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Tail spaces estimates on Hamming cube and Bernstein–Markov inequality ☆

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ABSTRACT

This note contains some estimates for tail spaces and some Bernstein–Markov inequalities for Banach space valued functions on Hamming cube. We use analytic paraproduct operator for tail spaces. For Bernstein–Markov inequalities the novelty is in getting rid of some irritating logarithms and in proving Bernstein–Markov inequalities for $|\nabla f|_X$ rather than for $\Delta^{1/2} f$ for X-valued polynomials on Hamming cube. There is an interesting difference for giving the estimates for $|\nabla f|_X$ rather than for $\Delta^{1/2} f$ as their comparison is equivalent to estimates of Riesz transforms on Hamming cube.

1. Introduction

We are interested in tail spaces $\mathcal{T}_d(X)$ of functions

$$f = \sum_{|S|>d} \hat{f}(S)\varepsilon^S$$

defined on the Hamming cube $\Omega_n = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n\}$. Here $\hat{f}(S)$ are coefficients belonging to Banach space X.

Later we will also need the space of polynomials $\mathcal{P}_d(X)$:

$$f = \sum_{|S|\le d} \hat{f}(S)\varepsilon^S$$

defined on the Hamming cube Ω_n. Here $\hat{f}(S)$ are coefficients belonging to Banach space X, which, in particular can be just $\mathbf{R}$.

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The reader will notice that in this paper we are mostly interested in bounding the expressions of the type $|\nabla f|_X$ for polynomials of fixed degree on Hamming cube. It would be interesting to get from this the estimates of the expressions of the type $\Delta^{1/2} f$. For Banach space valued functions the comparison of $|\nabla f|_X$ and $\Delta^{1/2} f$ is mostly an open task.

For Banach space valued functions $f : \Omega_n \to X$ it is not quite clear who majorized whom if we deal with $\|\Delta^{1/2} f\|_{L^p(X)}$ and $\||\nabla f|_X\|_p$.

It would be of interest to decide for exactly what class of Banach spaces X

$$\|\Delta^{1/2} f\|_{L^p(X)} \leq C_p \||\nabla f|_X\|_p \,.$$

We think that this is the class of spaces of finite co-type. In fact, this is one way to express the boundedness *from below* of Riesz transform of Hamming cube in spaces $L^p(\Omega_n, X)$, $1 < p < \infty$. For $X = \mathbf{R}$ this boundedness from below is always true for $1 < p < \infty$, see the result of E. Ben Efraim and F. Lust-Piquard [1] proved by non-commutative technique.

On the other hand, the converse inequality, that is the boundedness of Riesz transform of Hamming cube *from above*,

$$\||\nabla f|_X\|_p \leq C_p \|\Delta^{1/2} f\|_{L^p(X)}$$

does not have a reasonably wide class of Banach spaces for which it holds [7], [8]. For $p \geq 2$ and $X = \mathbf{R}$ this boundedness from above holds, but for $1 < p < 2$ it fails even for $X = \mathbf{R}$, see [1].

Moreover, it can fail even for $p > 2$ for a very nice UMD space X. Morally this means that it is more difficult to estimate *from above* $|\nabla f|_X$ than $\Delta^{1/2} f$.

We give Bernstein–Markov type inequalities for $|\nabla f|_X$ in Theorems 3.5, 3.7, 4.1. The proofs are based on the novel formula used in [11] to prove Enflo's conjecture (one can familiarize with the conjecture in [18]).

In Theorem 2.6 we reprove a theorem of Mendel and Naor [17] concerning the estimate from below of $\|\Delta f\|_{L^p(X)}$ for f in the tail space $\mathcal{T}_d(X)$ for K-convex Banach spaces X. Our proof is quite different from the one in [17]. The main idea is to use analytic paraproduct operators. A lot is known about these operators first introduced by Pommerenke, see e.g. [21], [2], [23], and the connection with the tail space estimates on Hamming cube seems to be new and worth to explore.

The first addendum (Section 6) is devoted to the proof of Lemma 2.8. The second addendum (Section 7) gives an explanation how to get rid of ε in Theorem 2.3 from [4]. This explanation combined with the proof of Theorem 2.3 in Eskenazis–Ivanisvili's [4] gives an easier proof of Theorem 2.2, which is one of the main results of Mendel–Naor's [17].

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2. Tail spaces estimates and dependence on K-convexity and d

We first cite the problem. In [17], Mendel and Naor asked if for every K-convex Banach space X and $p \in (1, \infty)$ there exists a finite positive $c(p, X)$ such that for every n and $d < n$, every function $f : \Omega_n \to X$ in the $\mathcal{T}_d(X)$ satisfies the estimate

$$\|\Delta f\|_{L^p(X)} \geq c(p, X) d \|f\|_{L^p(X)} \,.$$

Moreover, they formulated the tail smoothing conjecture: for every K-convex Banach space X and $p \in (1, \infty)$ there exists a finite positive $c(p, X), C(p, X)$ such that for every n and $d < n$, every function $f : \Omega_n \to X$ in the $\mathcal{T}_d(X)$ satisfies the estimate

$$\|P_t f\|_{L^p(X)} \le c(p, X) e^{-C(p,X)d\,t} \|f\|_{L^p(X)} ,$$

where $P_t = e^{-t\Delta}$ is the heat semi-group on Hamming cube. The estimate for $\|\Delta f\|_{L^p(X)}$ would follow just by integrating from 0 to ∞. In Theorem 5.1 of [17] they proved that there exists $A(p, X) \ge 1$ such that

$$\|P_t f\|_{L^p(X)} \le c(p, X) e^{-C(p,X)d\, \min(t, t^{A(p,X)})} \|f\|_{L^p(X)} .$$

Before continuing let us cite a crucial result of Pisier [19].

Theorem 2.1 *(Pisier). Let X be a K-convex space. Then for any $p \in (1, \infty)$ there exists angle $\alpha\pi \in (0, \pi)$ such that operators $e^{-z\Delta}$ are well defined and uniformly bounded in $A_\alpha := \{z \in \mathbf{C} : |\arg z| \le \frac{\alpha\pi}{2}\}$.*

In what concerns Mendel–Naor question, they proved in [17] the following result.

Theorem 2.2 *(Mendel–Naor).*

$$\|P_t f\|_{L^p(X)} \le c(p, X) e^{-C(p,X)d\, \min(t, t^{\frac{1}{\alpha}})} \|f\|_{L^p(X)} ,$$

$$\|\Delta f\|_{L^p(X)} \ge c(p, X) d^\alpha \|f\|_{L^p(X)} .$$

In [4] the following weaker results were proved:

Theorem 2.3 *(Eskenazis–Ivanisvili). Let X be a K-convex Banach space and let its angle be provided by Theorem 2.1. Then for every $\varepsilon > 0$ we have*

$$\|P_t f\|_{L^p(X)} \le c(p, X, \varepsilon) e^{-C(p,X,\varepsilon)d\, \min(t, t^{\frac{1}{\alpha}+\varepsilon})} \|f\|_{L^p(X)} ,$$

$$\|\Delta f\|_{L^p(X)} \ge c(p, X, \varepsilon) d^{\alpha-\varepsilon} \|f\|_{L^p(X)} \quad \forall \varepsilon > 0.$$

The proof of the latter theorem was easier than that of Theorem 2.2, but there was a price to pay by the extra ε (the constants involved depend badly on ε).

Remark 2.4. In Theorem 2.6 we will prove differently the second inequality of Theorem 2.2. In what concerns the first inequality of Theorem 2.2 we explain in the Section 7 how one can adapt the simpler reasoning of [4,5] to get rid of ε as well.

Theorem 2.5 *(Eskenazis–Ivanisvili). Let $f \in \mathcal{P}_{d+m}(X) \cap \mathcal{T}_d(X)$. Let X be a K-convex Banach space. Then*

$$\|\Delta f\|_{L^p(X)} \ge c(p, X) \frac{d}{m} \|f\|_{L^p(X)} .$$

Moreover,

$$\|\Delta^{1/2} f\|_{L^p(X)} \ge c(p, X) \Big(\frac{d}{m}\Big)^{1/2} \|f\|_{L^p(X)} .$$

It is well known that the second inequality above implies the first one, just because of [14]:

$$\|\Delta^\beta f\|_{L^p(X)} \le 4\|\Delta f\|^\beta_{L^p(X)} \cdot \|f\|^{1-\beta}_{L^p(X)}\,.$$

Now we will prove the result that falls a bit short of the conjecture, but at least it gets rid of $1/m$ in the Theorem 2.5 above and gets rid of ε in Theorem 2.3.

Theorem 2.6. *Let X be a K-convex space, $p \in (1,\infty)$, and α from Pisier theorem be in $(0,1]$. Then*

$$\|\Delta f\|_{L^p(X)} \ge c(p,X) d^\alpha \|f\|_{L^p(X)}\,.$$

Remark 2.7. This theorem is not new, see [17] Theorem 5.1. But the proof is different and it uses the so-called analytic paraproduct operators, about which a lot is known.

Proof. Given f on Ω_n from the tail space $\mathcal{T}_d(X)$ let us introduce a new function F of one more variable:

$$F(w,\varepsilon) := \sum_{|S|>d} w^{|S|}\hat f(S)\varepsilon^S,$$

and let us recognize that it is a bounded $L^p(X)$-valued function of w in 2-gone $G_\alpha := \{w : w = e^{-z}, z \in A_\alpha\}$. This is just reformulation of Pisier's theorem. Moreover it is not only bounded but also holomorphic in G_α,

$$F \in H^\infty(G_\alpha; L^p(X))\,.$$

Notice that the same works for $-G_\alpha$, as the flip $w \to -w$ can be absorbed by the flip $\varepsilon \to -\varepsilon$. Let us consider a domain $O_\alpha := G_\alpha \cup -G_\alpha$. It is easy to see that its boundary is smooth at all points except -1 and 1, where O_α forms angle $\pi\alpha$ by two real analytic curves (actually its boundary is real analytic except for ± 1, where real analytic curves form an angle $\alpha\pi$). Operator

$$\sum_{|S|} \hat f(S)\varepsilon^S \to \sum_{|S|} w^{|S|}\hat f(S)\varepsilon^S$$

is uniformly bounded from $L^p(X)$ to itself by Theorem 2.1.

Notice that $wF'_w = \sum_S w^{|S|}|S|\hat f(S)\varepsilon^S$ and $\Delta f = \sum_S |S|\hat f(S)\varepsilon^S$. So $wF'(w)$ is obtained by applying Pisier's Fourier multiplier operator to Δf. Therefore we have

$$\|F'_w(w)\|_{H^\infty(O_\alpha,L^p(X))} \le C\|wF'_w(w)\|_{H^\infty(O_\alpha,L^p(X))} \le M\|\Delta f\|_{L^p(X)}\,. \tag{2.1}$$

The first inequality is a trivial maximal principle for holomorphic functions, the second one is again the same Pisier's theorem.

By the next step we want the estimate of the following type (for $F'(w)$ built by $f \in \mathcal{T}_d(X)$)

$$\|F(w)\|_{H^\infty(O_\alpha,L^p(X))} \le \varepsilon_d \|F'_w(w)\|_{H^\infty(O_\alpha,L^p(X))} \tag{2.2}$$

with a small ε_d for large d.

Let φ be the conformal map from the unit disc $\mathbf{D}$ onto O_α, $\varphi(0) = 0$. Then (2.2) is equivalent to

$$\|F\circ\varphi\|_{H^\infty(\mathbf{D},L^p(X))} \le \varepsilon_d \|F'\circ\varphi\|_{H^\infty(\mathbf{D},L^p(X))} \tag{2.3}$$

Obviously, for $z \in \mathbf{D}$

$$F \circ \varphi(z) = \int_0^z F' \circ \varphi(\zeta) \cdot \varphi'(\zeta) d\zeta \,.$$

Denote $g(\zeta) := F' \circ \varphi(\zeta)$ and introduce the operator (sometimes called *analytic paraproduct with symbol* φ)

$$T_\varphi g := \int_0^z g(\zeta) \cdot \varphi'(\zeta) d\zeta \,.$$

Seems like it was Pommerenke who studied this operator first. Then it was widely researched, see e.g. [21], [2], [23] and the references therein. We need to understand the estimate of this operator on the space of (vector-valued) functions with the property that all their Taylor coefficients at 0 vanish till the order $d-1$. In other words we need to understand the norm of the operator T_ψ, where $\psi = \int_0^\zeta \zeta^{d-1} \varphi'(\zeta)\, d\zeta$. $\square$

Let us write the Taylor expansion of conformal map φ:

$$\varphi(z) = c_1^\alpha z + c_2^\alpha z^2 + \dots \,.$$

Then it is possible to prove

Lemma 2.8.

$$|\varphi'(z)| \asymp \frac{1}{|1-z^2|^{1-\alpha}} \,. \tag{2.4}$$

And $|c_n^\alpha| \asymp n^{-1-\alpha}$.

See Section 6 for the proof.

We can write

$$T_\varphi g = \int_0^z g(\zeta) \cdot \varphi'(\zeta) d\zeta = c_1^\alpha \int_0^z g d\zeta + 2c_2^\alpha \int_0^z g \cdot \zeta d\zeta + \dots +$$
$$m c_m^\alpha \int_0^z g \cdot \zeta^{m-1} d\zeta + \dots \,.$$

Function $g(\zeta) \cdot \zeta^{m-1}$ in our case is from $\mathcal{T}_k, k \geq d$, and its $H^\infty(\mathbf{D})$ norm is exactly the $H^\infty(\mathbf{D})$ norm of g itself.

So we need to understand how to estimate the norm of integration operator on $\mathcal{T}_{d+m}, m \geq 0$, for all m in $H^\infty(\mathbf{D})$ norm (vector-valued, but this will not ne important).

Lemma 2.9. *Let k be positive integer. There exists an $L^1(\mathbf{T})$ function s with $L^1(\mathbf{T})$ norm at most $\frac{C_0}{k}$ such that $\hat{s}(k+j), j \geq 0$, is $\frac{1}{k+j}$.*

Suppose this Lemma is proved. The integration operator maps ζ^{k+j} into $\frac{\zeta^{k+j+1}}{k+j+1}$. The convolution operator $s(\zeta)\star\cdot$ maps ζ^{k+j} into $\frac{\zeta^{k+j}}{k+j}$. Thus the integration operator is multiplication by ζ composed with convolution with s. But multiplication on ζ does not change the norm on tail spaces of $H^\infty(\mathbf{D})$ (regardless of whether it is vector valued or scalar valued). Hence, if Lemma 2.9 is proved then we can estimate $g \in H^\infty(\mathbf{D}; L^p(X))$, $g(0) = 0, g'(0) = 0, \dots, g^{(d-1)}(0) = 0$:

$$\|Tg\|_{H^\infty(\mathbf{D};L^p(X))} \le C_0 \|g\|_{H^\infty(\mathbf{D};L^p(X))} \sum_{m=1}^{\infty} m|c_m^\alpha| \frac{1}{d+m-1}\,.$$

Using Lemma 2.8 we conclude

$$\|Tg\|_{H^\infty(\mathbf{D};L^p(X))} \le C \|g\|_{H^\infty(\mathbf{D};L^p(X))} \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \frac{1}{d+m-1} \le$$
$$C_\alpha \|g\|_{H^\infty(\mathbf{D};L^p(X))} d^{-\alpha}\,.$$

Hence we proved that

$$\|F \circ \varphi\|_{H^\infty(\mathbf{D};L^p(X))} \le C_\alpha \|F'_w \circ \varphi\|_{H^\infty(\mathbf{D};L^p(X))} d^{-\alpha}\,. \tag{2.5}$$

Then we get (2.2):

$$\|F\|_{H^\infty(O_\alpha;L^p(X))} \le C_\alpha \|F'_w\|_{H^\infty(O_\alpha;L^p(X))} d^{-\alpha}\,.$$

We can now combine this with (2.1) and obtain

$$\|F\|_{H^\infty(O_\alpha;L^p(X))} \le C_\alpha\, M \|\Delta F\|_{L^p(X))} d^{-\alpha}\,.$$

But $F(1) = f$. Hence we get the proof of the theorem modulo Lemma 2.9:

$$\|f\|_{L^p(X))} \le C_\alpha\, M \|\Delta F\|_{L^p(X))} d^{-\alpha}, \quad \forall f \in \mathcal{T}_d(X)\,.$$

Proof. The proof of Lemma 2.9. We assume that k is even, which is enough for the proof. We consider first $S(x) = \sum_{j=1}^\infty \frac{\sin jx}{j}$, it is bounded and has Fourier coefficients as we wish: $1/j$, $j \ne 0$. Now we wish to change its Fourier coefficients in the interval $j \in [-k, k]$ and not change the Fourier coefficients outside this interval, and make the $L^1(-\pi,\pi)$ norm of modified function to be at most C_0/k. Consider nodes $x_r := \frac{\pi r}{k+1}$, $r = -k+1, \dots, -3, -1, 1, 3, \dots, k-1$. The number of nodes is k. Construct the Lagrange trigonometric interpolation polynomial $L_k(x)$,

$$L_k(x) = \sum_r S(x_r) \frac{\Pi_{m\ne r}(\sin x - \sin x_m)}{\Pi_{m\ne r}(\sin x_r - \sin x_m)}\,.$$

Clearly $L_k(x)$ has non-zero Fourier coefficients only on $[-k+1, k-1]$. It is easy to check that it is an odd function. Notice that the sign of $S(x) - L_k(x)$ alternates, it is a positive function on $[0, \frac{\pi}{k+1})$, negative on $(\frac{\pi}{k+1}, \frac{3\pi}{k+1})$, et cetera.

Consider "triangular" cos, call it $c(x)$, it is linear on $[-\pi, 0]$, linear on $[0, \pi]$ and $c(0) = 1, c(\pm\pi) = -1$. Its integral vanishes. So if we consider $c((k+1)x)$ its Fourier coefficients vanish on $[-k, k]$. Then $c'((k+1)x)$ also has its Fourier coefficients vanishing on $[-k, k]$, in particular, the scalar product in $L^2(-\pi,\pi)$

$$\langle L_k(x), c'((k+1)x)\rangle = 0\,.$$

Therefore,

$$\langle S(x) - L_k(x), c'((k+1)x)\rangle = \langle S(x), c'((k+1)x)\rangle .$$

But $c'((k+1)x = \pm 1$, moreover the pattern of signs repeats the pattern of signs of $S(x) - L_k(x)$, namely it is a positive function on $[0, \frac{\pi}{k+1})$, negative on $(\frac{\pi}{k+1}, \frac{3\pi}{k+1})$, et cetera. We conclude that

$$\|S(x) - L_k(x)\|_1 = \langle S(x) - L_k(x), c'((k+1)x)\rangle = \langle S(x), c'((k+1)x)\rangle .$$

But obviously the right hand side here is at most C_0/k again by noticing that the oscillation of S on intervals of order $\asymp 1/k$ is c/k. Lemma 2.9 is proved and thus the tail theorem is proved as well. □

Remark 2.10. Lemma 2.9 should be very well known and widely used, but I am grateful to Rostislav Matveev [16] for this elegant proof of Lemma 2.9.

3. On Bernstein–Markov inequality and the dependence on X and p

By $L^p(X)$ we always mean $L^p(\Omega_n; X)$.

We first cite four theorems from [4].

Theorem 3.1 *(Eskenazis–Ivanisvili). Let X be an arbitrary Banach space and $p \in [1, \infty]$. Then*

$$\|\Delta f\|_{L^p(X)} \le d^2 \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X) .$$

Theorem 3.2 *(Eskenazis–Ivanisvili). Let X be a Banach space and $p \in [1, \infty]$. Then if for all n*

$$\|\Delta f\|_{L^p(X)} \le (1-\eta) d^2 \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X),$$

then X is of finite co-type.

Theorem 3.3 *(Eskenazis–Ivanisvili). Let X be a K-convex Banach space and $p \in (1, \infty)$. Then*

$$\|\Delta f\|_{L^p(X)} \le C(p, X) d^{2-\varepsilon(p,X)} \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X),$$

Theorem 3.4 *(Eskenazis–Ivanisvili). Let $X = \mathbf{R}$ and $p \in (1, \infty)$. Then*

$$\|\Delta f\|_{L^p} \le C(p) d^{2-\frac{2}{\pi}\arcsin\frac{2\sqrt{p-1}}{p}} \|f\|_{L^p}, \quad \forall f \in \mathcal{P}_d(X) .$$

Now we will prove the following results.

Theorem 3.5. *Let $X = \mathbf{R}$, $p \in [2, \infty)$. Then*

$$\|\nabla f\|_{L^p} \le C(p) d^{1-\frac{1}{\pi}\arcsin\frac{2\sqrt{p-1}}{p}} \|f\|_{L^p}, \quad \forall f \in \mathcal{P}_d(X) .$$

If $p \in (1, 2)$ then

$$\|\nabla f\|_{L^p} \le C(p) d^{\frac{2}{p} - \frac{2\arcsin\frac{2\sqrt{p-1}}{p}}{p\pi}} \|f\|_{L^p}, \quad \forall f \in \mathcal{P}_d(X) .$$

Remark 3.6. It is almost Theorem 15 of [4], but we get rid of $\log d$ term in the estimate (30) of Theorem 15 of [4].

We recall the reader that if $1 < p < \infty$ then $\|\Delta^{1/2} f\|_{L^p} \le C(p) \|\nabla f\|_{L^p}$ for scalar valued functions (the result of E. Ben Efraim and F. Lust-Piquard [1]). But the opposite inequality true only for $2 \le p < \infty$, [1]. Morally this means that it is more difficult to estimate from above $|\nabla f|$ than $\Delta^{1/2} f$ (even for scalar valued f). Also clearly the power of d doubles up when we pass the estimate from $\|\Delta^{1/2} f\|_{L^p}$ to the estimate of $\|\Delta f\|_{L^p}$.

When we deal with the $\mathcal{P}_d(X)$ and X^* has type 2, it also has finite co-type $r \in [2, \infty)$ by König–Tzafriri theorem (see [9], Theorem 7.1.14). Then we have the following result.

Theorem 3.7. *Let X^* be of type 2 (and automatically of certain co-type $r < \infty$), and $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\||\nabla f|_X\|_{L^p} \le C(p) d^{2 - \frac{2}{\max(q,r)}} \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X),$$

where

$$|\nabla f|_X = \big(\sum_{i=1}^n \|D_i f\|_X^2\big)^{1/2}.$$

In Section 4 we will prove another version of this theorem. Theorem 4.1 below establishes the similar estimate but with a different power of d.

Proof. We recall the formula from [11]:

$$D_j e^{-t\Delta} f(\varepsilon) = \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \mathbf{E}_\xi \Big(\delta_j(t) f(\varepsilon_1 \cdot \xi_1(t), \dots, \varepsilon_n \cdot \xi_n(t))\Big). \tag{3.1}$$

Here

$$\delta_j(t) := \frac{\xi_j(t) - e^{-t}}{(1 - e^{-2t})^{1/2}},$$

where $\xi_j(t)$ are independent random variables having values ± 1 with probabilities $\frac{1 \pm e^{-t}}{2}$.

From (3.1) for every $\varepsilon \in \Omega_n$ we can write

$$|\nabla e^{-t\Delta} f(\varepsilon)| = \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{\ell^2_n} = 1} \Big| \mathbf{E}_\xi \sum_{j=1}^n \lambda_j \delta_j(t) f(\varepsilon \cdot \xi(t)) \Big|.$$

Hence,

$$|\nabla e^{-t\Delta} f(\varepsilon)| \le \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{\ell^2_n} = 1} \mathbf{E}_\xi \Big| \sum_{j=1}^n \lambda_j \delta_j(t) f(\varepsilon \cdot \xi(t)) \Big| \le$$

$$\frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{\ell^2_n} = 1} \Big(\mathbf{E}_\xi \Big| \sum_{j=1}^n \lambda_j \delta_j(t) \Big|^q \Big)^{1/q} \Big(\mathbf{E}_\xi |f(\varepsilon \cdot \xi)|^p \Big)^{1/p}.$$

Raise it to the power p and integrate:

$$\||\nabla e^{-t\Delta} f(\varepsilon)|\|_p^p \le \Big(\frac{e^{-t}}{(1 - e^{-2t})^{1/2}}\Big)^p \mathbf{E}_\varepsilon \mathbf{E}_\xi |f(\varepsilon \cdot \xi)|^p \cdot \max_{\lambda : \|\lambda\|_{\ell^2_n} = 1} \Big(\mathbf{E}_\xi \Big| \sum_{j=1}^n \lambda_j \delta_j(t) \Big|^q \Big)^{p/q} =$$

$$\mathbf{E}_\xi \mathbf{E}_\varepsilon |f(\varepsilon\cdot\xi)|^p \cdot \Big(\frac{e^{-t}}{(1-e^{-2t})^{1/2}}\Big)^p \cdot \max_{\lambda:\|\lambda\|_{\ell^2_n}=1} \Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \delta_j(t)\Big|^q\Big)^{p/q} =$$

$$\Big(\frac{e^{-t}}{(1-e^{-2t})^{1/2}}\Big)^p \|f\|_p^p \cdot \max_{\lambda:\|\lambda\|_{\ell^2_n}=1} \Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \delta_j(t)\Big|^q\Big)^{p/q}.$$

Consider the case $1 < q \le 2$. Then we just use

$$\Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \delta_j(t)\Big|^q\Big)^{p/q} \le \Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \delta_j(t)\Big|^2\Big)^{p/2} = 1,$$

because $\{\delta_j(t)\}_{j=1}^n$ is an orthonormal system and $\|\lambda\|_{\ell^2_n} = 1$.

We will use this later:

$$1 < q \le 2 \Rightarrow \||\nabla e^{-t\Delta} f(\varepsilon)|\|_p^p \le \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \|f\|_p^p \tag{3.2}$$

Now let us consider the case $q > 2$. In this case we need to estimate $\Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \delta_j(t)\Big|^q\Big)^{1/q}$ differently. First of all we can replace $\delta_j(t)$ by

$$\tilde\delta_j(t) := \frac{\xi_j(t) - \xi_j'(t)}{(1-e^{-2t})^{1/2}}$$

with $\xi_j'(t)$ be an independent copy of $\xi_j(t)$. This is just by Jensen inequality and $\mathbf{E}\xi_j'(t) = e^{-t}$. Random variables are symmetric and we use the following result.

The following contraction principle is a classical result of Maurey and Pisier (see, e.g., [20, Proposition 3.2]). We spell out a version with explicit constants. □

Theorem 3.8. *Let $(X, \|\cdot\|)$ be a Banach space of cotype $r < \infty$, let $\tilde\delta_1, \dots, \tilde\delta_n$ be i.i.d. symmetric random variables, and let ε be uniformly distributed on $\{-1,1\}^n$. Then for any $n \ge 1$, $\lambda_1, \dots, \lambda_n \in X$, and $1 \le q < \infty$, we have*

$$\Big(\mathbf{E}\Big\|\sum_{j=1}^n \lambda_j\ \tilde\delta_j\Big\|^q\Big)^{1/q} \le L_{r,q} \int_0^\infty \mathbf{P}\{|\tilde\delta(t)_1| > s\}^{\frac{1}{\max(q,r)}} ds \Big(\mathbf{E}\Big\|\sum_{j=1}^n \lambda_j \varepsilon_j\Big\|^q\Big)^{1/q}$$

with $L_{r,q} = L\, C_q(X) \max(1, (r/q)^{1/2})$, where L is a universal constant.

In the current situation $X = \mathbf{R}$, so $r = 2$ and $\max(q,2) = q$. Notice also that

$$\int_0^\infty \mathbf{P}\{|\xi_j(t) - \xi_j'(t)| > s\}^{1/q} ds = 2^{1-1/r}(1-e^{-2t})^{1/q}.$$

Therefore,

$$\Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \delta_j(t)\Big|^q\Big)^{1/q} \le C(1-e^{-2t})^{1/q-1/2} \Big(\mathbf{E}_\xi \Big|\sum_{j=1}^n \lambda_j \varepsilon_j\Big|^q\Big)^{1/q} \le$$

$$\frac{C(q)}{(1-e^{-2t})^{1/2-1/q}}$$

by Khintchine inequality and by $\|\lambda\|_{\ell^2_n} = 1$.

We will use this later: if $q > 2$ then

$$\||\nabla e^{-t\Delta} f(\varepsilon)|\|_p \le \frac{C(q)e^{-t}}{(1-e^{-2t})^{1-1/q}}\|f\|_p = \frac{C(q)e^{-t}}{(1-e^{-2t})^{1/p}}\|f\|_p\,. \tag{3.3}$$

Now let us use (3.2) for $p \ge 2$ and (3.3) for $1 < p < 2$ to finish the proof. We can consider $x = e^{-t}$ and write those inequalities as the estimate of p-th norm of

$$F_f(x,\varepsilon) := \sum_S x^{|S|}\hat{f}(S)\varepsilon_S, \quad \text{where } f = \sum_S \hat{f}(S)\varepsilon_S,\ 0 \le x \le 1\,.$$

We get

$$\||\nabla F(x,\cdot)|\|_p \le \frac{|x|}{(1-x^2)^{1/2}}\|f\|_p, \quad \forall x \in [-1,1], p \ge 2 \tag{3.4}$$

and

$$\||\nabla F(x,\cdot)|\|_p \le \frac{|x|}{(1-x^2)^{1/p}}\|f\|_p, \quad \forall x \in [-1,1], 1 < p < 2\,. \tag{3.5}$$

We initially have this estimates only for $0 \le x \le 1$ but flipping $x \to -x$ is absorbed by flipping $\varepsilon \to -\varepsilon$. By other methods these estimates were obtained also in [4], see (229) and (203) there.

Now we consider an auxiliary domain of the type considered in [4]. Let us fix $\beta \in (1,2)$ to be chosen later. Fix $r > 1$. Consider lens domain $\Omega(r) = \{z : |z - i\sqrt{r^2-1}| \le r,\ |z + i\sqrt{r^2-1}| \le r\}$. Consider

$$\Omega(r,\beta) := \Big(1 - \frac{1}{d^\beta}\Big)\Omega(r)\,.$$

Let $G_{\beta,r}$ denote Green's function with pole at infinity of $\mathbf{C}\setminus\Omega(r,\beta)$. It is rather easy to see that

$$G_{\beta,r}(1) \asymp d^{-\beta\frac{\pi}{2\pi - 2\arcsin\frac{2\sqrt{p-1}}{p}}}\,, \tag{3.6}$$

(notice that $2\pi - 2\arcsin\frac{2\sqrt{p-1}}{p}$ is the exterior angle for $\Omega(r,\beta)$ at corner points of the lens).

We choose β in (3.6) to have $G_{\beta,r}(1) \asymp \frac{1}{d}$, that is

$$\beta = 2 - \frac{2}{\pi}\arcsin\frac{2\sqrt{p-1}}{p}\,. \tag{3.7}$$

3.1. Complex variable

Below we repeatedly use estimates on Green's function and harmonic measure of relatively simple domains that can be found in [3], [13], [6], [24]. In fact, all those domains below can be transformed to strip domains considered in [24] by logarithmic mapping. Thorough estimates of harmonic measure on strip domains are given in [24]. Green's function estimates and harmonic measure estimates are basically the same things for simply connected domains [6].

Consider a new function in the complex domain:

$$H(z) := \log\||\nabla F(z,\cdot)|\|_p\,.$$

Notice that this function is subharmonic in the whole $\mathbf{C}$. To see this one should write the norm of the gradient as the supremum over the dual space $L^q(\Omega_n, \ell_n^2)$. Then we will get that $\||\nabla F(z,\cdot)|\|_p$ is the supremum over the unit ball of this dual space of the absolute values of linear combinations of $D_j F(z,\varepsilon)$. Each such term is analytic and logarithm of absolute value of linear combination of such terms is subharmonic. The supremum can be interchanged with logarithm and we get that $H(z)$ is subharmonic.

Let us collect properties of H. As f is a polynomial of degree d, we get that the growth of H at infinity is majorized by $d\log|z|$.

In the other hand, we can always think that $\|f\|_p = 1$, and then we just saw that on the interval $[-1+\frac{1}{d^\beta}, 1-\frac{1}{d^\beta}]$ function $H(x)$ has the estimate:

$$F(x)/d^{\beta/2} \le 1, \quad \text{if } p \ge 2; \quad F(x)/Cd^{\beta/p} \le 1, \quad \text{if } 1 < p < 2\,.$$

Then, say, $H(z) - \frac{\beta}{2}\log d$ is non positive on $[-1+\frac{1}{d^\beta}, 1-\frac{1}{d^\beta}]$ and is of order $d\log|z|$ at infinity.

But we can say much more by Weissler [25] and Ivanisvili–Nazarov [10]. It tuns out that then $H(z) - \frac{\beta}{2}\log d$ is non positive on $\mathbf{C}\setminus\Omega_{r,\beta,}$. These are the complex hypercontractivity results.

Hence, using Green's function $G_{\beta,r}$ of $\mathbf{C}\setminus\Omega_{r,\beta}$ with pole at infinity we get that

$$H(z) - \frac{\beta}{2}\log d \le dG_{\beta,r}(z)$$

uniformly in $\mathbf{C}\setminus G_{\beta,r}$. Hence,

$$\frac{\||\nabla F(z,\cdot)|\|_p}{d^{\beta/2}} \le e^{dG_{\beta,r}(z)}\,.$$

We are interested in this inequality for just one particular $z = 1$. Now we use (3.7) to have $e^{dG_{\beta,r}(1)} \asymp 1$. Hence we proved that for $p \ge 2$

$$\frac{\||\nabla F(1,\cdot)|\|_p}{d^{\beta/2}} \le C\,.$$

Exactly the same reasoning shows that for $1 < p < 2$

$$\frac{\||\nabla F(1,\cdot)|\|_p}{d^{\beta/p}} \le C\,.$$

Theorem 3.5 is completely proved just by plugging formula (3.7) for β.

Proof. The proof of Theorem 3.7 follows the same lines, but we need to use Theorem 3.8 for Banach spaces X^*. This is where we use that if X^* is of type 2 then is of finite co-type by König–Tzafriri theorem 7.1.14 in [9]. Type 2 is needed to conclude (using Khintchine–Kahane's inequality, see e.g. [9]):

$$\mathbf{E}_\varepsilon\Big\|\sum \varepsilon_j\lambda_j\Big\|_p \le C_p\mathbf{E}_\varepsilon\Big\|\sum \varepsilon_j\lambda_j\Big\|_2 \le C_p\big(\sum\|\lambda_j\|_{X^*}^2\big)^{1/2} \le C_p\,. \quad \square$$

3.2. Comparison of $\Delta^{1/2}\cdot$ and $|\nabla\cdot|_X$

The reader can notice that in this paper we are mostly interested in bounding the expressions of the type $|\nabla f|_X$. It would be interesting to get from this the estimates of the expressions of the type $\Delta^{1/2} f$. But for Banach space valued functions it is mostly an open task.

For Banach space valued functions $f : \Omega_n \to X$ it is not quite clear who majorized whom if we deal with $\|\Delta^{1/2} f\|_{L^p(X)}$ and $\||\nabla f|_X\|_p$.

We would like to decide (but this is still an open problem) for exactly what class of Banach spaces X

$$\|\Delta^{1/2} f\|_{L^p(X)} \le C_p \||\nabla f|_X\|_p \,.$$

We think that this is the class of spaces of finite co-type. In fact, this is one way to express the boundedness *from below* of Riesz transform of Hamming cube in spaces $L^p(\Omega_n, X)$, $1 < p < \infty$. For $X = \mathbf{R}$ this boundedness from below is always true, see [1].

On the other hand, the converse inequality, that is the boundedness of Riesz transform of Hamming cube *from above*,

$$\||\nabla f|_X\|_p \le C_p \|\Delta^{1/2} f\|_{L^p(X)}$$

does not have a reasonably wide class of Banach spaces for which it holds. For $p \ge 2$ and $X = \mathbf{R}$ this boundedness from above holds, but for $1 < p < 2$ it fails, see [1].

It can fail for very nice UMD space X even for $p > 2$.

4. Another exponent in Theorem 3.7

By $L^p(X)$ we always mean $L^p(\Omega_n; X)$, where Ω_n is Hamming cube. Let $1/q + 1/p = 1$. Let $\mathcal{P}(d, X)$ be the collection of polynomials with coefficients in Banach space X and of degree at most d. We prove that Theorem 3.7 can be given a different formulation in the sense of the power of d. Next theorem deals again with X such that X^* is of type 2. In particular, X^* and X are K-convex. Let $\pi\alpha$ denote the angle from Theorem 2.1.

Theorem 4.1. *If $f \in \mathcal{P}(d, X)$ and X^* is of type 2, then $\||\nabla f|_X\|_{L^p} \le C d^{\frac{2-\alpha}{p}} \|f\|_{L^p(X)}$ for $1 < p < 2$, and $\||\nabla f|_X\|_{L^p} \le C d^{1-\frac{\alpha}{2}} \|f\|_{L^p(X)}$ for $p \ge 2$.*

Proof. We recall the formula from [11]:

$$D_j e^{-t\Delta} f(\varepsilon) = \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \mathbf{E}_\xi \Big(\delta_j(t) f(\varepsilon_1 \cdot \xi_1(t), \dots, \varepsilon_n \cdot \xi_n(t)) \Big) \,. \tag{4.1}$$

Here

$$\delta_j(t) := \frac{\xi_j(t) - e^{-t}}{(1-e^{-2t})^{1/2}},$$

where $\xi_j(t)$ are independent random variables having values ± 1 with probabilities $\frac{1 \pm e^{-t}}{2}$.

The symmetric counterpart is

$$\delta'_j(t) := \frac{\xi_j(t) - \xi'_j(t)}{(1-e^{-2t})^{1/2}},$$

where vector $\{\xi'_j(t)\}$ is independent copy of $\{\xi_j(t)\}$.

We use the notation ℓ^2_n for $\ell^2_n(X^*)$ with the norm $(\sum_{j=1}^n \|\lambda_j\|^2_{X^*})^{1/2}$. From (4.1) for every $\varepsilon \in \Omega_n$ we can write

$$|\nabla e^{-t\Delta} f(\varepsilon)| = \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{\ell^2_n} = 1} \Big(\mathbf{E}_\xi \Big| \sum_{j=1}^n \delta_j(t) \langle \lambda_j, f(\varepsilon \cdot \xi) \rangle \Big| \Big) \,.$$

Hence,

$$|\nabla e^{-t\Delta} f(\varepsilon)| \le \frac{e^{-t}}{(1-e^{-2t})^{1/2}} \max_{\lambda:\|\lambda\|_{\ell^2_n}=1} \mathbf{E}_\xi \Big|\sum_{j=1}^n \delta_j(t)\langle \lambda_j, f(\varepsilon\cdot\xi(t))\rangle\Big| =$$

$$\frac{e^{-t}}{(1-e^{-2t})^{1/2}} \max_{\lambda:\|\lambda\|_{\ell^2_n}=1} \Big(\mathbf{E}_\xi\Big|\langle\sum_{j=1}^n \delta_j(t)\lambda_j, f(\varepsilon\cdot\xi)\rangle\Big|\Big) \le$$

$$\frac{e^{-t}}{(1-e^{-2t})^{1/2}} \max_{\lambda:\|\lambda\|_{\ell^2_n}=1} \mathbf{E}_\xi\Big(\Big\|\sum_{j=1}^n \delta_j(t)\lambda_j\Big\|_{X^*} \|f(\varepsilon\cdot\xi)\|_X\Big) \le$$

$$\frac{e^{-t}}{(1-e^{-2t})^{1/2}} \max_{\lambda:\|\lambda\|_{\ell^2_n}=1} \Big(\mathbf{E}_\xi\Big\|\sum_{j=1}^n \lambda_j\delta_j(t)\Big\|^q\Big)^{1/q} \Big(\mathbf{E}_\xi\|f(\varepsilon\cdot\xi)\|^p\Big)^{1/p}.$$

We wish to prove that if $q > 2$ then

$$\||\nabla e^{-t\Delta} f(\varepsilon)|\|_p \le \frac{C(q)e^{-t}}{(1-e^{-2t})^{1/p}} \|f\|_{L^p(X)}, \tag{4.2}$$

and if $1 \le q \le 2$ then

$$\||\nabla e^{-t\Delta} f(\varepsilon)|\|_p \le \frac{C(q)e^{-t}}{(1-e^{-2t})^{1/2}} \|f\|_{L^p(X)}. \tag{4.3}$$

For that let us work now with the term $\mathbf{E}_\xi\Big\|\sum_{j=1}^n \lambda_j\delta_j(t)\Big\|^q_{X^*}$ for a fixed $\{\lambda_j\} \in \ell^2_n$ of norm 1.

$$B^q := \mathbf{E}_\xi\Big\|\sum_{j=1}^n \lambda_j\delta_j(t)\Big\|^q \le \mathbf{E}_{\xi,\xi'}\Big\|\sum_{j=1}^n \lambda_j\delta_j'(t)\Big\|^q =$$

$$\mathbf{E}_{\xi,\xi'}\mathbf{E}_r\Big\|\sum_{j=1}^n r_i\lambda_j\delta_j'(t)\Big\|^q,$$

where r_i are independent Rademacher random variables.

The next lemma was provided by A. Borichev.

Lemma 4.2. *Let $a, b \ge 0$ and $Q \ge 2$ be a large number. Then*

$$(a+b)^Q \le 6a^Q + Q^Q b^Q.$$

Proof. We need to show that for all positive t, $(t+1)^Q \le 6t^Q + Q^Q$. If $t \le Q-1$ this is immediate. If $t \ge Q-1 \ge 1$, we write

$$(t+1)^Q = t^Q\Big(\frac{t+1}{t}\Big)^Q \le t^Q\Big(\frac{t+1}{t}\Big)^{t+1} \le 2t^Q\Big(1+\frac{1}{t}\Big)^t \le 2et^Q. \quad \square$$

We continue to estimate B^q:

$$B^q \le \mathbf{E}_{\xi,\xi'}\mathbf{E}_r\Big\|\sum_{j=1}^n r_i\lambda_j\delta_j'(t)\Big\|^q_{X^*} \le C_q\mathbf{E}_{\xi,\xi'}\Big(\mathbf{E}_r\Big\|\sum_{j=1}^n r_i\lambda_j\delta_j'(t)\Big\|^2_{X^*}\Big)^{q/2} \le$$

$$C_q D^{q/2}\mathbf{E}_{\xi,\xi'}\Big(\sum_{j=1}^n |\delta_j'(t)|^2\|\lambda_j\|^2_{X^*}\Big)^{q/2}.$$

In the last inequality we used that X^* is of type 2. The penultimate inequality is Kahane–Khintchine's inequality, see [9].

Notice that if $1 \le q \le 2$ then the above inequality gives

$$B^q \le C_q D^{q/2} \Big(\mathbf{E}_{\xi,\xi'} \sum_{j=1}^n |\delta'_j(t)|^2 \|\lambda_j\|^2_{X^*} \Big)^{q/2}$$

Hence,

$$1 \le q \le 2 \Rightarrow B \le C_q D^{1/2} \Big(\sum_{j=1}^n \|\lambda_j\|^2_{X^*} \Big)^{1/2} \le C_q D^{1/2} \,. \tag{4.4}$$

The estimate in case $2 < q < \infty$ is much more interesting.

Now we will continue by thinking that q is an even integer, $q = 2k$ (it is not important, just convenient). Let us now estimate

$$E := \mathbf{E}_{\xi,\xi'} \Big(\sum_{j=1}^n |\delta'_j(t)|^2 \|\lambda_j\|^2 \Big)^k \tag{4.5}$$

We denote

$$f_j := |\delta'_j(t)|^2 \|\lambda_j\|^2 \,. \tag{4.6}$$

Below we use Lemma 4.2 with $Q := q/2 - 1 = k - 1$:

$$\begin{gathered}
\mathbf{E}_{\xi,\xi'} (\textstyle\sum_{j=1}^n f_j)^k = \mathbf{E}_{\xi,\xi'} \sum_{i=1}^n (\sum_{j=1}^n f_j)^{k-1} f_i = \\
\mathbf{E}_{\xi,\xi'} \textstyle\sum_{i=1}^n ((f_1 + \dots f_{i-1} + f_{i+1} + \dots f_n) + f_i)^{k-1} f_i \le^{Lemma\ 4.2} \\
(k-1)^{k-1} \mathbf{E}_{\xi,\xi'} \textstyle\sum_{i=1}^n f_i^k + 6 \mathbf{E}_{\xi,\xi'} \sum_{i=1}^n (f_1 + \dots f_{i-1} + f_{i+1} + \dots f_n)^{k-1} \mathbf{E}_{\xi,\xi'} f_i \le \\
(k-1)^{k-1} \mathbf{E}_{\xi,\xi'} \textstyle\sum_{i=1}^n f_i^k + 6 \mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j)^{k-1} \mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j) \le \\
(k-1)^{k-1} \mathbf{E}_{\xi,\xi'} \textstyle\sum_{i=1}^n f_i^k + 6(k-2)^{k-2} \mathbf{E}_{\xi,\xi'} \sum_{i=1}^n f_i^{k-1} \mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j) + \\
6^2 \mathbf{E}_{\xi,\xi'} (\textstyle\sum_{j=1}^n f_j)^{k-2} \big[\mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j) \big]^2 \le \\
(k-1)^{k-1} \mathbf{E}_{\xi,\xi'} \textstyle\sum_{i=1}^n f_i^k + \dots + 6^\ell (k-\ell)^{k-\ell} \mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j^{k-\ell}) \big[\mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j) \big]^\ell + \dots \\
+ 6^{k-1} \big[\mathbf{E}_{\xi,\xi'} (\textstyle\sum_{j=1}^n f_j) \big]^k \,.
\end{gathered}$$

We used the fact that $\delta'_j(t)$, $j = 1, \dots, n$, are independent exactly as this has been done in Rosenthal's [22].

Now coming back to our notation (4.6) we see that as

$$\mathbf{E} |\delta'_j(t)|^2 = 2, \quad \mathbf{E} |\delta'_j(t)|^m \le \frac{2^{m-1}}{\sqrt{1 - e^{-2t}}^{m-2}} \,. \tag{4.7}$$

$$\mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j) \le 2 \|\{\lambda_j\}\|^2_{\ell^2_n} ,$$

$$\mathbf{E}_{\xi,\xi'} (\sum_{j=1}^n f_j^{k-\ell}) \le \frac{2^{2k-2\ell}}{\sqrt{1 - e^{-2t}}^{2k-2\ell-2}} \sum_{j=1}^n \|\lambda_j\|^{2k-2\ell} \,.$$

Therefore, we can estimate E from (4.5) as follows:

$$E \le 24^k \sum_{\ell=0}^{k-2} \Big[\frac{(k-\ell)^{k-\ell}}{\sqrt{1-e^{-2t}}^{2k-2\ell-2}} \sum_{j=1}^n \|\lambda_j\|^{2k-2\ell} \|\{\lambda_j\}\|^{2\ell} \Big] + 24^k \|\{\lambda_j\}\|^{2k} .$$

This obviously gives

$$E \le 2(24)^k \sum_{\ell=0}^{k-2} (k-\ell)^{k-\ell} \Big(\frac{1}{\sqrt{1-e^{-2t}}} \Big)^{2k-2\ell-2} \Big(\sum_{j=1}^n \|\lambda_j\|^2 \Big)^{k-\ell} \|\{\lambda_j\}\|^{2\ell} + 24^k \|\{\lambda_j\}\|^{2k} .$$

And so,

$$E \le C'(q) \|\{\lambda_j\}\|^q \sum_{\ell=0}^{k-2} (k-\ell)^{k-\ell} \Big(\frac{1}{\sqrt{1-e^{-2t}}} \Big)^{2k-2\ell-2}$$

Then

$$B \le \frac{C(q)}{(1-e^{-2t})^{\frac12 - \frac1q}} \|\{\lambda_j\}\|_{\ell^2_n(X^*)} = \frac{C(q)}{(1-e^{-2t})^{\frac12 - \frac1q}} . \tag{4.8}$$

Now let us use (4.2) for $1 < p < 2$ and (4.3) for $p \ge 2$ to finish the proof. We can consider $x = e^{-t}$ and write those inequalities as the estimate of p-th norm of

$$F_f(x, \varepsilon) := \sum_S x^{|S|} \hat{f}(S) \varepsilon_S, \quad \text{where } f = \sum_S \hat{f}(S) \varepsilon_S,\ 0 \le x \le 1 .$$

We get from (4.8) and (4.4) correspondingly that

$$\| |\nabla F(x, \cdot)| \|_p \le \frac{|x|}{(1-x^2)^{1/p}} \|f\|_{L^p(X)}, \quad \forall x \in [-1,1], 1 < p < 2 . \tag{4.9}$$

$$\| |\nabla F(x, \cdot)| \|_p \le \frac{|x|}{(1-x^2)^{1/2}} \|f\|_{L^p(X)}, \quad \forall x \in [-1,1], p \ge 2 . \tag{4.10}$$

We initially have this estimates only for $0 \le x \le 1$ but flipping $x \to -x$ is absorbed by flipping $\varepsilon \to -\varepsilon$.

Now consider a new function in the complex domain:

$$H(z) := \log \| |\nabla F(z, \cdot)|_X \|_p .$$

We repeat verbatim the reasoning of Section 3.1 but instead of domain $\Omega \setminus \Omega(\beta, r)$ and its Green's function, we consider domain $\mathbf{C} \setminus [-1 + \frac{1}{d^2}, 1 - \frac{1}{d^2}]$, whose Green's function G_d satisfies

$$G_d(1) \asymp \frac{1}{d} .$$

This proves

$$\frac{\| |\nabla F(1, \cdot)|_X \|_p}{d^{\max(2/p, 1)}} \le C .$$

But (4.8) and (4.4) can be used more efficiently if we use Pisier's Theorem 2.1 again. In fact, it can be used. As X^* has type 2, it is K convex. Then X is K-convex. Let us fix β to be chosen later and consider domain

$$O_{\beta,\alpha} := \Big(1 - \frac{1}{d^\beta}\Big) O_\alpha,$$

where O_α was introduced in the previous Section.

As X is K-concave, so is $\ell^2(X)$. Consequently (4.8) and (4.4) and Pisier's Theorem 2.1 applied to $L^p(\Omega_n, \ell^2(X))$ show that

$$\| |\nabla F(z, \cdot)|_X \| \le C d^{\beta/p},\ 1 < p < 2, \quad \le C d^{\beta/2},\ p \ge 2, \quad z \in O_{\beta,\alpha}\,.$$

Let $G_{\beta,\alpha}$ denote Green's function of $\mathbf{C} \setminus \bar{O}_{\beta,\alpha}$.

We repeat verbatim the reasoning of Section 3.1 but instead of domain $\Omega \setminus \Omega(\beta, r)$ and its Green's function, we consider domain $\mathbf{C} \setminus \bar{O}_{\beta,\alpha}$ whose Green's function $G_{\beta,\alpha}$ satisfies

$$G_{\beta,\alpha}(1) \asymp \Big(\frac{1}{d^\beta}\Big)^{\frac{\pi}{2\pi - \pi\alpha}} \asymp \frac{1}{d}\,,$$

if

$$\beta = 2 - \alpha\,.$$

This proves

$$\frac{\| |\nabla F(1, \cdot)|_X \|_p}{d^{\max(\frac{2-\alpha}{p}, \frac{2-\alpha}{2})}} \le C\,.$$

Theorem 4.1 is proved. □

Remark 4.3. We already mentioned in Section 3.2 that the boundedness of Riesz transform of Hamming cube *from above*,

$$\| |\nabla f|_X \|_p \le C_p \| \Delta^{1/2} f \|_{L^p(X)}$$

does not have a reasonably wide class of Banach spaces for which it holds. For $p \ge 2$ and $X = \mathbf{R}$ this boundedness from above holds, but for $1 < p < 2$ it fails, see [1]. It can fail for a very nice UMD space X even for $p > 2$. Therefore, Theorem 3.3 or other Bernstein–Markov type estimates of $\Delta^{1/2} f$ in [4] for X-valued polynomials f with X being K-convex cannot help to prove the estimates of the type of Theorem 4.1 or Theorem 3.7.

5. Non-commutative random variables and Bernstein–Markov inequalities on Hamming cube

We wish to demonstrate how the technique of non-commutative random variables can be used to prove certain Bernstein–Markov inequalities on Hamming cube. The estimates below are not as good as in the previous Section, and what follows serves only illustrative purpose of showing a beautiful approach.

To the best of our knowledge this approach was introduced by Francoise Lust-Piquard in [14], [1]. Almost all results of those papers are commutative (with the exception of results on CAR algebra), all methods are non-commutative. And even though many non-commutative proofs of those papers are by now made commutative (see, e.g. [12]), still some non-commutative proofs did not get a commutative analog up to now.

The non-commutative proof of a certain Bernstein–Markov inequality below is given not because of its efficiency, but because of its beauty.

We will prove now that for $p \geq 2$

$$f : \Omega_n \to \mathbf{R}, \deg f \leq d \Rightarrow \||\nabla f|\|_p \leq C_p\, d \|f\|_p, \tag{5.1}$$

which is worse than Theorem 3.5.

Let

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; P = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \; U = iQP.$$

They have anti-commutative relationship

$$QP = -PQ\,. \tag{5.2}$$

Let $Q_j = I \otimes \dots Q \otimes I \cdots \otimes I$, $P_j = I \otimes \dots P \otimes I \cdots \otimes I$, on j-th place. These are independent non-commutative random variables in the sense of tr = sum of diagonal elements divided by 2^n.

Put $Q_A = \Pi_{i \in A} Q_i$, $P_A = \Pi_{i \in A} P_i$

Now one considers algebra generated by Q_j, P_j (this is algebra of all matrices $\mathcal{M}_{2^n}$). We have a projection $\mathcal{P}$ from multi-linear polynomials in P_j, Q_j (notice $P^2 = I, Q^2 = I$) that kills everything except terms having only $Q's$.

Small (really easy) algebra shows (see [1]) that $\mathcal{P}$ can be written as $\rho\, Diag\, \rho^*$, where ρ is a conjugation by a unitary operator, and $Diag$, is an operator on matrices that just kills all matrix elements except the diagonal. This $Diag$ is obviously the contraction on Schatten-von Neumann class S_p for any $p \in [1, \infty]$ (obvious for Hilbert–Schmidt, $p = 2$, class and for bounded operators–so interpolation does that).

$$\mathcal{R}(\theta) Q_A = \Pi_{j \in A} (Q_j \cos\theta + P_j \sin\theta)\,, \; \mathcal{R}(\theta) P_A = \Pi_{j \in A} (P_j \cos\theta - Q_j \sin\theta)\,.$$

One can easily check that the action of $\mathcal{R}(\theta)$ is $R(\theta)^* T R(\theta)$ where $R(\theta)$ is a unitary matrix which is n-fold tensor product of

$$\rho_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

Extend it by linearity onto the whole algebra $\mathcal{M}_{2^n}$. Then it is obvious that automorphism $\mathcal{R}(\theta)$ preserves all Schatten–von Neumann S_p norms.

For any $f = \sum_{A \subset [n]} \hat{f}(A) \varepsilon^A$, the reasoning of [1] dictates to assign a non-commutative object, a matrix from $\mathcal{M}_{2^n}$ given by

$$T_f = \sum_{A \subset [n]} \hat{f}(A) Q_A\,.$$

Such matrices form commutative sub-algebra $M_{2^n} \subset \mathcal{M}_{2^n}$. Operators ∂_j, D_j can be considered on M_{2^n}, acting in a canonical way. For example,

$$\partial_i Q_A = \begin{cases} Q_{A \setminus i}, & \text{if } i \in A; \\ 0, & \text{if } i \notin A\,. \end{cases}$$

And $D_i := Q_i \partial_i$.

Consider now a matrix valued function

$$A_f(\theta) = \mathcal{R}(\theta) T_f \,.$$

It is a trigonometric polynomial of degree at most d with matrix coefficients. Bernstein–Markov inequality (its proof) works for such matrix valued polynomials in exactly the same way as for scalar polynomials. The easiest way to see that is to prove Bernstein–Markov estimate by convolution with Fejer kernels. Then we get

$$\Big\| \frac{d}{d\theta} A_f(\theta) \Big\|_{S_p} \le 2d \Big\| A_f(\theta) \Big\|_{S_p}, \quad 1 \le p \le \infty.$$

On the other hand we can calculate easily $\frac{d}{d\theta}\mathcal{R}(\theta)(Q_A) =$

$$-\sum_{j\in A} \Pi_{i\in A, i<j}(\cos\theta Q_i + \sin\theta P_i)((-\sin\theta Q_j + \cos\theta P_j)\Pi_{i\in A, i>j}(\cos\theta Q_i + \sin\theta P_i)\,.$$

By commutativity relations between P_j, Q_i, we observe that this is nothing else but $-\sum_{j\in A} \mathcal{R}(\theta)\big(P_j \partial_j Q_A\big)$. Hence

$$\frac{d}{d\theta} A_f(\theta) = \frac{d}{d\theta}\mathcal{R}(\theta) T_f = -\mathcal{R}(\theta)\Big(\sum_{j=1}^n P_j \partial_j T_f\Big)\,. \tag{5.3}$$

Therefore,

$$\|\mathcal{R}(\theta)\Big(\sum_{j=1}^n P_j \partial_j T_f\Big)\|_{S_p} \le 2d\, \|\mathcal{R}(\theta) T_f\|_{S_p}\,.$$

Transformation $\mathcal{R}_\theta$ preserves S_p norms (see above), and so

$$\|\sum_{j=1}^n P_j \partial_j T_f\|_{S_p} \le 2d\, \|T_f\|_{S_p}\,. \tag{5.4}$$

Let $\varepsilon_j^{(k)} = -1$ if $j = k$ and $= 1$ otherwise. Following [1] we see that $\|\sum_{j=1}^n P_j \partial_j T_f\|_{S_p} = \|\sum_{j=1}^n \varepsilon_j^{(k)} P_j \partial_j T_f\|_{S_p}$. This is because

$$Q_k\big(\sum_{j=1}^n P_j \partial_j Q_A\big) Q_k = \sum_{j=1}^n \varepsilon_j^{(k)} P_j \partial_j Q_A, \quad \forall A,$$

by anti-commutative relation $PQ = -QP$. Hence, for any sequence of signs

$$\|\sum_{j=1}^n P_j \partial_j T_f\|_{S_p} = \|\sum_{j=1}^n \varepsilon_j P_j \partial_j T_f\|_{S_p}\,.$$

Now one should use a non-commutative Khintchine inequality of Lust-Piquard and Pisier [15] and $2 \le p \le \infty$:

$$\mathbf{E}_\varepsilon \|\sum_{j=1}^n \varepsilon_j P_j \partial_j T_f\|_{S_p} \asymp_p \|\big(\sum_{j=1}^n (\partial_j T_f)^* P_j^* P_j \partial_j T_f\big)^{1/2}\|_{S_p} +$$

$$\Big(\sum_{j=1}^n P_j \partial_j T_f (\partial_j T_f)^* P_j^*\Big)^{1/2} \|_{S_p}\,.$$

But $P_j^* P_j = P_j^2 = I$, and in the second term P_j and $\partial_j T_f$ commute (as there is identity matrix on the j-th place of $\partial_j T_f$). Therefore

$$\|\Big(\sum_{j=1}^n (\partial_j T_f)^* P_j^* P_j \partial_j T_f\Big)^{1/2}\|_{S_p} + \Big(\sum_{j=1}^n P_j \partial_j T_f (\partial_j T_f)^* P_j^*\Big)^{1/2}\|_{S_p} =$$

$$2\|\Big(\sum_{j=1}^n (\partial_j T_f)^* \partial_j T_f\Big)^{1/2}\|_{S_p}\,.$$

Using (5.4) we conclude that for $p \in [2, \infty)$

$$\|\Big(\sum_{j=1}^n (\partial_j T_f)^* \partial_j T_f\Big)^{1/2}\|_{S_p} \le C_p d \|T_f\|_{S_p}\,.$$

Both matrices in the left hand side and the right hand side are form commutative algebra M_n. They are T_f and $T_{|\nabla f|}$. It is left to notice that for any scalar function f on Ω_n we have $\|T_f\|_{S_p} = \|f\|_{L^p(\Omega_n}$. This is just by using the basis of characteristic function of point sets $\{\varepsilon\}$ on Ω_n to compute the S_p norm of T_f. This basis consists of eigenfunctions of T_f with eigenvalues $f(\varepsilon)$. This is easy, see in [1].

We finally proved (5.1) by non-commutative approach of Francoise Lust-Piquard.

6. Addendum 1: Fourier coefficients of conformal map φ

We consider the domain

$$O_\alpha := -G_\alpha \cup G_\alpha,$$

where $G_\alpha = \{w : w = e^{-z}, |\arg z| \le \frac{\pi\alpha}{2}\}$. It is not very difficult to write down the boundary of this domain (see Section 7 below, where we partially do this). Then one can notice that it consists of two real analytic curve Γ_+, Γ_-, symmetric with respect to $\mathbf{R}$ and forming angle $\pi\alpha$ at $-1, 1$.

Hence, the conformal map $\varphi^{-1} : O_\alpha \to \mathbf{D}$ can be extended to a slightly wider domain bounded by real analytic curves γ_+, γ_-, such that γ_+ lies a bit higher than Γ_+ and meets Γ_+ at ± 1, and forms angle $\tau\pi$ with Γ_+ at points ± 1, where τ is a small strictly positive number. Symmetrically for γ_-, Γ_-.

Then conformal map φ is extended to domain $\mathcal{R}$ bounded by two symmetric real analytic curves, intersecting $\mathbf{T}$ only at ± 1 and making angle $(\alpha + \tau)\pi$ with $\mathbf{T}$ at those points.

Then

$$c_m = \int_{\mathbf{T}} \frac{1}{z^{m+1}} \varphi(z) dz = -\frac{1}{m} \int_{\partial\mathcal{R}} \varphi(z) d\frac{1}{z^m} = \frac{1}{m} \int_{\partial\mathcal{R}} \frac{1}{z^m} \varphi'(z) dz$$

Now we us that on $\partial\mathcal{R}$, ± 1 we have $|\frac{1}{z}| \le \frac{1}{1 + a(\tau)|y|}$ for $z = x + iy$. We get

$$|c_m| \lesssim \int_0^2 \frac{1}{(1 + ay)^m} \frac{1}{y^{1-\alpha}} dy \lesssim \int_0^2 e^{-a_1 m\, y}\, dy^\alpha = \int_0^{2^\alpha} e^{-b(m^\alpha t)^{1/\alpha}}\, dt\,.$$

The last integral is $\le \frac{1}{m^\alpha} \int_0^\infty e^{-bs^{\frac{1}{\alpha}}}\, ds \lesssim \frac{1}{m^\alpha}$.

7. Addendum 2: boundary of O_α and getting rid of ε in the proof of Theorem 6 of [4]

This section explains how to get rid of an ε in a theorem of Eskenazis and Ivanisvili [4] This ε was also avoided by a different approach already by Mendel and Naor [17]. But the proof in [4] is easier and it seems to be nice to push it to the level of result of [17] by eliminating this irritating ε that otherwise blows up the constants. This is what we will do now.

We consider two domains

$$\Omega(r) := \{z \in \mathbf{C} : \max\{|z - i\sqrt{r^2-1}|, |z + i\sqrt{r^2-1}|\} < r\},$$

and

$$O_\alpha := -G_\alpha \cup G_\alpha,$$

where $G_\alpha = \{w : w = e^{-z}, |\arg z| \le \frac{\pi\alpha}{2}\}$. We would like to compare those two domains for

$$\pi\alpha = 2\arcsin\frac{1}{r}\,. \tag{7.1}$$

The choice of r is dictated by the fact that for this choice the angle that the boundaries have at point 1 is the same (and symmetrically at -1).

It is not very difficult to write down the boundary of O_α, we will do this now for its parts near points ± 1.

Define a as follows $\tan\frac{\pi\alpha}{2} = \frac{\pi}{a}$. Let us consider $G_\alpha \cap \{\Re z \in [0, \frac{a}{2}]\}$. Consider $G_\alpha(a/2) := e^{-G_\alpha \cap \{\Re z \in [0,\frac{a}{2}]\}} = \{w = u + iv = e^{-z},\ z \in G_\alpha \cap \Re z \in [0, \frac{a}{2}]\}$. It consists of arcs S_t of the circles centered at point $(0,0)$ of radii $e^{-t}, 0 \le t \le \frac{a}{2}$, and each arc is symmetric (w.r. to $\mathbf{R}$), and has angle $2\arctan\frac{\pi}{a}t$. In particular, $S_{a/2}$ is a half-circle that intercepts v-axis at points $\pm e^{-a/2}$. The boundary of the domain $G_\alpha(a/2)$ consists of $S_{a/2}$ and of two real analytic symmetric (w.r. to $\mathbf{R}$) arcs, one of them $\Gamma(a/2)$ (the one in $\mathbf{C}_+$) being given by parametric equation:

$$\Gamma(a/2): \quad u = e^{-t}\cos\frac{\pi}{a}t, \quad v = e^{-t}\sin\frac{\pi}{a}t, \quad 0 \le t \le a/2\,.$$

We also have an interesting circle of radius $r := \sqrt{1 + \frac{a^2}{\pi^2}} = \frac{1}{\sin(\frac{\pi\alpha}{2})}$, with center at $-i\sqrt{r^2-1} = -i\frac{a}{\pi} = -i\,\mathrm{cotan}(\frac{\pi\alpha}{2})$.

Let us check that $\Gamma(a/2)$ lies below the circle, in other words that

$$\left(e^{-t}\cos\frac{\pi}{a}t\right)^2 + \left(e^{-t}\sin\frac{\pi}{a}t + \frac{a}{\pi}\right)^2 < 1 + \frac{a^2}{\pi^2}, \text{ for small } t > 0\,.$$

This is the same as

$$e^{-2t} + 2e^{-t}\frac{a}{\pi}\sin\frac{\pi}{a}t < 1\,.$$

We write

$$1 - 2t + \frac{4t^2}{2} - \frac{8t^3}{6} + \cdots + 2(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + ...)(t - ct^3 + \dots) =$$
$$1 + t^3 - \frac{4}{3}t^3 - 2ct^3 + \cdots < 1,$$

if t is small as c is positive. So the lens domain $\Omega(r)$ of [4] seems to contain O_α, at least it is not contained in it as $\Gamma(a/2)$ lies inside $\Omega(r)$.

That represents a small problem for [4] because inclusion (103) there is not valid if one chooses r according to our preferred choice (7.1). In its turn this is reflected in the formulas for conformal mapping one uses around (103). But the formula for conformal mapping of the unit disc onto $\Omega(r)$ is straightforward.

But if one chooses r not according to (7.1) but smaller, than the angle of the lens domain at ± 1 is smaller than $\pi\alpha$ and inclusion (103) holds. Thus Theorem 6 of [4] reproves the heat smoothing result of [17] with $A(p, X) > \frac{1}{\alpha}$, where α is the angle from Pisier's Theorem 2.1. But one can notice that just a small improvement in [4] reasoning gives the heat smoothing result with $A(p, X) = \frac{1}{\alpha}$.

Let us indicate this small change that should be implemented to get $A(p, X) = \frac{1}{\alpha}$ in Theorem 6 of [4].

As, in the contrast to (103) of [4], we have $\Omega_\alpha \subset \Omega(r)$ with r as in (7.1), then one need the estimates of conformal mapping of the disk onto O_α (the smaller of two domains). Of course the angle that boundary of O_α form at point 1 (and -1) is just $\pi\alpha$ (in notations of [4] it is θ). This angle is the same for $\Omega(r)$. But this observation is not enough to conclude the same asymptotic for conformal maps on these two domains.

However, this is a not a real problem. It is easy to see that asymptotic is in fact the same. To see that one transforms Ω_α and $\Omega(r)$ to strips by logarithmic map and then one uses Warschawski's estimate from [24], pages 280–281. It shows that asymptotic is the same because one can easily compute that $\int^\infty \Theta'(u)^2/\Theta(u)\, du$ converges, see [24], pages 280–281, for the explanation what is $\Theta(u)$ for strips.

The heat smoothing conjecture of [17] claims that $A(p, X) = 1$ for K-convex X, but it is still a conjecture. The important time is $t_0 = \frac{1}{d^\alpha}$. The estimate of Theorem 2.6, or slightly strengthened estimate of Theorem 6 of [4] or Theorem 5.1 of [17], all those estimates show that if X is K convex, then for X-valued f in the d-tail space

$$\|e^{-t_0\Delta} f\|_{L^p(X)} \le C\|f\|_{L^p(X)}\,.$$

This does not give us any interesting information. What the heat smoothing conjecture basically says is the following, let α be the angle from Pisier's Theorem 2.1, then

$$t_0 = \frac{1}{d^\alpha} \Rightarrow \|e^{-t_0\Delta} f\|_{L^p(X)} \le \varepsilon(d)\|f\|_{L^p(X)}, \quad \varepsilon(d) \to 0,\, d \to \infty\,.$$

This is still open.

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