

Optical conductivity of a metal near an Ising-nematic quantum critical point

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We study the optical conductivity of a pristine two-dimensional electron system near an Ising-nematic quantum critical point. We discuss the relation between the frequency scaling of the conductivity and the shape of the Fermi surface, namely, whether it is isotropic, convex, or concave. We confirm the cancellation of the leading order terms in the optical conductivity for the cases of isotropic and convex Fermi surfaces and show that the remaining contribution scales as $|\omega|^{2/3}$ at $T = 0$. On the contrary, the leading term, $\propto |\omega|^{-2/3}$, survives for a concave FS. We also address the frequency dependence of the optical conductivity near the convex-to-concave transition. Explicit calculations are carried out for the Fermi-liquid regime using the modified (but equivalent to the original) version of the Kubo formula, while the quantum-critical regime is accessed by employing the space-time scaling of the $Z = 3$ critical theory.

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I. INTRODUCTION

The optical conductivity of a correlated electron system contains important information about the strength of electron-electron (ee) interaction. This information is encoded both in the renormalization of the Drude weight and in the frequency and/or temperature scaling of the conductivity. Experimental and theoretical studies of optical conductivity is a very active research area [1–5].

Unlike the dc conductivity, which is rendered finite only either by umklapp ee scattering [6,7] or Baber (electron-hole) scattering in compensated metals [7,8], the optical conductivity is relatively free of the constraints imposed by momentum conservation: as long as the electron spectrum is not parabolic, the optical conductivity is rendered finite by ee interaction even in the absence of umklapps/compensation. In the Fermi liquid (FL) regime, the optical conductivity so far has been shown to adhere to two scaling forms. The first one is due to Gurzhi [9]:

$$\sigma'(\omega, T) \propto \frac{\omega^2 + 4\pi^2 T^2}{\omega^2}. \quad (1)$$

This form occurs in the presence of umklapp scattering or compensation both in two dimensions (2D) and three dimensions (3D), but also in 3D even without umklapps/compensation, as long as the electron dispersion contains higher than quadratic terms [10]. The second scaling

form,

$$\sigma'(\omega, T) \propto \frac{\omega^2 + 4\pi^2 T^2}{\omega^2} (3\omega^2 + 8\pi^2 T^2) \times \begin{cases} \ln \max\{|\omega|, T\}, & 2D \\ 1, & 3D \end{cases}, \quad (2)$$

is believed to describe systems with isotropic but nonparabolic spectrum, e.g., Dirac metals [11,12], in the absence of umklapp scattering/compensation.¹ At $T = 0$, Eqs. (1) and (2) are reduced to const and ω^2 , respectively (modulo a $\ln |\omega|$ factor in 2D). A suppression of $\sigma'(\omega, 0)$ in Eq. (2) compared to Eq. (1) reflects partial Galilean invariance of an isotropic system. The limiting forms of Eq. (2), i.e., ω^2 and T^4/ω^2 , have been derived by a large number of authors in a variety of contexts [13–19]. Both scaling forms are valid for $\omega \gg 1/\tau_j(T)$, where $\tau_j(T)$ is the appropriate current relaxation time at finite T , which behaves as $1/T^2$ under the conditions when Eq. (1) is applicable or as $1/T^4$ (modulo a $\ln T$ factor in 2D) under the conditions when Eq. (2) is applicable.

The case of a 2D anisotropic Fermi surface (FS) (again, in the absence of umklapp scattering/compensation) is more complicated. There, the conductivity depends on whether the FS is convex or concave, i.e., on whether it has inflection points. In the dc case, the FL-like T^2 correction to the residual resistivity vanishes for a convex FS, while the surviving term behaves as $T^4 \ln T$ [10,13–15,20]. On the other hand, the correction to the residual resistivity for a concave 2D FL is free of such cancellations and behaves as T^2 . A similar cancellation for a convex FS is expected to occur for the optical conductivity [3] and, indeed, it has recently been shown to be the case for a non-FL at the Ising-nematic quantum critical point

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¹Although the primary focus of Ref. [11] is on Dirac metals, the analysis there is applicable to any nonparabolic spectrum.

TABLE I. Optical conductivity of a 2D electron system near an Ising-nematic quantum critical point (QCP) for different types of Fermi surfaces. FL stands for the Fermi-liquid region.

$\sigma'(\omega, T)$	Fermi surface			
	Isotropic with nonparabolic spectrum or convex		Concave	
	FL	QCP	FL	QCP
$\omega \gg T$	$\omega^2 \ln \omega $	$ \omega ^{2/3}$	const	$ \omega ^{-2/3}$
$\omega \ll T$	$T^4 \ln T / \omega^2$	$T^{8/3} / \omega^2$	T^2 / ω^2	$T^{4/3} / \omega^2$

(QCP) [22], while Ref. [23] arrived at the same result regardless of the shape (convex vs concave) of the FS. The issues pertaining to the surviving term in the optical conductivity for a convex FS as well as the leading term for a concave FS have not been analyzed even in the FL regime. One of the goals of this paper is to resolve these outstanding issues. We will show explicitly that $\sigma'(\omega, T)$ for a 2D FL with convex/concave FS scales as predicted by Eqs. (2) and (1), respectively.

Another goal of this paper is to generalize the results for an electron system at the Ising-nematic QCP, which is an example of the $D = 2, Z = 3$ criticality [24,25]. This subject has a long and somewhat controversial history. The optical conductivity of such a system [in the context of fermions coupled to a $U(1)$ gauge field] was claimed in Ref. [26] to behave as

$$\sigma'(\omega, 0) \propto |\omega|^{-2/3}. \quad (3)$$

Later, it was realized, however, that the current-relaxing mechanism in Ref. [26] was not properly specified [3,22,23,27,28]. As we already mentioned, subsequent studies demonstrated the vanishing of the leading $|\omega|^{-2/3}$ term for a convex FS [22]. In this paper, we revisit the issue of the $-2/3$ scaling of the optical conductivity near an Ising-nematic QCP. We confirm that the $|\omega|^{-2/3}$ term vanishes both for an isotropic but nonparabolic spectrum and for a convex FS. We also show that the surviving term in the FL regime behaves according to Eq. (2) for both cases.² When extrapolated to the immediate vicinity of the Ising-nematic QCP, our result translates into

$$\sigma'(\omega, T) \propto |\omega|^{2/3} \max\{1, (T/|\omega|)^{8/3}\}. \quad (4)$$

The $T = 0$ limit of our result, $\sigma'(\omega, 0) \propto |\omega|^{2/3}$, contradicts the $\sigma'(\omega, 0) = \text{const}$ result of Ref. [22]. With respect to a concave FS, we show explicitly that the conductivity is restored back to the original form, Eq. (3). For finite T , we find

$$\sigma'(\omega, T) \propto |\omega|^{-2/3} \max\{1, (T/|\omega|)^{4/3}\}. \quad (5)$$

The $T = 0$ limit of this result was also obtained in Ref. [28] for a model of fermions coupled to loop current fluctuations. Following Refs. [10] and [29], we also obtain the scaling form of the conductivity near the convex-to-concave transition. For the reader's convenience, we summarize the results for the optical conductivity in various regimes in Table I.

The rest of the paper is organized as follows. In Sec. II, we formulate the model and introduce the modified version of the

Kubo formula that offers a practical advantage in calculation of the optical conductivity. In Sec. III, we apply this formalism to an isotropic electron system with a nonparabolic energy spectrum. In Secs. IV and V, we derive the optical conductivity of metals with convex and concave FSs, respectively, and also near the convex-to-concave transition. For each case, we discuss the frequency scaling of the conductivity both in the FL and quantum-critical regions. In Sec. VI we summarize our main findings and provide a broader discussion. Appendices A–C present some technical details of our calculations.

II. GENERAL FORMALISM

A. Hamiltonian and charge currents

We consider the following model Hamiltonian:

$$H = H_0 + gH_{\text{int}}, \quad (6a)$$

$$H_0 = \sum_{\mathbf{k}s} \epsilon_{\mathbf{k}} c_{\mathbf{k},s}^\dagger c_{\mathbf{k},s}, \quad (6b)$$

$$H_{\text{int}} = \frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) d_{\mathbf{q}} d_{-\mathbf{q}} = \frac{1}{2} \sum_{\mathbf{k}\mathbf{p}\mathbf{q}s s'} U(\mathbf{k}, \mathbf{p}, \mathbf{q}) c_{\mathbf{k},s}^\dagger c_{\mathbf{p},s'}^\dagger c_{\mathbf{p}+\mathbf{q},s'} c_{\mathbf{k}-\mathbf{q},s}, \quad (6c)$$

where $c_{\mathbf{k},s}^\dagger (c_{\mathbf{k},s})$ are the fermion creation (annihilation) operators for a particle in a state with momentum $\mathbf{k}$ and spin projection s ,

$$\mathbf{k}_\pm = \mathbf{k} \pm \mathbf{q}/2, \text{ and } \mathbf{p}_\pm = \mathbf{p} \pm \mathbf{q}/2. \quad (7)$$

The coupling constant g is factored out for convenience. Furthermore, $\epsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}} - E_F$, $\epsilon_{\mathbf{k}}$ is the band dispersion, E_F is the Fermi energy,

$$d_{\mathbf{q}} = \sum_{\mathbf{k},s} F(\mathbf{k}) c_{\mathbf{k},s}^\dagger c_{\mathbf{k}-\mathbf{q},s} \quad (8)$$

is the charge density in the angular momentum channel with a form-factor $F(\mathbf{k})$, and $U(\mathbf{k}, \mathbf{p}, \mathbf{q}) = F(\mathbf{k})F(\mathbf{p})V(\mathbf{q})$. For simplicity, we assume that the FS has at least fourfold symmetry, although this assumption is by no means crucial. The form factor is normalized such that

$$\int \frac{d\phi_{\mathbf{k}}}{2\pi} F^2(\mathbf{k}) = 1, \quad (9)$$

where $\phi_{\mathbf{k}}$ is the azimuthal angle of $\mathbf{k}$. For a purely density-density interaction, $F(\mathbf{k}) = 1$ and $U(\mathbf{k}, \mathbf{p}, \mathbf{q}) = V(\mathbf{q})$. The interaction responsible for a quantum phase transition is

²The actual form of the scaling function is somewhat different for the nematic system due to another contribution to the charge current; see Sec. II A.

modeled by the standard Orenstein-Zernike form

$$V(\mathbf{q}) = \frac{1}{q^2 + q_B^2}, \quad (10)$$

where q_B is the bosonic mass, equal to the inverse correlation length of the order parameter fluctuations. We assume that the system is close to the QCP, such that $q_B \ll k_F$. At finite q_B , the system is in a FL regime at low enough energies, namely, for $\max\{\omega, T\} \ll \omega_{FL}$, where

$$\omega_{FL} = v_F q_B^3 / g N_F, \quad (11)$$

v_F is the Fermi velocity, appropriately averaged over the FS, and N_F is the density of states at the Fermi energy. In the FL regime, the dynamic interaction can be replaced by the static one, which is what we did in Eq. (10). Our explicit calculations will be carried out in the FL regime only. To extend the results to the immediate vicinity of the QCP, we will invoke the space-time scaling of the $Z = 3$ critical theory and replace q_B by $\max\{|\omega|^{1/3}, T^{1/3}\}$ [27].

As usual, the (longitudinal part of) charge current is deduced from the continuity equation

$$\mathbf{q} \cdot \mathbf{j}_q = -[\rho_q, H], \quad (12)$$

where $\rho_q = e \sum_{\mathbf{k}s} c_{\mathbf{k}+,s}^\dagger c_{\mathbf{k}-,s}$ is the charge density operator. The commutator $[\rho, H_0]$ yields the single-particle part of the current, the $\mathbf{q} = 0$ component of which is given by

$$\mathbf{j}_0 = e \sum_{\mathbf{k}s} \mathbf{v}_k c_{\mathbf{k},s}^\dagger c_{\mathbf{k},s}, \quad (13)$$

where $\mathbf{v}_k = \nabla_k \varepsilon_k$ is the group velocity. However, the interaction part of the Hamiltonian Eq. (6c) does not commute with the charge density and, therefore, there is one more contribution to the current. Computing the commutator $[\rho_q, H_{int}]$, we obtain for the interaction part of the current at $\mathbf{q} = 0$ (see Appendix A for details)

$$\mathbf{j}_{int} = e \sum_{\mathbf{k}\mathbf{p}\mathbf{q}\mathbf{s}\mathbf{s}'} [(\nabla_k + \nabla_p)U(\mathbf{k}, \mathbf{p}, \mathbf{q})] c_{\mathbf{k}+,s}^\dagger c_{\mathbf{p}-,s'}^\dagger c_{\mathbf{p}+,s'} c_{\mathbf{k}-,s}, \quad (14)$$

where $\nabla_k = (\hat{\phi}/k)\partial_{\phi_k}$. The total current is given by the sum

$$\mathbf{j} = \mathbf{j}_0 + g\mathbf{j}_{int}. \quad (15)$$

B. Modified Kubo formula for the optical conductivity

Due to the fourfold symmetry of our model, $\sigma_{xx} = \sigma_{yy}$, and one can define the conductivity as $\sigma \equiv (\sigma_{xx} + \sigma_{yy})/2$. To calculate the conductivity, we will be using the modified version of the Kubo formula. The standard Kubo formula relates the real part of the conductivity to the imaginary part of the (retarded) current-current correlation function,

$$\sigma'(\omega, T) \equiv \text{Re } \sigma(\omega, T) = -\frac{1}{\omega} \text{Im } \Pi_j(\omega, T), \quad (16)$$

where

$$\begin{aligned} \Pi_j(\omega, T) &= -\frac{i}{2} \int_0^\infty dt e^{i\omega t} \langle [\mathbf{j}(t) ; \mathbf{j}(0)] \rangle \\ &\equiv -\frac{i}{2} \langle [\mathbf{j}(t) ; \mathbf{j}(0)] \rangle_\omega \end{aligned} \quad (17)$$

and $[\mathbf{a}; \mathbf{b}] = \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a}$. In this approach, the dissipative (real) part of the conductivity is nonzero only if the interaction

is dynamic, which means that the bare static interaction in Eq. (6c) needs to be renormalized by dynamic particle-hole pairs. An equivalent version of the Kubo formula is obtained from integrating Eq. (17) by parts [11,16,17]

$$\sigma'(\omega, T) = \frac{1}{2\omega^3} \text{Im } \langle [\mathbf{K}(t) ; \mathbf{K}(0)] \rangle_\omega, \quad (18)$$

where $\mathbf{K}(t) = i\partial_t \mathbf{j} = [\mathbf{j}, H]$. While Eq. (16) contains a two-particle Green's function, Eq. (18) contains a four-particle Green's function which, in general, is a more complicated object. However, to calculate the optical conductivity to order $O(g^2)$, it suffices to find $\mathbf{K}$ to order $O(g^2)$:

$$\mathbf{K}(t) = [\mathbf{j}_0 + g\mathbf{j}_{int}, H_0 + gH_{int}] = g\mathbf{K}_1(t) + g\mathbf{K}_2(t) + O(g^2), \quad (19)$$

where

$$\mathbf{K}_1(t) = [\mathbf{j}_0, H_{int}], \quad (20a)$$

$$\mathbf{K}_2(t) = [\mathbf{j}_{int}, H_0]. \quad (20b)$$

Because the product of $\mathbf{K}(t)$ and $\mathbf{K}(0)$ in Eq. (18) is already of order g^2 , the averaging in this equation can be performed over the free-fermion states. The advantage of Eq. (18) is that it accounts only for current-relaxing processes without the need of combining diagrams for the current-current correlation function. In this approach, the renormalization of the bare static interaction by dynamic particle-hole pairs is accounted for automatically, when the product of two four-fermion correlators in Eq. (18) is averaged over the quantum states of a noninteracting system.

Substituting Eq. (19) into Eq. (18), we obtain the conductivity as the sum of three terms:

$$\sigma'(\omega, T) = \sigma'_1(\omega, T) + \sigma'_2(\omega, T) + \sigma'_{12}(\omega, T), \quad (21)$$

where

$$\sigma'_1(\omega, T) = \frac{g^2}{2\omega^3} \text{Im } \langle [\mathbf{K}_1(t) ; \mathbf{K}_1(0)] \rangle_\omega, \quad (22a)$$

$$\sigma'_2(\omega, T) = \frac{g^2}{2\omega^3} \text{Im } \langle [\mathbf{K}_2(t) ; \mathbf{K}_2(0)] \rangle_\omega, \quad (22b)$$

$$\begin{aligned} \sigma'_{12}(\omega, T) &= \frac{g^2}{2\omega^3} \text{Im} \{ \langle [\mathbf{K}_1(t) ; \mathbf{K}_2(0)] \rangle_\omega \\ &\quad - \langle [\mathbf{K}_1(t) ; \mathbf{K}_2(0)] \rangle_{-\omega} \}. \end{aligned} \quad (22c)$$

For the commutators in Eqs. (20a) and (20b) we obtain (see Appendix B for details)

$$\mathbf{K}_1 = -\frac{e}{2} \sum_{\mathbf{k}\mathbf{p}\mathbf{q}\mathbf{s}\mathbf{s}'} \Delta \mathbf{v} U(\mathbf{k}, \mathbf{p}, \mathbf{q}) c_{\mathbf{k}+,s}^\dagger c_{\mathbf{p}-,s'}^\dagger c_{\mathbf{p}+,s'} c_{\mathbf{k}-,s}, \quad (23a)$$

$$\mathbf{K}_2 = e \sum_{\mathbf{k}\mathbf{p}\mathbf{q}\mathbf{s}\mathbf{s}'} \Delta \varepsilon (\nabla_k + \nabla_p) U(\mathbf{k}, \mathbf{p}, \mathbf{q}) c_{\mathbf{k}+,s}^\dagger c_{\mathbf{p}-,s'}^\dagger c_{\mathbf{p}+,s'} c_{\mathbf{k}-,s}, \quad (23b)$$

where

$$\Delta \mathbf{v} = \mathbf{v}_{\mathbf{k}+} + \mathbf{v}_{\mathbf{p}-} - \mathbf{v}_{\mathbf{k}-} - \mathbf{v}_{\mathbf{p}+} \quad (24)$$

and

$$\Delta \varepsilon = \varepsilon_{\mathbf{k}+} + \varepsilon_{\mathbf{p}-} - \varepsilon_{\mathbf{k}-} - \varepsilon_{\mathbf{p}+} \quad (25)$$

are the changes in the total velocity and total energy of two fermions, respectively, due to a collision. The commutator

in Eq. (20a) vanishes for a Galilean-invariant system, i.e., for $\mathbf{v}_{\mathbf{k}} = \mathbf{k}/m$, and thus the conductivity for this case is not affected by the electron-electron interaction, as it should be.

Substituting Eqs. (23a) and (23b) into Eq. (18) and applying Wick's theorem, we obtain for the three components of the conductivity

$$\sigma'_1(\omega, T) = \pi e^2 g^2 \frac{1 - e^{-\omega/T}}{\omega^3} \int_{\mathbf{k}, \mathbf{p}, \mathbf{q}} U^2(\mathbf{k}, \mathbf{p}, \mathbf{q}) (\Delta \mathbf{v})^2 M(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\epsilon_{\mathbf{k}_+} + \epsilon_{\mathbf{p}_-} - \epsilon_{\mathbf{k}_-} - \epsilon_{\mathbf{p}_+} + \omega), \quad (26a)$$

$$\sigma'_2(\omega, T) = 4\pi e^2 g^2 \frac{1 - e^{-\omega/T}}{\omega^3} \int_{\mathbf{k}, \mathbf{p}, \mathbf{q}} [(\nabla_{\mathbf{k}} + \nabla_{\mathbf{p}})U(\mathbf{k}, \mathbf{p}, \mathbf{q})]^2 (\Delta \epsilon)^2 M(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\epsilon_{\mathbf{k}_+} + \epsilon_{\mathbf{p}_-} - \epsilon_{\mathbf{k}_-} - \epsilon_{\mathbf{p}_+} + \omega), \quad (26b)$$

$$\sigma'_{12}(\omega, T) = 4\pi e^2 g^2 \frac{1 - e^{-\omega/T}}{\omega^3} \int_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \Delta \epsilon \Delta \mathbf{v} \cdot (\nabla_{\mathbf{k}} + \nabla_{\mathbf{p}}) U^2(\mathbf{k}, \mathbf{p}, \mathbf{q}) M(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\epsilon_{\mathbf{k}_+} + \epsilon_{\mathbf{p}_-} - \epsilon_{\mathbf{k}_-} - \epsilon_{\mathbf{p}_+} + \omega), \quad (26c)$$

where $\int_{\mathbf{k}}$ is a shorthand for $\int d^2k/(2\pi)^2$, while

$$M(\mathbf{k}, \mathbf{p}, \mathbf{q}) = n_F(\epsilon_{\mathbf{k}_+}) n_F(\epsilon_{\mathbf{p}_-}) [1 - n_F(\epsilon_{\mathbf{k}_-})] [1 - n_F(\epsilon_{\mathbf{p}_+})] \quad (27)$$

with $n_F(\epsilon)$ being the Fermi function. In the equations above, we neglected the exchange part of the interaction, which is small compared to the direct part for a long-range interaction, considered in this paper.

The delta functions in Eqs. (26b) and (26c) impose a constraint $\Delta \epsilon = -\omega$. Applying this constraint, we obtain instead of Eqs. (26b) and (26c)

$$\sigma'_2(\omega, T) = 4\pi e^2 g^2 \frac{1 - e^{-\omega/T}}{\omega} \int_{\mathbf{k}, \mathbf{p}, \mathbf{q}} [(\nabla_{\mathbf{k}} + \nabla_{\mathbf{p}})U(\mathbf{k}, \mathbf{p}, \mathbf{q})]^2 M(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\epsilon_{\mathbf{k}_+} + \epsilon_{\mathbf{p}_-} - \epsilon_{\mathbf{k}_-} - \epsilon_{\mathbf{p}_+} + \omega), \quad (28a)$$

$$\sigma'_{12}(\omega, T) = -4\pi e^2 g^2 \frac{1 - e^{-\omega/T}}{\omega^2} \int_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \Delta \mathbf{v} \cdot (\nabla_{\mathbf{k}} + \nabla_{\mathbf{p}}) U^2(\mathbf{k}, \mathbf{p}, \mathbf{q}) M(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\epsilon_{\mathbf{k}_+} + \epsilon_{\mathbf{p}_-} - \epsilon_{\mathbf{k}_-} - \epsilon_{\mathbf{p}_+} + \omega). \quad (28b)$$

Equations (26a), (28a), and (28b) form the basis of further analysis. In what follows, we will consider three examples: the case of an isotropic but nonparabolic spectrum, as well as of convex and concave FSs. In all cases, we focus on a single-band system and neglect umklapp scattering. The reason for the last assumption is that a long-range interaction, characteristic for an Ising-nematic QCP, strongly suppresses umklapp scattering even if the FS occupies a substantial part of the Brillouin zone [10,20].

III. ISOTROPIC BUT NONPARABOLIC SPECTRUM

In this section, we consider the case of an isotropic but nonparabolic spectrum, $\epsilon_{\mathbf{k}} = \epsilon(k)$, which is encountered, e.g., in Dirac metals. We begin with $\sigma'_1(\omega, T)$ in Eq. (26a), the analysis of which follows along the same lines as in Ref. [11]. Relabeling the momenta as $\mathbf{k} + \mathbf{q}/2 \rightarrow \mathbf{k}$ and $\mathbf{p} - \mathbf{q}/2 \rightarrow \mathbf{p}$, introducing the energy transfer as $\Omega = \epsilon_{\mathbf{k}-\mathbf{q}} - \epsilon_{\mathbf{k}} = \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}+\mathbf{q}} + \omega$, and restricting the integrals over the fermionic momenta to narrow regions near the FS, we rewrite Eq. (26a) as

$$\begin{aligned} \sigma'_1(\omega, T) = & e^2 g^2 \frac{\pi N_F^2}{\omega^3} (1 - e^{-\omega/T}) \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^{+\infty} d\epsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\epsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \int_0^{2\pi} \frac{d\phi_{\mathbf{k}\mathbf{q}}}{2\pi} \int_0^{2\pi} \frac{d\phi_{\mathbf{p}\mathbf{q}}}{2\pi} F^2\left(\mathbf{k} - \frac{\mathbf{q}}{2}\right) F^2\left(\mathbf{p} + \frac{\mathbf{q}}{2}\right) V^2(\mathbf{q}) \\ & \times (\Delta \mathbf{v})^2 n_F(\epsilon_{\mathbf{k}}) n_F(\epsilon_{\mathbf{p}}) [1 - n_F(\epsilon_{\mathbf{k}} + \Omega)] [1 - n_F(\epsilon_{\mathbf{p}} - \Omega + \omega)] \delta(\Omega - \epsilon_{\mathbf{k}-\mathbf{q}} + \epsilon_{\mathbf{k}}) \delta(\Omega - \omega + \epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{p}}), \end{aligned} \quad (29)$$

where $\phi_{\mathbf{nm}}$ is the angle between vectors $\mathbf{n}$ and $\mathbf{m}$, and

$$\Delta \mathbf{v} = \mathbf{v}_{\mathbf{k}} + \mathbf{v}_{\mathbf{p}} - \mathbf{v}_{\mathbf{k}-\mathbf{q}} - \mathbf{v}_{\mathbf{p}+\mathbf{q}}. \quad (30)$$

Note that $\mathbf{v}_{\mathbf{k}} = \epsilon'(k)\mathbf{k}/k$ for an isotropic dispersion. Therefore, if we project all the momenta onto the FS, i.e., put $k = p = |\mathbf{k} - \mathbf{q}| = |\mathbf{p} + \mathbf{q}| = k_F$ in $\Delta \mathbf{v}$, then $\Delta \mathbf{v}$ vanishes and so does the conductivity. To get a finite result, we need to expand $\Delta \mathbf{v}$ around the FS. Performing such an expansion, we obtain

$$\Delta \mathbf{v} = \frac{w_{\text{np}}}{k_F} \left[(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}}) \hat{\mathbf{k}} + (\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}+\mathbf{q}}) \hat{\mathbf{p}} + (\epsilon_{\mathbf{k}-\mathbf{q}} - \epsilon_{\mathbf{p}+\mathbf{q}}) \frac{\mathbf{q}}{k_F} \right], \quad (31)$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$, etc., and

$$w_{\text{np}} = 1 - \frac{k_F \epsilon''(k)}{\epsilon'(k)} \Big|_{k=k_F} \quad (32)$$

is the nonparabolicity coefficient [11]. A Galilean-invariant system has a parabolic spectrum, in which case $w_{\text{np}} = 0$. For a nonparabolic spectrum $w_{\text{np}} \neq 0$; for example, $w_{\text{np}} = 1$ for the Dirac spectrum.

Near quantum criticality, momentum transfers are small: $q \lesssim q_B \ll k_F$. Therefore, we can neglect the last term, proportional to $\mathbf{q}$, in Eq. (31), and also replace $F(\mathbf{k} - \mathbf{q}/2) \approx F(\mathbf{k})$ and $F(\mathbf{p} + \mathbf{q}/2) \approx F(\mathbf{p})$ in Eq. (29). Next, we express the dispersions entering Eq. (31) via ω and Ω , using the constraints

imposed by the delta functions in Eq. (29). This yields

$$(\Delta \mathbf{v})^2 = \frac{w_{\text{np}}^2}{k_{\text{F}}^2} [\Omega^2 + (\Omega + \omega)^2 - 2\Omega(\Omega + \omega) \cos \phi_{\mathbf{k}\mathbf{p}}]. \quad (33)$$

It is to be expected (and will be shown below to be the case) that typical values of $|\Omega|$ are on the order of $\max\{|\omega|, T\}$. Then $(\Delta \mathbf{v})^2 \sim w_{\text{np}}^2 \max\{\omega^2, T^2\}/k_{\text{F}}^2$, which implies that the conductivity is suppressed compared to its canonical FL value, Eq. (1). To obtain the leading term in the conductivity, it suffices to neglect Ω and ω in the delta functions in Eq. (29). For small q , these delta functions are then reduced to the geometric constraints $\cos \phi_{\mathbf{k}\mathbf{q}} = 0$ and $\cos \phi_{\mathbf{p}\mathbf{q}} = 0$, which imply that $\mathbf{k}$ and $\mathbf{p}$ are either parallel or antiparallel to each other, i.e., that $\cos \phi_{\mathbf{k}\mathbf{p}} = \pm 1$. Recalling the fourfold symmetry of the interaction, we find that the contributions from $\cos \phi_{\mathbf{k}\mathbf{p}} = \pm 1$ in Eq. (33) cancel each other, while the integral of the form factors over $\phi_{\mathbf{q}}$ gives a constant

$$a = \int_0^{2\pi} \frac{d\phi_{\mathbf{q}}}{2\pi} F^4(\mathbf{q}). \quad (34)$$

Next, the triple integral over $\varepsilon_{\mathbf{k}}$, $\varepsilon_{\mathbf{p}}$, and Ω is solved as [11]

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} d\varepsilon_{\mathbf{p}} d\Omega [(\Omega + \omega)^2 + \Omega^2] \\ & \times n_{\text{F}}(\varepsilon_{\mathbf{k}}) n_{\text{F}}(\varepsilon_{\mathbf{p}}) [1 - n_{\text{F}}(\varepsilon_{\mathbf{k}} + \Omega)] [1 - n_{\text{F}}(\varepsilon_{\mathbf{p}} - \Omega + \omega)] \\ & = \frac{\omega^5}{30(1 - e^{-\omega/T})} \left(1 + \frac{4\pi T^2}{\omega^2}\right) \left(3 + \frac{8\pi^2 T^2}{\omega^2}\right). \end{aligned} \quad (35)$$

Note that typical values of the variables in this integral are $|\varepsilon_{\mathbf{k}}| \sim |\varepsilon_{\mathbf{p}}| \sim |\Omega| \sim \max\{|\omega|, T\}$, which proves our earlier assertion. Finally, we notice that the integral over q diverges logarithmically at the lower limit because each of the delta functions in Eq. (29) brings in a factor of $1/q$. To regularize the divergence, one needs to return to the dynamic form of the interaction and ask how large q should be for the Landau-damping term to be neglected. The answer is that $q \gg q_{\text{B}}|\omega|/\omega_{\text{FL}}$, where ω_{FL} is the upper boundary of the FL region [30]³, given by Eq. (11). Therefore,

$$\int_{q_{\text{B}}|\omega|/\omega_{\text{FL}}}^{\infty} \frac{dq}{q} V^2(q) \approx \frac{1}{q_{\text{B}}} \ln \frac{\omega_{\text{FL}}}{|\omega|}. \quad (36)$$

Collecting everything together, we arrive at

$$\begin{aligned} \sigma'_1(\omega, T) &= \frac{e^2}{60\pi^2} a w_{\text{np}}^2 \frac{g^2 N_{\text{F}}^2}{q_{\text{B}}^4} \left(\frac{\omega}{v_{\text{F}} k_{\text{F}}}\right)^2 \left(1 + \frac{4\pi^2 T^2}{\omega^2}\right) \\ & \times \left(3 + \frac{8\pi^2 T^2}{\omega^2}\right) \ln \left(\frac{\omega_{\text{FL}}}{\max\{|\omega|, T\}}\right), \end{aligned} \quad (37)$$

which is the scaling form advertised in Eq. (2).

We now turn to $\sigma'_2(\omega, T)$ in Eq. (28a). The integrand of $\sigma'_2(\omega, T)$ can be projected right onto the FS without further expansions. The rest of the calculation is the same as for

$\sigma'_1(\omega, T)$, except for the energy integral, which yields

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} d\varepsilon_{\mathbf{p}} d\Omega [1 - n_{\text{F}}(\varepsilon_{\mathbf{k}} + \Omega)] [1 - n_{\text{F}}(\varepsilon_{\mathbf{p}} - \Omega + \omega)] \\ & \times n_{\text{F}}(\varepsilon_{\mathbf{k}}) n_{\text{F}}(\varepsilon_{\mathbf{p}}) = \frac{\omega^3}{6(1 - e^{-\omega/T})} \left(1 + \frac{4\pi^2 T^2}{\omega^2}\right). \end{aligned} \quad (38)$$

A straightforward calculation then leads to

$$\begin{aligned} \sigma'_2(\omega, T) &= \frac{e^2}{24\pi^2} \frac{g^2 N_{\text{F}}^2}{q_{\text{B}}^4} w_{\text{nem}}^2 \left(\frac{\omega}{v_{\text{F}} k_{\text{F}}}\right)^2 \left(1 + \frac{4\pi^2 T^2}{\omega^2}\right) \\ & \times \ln \left(\frac{\omega_{\text{FL}}}{\max\{|\omega|, T\}}\right), \end{aligned} \quad (39)$$

where

$$w_{\text{nem}}^2 = \int_0^{2\pi} \frac{d\phi_{\mathbf{q}}}{2\pi} \left(\frac{\partial F^2(\mathbf{q})}{\partial \phi_{\mathbf{q}}}\right)^2 \quad (40)$$

is the nematicity coefficient.

Finally, we come to the $\sigma'_{12}(\omega, T)$ contribution to the conductivity, Eq. (28b). By power counting, it is of the same order as σ'_1 and σ'_2 because $\Delta v \Delta \varepsilon$ scales as ω^2 [cf. Eq. (26)]. However $\sigma'_{12}(\omega, T)$ vanishes within the approximation of this section. The reason is that, once the last term in Eq. (31) is neglected, $\Delta \mathbf{v}$ becomes a sum of two radial vectors, directed along $\hat{\mathbf{k}}$ and $\hat{\mathbf{p}}$, respectively. As we already know, the leading contribution to the integral comes from almost collinear momenta $\mathbf{k}$ and $\mathbf{p}$. Therefore, $\Delta \mathbf{v}$ is also a radial vector, collinear with $\mathbf{k}$ (or $\mathbf{p}$). Now, in Eq. (28b) $\Delta \mathbf{v}$ is dotted into $\mathbf{w} \equiv (\nabla_{\mathbf{k}} + \nabla_{\mathbf{p}}) U^2(\mathbf{k}, \mathbf{p}, \mathbf{q})$, which is a vector sum of two tangential vectors, proportional to $\hat{\phi}_{\mathbf{k}}$ and $\hat{\phi}_{\mathbf{p}}$, respectively. Again, due to collinearity of $\mathbf{k}$ and $\mathbf{p}$, vector $\mathbf{w}$ is perpendicular to both. Thus $\Delta \mathbf{v} \cdot \mathbf{w} = 0$, and the leading term in $\sigma'_{12}(\omega, T)$ vanishes.

Therefore, the final result for the conductivity is the sum of Eqs. (37) and (39):

$$\begin{aligned} \sigma'(\omega, T) &= \frac{e^2}{12\pi^2} \frac{g^2 N_{\text{F}}^2}{q_{\text{B}}^4} \left(\frac{\omega}{v_{\text{F}} k_{\text{F}}}\right)^2 \left(1 + \frac{4\pi^2 T^2}{\omega^2}\right) \\ & \times \left[\frac{a w_{\text{np}}^2}{5} \left(3 + \frac{8\pi^2 T^2}{\omega^2}\right) + \frac{w_{\text{nem}}^2}{2} \right] \\ & \times \ln \left(\frac{\omega_{\text{FL}}}{\max\{|\omega|, T\}}\right). \end{aligned} \quad (41)$$

The nonparabolic and nematic contributions to the conductivity are of the same order for $\omega \gtrsim T$, while the nonparabolic contribution is the dominant one for $\omega \ll T$. Schematically,

$$\sigma'(\omega, T) \propto q_{\text{B}}^{-4} \omega^2 \max \left\{ \ln \frac{\omega_{\text{FL}}}{|\omega|}, \frac{T^4}{\omega^4} \ln \frac{\omega_{\text{FL}}}{T} \right\}. \quad (42)$$

As explained in Sec. II A, a crossover to the QCP is achieved by replacing $q_{\text{B}} \rightarrow \max\{|\omega|^{1/3}, T^{1/3}\}$. At the same time, ω_{FL} is replaced by $\max\{|\omega|, T\}$, and thus the logarithmic factors are replaced by constants of order one. As a result, Eq. (42) is replaced by Eq. (4).

Next, we turn our attention to anisotropic FSs.

³We thank A. Chubukov for clarifying the cutoff procedure for us and for bringing Ref. [30] to our attention.

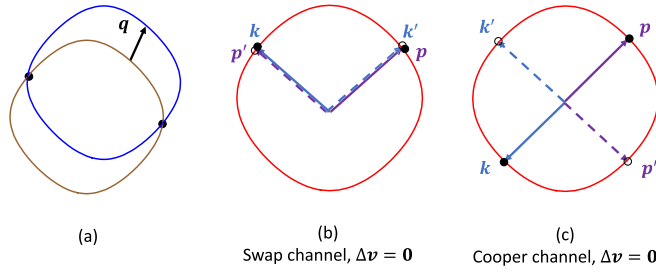


FIG. 1. (a) Graphic solution of the equation $\varepsilon_{k-q} = \varepsilon_k$. As shown, a convex FS contour has no more than two self-intersection points (black dots). (b) Swap scattering process, in which $\mathbf{k} \rightarrow \mathbf{p}'$ and $\mathbf{p} \rightarrow \mathbf{k}'$. (c) Cooper scattering process, in which $\mathbf{k} + \mathbf{p} = 0 = \mathbf{k}' + \mathbf{p}'$. Neither swap nor Cooper channel leads to current relaxation.

IV. CONVEX FERMI SURFACE

We proceed with the optical conductivity for a generic convex FS. As has been shown by many authors (see Sec. I), the leading term in the conductivity, i.e., $\max\{\text{const}, T^2/\omega^2\}$ in the FL regime, vanishes in this case. Our goal is to derive the surviving term. Our main interest will be in the $\sigma'_1(\omega, T)$ contribution [Eq. (26a)], because the general form of the $\sigma'_2(\omega, T)$ contribution [Eq. (28a)] does not depend on the shape of the FS.

We focus on the $T = 0$ case first and begin with a brief review of the arguments for the vanishing of the leading term, as given in Refs. [10] and [20]. To leading order in $\max\{\omega, T\}$, we project the integrand in Eq. (29) onto the FS and neglect both frequencies in the delta functions (Ω and ω). To ensure that the both delta functions are nonzero within the domain of integration, we need to find the solutions of two equations for the initial momenta $\mathbf{k}$ and $\mathbf{p}$ at fixed $\mathbf{q}$:

$$\varepsilon_{\mathbf{k}-\mathbf{q}} = \varepsilon_{\mathbf{k}} = 0, \quad (43a)$$

$$\varepsilon_{\mathbf{p}+\mathbf{q}} = \varepsilon_{\mathbf{p}} = 0. \quad (43b)$$

Excluding umklapps, the solutions of Eqs. (43a) and (43b) are the points of intersection between the original FS contour

and its two copies translated by $\pm\mathbf{q}$, see Fig. 1(a). Denoting $\bar{\mathbf{p}} = -\mathbf{p}$ and using that $\varepsilon_{-\mathbf{k}} = \varepsilon_{\mathbf{k}}$, we rewrite Eq. (43b) as $\varepsilon_{\bar{\mathbf{p}}-\mathbf{q}} = \varepsilon_{\bar{\mathbf{p}}} = 0$, upon which it becomes identical to Eq. (43a). For a convex FS, the first equation has (at most) two solutions, see Fig. 1(a). If we denote one of the solutions by $\mathbf{k}_0$, then the second one is $-\mathbf{k}_0 + \mathbf{q}$. Indeed, $\varepsilon_{-\mathbf{k}_0+\mathbf{q}-\mathbf{q}} = \varepsilon_{-\mathbf{k}_0+\mathbf{q}} \rightarrow \varepsilon_{\mathbf{k}_0} = \varepsilon_{\mathbf{k}_0-\mathbf{q}}$. However, the equation for $\bar{\mathbf{p}}$ also has (at most) two solutions: $(\bar{\mathbf{p}}_0, -\bar{\mathbf{p}}_0 + \mathbf{q}) = (-\mathbf{p}_0, \mathbf{p}_0 + \mathbf{q})$. Because the equations are the same, the solutions must also coincide, which leaves us with two choices: either $\mathbf{p}_0 = \mathbf{k}_0 - \mathbf{q}$ or $\mathbf{p}_0 = -\mathbf{k}_0$. The first choice corresponds to a “swap” scattering process, in which two electrons swap their initial momenta: $\mathbf{k}_0 \rightarrow \mathbf{k}_0 - \mathbf{q} = \mathbf{p}_0$, $\mathbf{p}_0 = \mathbf{k}_0 - \mathbf{q} \rightarrow \mathbf{p}_0 + \mathbf{q} = \mathbf{k}_0$, see Fig. 1(b). The second choice is a Cooper (or head-on) scattering process: $\mathbf{k}_0 \rightarrow \mathbf{k}_0 - \mathbf{q}$, $\mathbf{p}_0 = -\mathbf{k}_0 \rightarrow \mathbf{k}_0 + \mathbf{q}$; see Fig. 1(c). According to Eqs. (26a)–(26c), a scattering process contributes to the conductivity only if $\Delta\mathbf{v} = \mathbf{v}_{\mathbf{k}'} + \mathbf{v}_{\mathbf{p}'} - \mathbf{v}_{\mathbf{k}} - \mathbf{v}_{\mathbf{p}} \neq 0$. However, in a swap process $\mathbf{v}_{\mathbf{k}'} = \mathbf{v}_{\mathbf{p}}$ and $\mathbf{v}_{\mathbf{p}'} = \mathbf{v}_{\mathbf{k}}$, such that $\Delta\mathbf{v} = 0$. Likewise, in a Cooper process $\mathbf{v}_{\mathbf{p}} = \mathbf{v}_{-\mathbf{k}} = -\mathbf{v}_{\mathbf{k}}$ and $\mathbf{v}_{\mathbf{p}'} = \mathbf{v}_{-\mathbf{k}'} = -\mathbf{v}_{\mathbf{k}'}$, such that $\Delta\mathbf{v} = 0$ again. Therefore, the leading term in the conductivity vanishes.

To obtain a nonzero result, we have to consider small deviations from the previously found solutions for the swap and Cooper channels, i.e.,

$$\mathbf{k} = \mathbf{k}_0 + \delta\mathbf{k}, \quad \mathbf{p} = \mathbf{p}_0 + \delta\mathbf{p} = \mathbf{k}_0 - \mathbf{q} + \delta\mathbf{p}, \quad (44a)$$

$$\mathbf{k} = \mathbf{k}_0 + \delta\mathbf{k}, \quad \mathbf{p} = \mathbf{p}_0 + \delta\mathbf{p} = -\mathbf{k}_0 + \delta\mathbf{p}. \quad (44b)$$

Expanding $\Delta\mathbf{v}$ to first order in $\delta\mathbf{k}$ and $\delta\mathbf{p}$ around $\mathbf{k}_0$ and $\mathbf{p}_0$, we obtain

$$\Delta\mathbf{v}_S = [(\delta\mathbf{k} - \delta\mathbf{p}) \cdot \nabla](\mathbf{v}_{\mathbf{k}_0} - \mathbf{v}_{\mathbf{k}_0-\mathbf{q}}), \quad (45a)$$

$$\Delta\mathbf{v}_C = [(\delta\mathbf{k} + \delta\mathbf{p}) \cdot \nabla](\mathbf{v}_{\mathbf{k}_0-\mathbf{q}} - \mathbf{v}_{\mathbf{k}_0}) \quad (45b)$$

for the swap (S) and Cooper (C) channels, respectively. Without a loss of generality, we choose $\omega > 0$. With this choice and at $T = 0$, the $\sigma'_1(\omega, 0)$ contribution to the conductivity is reduced to

$$\begin{aligned} \sigma'_1(\omega, 0) = & \frac{\pi e^2 g^2}{(2\pi)^6 \omega^3} \int d^2 q \int d^2 \delta k \int d^2 \delta p \int_{-\omega}^0 d\Omega F^2\left(\mathbf{k} + \frac{\mathbf{q}}{2}\right) F^2\left(\mathbf{p} - \frac{\mathbf{q}}{2}\right) V^2(\mathbf{q}) (\Delta\mathbf{v})^2 \\ & \times \theta(\varepsilon_{\mathbf{k}}) \theta(-\varepsilon_{\mathbf{k}} - \Omega) \theta(\varepsilon_{\mathbf{p}}) \theta(-\varepsilon_{\mathbf{p}} + \Omega + \omega) \delta(\Omega - \varepsilon_{\mathbf{k}-\mathbf{q}} + \varepsilon_{\mathbf{k}}) \delta(\Omega + \omega + \varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}}). \end{aligned} \quad (46)$$

For any given $\mathbf{q}$, we need to find solutions $\mathbf{k}_0$ and $\mathbf{p}_0$ of Eqs. (43a) and (43b), integrate over $\delta\mathbf{k}$ and $\delta\mathbf{p}$ near these solutions, and then integrate over $\mathbf{q}$.

We start with the swap channel, in which $\mathbf{p}_0 = \mathbf{k}_0 - \mathbf{q}$. We expand the integrand of Eq. (46) in $\delta\mathbf{k}$ and $\delta\mathbf{p}$ and also use the condition $q \ll \langle k_F \rangle$, where $\langle k_F \rangle$ is a characteristic size of the FS. Then

$$\begin{aligned} \sigma'_1(\omega, 0) = & \frac{\pi e^2 g^2}{(2\pi)^6 \omega^3} \int d^2 q F^4(\mathbf{k}_0) V^2(\mathbf{q}) \int d^2 \delta k \int d^2 \delta p \int_{-\omega}^0 d\Omega (\Delta\mathbf{v}_S)^2 \\ & \times \theta(\mathbf{v}_{\mathbf{k}_0} \cdot \delta\mathbf{k}) \theta(-\mathbf{v}_{\mathbf{k}_0} \cdot \delta\mathbf{k} - \Omega) \theta(\mathbf{v}_{\mathbf{k}_0-\mathbf{q}} \cdot \delta\mathbf{p}) \theta(-\mathbf{v}_{\mathbf{k}_0-\mathbf{q}} \cdot \delta\mathbf{p} + \Omega + \omega) \delta(\delta\mathbf{k} \cdot \mathbf{u} - \Omega) \delta(\delta\mathbf{p} \cdot \mathbf{u} - \Omega - \omega), \end{aligned} \quad (47)$$

where $\Delta\mathbf{v}_S$ is given by Eq. (45a) and

$$\mathbf{u} = \mathbf{v}_{\mathbf{k}_0-\mathbf{q}} - \mathbf{v}_{\mathbf{k}_0} \approx -(\mathbf{q} \cdot \nabla) \mathbf{v}_{\mathbf{k}_0}. \quad (48)$$

The delta functions in Eq. (47) imply that the 2D integrals over $\delta\mathbf{k}$ and $\delta\mathbf{p}$ are, in fact, one-dimensional integrals along the straight lines:

$$\delta\mathbf{k} \cdot \mathbf{u} = \Omega, \quad (49a)$$

$$\delta \mathbf{p} \cdot \mathbf{u} = \Omega + \omega. \quad (49b)$$

It is convenient to choose δk_x and δp_x as independent integration variables and exclude δk_y and δp_y via

$$\delta k_y = \frac{\Omega}{u_y} - \delta k_x \frac{u_x}{u_y}, \quad (50a)$$

$$\delta p_y = \frac{\Omega + \omega}{u_y} - \delta p_x \frac{u_x}{u_y}. \quad (50b)$$

The Pauli principle (imposed by the theta functions) and energy conservation (imposed by the delta-functions), confine δk_x and δp_x to the following intervals:

$$k_{\min} \leq \delta k_x \leq k_{\max}, \quad (51a)$$

$$p_{\min} \leq \delta p_x \leq p_{\max}, \quad (51b)$$

where

$$k_{\min} = \theta\left(\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0 - \mathbf{q}, y}}{\nu} \Omega + \theta\left(-\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0, y}}{\nu} \Omega, \quad (52a)$$

$$k_{\max} = \theta\left(\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0, y}}{\nu} \Omega + \theta\left(-\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0 - \mathbf{q}, y}}{\nu} \Omega, \quad (52b)$$

$$p_{\min} = \theta\left(\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0, y}}{\nu} (\Omega + \omega) + \theta\left(-\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0 - \mathbf{q}, y}}{\nu} (\Omega + \omega), \quad (52c)$$

$$p_{\max} = \theta\left(\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0 - \mathbf{q}, y}}{\nu} (\Omega + \omega) + \theta\left(-\frac{\nu}{u_y}\right) \frac{v_{\mathbf{k}_0, y}}{\nu} (\Omega + \omega), \quad (52d)$$

and

$$\begin{aligned} \nu &= v_{\mathbf{k}_0 - \mathbf{q}, x} v_{\mathbf{k}_0, y} - v_{\mathbf{k}_0 - \mathbf{q}, y} v_{\mathbf{k}_0, x} \\ &\approx v_{\mathbf{k}_0, x} (\mathbf{q} \cdot \nabla) v_{\mathbf{k}_0, y} - v_{\mathbf{k}_0, y} (\mathbf{q} \cdot \nabla) v_{\mathbf{k}_0, x}, \end{aligned} \quad (53)$$

with $\mathbf{u}$ defined by Eq. (48). The last step in Eq. (53) again accounts for the smallness of q .

Substituting Eqs. (50a) and (50b) into Eq. (45a), we obtain for $(\Delta v)^2$ near the swap solution:

$$\begin{aligned} (\Delta v_S)^2 &= \left\{ \left[\delta k_x \left(\partial_{k_x} - \frac{u_x}{u_y} \partial_{k_y} \right) + \delta p_y \left(\frac{u_y}{u_x} \partial_{k_x} - \partial_{k_y} \right) \right. \right. \\ &\quad \left. \left. - \frac{\Omega + \omega}{u_x} \partial_{k_x} + \frac{\Omega}{u_y} \partial_{k_y} \right] \mathbf{u} \right\}^2. \end{aligned} \quad (54)$$

With all the constraints having been resolved, Eq. (47) is reduced to

$$\begin{aligned} \sigma'_1(\omega, 0) &= \frac{\pi e^2 g^2}{(2\pi)^6 \omega^3} \int d^2 q F^4(\mathbf{k}_0) V^2(\mathbf{q}) \frac{1}{u_y^2} \\ &\quad \times \int_{-\omega}^0 d\Omega \int_{k_{\min}}^{k_{\max}} d\delta k_x \int_{p_{\min}}^{p_{\max}} d\delta p_x (\Delta v_S)^2. \end{aligned} \quad (55)$$

The power counting of $\sigma'_1(\omega, 0)$ is already obvious at this point. Indeed, the integrals over δk_x and δp_x give a factor of ω each. Another factor of ω^2 comes from $(\Delta v_S)^2$. Finally, the integral over Ω gives one more factor of ω . Therefore, $\sigma'_1(\omega, 0) \propto \omega^{-3} \times \omega^2 \times \omega^2 \times \omega = \omega^2$.

To find the dependence of $\sigma'_1(\omega, 0)$ on the bosonic mass (q_B), we need to analyze the behavior of the integrand in

Eq. (55) at $q \rightarrow 0$. For simplicity, let's assume that $\nu/u_y > 0$. Then $k_{\max} - k_{\min} = (v_{\mathbf{k}_0, y} - v_{\mathbf{k}_0 - \mathbf{q}, y})\Omega/\nu$. The numerator in the last formula vanishes as q at $q \rightarrow 0$ but, according to Eq. (53), ν also vanishes as q and, therefore, the range of integration over δk_x remains finite at $q \rightarrow 0$. The same is true for $p_{\max} - p_{\min}$. Next, $(\Delta v)^2$ in Eq. (54) contains only the ratios u_i/u_j and $\partial_{k_i} u_j/u_l$ with $i, j, l = x, y$. While u_i by itself vanishes as q [cf. Eq. (48)], the ratios quoted above remain finite and, therefore, $(\Delta v_S)^2$ remains finite at $q \rightarrow 0$. Finally, the factor of $1/u_y^2$ in Eq. (55) diverges as $1/q^2$, and thus the integral over q diverges logarithmically at $q \rightarrow 0$. This is exactly the same divergence that we encountered in Sec. III. Cutting the divergence off in the same way as before, we arrive at the final result:

$$\sigma'_1(\omega, 0) \propto q_B^{-4} \omega^2 \ln \frac{\omega_{\text{FL}}}{|\omega|}. \quad (56)$$

The frequency dependence of the conductivity is the same as for an isotropic but nonparabolic spectrum at $T = 0$ [cf. Eq. (42)].

The contribution from the Cooper channel is analyzed along the same lines. It can be readily shown that the limits of integration over δk_x and δp_x for this case remain the same as for the swap channel [cf. Eqs. (52c)–(52d)]. The only change compared to the swap case is in the relative sign of $\delta \mathbf{k}$ and $\delta \mathbf{p}$ in Eq. (45b), which is irrelevant for scaling. Therefore, the combined contribution of the swap and Cooper channels is still given by Eq. (56).

The finite-temperature case does not require a special analysis because, in the FL regime, the integrals over the energies and over momenta tangential to the FS factorize. Therefore, the energy integration gives the same result as in Eq. (35).

As we already said, the scaling form of the $\sigma'_2(\omega, 0)$ contribution is insensitive to the shape of the FS, hence Eq. (39) applies to a convex FS as well. Next, the $\sigma'_{12}(\omega, 0)$ contribution already contains an extra factor of ω [cf. Eq. (28b)], while another factor of ω comes from the expansion of Δv around the swap and Cooper solutions. Therefore, the total conductivity for a convex FS can be written as

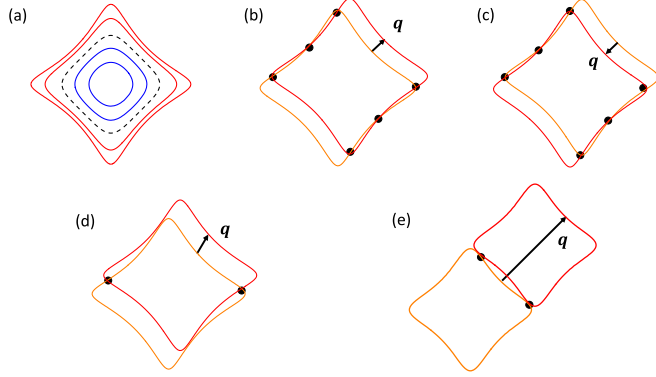
$$\begin{aligned} \sigma'(\omega, T) &= e^2 \frac{g^2 N_F^2}{q_B^4} \left(\frac{\omega}{v_F k_F} \right)^2 \left(1 + \frac{4\pi^2 T^2}{\omega^2} \right) \\ &\quad \times \left[\alpha \left(3 + \frac{8\pi^2 T^2}{\omega^2} \right) + \beta \right] \ln \left(\frac{\omega_{\text{FL}}}{\max\{|\omega|, T\}} \right), \end{aligned} \quad (57)$$

where α and β are the dimensionless coefficients which depend on the details of the convex FS. The overall form of the conductivity is the same as for an isotropic FS, cf. Eq. (41).

Extrapolating Eq. (57) to the vicinity of the QCP in the same way as in Sec. III, we arrive again at Eq. (4). At $T = 0$, the conductivity scales as

$$\sigma'(\omega, 0) \propto |\omega|^{2/3}. \quad (58)$$

This result disagrees with that by Guo *et al.* [22], who argued that $\sigma'(\omega, 0) = \text{const.}$



V. CONCAVE FERMI SURFACE

A. Generic concave Fermi surface

A concave FS can have more than two self-intersection points. If the FS has mirror symmetry, the maximum number

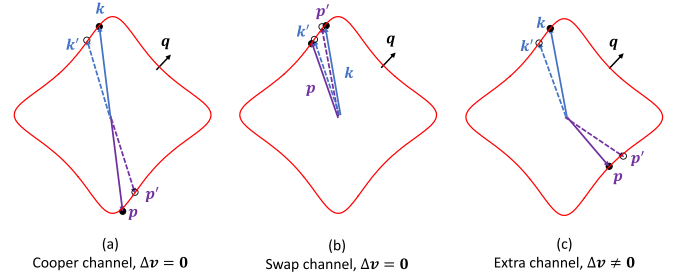


FIG. 3. Scattering processes on a concave Fermi surface. (a) An example of Cooper process with $\Delta v = 0$. (b) An example of swap process, also with $\Delta v = 0$. (c) A current-relaxing scattering process with $\Delta v \neq 0$.

of such points is six, and it is achieved if the FS is translated along the mirror axis, as illustrated in Fig. 2(b). One such point is the inflection point, two more are located on the same side of the curvilinear polygon, and the other three are located on the opposite side. Therefore, for given $\mathbf{q}$, Eq. (43a) for the allowed initial momenta $\mathbf{k}$ has up to six solutions. Solutions of Eq. (43b) for the initial momenta $\mathbf{p}$ are obtained by translating the FS by $-\mathbf{q}$, which yields another up to six solutions, see Fig. 2(c). The two sets of up to six solutions each generate up to 36 pairs of the initial momenta $\mathbf{k}$ and $\mathbf{p}$ at given $\mathbf{q}$. These pairs contain not only Cooper [Fig. 3(a)] and swap [Fig. 3(b)] channels, but also scattering processes that do relax the current. An example of such a process is shown in Fig. 3(c). For any such process $\Delta v \neq 0$, and the conductivity in Eqs. (26a)–(26c) remains finite after projecting the integrands onto the FS. Equation (26a) for $\sigma'_1(\omega, T)$ is then reduced to

$$\sigma'_1(\omega, T) = \frac{\pi e^2 g^2}{(2\pi)^4} \frac{1 - e^{-\omega/T}}{\omega^3} \int_{-\infty}^{\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{\infty} d\Omega n_F(\varepsilon_{\mathbf{k}} + \Omega) n_F(\varepsilon_{\mathbf{p}} - \omega - \Omega) [1 - n_F(\varepsilon_{\mathbf{k}})] [1 - n_F(\varepsilon_{\mathbf{p}})]$$

$$\times \int \frac{d^2 q}{(2\pi)^2} \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} (\Delta v)^2 U^2(\mathbf{k}, \mathbf{p}, \mathbf{q}) \delta(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}} + \Omega) \delta(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}} + \omega + \Omega)|_{\text{FS}}, \quad (59)$$

where $d\ell_{\mathbf{k}}$ is the line element of the FS contour, and it is understood that the fermionic momenta in the second line belong to the FS. As before, one can neglect the frequencies inside the delta functions. In contrast to the convex case of Sec. IV, Δv does not contain additional dependence on the frequencies Ω and ω but, instead, it vanishes at $q \rightarrow 0$ as q^2 . Therefore, the integral over q is convergent at the lower limit (as opposed to being logarithmically divergent in the convex case) and gives $\int dq q q^2 / (q^2 + q_B^2)^2 \sim q_B^{-2}$. Next, the triple integral over energies in the first line of Eq. (59) is given by Eq. (38). We thus obtain

$$\sigma'_1(\omega, T) \propto q_B^{-2} \frac{\omega^2 + 4\pi^2 T^2}{\omega^2}, \quad (60)$$

which is just the Gurzhi scaling in Eq. (1).

Up to an overall factor, the σ'_2 contribution is still given by Eq. (39), which is subleading to Eq. (60) and can be neglected.

The $\sigma'_{12}(\omega, T)$ contribution requires a bit of care. Rewriting Eq. (28b) in the same way as Eq. (59), we obtain

$$\sigma'_{12}(\omega, T) = -\frac{e^2 g^2}{4\pi^3} \frac{1 - e^{-\omega/T}}{\omega^2} \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}}$$

$$\times (\Delta v \cdot \mathbf{w}) n_F(\varepsilon_{\mathbf{k}}) n_F(\varepsilon_{\mathbf{p}}) [1 - n_F(\varepsilon_{\mathbf{k}} + \Omega)] [1 - n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega)] \delta(\Omega - \varepsilon_{\mathbf{k}-\mathbf{q}} + \varepsilon_{\mathbf{k}}) \delta(\Omega - \omega + \varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}}), \quad (61)$$

where

$$\mathbf{w} = F^2(\mathbf{p} + \mathbf{q}/2) \nabla_{\mathbf{k}} F^2|_{\mathbf{k}=\mathbf{q}/2} + F^2(\mathbf{k} - \mathbf{q}/2) \nabla_{\mathbf{p}} F^2|_{\mathbf{p}=\mathbf{q}/2}. \quad (62)$$

Suppose that we project all the momenta in the last equation onto the FS and set $T = 0$. Then the triple integral over energies gives a factor of ω^3 [cf. Eq. (35)], and thus $\sigma'_{12}(\omega, 0)$ appears to scale as ω . However, this is impossible because $\sigma'_{12}(\omega, 0)$ must be even in ω . Therefore, the leading ω term must vanish identically, while the subleading term comes from expanding $\Delta \mathbf{v}$ near the FS, as it was the case for an isotropic FS, cf. Sec. III. This implies that $\sigma'_{12}(\omega, 0)$ scales at least as ω^2 and is thus subleading to $\sigma'(\omega, 0)$. An expanded discussion of this point is presented in Appendix C.

As before, a crossover to the QCP is achieved by replacing $q_B \rightarrow \max\{|\omega|^{1/3}, T^{1/3}\}$ in Eq. (60), which yields

$$\sigma'(\omega, T) \propto \max \left\{ \frac{1}{|\omega|^{2/3}}, \frac{T^{4/3}}{\omega^2} \right\}. \quad (63)$$

This result restores the naive scaling form in Eq. (3), obtained by substituting the current relaxation rate, $1/\tau_j(\omega, T) \propto \max\{|\omega|^{4/3}, T^{4/3}\}$, into the Drude formula $\sigma'(\omega, T) \propto 1/\omega^2 \tau_j(\omega, T)$.

B. Optical conductivity near a convex-to-concave transition

Suppose that at a certain critical value of Fermi energy, $E_F = E_c$, the shape of the FS changes from convex ($\Delta \equiv E_F - E_c < 0$) to concave ($\Delta > 0$), see Fig. 2(b). The results presented in Secs. IV and V A are valid either well below or well above the convex-concave transition. Now we consider the vicinity of the transition.

Even if the FS is concave, the kinematic constraint $\varepsilon_{\mathbf{k}-\mathbf{q}} = \varepsilon_{\mathbf{k}}$ has more than two solutions only if $\mathbf{q}$ is along one of the high-symmetry axes and its magnitude is sufficiently small. If these conditions are not satisfied, then, as shown in Figs. 2(d) and 2(e), the number of self-intersection points goes back to two and, again, only the Cooper and swap channels are possible. The width of the angular interval near a high-symmetry direction, $\delta\phi_q$, and the maximum value of q , $q_{\max}$, depend on Δ in a critical manner. In what follows, we will determine these parameters for the fourfold symmetric case and analyze the dependence of optical conductivity on ω and Δ . In doing so, we will follow closely a derivation of the dc resistivity near the convex-to-concave transition in the presence of disorder, presented in Ref. [29].

Let us first find $\delta\phi_q$. For $q \ll k_F$, the kinematic constraint $\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}-\mathbf{q}}$ is equivalent to $\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q} = 0$. Since $\mathbf{v}_{\mathbf{k}}$ is along the normal to the FS at point $\mathbf{k}$, this condition implies that the normals to the FS at the self-intersection points are perpendicular to $\mathbf{q}$, see Fig. 4(a). Let $\phi_{\mathbf{k}}^*$ be an azimuthal angle of the normal to the FS at any given point $\mathbf{k}$. Then the constraint $\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q} = 0$ can be written as $\phi_{\mathbf{k}}^* = \phi_{\mathbf{q}} + \pi/2$. (By symmetry, it suffices to consider only the interval $\phi_{\mathbf{k}}^* - \phi_{\mathbf{q}} \in \{0, \pi\}$). Given the relation between $\phi_{\mathbf{k}}^*$ and $\phi_{\mathbf{q}}$, one can find the corresponding angular interval $\delta\phi_{\mathbf{k}}^*$ instead of $\delta\phi_{\mathbf{q}}$. As one goes around the FS, $\phi_{\mathbf{k}}^*$ varies with $\phi_{\mathbf{k}}$. The difference between the convex and concave FSs is in that the dependence of $\phi_{\mathbf{k}}^*$ on $\phi_{\mathbf{k}}$ is monotonic for the former and nonmonotonic for the latter;

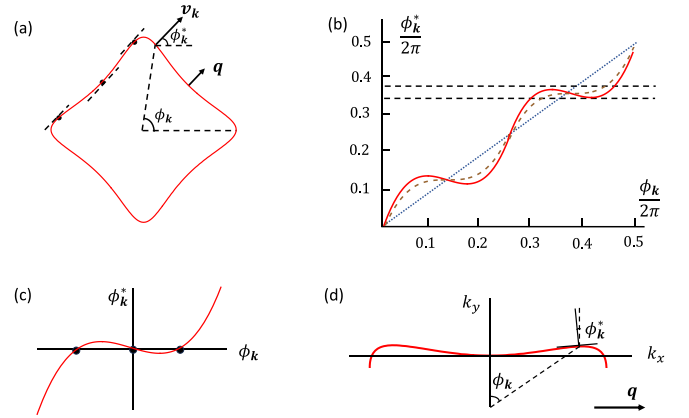


FIG. 4. (a) At the self-intersection points (black dots), the momentum transfer $\mathbf{q}$ is a tangent to the Fermi surface. $\mathbf{v}_{\mathbf{k}}$ is the Fermi velocity at point $\mathbf{k}$ on the Fermi surface (FS). (b) The dependence of $\phi_{\mathbf{k}}^*$ on $\phi_{\mathbf{k}}$ for various types of the FS. Dashed curve: $E < E_c$, convex FS; dotted line: at the convex-concave transition; solid curve: $E > E_c$, concave FS. (c) An enlarged view of the nonmonotonic dependence of $\phi_{\mathbf{k}}^*$ on $\phi_{\mathbf{k}}$. (d) A portion of the FS contour in local Cartesian coordinates.

see Fig. 4(b). The equation $\phi_{\mathbf{k}}^* = \phi_{\mathbf{q}} + \pi/2$ has multiple roots only for a concave FS. Note that the curves $\phi_{\mathbf{k}}^*(\phi_{\mathbf{k}})$ below and above the convex-to-concave transition coincide at inflection points. Near any of such points, the function $\phi_{\mathbf{k}}^*(\phi_{\mathbf{k}})$ is described by a cubic polynomial [see Fig. 4(c)],

$$\phi_{\mathbf{k}}^*(\phi_{\mathbf{k}}) = b(\phi_{\mathbf{k}} - \phi_{\text{inf}})^3 - c\Delta(\phi_{\mathbf{k}} - \phi_{\text{inf}}), \quad (64)$$

where ϕ_{inf} is the azimuthal coordinate of the inflection point, while b and $c > 0$ are constants that are specific for a given FS. The interval $\delta\phi_{\mathbf{k}}^*$ is then equal to the vertical distance between the maximum and minimum of the curve, which gives

$$\delta\phi_{\mathbf{k}}^* = \frac{4c^{3/2}}{3\sqrt{3}b^{1/2}} \Delta^{3/2} \sim \Delta^{3/2}. \quad (65)$$

Next, we need to find $\Delta \mathbf{v}$. For small q , we can write $\Delta \mathbf{v} \approx (\mathbf{q} \cdot \nabla)(\mathbf{v}_{\mathbf{k}_1} - \mathbf{v}_{\mathbf{k}_2})$, where $\mathbf{k}_2$ and $\mathbf{k}_1$ are any two solutions of the kinematic constraint $\mathbf{v}_{\mathbf{k}} \cdot \mathbf{q} = 0$. In terms of angles $\phi_{\mathbf{k}}^*$ and $\phi_{\mathbf{k}}$, we can rewrite $(\Delta \mathbf{v})^2 \approx q^2 (\frac{\partial \phi_{\mathbf{k}}^*}{\partial \phi_{\mathbf{k}}}|_{\phi_{\mathbf{k}_1}} - \frac{\partial \phi_{\mathbf{k}}^*}{\partial \phi_{\mathbf{k}}}|_{\phi_{\mathbf{k}_2}})^2$; see Fig. 4(d). For the $\phi_{\mathbf{k}}^*(\phi_{\mathbf{k}})$ curve in Eq. (64), we immediately find

$$(\Delta \mathbf{v})^2 \propto (q\Delta)^2. \quad (66)$$

Finally, to find $q_{\max}$, we rewrite Eq. (64) in a local Cartesian system with the x axis along the tangent to the FS at the inflection point, see Fig. 4(d). As shown in Ref. [29], such an equation reads $k_y = -bk_x^4/12 + c\Delta k_x^2/2$, where $k_{x,y}$ are measured from the inflection point [29]. Using this relation to solve $\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}-\mathbf{q}}$, we obtain a cubic equation of k_x which admits three different real roots only if $q \leq 2\sqrt{c\Delta/b}$. We thus arrive at the following result:

$$q_{\max} \propto \Delta^{1/2}. \quad (67)$$

We now come back to the integral over $\mathbf{q}$ in the second line of Eq. (59). Approximating $\int d^2q$ by $\int_0^{q_{\max}} dq q \delta\phi_{\mathbf{q}} \sim \int_0^{q_{\max}} dq q \delta\phi_{\mathbf{k}}^*$, using Eqs. (65)–(67), and substituting the interaction from Eq. (10), we obtain for the contribution to the

conductivity from current-relaxing channels:

$$\begin{aligned}\tilde{\sigma}'(\omega, 0) &\propto \delta\phi_{\mathbf{k}}^* \Delta^2 \int_0^{q_{\max}} \frac{dq q}{(q^2 + q_B^2)^2} \\ &\propto \Delta^{7/2} \max \left\{ \frac{1}{q_B^2}, \frac{q_{\max}^2}{q_B^4} \right\}.\end{aligned}\quad (68)$$

Adding up the contributions from the Cooper and swap channels, we obtain the conductivity in the FL regime,

$$\begin{aligned}\sigma'(\omega, 0) &\sim e^2 g^2 \left[\left(\frac{\Delta}{E_F} \right)^{7/2} \max \left\{ \frac{k_F^2}{q_B^2}, \frac{m \Delta k_F^2}{q_B^4} \right\} \Theta(\Delta) \right. \\ &\quad \left. + \frac{k_F^4}{q_B^4} \left(\frac{\omega}{E_F} \right)^2 \ln \frac{\omega_{\text{FL}}}{|\omega|} \right],\end{aligned}\quad (69)$$

where $\Theta(x)$ is the Heaviside function. The first term in the equation above is Gurzhi-like [cf. Eq. (1)], while the second one is the same as for a convex FS.

Near the QCP, the first option for the Gurzhi-like term in Eq. (69), $\propto \Delta^{7/2}/q_B^2$, becomes $\Delta^{7/2}/|\omega|^{2/3}$. The second option, $\propto \Delta^{9/2}/q_B^4$, formally becomes $\Delta^{9/2}/|\omega|^{4/3}$ but it occurs only in the regime where it is already subleading to the last term, and thus can be neglected. Finally, the optical conductivity near a convex-to-concave transition acquires the following scaling form:

$$\sigma'(\omega, 0) \propto \Theta(\Delta) \frac{\Delta^{7/2}}{|\omega|^{2/3}} + |\omega|^{2/3}.\quad (70)$$

Note that the conductivity has a minimum at $\omega = \omega_{\min} \propto \Delta^{21/8}$.

VI. SUMMARY

In this paper, we presented detailed calculations for the optical conductivity of a 2D metal near the Ising-nematic QCP. The focus of our attention was on the effect of the FS geometry that constrains the kinematics of possible scattering processes and thus ultimately determines the frequency dependence of the optical response. We identified the swap and Cooper channels for the case of a convex FS, which do not relax current, and extra channels for the case of a concave FS, which do. On a technical level, we used the modified Kubo formula, which expresses the conductivity via the correlation function of the time derivatives of the current rather than that

of the currents itself [11,16,17]. This method gives certain advantages as compared to a direct diagrammatic computation, namely, it (i) bypasses the need to consider the dynamic interaction and (ii) automatically accounts for all current-relaxing processes. This obviates the need for the usual bookkeeping of various diagrams that tend to partially cancel each other for a generic non-Galilean-invariant FL.

First, we considered an electron system with a non-parabolic but isotropic dispersion relation. Our main result is given by Eq. (41), which generalizes prior related calculations [11,12] by going beyond the model of a purely density-density interaction. Extrapolating Eq. (41) to the quantum-critical regime of an Ising-nematic phase transition, we find the frequency scaling of the optical conductivity as given by Eq. (4).

Next, we showed explicitly that Eqs. (41) and (4) also hold for a generic convex FS. In doing so, we confirmed the vanishing of the leading term in the conductivity [3,22,23]. However, we disagree with the conclusion of Ref. [22] that the remaining term is independent of the frequency.

Finally, we showed that the “naive” scaling form of the conductivity, $\sigma'(\omega) \propto 1/|\omega|^{2/3}$, is restored for a concave FS. In this respect, our results disagree with those of Ref. [23], in which the conductivity was found to contain only the $\delta(\omega)$ term, regardless of the shape of the FS. We also derived the frequency dependence of conductivity at the onset of a convex-to-concave transition, see Eq. (69).

Note added in proof. Recently, H. Guo showed [21] that the result of Ref. [22] $\sigma = \text{const}$ for a convex Fermi surface is not valid.

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APPENDIX A: COMMUTATOR ALGEBRA

In this Appendix, we will make use of the following two identities:

$$[A, BC] = [A, B]C + B[A, C]\quad (A1)$$

and

$$[A, BC] = \{A, B\}C - B\{A, C\},\quad (A2)$$

where $[x, y]$ and $\{x, y\}$ denote a commutator and anticommutator, respectively, of x and y . To simplify notations, we will suppress spin indices of the fermionic operators, as they are not essential for the commutator algebra.

1. Interaction part of the charge current

According to Eq. (12), we need to calculate the following commutator:

$$[\rho_{\mathbf{q}}, H_{\text{int}}] = \sum_{\mathbf{q}'} V(\mathbf{q}') [\rho_{\mathbf{q}}, d_{\mathbf{q}'} d_{-\mathbf{q}'}], \quad (\text{A3})$$

where $d_{\mathbf{q}}$ is given by Eq. (8). According to Eq. (A1):

$$[\rho_{\mathbf{q}}, d_{\mathbf{q}'} d_{-\mathbf{q}'}] = [\rho_{\mathbf{q}}, d_{\mathbf{q}'}] d_{-\mathbf{q}'} + d_{\mathbf{q}'} [\rho_{\mathbf{q}}, d_{-\mathbf{q}'}]. \quad (\text{A4})$$

Next, applying Eq. (A1) and then Eq. (A2), we obtain

$$\begin{aligned} [\rho_{\mathbf{q}}, d_{\mathbf{q}'}]/e &= \sum_{\mathbf{k}, \mathbf{k}'} F(\mathbf{k}') [c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}, c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger c_{\mathbf{k}'-\mathbf{q}'/2}] \\ &= - \sum_{\mathbf{k}, \mathbf{k}'} F(\mathbf{k}') ([c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger, c_{\mathbf{k}+\mathbf{q}/2}^\dagger] c_{\mathbf{k}'-\mathbf{q}'/2} + c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger [c_{\mathbf{k}'-\mathbf{q}'/2}, c_{\mathbf{k}+\mathbf{q}/2}^\dagger]) \\ &= - \sum_{\mathbf{k}, \mathbf{k}'} F(\mathbf{k}') ((\{c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger, c_{\mathbf{k}+\mathbf{q}/2}^\dagger\} c_{\mathbf{k}-\mathbf{q}/2} - c_{\mathbf{k}+\mathbf{q}/2}^\dagger \{c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger, c_{\mathbf{k}-\mathbf{q}/2}\}) c_{\mathbf{k}'-\mathbf{q}'/2} \\ &\quad - c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger (\{c_{\mathbf{k}'+\mathbf{q}'/2}^\dagger, c_{\mathbf{k}+\mathbf{q}/2}^\dagger\} c_{\mathbf{k}-\mathbf{q}/2} - c_{\mathbf{k}+\mathbf{q}/2}^\dagger \{c_{\mathbf{k}'-\mathbf{q}'/2}, c_{\mathbf{k}-\mathbf{q}/2}\})) \\ &= \sum_{\mathbf{k}} F(\mathbf{k} - \mathbf{q}/2 - \mathbf{q}'/2) c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2-\mathbf{q}'} - F(\mathbf{k} + \mathbf{q}/2 + \mathbf{q}'/2) c_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}'}^\dagger c_{\mathbf{k}-\mathbf{q}/2} \\ &= \sum_{\mathbf{k}} (F(\mathbf{k} - \mathbf{q}/2 + \mathbf{q}'/2) - F(\mathbf{k} + \mathbf{q}/2 + \mathbf{q}'/2)) c_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}'}^\dagger c_{\mathbf{k}-\mathbf{q}/2}, \end{aligned} \quad (\text{A5})$$

where we relabeled $\mathbf{k} - \mathbf{q}' \rightarrow \mathbf{k}$ at the last step. For small $\mathbf{q}$:

$$[\rho_{\mathbf{q}}, d_{\mathbf{q}'}]/e = - \sum_{\mathbf{k}} \mathbf{q} \cdot \nabla_{\mathbf{k}} F(\mathbf{k} + \mathbf{q}'/2) c_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}'}^\dagger c_{\mathbf{k}-\mathbf{q}/2}. \quad (\text{A6})$$

Likewise,

$$[\rho_{\mathbf{q}}, d_{-\mathbf{q}'}]/e = - \sum_{\mathbf{k}} \mathbf{q} \cdot \nabla_{\mathbf{k}} F(\mathbf{k} - \mathbf{q}'/2) c_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}'}^\dagger c_{\mathbf{k}-\mathbf{q}/2}. \quad (\text{A7})$$

The corresponding current operator is read off from Eq. (12) as

$$\mathbf{j}_{\text{int}}(\mathbf{q}) = e \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}'} (\nabla_{\mathbf{k}} F(\mathbf{k} + \mathbf{q}'/2) F(\mathbf{p}) c_{\mathbf{k}+\mathbf{q}/2+\mathbf{q}'}^\dagger c_{\mathbf{p}-\mathbf{q}'/2}^\dagger c_{\mathbf{p}+\mathbf{q}'/2} c_{\mathbf{k}-\mathbf{q}/2} + \nabla_{\mathbf{p}} F(\mathbf{p} - \mathbf{q}'/2) F(\mathbf{k}) c_{\mathbf{k}+\mathbf{q}'/2}^\dagger c_{\mathbf{p}+\mathbf{q}/2-\mathbf{q}'}^\dagger c_{\mathbf{p}-\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}'/2}). \quad (\text{A8})$$

On putting $\mathbf{q} = 0$, relabeling $\mathbf{k} + \mathbf{q}'/2 \rightarrow \mathbf{k}$ in the first term and $\mathbf{p} - \mathbf{q}'/2 \rightarrow \mathbf{p}$ in the second one, and restoring spin indices, the last equation is reduced to Eq. (14) of the main text.

2. Commutators of currents with the Hamiltonian

We start with $\mathbf{K}_1 = [\mathbf{j}_0, H_{\text{int}}]$, where H_0 and $\mathbf{j}_0$ are given by Eqs. (6b) and (13) of the main text, respectively. Applying Eq. (A1), we obtain

$$\mathbf{K}_1 = \frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) ([\mathbf{j}_0, d_{\mathbf{q}}] d_{-\mathbf{q}} + d_{\mathbf{q}} [\mathbf{j}_0, d_{-\mathbf{q}}]). \quad (\text{A9})$$

Evaluating the commutators in the above equation with the help of Eqs. (A1) and (A2), we find

$$[\mathbf{j}_0, d_{\mathbf{q}}] = e \sum_{\mathbf{k}, \mathbf{s}} (\mathbf{v}_{\mathbf{k}+\mathbf{q}/2} - \mathbf{v}_{\mathbf{k}-\mathbf{q}/2}) F(\mathbf{k}) c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} \quad (\text{A10})$$

and, correspondingly,

$$[\mathbf{j}_0, d_{-\mathbf{q}}] = e \sum_{\mathbf{p}} (\mathbf{v}_{\mathbf{p}-\mathbf{q}/2} - \mathbf{v}_{\mathbf{p}+\mathbf{q}/2}) F(\mathbf{p}) c_{\mathbf{p}-\mathbf{q}/2}^\dagger c_{\mathbf{p}+\mathbf{q}/2}. \quad (\text{A11})$$

Combining the last two equations, we have

$$\mathbf{K}_1 = -\frac{e}{2} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} (\mathbf{v}_{\mathbf{k}+\mathbf{q}/2} + \mathbf{v}_{\mathbf{p}-\mathbf{q}/2} - \mathbf{v}_{\mathbf{k}-\mathbf{q}/2} - \mathbf{v}_{\mathbf{p}+\mathbf{q}/2}) U(\mathbf{k}, \mathbf{p}, \mathbf{q}) c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{p}-\mathbf{q}/2}^\dagger c_{\mathbf{p}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}. \quad (\text{A12})$$

Restoring spin indices, we obtain Eq. (23a) of the main text.

Next, we compute $\mathbf{K}_2 = [\mathbf{j}_{\text{int}}, H_0]$, where $\mathbf{j}_{\text{int}}$ is defined in Eq. (14) of the main text. With nematic density d_q , defined in Eq. (8), and nematic current, defined as

$$\mathcal{J}_q = e \sum_{\mathbf{k}} \nabla_{\mathbf{k}} F(\mathbf{k}) c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2}, \quad (\text{A13})$$

the interacting part of the current $\mathbf{j}_{\text{int}}$ can be rewritten as

$$\mathbf{j}_{\text{int}} = \frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) (\mathcal{J}_q d_{-\mathbf{q}} + \mathcal{J}_{-\mathbf{q}} d_{\mathbf{q}}). \quad (\text{A14})$$

With the help of Eq. (A1), $\mathbf{K}_2$ becomes

$$\mathbf{K}_2 = -\frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) ([H_0, \mathcal{J}_q] d_{-\mathbf{q}} + \mathcal{J}_q [H_0, d_{-\mathbf{q}}] + [H_0, \mathcal{J}_{-\mathbf{q}}] d_{\mathbf{q}} + \mathcal{J}_{-\mathbf{q}} [H_0, d_{\mathbf{q}}]). \quad (\text{A15})$$

Calculating the commutators in the above equation, we obtain

$$[H_0, \mathcal{J}_q] = e \sum_{\mathbf{k}} \nabla_{\mathbf{k}} F(\mathbf{k}) (\varepsilon_{\mathbf{k}+\mathbf{q}/2} - \varepsilon_{\mathbf{k}-\mathbf{q}/2}) c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{k}-\mathbf{q}/2} \quad (\text{A16})$$

and

$$[H_0, d_{\mathbf{q}}] = \sum_{\mathbf{p}} F(\mathbf{p}) (\varepsilon_{\mathbf{p}+\mathbf{q}/2} - \varepsilon_{\mathbf{p}-\mathbf{q}/2}) c_{\mathbf{p}+\mathbf{q}/2}^\dagger c_{\mathbf{p}-\mathbf{q}/2}. \quad (\text{A17})$$

Combining all the terms in Eq. (A15), we obtain

$$\mathbf{K}_2 = -e \sum_{\mathbf{k}\mathbf{p}\mathbf{q}} (\nabla_{\mathbf{k}} + \nabla_{\mathbf{p}}) U(\mathbf{k}, \mathbf{p}, \mathbf{q}) (\varepsilon_{\mathbf{k}-\mathbf{q}/2} + \varepsilon_{\mathbf{p}+\mathbf{q}/2} - \varepsilon_{\mathbf{k}+\mathbf{q}/2} - \varepsilon_{\mathbf{p}-\mathbf{q}/2}) c_{\mathbf{k}+\mathbf{q}/2}^\dagger c_{\mathbf{p}-\mathbf{q}/2}^\dagger c_{\mathbf{p}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}. \quad (\text{A18})$$

On restoring spin indices, the last result gives Eq. (23b) of the main text.

APPENDIX B: AVERAGING OF COMMUTATORS OVER THE NONINTERACTING SYSTEM

In this Appendix, we derive Eqs. (26a)–(26c). We begin with Eq. (22a), which reads explicitly

$$\begin{aligned} \langle [\mathbf{K}_1(t), \mathbf{K}_1(0)] \rangle_\omega &= -i \frac{e^2}{4} \int_0^\infty dt e^{i\omega t} \sum_{s_1 s'_1 s_2 s'_2} \sum_{\mathbf{k}_1 \mathbf{p}_1 \mathbf{q}_1} \sum_{\mathbf{k}_2 \mathbf{p}_2 \mathbf{q}_2} (v_{\mathbf{k}_{1+}} + v_{\mathbf{p}_{1-}} - v_{\mathbf{k}_{1-}} - v_{\mathbf{p}_{1+}}) (v_{\mathbf{k}_{2+}} + v_{\mathbf{p}_{2-}} - v_{\mathbf{k}_{2-}} - v_{\mathbf{p}_{2+}}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) \\ &\quad \times U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) \left[c_{\mathbf{k}_{1+}, s_1}^\dagger(t) c_{\mathbf{p}_{1-}, s'_1}^\dagger(t) c_{\mathbf{p}_{1+}, s'_1}(t) c_{\mathbf{k}_{1-}, s_1}(t), c_{\mathbf{k}_{2+}, s_2}^\dagger(0) c_{\mathbf{p}_{2-}, s'_2}^\dagger(0) c_{\mathbf{p}_{2+}, s'_2}(0) c_{\mathbf{k}_{2-}, s_2}(0) \right]. \end{aligned} \quad (\text{B1})$$

Since Eq. (B1) already contains a square of the interaction, we can safely put $H = H_0$ in the evolution operators $e^{iHt} \approx e^{iH_0 t}$, which leads to

$$c_{\mathbf{k}_{1+}, s_1}^\dagger(t) c_{\mathbf{p}_{1-}, s'_1}^\dagger(t) c_{\mathbf{p}_{1+}, s'_1}(t) c_{\mathbf{k}_{1-}, s_1}(t) = c_{\mathbf{k}_{1+}, s_1}^\dagger c_{\mathbf{p}_{1-}, s'_1}^\dagger c_{\mathbf{p}_{1+}, s'_1} c_{\mathbf{k}_{1-}, s_1} e^{i(\varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}})t}. \quad (\text{B2})$$

Adding an infinitesimally small imaginary part to ω , we solve the time integral in Eq. (B1) as

$$\int_0^\infty dt e^{i(\omega + i0^+ + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}})t} = \frac{i}{\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}} + i0^+}. \quad (\text{B3})$$

Therefore,

$$\begin{aligned} \text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_1(0)] \rangle_\omega &= -\frac{\pi e^2}{4} \sum_{s_1 s'_1 s_2 s'_2} \sum_{\mathbf{k}_1 \mathbf{p}_1 \mathbf{q}_1} \sum_{\mathbf{k}_2 \mathbf{p}_2 \mathbf{q}_2} \delta(\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) \\ &\quad \times (v_{\mathbf{k}_{1+}} + v_{\mathbf{p}_{1-}} - v_{\mathbf{k}_{1-}} - v_{\mathbf{p}_{1+}}) (v_{\mathbf{k}_{2+}} + v_{\mathbf{p}_{2-}} - v_{\mathbf{k}_{2-}} - v_{\mathbf{p}_{2+}}) (\langle \dots \rangle^{\text{I}} - \langle \dots \rangle^{\text{II}}), \end{aligned} \quad (\text{B4})$$

where we defined

$$\langle \dots \rangle^{\text{I}} = \langle c_{\mathbf{k}_{1+}, s_1}^\dagger c_{\mathbf{p}_{1-}, s'_1}^\dagger c_{\mathbf{p}_{1+}, s'_1} c_{\mathbf{k}_{1-}, s_1} c_{\mathbf{k}_{2+}, s_2}^\dagger c_{\mathbf{p}_{2-}, s'_2}^\dagger c_{\mathbf{p}_{2+}, s'_2} c_{\mathbf{k}_{2-}, s_2} \rangle, \quad (\text{B5a})$$

$$\langle \dots \rangle^{\text{II}} = \langle c_{\mathbf{k}_{2+}, s_2}^\dagger c_{\mathbf{p}_{2-}, s'_2}^\dagger c_{\mathbf{p}_{2+}, s'_2} c_{\mathbf{k}_{2-}, s_2} c_{\mathbf{k}_{1+}, s_1}^\dagger c_{\mathbf{p}_{1-}, s'_1}^\dagger c_{\mathbf{p}_{1+}, s'_1} c_{\mathbf{k}_{1-}, s_1} \rangle. \quad (\text{B5b})$$

We now proceed by evaluating the average of the first term in the commutator, $\langle \dots \rangle^I$, using Wick's theorem:

$$\begin{aligned} \langle \dots \rangle^I = & -\langle c_{\mathbf{k}_{1+},s_1}^\dagger c_{\mathbf{k}_{2-},s_2} \rangle \langle c_{\mathbf{p}_{1-},s'_1}^\dagger c_{\mathbf{p}_{2+},s'_2} \rangle \langle c_{\mathbf{p}_{1+},s'_1} c_{\mathbf{k}_{2+},s_2}^\dagger \rangle \langle c_{\mathbf{k}_{1-},s_1} c_{\mathbf{p}_{2-},s'_2}^\dagger \rangle \\ & + \langle c_{\mathbf{k}_{1+},s_1}^\dagger c_{\mathbf{k}_{2-},s_2} \rangle \langle c_{\mathbf{p}_{1-},s'_1}^\dagger c_{\mathbf{p}_{2+},s'_2} \rangle \langle c_{\mathbf{p}_{1+},s'_1} c_{\mathbf{p}_{2-},s'_2}^\dagger \rangle \langle c_{\mathbf{k}_{1-},s_1} c_{\mathbf{k}_{2+},s_2}^\dagger \rangle \\ & + \langle c_{\mathbf{k}_{1+},s_1}^\dagger c_{\mathbf{p}_{2+},s'_2} \rangle \langle c_{\mathbf{p}_{1-},s'_1}^\dagger c_{\mathbf{k}_{2-},s_2} \rangle \langle c_{\mathbf{p}_{1+},s'_1} c_{\mathbf{k}_{2+},s_2}^\dagger \rangle \langle c_{\mathbf{k}_{1-},s_1} c_{\mathbf{p}_{2-},s'_2}^\dagger \rangle \\ & - \langle c_{\mathbf{k}_{1+},s_1}^\dagger c_{\mathbf{p}_{2+},s'_2} \rangle \langle c_{\mathbf{p}_{1-},s'_1}^\dagger c_{\mathbf{k}_{2-},s_2} \rangle \langle c_{\mathbf{p}_{1+},s'_1} c_{\mathbf{p}_{2-},s'_2}^\dagger \rangle \langle c_{\mathbf{k}_{1-},s_1} c_{\mathbf{k}_{2+},s_2}^\dagger \rangle. \end{aligned} \quad (\text{B6})$$

Using the equal-time averages $\langle c_{\mathbf{m},s_1}^\dagger c_{\mathbf{n},s_2} \rangle = \delta_{\mathbf{m},\mathbf{n}} \delta_{s_1,s_2} n_F(\varepsilon_{\mathbf{m}})$ and $\langle c_{\mathbf{m},s_1} c_{\mathbf{n},s_2}^\dagger \rangle = \delta_{\mathbf{m},\mathbf{n}} \delta_{s_1,s_2} [1 - n_F(\varepsilon_{\mathbf{m}})]$, we sum over over $\mathbf{k}_2, \mathbf{p}_2$, and all the spin indices in Eq. (B4) to obtain

$$\begin{aligned} & \sum_{s_1,s_2,s'_1,s'_2} \sum_{\mathbf{k}_2,\mathbf{p}_2} U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) (v_{\mathbf{k}_{2+}} + v_{\mathbf{p}_{2-}} - v_{\mathbf{k}_{2-}} - v_{\mathbf{p}_{2+}}) \langle \dots \rangle^I \\ & = (v_{\mathbf{k}_{1+}} + v_{\mathbf{p}_{1-}} - v_{\mathbf{k}_{1-}} - v_{\mathbf{p}_{1+}}) \left[2U\left(\frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}, \mathbf{p}_1 - \mathbf{k}_1\right) - 4U(\mathbf{k}_1, \mathbf{p}_1, -\mathbf{q}_1) - 4U(\mathbf{p}_1, \mathbf{k}_1, \mathbf{q}_1) \right. \\ & \quad \left. + 2U\left(\frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}, \frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \mathbf{k}_1 - \mathbf{p}_1\right) \right] n_F(\varepsilon_{\mathbf{k}_{1+}}) n_F(\varepsilon_{\mathbf{p}_{1-}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1+}})]. \end{aligned} \quad (\text{B7})$$

All other terms vanish due to incompatible conditions on the electron momenta. Recalling that $U(\mathbf{k}, \mathbf{p}, -\mathbf{q}) = F(\mathbf{k})F(\mathbf{p})V(-\mathbf{q}) = F(\mathbf{k})F(\mathbf{p})V(\mathbf{q}) = U(\mathbf{k}, \mathbf{p}, \mathbf{q})$, we simplify the last expression to

$$\begin{aligned} & \sum_{s_1,s_2,s'_1,s'_2} \sum_{\mathbf{k}_2,\mathbf{p}_2} U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) (v_{\mathbf{k}_{2+}} + v_{\mathbf{p}_{2-}} - v_{\mathbf{k}_{2-}} - v_{\mathbf{p}_{2+}}) \langle \dots \rangle^I \\ & = -8(v_{\mathbf{k}_{1+}} + v_{\mathbf{p}_{1-}} - v_{\mathbf{k}_{1-}} - v_{\mathbf{p}_{1+}}) \left[U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) - \frac{1}{2}U\left(\frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}, \mathbf{p}_1 - \mathbf{k}_1\right) \right] n_F(\varepsilon_{\mathbf{k}_{1+}}) n_F(\varepsilon_{\mathbf{p}_{1-}}) \\ & \quad \times [1 - n_F(\varepsilon_{\mathbf{k}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1+}})]. \end{aligned} \quad (\text{B8})$$

Likewise, we obtain for the second, $\langle \dots \rangle^{\text{II}}$, term in Eq. (B4):

$$\begin{aligned} & \sum_{s_1,s_2,s'_1,s'_2} \sum_{\mathbf{k}_2,\mathbf{p}_2} U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) (v_{\mathbf{k}_{2+}} + v_{\mathbf{p}_{2-}} - v_{\mathbf{k}_{2-}} - v_{\mathbf{p}_{2+}}) \langle \dots \rangle^{\text{II}} \\ & = -8(v_{\mathbf{k}_{1+}} + v_{\mathbf{p}_{1-}} - v_{\mathbf{k}_{1-}} - v_{\mathbf{p}_{1+}}) \left[U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) - \frac{1}{2}U\left(\frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}, \mathbf{p}_1 - \mathbf{k}_1\right) \right] \\ & \quad \times n_F(\varepsilon_{\mathbf{k}_{1-}}) n_F(\varepsilon_{\mathbf{p}_{1+}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1+}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1-}})]. \end{aligned} \quad (\text{B9})$$

Substituting Eqs. (B8) and (B9) into Eq. (B4), using the property $1 - n_F(\varepsilon) = e^{\varepsilon/T} n_F(\varepsilon)$, and relabeling $\mathbf{k}_1 \rightarrow \mathbf{k}, \mathbf{p}_1 \rightarrow \mathbf{p}, \mathbf{q}_1 \rightarrow \mathbf{q}$, we get

$$\begin{aligned} \text{Im}\langle [\mathbf{K}_1(t), \mathbf{K}_1(0)] \rangle_\omega = & 2\pi e^2 (1 - e^{-\omega/T}) \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \delta(\omega + \varepsilon_{\mathbf{k}_+} + \varepsilon_{\mathbf{p}_-} - \varepsilon_{\mathbf{p}_+} - \varepsilon_{\mathbf{k}_-}) (v_{\mathbf{k}_+} + v_{\mathbf{p}_-} - v_{\mathbf{k}_-} - v_{\mathbf{p}_+})^2 \\ & \times U(\mathbf{k}, \mathbf{p}, \mathbf{q}) \left[U(\mathbf{k}, \mathbf{p}, \mathbf{q}) - \frac{1}{2}U\left(\frac{\mathbf{k} + \mathbf{p} + \mathbf{q}}{2}, \frac{\mathbf{k} + \mathbf{p} - \mathbf{q}}{2}, \mathbf{p} - \mathbf{k}\right) \right] M(\mathbf{k}, \mathbf{p}, \mathbf{q}), \end{aligned} \quad (\text{B10})$$

where $M(\mathbf{k}, \mathbf{p}, \mathbf{q})$ is given by Eq. (27). The second term in [...] in the last expression is the exchange part of the interaction, which is small compared to the first (direct) term for our case of a long-range $U(\mathbf{q})$. Neglecting the exchange term and substituting Eq. (B10) into Eq. (22a), we obtain Eq. (26a) of the main text.

We now turn to Eq. (26b). Following the same steps as above up to Eq. (B6), we arrive at

$$\begin{aligned} \text{Im}\langle [\mathbf{K}_2(t), \mathbf{K}_2(0)] \rangle_\omega = & -\pi e^2 \sum_{s_1,s'_1,s_2,s'_2} \sum_{\mathbf{k}_1,\mathbf{p}_1,\mathbf{q}_1} \sum_{\mathbf{k}_2,\mathbf{p}_2} \delta(\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}}) (\nabla_{\mathbf{k}_1} + \nabla_{\mathbf{p}_1}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) \\ & \times U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) (\varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{k}_{1-}} - \varepsilon_{\mathbf{p}_{1+}}) (\varepsilon_{\mathbf{k}_{2+}} + \varepsilon_{\mathbf{p}_{2-}} - \varepsilon_{\mathbf{k}_{2-}} - \varepsilon_{\mathbf{p}_{2+}}) (\langle \dots \rangle^I - \langle \dots \rangle^{\text{II}}), \end{aligned} \quad (\text{B11})$$

where $\langle \dots \rangle^I$ and $\langle \dots \rangle^{II}$ are the same as in Eqs. (B5a) and (B5b). Summing the $\langle \dots \rangle^I$ term in Eq. (B11) over $\mathbf{k}_2, \mathbf{p}_2$, and over the spins, we find

$$\begin{aligned} & \sum_{s_1, s_2, s'_1, s'_2} \sum_{\mathbf{k}_2 \mathbf{p}_2 \mathbf{q}_2} (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) (\varepsilon_{\mathbf{k}_{2+}} + \varepsilon_{\mathbf{p}_{2-}} - \varepsilon_{\mathbf{k}_{2-}} - \varepsilon_{\mathbf{p}_{2+}}) \langle \dots \rangle^I \\ &= -8(\varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{k}_{1-}} - \varepsilon_{\mathbf{p}_{1+}}) \left[(\nabla_{\mathbf{k}_1} + \nabla_{\mathbf{p}_1}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) - \frac{1}{2} (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{p}_1 - \mathbf{k}_1) \Big|_{\mathbf{k}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \mathbf{p}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}} \right] \\ & \quad \times n_F(\varepsilon_{\mathbf{k}_{1+}}) n_F(\varepsilon_{\mathbf{p}_{1-}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1+}})]. \end{aligned} \quad (\text{B12})$$

Following similar steps for the $\langle \dots \rangle^{II}$ term in Eq. (B11) and using the same property of the Fermi functions, we arrive at

$$\begin{aligned} \text{Im} \langle [\mathbf{K}_2(t), \mathbf{K}_2(0)] \rangle_\omega &= 8\pi e^2 (1 - e^{-\omega/T}) \sum_{\mathbf{k}_1 \mathbf{p}_1 \mathbf{q}_1} \delta(\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}}) (\varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{k}_{1-}} - \varepsilon_{\mathbf{p}_{1+}})^2 (\nabla_{\mathbf{k}_1} + \nabla_{\mathbf{p}_1}) \\ & \quad \times U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) \left[(\nabla_{\mathbf{k}_1} + \nabla_{\mathbf{p}_1}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) - \frac{1}{2} (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) \Big|_{\mathbf{k}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \mathbf{p}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}, \mathbf{q}_2 = \mathbf{p}_1 - \mathbf{k}_1} \right] \\ & \quad \times n_F(\varepsilon_{\mathbf{k}_{1+}}) n_F(\varepsilon_{\mathbf{p}_{1-}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1+}})]. \end{aligned} \quad (\text{B13})$$

Neglecting the second (exchange) term in [...] in the last expression and substituting the result into Eq. (22b), we obtain Eq. (26b) of the main text.

Finally, we turn to Eq. (22c) which contains the cross-correlators, $\text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_2(0)] \rangle_{\pm\omega}$. The same steps as before lead us to

$$\begin{aligned} \text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_2(0)] \rangle_\omega &= \frac{\pi e^2}{2} \sum_{s_1, s'_1, s_2, s'_2} \sum_{\mathbf{k}_1 \mathbf{p}_1 \mathbf{q}_1} \sum_{\mathbf{k}_2 \mathbf{p}_2 \mathbf{q}_2} \delta(\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) (\mathbf{v}_{\mathbf{k}_{1+}} + \mathbf{v}_{\mathbf{p}_{1-}} - \mathbf{v}_{\mathbf{k}_{1-}} - \mathbf{v}_{\mathbf{p}_{1+}}) \\ & \quad \cdot (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{q}_2) (\varepsilon_{\mathbf{k}_{2+}} + \varepsilon_{\mathbf{p}_{2-}} - \varepsilon_{\mathbf{k}_{2-}} - \varepsilon_{\mathbf{p}_{2+}}) (\langle \dots \rangle^I - \langle \dots \rangle^{II}) \end{aligned} \quad (\text{B14})$$

and

$$\begin{aligned} \text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_2(0)] \rangle_\omega &= 4\pi e^2 \omega (1 - e^{-\omega/T}) \sum_{\mathbf{k}_1 \mathbf{p}_1 \mathbf{q}_1} \delta(\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}}) (\mathbf{v}_{\mathbf{k}_{1+}} + \mathbf{v}_{\mathbf{p}_{1-}} - \mathbf{v}_{\mathbf{k}_{1-}} - \mathbf{v}_{\mathbf{p}_{1+}}) \\ & \quad \cdot \left[(\nabla_{\mathbf{k}_1} + \nabla_{\mathbf{p}_1}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) - \frac{1}{2} (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{p}_1 - \mathbf{k}_1) \Big|_{\mathbf{k}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \mathbf{p}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}} \right] U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) \\ & \quad \times n_F(\varepsilon_{\mathbf{k}_{1+}}) n_F(\varepsilon_{\mathbf{p}_{1-}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1+}})], \end{aligned} \quad (\text{B15})$$

where we used the energy-conserving delta-function to express the energy difference in terms of ω . Replacing $\omega \rightarrow -\omega$, relabeling $\mathbf{k}_1 \leftrightarrow \mathbf{p}_1$, and taking into account that $U(\mathbf{k}, \mathbf{p}, \mathbf{q}) = U(\mathbf{p}, \mathbf{k}, \mathbf{q})$, we find

$$\begin{aligned} \text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_2(0)] \rangle_\omega &= 4\pi e^2 \omega (1 - e^{\omega/T}) \sum_{\mathbf{k}_1 \mathbf{p}_1 \mathbf{q}_1} \delta(\omega + \varepsilon_{\mathbf{k}_{1+}} + \varepsilon_{\mathbf{p}_{1-}} - \varepsilon_{\mathbf{p}_{1+}} - \varepsilon_{\mathbf{k}_{1-}}) (\mathbf{v}_{\mathbf{k}_{1+}} + \mathbf{v}_{\mathbf{p}_{1-}} - \mathbf{v}_{\mathbf{k}_{1-}} - \mathbf{v}_{\mathbf{p}_{1+}}) \\ & \quad \cdot \left[(\nabla_{\mathbf{k}_1} + \nabla_{\mathbf{p}_1}) U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) - \frac{1}{2} (\nabla_{\mathbf{k}_2} + \nabla_{\mathbf{p}_2}) U(\mathbf{k}_2, \mathbf{p}_2, \mathbf{p}_1 - \mathbf{k}_1) \Big|_{\mathbf{k}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 + \mathbf{q}_1}{2}, \mathbf{p}_2 = \frac{\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{q}_1}{2}} \right] U(\mathbf{k}_1, \mathbf{p}_1, \mathbf{q}_1) \\ & \quad \times n_F(\varepsilon_{\mathbf{p}_{1+}}) n_F(\varepsilon_{\mathbf{k}_{1-}}) [1 - n_F(\varepsilon_{\mathbf{p}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{k}_{1+}})]. \end{aligned} \quad (\text{B16})$$

Now we apply the identity

$$(1 - e^{\omega/T}) n_F(\varepsilon_{\mathbf{k}_{1+}}) n_F(\varepsilon_{\mathbf{p}_{1-}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1-}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1+}})] = -(1 - e^{-\omega/T}) n_F(\varepsilon_{\mathbf{k}_{1-}}) n_F(\varepsilon_{\mathbf{p}_{1+}}) [1 - n_F(\varepsilon_{\mathbf{k}_{1+}})] [1 - n_F(\varepsilon_{\mathbf{p}_{1-}})] \quad (\text{B17})$$

to get

$$\text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_2(0)] \rangle_\omega = -\text{Im} \langle [\mathbf{K}_1(t), \mathbf{K}_2(0)] \rangle_{-\omega}. \quad (\text{B18})$$

According to Eq. (22c), we then have

$$\sigma'_{12}(\omega, T) = \frac{g^2}{\omega^3} \text{Im} \langle [\mathbf{K}_1(t); \mathbf{K}_2(0)] \rangle_\omega. \quad (\text{B19})$$

Substituting Eq. (B15) into Eq. (B19) and neglecting the exchange term, we obtain Eq. (26c) of the main text.

APPENDIX C: PROPERTIES OF $\sigma'_{12}(\omega, T)$

In Sec. V A, we encountered the following expression for the cross-term in the optical conductivity [Eq. (61)]:

$$\sigma'_{12}(\omega, T) = e^2 g^2 \frac{\pi}{(2\pi)^4 \omega^2} (1 - e^{-\omega/T}) \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} (\mathbf{w} \cdot \Delta \mathbf{v}) n_F(\varepsilon_{\mathbf{k}}) n_F(\varepsilon_{\mathbf{p}}) \times [1 - n_F(\varepsilon_{\mathbf{k}} + \Omega)][1 - n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega)] \delta(\Omega - \varepsilon_{\mathbf{k}-\mathbf{q}} + \varepsilon_{\mathbf{k}}) \delta(\Omega - \omega + \varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}}). \quad (\text{C1})$$

To check that $\sigma'_{12}(\omega, T)$ is even in ω , we flip the sign of frequency ($\omega \rightarrow -\omega$) and relabel $\mathbf{k} - \mathbf{q} \leftrightarrow \mathbf{p}$ and $\mathbf{p} + \mathbf{q} \leftrightarrow \mathbf{k}$. (Note that the last transformation results in $\Delta \mathbf{v} \rightarrow -\Delta \mathbf{v}$.) Therefore,

$$\sigma'_{12}(-\omega, T) = -e^2 g^2 \frac{\pi}{(2\pi)^4 \omega^2} (1 - e^{\omega/T}) \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} (\mathbf{w} \cdot \Delta \mathbf{v}) n_F(\varepsilon_{\mathbf{p}+\mathbf{q}}) \times n_F(\varepsilon_{\mathbf{k}-\mathbf{q}})[1 - n_F(\varepsilon_{\mathbf{p}+\mathbf{q}} + \Omega)][1 - n_F(\varepsilon_{\mathbf{k}-\mathbf{q}} - \Omega - \omega)] \delta(\Omega - \varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}+\mathbf{q}}) \delta(\Omega + \omega + \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}}). \quad (\text{C2})$$

Next, we shift the integration variable as $\Omega + \omega \rightarrow \Omega$ and eliminate $\varepsilon_{\mathbf{k}-\mathbf{q}}$ and $\varepsilon_{\mathbf{p}+\mathbf{q}}$ in favor of $\varepsilon_{\mathbf{k}}$ and $\varepsilon_{\mathbf{p}}$, using the delta functions. This gives

$$\sigma'_{12}(-\omega, T) = -e^2 g^2 \frac{\pi}{(2\pi)^4 \omega^2} (1 - e^{\omega/T}) \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} (\mathbf{w} \cdot \Delta \mathbf{v}) \times n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega) n_F(\varepsilon_{\mathbf{k}} + \Omega) [1 - n_F(\varepsilon_{\mathbf{p}})] [1 - n_F(\varepsilon_{\mathbf{k}})] \delta(\Omega - \omega - \varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}+\mathbf{q}}) \delta(\Omega + \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}-\mathbf{q}}). \quad (\text{C3})$$

Now the delta functions are the same as in Eq. (C1). Further, we rewrite the product of the Fermi functions in the last equation as

$$\begin{aligned} & n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega) n_F(\varepsilon_{\mathbf{k}} + \Omega) [1 - n_F(\varepsilon_{\mathbf{p}})] [1 - n_F(\varepsilon_{\mathbf{k}})] \\ &= n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega) n_F(\varepsilon_{\mathbf{k}} + \Omega) n_F(\varepsilon_{\mathbf{p}}) n_F(\varepsilon_{\mathbf{k}}) e^{\beta(\varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{k}})} \\ &= n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega) n_F(\varepsilon_{\mathbf{k}} + \Omega) n_F(\varepsilon_{\mathbf{p}}) n_F(\varepsilon_{\mathbf{k}}) e^{\beta(\varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{k}})} e^{\beta(\varepsilon_{\mathbf{p}} - \Omega + \omega)} e^{-\beta(\varepsilon_{\mathbf{p}} - \Omega + \omega)} e^{\beta(\varepsilon_{\mathbf{k}} + \Omega)} e^{-\beta(\varepsilon_{\mathbf{k}} + \Omega)} \\ &= [1 - n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega)] [1 - n_F(\varepsilon_{\mathbf{k}} + \Omega)] n_F(\varepsilon_{\mathbf{p}}) n_F(\varepsilon_{\mathbf{k}}) e^{-\beta\omega}. \end{aligned} \quad (\text{C4})$$

Substituting this back into Eq. (C1), we see that indeed $\sigma'_{12}(-\omega, T) = \sigma'_{12}(\omega, T)$, as it should.

Nevertheless, if we neglect the frequencies in the delta functions and use Eq. (35) for the energy integrals, we get an *odd* function of frequency, i.e., $\sigma'_{12}(\omega, T) \propto \omega(1 + 4\pi^2 T^2/\omega^2)$. Therefore, one cannot neglect the frequencies in this case. Note that $\sigma'_{12}(\omega, T)$ is even only because $\Delta \mathbf{v}$ changes signs on the transformations employed. Now, let's expand the dispersions in the delta functions in Eq. (C1) to $O(q)$ but keep the frequencies in there, and see if the evenness of $\sigma'_{12}(\omega, T)$ is preserved by these simplifications. Performing the steps indicated above, we obtain

$$\sigma'_{12}(\omega, T) = e^2 g^2 \frac{\pi}{(2\pi)^4 \omega^2} (1 - e^{-\omega/T}) \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} \times (\mathbf{w} \cdot \Delta \mathbf{v}) n_F(\varepsilon_{\mathbf{k}}) n_F(\varepsilon_{\mathbf{p}}) [1 - n_F(\varepsilon_{\mathbf{k}} + \Omega)] [1 - n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega)] \delta(\Omega + \mathbf{v}_{\mathbf{k}}^F \cdot \mathbf{q}) \delta(\Omega - \omega + \mathbf{v}_{\mathbf{p}}^F \cdot \mathbf{q}), \quad (\text{C5})$$

where the subscript *F* indicates that the velocities are taken right on the FS, i.e., they do not depend on $\varepsilon_{\mathbf{k}}$ and $\varepsilon_{\mathbf{p}}$. On $\omega \rightarrow -\omega$, the last expression becomes

$$\sigma'_{12}(-\omega, T) = e^2 g^2 \frac{\pi}{(2\pi)^4 \omega^2} (1 - e^{\omega/T}) \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} \times (\mathbf{w} \cdot \Delta \mathbf{v}^F) n_F(\varepsilon_{\mathbf{k}}) n_F(\varepsilon_{\mathbf{p}}) [1 - n_F(\varepsilon_{\mathbf{k}} + \Omega)] [1 - n_F(\varepsilon_{\mathbf{p}} - \Omega - \omega)] \delta(\Omega + \mathbf{v}_{\mathbf{k}}^F \cdot \mathbf{q}) \delta(\Omega + \omega + \mathbf{v}_{\mathbf{p}}^F \cdot \mathbf{q}). \quad (\text{C6})$$

Now we perform the following sequence of transformations: relabel $\mathbf{k} \leftrightarrow \mathbf{p}$ (note that $\Delta \mathbf{v}$ does not change its sign on this transformation), shift the variable as $\Omega + \omega \rightarrow \Omega$, and replace $\varepsilon_{\mathbf{k}, \mathbf{p}} \rightarrow -\varepsilon_{\mathbf{k}, \mathbf{p}}$ (note that the velocities are not affected by this transformation). Then we obtain

$$\sigma'_{12}(-\omega, T) = e^2 g^2 \frac{\pi}{(2\pi)^4 \omega^2} (1 - e^{\omega/T}) \int \frac{d^2 q}{(2\pi)^2} V^2(\mathbf{q}) \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{k}} \int_{-\infty}^{+\infty} d\varepsilon_{\mathbf{p}} \int_{-\infty}^{+\infty} d\Omega \oint \frac{d\ell_{\mathbf{k}}}{v_{\mathbf{k}}} \oint \frac{d\ell_{\mathbf{p}}}{v_{\mathbf{p}}} \times (\mathbf{w} \cdot \Delta \mathbf{v}^F) n_F(-\varepsilon_{\mathbf{p}}) n_F(-\varepsilon_{\mathbf{k}}) [1 - n_F(-\varepsilon_{\mathbf{p}} + \Omega - \omega)] [1 - n_F(-\varepsilon_{\mathbf{k}} - \Omega)] \delta(\Omega - \omega + \mathbf{v}_{\mathbf{p}}^F \cdot \mathbf{q}) \delta(\Omega + \mathbf{v}_{\mathbf{k}}^F \cdot \mathbf{q}). \quad (\text{C7})$$

The rest of the transformations affect only the Fermi functions:

$$\begin{aligned} & n_F(-\varepsilon_{\mathbf{p}}) n_F(-\varepsilon_{\mathbf{k}}) [1 - n_F(-\varepsilon_{\mathbf{p}} + \Omega - \omega)] [1 - n_F(-\varepsilon_{\mathbf{k}} - \Omega)] \\ &= [1 - n_F(\varepsilon_{\mathbf{p}})] [1 - n_F(\varepsilon_{\mathbf{k}})] n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega) n_F(\varepsilon_{\mathbf{k}} + \Omega) \\ &= n_F(\varepsilon_{\mathbf{p}}) n_F(\varepsilon_{\mathbf{k}}) e^{\beta(\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{p}})} n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega) e^{\beta(\varepsilon_{\mathbf{p}} - \Omega + \omega)} e^{-\beta(\varepsilon_{\mathbf{p}} - \Omega + \omega)} n_F(\varepsilon_{\mathbf{k}} + \Omega) e^{\beta(\varepsilon_{\mathbf{k}} + \Omega)} e^{-\beta(\varepsilon_{\mathbf{k}} + \Omega)} \\ &= n_F(\varepsilon_{\mathbf{p}}) n_F(\varepsilon_{\mathbf{k}}) [1 - n_F(\varepsilon_{\mathbf{p}} - \Omega + \omega)] [1 - n_F(\varepsilon_{\mathbf{k}} + \Omega)] e^{-\beta\omega}. \end{aligned} \quad (\text{C8})$$

Substituting Eq. (C8) into Eq. (C7) yields $\sigma'_{12}(-\omega, T) = -\sigma'_{12}(\omega, T)$, which is incorrect.

Let's summarize: projecting the momenta onto the FS yields $\sigma'_{12}(\omega, T) = A\omega$ (at $T = 0$). However, such a term is not allowed by time-reversal symmetry; thus we must have $A = 0$, regardless of the shape of a FS. Therefore, even for a concave FS, one needs to expand Δv in deviations from the FS, as done for the isotropic case. That should give $\sigma'_{12}(\omega, T) = B\omega^2$, which is subleading to $\sigma'_1(\omega, T)$.

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