of the Lie algebra $\mathfrak{g}$. Finally, a representation $\psi' : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ makes V a module for the universal enveloping algebra $U(\mathfrak{g})$, and we make no distinction between such representations and modules. Given dominant $\lambda \in \Lambda$, we refer to $V(\lambda)$ as a **highest weight module** for G , or equivalently $U(\mathfrak{g})$.

3 Highest Weight Hessenberg Varieties

In this section we study Questions 1.3, 1.4, and 1.5 for *general* Hessenberg varieties of Definition 1.2. We prove Theorem 1.6 (see Theorem 3.6 below), establishing that the answer to all three questions in this context is “Yes.” Then we show that the answer to the equality problem, Question 1.3, for adjoint Hessenberg varieties is “No,” by proving Theorem 1.7 (see Theorem 3.7 below).

Let $\psi : G \rightarrow GL(V(\lambda))$ be the irreducible representation of G with highest weight λ , and fix a highest weight vector $v_\lambda \in V(\lambda)_\lambda$. Since $\mathbb{C}\{v_\lambda\}$ is a $\psi(B)$ -invariant subspace of $V(\lambda)$, it follows that the Hessenberg variety $\mathcal{B}(v_\lambda, H)$ is invariant under left translation by B and therefore a union of Schubert varieties. We call the Hessenberg variety $\mathcal{B}(v_\lambda, H)$ a **highest weight Hessenberg variety**.

Highest weight Hessenberg varieties defined using the adjoint representation have been studied by Tymoczko and by Abe–Crooks in [1, 27]. Suppose $\mathfrak{g}$ is simple and consider the adjoint representation of G on $\mathfrak{g}$, which is irreducible. The highest weight for the adjoint representation is the highest root $\theta \in \Phi$. We fix nonzero $E_\theta \in \mathfrak{g}_\theta$, so E_θ is a highest weight vector for the adjoint representation of G . Abe and Crooks give an explicit description of the highest weight Hessenberg variety $\mathcal{B}(E_\theta, H)$ as a union of Schubert varieties in the type A case whenever $\mathfrak{b} \subseteq H$. The following result, due to Tymoczko (see [27, Prop. 4.5]), describes a collection of Schubert varieties equal to highest weight adjoint Hessenberg varieties.

Proposition 3.1 (Tymoczko). *Suppose $\gamma \in \Phi$ is a root in the same W -orbit as θ and let w be the maximal length element of W such that $w^{-1}(\theta) = \gamma$. Let H_γ be the B -submodule of $\mathfrak{g}$ generated by $E_{w^{-1}(\theta)} = E_\gamma$. Then, $X_w = \mathcal{B}(E_\theta, H_\gamma)$.*

The B -module H_γ defined in Proposition 3.1 is known as a Demazure module. Such modules are defined similarly for arbitrary irreducible representations of G , and will be used below to prove Theorem 1.6.

3.1 Demazure Modules

Throughout this section, we let λ denote a fixed dominant weight and $V(\lambda)$ the associated highest weight module for G , or equivalently, for $U(\mathfrak{g})$. For each $w \in W$, fix a nonzero vector $v_{w(\lambda)}$ in the (one-dimensional) weight space $V(\lambda)_{w(\lambda)}$. The **Demazure module** $H_{w(\lambda)}$ is the $U(\mathfrak{b})$ -submodule of $V(\lambda)$ generated by $v_{w(\lambda)}$. As remarked in [19, Definition 8.1.22], $H_{w(\lambda)}$ is B -invariant and so is a Hessenberg space in $V(\lambda)$. Indeed, $H_{w(\lambda)}$ is the B -submodule of $V(\lambda)$ generated by $v_{w(\lambda)}$.

Example 3.2 We record here some observations regarding these constructions in the adjoint case that will be useful in Sects. 4 and 6. Given a root $\gamma \in \Phi$ in the same W -orbit as θ , the Demazure module $H_\gamma \subseteq \mathfrak{g}$ is the B -submodule generated by a root vector $E_\gamma \in \mathfrak{g}_\gamma$. Since each H_γ is B -invariant, each is also T -invariant and it follows that $\mathfrak{g}_\beta \cap H_\gamma \neq \emptyset$ implies $\mathfrak{g}_\beta \subseteq H_\gamma$.

Motivated by this property, we define a second partial order on Φ by

$$\gamma \leq \beta \text{ whenever } \mathfrak{g}_\beta \subseteq H_\gamma. \quad (3.1)$$

We note that $\gamma \leq \beta$ implies $\gamma \preceq \beta$, where $\preceq$ is defined in (2.1), but not vice versa. For example, say $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ so Φ is of type A_3 . Set

$$H_{21} := \mathfrak{sl}_4(\mathbb{C}) \cap \mathbb{C}\{E_{ij} : i \in \{1, 2\}\},$$

Then $H_{\epsilon_2 - \epsilon_1} \subseteq H_{21}$, since conjugation by any upper triangular matrix $b \in B$ maps the root vector $E_{21} \in \mathfrak{g}_{\epsilon_2 - \epsilon_1}$ into H_{21} . Since $E_{34} \notin H_{21}$, this shows $\epsilon_2 - \epsilon_1 \not\preceq \epsilon_3 - \epsilon_4$. On the other hand, $\epsilon_2 - \epsilon_1 \preceq \epsilon_3 - \epsilon_4$, since $\epsilon_2 - \epsilon_1 \in \Phi^-$ and $\epsilon_3 - \epsilon_4 \in \Phi^+$.

The partial order $\leq$ on Φ defined in (3.1) is a special case of the partial order on $\{w(\lambda) : w \in W\}$ for $\lambda \in \Lambda$ a dominant weight defined and studied by Proctor in [24]. The following lemma summarizes [24, Proposition 3] in the adjoint case.

Lemma 3.3 (Proctor). *Given $\gamma, \gamma' \in \Phi$ we have $\gamma \leq \gamma'$ if and only if there exist positive roots $\gamma_{i_1}, \dots, \gamma_{i_k} \in \Phi^+$ and nonnegative integers $n_1, n_2, \dots, n_k$ such that $\gamma' = \gamma + n_1\gamma_{i_1} + n_2\gamma_{i_2} + \dots + n_k\gamma_{i_k}$ and $\gamma + n_1\gamma_{i_1} + \dots + n_m\gamma_{i_m} \in \Phi$ for all $1 \leq m \leq k$.*

3.2 Proof of Theorem 1.6

Let J denote the set of simple reflections stabilizing λ , i.e., $J = \{s_\alpha : s_\alpha(\lambda) = \lambda\}$. The subgroup W_J generated by J is the stabilizer of λ in W . Recall that the set of left cosets of W_J in W can be identified with the set of shortest left coset representatives, denoted herein by W^J (see [4, Section 2.4]). The Bruhat order on W induces an order on W^J , and we have $\tau \leq_{\text{Br}} w$ implies $\tau^J \leq_{\text{Br}} w^J$ where τ^J and w^J denote the shortest coset representatives for τW_J and $w W_J$, respectively (see [4, Proposition 2.5.1]). Note that there is a bijection between W^J and the set $\{w(\lambda) : w \in W\}$ given by $\tau \mapsto \tau(\lambda)$ and thus $H_{w(\lambda)} = H_{w^J(\lambda)}$.

The next result is essentially Theorem 2.9 of the paper [6] of Bernšteĭn, Gel'fand and Gel'fand. In [6] it is assumed that $\mathfrak{g}$ is the Lie algebra of a simply connected semisimple group, that $H_{w(\lambda)}$ is the module for the nilpotent radical of $\mathfrak{b}$ generated by $v_{w(\lambda)}$, and that λ is strictly dominant. As we explain below, the proof of [6, Theorem 2.9] given therein remains valid under the weaker assumptions stated in Theorem 3.4 below.

Theorem 3.4 (Bernšteĭn-Gel'fand-Gel'fand). *Suppose λ is a dominant weight for the reductive Lie algebra $\mathfrak{g}$, with associated irreducible representation $\psi' : \mathfrak{g} \rightarrow \mathfrak{gl}(V(\lambda))$ and highest weight vector v_λ . Let τ, w be distinct elements of W^J . Then $\tau <_{\text{Br}} w$ if and only if $H_{\tau(\lambda)} \subset H_{w(\lambda)}$.*

A key point in the proof of [6, Theorem 2.9] is [6, Proposition 2.7], which says that of $\preceq$ is a partial order on W satisfying

- (a) if $w \in W$, $s \in S$, and $\ell(sw) = \ell(w) + 1$ then $w \preceq sw$; and
- (b) if $\tau \preceq w$ and $s \in S$ then either $s\tau \preceq w$ or $s\tau \preceq sw$;
then $\preceq = \leq_{\text{Br}}$. Using the same ideas used to prove [6, Proposition 2.7], one can prove that if $\preceq$ is a preorder on W satisfying (a), (b), and

(c) if $\tau^J = w^J$ then $\tau \preceq w$ and $w \preceq \tau$

then $\tau \preceq w$ if and only if $\tau^J \leq_{\text{Br}} w^J$. The proof by Bernšteĭn, Gel'fand, and Gel'fand of their Theorem 2.9 can then be followed directly to prove our Theorem 3.4: one observes that [6, Lemma 2.10] holds in reductive groups and uses that lemma to show that if W_J is the stabilizer in W of dominant weight λ then the preorder $\preceq$ on W given by $\tau \preceq w$ if and only if $H_{\tau(\lambda)} \subseteq H_{w(\lambda)}$ satisfies (a), (b) and (c). Note that we must assume $\tau^J \neq w^J$ for [6, Lemma 2.10] to apply (with τ being replaced by w' in their notation), but this is captured in the assumptions of Theorem 3.4 and our claims hold trivially in the case of equality.

With this terminology in place, we can now prove the following result, generalizing Proposition 3.1 above.

Theorem 3.5 *Suppose λ is a dominant weight for G , with associated highest weight representation $\psi : G \rightarrow GL(V(\lambda))$. Let μ be a weight of $V(\lambda)$ such that μ and λ are in the same W -orbit and let $w \in W$ be the longest element satisfying $w(\lambda) = \mu$. Then $X_{w^{-1}} = \mathcal{B}(v_\lambda, H_{w^J(\lambda)})$.*

Proof Our assumptions on w imply $w = w^J y_0$ where $w^J \in W^J$ and y_0 is the longest element of W_J . We have $\tau \leq_{\text{Br}} w$ if and only if $\tau^J \leq_{\text{Br}} w^J$ in this case.

As $\mathbb{C}\{v_\lambda\}$ is $\psi(B)$ -invariant, so is the general highest weight Hessenberg variety $\mathcal{B}(v_\lambda, H_{w^J(\lambda)})$. Therefore, $\mathcal{B}(v_\lambda, H_{w^J(\lambda)})$ is a union of Schubert cells. We see now that

$$\mathcal{B}(v_\lambda, H_{w^J(\lambda)}) = \bigcup_{\substack{\tau \in W \\ \psi(\tau)(v_\lambda) \in H_{w^J(\lambda)}}} C_{\tau^{-1}}. \quad (3.2)$$

Note that $\psi(\tau)(v_\lambda) \in H_{w^J(\lambda)}$ if and only if $H_{\tau^J(\lambda)} \subseteq H_{w^J(\lambda)}$. Combining (3.2) and the fact that $\tau \leq_{\text{Br}} w$ if and only if $\tau^{-1} \leq_{\text{Br}} w^{-1}$ with Theorem 3.4 (applied to $\psi' = d\psi$), we see that

$$\mathcal{B}(v_\lambda, H_{w^J(\lambda)}) = X_{w^{-1}},$$

as desired. $\square$

We are ready to prove Theorem 1.6, restated here for convenience.

Theorem 3.6 *Let λ be a strictly dominant weight for G , with associated highest weight representation $\psi : G \rightarrow GL(V(\lambda))$ and highest weight vector v_λ . Then, $X_w = \mathcal{B}(v_\lambda, H_{w^{-1}(\lambda)})$.*

Proof Suppose λ is a strictly dominant weight of G . Then, $J = \{e\}$ and $W^J = W$. Theorem 3.6 now follows immediately from Theorem 3.5 since the weights $w(\lambda)$ for $w \in W$ are pairwise distinct. $\square$

3.3 Schubert Varieties that are Not Adjoint Hessenberg Varieties

We prove herein that for “most” reductive G there are Schubert varieties in $\mathcal{B}$ that are not adjoint Hessenberg varieties. The following is a restatement of Theorem 1.7 above.

Theorem 3.7 *If some nonabelian simple ideal of $\mathfrak{g}$ is isomorphic with neither $\mathfrak{sl}_2(\mathbb{C})$ nor $\mathfrak{sl}_3(\mathbb{C})$, then there is some $w \in W$ such that no adjoint Hessenberg variety in $\mathcal{B}$ is equal to X_w . In particular, assume that the root system Φ for G is irreducible and not of type A_1 or A_2 . Let θ be the highest root in Φ and let W_θ be the stabilizer of θ in W . Let w_0 be the longest element of W . If y is a nonidentity element of W_θ , then no adjoint Hessenberg variety in $\mathcal{B}$ is equal to X_{yw_0} .*

We begin by recalling some well-known facts about the structure of reductive groups. Given reductive G , let Z^0 be the connected component of the identity in the center $Z(G)$. Then $G = Z^0 G'$, with the commutator subgroup $G' \leq G$ being semisimple. Moreover, $Z^0 \cap G'$ is finite. (See for example [26, Proposition 7.3.1, Corollary 8.1.6].) As G' is semisimple, there exist nonabelian simple algebraic groups $L_1, \dots, L_k$ such that G' is the central product $L_1 \circ \dots \circ L_k$ (see for example [26, Theorem 8.1.5]). Set $K := Z^0 \cap G'$ and $\overline{G} := G/K \cong G'/K \times Z^0/K$. The projection of G onto $\overline{G}$ induces a surjection of the corresponding Lie algebras. This surjection of Lie algebras has trivial kernel since K is finite and thus its Lie algebra is trivial. So, if $\mathfrak{g}, \mathfrak{g}'$ and $\mathfrak{z}$ are the respective Lie algebras of G, G' and Z^0 , then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$. Each L_i has finite center $Z(L_i) \leq Z(G')$, and $Z(G') = \prod_{i=1}^k Z(L_i)$. For $i \in [k]$, let $\mathfrak{l}_i$ be the Lie algebra of L_i . Considering the projection of G' onto $G'/Z(G') \cong L_1/Z(L_1) \times \dots \times L_k/Z(L_k)$, we see that $\mathfrak{g}' = \bigoplus_{i=1}^k \mathfrak{l}_i$. The upshot of all this is that

- $\mathfrak{g} = \mathfrak{z} \oplus \bigoplus_{i=1}^k \mathfrak{l}_i$;
- if $i, j \in [k]$ with $i \neq j$, then $\text{Ad}(L_i)$ acts trivially on both $\mathfrak{z}$ and $\mathfrak{l}_j$;
- if, for each $i \in [k]$, $\mathfrak{l}_i = \mathfrak{h}_i \oplus \bigoplus_{\alpha \in \Phi_i} \mathfrak{l}_{i,\alpha}$ is a Cartan decomposition of $\mathfrak{l}_i$, then $\mathfrak{h} := \mathfrak{z} \oplus \bigoplus_{i=1}^k \mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i=1}^k \bigoplus_{\alpha \in \Phi_i} \mathfrak{l}_{i,\alpha}$ is a Cartan decomposition.

For each i , let T_i and B_i be, respectively, a maximal torus and Borel subgroup of L_i with $T_i \leq T \cap B_i$ and $B_i \leq B$. We may assume that the Cartan subalgebra $\mathfrak{h}_i$ described above is the Lie algebra of T_i , that $\mathfrak{b}_i := \mathfrak{h}_i \oplus \bigoplus_{\alpha \in \Phi_i^+} \mathfrak{l}_{i,\alpha}$ is the Lie algebra of B_i , and that $\Phi = \bigcup_{i=1}^k \Phi_i$. Set $W_i = N_{L_i}(T_i)/T_i$. Then $W = \prod_{i=1}^k W_i$. Let θ_i be the highest root in the (irreducible, crystallographic) root system Φ_i .

Lemma 3.8 *Let Γ be an irreducible, crystallographic root system with associated Weyl group X , and let $\alpha \in \Gamma$. The stabilizer of α in X is trivial if and only if Γ is of type A_1 or A_2 .*

Proof We observe that $\alpha \in \Gamma$ has trivial stabilizer in X if and only if the X -orbit $X(\alpha)$ has size $|X|$. If Γ has rank three or more, then $|X(\alpha)| \leq |\Gamma| < |X|$ (see for example [8, p. 43]). If Γ is of type B_2 or G_2 , then $|\Gamma| = |X|$, but since Γ contains roots of two different lengths, $|X(\alpha)| < |\Gamma|$. Inspection shows that if Γ is of type A_1 or A_2 then every root in Γ has trivial stabilizer in X . $\square$

Lemma 3.9 *Let λ be a strictly dominant weight for G , with associated highest weight representation $\psi : G \rightarrow GL(V(\lambda))$ and $0 \neq v \in V(\lambda)$. Then there is some $b \in B$ such that $\psi(b)v$ projects nontrivially onto $V(\lambda)_\lambda$.*

Proof By the Lie-Kolchin Theorem (see for example [17, Theorem 17.6]) there is some 1-dimensional B -invariant subspace of the B -submodule of $V(\lambda)$ generated by v . The only 1-dimensional B -invariant subspace of $V(\lambda)$ is the highest weight space $V(\lambda)_\lambda$. $\square$

We are ready to prove Theorem 3.7; recall that $\dot{w}$ is a fixed representative in $N_G(T)$ for the Weyl group element $w \in W$.

Proof of Theorem 3.7 We pick $j \in [k]$ such that Φ_j is neither of type A_1 nor of type A_2 . Applying Lemma 3.8 with $(\Gamma, X, \alpha) = (\Phi_j, W_j, \theta_j)$, we see that θ_j has nontrivial stabilizer in W_j . So, fix $1 \neq y \in W_j$ with $y(\theta_j) = \theta_j$. Let $w_{0,j}$ be the longest element of W_j . Note that $\ell(yw_{0,j}) < \ell(w_{0,j})$ so in particular, $w_{0,j} \not\leq_{\text{Br}} yw_{0,j}$.

Assume for contradiction that there exist $x \in \mathfrak{g}$ and a Hessenberg space $H \subseteq \mathfrak{g}$ such that $X_{yw_{0,j}} = \mathcal{B}(x, H)$. Write

$$x = x_0 + \sum_{i=1}^k x_i,$$

with $x_0 \in \mathfrak{z}$ and $x_i \in \mathfrak{l}_i$ for all $i \in [k]$. As $B = eB \in X_{yw_{0,j}}$, we see that $x \in H$. If $x_j = 0$ then $\dot{w} \cdot x = x$ for all $w \in W_j$. In particular, $\dot{w}_{0,j}B \in \mathcal{B}(x, H)$, which is impossible since $w_{0,j} \not\leq_{\text{Br}} yw_{0,j}$. So, we assume now that $x_j \neq 0$.

By Lemma 3.9, there is some $b \in B_j$ such that $0 \neq b^{-1} \cdot x$ projects nontrivially onto $\mathfrak{g}_{\theta_j}$. As $X_{yw_{0,j}}$ is B -invariant and contains $\dot{y}\dot{w}_{0,j}B$, we see that $(b\dot{y}\dot{w}_{0,j})^{-1} \cdot x \in H$. Now

$$(b\dot{y}\dot{w}_{0,j})^{-1} \cdot x = \dot{w}_{0,j}^{-1} \cdot (\dot{y}^{-1} \cdot (b^{-1} \cdot x)).$$

As $\mathfrak{g}_{\theta_j}$ is B_j -invariant, we see that $b^{-1} \cdot x$ projects nontrivially onto $\mathfrak{g}_{\theta_j}$. As $y \in (W_j)_{\theta_j}$, also $\dot{y}^{-1} \cdot (b^{-1} \cdot x)$ projects nontrivially onto $\mathfrak{g}_{\theta_j}$. Finally, $w_{0,j}^{-1} = w_{0,j}$ maps θ_j to $-\theta_j$ and it follows that $(b\dot{y}\dot{w}_{0,j})^{-1} \cdot x$ projects nontrivially onto $\mathfrak{g}_{-\theta_j}$.

As H projects nontrivially onto the 1-dimensional root space $\mathfrak{g}_{-\theta_j}$, we see that $\mathfrak{g}_{-\theta_j} \subseteq H$. Since $\mathfrak{g}_{-\theta_j}$ generates the B_j -module $\mathfrak{l}_j$, we must have $\mathfrak{l}_j \subseteq H$. In particular, $x_j \in H$, hence $x - x_j \in H$. As $Ad(L_j)$ fixes $x - x_j$, it follows that for all $w \in W_j$,

$$\dot{w}^{-1} \cdot x = x - x_j + \dot{w}^{-1} \cdot x_j \in H.$$

In particular, $\dot{w}_{0,j}^{-1} \cdot x \in H$, leading again to the impossibility $\dot{w}_{0,j}B \in X_{yw_{0,j}}$. $\square$

Example 3.10 Consider the Adjoint representation of $SL_n(\mathbb{C})$ on $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. In this case $\theta = \epsilon_1 - \epsilon_n$ and the stabilizer of θ is the subgroup $W_\theta = \langle s_2, \dots, s_{n-2} \rangle$. Thus, $W_\theta \neq \{e\}$ whenever $n \geq 4$ (that is, whenever the rank of $\mathfrak{sl}_n(\mathbb{C})$ is at least 3). By

Theorem 3.7, every element of the right coset $W_\theta w_0$, except for w_0 , has the property that the corresponding Schubert variety cannot be realized as a Hessenberg variety. Note that the reflection

$$s_\theta = s_{n-1} \cdots s_2 s_1 s_2 \cdots s_{n-1} = (1, n)$$

is the minimal length coset representative for $W_\theta w_0$. Thus,

$$W_\theta w_0 \setminus \{w_0\} = W_\theta s_\theta \setminus \{w_0\} = \{ys_\theta : y \in W_\theta, y \neq y_0\}$$

where y_0 denotes the longest element of W_θ . This shows X_{ys_θ} is not equal to any adjoint Hessenberg variety $\mathcal{B}(x, H)$ for all $y \in W_\theta \setminus \{y_0\}$.

Note the set $W_\theta s_\theta \setminus \{w_0\}$ from Example 3.10 consists of all $w \in S_n$ such that $w_1 = n$, $w_n = 1$, and there exist $k, \ell \in \{2, \dots, n-1\}$ such that $k < \ell$ and $w_k < w_\ell$. In particular, all such w contain the pattern [4231]. The results of Section 4 extend Theorem 3.7 to a general statement about pattern avoidance in the type A case.

To conclude, we note that our proof of Theorem 3.7 can be adjusted easily to obtain the following converse to Theorem 3.6.

Proposition 3.11 *Let λ be a dominant but not strictly dominant weight for G , with associated highest weight representation $\psi : G \rightarrow GL(V(\lambda))$. Then there exists $w \in W$ such that no pair $(x \in V(\lambda), H \subseteq V(\lambda))$ with H a Hessenberg space satisfies $\mathcal{B}(x, H) = X_w$.*

4 The Equality Question for Type A Adjoint Hessenberg Varieties and Pattern Avoidance

Assume that $G = SL_n(\mathbb{C})$ throughout. In this section we prove Theorem 1.8, which says that if $w \in S_n$ contains the pattern [4231], then there is no adjoint Hessenberg variety in $\mathcal{B}$ equal to X_w . We will see in Example 4.7 that there is no pattern avoidance criterion characterizing the set of type A Schubert varieties that are adjoint Hessenberg varieties. Theorem 1.8 remains true if we take $G = GL_n(\mathbb{C})$, as can be shown with minor modifications, omitted herein, to our proofs.

As mentioned earlier, Theorem 1.8 is much more powerful than Theorem 1.7 when only type A is considered. Indeed, Theorem 1.7 implies that the number of $w \in S_n$ such that X_w is not equal to an adjoint Hessenberg variety grows at least quadratically with n , while Theorem 1.8 says that the number of such w is at least $n! - c^n$ for some constant c .

We begin by recording two basic facts that we will use repeatedly. Each matrix $x \in \mathfrak{sl}_n(\mathbb{C})$ can be written as a linear combination of elementary matrices $\{E_{ij} : (i, j) \in [n] \times [n]\}$. We define $c_{ij} : \mathfrak{sl}_n(\mathbb{C}) \rightarrow \mathbb{C}$ to be the coordinate function returning the coefficient of E_{ij} in such a combination. Throughout this section, we write $g \cdot x$ for $Ad(g)(x)$ whenever $g \in SL_n(\mathbb{C})$ and $x \in \mathfrak{sl}_n(\mathbb{C})$.

Lemma 4.1 *Let $x \in \mathfrak{sl}_n(\mathbb{C})$ with $\mathbf{c}_{ij}(x) \neq 0$ for some $(i, j) \in [n] \times [n]$ such that $i \neq j$. If $(k, \ell) \in [n] \times [n]$ with $k \leq i$ and $\ell \geq j$ then there exists $b \in B$ such that $\mathbf{c}_{k\ell}(b \cdot x) \neq 0$.*

Proof First suppose that $(k, \ell) = (j, i)$. This implies $j < i$ and so $b_\alpha := I + \alpha E_{ji}$ is an element of B for all $\alpha \in \mathbb{C}$. A direct computation shows that $\mathbf{c}_{ji}(b_\alpha \cdot x) = \mathbf{c}_{ji}(x) + \alpha(\mathbf{c}_{ii}(x) - \mathbf{c}_{jj}(x)) - \alpha^2 \mathbf{c}_{ij}(x)$. Thus, there exists $\alpha \in \mathbb{C}$ such that $\mathbf{c}_{ji}(b_\alpha \cdot x) \neq 0$. We now consider $(k, \ell) \neq (j, i)$, i.e., the case that $i \neq \ell$ or $k \neq j$. When $i \neq \ell$ the lemma follows from the existence of $b_1, b_2 \in B$ such that $\mathbf{c}_{i\ell}(b_1 \cdot x) \neq 0$ and $\mathbf{c}_{k\ell}(b_2 \cdot (b_1 \cdot x)) \neq 0$. Symmetrically, when $j \neq k$ the lemma follows from the existence of $b_1, b_2 \in B$ such that $\mathbf{c}_{kj}(b_1 \cdot x) \neq 0$ and $\mathbf{c}_{k\ell}(b_2 \cdot (b_1 \cdot x)) \neq 0$. Therefore, to settle the case $(k, \ell) \neq (j, i)$, it suffices to assume that $i = k$ or $j = \ell$.

When $i = k$ and $j = \ell$ the statement of the lemma is immediate so let us assume that this is not the case. If $i = k$ then for each $\alpha \in \mathbb{C}$, $b_\alpha := I + \alpha E_{j\ell}$ is an element of B and a direct computation shows that $\mathbf{c}_{i\ell}(b_\alpha \cdot x) = \mathbf{c}_{i\ell}(x) - \alpha \mathbf{c}_{ij}(x)$. Thus, there exists $\alpha \in \mathbb{C}$ such that $\mathbf{c}_{i\ell}(b_\alpha \cdot x) \neq 0$. Similarly, If $\ell = j$ then $b_\alpha := I + \alpha E_{ki}$ is an element of B and $\mathbf{c}_{kj}(b_\alpha \cdot x) = \mathbf{c}_{kj}(x) + \alpha \mathbf{c}_{ij}(x)$, which implies that there exists $\alpha \in \mathbb{C}$ such that $\mathbf{c}_{kj}(b_\alpha \cdot x) \neq 0$. $\square$

Note that Lemma 4.1 gives us an explicit description of the partial order $\leq$ on Φ defined in Section 3.1. Indeed, we have $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}\{E_{ij}\}$ and the lemma tells us that there exists $b \in B$ such that $\mathbf{c}_{k\ell}(b \cdot E_{ij}) \neq 0$ so $\mathfrak{g}_{\epsilon_k - \epsilon_\ell}$ is contained in the Demazure module $H_{\epsilon_i - \epsilon_j}$. In summary,

$$\epsilon_i - \epsilon_j \leq \epsilon_k - \epsilon_\ell \Leftrightarrow k \leq i \text{ and } \ell \geq j.$$

Note that this description of $\leq$ can also be obtained from Lemma 3.3.

Lemma 4.2 *Say $H \subseteq \mathfrak{sl}_n(\mathbb{C})$ is a Hessenberg space and $E_{ij} \in H$ with $i \neq j$. If $k, \ell \in [n]$ with $k \neq \ell$ and both $k \leq i$ and $\ell \geq j$ hold, then $E_{k\ell} \in H$. In other words, if $\mathfrak{g}_\beta \subseteq H$ then $\mathfrak{g}_\gamma \subseteq H$ for all $\gamma \in \Phi$ such that $\beta \leq \gamma$.*

Proof By Lemma 4.1 there exists $b \in B$ such that $\mathbf{c}_{k\ell}(b \cdot E_{ij}) \neq 0$. Since H is B -invariant, we have $b \cdot E_{ij} \in H$ implying $\mathfrak{g}_{\epsilon_k - \epsilon_\ell} \subseteq H$ and $E_{k\ell} \in H$. $\square$

Theorem 1.8 will follow directly from the next result.

Lemma 4.3 *Assume $w \in S_n$ and $1 \leq i < j < k < \ell \leq n$ with $w_\ell < w_j < w_k < w_i$. Define $\tau \in S_n$ by*

$$\tau_m := \begin{cases} w_m & m \notin \{j, k\}, \\ w_k & m = j, \\ w_j & m = k. \end{cases}$$

Let $x \in \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and let $H \subseteq \mathfrak{sl}_n(\mathbb{C})$ be a Hessenberg space. If the Schubert cell $C_{w^{-1}}$ is contained in the adjoint Hessenberg variety $\mathcal{B}(x, H)$, then $\tau^{-1}B \in \mathcal{B}(x, H)$.

Proof Given w, x , and H as in the statement, we assume $C_{w^{-1}} \subseteq \mathcal{B}(x, H)$, and write $\mathbf{c}_{pq}$ for $\mathbf{c}_{pq}(x)$ whenever $p, q \in [n]$ with $p \neq q$. Set

$$x' := \sum_{p \neq q} \mathbf{c}_{pq} E_{pq}, \quad \text{and} \quad s := x - x' \in \mathfrak{h}.$$

Then $x = x' + s$. We will show that both $\dot{\tau} \cdot x'$ and $\dot{\tau} \cdot s$ lie in H , thus proving the lemma.

We know that $\dot{w} \cdot x \in H$, since $\dot{w}^{-1} B \in C_{w^{-1}} \subseteq \mathcal{B}(x, H)$. In particular, $E_{pq} \in H$ whenever $p \neq q$ and $\mathbf{c}_{pq}(\dot{w} \cdot x) \neq 0$. As $\mathbf{c}_{pq}(\dot{w} \cdot x') = \mathbf{c}_{pq}(\dot{w} \cdot x)$ whenever $p \neq q$, it follows that $\dot{w} \cdot x' \in H$. Therefore, $\dot{w} \cdot s = \dot{w} \cdot x - \dot{w} \cdot x' \in H$.

We see now that to show $\dot{\tau} \cdot s \in H$, it suffices to show

$$\dot{w} \cdot s - \dot{\tau} \cdot s \in H. \quad (4.1)$$

For $p \in [n-1]$, we write h_p for $E_{pp} - E_{p+1\ p+1} \in \mathfrak{h}$. There exist $c_p \in \mathbb{C}$ such that

$$s = \sum_{p=1}^{n-1} c_p h_p.$$

We observe that $\dot{w} \cdot E_{mm} = E_{w_m w_m}$ for all $m \in [n]$, and similar for $\dot{\tau}$. A direct computation shows that, whether or not $k = j+1$, we obtain

$$\dot{w} \cdot s - \dot{\tau} \cdot s = (c_{j-1} - c_j - c_{k-1} + c_k)(E_{w_k w_k} - E_{w_j w_j}).$$

If $(c_{j-1} - c_j - c_{k-1} + c_k) = 0$, then (4.1) holds. So, we assume from now on that

$$c_{j-1} - c_j - c_{k-1} + c_k \neq 0. \quad (4.2)$$

In order to show that (4.1) holds, we must show that

$$E_{w_k w_k} - E_{w_j w_j} \in H. \quad (4.3)$$

We claim that there is some $b \in B$ with $\mathbf{c}_{jk}(b \cdot x) \neq 0$. Given this claim, as $i < j$ and $\ell > k$, Lemma 4.1 implies that there exists $b' \in B$ such that $\mathbf{c}_{i\ell}(b' b \cdot x) \neq 0$. It follows that

$$\mathbf{c}_{w_i w_\ell}(\dot{w} b' b \cdot x) \neq 0. \quad (4.4)$$

Since $(\dot{w} b' b)^{-1} B \in C_{w^{-1}} \subseteq \mathcal{B}(x, H)$, we see that $\dot{w} b' b \cdot x \in H$. It follows from (4.4) that $E_{w_i w_\ell} \in H$. Using Lemma 4.2, we see that $E_{w_k w_j} \in H$. Finally, as $E_{w_j w_k} \in \mathfrak{b}$ and H is a Hessenberg space, we get

$$E_{w_k w_k} - E_{w_j w_j} = -[E_{w_j w_k}, E_{w_k w_j}] \in H.$$

Thus (4.3) and (4.1) hold.

We aim now to prove the claim that there exists $b \in B$ with $\mathbf{c}_{jk}(b \cdot x) \neq 0$. By Lemma 4.1, such b exists if there are p, q such that all of $p \neq q$, $j \leq p$, $q \leq k$, and $\mathbf{c}_{pq} \neq 0$ hold. We may therefore assume no such p, q exist. In that case (see for example [8, p.61]) $\mathbf{c}_{jk}(b \cdot x') = 0$ for all $b \in B$. Now we choose

$$b := I + E_{jk} \in B.$$

From direct calculation (or by [8, p.61]), we see that

$$b \cdot s = s + (c_{j-1} - c_j - c_{k-1} + c_k)E_{jk}.$$

It follows now from (4.2) that

$$\mathbf{c}_{jk}(b \cdot x) = \mathbf{c}_{jk}(b \cdot x') + \mathbf{c}_{jk}(b \cdot s) \neq 0$$

as desired. This completes our proof that $\dot{\tau} \cdot s \in H$.

It remains to show that $\dot{\tau} \cdot x' \in H$. Equivalently, we must show that, for all $p, q \in [n]$ with $p \neq q$,

$$\text{if } \mathbf{c}_{pq} \neq 0 \text{ then } E_{\tau_p \tau_q} \in H. \quad (4.5)$$

Since $\dot{w}^{-1}B \in \mathcal{B}(x, H)$, we know that $E_{w_p w_q} \in H$ whenever $p \neq q$ and $\mathbf{c}_{pq} \neq 0$. Therefore, (4.5) holds whenever $\{p, q\} \cap \{j, k\} = \emptyset$. For all remaining pairs (p, q) , we will choose appropriate pairs (d, e) and (s, t) of indices such that

$$(w_s, w_t) = (\tau_p, \tau_q), \quad d \leq p, \quad e \geq q, \quad w_s \leq w_d, \quad \text{and} \quad w_t \geq w_e.$$

Assertion (4.5) follows in each case from an argument of the following form:

“Since $\mathbf{c}_{pq} \neq 0$, it follows from Lemma 4.1 that there is some $b \in B$ such that $\mathbf{c}_{de}(b \cdot x) \neq 0$. Therefore, $\mathbf{c}_{w_d w_e}(\dot{w}b \cdot x) \neq 0$. As $(\dot{w}b)^{-1}B \in C_{w^{-1}} \subseteq \mathcal{B}(x, H)$, we see that $E_{w_d w_e} \in H$. By Lemma 4.2, $E_{w_s w_t} \in H$.”

An exhaustive list of pairs (p, q) and corresponding pairs (d, e) and (s, t) is given in the table below.

(p, q)	(d, e)	(s, t)
(j, k)	(i, ℓ)	(k, j)
(j, i)	(i, j)	(k, i)
$(j, q), q \notin \{i, k\}$	(i, q)	(k, q)
(ℓ, j)	(k, ℓ)	(ℓ, k)
$(p, j), p \notin \{k, \ell\}$	(p, k)	(p, k)
(k, j)	(j, k)	(j, k)
$(k, q), q \neq j$	(j, q)	(j, q)
(ℓ, k)	(i, ℓ)	(ℓ, j)
$(p, k), p \notin \{j, \ell\}$	(p, ℓ)	(p, j)

We conclude that (4.5) holds and our proof is complete. $\square$

We can now prove Theorem 1.8, restated here for the reader's convenience.

Theorem 4.4 *Let $w \in S_n$. If some adjoint Hessenberg variety $\mathcal{B}(x, H) \subseteq SL_n(\mathbb{C})/B$ is equal to the Schubert variety X_w , then w avoids the pattern [4231].*

Proof It is well-known and straightforward to show that, given permutations σ and w , w contains the pattern σ if and only if w^{-1} contains σ^{-1} . Since [4231] is its own inverse, the theorem will follow once we show that if $\mathcal{B}(x, H) = X_{w^{-1}}$, then w avoids the pattern [4231].

If $\mathcal{B}(x, H) = X_{w^{-1}}$ then $C_{w^{-1}} \subseteq \mathcal{B}(x, H)$. If w contains the pattern [4231] then $\dot{\tau}^{-1}B \in \mathcal{B}(x, H)$, with τ as in Lemma 4.3. We observe that $\ell(\tau) > \ell(w)$, since τ is obtained from w by exchanging the positions of two letters that appear in ascending order in w . It follows that $\tau \not\leq_{\text{Br}} w$ and $\tau^{-1} \not\leq_{\text{Br}} w^{-1}$. This forces $\mathcal{B}(x, H) \neq X_{w^{-1}}$. $\square$

As mentioned in the introduction, the Marcus–Tardos Theorem guarantees that the number $|S_n(4231)|$ of permutations in S_n avoiding [4231] is bounded above by some exponential function of n . In particular,

$$\lim_{n \rightarrow \infty} \frac{|S_n(4231)|}{|S_n|} = 0.$$

So, for large n , Schubert varieties in $SL_n(\mathbb{C})/B$ that are also Hessenberg varieties in $SL_n(\mathbb{C})/B$ are extremely rare. To our knowledge, the best exponential bounds for $|S_n(4231)|$ known currently are (for all large enough n)

$$10.271^n \leq |S_n(4231)| \leq 13.5^n,$$

due to Bevan, Brignall, Elvey Price and Pantone in [5]. It is worth remarking that we do not know if there is any constant $c > 1$ such that the number of Schubert varieties in $SL_n(\mathbb{C})/B$ that are (equal to) Hessenberg varieties in $SL_n(\mathbb{C})/B$ is at least c^n .

We can use similar reasoning as that used in the proof of Lemma 4.3 to argue that certain Schubert varieties in $SL_4(\mathbb{C})/B$ are not equal to any Hessenberg variety in $SL_4(\mathbb{C})/B$. These arguments do not extend as easily to pattern avoidance results, but we include one example here for the sake of clarity. Example 4.6 below completely characterizes which Schubert varieties are equal to Hessenberg varieties in $SL_4(\mathbb{C})/B$.

Lemma 4.5 *Assume $n = 4$. If $w \in \{[4123], [2341]\}$ then no adjoint Hessenberg variety is equal to $X_{w^{-1}}$.*

Proof First, we consider $w = [4123]$. Note that permutations $\sigma = [2314]$ and $\tau = [1342]$ are the Bruhat-minimal elements of S_4 such that $\sigma \not\leq w$ and $\tau \not\leq w$. We argue that given $x \in \mathfrak{sl}_4(\mathbb{C})$ and Hessenberg space $H \subseteq \mathfrak{sl}_4(\mathbb{C})$ such that $\mathcal{B}(x, H) \subseteq SL_4(\mathbb{C})/B$ is a B -invariant Hessenberg variety with $X_{w^{-1}} \subseteq \mathcal{B}(x, H)$, then either $\dot{\sigma}^{-1}B \in \mathcal{B}(x, H)$ or $\dot{\tau}^{-1}B \in \mathcal{B}(x, H)$. This proves $X_{w^{-1}} \subsetneq \mathcal{B}(x, H)$ and thus $X_{w^{-1}}$ is not equal to a Hessenberg variety.

To begin, write

$$x = \sum_{(p,q) \in [4] \times [4]} \mathbf{c}_{pq} E_{pq}$$

where $\mathbf{c}_{pq} = \mathbf{c}_{pq}(x)$ for all p, q .

We define an integer $q_0 \in [5]$ as follows. If $\{q \in [4] : \mathbf{c}_{pq} \neq 0 \text{ for some } p \neq q\} = \emptyset$ (i.e. if x is a diagonal matrix) then we set $q_0 = 5$. Otherwise, set $q_0 := \min\{q \in [4] : \mathbf{c}_{pq} \neq 0 \text{ for some } p \neq q\}$.

Suppose first that $q_0 = 1$ or $q_0 = 2$. By Lemma 4.1, there exists $b \in B$ such that $\mathbf{c}_{12}(b \cdot x) \neq 0$ and thus $\mathbf{c}_{41}(\dot{w} \cdot b \cdot x) \neq 0$. Since $X_{w^{-1}} \subseteq \mathcal{B}(x, H)$, we know $b^{-1}\dot{w}^{-1}B \in \mathcal{B}(x, H)$ and therefore $E_{41} \in H$. It follows that $H = \mathfrak{sl}_4(\mathbb{C})$. So, $\mathcal{B}(x, H) = SL_4(\mathbb{C})/B$ contains both $\dot{\sigma}^{-1}B \in \mathcal{B}(x, H)$ and $\dot{\tau}^{-1}B \in \mathcal{B}(x, H)$ in this case.

We assume now that $q_0 \geq 3$. This implies, in particular, that

$$\mathbf{c}_{21} = \mathbf{c}_{31} = \mathbf{c}_{41} = \mathbf{c}_{12} = \mathbf{c}_{32} = \mathbf{c}_{42} = 0,$$

hence

$$x = \mathbf{c}_{11}E_{11} + \mathbf{c}_{22}E_{22} + \sum_{(p,q) \in [4] \times \{3,4\}} \mathbf{c}_{pq}E_{pq}. \quad (4.6)$$

If $\mathbf{c}_{11} \neq \mathbf{c}_{22}$ then there exists $b \in B$ such that $\mathbf{c}_{12}(b \cdot x) \neq 0$. (This is easily verified by direct calculation.) Since $\mathcal{B}(x, H)$ is B -invariant, we see that $\mathcal{B}(x, H) = \mathcal{B}(b \cdot x, H)$. Thus, if $\mathbf{c}_{11} \neq \mathbf{c}_{22}$ we can argue $\mathcal{B}(x, H) = SL_4(\mathbb{C})/B$ as above. So, we assume $\mathbf{c}_{11} = \mathbf{c}_{22}$ for the remainder of the proof.

If $q_0 = 3$, then $\mathbf{c}_{p3} \neq 0$ for some $p \in \{1, 2, 4\}$. By Lemma 4.1, there exists $b \in B$ such that $\mathbf{c}_{13}(b \cdot x) \neq 0$ and therefore $\mathbf{c}_{42}(\dot{w} \cdot b \cdot x) \neq 0$. Since $b^{-1}\dot{w}^{-1}B \in \mathcal{B}(x, H)$, we get $E_{42} \in H$. Thus,

$$\mathfrak{sl}_4(\mathbb{C}) \cap \mathbb{C}\{E_{pq} : p \in [4], q \in \{2, 3, 4\}\} \subseteq H.$$

Since we are working under the assumptions of (4.6) above and $\mathbf{c}_{11} = \mathbf{c}_{22}$, we have

$$\dot{\tau} \cdot x - x \in \mathfrak{sl}_4(\mathbb{C}) \cap \mathbb{C}\{E_{pq} : p \in [4], q \in \{2, 3, 4\}\} \subseteq H$$

and since $x \in H$ (because $eB \in X_{w^{-1}} \subseteq \mathcal{B}(x, H)$), we get $\dot{\tau} \cdot x \in H$ in this case.

Now assume $q_0 \geq 4$. So, $\mathbf{c}_{pq} = 0$ for all $q \in \{1, 2, 3\}$ and $p \neq q$. Thus (4.6) becomes:

$$x = \mathbf{c}_{11}E_{11} + \mathbf{c}_{11}E_{22} + \mathbf{c}_{33}E_{33} + \sum_{p \in [4]} \mathbf{c}_{p4}E_{p4} \quad (4.7)$$

As above, a direct computation shows that if $\mathbf{c}_{11} \neq \mathbf{c}_{33}$ then there exists $b \in B$ such that $\mathbf{c}_{13}(b \cdot x) \neq 0$. The arguments of the previous paragraph then imply $\dot{\tau}^{-1}B \in$

$\mathcal{B}(b \cdot x, H) = \mathcal{B}(x, H)$. We therefore assume that $\mathbf{c}_{11} = \mathbf{c}_{33}$ for the remainder of the proof.

If $q_0 = 4$ then $\mathbf{c}_{p4} \neq 0$ for some $p \in \{1, 2, 3\}$. By Lemma 4.1 there exists $b \in B$ such that $\mathbf{c}_{14}(b \cdot x) \neq 0$ and therefore $\mathbf{c}_{43}(\dot{w} \cdot b \cdot x) \neq 0$. This implies $E_{43} \in H$ and thus

$$\mathfrak{sl}_4(\mathbb{C}) \cap \mathbb{C}\{E_{pq} : p \in [4], q \in \{3, 4\}\} \subseteq H.$$

The assumptions (4.7) and $\mathbf{c}_{11} = \mathbf{c}_{33}$ now imply $\dot{\sigma} \cdot x - x \in H$ and thus $\dot{\sigma}^{-1}B \in \mathcal{B}(x, H)$ in this case.

Finally, if $q_0 = 5$ then we must have

$$x = \mathbf{c}_{11}E_{11} + \mathbf{c}_{11}E_{22} + \mathbf{c}_{11}E_{33} + \mathbf{c}_{44}E_{44}.$$

If $\mathbf{c}_{11} \neq \mathbf{c}_{44}$ then there exists $b \in B$ such that $\mathbf{c}_{14}(b \cdot x) \neq 0$ and the argument of the previous paragraph implies $\dot{\sigma}^{-1}B \in \mathcal{B}(b \cdot x, H) = \mathcal{B}(x, H)$. We may therefore assume $\mathbf{c}_{11} = \mathbf{c}_{44}$. Since $x \in \mathfrak{sl}_4(\mathbb{C})$, this is possible if and only if $x = 0$, which in turn implies $\mathcal{B}(x, H) = \mathcal{B}$ and so $\dot{\sigma}^{-1}B, \dot{\tau}^{-1}B \in \mathcal{B}(x, H)$.

The proof for $w = [2341]$ follows from similar reasoning using $\sigma = [3124]$ and $\tau = [1423]$. Exchanging the roles of rows and columns in the proof above yields the desired result. For brevity, we omit the details here. $\square$

Example 4.6 Let $n = 4$. In this case, $X_{w^{-1}}$ is an adjoint Hessenberg variety if and only if

$$w \notin \{[4231], [2341], [4123], [1342], [3124]\}.$$

Indeed, the first three elements of the set above are exactly the patterns from Theorem 4.4 and Lemma 4.5. The proof that $X_{w^{-1}}$ is not equal to an adjoint Hessenberg variety when $w = [1342]$ or $w = [3124]$ is similar to that of Lemma 4.5. Finally, the remaining 19 permutations can each be realized as an adjoint Hessenberg variety. Indeed, 12 of these permutations have the property that $X_{w^{-1}}$ can be realized as highest weight adjoint Hessenberg varieties by Proposition 3.1. Of the remaining 12 permutations that are not maximal length coset representatives for $S_4/\langle s_2 \rangle$, five appear above and the remaining 7 can be realized as Hessenberg varieties using the data from Table 1. In this table we denote H as a matrix with starred entries; this means that

$$H = \mathfrak{sl}_4(\mathbb{C}) \cap \mathbb{C}\{E_{ij} : \text{entry } (i, j) \text{ contains a } *\}.$$

Given Theorem 1.8, it is natural to ask whether the property that X_w is an adjoint Hessenberg variety is characterized by pattern avoidance. This is not the case, as the next example shows.

Example 4.7 Let $n = 5$ and $w = [13425] \in S_5$. A direct computation shows when

$$H = \mathfrak{sl}_5(\mathbb{C}) \cap \mathbb{C}\{E_{12}, E_{13}, E_{14}, E_{15}, E_{25}, E_{35}\}$$

Table 1 Answer to the equality question for all Schubert varieties X_{w-1} where $w \in S_4$ is such that $\ell(ws_2) > \ell(w)$

$w \in S_4$ such that $\ell(ws_2) > \ell(w)$	$X_{w-1} = \mathcal{B}(x, H)$ in $SL_4(\mathbb{C})/B$?	x	H
[1234]	yes	$E_{12} + E_{24}$	$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
[1243]	yes	$E_{12} + E_{24}$	$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
[1342]	no	—	—
[2134]	yes	$E_{13} + E_{34}$	$\begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$
[2143]	yes	$E_{13} + E_{24}$	$\begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
[2341]	no	—	—
[3124]	no	—	—
[3142]	yes	$E_{13} + E_{24}$	$\begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$
[3241]	yes	$E_{13} + E_{24}$	$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$
[4123]	no	—	—
[4132]	yes	$E_{13} + E_{24}$	$\begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$
[4231]	no	—	—

we get that $\mathcal{B}(E_{12} + E_{25}, H) = X_{w-1}$. Therefore, although w must avoid the pattern [1342] when $n = 4$ in order to be realized as an adjoint Hessenberg variety, this is not the case when $n = 5$.

As we have seen in Section 3 every highest weight Hessenberg variety is B -invariant. The question of whether or not an arbitrary adjoint Hessenberg variety is B -invariant is much more nuanced than in the highest weight case. Indeed, Examples 4.6 and 4.7 exhibit B -invariant adjoint Hessenberg varieties that are not highest weight Hessenberg varieties. This motivates the following open question.

Question 4.8 Given an adjoint Hessenberg variety $\mathcal{B}(x, H)$ such that $x \notin \mathbb{C}\{E_\theta\}$, what conditions on x and Hessenberg space H guarantee that $\mathcal{B}(x, H)$ is B -invariant? In the case that $\mathcal{B}(x, H)$ is B -invariant, what conditions guarantee it is irreducible?

5 The Isomorphism Question for Type A Adjoint Hessenberg Varieties

Our goal is to prove the following result, which is a restatement of Theorem 1.9 above.

Theorem 5.1 Suppose $n \geq 6$. Assume that $G = GL_n(\mathbb{C})$ or $G = SL_n(\mathbb{C})$ and B is the Borel subgroup of G consisting of upper triangular matrices. Let w_0 be the longest element of the Weyl group $W = S_n$, and for $i \in [n-1]$, let $s_i \in W$ be the transposition $(i, i+1)$. If $3 \leq i \leq n-3$, then there do not exist $x \in \mathfrak{g}$ and subspace $H \subseteq \mathfrak{g}$ such that $[\mathfrak{b}, H] \subseteq H$ and $\mathcal{B}(x, H)$ is isomorphic with $X_{s_i w_0}$.

Assume that we have fixed $n \geq 4$. As each Schubert variety $X_{s_i w_0}$ is irreducible of codimension one in the flag variety $\mathcal{B}$, our plan is to study closely the structure of adjoint Hessenberg varieties of codimension one in $\mathcal{B}$. In particular, we study the Euler characteristic of irreducible adjoint Hessenberg varieties of codimension one. We will see that in all but two cases, if $3 \leq i \leq n-3$, no such variety has the same Euler characteristic as $X_{s_i w_0}$. The remaining two cases will be handled by examining Betti numbers.

A few remarks are in order. First, Proposition 3.1 tells us that for all $n \leq 3$, every Schubert variety is equal to an adjoint Hessenberg variety and that for all n , $X_{s_1 w_0}$ and $X_{s_{n-1} w_0}$ are adjoint Hessenberg varieties. Second, Theorem 5.1 does not resolve the isomorphism question for the Schubert varieties $X_{s_2 w_0}$ and $X_{s_{n-2} w_0}$ whenever $n \geq 4$. In fact, the existence of Hessenberg varieties in $\mathcal{B}$ with the same Betti numbers as $X_{s_2 w_0}$ and $X_{s_{n-2} w_0}$ renders the methods used to prove Theorem 5.1 useless. Recall that $\theta = \varepsilon_1 - \varepsilon_n$ is the highest root in Φ . We define the Hessenberg space

$$H(\overline{-\theta}) := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi \setminus \{-\theta\}} \mathfrak{g}_\alpha.$$

As is standard, we define the Poincaré polynomial of a space X as

$$\text{Poin}(X, q) := \sum_{i \geq 0} \dim_{\mathbb{C}} H^i(X; \mathbb{C}) q^i.$$

Example 5.2 We observe first for use here and later that for $1 \leq i \leq n-1$, $X_{s_i w_0}$ and $X_{s_{n-i} w_0}$ are isomorphic. Indeed, it follows from [25, Theorem 1.3] that for any $y \in S_n$, $X_{w_0 y w_0} \cong X_y$, and direct calculation shows that $w_0(s_i w_0)w_0 = s_{n-i} w_0$. Let $n \geq 4$ and suppose $v = s_2 w_0$ or $v = s_{n-2} w_0$. For either choice for v one can use the Tableau Criterion of Theorem 2.1 to describe efficiently those $y \in S_n$ that do not live below v in the Bruhat order and then calculate

$$\text{Poin}(X_v, \sqrt{q}) = [n-2]_q! \left([n]_q [n-1]_q - q^{2n-3} - q^{2n-4} \right). \quad (5.1)$$

(Here $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = \prod_{j=1}^n [j]_q$.) Using either [28, Theorem 1.1] or [23, Theorem 5.4 and Corollary 5.5] along with (5.1), one can verify that if $x = E_{1n-1} + E_{2n}$ or $x = E_{12} + E_{2n}$ then $\text{Poin}(X_v, q) = \text{Poin}(\mathcal{B}(x, H(\overline{-\theta})), q)$. We outline such a verification when $x = E_{12} + E_{2n}$. In the language of [28], $\lambda_x = (3, 1, \dots, 1)$. Given a filling T of the (English) Young diagram of shape λ_x with the numbers 1 to n (without repetitions), let $w_T \in S_n$ be the permutation obtained from reading the entries in T first from bottom to top in the first column and then from left to right in the first row. Define sets

$$Y_n := \{w \in S_n : (w_{n-2}, w_{n-1}) = (n, 1) \text{ or } (w_{n-1}, w_n) = (n, 1)\}$$

and

$$Z_n := \{w \in S_n \setminus Y_n : 1 \in \{w_{n-1}, w_n\}\}.$$

According to [28, Theorem 1.1], $\mathcal{B}(x, H(\overline{-\theta}))$ admits an affine paving with cells indexed by those T such that $w_T \in S_n \setminus Y_n$. Moreover, if $w_T \in S_n \setminus Y_n$ then the dimension of the cell indexed by T is $\ell(w_T)$ if $w_T \notin Z_n$. This dimension is $\ell(w_T) - 1$ if $w_T \in Z_n$. We conclude that

$$\text{Poin}(\mathcal{B}(x, H(\overline{-\theta})), \sqrt{q}) = \sum_{w \in S_n \setminus (Y_n \cup Z_n)} q^{\ell(w)} + \frac{1-q}{q} \sum_{w \in Z_n} q^{\ell(w)}.$$

Our claim now follows from direct calculations, using the identity $\sum_{0 \leq a < b \leq m} q^{a+b} = q \binom{m+1}{2}_q$. A similar argument works when $x = E_{1n-1} + E_{2n}$.

We do not know if either of $X_{s_2 w_0}$ and $X_{s_{n-2} w_0}$ is isomorphic to either of the Hessenberg varieties appearing in Example 5.2.

We now commence our study of codimension one adjoint Hessenberg varieties. Our main goal is to prove Proposition 1.10, restated below for the reader's convenience.

Proposition 5.3 *If $x \in \mathfrak{g}$ and $H \subseteq \mathfrak{g}$ is a Hessenberg space such that $\mathcal{B}(x, H)$ is irreducible and has codimension one in $\mathcal{B}$, then the Euler characteristic $\chi(\mathcal{B}(x, H))$ is divisible by $(n-2)!$.*

Proposition 5.3 will be combined with Lemma 5.4 below to reduce the proof of Theorem 5.1 to the examination of two special cases.

Lemma 5.4 *Let $S = \{s_1, \dots, s_{n-1}\}$ be the set of simple reflections in S_n . For $i \in [n-1]$, let $W(i)$ be the subgroup of S_n generated by $S \setminus \{s_i\}$. Then*

$$\chi(X_{s_i w_0}) = n! - |W(i)| = n! - i!(n-i)!.$$

Proof It follows directly from the definitions that $W(i) \cong S_i \times S_{n-i}$ so $|W(i)| = i!(n-i)!$. The bijection on S_n sending w to ww_0 is an anti-automorphism of the Bruhat order (see for example [4, Proposition 2.3.4]). It follows that the number of

Schubert cells C_u not contained in $X_{s_i w_0}$ is equal to the number of $u \in S_n$ such that $s_i \not\leq u$, which is the number of elements of $W(i)$. $\square$

The remainder of this section consists of the proof of Proposition 5.3 followed by the proof of Theorem 5.1. The key step is the next result, which we will prove after stating a preliminary lemma.

Lemma 5.5 *Assume that $x \in \mathfrak{g}$ and that $H \subseteq \mathfrak{g}$ is a Hessenberg space. If $\mathcal{B}(x, H)$ has dimension $\dim \mathcal{B} - 1 = \binom{n}{2} - 1$, then at least one of*

- $H = H(\overline{-\theta})$, or
- x has an eigenspace of dimension $n - 1$

must hold.

In order to prove Lemma 5.5, we define two more “large” Hessenberg spaces. Set $\beta_1 := \varepsilon_{n-1} - \varepsilon_1$ and $\beta_2 := \varepsilon_n - \varepsilon_2$. For $i \in \{1, 2\}$ we define the Hessenberg space

$$H(\overline{\beta_i}) := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi \setminus \{-\theta, \beta_i\}} \mathfrak{g}_\alpha.$$

Assume temporarily that $G = GL_n(\mathbb{C})$, $x \in \mathfrak{g}$ and $H \subseteq \mathfrak{g}$ is a Hessenberg space. The map $\iota : SL_n(\mathbb{C})/(B \cap SL_n(\mathbb{C})) \rightarrow \mathcal{B}$ sending $g(B \cap SL_n(\mathbb{C}))$ to gB is an isomorphism. Note that ι factors through the identification of each of G/B and $SL_n(\mathbb{C})/(B \cap SL_n(\mathbb{C}))$ with the variety of full flags in $\mathbb{C}^n$. Thus, if $x \in \mathfrak{sl}_n(\mathbb{C})$ then $\mathcal{B}(x, H) = \iota(\mathcal{B}(x, H \cap \mathfrak{sl}_n(\mathbb{C})))$, while if $x \notin \mathfrak{sl}_n(\mathbb{C})$ and $H \subseteq \mathfrak{sl}_n(\mathbb{C})$ then $\mathcal{B}(x, H) = \emptyset$. Therefore, the next assumption is harmless.

Assumption 5.6 If $G = GL_n(\mathbb{C})$ and $H \subseteq \mathfrak{g}$ is a Hessenberg space, then $H \not\subseteq \mathfrak{sl}_n(\mathbb{C})$.

We continue now under our original assumption that $G = GL_n(\mathbb{C})$ or $G = SL_n(\mathbb{C})$. Given Assumption 5.6, the next lemma holds for Hessenberg spaces in either $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

Lemma 5.7 *Let $H \subseteq \mathfrak{g}$ be a Hessenberg space satisfying Assumption 5.6.*

1. *If $\dim_{\mathbb{C}} \mathfrak{g}/H = 1$ then $H = H(\overline{-\theta})$.*
2. *If $\dim_{\mathbb{C}} \mathfrak{g}/H = 2$ then $H = H(\overline{\beta_i})$ for some $i \in \{1, 2\}$.*
3. *If $\dim_{\mathbb{C}} \mathfrak{g}/H > 2$ then $H \subset H(\overline{\beta_i})$ for some $i \in \{1, 2\}$.*

Proof If $\mathfrak{g}_\gamma \subseteq H$, then H contains the Demazure module in $\mathfrak{g}$ generated by the root vector E_γ so $\mathfrak{g}_\beta \subseteq H$ for all $\beta \geq \gamma$ by (3.1). Now if $\pi_{-\theta}(H) \neq 0$ then $\mathfrak{g}_{-\theta} \subseteq H$ and, as $E_{n1} \in \mathfrak{g}_{-\theta}$ generates the Demazure module $\mathfrak{g} \cap \mathfrak{sl}_n(\mathbb{C})$, we have $H = \mathfrak{g}$. Therefore every proper Hessenberg subspace of $\mathfrak{g}$ is contained in $H(\overline{-\theta})$ and (1) follows. Moreover, if $\dim_{\mathbb{C}} \mathfrak{g}/H = 2$ then $H \subset H(\overline{-\theta})$. For all $\gamma \in \Phi \setminus \{-\theta\}$, we have $\gamma \geq \beta_1$ or $\gamma \geq \beta_2$. Thus $E_{n-1,1} \in \mathfrak{g}_{\beta_1}$ and $E_{n2} \in \mathfrak{g}_{\beta_2}$ together generate the B -module $H(\overline{-\theta}) \cap \mathfrak{sl}_n(\mathbb{C})$, and assertion (2) follows. The poset of Hessenberg spaces in $\mathfrak{g}$ satisfying Assumption 5.6, ordered by inclusion, is graded by dimension. Now (3) follows from (2). $\square$

For $x \in \mathfrak{g}$ and $c \in \mathbb{C}$, set

$$\text{mult}_c(x) := \dim_{\mathbb{C}} \ker(x - cI).$$

Lemma 5.5 follows directly from Lemma 5.7 and the next result. Indeed, if $H \neq H(\overline{-\theta})$ then by Lemma 5.7 we have $H \subseteq H(\overline{\beta_i})$ for some $i \in \{1, 2\}$. Since $\dim \mathcal{B}(x, H) \leq \dim \mathcal{B}(x, H(\overline{\beta_i}))$, Proposition 5.8 below implies that $\max_{c \in \mathbb{C}} (\text{mult}_c(x)) \geq n - 1$ whenever $\dim \mathcal{B}(x, H) = \binom{n}{2} - 1$. As $\mathcal{B}(x, H) = \emptyset$ or $\mathcal{B}(x, H) = \mathcal{B}$ whenever x is scalar, Lemma 5.5 follows.

Proposition 5.8 *For $i \in \{1, 2\}$ and $x \in \mathfrak{g}$, $\mathcal{B}(x, H(\overline{\beta_i}))$ has dimension at most*

$$\max \left(\binom{n}{2} - 2, \binom{n-1}{2} - 1 + \max_{c \in \mathbb{C}} (\text{mult}_c(x)) \right).$$

Proof We claim first that it suffices to consider the case $i = 1$. Let $\sigma_0 \in GL_n(\mathbb{C})$ be the involution mapping e_i to e_{n+1-i} for all $i \in [n]$. Define an automorphism $r : G \rightarrow G$ by

$$r(g) = \sigma_0(g^{-1})^{\text{tr}} \sigma_0.$$

We observe that if $b \in B$ then $(b^{-1})^{\text{tr}}$ is lower triangular, and thus $r(b) \in B$. Therefore the map $\bar{r} : \mathcal{B} \rightarrow \mathcal{B}$ sending gB to $r(g)B$ is a well defined automorphism of $\mathcal{B}$. A direct calculation shows that

$$\bar{r}(\mathcal{B}(x, H(\overline{\beta_2}))) = \mathcal{B}(\sigma_0 x^{\text{tr}} \sigma_0, H(\overline{\beta_1})).$$

Since x and $\sigma_0 x^{\text{tr}} \sigma_0$ are conjugate, we see that the dimensions of $\mathcal{B}(x, H(\overline{\beta_1}))$ and $\mathcal{B}(x, H(\overline{\beta_2}))$ are equal. Our claim follows.

Noting that $h(H(\overline{\beta_1})) = (n-2, n, \dots, n)$, we consider the map $\pi : \mathcal{B}(x, H(\overline{\beta_1})) \rightarrow \mathbb{P}^{n-1}$ given by

$$\pi(\mathcal{V}_{\bullet}) = V_1.$$

As π is the restriction to $\mathcal{B}(x, H(\overline{\beta_1}))$ of the projection of the product of Grassmannians $\prod_{k=1}^{n-1} \mathbb{G}(k, n)$ onto its first factor, π is a regular map. Let $\mathcal{C}$ be an irreducible component of $\mathcal{B}(x, H(\overline{\beta_1}))$ and let $Y \in \pi(\mathcal{C})$. According to [16, Corollary 11.13],

$$\dim \mathcal{C} \leq \dim \pi(\mathcal{C}) + \dim \pi^{-1}(Y). \quad (5.2)$$

The fiber $\pi^{-1}(Y)$ consists of all flags $\mathcal{V}_{\bullet}$ such that $V_1 = Y$ and $Y + xY \subseteq V_{n-2}$. If Y is not x -invariant, then $Y + xY$ is two-dimensional. In this case, we define the (surjective) map ζ from $\pi^{-1}(Y)$ to the Grassmannian of $(n-4)$ -dimensional subspaces of $\mathbb{C}^n/(Y + xY)$ (a projective variety) sending $\mathcal{V}_{\bullet}$ to $V_{n-2}/(Y + xY)$. For every such subspace $Z' = Z/(Y + xY)$, $\zeta^{-1}(Z')$ is isomorphic to the product

$\text{Flag}_{n-3} \times \mathbb{P}^1$. Indeed, $\zeta^{-1}(Z')$ consists of all $\mathcal{V}_\bullet$ such that $V_1 = Y$ and $V_{n-2} = Z$. Choosing $V_2, \dots, V_{n-3}$ is equivalent to choosing a flag in V_{n-2}/V_1 , and choosing V_{n-1} is equivalent to choosing a 1-dimensional subspace of $\mathbb{C}^n/V_{n-2}$. It follows from [16, Theorem 11.14] that $\pi^{-1}(Y)$ is irreducible. We may apply [16, Theorem 11.12] and well-known facts about Grassmannians and flag varieties to get

$$\begin{aligned} \dim \pi^{-1}(Y) &= \dim \mathbb{G}(n-4, n-2) + \dim(\text{Flag}_{n-3} \times \mathbb{P}^1) \\ &= 2(n-4) + \binom{n-3}{2} + 1. \end{aligned} \quad (5.3)$$

Combining (5.2) and (5.3), we get

$$\dim \mathcal{C} \leq n-1 + 2(n-4) + \binom{n-3}{2} + 1 = \binom{n}{2} - 2,$$

and the claim of the Proposition follows in this case.

We are left with the case where every $Y \in \pi(\mathcal{C})$ is x -invariant. We consider the map from $\mathcal{C}$ to $\mathbb{C}$ sending $\mathcal{V}_\bullet$ to the eigenvalue of x on V_1 . As $\mathcal{C}$ is irreducible and therefore connected, and the set of eigenvalues of x is discrete, there exists some eigenvalue c of x such that every $Y \in \pi(\mathcal{C})$ is spanned by an element of $\ker(x - cI)$. It follows that

$$\dim \pi(\mathcal{C}) \leq \text{mult}_c(x) - 1. \quad (5.4)$$

For any $Y \in \pi(\mathcal{C})$, $\pi^{-1}(Y)$ consists of all flags $\mathcal{V}_\bullet$ such that $V_1 = Y$. It follows that $\pi^{-1}(Y)$ is isomorphic with Flag_{n-1} . Combining (5.2) and (5.4), we get

$$\dim \mathcal{C} \leq \binom{n-1}{2} + \text{mult}_c(x) - 1.$$

This concludes the proof. $\square$

With Lemma 5.5 in hand, we analyze the two cases arising from its conclusion starting with the case in which x has an eigenspace of dimension $n-1$. Recall from Example 5.2 that $X_{s_1 w_0}$ and $X_{s_{n-1} w_0}$ are isomorphic.

Lemma 5.9 *Assume that $x \in \mathfrak{g}$ has an eigenspace of dimension $n-1$ and $H \subseteq \mathfrak{g}$ is a Hessenberg space. If $\mathcal{B}(x, H)$ is irreducible of codimension one in $\mathcal{B}$, then $\mathcal{B}(x, H)$ is isomorphic with $X_{s_1 w_0}$ and $X_{s_{n-1} w_0}$.*

Proof We observe that if x has an eigenspace of dimension $n-1$, then x is either semisimple or conjugate to $cI + E_{1n}$ for some scalar c . We consider first the case where x is semisimple. We may assume without loss of generality that there exist distinct constants c, d such that $xe_i = ce_i$ for $i \in [n-1]$, and $xe_n = de_n$. If $H = \mathfrak{g}$, then $\mathcal{B}(x, H) = \mathcal{B}$. So, we assume that $H \neq \mathfrak{g}$. By Lemma 5.7(1), $H \subseteq H(-\bar{\theta})$ and so $\mathcal{B}(x, H) \subseteq \mathcal{B}(x, H(-\bar{\theta}))$. We will show that $\mathcal{B}(x, H(-\bar{\theta}))$ has two irreducible components, one of which is $X_{s_{n-1} w_0}$ with the other isomorphic to $X_{s_1 w_0}$. The claim of the lemma in this case then follows immediately.

Since $h(H(\overline{-\theta})) = (n-1, n, \dots, n)$, we see that $\mathcal{B}(x, H(\overline{-\theta}))$ consists of all flags $\mathcal{V}_\bullet$ such that $xV_1 \subset V_{n-1}$. In particular, if V_1 is x -invariant, then $\mathcal{V}_\bullet \in \mathcal{B}(x, H(\overline{-\theta}))$. Define

$$\mathcal{C}_1 := \{\mathcal{V}_\bullet : V_1 \subset \mathbb{C}\{e_i : i \in [n-1]\}\}.$$

As $\mathbb{C}\{e_i : i \in [n-1]\} = \ker(x - cI)$, we see that $\mathcal{C}_1 \subseteq \mathcal{B}(x, H(\overline{-\theta}))$. By the flag description of Schubert varieties and the fact that $s_{n-1}w_0 = [n-1 \ n \ n-2 \ \dots \ 2 \ 1]$, it follows that

$$\mathcal{C}_1 = X_{s_{n-1}w_0}.$$

Now set

$$\mathcal{C}_2 := \{\mathcal{V}_\bullet : e_n \in V_{n-1}\}.$$

We will show that $\mathcal{B}(x, H(\overline{-\theta})) = \mathcal{C}_1 \cup \mathcal{C}_2$, and that, with σ_0 the involution mapping e_i to e_{n+1-i} as above, $\mathcal{C}_2 = \sigma_0(X_{s_1w_0})$, thereby completing our examination of the case where x is semisimple. Let $\mathcal{V}_\bullet \in \mathcal{C}_2$. If V_1 is x -invariant, then $\mathcal{V}_\bullet \in \mathcal{B}(x, H(\overline{-\theta}))$. If V_1 is not x -invariant, then $V_1 = \mathbb{C}\{e_n + y\}$ for some nonzero $y \in \mathbb{C}\{e_i : i \in [n-1]\}$. Now $xV_1 = \mathbb{C}\{de_n + cy\}$. If $c = 0$ then $d \neq 0$ and $xV_1 = \mathbb{C}\{e_n\} \subset V_{n-1}$. If $c \neq 0$ then

$$V_1 + xV_1 = \mathbb{C}\left\{e_n + y, e_n + y - \frac{1}{c}(de_n + cy)\right\} = \mathbb{C}\{e_n + y, e_n\},$$

the second equality holding since $d \neq c$. Now $xV_1 \subset V_1 + xV_1 \subset V_{n-1}$. In either case, $\mathcal{V}_\bullet \in \mathcal{B}(x, H(\overline{-\theta}))$ so $\mathcal{C}_2 \subseteq \mathcal{B}(x, H(\overline{-\theta}))$.

Conversely, say $\mathcal{V}_\bullet \in \mathcal{B}(x, H(\overline{-\theta})) \setminus \mathcal{C}_1$. If $V_1 = \mathbb{C}\{e_n\}$, then $e_n \in V_{n-1}$. Otherwise, $V_1 = \mathbb{C}\{e_n + y\}$ for some nonzero $y \in \mathbb{C}\{e_i : i \in [n-1]\}$. Arguing as in the paragraph just above, we see that $e_n \in V_1 + xV_1$ and therefore $e_n \in V_{n-1}$. We conclude that $\mathcal{V}_\bullet \in \mathcal{C}_2$ and $\mathcal{B}(x, H(\overline{-\theta})) = \mathcal{C}_1 \cup \mathcal{C}_2$ as claimed.

Now $s_1w_0 = [nn-1 \ \dots \ 3 \ 1 \ 2]$. By the flag description of Schubert varieties, $X_{s_1w_0} = \{\mathcal{V}_\bullet : e_1 \in V_{n-1}\}$. Now

$$\sigma_0(X_{s_1w_0}) = \sigma_0(\{\mathcal{V}_\bullet : e_1 \in V_{n-1}\}) = \{\mathcal{F}_\bullet : e_n \in F_{n-1}\} = \mathcal{C}_2,$$

the second equality following from $\sigma_0e_1 = e_n$.

It remains to examine the case where x is conjugate to $cI + E_{1n}$ for some $c \in \mathbb{C}$ (and $H \subseteq \mathfrak{g}$ is an arbitrary Hessenberg space). We claim that in this case, either $\mathcal{B}(x, H) = \emptyset$ or $\mathcal{B}(x, H) \simeq \mathcal{B}(E_{1n}, H)$. Without loss of generality, we may assume $x = cI + E_{1n}$. If $\mathcal{B}(x, H) \neq \emptyset$ then there is some $g \in G$ such that $g^{-1} \cdot x \in H$. Write $g = b_1wb_2$ with $b_1, b_2 \in B$ and $w \in S_n$. We see that $\dot{w}^{-1}b_1^{-1} \cdot x \in b_2 \cdot H = H$. Now

$$\dot{w}^{-1}b_1^{-1} \cdot x = cI + \dot{w}^{-1}b_1^{-1} \cdot E_{1n} = cI + dE_{w_1^{-1}w_n^{-1}}$$

for some $d \in \mathbb{C}^*$. In particular, we see that $\pi_{\varepsilon_{w_1^{-1}} - \varepsilon_{w_n^{-1}}}(H) \neq 0$, hence $E_{w_1^{-1}w_n^{-1}} \in H$. It follows that $cI \in H$. For arbitrary $g \in G$, the matrix $g^{-1} \cdot x = cI + g^{-1} \cdot E_{1n}$ lies in H if and only if $g^{-1} \cdot E_{1n} \in H$. The claim follows.

Now $\mathcal{B}(E_{1n}, H)$ is a highest weight Hessenberg variety for the adjoint representation and is therefore a union of Schubert cells. In particular, if $\mathcal{B}(x, H)$ is irreducible and of codimension one, then $\mathcal{B}(x, H) \simeq X_{s_i w_0}$ for some $i \in [n-1]$. We observe that if $1 < i < n-1$ then $s_i w_0$ maps 1 to n and n to 1 and so lies in the same coset of the stabilizer of $\theta = \varepsilon_1 - \varepsilon_n$ as w_0 . On the other hand, neither $s_1 w_0$ nor $s_{n-1} w_0$ lies in the coset and so both are longest representatives of the coset containing them. The lemma follows from Proposition 3.1 and Theorem 3.7 (and Lemma 5.4). $\square$

We record the novel geometric results from the proof of Lemma 5.9 below.

Corollary 5.10 *Let x be a semisimple matrix such that there exists distinct constants c, d with $xe_i = ce_i$ for $i \in [n-1]$ and $xe_n = de_n$. The adjoint Hessenberg variety $\mathcal{B}(x, H(\overline{-\theta}))$ is a union of two irreducible components, one equal to the Schubert variety $X_{s_{n-1}w_0}$ and the other isomorphic to the Schubert variety $X_{s_1w_0}$.*

The following lemma, when combined with Lemmas 5.4, 5.5 and 5.9, will complete the proof of Proposition 5.3.

Lemma 5.11 *For every $x \in \mathfrak{g}$, the Euler characteristic $\chi(\mathcal{B}(x, H(\overline{-\theta})))$ is divisible by $(n-2)!$.*

Proof We assume without loss of generality that $x = x_s + x_n$ with x_s diagonal and x_n an upper triangular, nilpotent matrix satisfying the assumptions of [23, Corollary 4.9]. Let $\mathcal{C}(x, H(\overline{-\theta}))$ be the set of all $w \in W$ such that the Schubert cell C_w has nonempty intersection with $\mathcal{B}(x, H(\overline{-\theta}))$. By [23, Theorem 5.4] and its proof, the affine spaces $\mathcal{B}(x, H(\overline{-\theta})) \cap C_w$ with $w \in \mathcal{C}(x, H)$ determine an affine paving of $\mathcal{B}(x, H(\overline{-\theta}))$. So,

$$\chi(\mathcal{B}(x, H(\overline{-\theta}))) = \dim_{\mathbb{C}} H^*(\mathcal{B}(x, H(\overline{-\theta}))) = |\mathcal{C}(x, H(\overline{-\theta}))|.$$

Now by [23, Proposition 3.7], $w \in \mathcal{C}(x, H(\overline{-\theta}))$ if and only if $Ad(\dot{w}^{-1})(x_n) \in H(\overline{-\theta})$. We now write

$$x_n = \sum_{j < k} c_{jk} E_{j,k}$$

for some $c_{jk} \in \mathbb{C}$. It follows that $\mathcal{C}(x, H(\overline{-\theta}))$ fails to contain exactly those $w \in W$ such that $w(-\theta) = \varepsilon_j - \varepsilon_k$ for pairs $j < k$ satisfying $c_{jk} \neq 0$. In particular, $W \setminus \mathcal{C}(x, H(\overline{-\theta}))$ is a union of cosets of the stabilizer of $-\theta$ in W . As this stabilizer is isomorphic with S_{n-2} , the lemma follows. $\square$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1 Observe that $X_{s_i w_0}$ is irreducible. So, if there is some adjoint Hessenberg variety $\mathcal{B}(x, H) \subseteq \mathcal{B}$ isomorphic with $X_{s_i w_0}$, then $i!(n-i)!$ is divisible by $(n-2)!$ by Proposition 5.3 and Lemma 5.4. However, if $n \geq 9$ and $3 \leq i \leq n-3$, then $0 < i!(n-i)! < (n-2)!$. Moreover, inspection shows that if $6 \leq n \leq 8$ and $3 \leq i \leq n-3$, then $(n-2)!$ does not divide $i!(n-i)!$ unless $(n, i) \in \{(8, 3), (8, 5)\}$.

We assume now that $n = 8$ and $i \in \{3, 5\}$. In this case, we have $\chi(X_{s_i w_0}) = 8! - 3!5!$. If $\mathcal{B}(x, H)$ is irreducible of codimension one in $\mathcal{B}$, then we have by Lemmas 5.5 and 5.9 that $H = H(\overline{-\theta})$. Let $x = x_s + x_n$ be the decomposition of x as in the proof Lemma 5.11. Arguing as in that proof, we get that

$$\chi(\mathcal{B}(x, H(\overline{-\theta}))) = |\mathcal{C}(x, H(\overline{-\theta}))| = 8! - 6! |\{(j < k) : c_{jk} \neq 0\}|.$$

It follows immediately that if $\chi(\mathcal{B}(x, H(\overline{-\theta}))) = 8! - 3!5!$, then $x_n = E_{jk}$ for some $1 \leq j < k \leq n$. Finally, one can use the formulas given in [23, Corollary 5.5] to check that there is no Hessenberg variety $\mathcal{B}(x_s + E_{jk}, H(\overline{-\theta}))$ in the flag variety $\mathcal{B} = SL_8(\mathbb{C})/B$ with Betti numbers equal to those of the Schubert varieties $X_{s_3 w_0}$ and $X_{s_5 w_0}$. For the sake of brevity, we omit these computations. $\square$

6 Type C Adjoint Hessenberg Varieties

Recall from the introduction that the type C flag variety can be identified as the fixed-point set of a type A flag variety under a certain automorphism σ . Similarly, the Type A and Type C Schubert varieties are closely connected as each type C Schubert variety is the variety of σ -fixed points of a type A Schubert variety (see [20, Chapter 6]). The first main result of this section is that the same is true of type C Hessenberg varieties. Namely Theorem 6.3 below says that every type C adjoint Hessenberg variety is the variety of σ -fixed points of a type A Hessenberg variety and implies Theorem 1.13. With this groundwork in place, we establish the type C pattern avoidance result stated in Theorem 1.12 (see Theorem 6.10 below). We remark that many of our proofs in this section would be considerably easier and shorter were we to assume that every Hessenberg space H contains the Borel subalgebra $\mathfrak{b}$. However, to prove the results in complete generality, we do not make this assumption below.

We begin by fixing the notation needed to define the automorphism σ . Let E be the $2n \times 2n$ block matrix

$$E = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix}$$

where J is the $n \times n$ matrix with 1's on the anti-diagonal, and 0's elsewhere. We follow [20, Chapter 6] and identify $Sp_{2n}(\mathbb{C})$ with the fixed point set of the involution $\sigma : SL_{2n}(\mathbb{C}) \rightarrow SL_{2n}(\mathbb{C})$ defined by $\sigma(A) = E(A^{\text{tr}})^{-1}E^{-1}$. Explicitly, consider the

embedding $\phi : Sp_{2n}(\mathbb{C}) \hookrightarrow SL_{2n}(\mathbb{C})$ whose image stabilizes the alternating form $\langle -, - \rangle$ defined by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = 2n + 1 - i \text{ and } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $e_1, \dots, e_{2n}$ is the standard basis of $\mathbb{C}^{2n}$.) Then $G_C := \phi(Sp_{2n}(\mathbb{C}))$ is the group of σ -fixed points in $SL_{2n}(\mathbb{C})$. Throughout this section we identify $Sp_{2n}(\mathbb{C})$ with G_C .

The maximal torus T in $SL_{2n}(\mathbb{C})$ consisting of diagonal matrices and fixed Borel subgroup B in $SL_{2n}(\mathbb{C})$ consisting of upper triangular matrices are stable under σ , and T^σ (respectively, $B_C := B^\sigma$) is a maximal torus (respectively, Borel subgroup) in G_C . For $i \in [2n]$, set

$$i' = 2n + 1 - i.$$

Let W be the Weyl group of $Sp_{2n}(\mathbb{C})$. The embedding ϕ induces an embedding $\phi^* : W \rightarrow S_{2n}$ with image $W_C := \phi^*(W)$ consisting of those $w \in S_{2n}$ satisfying $w(i)' = w(i')$ for all $i \in [2n]$. We call such w **signed permutations**.

We will write σ for the differential $d\sigma$ which is the involution of the Lie algebra given by

$$\sigma : \mathfrak{sl}_{2n}(\mathbb{C}) \rightarrow \mathfrak{sl}_{2n}(\mathbb{C}), \quad \sigma(x) = Ex^{\text{tr}}E.$$

We identify $\mathfrak{sp}_{2n}(\mathbb{C})$ with $\mathfrak{g}_C := \mathfrak{sl}_{2n}(\mathbb{C})^\sigma$. We observe that $\mathfrak{h}^\sigma$ is the Lie algebra of T^σ , and

$$\mathfrak{h}^\sigma = \{\text{diag}(d_1, \dots, d_{2n}) \in \mathfrak{sl}_{2n}(\mathbb{C}) : d_i = -d_{i'}\}. \quad (6.1)$$

The involution σ also induces an involution of $\mathfrak{h}^*$ defined by

$$\sigma : \mathfrak{h}^* \rightarrow \mathfrak{h}^*, \quad \sigma(\epsilon_i) = -\epsilon_{i'}.$$

We describe now a surjective map from the type A root system Φ_A to the type C root system Φ_C known as the **folding map**. Set $\bar{\epsilon}_i := \epsilon_i - \epsilon_{i'}$ and note that, by definition, $\bar{\epsilon}_i = -\bar{\epsilon}_{i'}$. The folding map is now defined to be

$$\varphi : \Phi_A \rightarrow \Phi_C, \quad \varphi(\epsilon_i - \epsilon_j) = \frac{1}{2}(\bar{\epsilon}_i - \bar{\epsilon}_j). \quad (6.2)$$

Let $H_C \subseteq \mathfrak{g}_C$ be a type C Hessenberg space. Our first goal is to construct a type A Hessenberg space H whose σ -fixed points are the elements of H_C . Define $\Phi_{H_C} := \{\gamma \in \Phi_C : \mathfrak{g}_\gamma \subseteq H_C\}$ and set

$$\Phi_H := \{\epsilon_i - \epsilon_j \in \Phi_A : \varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}\} \subseteq \Phi_A. \quad (6.3)$$

With $H \cap \mathfrak{h}$ to be described below, we define $H \subseteq \mathfrak{sl}_{2n}(\mathbb{C})$ to be the subspace such that

$$H = (H \cap \mathfrak{h}) \oplus \bigoplus_{\epsilon_i - \epsilon_j \in \Phi_H} \mathbb{C}\{E_{ij}\}. \quad (6.4)$$

Given $i, j \in [2n]$, we set

$$h_{ij} := [E_{ij}, E_{ji}] = E_{ii} - E_{jj},$$

and define

$$H \cap \mathfrak{h} := \mathbb{C}\{h, h_{ij} : h \in H_C \cap \mathfrak{h}^\sigma \text{ and } i, j \in [2n] \text{ such that } \epsilon_i - \epsilon_j, \epsilon_j - \epsilon_i \in \Phi_H\}.$$

The example below shows that the intersection of the type A Hessenberg space defined in (6.4) with the Cartan subalgebra depends on the roots Φ_C in a nuanced way.

Example 6.1 Suppose $n = 2$, and hence Φ_A is a root system of type A_3 and Φ_C a root system of type C_2 . Write α_i for the simple root $\epsilon_i - \epsilon_{i+1} \in \Phi_A$. The simple roots in Φ_C are $\beta_1 = \phi(\alpha_1) = \phi(\alpha_3)$ and $\beta_2 = \phi(\alpha_2)$. Now $(\Phi_C)_+ = \{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}$. In the embedding of $\mathfrak{sp}_4(\mathbb{C})$ in $\mathfrak{sl}_4(\mathbb{C})$ described above, the positive root spaces $(\mathfrak{g}_C)_{\beta_1}$, $(\mathfrak{g}_C)_{\beta_2}$, $(\mathfrak{g}_C)_{\beta_1 + \beta_2}$, and $(\mathfrak{g}_C)_{2\beta_1 + \beta_2}$ are spanned respectively by $E_{12} - E_{34}$, E_{23} , $E_{13} + E_{24}$, and E_{14} . The corresponding negative root spaces are spanned by the transposes of the given matrices. Also, the Cartan subalgebra $\mathfrak{h}^\sigma$ of $\mathfrak{g}_C$ is spanned by $h_{\beta_1} = E_{11} - E_{22} + E_{33} - E_{44}$ and $h_{\beta_2} = E_{22} - E_{33}$.

Say H_C is the $ad(\mathfrak{b}_C)$ -module generated by $(\mathfrak{g}_C)_{-\beta_1}$, that is, H_C is the Demazure module in $\mathfrak{g}_C$ generated by $E_{-\beta_1}$. Using direct calculation (or [8, p. 61]), we see that

$$H_C = \mathbb{C}\{h_{\beta_1}\} \oplus (\mathfrak{g}_C)_{-\beta_1} \oplus \bigoplus_{\beta \in (\Phi_C)_+} (\mathfrak{g}_C)_\beta.$$

It follows that

$$\Phi_H = \{-\alpha_1, -\alpha_3\} \cup (\Phi_A)_+$$

and

$$H = \mathbb{C}\{h_{12}, h_{34}\} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_3} \oplus \bigoplus_{\alpha \in (\Phi_A)_+} \mathfrak{g}_\alpha.$$

In particular, $H \cap \mathfrak{h}$ contains $H_C \cap \mathfrak{h}^\sigma$ strictly.

On the other hand, if H_C is the Demazure module generated by $(\mathfrak{g}_C)_{-\beta_2}$ then

$$H_C = \mathbb{C}\{h_{\beta_2}\} \oplus (\mathfrak{g}_C)_{-\beta_2} \oplus \bigoplus_{\beta \in (\Phi_C)_+} (\mathfrak{g}_C)_\beta,$$

$$\Phi_H = \{-\alpha_2\} \cup (\Phi_A)_+,$$

and

$$H = \mathbb{C}\{h_{23}\} \oplus (\mathfrak{g}_C)_{-\alpha_2} \oplus \bigoplus_{\alpha \in (\Phi_A)_+} \mathfrak{g}_\alpha.$$

Thus this case, $H \cap \mathfrak{h} = H_C \cap \mathfrak{h}^\sigma$.

Remark 6.2 If Φ_{H_C} contains all negative simple roots of Φ_C , then $H_C \cap \mathfrak{h}^\sigma = \mathfrak{h}^\sigma$ is the Cartan subalgebra of $\mathfrak{sl}_{2n}(\mathbb{C})$ and in this case, $H \cap \mathfrak{h} = \mathfrak{h}$ also.

We observe that the action of σ on $SL_{2n}(\mathbb{C})$ induces an automorphism of the type A flag variety $\mathcal{B}_A := SL_{2n}(\mathbb{C})/B$, which will also be denoted by σ . The map ϕ' from $\mathcal{B}_C := G_C/B_C$ to $\mathcal{B}_A$ sending gB_C to gB is a well-defined embedding, and $(\mathcal{B}_A)^\sigma = \phi'(\mathcal{B}_C)$ (see, for example, [20, Proposition 6.1.1.1]).

We can now state the main theorem of this section.

Theorem 6.3 *Given a type C Hessenberg space $H_C \subseteq \mathfrak{g}_C$, let H be the subspace of $\mathfrak{sl}_{2n}(\mathbb{C})$ defined as in (6.4) above.*

- (1) *The subspace H is a type A Hessenberg space such that $H^\sigma = H_C$.*
- (2) *Let $x \in \mathfrak{g}_C$. The image under ϕ' of the type C Hessenberg variety $\mathcal{B}_C(x, H_C)$ is $\mathcal{B}_A(x, H)^\sigma$.*

We prove first that in Theorem 6.3, (1) implies (2). The proof of Theorem 6.3(1) is delayed until after Lemma 6.7.

Proof of Theorem 6.3(2) We wish to show that $\phi'(\mathcal{B}_C(x, H_C)) = \mathcal{B}_A(x, H)^\sigma$ as varieties. The set $\mathcal{G}_C(x, H_C) := \{g \in G_C : g^{-1} \cdot x \in H_C\}$ is a subvariety of G_C invariant under the right action of B_C on G_C whose image under the morphism $\mu : G_C \rightarrow \mathcal{B}_C$ is $\mathcal{B}_C(x, H_C)$. Let $\mathcal{G}_A(x, H) := \{g \in SL_{2n}(\mathbb{C}) : g^{-1} \cdot x \in H\}$. Since $G_C = SL_{2n}(\mathbb{C})^\sigma$, we have that $\mathcal{G}_A(x, H)^\sigma = \mathcal{G}_A(x, H) \cap G_C$ is a subvariety of G_C .

Let us show that $\mathcal{G}_A(x, H)^\sigma = \mathcal{G}_C(x, H_C)$, as subvarieties of G_C . By (1) of Theorem 6.3 we have

$$\mathcal{G}_C(x, H_C) = \{g \in G_C : g^{-1} \cdot x \in H, g^{-1} \cdot x \in \mathfrak{sl}_{2n}(\mathbb{C})^\sigma\}.$$

Given $g \in G_C$, since $x \in \mathfrak{g}_C$, we have $g^{-1} \cdot x \in \mathfrak{g}_C$ and thus $g^{-1} \cdot x$ is σ -stable. It follows that the constraints imposed by $g^{-1} \cdot x \in \mathfrak{sl}_{2n}(\mathbb{C})^\sigma$ are redundant, so indeed $\mathcal{G}_C(x, H_C) = \mathcal{G}_A(x, H)^\sigma$. In particular,

$$\phi'(\mathcal{B}_C(x, H_C)) = \phi'(\mu(\mathcal{G}_C(x, H_C))) = \phi'(\mu(\mathcal{G}_A(x, H)^\sigma))$$

as subvarieties of $\mathcal{B}$. Finally, note that by definition of the maps,

$$\begin{aligned} \phi'(\mu(\mathcal{G}_A(x, H)^\sigma)) &= \phi'(\{gB_C \in \mathcal{B}_C : \sigma(g) = g, g^{-1} \cdot x \in H\}) \\ &= \{gB \in \mathcal{B} : \sigma(g) = g, g^{-1} \cdot x \in H\} \\ &= \mathcal{B}_A(x, H)^\sigma. \end{aligned}$$

This concludes the proof. $\square$

Now we develop the tools needed to prove Theorem 6.3(1). Recall the partial order $\leq$ on the root system Φ defined in Section 3.1 above, and the concrete description of that order given in Lemma 3.3. Our next result tells us that the folding map interacts nicely with σ , is compatible with $\leq$, and is well-behaved with respect to the W_C -action. A proof can be found in [20, Chapter 6.1].

Lemma 6.4 *Let $\varphi : \Phi_A \rightarrow \Phi_C$ be the folding map defined as in (6.2). This map satisfies each of the following conditions.*

1. $\varphi(\Phi_A^+) = \Phi_C^+$.
2. Given $\gamma \in \Phi_C$, $\varphi^{-1}(\gamma)$ is precisely the σ -orbit of any $\gamma' \in \Phi_A$ such that $\varphi(\gamma') = \gamma$.
3. The map φ is compatible with the partial ordering $\leq$ on Φ_A and Φ_C , that is, given $\gamma_1, \gamma_2 \in \Phi_A$, we have $\gamma_1 \leq \gamma_2$ implies $\varphi(\gamma_1) \leq \varphi(\gamma_2)$.
4. φ is equivariant with respect to the canonical action of W_C on Φ_A and Φ_C .

For use below, we recall that there is a simple description of the partial order $\leq$ on the root system Φ_A , given by

$$\epsilon_i - \epsilon_j \leq \epsilon_k - \epsilon_\ell \Leftrightarrow k \leq i \text{ and } \ell \geq j. \quad (6.5)$$

Consider the surjective linear map

$$\bar{\sigma} : \mathfrak{sl}_{2n}(\mathbb{C}) \rightarrow \mathfrak{g}_C, \quad \bar{\sigma}(x) = x + \sigma(x).$$

Note that $\bar{\sigma}$ is not a Lie algebra homomorphism. However, the next lemma tells us that $\bar{\sigma}$ maps the root spaces of $\mathfrak{sl}_{2n}(\mathbb{C})$ onto those of $\mathfrak{sp}_{2n}(\mathbb{C})$.

Lemma 6.5 *Let $h \in \mathfrak{h}^\sigma$. For all $1 \leq k, \ell \leq 2n$ with $k \neq \ell$, we have $\bar{\sigma}(E_{k\ell}) \in \mathfrak{g}_{\varphi(\epsilon_k - \epsilon_\ell)}$ and $\varphi(\epsilon_k - \epsilon_\ell)(h) = (\epsilon_k - \epsilon_\ell)(h)$.*

Proof Since $E_{k\ell}$ is a root vector corresponding to the root $\epsilon_k - \epsilon_\ell \in \Phi_A$ we have $[h, E_{k\ell}] = (\epsilon_k - \epsilon_\ell)(h)E_{k\ell}$. As σ is an involution of $\mathfrak{sl}_{2n}(\mathbb{C})$ and $\sigma(h) = h$ we get

$$\sigma([h, \sigma(E_{k\ell})]) = [h, E_{k\ell}] = (\epsilon_k - \epsilon_\ell)(h)E_{k\ell} \Rightarrow [h, \sigma(E_{k\ell})] = (\epsilon_k - \epsilon_\ell)(h)\sigma(E_{k\ell}).$$

Thus,

$$\begin{aligned} [h, \bar{\sigma}(E_{k\ell})] &= [h, E_{k\ell}] + [h, \sigma(E_{k\ell})] \\ &= (\epsilon_k - \epsilon_\ell)(h)E_{k\ell} + (\epsilon_k - \epsilon_\ell)(h)\sigma(E_{k\ell}) \\ &= (\epsilon_k - \epsilon_\ell)(h)\bar{\sigma}(E_{k\ell}). \end{aligned}$$

To conclude the argument, we have only to show that $\varphi(\epsilon_k - \epsilon_\ell)(h) = (\epsilon_k - \epsilon_\ell)(h)$. Since $h \in \mathfrak{h}^\sigma$, we have $\epsilon_k(h) = -\epsilon_{k'}(h)$ for all $k \in [2n]$ (see (6.1)). Thus,

$$\varphi(\epsilon_k - \epsilon_\ell)(h) = \frac{1}{2}(\bar{\epsilon}_k - \bar{\epsilon}_\ell)(h) = \frac{1}{2}(\epsilon_k - \epsilon_{k'} - \epsilon_\ell + \epsilon_{\ell'})(h) = (\epsilon_k - \epsilon_\ell)(h)$$

as desired. $\square$

Before arguing that H is a type A Hessenberg space, we prove it is σ -invariant.

Lemma 6.6 *Let $H_C \subseteq \mathfrak{sp}_{2n}(\mathbb{C})$ be a type C Hessenberg space and $H \subseteq \mathfrak{sl}_{2n}(\mathbb{C})$ be the subspace defined as in (6.4) above. Then $\sigma(H) = H$.*

Proof It suffices to show that $\sigma(H) \subseteq H$. To do this, we check that

1. $\sigma(E_{ij}) \in H$ for all i, j such that $\epsilon_i - \epsilon_j \in \Phi_H$, i.e., such that $\varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}$,
2. $\sigma(h_{ij}) \in H$ for all i, j such that $\epsilon_i - \epsilon_j, \epsilon_j - \epsilon_i \in \Phi_H$, and
3. $\sigma(h) \in H$ for all $h \in H_C \cap \mathfrak{h}^\sigma$.

Condition (3) follows immediately from the facts that $h \in \mathfrak{h}^\sigma$ and $H_C \cap \mathfrak{h}^\sigma \subseteq H$. Note that $\sigma(E_{ij}) \in \mathbb{C}\{E_{j'i'}\}$, so to prove (1) it suffices to show that $\epsilon_{j'} - \epsilon_{i'} \in \Phi_H$, i.e., that $\varphi(\epsilon_{j'} - \epsilon_{i'}) \in \Phi_{H_C}$. But since

$$\varphi(\epsilon_{j'} - \epsilon_{i'}) = \frac{1}{2}(\bar{\epsilon}_{j'} - \bar{\epsilon}_{i'}) = \frac{1}{2}(\bar{\epsilon}_i - \bar{\epsilon}_j) = \varphi(\epsilon_i - \epsilon_j) \quad (6.6)$$

we do indeed get $\varphi(\epsilon_{j'} - \epsilon_{i'}) = \varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}$.

Finally, we prove (2). Suppose $i, j \in [2n]$ such that $\pm(\epsilon_i - \epsilon_j) \in \Phi_H$, and consider

$$\sigma(h_{ij}) = \sigma(E_{ii} - E_{jj}) = -E_{i'i'} + E_{j'j'} = h_{j'i'}.$$

By (6.6), we have $\varphi(\epsilon_{j'} - \epsilon_{i'}) = \varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}$ so $\epsilon_{j'} - \epsilon_{i'} \in \Phi_H$ and similarly $\epsilon_{i'} - \epsilon_{j'} \in \Phi_H$ also. By the definition of H , this implies $h_{j'i'} \in H$, as desired. $\square$

Next, we argue that Φ_H is an upper order ideal with respect to the partial order $\leq$ on Φ_A .

Lemma 6.7 *Let $H_C \subseteq \mathfrak{g}_C$ be a type C Hessenberg space. The corresponding subset $\Phi_H \subset \Phi_A$ defined as in (6.3) is an upper order ideal with respect to the partial order $\leq$ on Φ_A .*

Proof Suppose $i, j \in [2n]$ such that $\epsilon_i - \epsilon_j \in \Phi_H$ and let $\epsilon_k - \epsilon_\ell \in \Phi_A$ with $\epsilon_k - \epsilon_\ell \geq \epsilon_i - \epsilon_j$. By Lemma 6.4, $\varphi(\epsilon_k - \epsilon_\ell) \geq \varphi(\epsilon_i - \epsilon_j)$. Since H_C is B_C -invariant, Φ_{H_C} is an upper order ideal with respect to the partial order $\leq$ on Φ_C (see Section 3.1). Thus as $\varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}$, we have $\varphi(\epsilon_k - \epsilon_\ell) \in \Phi_{H_C}$ and so $\epsilon_k - \epsilon_\ell \in \Phi_H$ as desired. $\square$

Proof of Theorem 6.3(1) We begin by arguing that $[\mathfrak{b}, H] \subseteq H$, which proves that H is a type A Hessenberg space. Since $\mathfrak{b} = \mathfrak{h} \oplus \mathbb{C}\{E_{k\ell} : 1 \leq k < \ell \leq 2n\}$, we have only to show that

- (A) $[h, x] \in H$ for all $h \in \mathfrak{h}$ and $x \in H$, and that
- (B) $[E_{k\ell}, x] \in H$ for all $1 \leq k < \ell \leq 2n$ and $x \in H$.

By the definition of H , we may write

$$x = h' + \sum_{\substack{1 \leq i < j \leq n \\ \pm(\epsilon_i - \epsilon_j) \in \Phi_H}} d_{ij} h_{ij} + \sum_{\epsilon_i - \epsilon_j \in \Phi_H} c_{ij} E_{ij}$$

for $c_{ij}, d_{ij} \in \mathbb{C}$ and $h' \in H_C \cap \mathfrak{h}^\sigma$. Assertion (A) admits a straightforward proof: since $h' + \sum_{\pm(\epsilon_i - \epsilon_j) \in \Phi_H} d_{ij} h_{ij} \in \mathfrak{h}$, we have

$$[h, x] = \sum_{\epsilon_i - \epsilon_j \in \Phi_H} c_{ij} [h, E_{ij}] = \sum_{\epsilon_i - \epsilon_j \in \Phi_H} c_{ij} (\epsilon_i - \epsilon_j)(h) E_{ij} \in H$$

by the definition of H .

We turn to assertion (B). Let $k, \ell \in [2n]$ with $k < \ell$. We have

$$[E_{k\ell}, x] = [E_{k\ell}, h'] + \sum_{\substack{1 \leq i < j \leq n \\ \pm(\epsilon_i - \epsilon_j) \in \Phi_H}} d_{ij} [E_{k\ell}, h_{ij}] + \sum_{\epsilon_i - \epsilon_j \in \Phi_H} c_{ij} [E_{k\ell}, E_{ij}]. \quad (6.7)$$

To prove $[E_{k\ell}, x] \in H$, we argue that every Lie bracket appearing in each summand on the right side of (6.7) is an element of H .

Case 1: Suppose $i \neq j$ with $\epsilon_i - \epsilon_j \in \Phi_H$. First if $\{i, j\} = \{k, \ell\}$, we have

$$[E_{k\ell}, E_{ij}] = \begin{cases} 0 & \text{if } i = k \text{ and } j = \ell \\ -h_{ij} & \text{if } i = \ell \text{ and } j = k. \end{cases}$$

Thus if $[E_{k\ell}, E_{ij}] \neq 0$, then $j = k < \ell = i$. In this case our assumption that $\epsilon_i - \epsilon_j \in \Phi_H$ implies $\epsilon_j - \epsilon_i \in \Phi_H$ since Φ_H is an upper-order ideal by Lemma 6.7. Now $h_{ij} = [E_{ij}, E_{ji}] \in H$, by definition of $H \cap \mathfrak{h}$. On the other hand, if $\{i, j\} \neq \{k, \ell\}$ and $[E_{k\ell}, E_{ij}] \neq 0$ then $[E_{k\ell}, E_{ij}] \in \mathfrak{g}_\gamma$ where $\gamma = (\epsilon_k - \epsilon_\ell) + (\epsilon_i - \epsilon_j)$. In particular, we have $\epsilon_i - \epsilon_j \leq \gamma$ since $\epsilon_k - \epsilon_\ell \in \Phi_A^+$. This implies $\gamma \in \Phi_H$ by Lemma 6.7, hence $[E_{k\ell}, E_{ij}] \in \mathfrak{g}_\gamma \subseteq H$.

Case 2: Let $i, j \in [2n]$ such that $i < j$ and $\epsilon_i - \epsilon_j, \epsilon_j - \epsilon_i \in \Phi_H$. By definition we must have $h_{ij} = E_{ii} - E_{jj} \in H \cap \mathfrak{h}$. Furthermore, we know $[E_{k\ell}, h_{ij}] = (\epsilon_\ell - \epsilon_k)(h_{ij})E_{k\ell}$. Thus, we have only to show that $\epsilon_k - \epsilon_\ell \in \Phi_H$ whenever $(\epsilon_\ell - \epsilon_k)(h_{ij}) \neq 0$. Since

$$(\epsilon_\ell - \epsilon_k)(h_{ij}) = \delta_{\ell i} - \delta_{\ell j} - \delta_{ki} + \delta_{kj},$$

where δ_{ab} is the Kronecker delta function, the condition $(\epsilon_\ell - \epsilon_k)(h_{ij}) \neq 0$ implies $\{i, j\} \cap \{k, \ell\} \neq \emptyset$. Now, we consider the various possibilities and show that $k \leq j$ and $\ell \geq i$ in each case. Recall that $i < j$ and $k < \ell$ by assumption.

- If $i = k$ then $k = i < j$ and $\ell > k = i$.
- If $i = \ell$, then it follows immediately that $k < \ell = i < j$. Therefore, $k < j$ and $\ell \geq i$.
- If $i \neq k$ and $i \neq \ell$ then our assumptions imply that either $j = k$ or $j = \ell$. In the first case, we get $i < j = k < \ell$, hence $k \leq j$ and $\ell > i$. In the second case, we get $k < \ell = j$ and $\ell = j > i$.

Since $k \leq j$ and $\ell \geq i$, we see that $\epsilon_j - \epsilon_i \leq \epsilon_k - \epsilon_\ell$. The assumption that $\epsilon_j - \epsilon_i \in \Phi_H$ implies $\epsilon_k - \epsilon_\ell \in \Phi_H$ by Lemma 6.7, as desired.

Case 3: Suppose $h' \in H_C \cap \mathfrak{h}^\sigma$. As above, we get $[E_{k\ell}, h'] = (\epsilon_\ell - \epsilon_k)(h')E_{k\ell}$ and thus $[E_{k\ell}, h'] \in H$ will follow if we are able to show $\epsilon_k - \epsilon_\ell \in \Phi_H$ whenever $(\epsilon_\ell - \epsilon_k)(h') \neq 0$. By Lemma 6.5, we have

$$[\bar{\sigma}(E_{k\ell}), h'] = -\varphi(\epsilon_k - \epsilon_\ell)(h')\bar{\sigma}(E_{k\ell}) = (\epsilon_\ell - \epsilon_k)(h')\bar{\sigma}(E_{k\ell}).$$

We observe that H_C is a type C Hessenberg space, $\bar{\sigma}(E_{k\ell}) \in \mathfrak{b}^\sigma$, and $h' \in H_C$. It follows that $[\bar{\sigma}(E_{k\ell}), h'] \in H_C$. Hence, the assumption $(\epsilon_\ell - \epsilon_k)(h') \neq 0$ implies $\bar{\sigma}(E_{k\ell}) \in H_C$. However, since $\bar{\sigma}(E_{k\ell})$ spans the root space corresponding to $\varphi(\epsilon_k - \epsilon_\ell)$ in $\mathfrak{g}_C$, we have $\varphi(\epsilon_k - \epsilon_\ell) \in \Phi_{H_C}$ and thus $\epsilon_k - \epsilon_\ell \in \Phi_H$ as desired.

Having settled all cases, we conclude that H is a type A Hessenberg space. To complete the proof, we establish now that $H^\sigma = H_C$. Since, by Lemma 6.6, H is σ -stable we know $H^\sigma = \bar{\sigma}(H)$ and we will show $\bar{\sigma}(H) = H_C$. Since $H_C \subseteq H$ by definition, the inclusion $H_C \subseteq \bar{\sigma}(H)$ is a consequence of $\bar{\sigma}(H_C) = H_C$.

We verify that $\bar{\sigma}(H) \subseteq H_C$. It is straightforward that $\bar{\sigma}(h) \in H_C$ if $h \in H_C \cap \mathfrak{h}^\sigma$. Next, if $\epsilon_i - \epsilon_j \in \Phi_H$ then $\varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}$ and Lemma 6.5 now implies $\bar{\sigma}(E_{ij}) \in H_C$. Last, consider $1 \leq i < j \leq 2n$ such that $\epsilon_i - \epsilon_j, \epsilon_j - \epsilon_i \in \Phi_H$. Then $\varphi(\epsilon_i - \epsilon_j), -\varphi(\epsilon_i - \epsilon_j) \in \Phi_{H_C}$ and by another application of Lemma 6.5, we get

$$\bar{\sigma}(E_{ij}) \in \mathfrak{b}^\sigma, \bar{\sigma}(E_{ji}) \in H_C \Rightarrow [\bar{\sigma}(E_{ij}), \bar{\sigma}(E_{ji})] \in H_C.$$

Using the facts that $\sigma(E_{ji}) \in \mathbb{C}\{E_{i'j'}\}$ and $i = j'$ if and only if $i' = j$, we obtain

$$[E_{ij}, \sigma(E_{ji})] = \begin{cases} 0 & \text{if } j \neq i' \\ h_{ij} & \text{if } j = i'. \end{cases}$$

Therefore we have

$$\begin{aligned} [\bar{\sigma}(E_{ij}), \bar{\sigma}(E_{ji})] &= [E_{ij}, E_{ji}] + [\sigma(E_{ij}), E_{ji}] + [E_{ij}, \sigma(E_{ji})] + [\sigma(E_{ij}), \sigma(E_{ji})] \\ &= [E_{ij}, E_{ji}] + \sigma([E_{ij}, E_{ji}]) + [E_{ij}, \sigma(E_{ji})] + \sigma([E_{ij}, \sigma(E_{ji})]) \\ &= \bar{\sigma}([E_{ij}, E_{ji}]) + \bar{\sigma}([E_{ij}, \sigma(E_{ji})]) \\ &= c \bar{\sigma}(h_{ij}) \end{aligned}$$

with $c \in \{1, 2\}$. This implies $\bar{\sigma}(h_{ij}) \in H_C$, and we conclude that $\bar{\sigma}(H) \subseteq H_C$. $\square$

6.1 Type C Pattern Avoidance

Recall that W_C denotes the subgroup in S_{2n} consisting of signed permutations. Thus the notion of pattern avoidance as defined in Section 2.4 makes sense for elements of W_C . The objective of this section is to prove Theorem 1.12 using Theorem 6.10 below, generalizing the pattern avoidance result of Theorem 4.4 to the type C setting.

We may write each matrix in $\mathfrak{g}_C$ in terms of the Chevalley basis of type C, fixed as in [20, Section 6.1]. Given $\gamma \in \Phi_C$, we denote by E_γ the corresponding root vector and by $\mathbf{c}_\gamma : \mathfrak{sp}_{2n}(\mathbb{C}) \rightarrow \mathbb{C}$ the coordinate function returning the coefficient of E_γ .

Note that if $\epsilon_i - \epsilon_j \in \Phi_A$ such that $\varphi(\epsilon_i - \epsilon_j) = \gamma$ then $\mathbb{C}\{E_\gamma\} = \mathbb{C}\{\bar{\sigma}(E_{ij})\}$ by Lemma 6.5. In particular, for all $y \in \mathfrak{g}_C$ we have

$$\mathbf{c}_\gamma(y) \neq 0 \Leftrightarrow \mathbf{c}_{ij}(y) \neq 0 \Leftrightarrow \mathbf{c}_{j'i'}(y) \neq 0 \quad (6.8)$$

where $\mathbf{c}_{ij} : \mathfrak{sl}_{2n}(\mathbb{C}) \rightarrow \mathbb{C}$ denotes the coordinate function returning the coefficient of E_{ij} as in Section 4 above. We begin with the following statement, which is a Type C analogue of Lemma 4.1.

Lemma 6.8 *Let $x \in \mathfrak{g}_C$ and suppose $\mathbf{c}_\gamma(x) \neq 0$ for some $\gamma \in \Phi_C$. For all $\beta \in \Phi_C$ such that $\beta \geq \gamma$ there exists $b \in B_C$ such that $\mathbf{c}_\beta(b \cdot x) \neq 0$.*

Proof Let $\gamma', \beta' \in \Phi_A$ be such that $\varphi(\gamma') = \gamma$ and $\varphi(\beta') = \beta$. By definition, $\gamma' \in \{\epsilon_i - \epsilon_j, \epsilon_{j'} - \epsilon_{i'}\}$ and $\beta' \in \{\epsilon_k - \epsilon_\ell, \epsilon_{\ell'} - \epsilon_{k'}\}$ for some $i, j, k, \ell \in [2n]$ such that $i \neq j$ and $k \neq \ell$. Since $B_C \subseteq B$, by definition of the partial order $\leq$, we may assume without loss of generality that $\gamma' = \epsilon_i - \epsilon_j$ and $\beta' = \epsilon_k - \epsilon_\ell$ with $\beta' \geq \gamma'$, i.e., $k \leq i$ and $\ell \geq j$. By (6.8), our assumption that $\mathbf{c}_\gamma(x) \neq 0$ implies $\mathbf{c}_{ij}(x) \neq 0$ and to complete the proof of the lemma, it suffices to verify that there exists $b \in B_C$ such that $\mathbf{c}_{k\ell}(b \cdot x) \neq 0$. Throughout the rest of the proof, α will be a parameter that can take any complex value we choose.

If $(k, \ell) = (j, i)$ then $j < i$ and so $b_\alpha = I + \alpha E_{ji} + \alpha \sigma(E_{ji}) \in B_C$. We have $b_\alpha^{-1} = I - \alpha E_{ji} - \alpha \sigma(E_{ji})$ and $\sigma(E_{ji}) = \pm E_{i'j'}$. Note that if $\sigma(E_{ji}) = -E_{i'j'}$, then we must have either $i, j \leq n$ or $i, j > n$ so $\delta_{ij'} = 0$ in that case. Thus for all $p, q \in [2n]$ we have

$$\mathbf{c}_{jp}(b_\alpha) = \begin{cases} 1 & \text{if } p = j \\ \alpha(1 + \delta_{ij'}) & \text{if } p = i \end{cases} \quad \text{and} \quad \mathbf{c}_{qi}(b_\alpha^{-1}) = \begin{cases} 1 & \text{if } q = i \\ -\alpha(1 + \delta_{ij'}) & \text{if } q = j \end{cases}$$

Using the fact that the (k, ℓ) -entry of the product of three $2n \times 2n$ matrices X, Y , and Z is

$\sum_{p=1}^{2n} \sum_{q=1}^{2n} X_{kp} Y_{pq} Z_{q\ell}$, we obtain

$$\mathbf{c}_{k\ell}(b_\alpha \cdot x) = \mathbf{c}_{ji}(b_\alpha \cdot x) = \mathbf{c}_{ji}(x) + \alpha(1 + \delta_{ij'}) (\mathbf{c}_{ii}(x) - \mathbf{c}_{jj}(x)) - \alpha^2(1 + 3\delta_{ij'}) \mathbf{c}_{ij}(x).$$

Since $(1 + 3\delta_{ij'}) \mathbf{c}_{ij}(x) \neq 0$, there exists $\alpha \in \mathbb{C}$ such that $\mathbf{c}_{ji}(b_\alpha \cdot x) \neq 0$.

If $i \neq \ell$ the lemma will follow from the existence of $b_1, b_2 \in B_C$ such that $\mathbf{c}_{i\ell}(b_1 \cdot x) \neq 0$ and $\mathbf{c}_{k\ell}(b_2 \cdot (b_1 \cdot x)) \neq 0$. Symmetrically, if $j \neq k$ the lemma will follow from the existence of $b_1, b_2 \in B_C$ such that $\mathbf{c}_{kj}(b_1 \cdot x) \neq 0$ and $\mathbf{c}_{k\ell}(b_2 \cdot (b_1 \cdot x)) \neq 0$. Therefore, to settle the case $(k, \ell) \neq (j, i)$ it suffices to consider $i = k$ or $j = \ell$. Since both follow from similar arguments, we only write the proof assuming $k = i$. Moreover, we assume $\ell \neq j$ since the case $(i, j) = (k, \ell)$ is trivial.

If $\ell = j'$ then $b = I + \alpha E_{j\ell} \in B_C$. Since $\mathbf{c}_{i\ell}(b \cdot x) = \mathbf{c}_{i\ell}(x) - \alpha \mathbf{c}_{ij}(x)$ is a nonzero polynomial in $\mathbb{C}[\alpha]$, we can choose α such that $\mathbf{c}_{i\ell}(b \cdot x) \neq 0$. If $\ell \neq j'$ define $b \in B_C$ by

$$b := \begin{cases} I + \alpha E_{j\ell} + \alpha E_{\ell'j'}, & |\{j, \ell\} \cap [n]| = 1 \\ I + \alpha E_{j\ell} - \alpha E_{\ell'j'}, & |\{j, \ell\} \cap [n]| \neq 1 \end{cases},$$

so

$$b^{-1} = \begin{cases} I - \alpha E_{j\ell} - \alpha E_{\ell'j'}, & |\{j, \ell\} \cap [n]| = 1 \\ I - \alpha E_{j\ell} + \alpha E_{\ell'j'}, & |\{j, \ell\} \cap [n]| \neq 1 \end{cases}.$$

Using the formula for the (k, ℓ) -entry of the product of three $n \times n$ matrices once more, we obtain,

$$\mathbf{c}_{i\ell}(b \cdot x) = \begin{cases} \mathbf{c}_{i\ell}(x) - \alpha \mathbf{c}_{ij}(x) + \alpha \delta_{i\ell'} \mathbf{c}_{j'\ell}(x) - \alpha^2 \delta_{i\ell'} \mathbf{c}_{j'j}(x), & |\{j, \ell\} \cap [n]| = 1 \\ \mathbf{c}_{i\ell}(x) - \alpha \mathbf{c}_{ij}(x) - \alpha \delta_{i\ell'} \mathbf{c}_{j'\ell}(x) + \alpha^2 \delta_{i\ell'} \mathbf{c}_{j'j}(x), & |\{j, \ell\} \cap [n]| \neq 1. \end{cases}$$

If $i \neq \ell'$ then it is immediate that we can choose α such that $\mathbf{c}_{i\ell}(b \cdot x) \neq 0$. Finally, let's suppose that $i = \ell'$ and note that $|\{j, \ell\} \cap [n]| = 1$ if and only if $|\{j, i\} \cap [n]| \neq 1$. Since $x \in \mathfrak{g}_C$ it follows that the coefficient of α in $\mathbf{c}_{i\ell}(b \cdot x)$ is $-2\mathbf{c}_{ij}(x) \neq 0$. Since $\mathbf{c}_{i\ell}(b \cdot x)$ is a nonzero polynomial in $\mathbb{C}[\alpha]$ we can choose α such that $\mathbf{c}_{i\ell}(b \cdot x) \neq 0$. $\square$

Proposition 6.9 *Assume that $x \in \mathfrak{g}_C$ and that $\mathcal{B}_C(x, H_C) \subseteq G_C/B_C$ is a type C Hessenberg variety. Let $w \in W_C$ with $B_C \dot{w} B_C \subseteq \mathcal{B}_C(x, H_C)$. If $H \subseteq \mathfrak{sl}_{2n}(\mathbb{C})$ is the type A Hessenberg space defined using H_C as in (6.4) above, then the type A Schubert cell $B \dot{w} B$ is contained in $\mathcal{B}_A(x, H)$.*

Proof We write

$$x = h + \sum_{\gamma \in \Phi_C} d_\gamma E_\gamma$$

where $d_\gamma \in \mathbb{C}$, E_γ is a nonzero root vector in $\mathfrak{g}_\gamma \subseteq \mathfrak{g}_C$, and $h \in \mathfrak{h}_C$. We set $x_1 := \sum_{\gamma \in \Phi_C} d_\gamma E_\gamma$. So, $x = h + x_1$. Note that $\mathbf{c}_\gamma(x) = d_\gamma$ for all $\gamma \in \Phi_C$.

To prove $B \dot{w} B \subseteq \mathcal{B}_A(x, H)$, it suffices to show that $u \dot{w} B \in \mathcal{B}_A(x, H)$ for all u in the unipotent radical U of B . In particular, we must show

$$(u \dot{w})^{-1} \cdot x = \dot{w}^{-1} u^{-1} \cdot h + \dot{w}^{-1} u^{-1} \cdot x_1 \in H. \quad (6.9)$$

Since U is unipotent, the exponential map $\exp : \mathfrak{u} \rightarrow U$ is a diffeomorphism. Therefore we may write $u = \exp(y)$ for some $y = \sum_{p < q} \mathbf{c}_{pq}(y) E_{pq} \in \mathfrak{u}$.

Now we compute $u^{-1} \cdot x_1$ and $u^{-1} \cdot h$. By properties of the adjoint representation, we obtain

$$\begin{aligned} u^{-1} \cdot x_1 &= \text{Ad}(\exp(-y))(x_1) = \exp(\text{ad}_{-y})(x_1) \\ &= x_1 + \sum_{m=1}^{\infty} \frac{1}{m!} \text{ad}_{-y}^m(x_1) = x_1 + \sum_{m=1}^{\infty} \sum_{\gamma \in \Phi_C} \frac{1}{m!} d_\gamma \text{ad}_{-y}^m(E_\gamma). \end{aligned}$$

Given $\gamma \in \Phi_C$ such that $d_\gamma \neq 0$, we write $\varphi^{-1}(\gamma) = \{\epsilon_i - \epsilon_j, \epsilon_{j'} - \epsilon_{i'}\}$ for some $i, j \in [2n]$ with $i \neq j$. By Lemma 6.5, we know that $E_\gamma \in \mathbb{C}\{E_{ij}, E_{j'i'}\}$ and thus for

all $m \geq 1$, we have

$$\begin{aligned} \text{ad}_{-y}^m(E_\gamma) &\in \mathbb{C}\{E_{k\ell} : k \leq i, \ell \geq j \text{ or } k \leq j', \ell \geq i'\} \\ &\oplus \mathbb{C}\{h_{k\ell} : j \leq k < \ell \leq i \text{ or } i' \leq k < \ell \leq j'\}. \end{aligned}$$

(As above, $h_{k\ell} = [E_{k\ell}, E_{\ell k}] = E_{kk} - E_{\ell\ell}$.) In particular, we see that

$$u^{-1} \cdot x_1 \in \bigoplus_{d_\gamma \neq 0} \bigoplus_{\substack{\epsilon_i - \epsilon_j \in \Phi_A \\ \varphi(\epsilon_i - \epsilon_j) = \gamma}} (\mathbb{C}\{E_{k\ell} : k \leq i, \ell \geq j\} \oplus \mathbb{C}\{h_{k\ell} : j \leq k < \ell \leq i\}). \quad (6.10)$$

Next, we have that

$$\begin{aligned} u^{-1} \cdot h &= \text{Ad}(\exp(-y))(h) = \exp(\text{ad}_{-y})(h) = h + \sum_{m=1}^{\infty} \frac{1}{m!} \text{ad}_{-y}^m(h) \\ &= h + \sum_{m=1}^{\infty} \sum_{p < q} \frac{1}{m!} \mathbf{c}_{pq}(y)(\epsilon_p - \epsilon_q)(h) \text{ad}_{-y}^{m-1}(E_{pq}). \end{aligned}$$

Applying similar reasoning as above and using the fact that $p < q$, we have

$$u^{-1} \cdot h - h \in \bigoplus_{\substack{\epsilon_p - \epsilon_q \in \Phi_A^+ \\ (\epsilon_p - \epsilon_q)(h) \neq 0}} \mathbb{C}\{E_{k\ell} : k \leq p, \ell \geq q\}. \quad (6.11)$$

Note that $\dot{w}^{-1} \cdot h \in H_C \subseteq H$ by assumption. Thus equations (6.9), (6.10) and (6.11) imply that to prove the proposition, it suffices to show

$$\bigoplus_{d_\gamma \neq 0} \bigoplus_{\substack{\epsilon_i - \epsilon_j \in \Phi_A \\ \varphi(\epsilon_i - \epsilon_j) = \gamma}} (\mathbb{C}\{E_{w_k^{-1}w_\ell^{-1}} : k \leq i, \ell \geq j\} \oplus \mathbb{C}\{h_{w_k^{-1}w_\ell^{-1}} : j \leq k < \ell \leq i\}) \subseteq H \quad (6.12)$$

and

$$\bigoplus_{\substack{\epsilon_p - \epsilon_q \in \Phi_A \\ (\epsilon_p - \epsilon_q)(h) \neq 0}} \mathbb{C}\{E_{w_k^{-1}w_\ell^{-1}} : k \leq p, \ell \geq q\} \subseteq H. \quad (6.13)$$

First we establish (6.12). If $d_\gamma \neq 0$, Lemma 6.8 implies that for each $\beta \in \Phi_C$ with $\beta \geq \gamma$ there exists $b \in B_C$ such that,

$$0 \neq \mathbf{c}_\beta(b \cdot x) = \mathbf{c}_{w^{-1}(\beta)}(\dot{w}^{-1}b \cdot x).$$

Our assumption that $B_C \dot{w} B_C \subseteq \mathcal{B}_C(x, H_C)$ now implies $E_{w^{-1}(\beta)} \in H_C$ and thus $w^{-1}(\beta) \in \Phi_{H_C}$ for all $\beta \geq \gamma$. Note that by Lemma 6.4, for all $\epsilon_k - \epsilon_\ell \in \Phi_A$ with $\epsilon_k - \epsilon_\ell \geq \epsilon_i - \epsilon_j$ we have $\varphi(\epsilon_k - \epsilon_\ell) \geq \gamma$. This implies $w^{-1}(\varphi(\epsilon_k - \epsilon_\ell)) \in \Phi_{H_C}$ and,

since φ is W_C -equivariant, we have $\varphi(\epsilon_{w_k^{-1}} - \epsilon_{w_\ell^{-1}}) \in \Phi_{H_C}$. Using the description of $\leq$ for Φ_A from (6.5) we have now proved:

$$\begin{aligned} d_{\varphi(\epsilon_i - \epsilon_j)} \neq 0 &\Rightarrow \epsilon_{w^{-1}(k)} - \epsilon_{w^{-1}(\ell)} \in \Phi_H \text{ for all } k \leq i \text{ and } \ell \geq j \\ &\Rightarrow E_{w_k^{-1}w_\ell^{-1}} \in H \text{ for all } k \leq i \text{ and } \ell \geq j. \end{aligned}$$

It follows that

$$\begin{aligned} d_{\varphi(\epsilon_i - \epsilon_j)} \neq 0 \text{ and } i > j &\Rightarrow E_{w_k^{-1}w_\ell^{-1}}, E_{w_\ell^{-1}w_k^{-1}} \in H \text{ for all } k, \ell \text{ such that } j \leq k < \ell \leq i \\ &\Rightarrow h_{w_k^{-1}w_\ell^{-1}} \in H \text{ for all } k, \ell \text{ such that } j \leq k < \ell \leq i, \end{aligned}$$

where the last implication follows from the fact that $[b, H] \subseteq H$. This concludes the proof of (6.12).

Next we prove (6.13). Fix $\epsilon_p - \epsilon_q \in \Phi_A$ such that $(\epsilon_p - \epsilon_q)(h) \neq 0$. First, we note that if $\varphi(\epsilon_p - \epsilon_q) \geq \gamma$ for some $\gamma \in \Phi_C$ such that $d_\gamma \neq 0$, then

$$\mathbb{C} \left\{ E_{w_k^{-1}w_\ell^{-1}} : k \leq p, \ell \geq q \right\} \subseteq H$$

by (6.12). Thus it suffices to consider the case in which $\varphi(\epsilon_p - \epsilon_q) \not\geq \gamma$ for every $\gamma \in \Phi_C$ such that $d_\gamma \neq 0$. This last assumption implies E_{pq} is not a summand of $u^{-1} \cdot x_1$ for any $u \in U$, i.e., $\mathbf{c}_{pq}(u^{-1} \cdot x_1) = 0$ for all $u \in U$. Consider $u_{pq} := I_n + \bar{\sigma}(E_{pq}) \in B_C$. Applying Lemma 6.5 and using properties of the adjoint action we have

$$u_{pq}^{-1} \cdot h = h - [\bar{\sigma}(E_{pq}), h] + \sum_{i=2}^{\infty} \frac{1}{i!} \text{ad}_{-\bar{\sigma}(E_{pq})}^i(h) = h + (\epsilon_p - \epsilon_q)(h) \bar{\sigma}(E_{pq}).$$

This implies

$$u_{pq}^{-1} \cdot x = u_{pq}^{-1} \cdot h + u_{pq}^{-1} \cdot x_1 = h + (\epsilon_p - \epsilon_q)(h) \bar{\sigma}(E_{pq}) + u_{pq}^{-1} \cdot x_1.$$

Since $\bar{\sigma}(E_{pq}) \in \mathfrak{g}_{\varphi(\epsilon_p - \epsilon_q)}$ and $\mathbf{c}_{pq}(u_{pq}^{-1} \cdot x_1) = 0$, it follows that $\mathbf{c}_{\varphi(\epsilon_p - \epsilon_q)}(u_{pq}^{-1} \cdot x) = (\epsilon_p - \epsilon_q)(h) \neq 0$. Since $u_{pq} \in B_C$, the assumption $B_C \dot{w} B_C \subseteq \mathcal{B}_C(x, H_C)$ implies $\dot{w}^{-1} b u_{pq}^{-1} \cdot x \in H_C$ for all $b \in B_C$. Furthermore, Lemma 6.8 implies that for each $\beta \geq \varphi(\epsilon_p - \epsilon_q)$ there exists $b \in B_C$ such that

$$0 \neq \mathbf{c}_\beta(b u_{pq}^{-1} \cdot x) = \mathbf{c}_{w^{-1}(\beta)}(\dot{w}^{-1} b u_{pq}^{-1} \cdot x)$$

and thus we have $w^{-1}(\beta) \in \Phi_{H_C}$ for all $\beta \geq \varphi(\epsilon_p - \epsilon_q)$. In particular, arguing as in the proof of (6.12), we have

$$E_{w_k^{-1}w_\ell^{-1}} \in H \text{ for all } k \leq p \text{ and } \ell \geq q.$$

This establishes (6.13) and completes the proof. $\square$

We conclude this section with the proof of Theorem 1.12.

Theorem 6.10 *Let $G = Sp_{2n}(\mathbb{C})$ and let $B \leq G$ be the Borel subgroup whose image under ϕ consists of upper triangular matrices. Fix $w \in W$. If there exist $x \in \mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ and a Hessenberg space $H_C \subseteq \mathfrak{g}$ such that $\mathcal{B}_C(x, H_C) = X_{w^{-1}}^C$, then $\phi^*(w)$ avoids the pattern [4231].*

We remark that since [4231] is its own inverse, this theorem implies that if $\mathcal{B}_C(x, H_C) = X_w^C$, then $\phi^*(w)$ avoids the pattern [4231]. So Theorem 6.10 implies Theorem 1.12.

Proof Suppose $w \in W_C$ contains the pattern [4231]. Seeking a contradiction, suppose there exists $x \in \mathfrak{g}_C$ and a type C Hessenberg space $H_C \subseteq \mathfrak{g}_C$ such that $X_{w^{-1}}^C = \mathcal{B}_C(x, H_C)$. By Proposition 6.9, $B\dot{w}^{-1}B \subseteq \mathcal{B}_A(x, H)$ where $H \subseteq \mathfrak{sl}_{2n}(\mathbb{C})$ is the type A Hessenberg space defined using H_C as in (6.4) above. To obtain a contradiction, we show that there exists $v \in W_C$ such that $v \not\leq_{Br} w$ and $\dot{v}^{-1}B \in \mathcal{B}_A(x, H)$. Given this statement, we would then have by Theorem 6.3 that $\dot{v}^{-1}B \in (\mathcal{B}_A(x, H))^\sigma = \mathcal{B}_C(x, H_C)$ contradicting our assumption that $X_{w^{-1}}^C = \mathcal{B}_C(x, H_C)$.

Since w contains the pattern [4231] there exist i, j, k, ℓ such that $1 \leq i < j < k < \ell \leq 2n$ and $w_\ell < w_j < w_k < w_i$. Consider $\tau = (w_j, w_k)w$ where (w_j, w_k) is the transposition exchanging w_j and w_k , so $\tau \in S_{2n}$ is as defined in the statement of Lemma 4.3. By Lemma 4.3, $\dot{\tau}^{-1}B \in \mathcal{B}_A(x, H)$, that is, $\dot{\tau} \cdot x \in H$. If $j' = k$ then $\tau \in W_C$ and taking $v = \tau$ accomplishes the desired goal. We may therefore assume $j' \neq k$ for the remainder of the proof.

We have that $1 \leq \ell' < k' < j' < i' \leq 2n$ and $w_{i'} < w_{k'} < w_{j'} < w_{\ell'}$. Let $v = (w_{j'}, w_{k'})\tau = (w_{j'}, w_{k'})(w_j, w_k)w$, where $(w_{j'}, w_{k'})$ is the transposition exchanging $w_{j'}$ and $w_{k'}$. Note that $v \not\leq_{Br} w$ since $\ell(v) > \ell(w)$ and $v \in W_C$. In order to argue that $\dot{v} \cdot x \in H$, we write $x = s + x'$ where

$$s = \sum_{p \in [2n-1]} c_p(E_{pp} - E_{p+1, p+1}) \quad \text{and} \quad x' = \sum_{\substack{(p, q) \in [2n] \times [2n] \\ p \neq q}} c_{pq}(x)E_{pq}$$

and show $\dot{v} \cdot s, \dot{v} \cdot x' \in H$. Note that $\dot{w} \cdot s, \dot{w} \cdot x \in H$ by assumption. From the proof of Lemma 4.3 we have that $\dot{\tau} \cdot s \in H$ and

$$\dot{w} \cdot s - \dot{\tau} \cdot s = (c_{j-1} - c_j - c_{k-1} + c_k)(E_{w_k w_k} - E_{w_j w_j}) \in H.$$

Since $\sigma(H) \subseteq H$, we see that

$$\begin{aligned} \sigma(\dot{w} \cdot s - \dot{\tau} \cdot s) &= (c_{j-1} - c_j - c_{k-1} + c_k)(-E_{w_{k'} w_{k'}} + E_{w_{j'} w_{j'}}) \\ &= (c_{k'-1} - c_{k'} - c_{j'-1} + c_{j'})(E_{w_{j'} w_{j'}} - E_{w_{k'} w_{k'}}) \in H, \end{aligned}$$

where the last equality follows from the fact that $s \in \mathfrak{h}_C$ (see (6.1)). A direct computation shows that

$$\dot{w} \cdot s - \dot{v} \cdot s = (c_{j-1} - c_j - c_{k-1} + c_k)(E_{w_k w_k} - E_{w_j w_j})$$

$$+(c_{k'-1} - c_{k'} - c_{j'-1} + c_{j'})(E_{w_{j'}w_{j'}} - E_{w_{k'}w_{k'}}),$$

which lies in H , and therefore $\dot{w} \cdot s \in H$ implies $\dot{v} \cdot s \in H$.

Next we show that $\dot{v} \cdot x' \in H$ by proving that $E_{v_p v_q} \in H$ whenever $\mathbf{c}_{pq}(x) \neq 0$. Let $p \neq q$ be such that $\mathbf{c}_{pq}(x) \neq 0$. Let $\tau' = (w_{j'}, w_{k'})w$, and note that we have $\dot{\tau}' \cdot x \in H$ by Lemma 4.3. In fact, by the proof of Lemma 4.3 applied to both τ and τ' we have

$$\{p, q\} \cap \{j', k'\} = \emptyset \Rightarrow E_{v_p v_q} = E_{\tau_p \tau_q} \in H$$

and

$$\{p, q\} \cap \{j, k\} = \emptyset \Rightarrow E_{v_p v_q} = E_{\tau'_p \tau'_q} \in H.$$

To complete the proof, suppose that $\{p, q\} \cap \{j', k'\} \neq \emptyset$ and $\{p, q\} \cap \{j, k\} \neq \emptyset$. If $p \in \{j', k'\}$ and $q \in \{j, k\}$ then $\ell' < p$ and $q < \ell$. By Lemma 4.1 there exists $b \in B$ such that $0 \neq \mathbf{c}_{\ell' \ell}(b \cdot x) = \mathbf{c}_{w_{\ell'} w_{\ell}}(\dot{w}b \cdot x)$. Since $b^{-1}\dot{w}^{-1}B \in C_{w^{-1}} \subset \mathcal{B}(x, H)$, we must have $E_{w_{\ell'} w_{\ell}} \in H$. Now by 4.2, since $v_p \leq w_{\ell'}$ and $v_q \geq w_{\ell}$, we have $E_{v_p v_q} \in H$, as desired. Next, we consider the case in which $p \in \{j, k\}$ and $q \in \{j', k'\}$. Since $i < p$ and $q < i'$, by Lemma 4.1 there exists $b \in B$ such that $0 \neq \mathbf{c}_{ii'}(b \cdot x) = \mathbf{c}_{w_i w_{i'}}(\dot{w}b \cdot x)$. Since $b^{-1}\dot{w}^{-1}B \in C_{w^{-1}} \subset \mathcal{B}(x, H)$, then $E_{w_i w_{i'}} \in H$. Now by Lemma 4.2, by $v_p \leq w_i$ and $v_q \geq w_{i'}$, we have $E_{v_p v_q} \in H$, as desired. This concludes the proof. $\square$

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Declarations

Conflict of interest The authors declare no competing interests.

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