

Getting the Lay of the Land in Discrete Space: A Survey of Metric Dimension and Its Applications*

Richard C. Tillquist[†]
Rafael M. Frongillo[†]
Manuel E. Lladser[‡]

Abstract. The metric dimension of a graph is the smallest number of nodes required to identify all other nodes uniquely based on shortest path distances. Applications of metric dimension include discovering the source of a spread in a network, canonically labeling graphs, and embedding symbolic data in low-dimensional Euclidean spaces. This survey gives a self-contained introduction to metric dimension and an overview of the quintessential results and applications. We discuss methods for approximating the metric dimension of general graphs, and specific bounds and asymptotic behavior for deterministic and random families of graphs. We conclude with related concepts and directions for future work.

Key words. metric dimension, graph embedding, multilateration, graph isomorphism, resolving set

MSC codes. 05C12, 05C60, 05C62, 05C85, 05C90, 68R10

DOI. 10.1137/21M1409512

Contents

I	Introduction	920
2	Formal Definition and Examples	921
2.1	Paths, Complete Graphs, and Cycles	922
2.2	Grids	923
2.3	Fans and Wheels	923
3	Computational Complexity and Approximation	925
3.1	NP-Completeness	925
3.2	Parameterized Complexity and Special Cases	927
3.3	Approximation and the Information Content Heuristic	927
3.4	Other Heuristics	928

*Received by the editors April 2, 2021; accepted for publication (in revised form) August 19, 2022; published electronically November 7, 2023.

<https://doi.org/10.1137/21M1409512>

Funding: This work was partially supported by NSF IIS grant 1836914. The BioFrontiers Computing Core at the University of Colorado–Boulder provided high-performance computing resources (funded by NIH grant 1S10OD012300), supported by BioFrontiers IT group.

[†]Department of Computer Science, University of Colorado, Boulder, CO 80309-0430 USA (richard.tillquist@colorado.edu, raf@colorado.edu).

[‡]Department of Applied Mathematics, University of Colorado, Boulder, CO 80309-0526 USA (manuel.lladser@colorado.edu).

4	Graph Features and Modifications	928
4.1	Diameter and Metric Dimension	928
4.2	Twin Nodes and Metric Dimension	929
4.3	Graphs with Extreme Metric Dimension	929
4.4	Vertex and Edge Deletions	930
5	Families of Graphs	931
5.1	Trees and Unicyclic Graphs	931
5.2	Hamming Graphs	934
5.3	Product Graphs	936
5.4	Connections to Abstract Algebra	938
5.5	Strongly Regular and Distance Regular Graphs	939
5.6	Other Graph Families	939
6	Random Graph Models	941
6.1	Erdős–Rényi Random Graphs	941
6.2	Stochastic Block Model	942
6.3	Random Trees and Forests	943
7	Applications	944
7.1	Game Theory	944
7.2	Source Localization	945
7.3	Detecting Network Motifs	946
7.4	Embedding Biological Sequence Data	947
8	Related Concepts	948
8.1	Doubly Resolving Sets	948
8.2	Strong Metric Dimension	949
8.3	Multilateration	949
8.4	Truncated Metric Dimension	950
8.5	Resolving Number, Upper Dimension, Random k -Dimensionality . . .	951
8.6	Directed Graphs	951
8.7	Other Related Notions	952
8.8	Metric Space Connections	953
9	Conclusion and Open Questions	953
	Acknowledgments	954
	References	954

1. Introduction. In the Euclidean plane, any set of three noncollinear points is enough to uniquely distinguish all points in the space based on distances. This process, called trilateration in $\mathbb{R}^2$, is the basic technique through which global positioning systems (GPS) are able to pinpoint a location on the surface of the Earth. More generally, if $\|\cdot\|$ denotes the Euclidean distance and $R = \{\mathbf{r}_1, \dots, \mathbf{r}_{n+1}\} \subset \mathbb{R}^n$ is a set of $n + 1$ affinely independent points, the vectors $(\|\mathbf{x} - \mathbf{r}_1\|, \dots, \|\mathbf{x} - \mathbf{r}_{n+1}\|)$ and $(\|\mathbf{y} - \mathbf{r}_1\|, \dots, \|\mathbf{y} - \mathbf{r}_{n+1}\|)$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are different when $\mathbf{x} \neq \mathbf{y}$. The situation becomes more complex, however, if the space of interest is discrete

instead of continuous. One class of discrete spaces of particular interest comprises those that can be represented as graphs coupled with shortest path distance. On a graph $G = (V, E)$ the notion of *metric dimension* is analogous to the number of satellites required for GPS to work effectively. The goal is to pick a minimal set of vertices $R \subseteq V$ capable of identifying every vertex based solely on shortest path distances to R . Solving this problem exactly in an arbitrary graph is computationally complex but provides information that is useful in a variety of settings. A small set of “satellites” or “landmarks” in a discrete space can be valuable in assisting robots navigating over a physical space or in tracking the progress of a disease as it spreads between cities. Such sets can also be used for identifying a source of misinformation in a social network, comparing network structure, categorizing chemical structures, or representing symbolic data numerically.

In this work we collate and interpret a number of theoretical results and approximation techniques associated with metric dimension, paying particular attention to specific types of graphs and applications. We survey recent work and describe promising directions for future work.

2. Formal Definition and Examples. Let $G = (V, E)$ be a graph, potentially with weighted edges, and let $d(u, v)$ denote the shortest path distance in G from $u \in V$ to $v \in V$. Unless otherwise stated, when no such path exists, $d(u, v) = \infty$.

DEFINITION 2.1 (resolving set). $R \subseteq V$ is *resolving* if, for all distinct $u, v \in V$, there exists $r \in R$ such that $d(r, u) \neq d(r, v)$. Such an r is said to *resolve* or *distinguish* u and v . Such sets are also sometimes called *locating sets* [193], *reference sets* [195], or *metric generators* [186].

In the context of $\mathbb{R}^n$, any set of $(n + 1)$ or more affinely independent points is analogous to a resolving set in a graph.

By definition, $R = \{r_1, \dots, r_k\}$ is resolving if and only if the transformation

$$d(u|R) := (d(r_1, u), \dots, d(r_k, u))$$

from V to $(\mathbb{R} \cup \infty)^{|R|}$ is injective, i.e., every vertex $u \in V$ is uniquely represented by the vector of distances from all vertices in R (listed in an arbitrary but specified order) to u . In many settings, minimizing the dimension $|R|$ of these vectors is a central goal.

DEFINITION 2.2 (metric dimension). The *metric dimension* $\beta(G)$ of G (also sometimes denoted $\dim(G)$ or $\mu(G)$) is the smallest cardinality of resolving sets on G . If R is a resolving set on G and $|R| = \beta(G)$, R is called a *minimal resolving set*, *basis set*, or *metric basis* of G .

In the context of graphs, the concept of metric dimension was introduced separately by Slater in 1975 [193] and by Harary and Melter in 1976 [99], though the dimension of graphs was discussed earlier by Erdős, Harary, and Tutte in 1965 [66] and the dimension of general metric spaces by Blumenthal in 1953 [28]. Both the Slater and Harary and Melter papers focus on the metric dimension of trees and describe equivalent exact formulae for graphs of this kind. Harary and Melter briefly discuss the metric dimension of several other types of graphs including cycles, complete graphs, and complete bipartite graphs, though the metric dimension of wheel graphs is incorrectly stated as two. They also give an algorithm to reconstruct a tree given distances from every node to the elements of a resolving set. This is not possible for general graphs as not all edges are guaranteed to be represented in a shortest path with an element of a resolving set as an endpoint (see section 8.2).

Before continuing, we examine several types of graphs for which minimal resolving sets are readily described and easily visualized. Through this examination, we hope to strengthen the reader’s intuitive grasp of metric dimension and to solidify concepts that are important in future sections.

2.1. Paths, Complete Graphs, and Cycles. For $G = (V, E)$ connected with $|V| = n \geq 2$, the path graph P_n and the complete graph K_n represent the two extremes of metric dimension. Indeed, $\beta(G) = 1$ if and only if $G \simeq P_n$, and $\beta(G) = n - 1$ if and only if $G \simeq K_n$ [42]. Resolving sets for P_n and K_n are readily apparent (see Figure 1). For a path, either of the end vertices resolves every vertex, as each distance 0 to $(n - 1)$ is attained exactly once. For a complete graph, every vertex is at distance 0 from itself and at distance 1 from all other vertices. This means that a single vertex $v \in V$ uniquely identifies itself but no other vertices. In order to distinguish all vertices, a resolving set of K_n must be of cardinality at least $(n - 1)$. Moreover, any such set is resolving as the excluded vertex is the only one at a strictly positive distance from all other vertices.

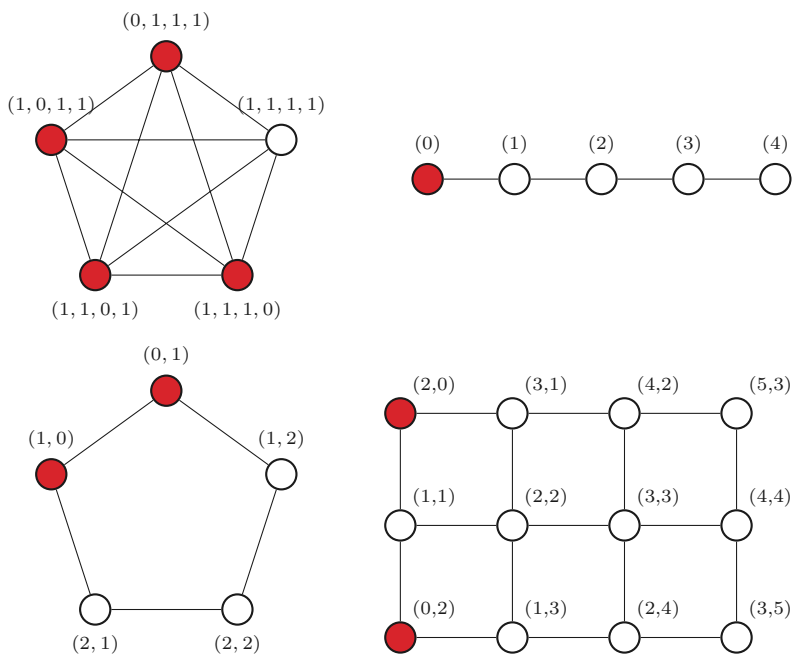


Fig. 1 Minimal resolving sets (shaded circles) for the complete graph K_5 (upper left), the path P_5 (upper right), the cycle C_5 (lower left), and the grid $G_{4,3}$ (lower right). Nodes are annotated with their distance vectors $d(u|R)$, which must be unique.

The cycle graph C_n on $n > 2$ vertices has metric dimension 2 [43]. No single vertex can resolve C_n because every vertex has degree 2. Next, we construct a set of cardinality 2 and show that it is resolving. Suppose that the vertex set of C_n is $\{0, \dots, n - 1\}$, where consecutive integers are neighbors and so are 0 and $n - 1$. Let $R = \{0, 1\}$. Then $d(x|R) = (\min\{x, n - x\}, \min\{x - 1, n + 1 - x\})$. But using that $\min\{a, b\} = (a + b - |a - b|)/2$, it follows that if $d(x|R) = d(y|R)$, then $|n - 2x| = |n - 2y|$ and $|n - 2(x - 1)| = |n - 2(y - 1)|$; in particular, x and y are at the same distance

from $n/2$, and so are $x - 1$ and $y - 1$, which is possible only when $x = y$. Hence, $\beta(C_n) = 2$ (see Figure 1).

2.2. Grids. Consider the two-dimensional grid $G_{m,n}$ with $m, n \geq 1$ integers. The vertices of this graph correspond to ordered pairs $(x, y) \in \mathbb{Z}^2$ such that $0 \leq x < m$ and $0 \leq y < n$. The edges correspond to pairs at Euclidean distance exactly 1 when considered as points in $\mathbb{R}^2$ (see Figure 1). In particular, every vertex has degree at least 2 and no singleton can be resolving. Clearly, $d((0, 0), (x, y)) = x + y$ and, more generally, $d((u, v), (x, y)) = |u - x| + |v - y|$. Thus, shortest path distance in this case is equivalent to the ℓ_1 -norm or Manhattan distance. Let $R = \{(0, 0), (0, n - 1)\}$. Observe that $d((a, b)|R) = (a + b, a + n - 1 - b)$. Then, for vertices (a, b) and (x, y) , we have $d((a, b)|R) = d((x, y)|R)$ if and only if $a = x$ and $b = y$. Hence, since no single vertex can resolve this graph, $\beta(G_{m,n}) = 2$ [134, 159]. Symmetric arguments show that $\{(0, 0), (m - 1, 0)\}$, $\{(0, n - 1), (m - 1, n - 1)\}$, and $\{(m - 1, 0), (m - 1, n - 1)\}$ are also resolving for $G_{m,n}$.

For the d -dimensional grid $G_{n_1, \dots, n_d}$ defined in an analogous manner for $n_i > 1$ with $1 \leq i \leq d$, $\beta(G_{n_1, \dots, n_d}) \leq d$ [134]. The key idea is that if $r_0 := (0, \dots, 0)$, and r_i is the vector of zeros with entry $(n_i - 1)$ in the i th position, then for any vertex $v = (x_1, \dots, x_d)$ in the grid we have

$$\begin{aligned} d(r_0, v) &= x_1 + \dots + x_d, \\ d(r_1, v) &= (n_1 - 1 - x_1) + x_2 + \dots + x_d, \\ &\dots \\ d(r_{d-1}, v) &= x_1 + \dots + (n_{d-1} - 1 - x_{d-1}) + x_d. \end{aligned}$$

Since this linear system of d equations and d unknowns is invertible, $\{r_0, \dots, r_{d-1}\}$ resolves the d -dimensional grid.¹

2.3. Fans and Wheels. Finally, we consider fans and wheels, which can be thought of as simple modifications of paths and cycles with the addition of a single fully connected vertex. Formally, the fan graph on $n + 1$ vertices, denoted F_n , consists of a path P_n on n vertices and one additional vertex, a , adjacent to all vertices on P_n . Similarly, the wheel graph W_n is a cycle C_n of cardinality n with an additional vertex, which we also denote as a , adjacent to all vertices on C_n . As the metric dimensions of P_n and C_n are elementary to determine (1 and 2, respectively), one might expect the metric dimensions of F_n and W_n to be similarly trivial, yet this is not the case. Part of this complexity stems from the diameter of these graphs shrinking from order n to at most 2, reducing the possible shortest path distances to the set $\{0, 1, 2\}$. Specifically, their metric dimensions are given as follows [189]:

$$(2.1) \quad \beta(F_{x+5k}) = \beta(W_{x+5k}) = \begin{cases} 3 + 2k & \text{when } x \in \{7, 8\}, \\ 4 + 2k & \text{when } x \in \{9, 10, 11\}, \end{cases}$$

for all $k \geq 0$. We find it instructive to provide a high level proof of this result. Accordingly, we assume in what follows that $n > 6$.

Focusing first on F_n , observe that node a appears in none of its minimal resolving sets. To see this, suppose for a contradiction that R is a minimal resolving set of F_n such that $a \in R$. Since R is minimal, there must be two different vertices $u, v \in V$

¹Theorem 2.5 in [134] suggests that $\beta(G_{m,n,\dots}) = d$. We note, however, that $G_{2,2,\dots,2} \simeq Q_d$, the d -dimensional hypercube, and $\beta(Q_k) < k$ for $k > 4$.

such that $d(u|R \setminus \{a\}) = d(v|R \setminus \{a\})$. Furthermore, without loss of generality we must have $u = a$; otherwise $d(u, a) = d(v, a) = 1$ and R would not resolve u and v . As $u = a$, and as $d(a|R \setminus \{a\})$ is the all-ones vector, v must be distance 1 from every node in $R \setminus \{a\}$. By definition of F_n , there are at most two nodes other than a at distance 1 from v , so $|R \setminus \{a\}| \leq 2$. One can check that there are now at most six possible values of $d(\cdot|R \setminus \{a\})$, namely, $(1, 2), (0, 2), (1, 1), (2, 0), (2, 1), (2, 2)$ if $|R \setminus \{a\}| = 2$, and $(0), (1), (2)$ if $|R \setminus \{a\}| = 1$. As a is distance 1 from all nodes on the path, and $n > 6$, we cannot have resolved F_n . Hence, no minimal resolving set of F_n contains the vertex a .

Let R be a minimal resolving set of F_n . Define $\beta := |R|$. Note that at most β vertices of P_n can be adjacent to exactly one vertex of R . Otherwise, if there were more, the pigeonhole principle would imply that at least two vertices are indistinguishable with respect to their distances to R . Since at most one vertex of P_n can be distance 2 from all vertices in R , all the remaining nodes of P_n must either be in R or be adjacent to exactly two vertices of R . Hence $n \leq 1 + 2\beta + \beta/2$ or, equivalently, $\lceil 2(n-1)/5 \rceil \leq \beta$. But, because $n > 6$, we may write $n = x + 5k$ with $7 \leq x \leq 11$ and $k \geq 0$. In particular [189]

$$\beta(F_{x+5k}) \geq \begin{cases} 3 + 2k & \text{when } x \in \{7, 8\}, \\ 4 + 2k & \text{when } x \in \{9, 10, 11\}. \end{cases}$$

It is easy to see that this lower bound on $\beta(F_{x+5k})$ is also an upper bound via a simple construction. In particular, there is a resolving set of F_{x+5k} of cardinality $3 + 2k$ when x is 7 or 8, and of cardinality $4 + 2k$ when x is 9, 10, or 11. Let $m = n \pmod{5}$ and let R be a minimal resolving set of F_n . For each consecutive, full block of five vertices, $v_x, v_y \in R$ where $x = 5j + 2$ so that v_x is the second vertex of the j th block and $y = 5j + 4$ so that v_y is the fourth vertex of the j th block for $j \geq 0$. If $m \in \{2, 3\}$, $v_n \in R$. If instead $m = 4$, $v_{(n-2)}, v_n \in R$ (see Figure 2).

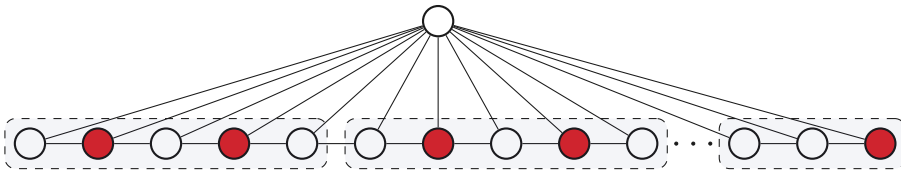


Fig. 2 A visualization of a minimal resolving set of the fan F_n when $(n \bmod 5) = 3$. Dashed lines group separate consecutive blocks of five nodes.

Finally, we claim that $\beta(W_n) = \beta(F_n)$ for $n > 6$. Indeed, suppose without loss of generality that R is a minimal resolving set of F_n such that $1, n \notin R$ (any resolving set of F_n can be made to satisfy this requirement by replacing 1 with 2 or 3, and n with $n - 1$ or $n - 2$, depending on $R \setminus \{1, n\}$). Since the only difference between the fan and wheel graphs is the inclusion of the edge $\{1, n\}$ in W_n , and since this edge is not required to determine shortest path distances between elements of R and any other vertices in W_n , R is also a resolving set for W_n . So $\beta(W_n) \leq \beta(F_n)$. Conversely, suppose that R is a minimal resolving set of W_n ; in particular, $|R| \leq \beta(F_n)$. Then there must be at least one edge $\{i, j\}$ in W_n such that $i, j \notin R$ and $i, j \neq a$. This follows from $\beta(W_n) < \lceil \frac{n-1}{2} \rceil$. Removing this edge, therefore, does not affect R as a resolving set. In particular, R also resolves F_n and $\beta(W_n) = \beta(F_n)$, which shows (2.1).

3. Computational Complexity and Approximation. Verifying that a given set of nodes R in $G = (V, E)$ constitutes a resolving set is straightforward in theory. For every $v \in V$, the vector of distances $d(v|R)$ can be generated in $O(|E| + |V| \log |V|)$ time. This collection of $|V|$ vectors then needs to be checked for duplicates. If all vectors are unique, the set is resolving; otherwise there is at least one pair of indistinguishable nodes in G based on R .

A brute force solution determining the exact metric dimension of a general graph, on the other hand, requires an exhaustive search over a very large solution space. For a fixed set of cardinality s , there are $\binom{|V|}{s}$ subsets of nodes that must be considered. Since we are interested in the smallest s for which a subset of nodes of this cardinality resolves G , increasing values of s starting at 1 must be tested until a resolving set is found. Indeed, for a positive integer k , deciding whether $\beta(G) \leq k$ is a computationally difficult problem. In light of this, researchers have turned to parameterized complexity and approximation methods designed to find small resolving sets on general graphs.

3.1. NP-Completeness. The decision problem associated with metric dimension is NP-complete (for background on computational complexity, see, e.g., Goldreich [88]). In this section, we present one proof of NP-completeness via reduction from 3-SAT, the problem of testing whether a given Boolean formula in conjunctive normal form, with three literals per clause, has a satisfying assignment [129]. An alternative reduction, from the three-dimensional matching problem, is cited in [82].

Formally, the 3-SAT problem is as follows. Let E be a Boolean expression in conjunctive normal form with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$. For instance, the formula $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \bar{x}_4)$ consists of two clauses and four variables. 3-SAT is the problem of determining, given such a formula E , whether there exists an assignment mapping variables to truth values making E true. For the previous formula, setting x_1 and x_2 to True and x_3 and x_4 to False is one such assignment.

For an arbitrary 3-SAT instance E , we will construct a graph G such that E is satisfiable if and only if $\beta(G) = n + m$. We follow the construction of [134]. For every variable x_i create a six cycle with nodes labeled $T_i, a_i^1, b_i^1, F_i, b_i^2, a_i^2$, listed clockwise (see Figure 3, left). For every clause C_j create a four star with nodes labeled $c_j^k, 1 \leq k \leq 5$, and central node c_j^2 (see Figure 3, right).

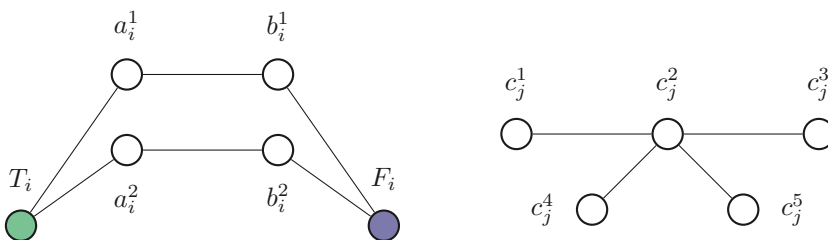


Fig. 3 Visualization of gadgets from [134] for a single variable (left) and a single clause (right) in a Boolean expression.

These cycles and stars are used to form a connected graph by including the edge $\{T_i, c_j^1\}$ for every variable x_i and every clause C_j . In addition, when x_i is used as a positive literal in C_j , the edges $\{F_i, c_j^1\}$ and $\{F_i, c_j^3\}$ are added to the graph. Instead,

when x_i is used as a negative literal in C_j the edges $\{F_i, c_j^1\}$ and $\{T_i, c_j^3\}$ are added. Otherwise, if x_i does not appear in C_j , the edges $\{F_i, c_j^1\}$, $\{F_i, c_j^3\}$, and $\{T_i, c_j^3\}$ are added (see Figure 4). Since there are a total of $6n + 5m$ nodes in the final graph, and there are at most four edges between the subgraphs representing x_i and C_j for all $1 \leq i \leq n$ and $1 \leq j \leq m$, this construction takes polynomial time.

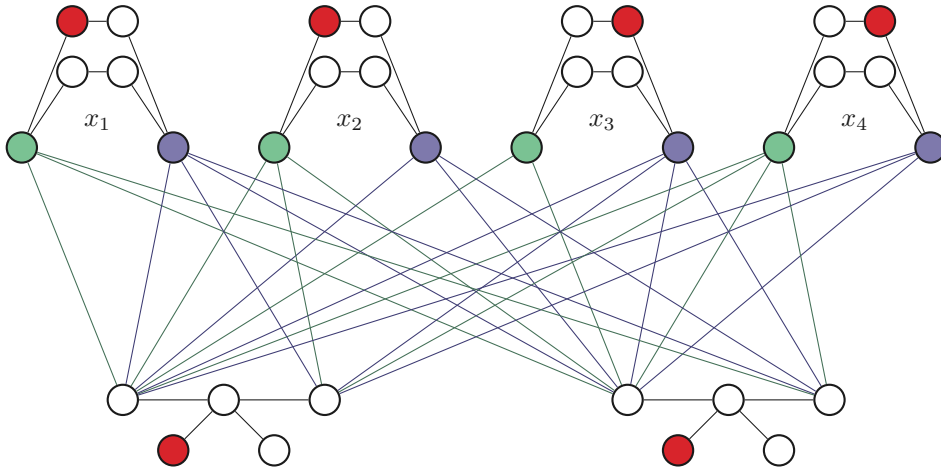


Fig. 4 The graph produced by the reduction for the Boolean formula $E = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3 \vee \overline{x_4})$. Gadgets for variables and clauses are oriented as in Figure 3. As E is satisfiable, the graph has a resolving set of cardinality $4 + 2 = 6$, shown in red.

Notice that any resolving set R of G must include at least one of $\{a_i^1, a_i^2, b_i^1, b_i^2\}$ for all $1 \leq i \leq n$ and at least one of $\{c_j^4, c_j^5\}$ for all $1 \leq j \leq m$. Hence $\beta(G) \geq n + m$. Showing that $\beta(G) = n + m$ if E is satisfiable is straightforward. Given an assignment of the variables, take $R = \{c_j^4 \mid 1 \leq j \leq m\} \cup \{a_i^1 \mid x_i \text{ is true}\} \cup \{b_i^1 \mid x_i \text{ is false}\}$ (one could also choose c_j^5 , a_i^2 , and/or b_i^2).

To see that R is resolving, first consider the variable gadget for x_i and any clause C_j . Recall that $c_j^4 \in R$. The vertices of the x_i gadget are split into two groups based on distances to c_j^4 : vertices $\{T_i, F_i\}$ at distance 3 and $\{a_i^1, a_i^2, b_i^1, b_i^2\}$ at distance 4. Any vertex in the set $\{a_i^1, a_i^2, b_i^1, b_i^2\}$, therefore, serves to disambiguate the elements of these groups: T_i and F_i will have distance 1 and 2, or vice versa, and the remaining nodes will attain every distance in $\{0, 1, 2, 3\}$. As we have either $a_i^1 \in R$ or $b_i^1 \in R$, the variable gadget is resolved. In fact, this statement holds regardless of whether or not the formula is satisfiable.

Now consider a clause C_j and a variable x_i that causes this clause to be satisfied. There are two cases: either x_i is a positive literal in C_j and is given a value of true, or x_i is a negative literal in C_j and is given a value of false. In the first case, we have $a_i^1 \in R$, and recall $c_j^4 \in R$. In the C_j gadget, c_j^4 is the unique vertex at distance 1 from a_i^1 , and c_j^5 is the unique vertex at distance 4 from a_i^1 , distinguishing these vertices from all others. Finally, we have $d(a_i^1, c_j^1) = 2$ and $d(a_i^1, c_j^3) = 3$, finishing the proof of the first case. The second case is symmetric to the first, with the edge $\{F_i, c_j^3\}$ becoming $\{T_i, c_j^3\}$ and the role of a_i^1 played by b_i^1 instead.

Conversely, it can be shown that E is satisfiable if $\beta(G) = n + m$ by setting x_i to true if either a_i^1 or a_i^2 is in the resolving set, and to false otherwise. Thus, this

construction reduces 3-SAT to the metric dimension decision problem in polynomial time.

3.2. Parameterized Complexity and Special Cases. Although metric dimension is a computationally difficult problem on arbitrary graphs, there are efficient algorithms in certain restricted settings. For example, efficient algorithms are known to compute metric dimension on trees [99, 193], cographs [65], outerplanar graphs [59], chain graphs [78], and cactus block graphs [107]. We discuss several of these cases in sections 4 and 5. On the other hand, metric dimension remains NP-complete even when restricting to planar graphs (including of bounded degree) [59], bipartite graphs and variations (split, cobipartite, and line graphs of bipartite graphs) [65], and interval graphs or permutation graphs (even of diameter 2) [81].

There are also fixed-parameter tractable algorithms with respect to certain parameters, such as vertex cover number [100], maximum leaf number [64], modular width [25], and the sum of treelength and maximum degree [25]. The phrase *fixed-parameter tractable* means that there is an algorithm whose runtime is polynomial in the input size when the corresponding parameter of the graph is taken as fixed. Such algorithms can be useful in practice when one needs a general algorithm, which does not make any assumption on the input graph, but one expects the inputs to have a low parameter value. Some dual parameterizations of metric dimension also have tractable algorithms [94]. Unfortunately, under some natural parameters, metric dimension remains intractable even in a parameterized sense. Most notable is treewidth, which has seen recent hardness results [30] culminating in the result that the metric dimension problem is NP-hard even for graphs of treewidth 24 [151]. It remains open at what treewidth (between 2 and 24) the problem becomes hard.

Finally, combining the two approaches above, there are also fixed-parameter tractable algorithms for certain restricted families of graphs. For example, interval graphs [81] and permutation graphs [25] have fixed-parameter tractable algorithms with respect to the minimum resolving set cardinality.

3.3. Approximation and the Information Content Heuristic. Faced with the hardness of the general metric dimension problem, it is natural to turn to approximation algorithms. Khuller, Raghavachari, and Rosenfeld [134] give the first such algorithm via reduction to set cover, where each node in the resolving set covers a subset of pairs of vertices. Leveraging the well-known greedy approximation guarantee for set cover, this algorithm achieves an approximation factor $2\ln(n) + O(1)$, where $n = |V|$ (an algorithm for metric dimension with approximation factor α means the cardinality of the resolving set returned is always within α times the true metric dimension). Hauptmann, Schmied, and Viehmann [101] later improved this factor to $1 + (1 + o(1)) \cdot \ln(n)$, via the *information content heuristic* (ICH) algorithm we describe below. They also showed that this approximation factor is essentially the best possible; specifically, it is impossible to $((1 - \epsilon)\ln n)$ -approximate the metric dimension for any $\epsilon > 0$ unless NP is a subset of $\text{DTIME}(n^{\log \log n})$, i.e., unless all problems in NP can be solved by deterministic algorithms in time $O(n^{\log \log n})$ in their input size n . Other approximation hardness results are known as well; for example, no $o(\log n)$ -approximation algorithm exists if $P \neq \text{NP}$ [24], even on graphs of maximum degree 3 [100].

The ICH algorithm of Hauptmann, Schmied, and Viehmann [101], based on one originally devised for the test set problem [27], uses a heuristic similar to information gain in the construction of decision trees [173]. Let $G = (V, E)$ be a graph with $n = |V|$. Consider the classification problem for which each vertex is a training

example from a unique class. The feature vector of each $v \in V$ is simply $d(v|V) \in \mathbb{R}^n$, the shortest path distances to all vertices in the graph. In particular, the $(n \times n)$ distance matrix $\mathbf{D}$ associated with G fully describes the training data. The choice of resolving set R can thus be thought of as the choice of some subset of features: $d(v|R)$ is simply a selection of entries of $d(v|V)$ with $v \in V$, corresponding to the “features” R .

In the usual greedy algorithm to construct a decision tree, at each node of the tree one selects the feature which maximizes the information gained about the true class at the child nodes. In the case of resolving sets, each node of the decision tree corresponds to the addition of some vertex u to R , with branches corresponding to the possible values of $d(\cdot, u)$. To measure information in our setting, consider the distribution p_R induced by the equivalence classes of $d(\cdot|R)$, which assigns probability $|\{v \in V : d(v|R) = d\}|/n$ to each possible value $d \in \{d(v|R) : v \in V\}$. Then the information of R is measured by the Shannon entropy of p_R , denoted $H(R)$. Letting R_t be the resolving set at iteration t , with $R_0 = \emptyset$, the ICH algorithm chooses a vertex $v_t \in V$ maximizing $H(R_t \cup \{v_t\})$. The algorithm terminates when $H(R_t) = \log(n)$, the maximum possible entropy over n items, indicating that all n vertices are uniquely represented by their distances to R_t .² The runtime of the ICH algorithm is $O(n^3)$, making it effective only for relatively small graphs.

3.4. Other Heuristics. Among many heuristic methods commonly deployed in nonconvex search problems, genetic algorithms and variable neighborhood search (VNS) in particular have been used to find small resolving sets on general graphs with some success. *Genetic algorithms*, inspired by the concept of biological evolution, seek optimal solutions to problems by incrementally changing a population of candidate solutions from one generation to the next through the biologically motivated operations of mutation and selective recombination [58]. This approach has been shown to perform quite well when applied to metric dimension in comparison to other state-of-the-art algorithms, including methods based on an integer programming formulations and the CPLEX [111] optimization package [140].

The VNS technique starts with an initial, nonoptimal solution and iteratively expands a neighborhood on which to perform a local search. When a point in the space which improves upon the initial solution is found, the search is restarted with this point at its center. In the context of searching for small resolving sets, VNS seems to outperform genetic algorithm based methods on many kinds of graphs and has been used to improve upon previous upper bounds for certain hypercubes [162].

4. Graph Features and Modifications. In this section, we overview some general observations about metric dimension and its relationship to other graph quantities. These observations are often useful in bounding or exactly determining the metric dimension of a given graph.

4.1. Diameter and Metric Dimension. The *diameter* of a graph $G = (V, E)$, denoted $\text{diam}(G)$, is the length of a longest shortest path in G . For ease of notation,

²In fact, the ICH algorithm is exactly the same as the information gain algorithm for decision trees, under the constraint that the decision made at every node of the tree at the same level must be the same, i.e., we choose the same vertex at iteration t for all decision nodes at level $t - 1$. To see the equivalence, let $n_d = |\{v \in V : d(v|R) = d\}|$ be the size of equivalence class d , and p_d be the distribution of labels within equivalence class d , i.e., the uniform distribution on all n_d vertices in the equivalence class. Then maximizing information gain, weighted by the size of each decision node (recall that we must choose the same feature to split at all nodes), is the same as maximizing $H(p_R)$ because $\min \sum_d \frac{1}{n_d} H(p_d) = \min \sum_d \frac{1}{n_d} \log(n_d) = \max - \sum_d \frac{1}{n_d} \log(\frac{1}{n_d}) = \max H(p_d)$.

let $\delta = \text{diam}(G)$ and $k = \beta(G)$. It is not surprising that a relationship exists between δ and k . Indeed, let $R \subset V$ be a minimum resolving set of G and consider $d(v|R)$ for each $v \in V$. Since such vectors can only contain a 0 when $v \in R$ and, for $v \notin R$, $1 \leq d(r, v) \leq \delta$, it follows that $|V| \leq \delta^k + k$ [134]. This bound is usually loose, though graphs with $|V| = \delta + k$ have been fully characterized [105]. The related bound

$$n \leq (\lfloor 2\delta/3 \rfloor + 1)^k + k \sum_{i=1}^{\lceil \delta/3 \rceil} (2i - 1)^{k-1}$$

is generally tighter [105]. This bound can be made tighter still for specific families of graphs. A partial list of such results follows:

- If G is a tree, $|V| \leq (\delta k + 4)(\delta + 2)/8$, with equality for trees with even diameter [23].
- If G is an outerplanar graph, $|V| = O(\delta^2 k)$ [23].
- If K_i is not a minor of G , $|V| \leq (\delta k + 1)^{i-1} + 1$ [23].
- If G has constant treewidth, $|V| = O(k\delta^{O(1)})$ [23].
- If the rankwidth of G is at most r , $|V| \leq (\delta k + 1)^{\delta(3(2^r)+2)} + 1$ [23].
- If G is an interval or permutation graph, $|V| = O(\delta k^2)$ [80].
- If G is a unit interval graph, $|V| = O(\delta k)$ [80].

4.2. Twin Nodes and Metric Dimension. Let $G = (V, E)$ be an undirected graph and, for each $v \in V$, define the closed-neighborhood of v as $N(v) = \{u \mid \{u, v\} \in E\}$. We call $u, v \in V$ *twins* when $N(u) \cup \{u\} = N(v) \cup \{v\}$. The *twin graph* of G , sometimes denoted G^* , includes a single vertex for each set of twins and an edge between two vertices when an edge exists between the corresponding sets of twins in G .

Twin nodes have an interesting relationship to metric dimension. In fact, when u and v are twins, $d(u, w) = d(v, w)$ for all $w \in V \setminus \{u, v\}$. As a result, any resolving set of G , minimal or not, must include at least one of u and v . More formally, define over V the equivalence relation: $u \equiv v$ if and only if u and v are twins. Let $\tau(G)$ be the set of twin equivalence classes of G . Then, if R is a resolving set of G , $|R \cap \tau| \geq |\tau| - 1$ for each $\tau \in \tau(G)$. In particular [105]

$$\beta(G) \geq \sum_{\tau \in \tau(G)} (|\tau| - 1).$$

Twin nodes have been used, for example, to study connections among metric dimension, diameter, and graph order [105].

4.3. Graphs with Extreme Metric Dimension. As we saw in section 2.1, for $G = (V, E)$, $1 \leq \beta(G) \leq n - 1$, where $\beta(G) = 1$ if and only if $G \simeq P_n$, and $\beta(G) = n - 1$ if and only if $G \simeq K_n$. In fact, for every $1 \leq k \leq n - 1$ there is a connected graph G with n vertices and $\beta(G) = k$ [42].

Suppose $G = (V, E)$ is such that $\beta(G) = 2$. Such graphs must have a set of simple properties [134]. In particular, if $\{u, v\} \subset V$ is a resolving set of G of minimum cardinality, the following properties hold:

1. G cannot contain K_5 as a subgraph.
2. G cannot contain $K_{3,3}$ as a subgraph ($K_{m,n}$ denotes the complete bipartite graph with partitions of cardinality m and n).
3. There is a unique shortest path between u and v .
4. $\deg(w) \leq 5$ for all nodes $w \in V$ on the shortest path between u and v .

5. $\deg(u) \leq 3$ and $\deg(v) \leq 3$.

Properties (1) and (2) may bring Wagner's theorem [211] to mind, a characterization of planar graphs forbidding K_5 and $K_{3,3}$ as minors, suggesting that G must be planar. On the contrary, there are nonplanar graphs with metric dimension 2 [134]. See [200] for other properties satisfied by graphs with metric dimension 2. See also [85] for constraints on graphs with any given metric dimension.

Graphs $G = (V, E)$ with $\beta(G) = (n - 2)$ have also been fully characterized [42]. For two graphs G_1 and G_2 , let $G_1 \cup G_2$ denote their disjoint union and let $G_1 + G_2$ denote the graph formed by taking a disjoint union and joining every node in G_1 with every node in G_2 . Furthermore, define $\bar{K}_n$ to be a graph with n nodes and no edges. Then, the metric dimension of a graph with n nodes is $n - 2$ if and only if the graph is one of the following:

- A complete bipartite graph, $K_{s,t}$ with $s, t \geq 1$.
- $K_s + \bar{K}_t$ with $s \geq 1$ and $t \geq 2$.
- $K_s + (K_1 \cup K_t)$ with $s, t \geq 1$.

There is also a complete characterization of graphs with $\beta(G) = (n - 3)$ based on an enumeration of possible structures that G^* , the twin graph of G , may have [123].

Finally, related to these results, one can ask more generally how metric dimension relates to containing complete graphs. Let G be a graph with $\beta(G) \leq k$. Then if G contains either K_n or $K_{n,n}$, we must have $n \leq 2^{\Theta(k)}$ [85].

4.4. Vertex and Edge Deletions. Given a graph $G = (V, E)$ and a vertex $v \in V$ or edge $e \in E$, we denote by $(G - v)$ and $(G - e)$ the graphs obtained by deleting v and e from G , respectively. A natural line of questioning is to ask what effect such deletions can have on the metric dimension.

Vertex deletions can yield arbitrarily large changes in metric dimension, both positive and negative. Buczkowski et al. [34] show that the difference $\beta(G) - \beta(G - v)$ can be arbitrarily large; for example, as we saw in section 2.3, the wheel graph W_n has metric dimension $\beta(W_n) \approx \frac{2}{5}n$, whereas removing the center vertex reduces the metric dimension to that of a cycle, $\beta(C_n) = 2$. Eroh et al. [69] show that the difference $\beta(G - v) - \beta(G)$ can also be arbitrarily large. Their example takes k trees, each with a root and three leaves, and connects the roots in a line. The resulting graph G' is a tree on $4k$ vertices; as we discuss in section 5.1, it has metric dimension $\beta(G') = 2k$, as two of the three leaves in every subtree must be added to the resolving set. Let G be given by adding a single vertex v to G' and connecting v to one leaf in each subtree. Eroh et al. [69] show that the metric dimension drops to $\beta(G) = k$, as now one only needs one leaf from each subtree.

Edge deletions can also arbitrarily decrease metric dimension, but can only increase it by at most two. Eroh et al. [69] give an intricate example of the former, given by chaining in parallel k copies of C_7 and P_3 , all linked to two vertices A and B . G is then given by adding an edge $e = \{A, B\}$. They show that $\beta(G) = 2k$ as one vertex from each C_7 and P_3 must be in the resolving set, whereas $\beta(G - e) = k + 1$ as the C_7 vertices in the resolving set can be replaced by A . On the other hand, the same authors show by a case analysis that $\beta(G - e) \leq \beta(G) + 2$ for any graph G (considering that vertex deletions entail as many as $n - 1$ edge deletions, this bound does not contradict the potentially large effect of vertex deletions). The effect of edge deletion and addition on trees has also been studied [42, 70]; in particular, for a tree T on at least three vertices, we have $\beta(T) - 2 \leq \beta(T + e) \leq \beta(T) + 1$.

A vertex is called *pendant* when it has degree 1. (An edge is called pendant when it has a pendant end vertex.) If G is nontrivial and connected and G' is obtained from

G by the addition of a pendant vertex, then $\beta(G) \leq \beta(G') \leq \beta(G) + 1$ [34]. However, if the added vertex has degree 2 or higher, the metric dimension of G' may be outside the interval $[\beta(G), \beta(G) + 1]$. This has motivated the study of the notion of *pendant graphs* [144].

These results show how sensitive metric dimension is to elementary graph operations. While some bounds discussed above are tight, others may not be. In particular, it would be interesting to see whether these bounds are tight in a multiplicative sense, e.g., whether an edge deletion could reduce metric dimension from $\Theta(n)$ to $\Theta(1)$.

5. Families of Graphs. While determining the metric dimension of arbitrary graphs is a computationally complex task, exact formulae, upper bounds, and polynomial time algorithms exist for certain types of graphs. In practical applications these analytic and algorithmic results are of crucial importance. Generic approximation algorithms tend to provide small resolving sets but do not scale well enough with network size to be useful on larger networks, which routinely include more than 10^6 total nodes [40, 160, 203]. Tailored algorithms for specific graph structures can therefore be useful in quickly discovering small resolving sets even on large networks.

In what follows, we discuss some of the more prominent graph families on which metric dimension has been studied. See Tables 1 and 2 for exact formulae and bounds for the metric dimension of various graph families. Tables 3 and 4 provide references to formulae or efficient computational methods for the metric dimension of diverse graph families.

5.1. Trees and Unicyclic Graphs. Problems that are computationally difficult on general graph structures often admit more efficient solutions on trees. This is the case for metric dimension. A simple formula giving the metric dimension of trees that are not also paths [42, 99, 193] leads immediately to a linear time algorithm for finding resolving sets of minimum cardinality on trees. To begin, we present several important definitions. Let $G = (V, E)$ be a general graph, not necessarily a tree. In

Table 1 Exact values for the metric dimension of several different families of graphs.

Graph Type	Symbol (Constraint)	Metric Dimension	Reference
Amalgamation of Cycles	$B_n, n \geq 2, e$ even cycles	$\beta(B_n) = \begin{cases} n, & e = 0 \\ n + e - 1, & e \geq 1 \end{cases}$	[118]
Antiprisms	$A_n (n \geq 3)$	3	[124]
Complete Graphs	K_n	$n - 1$	§2.1, [42]
Cycles	C_n	2	§2.1, [43]
De Bruijn Graphs	$B_{d,n}$	$d^{n-1}(d - 1)$	[77, 174]
Fans	$F_n (n \notin \{1, 2, 3, 6\})$	$\lfloor \frac{2n+2}{5} \rfloor$	§2.3, [104]
Hexagonal Graphs	$HX(n) (n > 1)$	3	§5.6, [154]
Honeycomb Graphs	$HC(n) (n > 1)$	3	§5.6, [154]
Jahangir Graphs	$J_{2n} (n \geq 4)$	$\lfloor \frac{2n}{3} \rfloor$	[206]
Kautz Graphs	$K_{d,n}$	$(d^{n-1} + d^{n-2})(d - 1)$	[77, 174]
Paths	P_n	1	§2.1, [42]
Petersen Graphs	$P_{n,2} (n \geq 5)$	3	[124]
Petersen Graphs	$P_{2m+1,m}, m \geq 1$	$\beta(P_{2m+1,m}) = \begin{cases} 2, & m = 1 \\ 3, & m > 1 \end{cases}$	[4]
Prisms	D_n	2 for odd n , 3 for even n	[124]
Sierpiński Graphs	$S_{K_n}^t, n, t \geq 1$	$n - 1$	[135]
Sierpiński Graphs over Stars	$S_{K_{1,n}}^t, n, t \geq 2$	$(n + 1)^{t-2}(n^2 - n - 1) + 1$	[6]
Trees	T_n	$\ell(T_n) - \text{ex}(T_n)$	§5.1, [42, 99, 193]
Wheels	$W_n (n > 6)$	$\beta(W_n) = \beta(F_n)$	§2.3, [189]

Table 2 *Bounds on the metric dimension of several different families of graphs.*

Graph Type	Symbol (Constraint)	Metric Dimension	Reference
Bilinear Forms Graphs	$Hq(n, d) \ (n \geq d \geq 2)$	$\beta(Hq(n, d)) \leq q^{n+d-1+\lfloor \frac{d+1}{n} \rfloor}$	[76]
Cartesian Product with K_2	$G \square K_2$	$\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1$	§5.2, [42]
Doubled Odd Graphs	$O(2e + 1, e, e + 1) \ (e \geq 2)$	$\beta(O(2e + 1, e, e + 1)) \leq 2e + 1$	[93]
Dual Polar Graphs	$\Gamma(q, d, e)$	$\beta(\Gamma(q, d, e)) \leq \frac{(q^{d+e-1}+1)(q^{d+e-1}-q^{e-1}+q-1)}{(q^{e-1}+1)(q-1)}$	[19]
Grids in d dimensions	$Gm, n, \dots$	$\beta(Gm, n, \dots) \leq d$	§2.2, [134, 159]
Hamming Graphs	$H_{k,a}$	$\beta(H_{k,a}) \leq \beta(H_{k+1,a}) \leq \beta(H_{k,a}) + \lfloor a/2 \rfloor$	§5.2, [203]
Johnson Graphs	$J(n, e)$	$\beta(J(n, e)) \leq (e + 1) \lfloor n/(e + 1) \rfloor$	[17, 93]
Paley Graph	F_p with p prime and $p \equiv 1 \pmod{4}$	$\lfloor \log_2(p) \rfloor \leq \beta(F_p) \leq \lfloor \log_2 p \rfloor$	[79]
Para-line Graphs	$G^*, n \geq 2$	$\lfloor \log_2(\max\{\deg(v) \mid v \in G^*\}) \rfloor \leq \beta(G^*) \leq n - 1$	[136]
Unicyclic Graphs	$T + e$	$\beta(T) - 2 \leq \beta(T + e) \leq \beta(T) + 1$	§5.1, [42]

Table 3 *Families of graphs for which metric dimension can be determined efficiently.*

Graph Type	Metric Dimension
Benes Networks	Polynomial time solvable [156]
Butterfly Networks	Polynomial time solvable [156]
Cactus Block Graphs	Linear time solvable [107]
Chain Graphs	Linear time solvable [78]
Cographs	Linear time solvable [65]
Outerplanar Graphs	Polynomial time solvable, §5.6, [59, 60]
Two-Connected Bipartite	
Distance-Hereditary Graphs	Polynomial time solvable [164]

Table 4 *Additional families of graphs for which metric dimension is known or can be determined efficiently. See [104] for more bounds.*

Graph Type	Metric Dimension
Cayley Digraphs	§5.4, [74, pp. 34–37]
Circulant Networks	See [49, 91, 113, 175, 209]
Complete k -partite Graphs	See [183]
Generalized Wheel Graphs	§5.6, [197]
Graphs with Pendant Edges	§4.4, [144]
Grassmann Graphs	See [158, p. 98]
Harary Graphs $H_{4,n}$	See [124, p. 9]
Kneser Graphs	See [17, p. 750]
Line Graphs	See [77, pp. 803–804]
Petersen Graphs $P(n, 3)$	See [114]
Regular Bipartite Graphs	See [12, pp. 16–17]
Torus Network	See [155, pp. 268, 271]
Twisted Grassmann Graphs	See [93, p. 4]

what follows, for $v \in V$, we use $\deg(v) = |\{u \in V \mid \{v, u\} \in E\}|$ to denote the degree of v .

DEFINITION 5.1 (leaf vertex). *A vertex $\ell \in V$ is called a leaf when $\deg(\ell) = 1$. The number of leaves in G is denoted $\ell(G)$.*

DEFINITION 5.2 (major vertex and terminal degree). *A vertex $v \in V$ is called a major vertex when $\deg(v) \geq 3$. The terminal degree of a major vertex $v \in V$ is the number of leaves $\ell \in V$ such that $d(\ell, v) < d(\ell, u)$ for all other major vertices $u \in V$.*

DEFINITION 5.3 (exterior major vertex). *A major vertex of G is called exterior when its terminal degree is strictly positive. The number of exterior major vertices in G is denoted $ex(G)$.*

From these definitions, we can write $\beta(G) \geq \ell(G) - ex(G)$, with equality when G is a tree such that $ex(G) > 0$ or, equivalently, when G is not a path [42]. Moreover, any set $R \subset V$ which contains every leaf, except one, associated with each exterior major vertex is a subset of a minimal resolving set in G . When G is a tree, any such R is resolving [42]. These observations permit an $O(|V| + |E|)$ algorithm for constructing minimal resolving sets on trees: after partitioning the leaves of a tree based on exterior major vertices using a depth first search, one element of each partition may be dropped to produce a resolving set of minimum cardinality (see Figure 5).

Similar ideas can be used to study the metric dimension of infinite trees. If T is an infinite tree with finite metric dimension, the number of major vertices in T must be finite [36]. Furthermore, if T is not an infinite path and has finite metric dimension, $\beta(T)$ can be determined exactly. In particular

$$\beta(T) = \sum_{v \in \{u \mid \deg(u) \geq 3\}} \max\{P_T(v) - 1, 0\},$$

where $P_T(v)$ is the number of finite and infinite paths consisting only of nodes of degree at most 2 with v as an endpoint [36]. Notably, this is analogous to the result on finite graphs where the sum would be over major vertices.

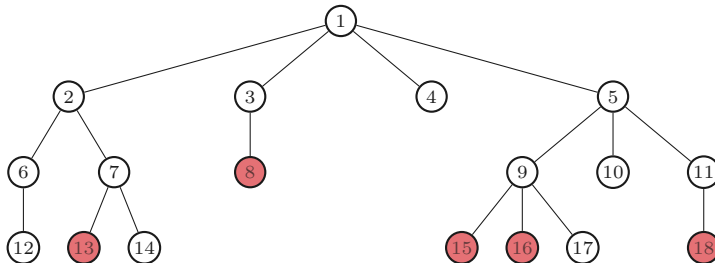


Fig. 5 A tree of order 18. Vertices 4, 8, 10, 12, 13, 14, 15, 16, 17, and 18 are leaves. The vertices 1, 2, 5, 7, and 9 are exterior major vertices with terminal degree 2, 1, 2, 2, and 3, respectively. $R = \{8, 13, 15, 16, 17, 18\}$ is a resolving set of minimum cardinality.

Let $G = (V, E)$ be a *unicyclic graph* (i.e., a graph that can be expressed as a tree with a single additional edge) with $|V| \geq 3$. Let T be any spanning tree of G and e the only edge in G that is not in T . Then $\beta(T) - 2 \leq \beta(G) \leq \beta(T) + 1$ [42, 70, 171].³ As illustrated in Figure 6, there are unicyclic graphs achieving each of the values in the integer interval $\{\beta(T) - 2, \dots, \beta(T) + 1\}$.

To justify the lower bound on $\beta(G)$ for unicyclic G , we consider three cases. First, if e is incident on leaves in T , then $\ell(G) = \ell(T) - 2$, and $ex(G) \leq ex(T)$ because G and T have the same major vertices but e reduces the terminal degree of at least one major vertex in T . In particular, $\beta(G) \geq \ell(G) - ex(G) \geq (\ell(T) - 2) - ex(T) = \beta(T) - 2$. Instead, if e is incident on exactly one leaf in T , then $\ell(G) = \ell(T) - 1$, and $ex(G) \leq ex(T) + 1$ because e may turn a vertex in T into an exterior major

³We note that [70] includes a complete proof of this result, correcting the outline in [42] and adding detail not present in [171].

vertex. So, $\beta(G) \geq \ell(G) - ex(G) \geq (\ell(T) - 1) - (ex(T) + 1) = \beta(T) - 2$. Finally, if e is not incident on any leaf in T , $ex(G) \leq ex(T) + 2$ because both vertices that e is incident on may become exterior major vertices. Hence, $\beta(G) \geq \ell(G) - ex(G) \geq \ell(T) - (ex(T) + 2) = \beta(T) - 2$.

Slightly more work is required to verify the upper bound on the metric dimension of unicyclic graphs. One approach is to focus on a subset of major vertices on the cycle in G . Let W contain every leaf in G , except one, associated with each exterior major vertex, and let m be the number of major vertices on the cycle in G with a branch to an element of W . Proceeding by cases, it can be shown that $\beta(G) \leq \beta(T) + 1$ whether $m \geq 3$, $m = 2$, $m = 1$, or $m = 0$ [42, 70].

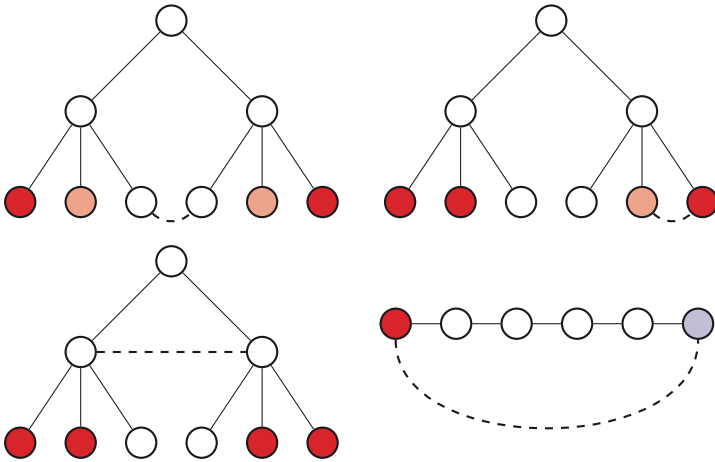


Fig. 6 Example of trees T such that, when a single edge (dashed) is added to form the unicyclic graph G , $\beta(G) = \beta(T) - 2$ (top left), $\beta(G) = \beta(T) - 1$ (top right), $\beta(G) = \beta(T)$ (bottom left), or $\beta(G) = \beta(T) + 1$ (bottom right) [42]. In each example, red nodes belong to a resolving set for both G and T , light red nodes are used to resolve T , and light blue nodes are used to resolve G .

5.2. Hamming Graphs. There are many ways to measure distance between pairs of strings. One of the simplest and most common is the Hamming distance [98]. Comparing two strings of the same length, the Hamming distance counts the number of positions in which the strings disagree. This distance can be used to define a graph.

DEFINITION 5.4 (Hamming graph). Let $V = A^k$ be the set of all strings of length k from the alphabet A of size $a = |A|$, and let $d(u, v)$ be the Hamming distance between $u, v \in V$. The Hamming graph $H_{k,a}$ contains a vertex associated with each $v \in V$ and the edge $\{u, v\} \in E$ only when $d(u, v) = 1$.

Despite the highly symmetric nature of Hamming graphs, an efficient algorithm to compute their metric dimension is not known. However, some upper bounds on $\beta(H_{k,a})$ have been established. See section 7.1 for additional results on the metric dimension of Hamming graphs motivated by game theory.

For the special case when $a = 2$, $Q_k := H_{k,2}$ is called a hypercube. The hypercube turns out to be an important structure when studying a particular coin weighing problem: given a set of k coins of two different weights, how many weighings are required to determine which coins are lighter [196]?

An asymptotic result tells us that $\lim_{k \rightarrow \infty} \beta(Q_k) \frac{\log_2(k)}{k} = 2$, suggesting a method

using $2^n - 1$ weighings to find the answer when $k = n2^{n-1}$ [67, 152]. This asymptotic behavior generalizes to arbitrary Hamming graphs as $\lim_{k \rightarrow \infty} \beta(H_{k,a}) \frac{\log_a(k)}{k} = 2$ [125].

Bounds related to the metric dimension of Cartesian products of connected graphs (see section 5.3 for the definition) have proven useful in studying Hamming graphs. In particular, $\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1$ [42] and, since $H_{k,a} = K_a^{\square k}$ is the Cartesian product of k copies of the complete graph K_a , it follows that $\beta(Q_k) \leq \beta(Q_{k+1}) \leq \beta(Q_k) + 1$. This approach yields the identity $\beta(H_{2,a}) = \lfloor \frac{2}{3}(2a - 1) \rfloor$ [38]. Table 5 shows exact values of $\beta(Q_k)$ verified via brute force for $1 \leq k \leq 10$, and upper bounds based on a variable neighborhood search for $11 \leq k \leq 17$ [162].

Table 5 Exact values of $\beta(Q_k)$ for $1 \leq k \leq 10$ and upper bounds for $11 \leq k \leq 17$ [162].

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\beta(Q_k)$	1	2	3	4	4	5	6	6	7	7	8	8	8	9	9	10	10

The bounds for the metric dimension of hypercubes have been generalized for arbitrary Hamming graphs as $\beta(H_{k,a}) \leq \beta(H_{k+1,a}) \leq \beta(H_{k,a}) + \lfloor \frac{a}{2} \rfloor$ [203]. The proof is constructive, allowing for the generation of R_{k+1} , a resolving set for $H_{k+1,a}$, from R_k , any resolving set for $H_{k,a}$.

Edge metric dimension, where the edges of a graph are distinguished based on distances to vertices (see section 8.7), and mixed metric dimension, where both vertices and edges are distinguished, have also been studied in the context of hypercubes. In particular, $\beta(Q_k)$ and the mixed metric dimension of Q_k are equivalent for $k \geq 3$. Furthermore, edge metric dimension and metric dimension are equivalent when k is odd and differ by at most 1 when k is even [132].

When dealing with very large Hamming graphs, verifying that a given subset of vertices is resolving becomes intractable via the simple brute force approach. However, the recursive structure and highly symmetric nature of Hamming graphs allows for far more efficient resolvability checks in practice. Indeed, by describing resolvability on Hamming graphs as a linear system, integer programming techniques can be used to verify quickly that a given set of vertices is resolving with high probability. A somewhat slower but deterministic solution can be implemented using Gröbner bases. These techniques have been used to discover a resolving set of cardinality 77 for the Hamming graph $H_{8,20}$, showing that $\beta(H_{8,20}) \leq 77$ [147]. With 25.6 billion vertices, traditional methods for finding small resolving sets are not computationally feasible in this setting.

There are other ways to measure distance between pairs of strings besides the Hamming distance. The *edit distance*, also called *Levenshtein distance* [150], is defined as the minimum number of character substitutions, deletions, or insertions needed to transform one string into another. Several notions analogous to Hamming graphs have been considered in the literature based on the edit distance [170, 182, 199, 208, 219]. In [179, 180], the *Levenshtein graph* $L_{k_1,k_2;a}$ is defined as having as vertex set all strings of length between k_1 and k_2 (inclusive), and two vertices are declared neighbors if and only if their edit distance is 1. It follows that $L_{k_1,k_2;a}$ is connected, and its geodesic distance equals the edit distance if and only if $k_1 = k_2 \leq 2$ or $k_1 < k_2$. Otherwise, if $k_1 = k_2 > 2$, then the corresponding graph is isomorphic to a Hamming graph. Using novel formulae for the edit distance between an arbitrary string and a single-

or double-run string, the authors in [180] characterized the automorphism group of $L_{k_1, k_2; a}$ and identified a resolving set of cardinality $O(ak_2(k_2 - k_1 + 1))$; in particular, $\beta(L_{k_1, k_2; a}) = O(ak_2(k_2 - k_1 + 1))$. Narrowing down on the metric dimension of Levenshtein graphs remains a challenge, particularly due to the comparatively small automorphism group of these graphs and difficulties arising from having to calculate the edit distance between arbitrary strings via dynamic programming.

5.3. Product Graphs. The *corona product* of two graphs G and H , denoted $G \odot H$, is the graph obtained by making one copy of G and $|G|$ copies of H and adding all possible edges between the i th vertex of G and the i th copy of H . In [119], it is shown that $\beta(C_n \odot K_1) = 3$ for n even and $\beta(C_n \odot K_1) = 2$ for n odd. Also, if $m \geq 2$, then $\beta(G \odot \bar{K}_m) = |G| \cdot (m - 1)$. In [144], which corrects some of the results in [119], it is shown that for $n \geq 3$ and $m \geq 2$, $\beta((P_n \times P_m) \odot K_1) = 3$, and for $m \geq 2$, $\beta((K_n \times P_m) \odot K_1) = 3$ for $n = 3$, but $\beta((K_n \times P_m) \odot K_1) = (n - 1)$ for $n \geq 4$.

One can also consider $G \odot \mathcal{H}$, where $\mathcal{H} = \{H_1, \dots, H_n\}$ is a family of not necessarily isomorphic graphs. Here, vertex i in G is adjacent to all vertices of H_i . In this case, if G is connected with at least two vertices and the graphs H_i are nontrivial, $\beta(G \odot \mathcal{H}) = \sum_{H_i \in \mathcal{H}_1} \beta(K_1 + H_i) + \sum_{H_i \in \mathcal{H}_2} (\beta(K_1 + H_i) - 1)$, where $H_i \in \mathcal{H}_1$ if the vertex in K_1 is not an element of any minimal resolving set of $K_1 + H_i$, and $H_i \in \mathcal{H}_2$ otherwise [145]. In the case that all $H_i \in \mathcal{H}$ are isomorphic to each other, this means that $\beta(G \odot \mathcal{H}) = n \cdot \beta(K_1 + H_1)$ if the vertex in K_1 does not belong to a minimal resolving set of $K_1 + H_1$, and $\beta(G \odot \mathcal{H}) = n \cdot (\beta(K_1 + H_1) - 1)$ otherwise [145].

The graph $G \odot^k H$ is defined recursively as $G \odot^k H = (G \odot^{k-1} H) \odot H$. The metric dimension of such structures has been studied with respect to different constraints on G and H . In particular, letting n_1 and n_2 be the order of two connected graphs G and H , respectively, $\beta(G \odot^k H) \geq n_1(n_2 + 1)^{k-1} \beta(H)$, with equality when the diameter of H is at most 2. In addition, if the diameter of H is at least 6 or H is a cycle, $\beta(G \odot^k H) = n_1(n_2 + 1)^{k-1} \beta(K_1 \odot H)$ [216].

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$. When H has a specified root vertex v , the *rooted product* $G \circ_v H$ includes $|V_1|$ isomorphic copies of H with edges between the roots such that these nodes induce a subgraph isomorphic to G . When v is not an element of any minimal resolving set and G is connected, $\beta(G \circ_v H) = |V_1| \cdot \beta(H)$. If v is an element of some minimal resolving set, H is not a path, and G is connected, then $\beta(G \circ_v H) = n \cdot (\beta(H) - 1)$ instead [215]. This result is directly connected to the fact that $G \odot H \simeq G \circ_v (K_1 + H)$. For the specific case $T \circ_v P_n$, where T is a tree with at least three vertices and v is an endpoint of P_n , $\beta(T \circ_v P_n) = \ell(T)$ follows from the general formula for the metric dimension of trees [215]. The rooted product of Harary graphs $H_{m,n}$, a family of m -connected graphs on n vertices with the fewest possible edges, with certain cycles and paths, has also been studied. Specifically, $\beta(H_{m,n} \circ_v C_k) = n$ for $k = 3, 4, 5$ and any v on the cycle. Also, for $n \geq 5$, $\beta(H_{4,n} \circ_v P_k) = 3$ when $n \bmod 4 \in \{0, 2, 3\}$ and $\beta(H_{4,n} \circ_v P_k) \leq 4$ when $n \bmod 4 = 1$ [117].

Results related to corona and rooted products have also been studied from the perspective of cut vertices. Let $G_1, \dots, G_k$ be a family of nontrivial, connected, and pairwise disjoint graphs. Consider a graph G constructed from this family by iteratively identifying or merging pairs of vertices in separate graphs G_i . These merged vertices are called cut vertices in G . The *attaching metric dimension* of G_i , denoted $\beta^*(G_i)$, is then defined as the cardinality of the smallest subset of vertices R in G_i such that $R \cup A(G_i)$ is a resolving set of G_i , where $A(G_i)$ is the set of vertices in G_i that were merged during the construction of G . For any graph G that can be realized

via this kind of construction, $\beta(G) \geq \sum_{i=1}^k \beta^*(G_i)$ [145]. Graphs that can be viewed as corona or rooted products fall into this category.

The *Cartesian product* of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, denoted $G \square H$, has vertex set $V_1 \times V_2$, and two nodes (u, v) and (s, t) are connected by an edge when $u = s$ and $\{v, t\} \in E_2$, or when $\{u, s\} \in E_1$ and $v = t$. Assume that G is connected. We have already mentioned in section 5.2 that $\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1$ [42]. More generally, for $n \geq 3$, $\beta(G \square K_n) \leq \beta(G) + n - 2$ [37]. On the other hand, $\beta(G) \leq \beta(G \square P_n) \leq \beta(G) + 1$, and $\beta(G \square C_n) \leq \beta(G) + 1$ if n is odd, whereas $\beta(G \square C_n) \leq \beta(G) + 2$ if n is even [37]. Exact formulae are also known for Cartesian products of graphs in special families, including for infinite graphs [134, 104, 36]. For instance, $\beta(P_m \square P_n) = 2$, $\beta(C_m \square C_n) = 3$ if mn is odd, but $\beta(C_m \square C_n) = 4$ if mn is even. In addition, for $m \leq n$, $\beta(K_m \square K_n) = (n - 1)$ if $2(m - 1) < n$, but $\beta(K_m \square K_n) = \lfloor \frac{2(m+n-1)}{3} \rfloor$ otherwise. For $n \geq 3$, $\beta(P_m \square K_n) = (n - 1)$. As for Cartesian products involving infinite graphs, we have, for instance, that $\beta(P_\infty \square P_n) = 2$ and $\beta(P_{2\infty} \square P_n) = 3$ as long as $n \geq 2$. Here, P_∞ and $P_{2\infty}$ are the one-way and two-way infinite paths, respectively. Instead, for $n \geq 4$, $\beta(P_\infty \square K_n) = \beta(P_{2\infty} \square K_n) = (n - 1)$. For simple, finite, and connected graphs G and H , it is known, for example, that [37]

$$\max\{\beta(G), \beta(H)\} \leq \beta(G \square H) \leq \min\{\beta(G) + |V_2|, \beta(H) + |V_1|\} - 1.$$

We emphasize that all the cited results concern undirected graphs. A roadblock for analogous results about directed graphs is that the Cartesian product of two (strongly) connected digraphs is not necessarily (strongly) connected.

The *strong product* of G and H , $G \boxtimes H$, again has vertex set $V_1 \times V_2$. (u, s) and (v, t) are adjacent in $G \boxtimes H$ if and only if $u = v$ and $\{s, t\} \in E_2$, $s = t$ and $\{u, v\} \in E_1$, or $\{u, v\} \in E_1$ and $\{s, t\} \in E_2$. In general,

$$\beta(G \boxtimes H) \leq |V_1| \cdot \beta(H) + |V_2| \cdot \beta(G) - \beta(G) \cdot \beta(H),$$

with equality when G and H are complete graphs [178]. The metric dimension of strong products of several families of graphs has been studied. For paths in particular, $\beta(P_{n_1} \boxtimes P_{n_1}) = 3$ [178] and $\beta(P_{n_1} \boxtimes P_{n_2}) = \lceil \frac{n_1+n_2-2}{n_1-1} \rceil$ [3] when $2 \leq n_1 < n_2$.

The *direct product* of two graphs G and H , denoted $G \times H$, has vertex set $V_1 \times V_2$ and an edge between (u, s) and (v, t) exactly when $\{u, v\} \in E_1$ and $\{s, t\} \in E_2$. The metric dimension of the direct product of several pairs of graph families is known. For instance, there are exact results for complete graphs of particular orders and for the direct product of isomorphic odd cycles. In particular, for $k \geq 1$, $\beta(C_{2k+1} \times C_{2k+1}) = 3$. In addition, $\beta(P_r \times K_t) = \lceil r/3 \rceil(t - 1)$ for $r \geq 4$ and $t \geq 3$. For the direct product of a cycle and a complete graph, $\lceil r(t - 1)/2 \rceil \leq \beta(C_r \times K_t) \leq \lceil r/3 \rceil(t - 1)$ for $r \geq 7$ and $t \geq 4$. When $r \pmod{3} = 0$, $\beta(C_r \times K_t) \leq r(t - 1)/3$ [143]. The strong metric dimension (see section 8.2) of direct products has also been studied [143].

The *lexicographic product* of G and H , denoted $G \cdot H$, has vertex set $V_1 \times V_2$. Two vertices (u, s) and (v, t) are adjacent when either $\{u, v\} \in E_1$ or $u = v$ and $\{s, t\} \in E_2$. It can be shown that if $|V_1|, |V_2| \geq 2$ and H has $k \geq 1$ components $H_1, \dots, H_k$,

$$n \cdot \left((-1) + \sum_{p=1}^k \beta(H_p) \right) \leq \beta(G \cdot H) \leq n \cdot \left((k - 1) + \sum_{p=1}^k \beta(H_p + K_1) \right) + (n - 2),$$

where $n = |V_1|$. Furthermore,

$$n \cdot \beta(H) \leq \beta(G \cdot H) \leq n \cdot \beta(H + K_1) + (n - 2),$$

and both sets of bounds are tight [184]. Adjacency metric dimension, i.e., the metric dimension when only neighbors of a node can be used to distinguish it from other nodes (see section 8.4), has also been used as a tool to study the metric dimension of lexicographic products of graphs [122].

5.4. Connections to Abstract Algebra. The *Cayley graph* associated with a group $(\Gamma, *)$ and subset $X \subset \Gamma$, which usually does not contain the identity, is a directed graph that has Γ as its vertex set, and an edge from a vertex u to another v if and only if there is $x \in X$ such that $u * x = v$. This graph is denoted as $\text{Cay}(\Gamma, X)$, and X is called its *generator*. Cayley graphs are strongly connected if and only if Γ is generated by X (i.e., every element in Γ may be represented as the product of elements in X).

For integers $n > 0$, let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group of order n . This Abelian group may be presented as the n -roots of unity. An extensive study of the metric dimension of Cayley digraphs associated with finite Abelian groups was accomplished in [74, 167]. For integers $k \geq 2$ and $0 < n_1 \leq n_2 \leq \dots \leq n_k$, if $e_1, \dots, e_k$ denotes the canonical basis of $\mathbb{R}^k$ and $G = (\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_k}; \{e_1, \dots, e_k\})$, then $n_{k-1} \leq \beta(G) \leq n_{k-1} + \sum_{i=1}^{k-2} (n_i - 1)$ [74]; in particular,

$$\beta(\text{Cay}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}, \{(1, 0), (0, 1)\})) = n_1.$$

The previous bounds are sharp though various improvements are possible. For instance, if $2 \leq m \leq n \leq k$, then $\beta(\text{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \mathbb{Z}_k, \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})) = n$ and $\beta(\text{Cay}(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_k, \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})) = 3$ [167].

A pioneering work on Cayley graphs associated with non-Abelian groups was initiated in [74] and extended significantly in [1]. If Γ is a group generated by X , with identity element e_Γ , then for any integer $m \geq 1$ [74],

$$\beta(\text{Cay}(\Gamma, X)) \leq \beta(\text{Cay}(\Gamma \oplus \mathbb{Z}_m, X \times \{0\} \cup \{(e_\Gamma, 1)\})) \leq \beta(\text{Cay}(\Gamma, X)) + m - 1.$$

The *dihedral group* D_{2n} of order $2n$ (i.e., group of symmetries of a regular polygon with n sides) admits the presentation $D_{2n} = \{r, s \text{ such that } r^n = 1, s^2 = 1, srs = r^{-1}\}$. Here, r and s are elements associated with the rotational and reflection symmetries of an n -gon. It turns out that $\beta(\text{Cay}(D_{2n}, \{r, s\})) = n$ and a minimal resolving set of $\text{Cay}(D_{2n}, \{r, s\})$ is $\{r^0, \dots, r^{n-1}\}$ [74]. More generally, the metric dimension and minimal resolving set for Cayley graphs associated with the (split metacyclic) group $\Gamma = \{a, b \text{ such that } a^n = b^{2s} = 1 \text{ and } ba = a^{-1}b\}$, where $s \geq 1$ is an integer, and generator $X = \{a, b\}$, were determined in [1].

The metric dimension of the so-called power graph associated with a group has also been studied. The *power graph* $\mathcal{P}_\Gamma$ associated with a group $(\Gamma, *)$ has vertex set Γ and there is an edge between two different vertices x and y if and only if there is an integer $m > 0$ such that $x = y^m$ or $y = x^m$. A general formula for the metric dimension of the power graph of a finite group was determined in [75]. As a corollary, the metric dimension of the power graph associated with the cyclic group $\mathbb{Z}_n$, where $n = \prod_{i=1}^t p_i^{r_i}$ with $p_1 < \dots < p_t$ primes and $r_1, \dots, r_t > 0$ integers, is given by [75]

$$\beta(\mathcal{P}_{\mathbb{Z}_n}) = \begin{cases} n-1, & t=1, \text{ i.e., } n \text{ is a prime;} \\ n-2r_2 & \text{if } (t, p_1, r_1) = (2, 2, 1); \\ n-2r_1 & \text{if } (t, p_1, r_2) = (2, 2, 1); \\ n+1 - \prod_{i=1}^t (r_i + 1) & \text{otherwise.} \end{cases}$$

Let $(R, +, \cdot)$ be a finite commutative ring. The *total graph* $\tau(R)$ associated with any such ring has vertex set R , and two different vertices $u, v \in R$ are connected by an edge if and only if $(u + v)$ is a divisor of 0 (i.e., there is $w \in R \setminus \{0\}$ such that $(u + v) \cdot w = 0$). Let $J(R)$ be the intersection of all the maximal ideals of R . If $J(R)$ is nontrivial, then $\beta(\tau(R)) = |J(R) - 1| \cdot |R/J(R)|$ [61]. Instead, if $J(R)$ is trivial, R is a direct product of fields and the metric dimension of $\tau(R)$ can be determined from the characteristic (i.e., smallest multiple of 1 that is a divisor of 0) of each field. For instance, if $R = F_1 \times F_2$ with F_1 and F_2 fields satisfying $\max\{2|F_1| - 1, |F_1|\} \leq |F_2|$, and either $\text{char}(F_2) \neq 2$ or $\text{char}(F_1) = \text{char}(F_2) = 2$, then $\beta(\tau(R)) = |F_2| - 1$ [61].

5.5. Strongly Regular and Distance Regular Graphs. A graph $G = (V, E)$ is *regular* when all vertices have the same degree. G is *strongly regular* if it is regular, every pair of adjacent vertices has the same number of common neighbors, and every pair of nonadjacent vertices has the same number of common neighbors. Strongly regular graphs have diameter 2 and are closed under complementation. They are called *trivial* when their connected components are complete graphs of the same dimension.

Interestingly, although the number of nodes in a strongly regular graph is uniquely determined by its three parameters, that is not necessarily the case for their metric dimension. Using an integer linear programming formulation [57] and the CPLEX optimizer [111], the metric dimension of all nontrivial strongly regular graphs with up to 45 vertices, a total of 43,759 unique structures, has been determined [139].

A graph G is said to be *distance regular* when it is regular and, for $u, v \in V$, the number of vertices at distance i from u and at distance j from v depends only on i, j , and $d(u, v)$. The metric dimension of all such graphs on at most 34 vertices and with vertex degree up to 13 (with the exception of three structures) has been computed. The metric dimensions of rank-3 strongly regular graphs (graphs that are both strongly regular and distance transitive) with up to 100 vertices and Hadamard graphs with up to 20 vertices have been computed as well [14].

Let $G = (V, E)$ be a distance regular graph with diameter δ and let $G_d = (V, E_d)$, where $\{u, v\} \in E_d$ if and only if $d(u, v) = d$. G is *primitive* if G_d is connected for all $0 < d \leq \delta$ and *imprimitive* otherwise. When G_δ is disconnected, G is *antipodal*. Distance regular graphs can be grouped further into 13 classes based on several characteristics including whether or not the graph is primitive, antipodal, or bipartite, and the graph's diameter [5, 15]. The metric dimension of graphs in many of these classes has been studied. For instance, primitive graphs with diameter at least 2 and vertex degree at least 3 have metric dimension $\beta(G) < 4\sqrt{n} \log(n)$ [8, 9]. Using similar techniques, the metric dimension of graphs in several other classes can be shown to have an upper bound of $O(\sqrt{n} \log(n))$ [16]. Studying halved and folded graphs allows for some insight into the metric dimension of graphs in other classes [15].

It is worth noting that studying metric dimension more directly with tools and techniques from group theory has proven quite fruitful. Indeed, beyond those mentioned previously in this section, there are connections between metric dimension and base size, permutation groups, coherent configurations, and association schemes. A thorough review of these connections is given in [18].

5.6. Other Graph Families. Metric dimension has been studied on a wide variety of graphs with diverse characteristics. Here we briefly mention results for a handful of graph families beyond those discussed previously. In section 2.3 we explored the metric dimension of wheels. A number of graph structures related to wheels have also been examined. In particular, $\beta(C_n + \overline{K}_m) = \beta(W_n) + m - 1$ when $n \neq 3$ or 6, and $\beta(C_n + \overline{K}_m) = m + 1$ otherwise. Here $C_n + \overline{K}_m$ is isomorphic to the *generalized*

wheel with m central vertices adjacent to all n vertices of a cycle [197]. The m -level wheel $W_{n,m}$ consisting of m cycles C_n sharing a common central vertex has metric dimension $\beta(W_{n,m}) = \beta(W_{n,1}) + (m - 1)\lfloor \frac{2n+4}{5} \rfloor$ for $n \geq 7$ and $m \geq 3$ [192].

The metric dimension of some convex polytopes related to wheels has also been studied. Indeed, several families of convex polytopes with constant [112, 115] and unbounded [116, 192] metric dimension exist.

While the problem of determining the metric dimension of planar graphs is NP-hard, its value can be found for arbitrary outerplanar graphs in $O(n^{12})$ time [60]. When attention is focused on maximal outerplanar graphs G where the addition of any edge produces a nonouterplanar graph, it can be shown that $2 \leq \beta(G) \leq \lceil \frac{2n}{5} \rceil$. Such graphs with $\beta(G) = 2$ have been fully characterized by considering an embedding of G in $P_n \boxtimes P_n$, the strong product of two paths. Furthermore, a linear time algorithm exists for finding resolving sets of cardinality $\lceil \frac{2n}{5} \rceil$ in any maximal outerplanar graph [51].

In the context of infinite graphs where all vertices have finite degree, if $\beta(G) = k$, then the maximum degree of any vertex is $3^k - 1$. Moreover, if G is an infinite graph such that $\beta(G)$ is finite, then $\deg(v) \leq M$ for all $v \in V$ and some positive integer M . The metric dimension of Cartesian products of P_∞ and $P_{2\infty}$, the two-way infinite path, with finite paths, cycles, and complete graphs, is also known [36]. Other lines of research have focused on characterizing infinite families of graphs with finite metric dimension; see [4, 192, 205].

In section 2.2, we examined two-dimensional grids, $G_{m,n}$, and argued that $\beta(G_{m,n}) = 2$. Naturally, these grids correspond to a square tiling of $\mathbb{R}^2$. Similarly, following the terminology and notation of [154], the *honeycomb networks* $HC(n)$ and the *hexagon networks* $HX(N)$ correspond to partial hexagonal and equilateral triangle tilings of $\mathbb{R}^2$, respectively.

By identifying a useful coordinate system for dealing with distances in hexagonal graphs, one can show that any three vertices of degree 3 forming adjacent corners of the underlying hexagon in $HX(n)$ serve as a resolving set. Since $HX(n)$ violates properties that any graph with metric dimension 2 must have [134], we conclude that $\beta(HX(n)) = 3$ for $n > 1$ [154]. Noting that $HC(n)$ is the so-called bounded dual of $HX(n + 1)$ and taking advantage of this relationship, it can be shown that $\beta(HC(n)) = 3$ for $n > 1$ as well [154] (see Figure 7).

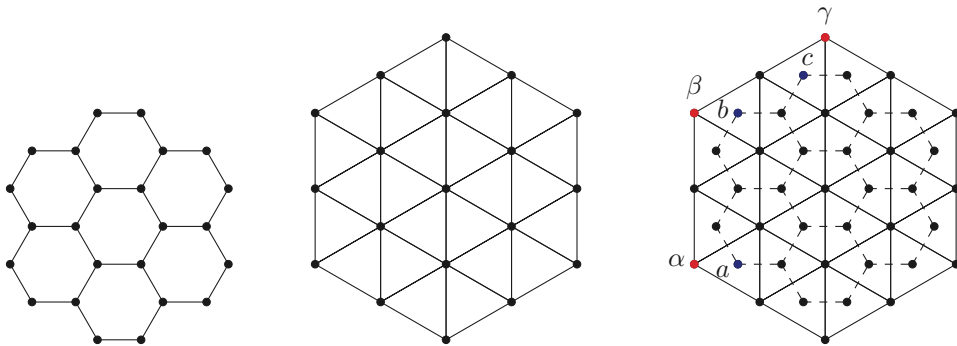


Fig. 7 Visualization of the honeycomb network $HC(2)$ (left), the hexagon network $HX(3)$ (center), and $HC(2)$ (dashed) as the bounded dual of $HX(3)$ (solid) with minimum resolving sets $\{a, b, c\}$ and $\{\alpha, \beta, \gamma\}$, respectively, following the construction in [154] (right).

Recall that *projective planes* are *incidence* systems of points and lines satisfying certain axioms. In a projective plane of order q , which must be of the form p^r , with p a prime and $r \geq 1$ an integer, any line contains $(q+1)$ points, and $(q+1)$ lines pass through each point. The metric dimension of the incidence graph of projective planes of order $q \geq 2$ has been completely characterized [14, 102, 103]. For $q = 2, 3, 4, 5, 7$, this is 5, 8, 10, 15, 23, respectively. For $q \geq 8$, the metric dimension is $4q - 4$. The metric dimension of the incidence graph of any affine plane of order $q \geq 2$ has also been determined [20, 102]. For $q = 2$, this is 3, whereas for $q > 2$, this is $3q - 4$.

Given two (disjoint) copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of a graph $G = (V, E)$, and a bijection $\sigma : V \rightarrow V$, the *permutation graph* G_σ has vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{\{u, v\} \text{ such that } \sigma(u) = v \text{ or } \sigma(v) = u\}$. If G is connected of order at least 2, then, for all permutations σ of V , $2 \leq \beta(G_\sigma) \leq n - 1$ [97]. These bounds are sharp: the lower bound is achieved by any connected graph G of order 2, whereas the upper bound is achieved by any permutation graph of K_n when $n \geq 3$.

6. Random Graph Models. Real world networks rarely fully conform to the requirements for structurally deterministic graph families. Random graph models, which define distributions over graph structures, often in terms of some generative process, are therefore useful in modeling real networks. Understanding the behavior of metric dimension as a random variable with respect to these models allows for the general study of metric dimension, resolving sets, and efficient means for finding small resolving sets in some situations. Though there has not been as much work on metric dimension in this context as compared to deterministic graphs, there have been several significant contributions in this area concerning Erdős–Rényi random graphs [29] and random trees and forests [161].

6.1. Erdős–Rényi Random Graphs. For each $0 \leq p \leq 1$, let $G_{n,p}$ denote a (simple) random graph with n vertices obtained by including each of the possible $\binom{n}{2}$ edges with probability p , independently of all other edges.

As $n \rightarrow \infty$, the set of $\lceil (3 \log n) / \log 2 \rceil$ highest degree vertices in $G_{n,1/2}$ suffices as a resolving set with high probability [11]. This upper bound on $\beta(G_{n,1/2})$ was originally used as part of a simple heuristic algorithm for canonically labeling graphs and determining whether or not two graphs are isomorphic [11]—see section 7.3. More recently, focusing solely on adjacency information in $G_{n,p}$, this highly likely bound has been generalized for arbitrary values of p to $\beta(G_{n,p}) \leq \frac{-3 \ln(n)}{\ln(p^2 + (1-p)^2)}$ [201, 204]. The proof of this generalization does not rely on choosing a resolving set based on any particular property. In fact, any subset of nodes in $G_{n,p}$ of cardinality at least $\frac{-3 \ln(n)}{\ln(p^2 + (1-p)^2)}$ is a resolving set with high probability for large n .

Detailed insight into the metric dimension of Erdős–Rényi graphs can be gained from the following result, where p is a function of n .

THEOREM 6.1 (adapted from [29]). *Let $d = (n - 1)p$ be the expected degree of $G_{n,p}$. Suppose that*

$$\log^5 n \ll d \leq n \left(1 - \frac{3 \log(\log n)}{\log n} \right).$$

Let $i \geq 0$ be the largest integer such that $d^i = o(n)$ and let $c = c(n) = e^{d^{i+1}/n}$. If β_n denotes the metric dimension of $G_{n,p}$, then the following holds asymptotically almost

surely:

$$\beta_n = \begin{cases} \Theta(\log n) & \text{if } c = \Theta(1), \\ \Theta(c \log n) & \text{if } c^{-1} = \Omega(d^i/n), \\ \Theta(\frac{n \log n}{d^i}) & \text{if } c^{-1} \ll d^i/n. \end{cases}$$

The regimes of p described in this theorem produce a zig-zag pattern in $\beta(G_{n,p})$ as a function of p . Indeed, it can be shown that $\log_n(\beta(G_{n,n^x-1}))$ for $0 < x < 1$ approaches the function $f(x) = 1 - x \lfloor 1/x \rfloor$ as $n \rightarrow \infty$ with high probability [29].

This zig-zag pattern can be anticipated by reasoning as follows. In a dense graph (i.e., with many edges and low path length entropy), consider picking a single vertex v to add to a growing resolving set. This vertex defines an equivalence relation on the graph: two vertices are equivalent if they are the same distance away from v . In terms of distances and cardinalities, the equivalence classes are nearly the same—regardless of the vertex chosen. The ratio between the sizes of the two largest equivalence classes has great influence on the overall metric dimension. When this ratio is close to 1, picking a new vertex to add to the growing resolving set from the largest class will, on average, lead to more new equivalence classes than when the largest equivalence class contains many more vertices than the second largest. So, the overall metric dimension might be smaller the closer the two largest equivalence classes are in terms of size [29].

The zig-zag pattern observed in $\beta(G_{n,p})$ comes from how this ratio evolves in $G_{n,p}$ with decreasing p . Let $D_v(i)$ denote the set of vertices at a distance i from a chosen vertex v . When $p = 1$, the graph is complete and $|D_v(0)| = 1$ and $|D_v(1)| = n - 1$ for all vertices v , so that $\beta(G_{n,1}) = n - 1$. As p decreases, $|D_v(i)|$ for $i > 1$ increases. At first, this increase is faster for smaller values of i . Eventually, $|D_v(1)| \approx |D_v(2)|$ and the metric dimension is small. Decreasing p further, the sizes of the two largest distance sets move away from one another and the ratio between their sizes increases. This pattern then repeats itself as the identities of the largest sets change [29].

Given two real-valued random variables, X and Y , define the following measure of similarity between their distributions:

$$d(X, Y) := \sup_h \frac{E(h(X)) - E(h(Y))}{\sup_x |h(x)| + \sup_x |h'(x)|},$$

where the supremum is taken over all bounded (test) functions $h : \mathbb{R} \rightarrow \mathbb{R}$ with bounded derivative, and $E(\cdot)$ is used to denote expectation. The behavior of $\beta(G_{n,p})$ when p is comparatively small is described by the following result (the previous result addressed the case when p is not as small).

THEOREM 6.2 ([161]). *Let β_n denote the metric dimension of $G_{n,p}$.*

- (i) *For $p = o(n^{-1})$, $\beta_n = n(1 + o(1))$ asymptotically almost surely.*
- (ii) *For $p = c/n$ with $0 < c < 1$, the sequence of random variables*

$$X_n = \frac{\beta_n - E(\beta_n)}{\sqrt{\text{Var}(\beta_n)}}$$

converges in distribution to a standard normal Z as $n \rightarrow \infty$, at a rate $d(X_n, Z) = O(n^{-1/2})$. Moreover, $E(\beta_n) = Cn(1 + o(1))$ and $\text{Var}(\beta_n) = \Theta(n)$, where C is an explicit constant that depends on c only.

6.2. Stochastic Block Model. The *stochastic block model (SBM)* is a generative graph model used frequently to study networks with simple community structure. In

its most basic form, the SBM has two main parameters, C and P . C is a partition of n vertices into $c \geq 1$ disjoint communities $C_1, \dots, C_c$, and P is a $(c \times c)$ symmetric matrix of adjacency probabilities. The communities are also sometimes defined stochastically using a probability vector of dimension c .

We say $G \sim SBM(n; C, P)$ when, for $u \in C_i$ and $v \in C_j$ with $u \neq v$, $\{u, v\} \in E$ with probability $P_{i,j}$, the entry in row i and column j of P , independently of all other pairs of nodes. Considered separately, each individual community is equivalent to an Erdős–Rényi random graph. This suggests using bounds on $\beta(G_{n,p})$ to study the metric dimension of the SBM. Intercommunity adjacency probabilities, however, complicate the situation and must be dealt with carefully. For example, given $c = 2$ and

$$P = \begin{bmatrix} o(1) & \frac{1}{2} \\ \frac{1}{2} & o(1) \end{bmatrix},$$

the communities of the resulting graph will be sparse and individual vertices may be difficult to distinguish without the help of vertices from both communities. Unfortunately, characterizing the precise interaction between vertices from different communities is not trivial, especially given the complicated dependencies among shortest path distances in such graphs.

Nevertheless, it can be shown that the adjacency metric dimension of a graph (see section 8.4) serves as an upper bound on metric dimension [122]. In particular, since the entries in the adjacency matrix of the SBM are by definition independent, a probabilistic upper bound on the metric dimension of these kinds of graphs can be established [201, 204]. Indeed, letting $\mathbb{P}(G; R)$ denote the probability that there are nodes in $G \sim SBM(n; C, P)$ with the same neighbors in R , where $R \subseteq \{1, \dots, n\}$ contains k_i nodes in community i , the first-moment method implies that

$$\mathbb{P}(G; R) \leq \sum_{1 \leq i \leq j \leq c} |V_i| |V_j| \prod_{\ell=1}^c P_{i,\ell} P_{j,\ell} + (1 - P_{i,\ell})(1 - P_{j,\ell})^{k_\ell}.$$

This inequality serves as the basis of an effective, fast algorithm for selecting vertices in G such that $\sum_{1 \leq i \leq c} k_i$ is minimized and $\mathbb{P}(G; R)$ is less than a given threshold value. In essence, this algorithm provides an intelligent strategy for determining how the vertices of small resolving sets should be distributed across communities for any graph $G \sim SBM(n; C, P)$ with fixed parameters [201, 204].

6.3. Random Trees and Forests. The metric dimension of a disconnected graph is, by definition, the sum of metric dimension of its connected components. This is because the distance between any two vertices from different components is regarded as ∞ . Accordingly, it is not surprising that the metric dimension of F_n , a forest on n vertices chosen uniformly at random, has the same limiting distribution as that of T_n [161], a tree on n vertices also chosen uniformly at random. Furthermore, if $\beta_n = \beta(T_n)$, then, as $n \rightarrow \infty$, the random variables

$$X_n = \frac{\beta_n - E(\beta_n)}{\sqrt{\text{Var}(\beta_n)}}$$

converge in distribution to a standard normal, where $E(\beta_n) = \mu n(1 + o(1))$ and $\text{Var}(\beta_n) = \sigma^2 n(1 + o(1))$, with $\mu \simeq 0.14076941$ and $\sigma^2 \simeq 0.063748151$ (see Figure 8).

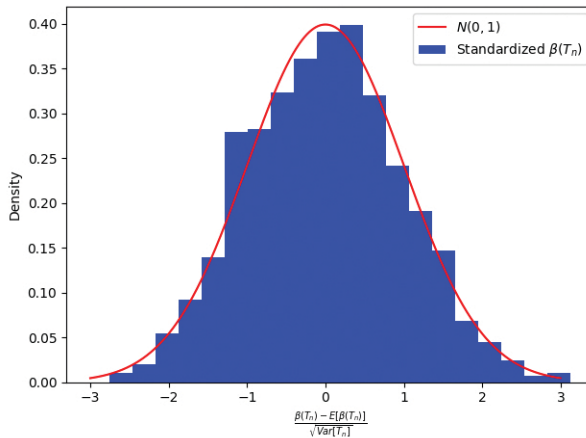


Fig. 8 Histogram of the metric dimensions of 1000 random trees on 2000 vertices (blue bars), along with the probability density function of a standard normal distribution (red curve). The Jarque–Bera test produces a p -value of 0.222; in particular, the data carries little to no evidence against the (null) hypothesis that it comes from a standard Gaussian.

7. Applications. Small resolving sets are useful in a variety of situations. The direct analogy between metric dimension and trilateration of the plane makes potential applications regarding navigation [134] and location detection [193] in discrete space immediately apparent. Resolving sets have also been used as a means of comparing graphs. The classification of chemical compounds based on general chemical structure can be accomplished using resolving sets [42, 126, 127]. One method for quickly determining whether many, though not all, pairs of graphs are isomorphic relies on comparing vertex representations based on presumed resolving sets [11]. In this section, we examine several applications of resolving sets including as a tool for studying certain games [50, 87, 120, 121, 128], as observers in detecting the source of a spread over a network [198], as a tool for detecting network motifs [110], and as the basis of a method for embedding DNA sequences in real space [203].

7.1. Game Theory. The *Mastermind game*, a game closely related to the coin weighing problem described in section 5.2, has been analyzed with the aid of Hamming graphs. Mastermind is played between two players, A and B . Player A starts by choosing $s = s_1 \dots s_k$, a sequence unknown to player B of k symbols from a reference alphabet of size c . Player B attempts to guess s as quickly possible. After each guess q , player A provides two values: $a(q, s)$, the number of positions where q and s agree, and $b(q, s)$, the total number of correct symbols at incorrect positions. Donald Knuth created an algorithm that solved the commercial version of the game with $k = 4$ and $c = 6$ with at most five questions [137]. For arbitrary values of k and c , a number of results and bounds exist [50, 87], though there is no known optimal solution.

A static variant of Mastermind, in which player B must make all guesses at once with no feedback and player A only provides $a(q, s)$, has been shown to be NP-complete [89]. Indeed, notice that $a(q, s) = k - d(q, s)$, where $d(q, s)$ is the Hamming distance between the two sequences. In particular, to guess s with the least number of questions, player B should make guesses that reveal the Hamming distance from s to a resolving set of $H_{k,c}$ of cardinality $\beta(H_{k,c})$. A deeper analysis of this situation yields an upper bound on $\beta(H_{k,c})$ when k is small in comparison to c . In particular,

if $\epsilon < 1$ and $c < k^{1-\epsilon}$, then $\beta(H_{k,c}) \leq (2 + \epsilon)k^{\frac{1+2\log_2(c)}{\log_2(k) - \log_2(c)}}$ [50]. Optimal strategies are known for $k = 2$ when $c \geq 3$, and for $k = 3$ when $c \geq 2$, showing that $\beta(H_{2,c}) = \lceil (4c - 1)/3 - 1 \rceil$ [120] and $\beta(H_{3,c}) = \lfloor 3c/2 \rfloor$ in these cases [121].

The Maker-Breaker domination game [62] is a hybrid between the Maker-Breaker game [68] and the domination game [33]. The analysis of a new version of the game called the *Maker-Breaker resolving game* (MBRG) is centered around metric dimension and resolving sets. Given a graph G , two opponents called Resolver and Spoiler alternate selecting vertices from G . The Resolver wins if they can eventually select a subset of nodes in G that resolves it; otherwise, the Spoiler wins. In [128], various general results relate the minimum number of moves to win the MBRG (assuming there is such a strategy) to the metric dimension of G , its order, the outcome of the game, and the optimal number of moves based on who starts the game. Specializations of these results are then given for nonpath trees, the Petersen graph, bouquets of cycles, complete multipartite graphs, and some grid-like graphs.

7.2. Source Localization. A variety of transmission processes, such as information and disease, occur on networks. News and rumors circulate over social networks like Facebook and Twitter, and businesses take advantage of “influencers” to maximize the effect of marketing campaigns [31, 131, 142, 207]. Physical interaction networks are often used as a tool for studying the spread of diseases across communities [163, 166, 168, 169]. In many contexts, it is valuable to locate the source of a transmission process, the node or set of nodes from which the spread began, for example, to better understand the process and to decrease/increase transmission speed. Existing approaches to solving this problem include dynamic message passing [153], time reversal [190], and maximum likelihood estimators [169, 188].

Small resolving sets can also provide an elegant solution to source localization. As a concrete example, suppose that a transmission process on the graph $G = (V, E)$ starts at an unknown source $s \in V$ at time t_0 and travels at unit speed across the edges. Suppose we can prespecify a set $R \subseteq V$ of observation nodes, such that we will observe the times t_r at which the process first reaches each node $r \in R$. A natural question is, therefore, for which sets R can we infer the source location s from the observation times $\{t_r\}_{r \in R}$ [169, 218]? The answer is exactly the sets R which are resolving for G . By definition of the process, we will have $t_r - t_0 = d(r, s)$, and thus the observation times uniquely identify s , for all possible sources s , exactly when R is a resolving set.

While resolving sets present a promising solution to source localization, there are two nontrivial difficulties that arise in real world transmission processes which require attention. First, it is very unlikely that the start time t_0 of the process will be known. Second, the speed at which information or diseases traverse a given edge, often called the edge length, is rarely deterministic or known precisely. Instead, edge lengths are often modeled as randomly drawn from some known distribution.

An unknown start time may be addressed by strengthening the notion of resolving sets to that of doubly resolving sets [38, 198]; see also section 8.1. We call $R \subseteq V$ *doubly resolving* when, for every pair of nodes $u, v \in V$, there is a pair $r, r' \in R$ such that $d(r, u) - d(r', u) \neq d(r, v) - d(r', v)$. If R is doubly resolving, it is also resolving. Furthermore, one easily calculates $d(r, v) - d(r', v) = t_r - t_0 - t_{r'} + t_0 = t_r - t_{r'}$ for all $v \in V$ and $r, r' \in R$. Thus, as doubly resolving sets are based on relative and not absolute distances, the source can once again be located from the set $\{t_r\}_{r \in R}$, even if the start time t_0 is unknown.

Random edge lengths are more difficult to address and solutions depend on the

details of the transmission process. When the edge length variance is low relative to its mean, observation times will be close to their expected value, and resolving sets allow for exact solutions with high probability. When the edge length variance is high, however, observation times will carry very little information about the expected distances, especially for nodes at greater distance from the source s . In this case, we can increase accuracy by adding nodes to R using a path covering strategy such as truncated metric dimension (see section 8.4). In this way, distances between observation nodes are small and thus noise accumulation is low, and we maintain resolvability of expected distances [198].

In some situations, knowledge of the network itself may be incomplete. This results in inaccurate shortest path information and potential misidentification of the source. In the restricted setting where a set of k distinct subgraphs is known and there are $k - 1$ unobserved edges connecting these subgraphs, extended resolving sets can be used to address this shortcoming. Let $\mathcal{H}(G)$ be the set of all possible connected graphs that can be formed by adding $k - 1$ edges to $G = (V, E)$, a graph with k separate components. R is an *extended resolving set* of G if, for all $u, v \in V$ and all $H_1, H_2 \in \mathcal{H}(G)$, there is $r \in R$ such that $d_{H_1}(u, r) \neq d_{H_2}(v, r)$, where $d_H(\cdot, \cdot)$ is the distance between two vertices in the graph H . When the components of G are all trees, complete graphs, grids, or cycles, minimal extended resolving sets have been characterized. More generally, extended metric dimension can be bounded by considering minimal resolving sets of each component along with their boundary vertices [217]. This definition provides an interesting solution to the problem of missing edges in the context of source localization and suggests a general framework for working with metric dimension when the structure of the underlying graph is uncertain.

7.3. Detecting Network Motifs. A common tool in network science to compare graphs is via *motifs*, i.e., subgraphs appearing with higher than expected frequency. Network motifs are believed to play important roles in the structure and underlying dynamics of networks in a variety of fields including social sciences [108, 109], biology [63, 90, 148, 191], chemistry [95, 214], and data mining more generally [52, 210, 212]. By finding and analyzing these motifs, researchers gain insight into the functional properties of different systems. The problem of discovering over- or under-represented subgraphs in a large network, however, poses significant computational challenges: the *subgraph isomorphism problem* (i.e., determining whether or not a given graph occurs as a subgraph in a larger graph) is NP-complete [53, 213]. On the other hand, *graph isomorphism*, the special case of subgraph isomorphism when the graphs are of the same order, is believed to be an easier problem [56, 185], especially given the recent quasi-polynomial time algorithm in [10]. A natural algorithm to count k -node motifs is therefore to enumerate subgraphs of order k and test whether they are isomorphic to the given motif. Resolving sets and metric dimension have been used as the foundation of tools to solve graph isomorphism [11] and, in the manner above, for motif detection [110].

One technique to solve graph isomorphism is through a *canonical labeling*, a way to assign unique labels to nodes which is invariant under graph isomorphism. Since resolving sets provide a unique label for each node in a graph, they can serve as the basis of such labelings. In particular, suppose that one could uniquely identify a resolving set R given a graph $G = (V, E)$ and, furthermore, uniquely identify an ordering $R = \{r_1, \dots, r_{|R|}\}$. Then assigning label $d(u|R) \in \mathbb{R}^{|R|}$ to each node $u \in V$ would give a canonical labeling, as both R and the ordering are uniquely determined

by G , and given R and this ordering, the distance vectors are unique since R is resolving.

Using this general approach, the following quadratic time canonical labeling algorithm provably solves graph isomorphism with high probability for Erdős–Rényi random graphs $G_{n,p}$, that is, graphs on n vertices such that each edge appears with independent probability p [11]. Consider a graph $G = (V, E)$ and let $n = |V|$ and $R = \{r_1, \dots, r_{|R|}\}$ be the set of the $\lceil (3 \log(n))/\log(2) \rceil$ highest degree vertices in V .⁴ The algorithm labels each vertex in G with the set of nodes in R which are adjacent (this step of the algorithm leverages the fact that the diameter of $G_{n,1/2}$ is 2 with high probability as n increases, so the set of adjacent R nodes is equivalent to the distance vector $d(\cdot|R)$). This labeling is canonical, i.e., invariant under isomorphism, as long as no pair of vertices in R has the same degree, and the labels are unique. Under $G_{n,1/2}$, the probability that this algorithm succeeds in finding a canonical labeling is at least $1 - \sqrt[3]{1/n}$ for large n [11].

Another recent algorithm uses the above technique more explicitly, by directly computing a canonical resolving set and ordering [110]. The approach is as follows. Enumerate all resolving sets R of cardinality $\beta(G)$ and permutations π and compute the corresponding adjacency matrix $A_{R,\pi}$ for each one. In $A_{R,\pi}$, the vertices of G are ordered lexicographically with respect to their distance vectors $d(\cdot|R)$ ordered by π . Now take R and π such that $A_{R,\pi}$ is lexicographically first (after flattening) among all such choices. Given this choice of R and π , labeling the vertices of G by their distance vector representations gives a canonical labeling.

This approach can be time consuming, as it requires computing (an upper bound for) $\beta(G)$ and enumerating all resolving sets R of this cardinality and all orderings on these sets. One can improve performance slightly by ignoring the relative ordering of twin vertices (see section 4.2) [110]. Tests on a wide variety of graphs show that this method is effective but somewhat slower than the graph isomorphism tool provided with the NAUTY program [157]. However, this method is faster than NAUTY on multidimensional mesh graphs and was the core algorithm in a tool for identifying and counting statistically significant subgraphs in the transcriptional regulation networks of *Saccharomyces cerevisiae* (yeast) and *Escherichia coli* (*E. coli*) [110].

7.4. Embedding Biological Sequence Data. High-throughput sequencing technologies have enabled biologists to collect a wealth of DNA, RNA, and amino acid sequence data. The abundance of this information makes computational analysis methods, including those based on machine learning algorithms, indispensable. The majority of these methods, however, cannot directly learn from symbolic data like biological sequences, and they deal instead with numerical feature vectors. Methods to embed symbolic sequences into real vector spaces are thus an important preprocessing step [203]. Low-dimensional embeddings are especially useful, both to reduce the computational cost of learning algorithms and to avoid overfitting.

Consider the task of embedding a sequence of length ℓ , composed of symbols from an alphabet of size a , into a real vector space. For example, DNA and RNA sequences have $a = 4$, while amino acids are composed of $a = 20$ possible symbols. A naïve approach to this embedding is the so-called *one-hot encoding*, which simply generates an indicator vector for each of the a^ℓ possible sequences of length ℓ . This approach is untenable for most biological sequence data, where ℓ can be quite large. One lower-

⁴In fact, any set of vertices of this cardinality will suffice as a resolving set with high probability, not just vertices of high degree. This fact is directly related to the fact that the degree distribution of $G_{n,p}$ is binomial and therefore concentrated around its mean [7, 55].

dimensional approach to embed such sequences uses *binary representations*, which indicate the presence or absence of each character in the alphabet at each position in the sequence, thus requiring $a \cdot \ell$ dimensions [39]. Another common approach uses *k-mer count vectors*, usually with $k \ll \ell$, which count the number of times that every possible contiguous subsequence of length k occurs in the larger sequence using a sliding window [149]. Unlike one-hot encodings and binary representations, *k-mer count vectors* are not guaranteed to produce an injective embedding.

Resolving sets can be applied to this problem as well. Often the domain of interest has some natural distance metric between sequences. For biological sequences and several other domains, one possible choice is Hamming distance, which simply counts the number of indices in which the two sequences differ. The Hamming distance induces the Hamming graph $H_{\ell,a}$ on length- ℓ sequences from a symbols, where there are edges between sequences which differ in only one entry, and hence the path length between two sequences is their Hamming distance (see section 5.2). Given a resolving set R on $H_{\ell,a}$, each sequence can be uniquely represented by its vector of distances to the elements of R . These distance vectors are therefore an injective $|R|$ -dimensional embedding, and the metric dimension of $H_{\ell,a}$ gives the smallest possible embedding dimension for this approach.

Embeddings based on resolving sets of $H_{3,4}$ were used as features to classify DNA sequences of length 20 as being centered, or not, at intron-exon boundaries in the fruit fly genome [203]. In this study, the resolving set embedding outperforms *k-mer count vector* and binary representation based features with respect to accuracy and specificity, and is competitive with features based on other state-of-the-art embedding techniques like node2vec [92] and multidimensional scaling [141].⁵ Resolving set embeddings are also generally more compact than those based on *k-mer count vectors* or binary representations. For example, part of the genome of the Dengue virus codes for a protease (i.e., an enzyme that digests proteins) that targets octapeptides (i.e., amino acid sequences of length 8) in human cells. While the space of all octapeptides is large, consisting of 25.6 billion sequences, no more than 82 are required for a resolving set. Based on such a set, $H_{8,20}$ may be embedded into $\mathbb{R}^{82}$. In comparison, 3-mer count vectors use 8,000 dimensions, while binary vector representations require 160 dimensions [203].

8. Related Concepts. There is a wide variety of concepts closely linked to metric dimension. Some strengthen or alter the constraints placed on the identifiability of nodes, while others are extensions of the concept itself. In this section we will briefly discuss several such notions. A survey by Chartrand and Zhang [48] contains more complete characterizations of several of these concepts as well as information on concepts not mentioned here. There is also a body of work on conditional resolvability which focuses on resolving sets that have some additional property. For example, one can consider the smallest resolving set of a graph that induces a connected subgraph or that is also an independent set. For these conditional variants, we direct the reader to a survey by Saenpholphat and Zhang [181]. See also another forthcoming survey on variants of metric dimension [146].

8.1. Doubly Resolving Sets. As discussed in section 7.2, while one can uniquely identify all nodes in a graph based on distances to a resolving set, this identification

⁵For this particular task, *k-mer count vectors* are not well suited. As positive examples consist of half intronic and half exonic DNA, we might expect the location of *k-mers* within the larger sequence to matter a great deal, yet *k-mer count vectors* do not directly encode this information.

can fail if one only knows distances up to an additive constant. Doubly resolving sets address this shortcoming and have proven useful in identifying the source of a spread in a network [198] and in determining bounds on the metric dimension of Cartesian products of graphs [38]. In particular, a set $R \subseteq V$ is called *doubly resolving* if, for every pair of different nodes $u, v \in V$, there is a pair $r, r' \in R$ such that $d(r, u) - d(r', u) \neq d(r, v) - d(r', v)$. Such sets differentiate nodes based on relative as opposed to absolute distances.

To fix ideas, consider the path graph P_n with nodes labeled consecutively from 1 to n and resolving set $\{1\}$. Suppose a signal is sent along the path from some node i at an arbitrary time t . Traversing each edge in one time unit, this signal reaches node 1 at time $(t + i - 1)$. Since t is unknown, node i cannot be distinguished as the source of the signal from this information alone. The set $\{1, n\}$, on the other hand, is doubly resolving on P_n . Now a signal sent from i at time t will arrive at node 1 at time $(t + i - 1)$ and at node n at time $(t + n - i)$. So for any node $j \neq i$, $d(i, 1) - d(i, n) = (t + i - 1) - (t + n - i) = 2i - n - 1 \neq 2j - n - 1 = (t + j - 1) - (t + n - j) = d(j, 1) - d(j, n)$, and the source of the signal can be uniquely determined.

A related notion is that of *k-metric dimension*, where one seeks a resolving set R such that for any two nodes $u, v \in V$, there are at least k elements $r \in R$ such that $d(r, u) \neq d(r, v)$. See [146, sect. 6] for a treatment of this concept, which we note is different from truncated metric dimension (see section 8.4).

8.2. Strong Metric Dimension. Since all vertices in a graph $G = (V, E)$ are distinguished based on distance to a resolving set $R \subseteq V$, it is tempting to think that G may be reconstructed using R . These distances can only recover shortest paths, however, and thus any edge in E that is not part of a unique shortest path from any $v \in V$ to any $r \in R$ could be excluded in such a reconstruction. For instance, consider the cycle C_6 with the minimum resolving set $R = \{1, 3\}$. Notice that R remains a minimum resolving set if the edge $\{2, 5\}$ is added. Furthermore, for each $v \in V$, $d(v|R)$ is the same with or without this extra edge. As a result, R is not sufficient to guarantee a faithful reconstruction of C_6 . Notably, any resolving set is enough to reconstruct a tree, as the only path between any pair of vertices is the shortest path.

A set $S \subseteq V$ is said to *strongly resolve* G if, for every distinct $u, v \in V$, there is a vertex $s \in S$ such that u lies on a shortest path from s to v or v lies on a shortest path from s to u [186]. The cardinality of smallest possible strongly resolving sets on a graph is its strong metric dimension. By definition, every edge in E must be accounted for by a shortest path distance $d(v, s)$ for some $v \in V$ and $s \in S$. This allows G to be reconstructed exactly based on a strong resolving set [186]. For a survey of results and approximation methods related to strong metric dimension, see [138].

8.3. Multilateration. The definition of metric dimension depends heavily on graph structure and a notion of edge distances, yet there are pairwise distance matrices that do not correspond to a graph or metric space. Consider the following matrix:

$$M = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{pmatrix} 0 & 10 & 100 \\ \infty & 0 & 10 \\ \infty & \infty & 0 \end{pmatrix} \end{matrix}.$$

Notice that the points in M do not abide by the triangle inequality. While we can circumvent the fact that values in this matrix are not symmetric by using directed edges, violating the triangle inequality would require redefining the distance between

pairs of nodes. Fortunately, metric dimension makes no use of the actual distance between two vertices beyond checking equality, and thus one can imagine a relaxation using a more general distance function. In fact, one could even allow the entries of M to come from an arbitrary set other than the reals.

This more general problem on arbitrary matrices is called *multilateration* [203]. To be more precise, let I be a set of items associated with the rows of a matrix M , and let F be a set of functions over I associated with columns so that $M(i, f) = f(i)$. Analogous to metric dimension, the goal of multilateration is to determine a resolving set $R \subseteq F$ of minimum size such that the vectors $(r(i))_{r \in R}$ are unique for all $i \in I$. One can equivalently think of R as a set of columns of M such that the row vectors of the induced submatrix are unique. Borrowing notation from metric dimension, we set $\beta(M) = |R|$ for a set R of minimum size. The entries $M(i, f)$ need not be numeric; one only needs a notion of equivalence on elements of $f(I)$ for each $f \in F$, i.e., for values in the same column of M . In the context of graphs, multilateration is equivalent to metric dimension: if G is a graph with pairwise distance matrix D , then we have $\beta(G) = \beta(D)$.

8.4. Truncated Metric Dimension. In some scenarios, complete distance information of a network is unavailable. In particular, it might be that only distances below a certain threshold are available, perhaps because collecting long-distance information is costly or prone to an excessive amount of noise. In such cases it might not be possible to determine the metric dimension of the full graph. Instead, given a graph $G = (V, E)$ and a maximum distinguishable distance $k \geq 1$, let $d_k(u, v) = \min\{d(u, v), k + 1\}$ be the k -truncated distance between $u, v \in V$ and let D_k be the k -truncated distance matrix of G . Then the k -truncated metric dimension of G is defined as $\beta_k(G) = \beta(D_k)$, where $\beta(D_k)$ is defined as in multilateration (see section 8.3) [86, 201, 202].

This notion is a generalization of the concept of adjacency metric dimension [122] where $k = 1$ and a constrained version of (k, t) -metric dimension [73]. Here t , instead of k , defines a maximum distinguishable distance, while k refers to the minimum number of elements of a resolving set that must distinguish each pair of vertices. As a result, (k, t) -metric dimension is equivalent to truncated metric dimension when $k = 1$. When $t \geq \text{diam}(G)$, this variation corresponds to k -metric dimension [2, 22, 72]. See also section 8.1. There are also connections between truncated metric dimension and *locating-dominating sets* [194], for which $k = 1$ and $d(u, r) \leq 1$ for all $u \in V$ and at least one r in the set, and to *identifying codes* [130], which have the additional constraint that $d(r', r) = 1$ for all vertices r in the code and at least one other r' in the code—so that no element can distinguish itself.

Beyond settings with restricted distance information, truncated metric dimension can also be an effective tool for studying metric dimension. For instance, it can be shown that for all graphs G and all $k \geq 1$, $\beta_k(G) \geq \beta_{k+1}(G)$ [86, 201, 202]; as $\beta(G) = \beta_{n-1}(G)$, this means truncated metric dimension can give upper bounds on $\beta(G)$. As one application, an asymptotically tight upper bound on the metric dimension of the Erdős–Rényi random graph $G_{n,p}$ can be determined by focusing on $\beta_1(G)$, i.e., on adjacency information alone [201, 204]. While $G_{n,p}$ contains dependencies between shortest path lengths, adjacencies are independent by definition, making the 1-truncated metric dimension of these graphs far easier to characterize than their standard metric dimension (see section 6.2).

There are exact formulae for the truncated metric dimension of certain families of graphs including paths, cycles, complete graphs, fans, wheels, trees with a particular structure, and complete m -partite graphs. Beside that, graphs for which the truncated

metric dimension is 1, $(n-2)$, and $(n-1)$ have been fully characterized, and the effect of vertex and edge removal has been studied [86, 202].

8.5. Resolving Number, Upper Dimension, Random k -Dimensionality. Given a resolving set R of $G = (V, E)$, it is natural to consider removing elements of R while keeping the set resolving, which some authors refer to as *minimal*. Let $\mathcal{R}(G)$ be the set of all resolving sets which cannot be made smaller in this way, i.e., for which no proper subset $S \subset R$ is also resolving.

Resolving sets $R \in \mathcal{R}(G)$, while minimal in the sense of set inclusion, are not guaranteed to be minimal in the sense that $|R| = \beta(G)$, the minimum possible cardinality of any resolving set of G . For example, on the path P_6 with vertices $\{1, \dots, 6\}$, the set $R = \{3, 4\}$ is in $\mathcal{R}(P_6)$, but $\beta(P_6) = 1$. Resolving sets of cardinality $\beta(G)$ are also elements of $\mathcal{R}(G)$; however, the converse is not necessarily true. The cardinality of largest set-inclusion-minimal resolving sets, on the other hand, is called the *upper dimension* $\dim^+(G)$, given by $\dim^+(G) = \max_{R \in \mathcal{R}(G)} |R|$.

To generate sets in $\mathcal{R}(G)$, one could start with $R = V$ and iteratively remove vertices while keeping R resolving. In some cases, starting with a smaller set is guaranteed to succeed as well. The *resolving number* of a graph, denoted $\text{res}(G)$, is the smallest integer such that all subsets $S \subset V$ with $|S| = \text{res}(G)$ are resolving sets. It is known that the number of graphs with a resolving number greater than or equal to four is finite [83]. Combined with the previous inequalities, we have $\beta(G) \leq \dim^+(G) \leq \text{res}(G) \leq n-1$ [43, 83]. Furthermore, for every pair of integers $2 \leq m \leq n$, there is a connected graph G such that $\beta(G) = m$ and $\dim^+(G) = n$ [83].

When $k = \beta(G) = \text{res}(G)$, every $R \in \mathcal{R}(G)$ with $|R| = k$ is resolving and is of minimum cardinality. In this case G is said to be *randomly k -dimensional*. The only known randomly k -dimensional connected graphs are complete graphs K_k with $k \geq 1$, and odd cycles C_n with $n \geq 3$ [83].

8.6. Directed Graphs. In directed graphs, or *digraphs*, the notion of (geodesic) distance between pairs of nodes requires some care. Some authors regard $d(u, v)$ as undefined for a pair of vertices u and v where there is no directed path from u to v , whereas others define $d(u, v) = +\infty$ in this case. Issues of this type also arise when studying the effect of edge orientations in undirected graphs.

If the geodesic distance is regarded as undefined when there is no directed path between a pair of nodes in a digraph, its metric dimension may also be undefined. The characterization of digraphs with a defined metric dimension remains elusive in the literature. Of course, if a digraph is strongly connected, then its metric dimension is defined. Also, if a (weakly) connected digraph D remains or becomes strongly connected upon the removal of a vertex, then its $\beta(D)$ is defined [44]. Interestingly, the metric dimension of oriented trees is either 1 or is undefined and, when defined, the tree must be a directed Hamiltonian path [44].

If $D = (V, E)$ is a (weakly) connected digraph with a defined metric dimension such that $|V| \geq 3$, and the outer degree of every vertex is at least 1, then $\beta(D) \leq n-2$. If instead $|V| \geq 5$ and the outer degree of every vertex is at least 2, then $\beta(D) \leq n-3$ [44]. This latter bound is sharp.

With the convention that $d(u, v) := +\infty$ if there is no directed path from u to v in a digraph D , the analogue of metric dimension has been denoted $\text{MD}(D)$, whereas the worst possible value of $\text{MD}(G)$ among all possible orientations of an undirected graph G has been denoted $\text{WOMD}(G)$. Upper bounds for the metric dimension of strongly connected digraphs can be established based on their maximum in- and out-degree, as well as the cardinality of a minimum vertex cover of certain auxiliary

undirected graphs associated with the digraph [26]. Upper bounds for strong Eulerian orientations (i.e., an orientation where the in- and out-degree of each vertex is the same), tori (i.e., Cartesian products of two cycles), and grids (i.e., Cartesian product of two paths) are also possible [26]. For a family $\mathcal{G}$ of undirected graphs which admit strong orientations, in particular, each graph in the family is 2-edge-connected, $\text{WOMD}(\mathcal{G}) := \max_{G \in \mathcal{G}} \text{WOMD}(G)/|V(G)|$. Asymptotic lower and upper bounds for this quantity when the maximum degree of a graph in $\mathcal{G}$ is either 2, 4, or is allowed to tend to infinity have also been obtained [26].

8.7. Other Related Notions. A graph is called *uniquely dimensional* when it has a unique minimal resolving set. A uniquely dimensional graph with metric dimension k is called a uniquely k -dimensional graph. In [13], upper bounds for the metric dimension of uniquely dimensional graphs based on order, diameter, and girth are obtained. The authors also construct uniquely k -dimensional graphs of specific diameters and orders and bound the order of the smallest graphs of particular characteristics. A more refined question is whether a particular vertex must be present in any minimal resolving set. Hakanen et al. [96] study such vertices as well as a dual notion, specifically, vertices which do not appear in any minimal resolving set.

The notion of *edge metric dimension* of a graph $G = (V, E)$ was introduced by Kelenc, Tratnik, and Yero [133]. For a $v \in V$ and $e = \{x, y\} \in E$, let $d(e, v) := \min\{d(x, v), d(y, v)\}$. A vertex v is said to distinguish two edges e_1 and e_2 when $d(e_1, v) \neq d(e_2, v)$. The edge metric dimension of G , denoted $\text{edim}(G)$, is the cardinality of a smallest edge resolving set of vertices in the graph. For a tree T that is not a path, $\beta(T) = \text{edim}(T)$, and for a unicyclic graph G , $|\beta(G) - \text{edim}(G)| \leq 1$. These inequalities may be regarded as specializations of a more general result about *cactus graphs*, i.e., graphs where all cycles have pairwise disjoint edges: a cactus graph G with c edge-pairwise disjoint cycles satisfies $|\beta(G) - \text{edim}(G)| \leq c$ [187]. Geneson [85] shows that d -dimensional grids have $\text{edim}(G) = d$ among other results and extremal constructions; see also Zubrilina [220].

The notion of metric dimension has also been extended from vertices to vertex partitions [45, 46]. Given an (ordered) partition $P = \{P_1, \dots, P_k\}$ of the vertex set V of a graph G , define $d(v|P) := (d(v, P_1), \dots, d(v, P_k))$ for each $v \in V$. A partition is called resolving when for all $u, v \in V$, if $d(u|P) = d(v|P)$, then $u = v$. The *partition metric dimension* of G , denoted $\text{dim}_P(G)$, is the cardinality of the smallest possible resolving partition of the graph. It turns out that $\text{dim}_P(G) \leq 1 + \beta(G)$, and graphs of any metric partition may be constructed as long as this relationship is preserved [41]. The partition dimension of a graph has been related to various parameters such as clique number, diameter, maximum degree, and order [41].

Other notions which place additional constraints on resolving sets have been studied. As one particularly natural example, Eroh, Kang, and Yi [71] introduce the *connected metric dimension*, which requires resolving sets to induce a connected subgraph. In general, the connected metric dimension of a graph G may be much larger than its metric dimension: if G is a path P_n with one leaf vertex added to each end, then $\beta(G) = 2$, whereas the connected metric dimension is $n + 2$ as one needs to add the entire path. The authors show that G having a connected metric dimension of 2, i.e., being resolved by an edge in the graph, implies that G is planar, an interesting contrast to metric dimension [134].

Finally, a subset D of nodes in a graph $G = (V, E)$ is called *determining* when any graph automorphism is uniquely determined by its action over the nodes in D . Equivalently, if σ is an automorphism of G and $\sigma(v) = v$ for all $v \in D$, then $\sigma(v) = v$

for all $v \in V$. The *determining number* $\text{Det}(G)$ of a graph G is the cardinality of its smallest determining set. It is well known that the determining number of a graph is always a lower bound for its metric dimension [32], though their difference can be arbitrarily large. Indeed, for $n \geq 8$ [84],

$$\left\lfloor \frac{2n}{5} \right\rfloor - 2 \leq \max_{G: |V|=n} (\beta(G) - \text{Det}(G)) \leq (n - 2).$$

These bounds, however, are not believed to be sharp. Indeed, it is conjectured that for all n sufficiently large, $\max_{G: |V|=n} (\beta(G) - \text{Det}(G)) = \lfloor n/2 \rfloor - 1$ [84]. On the other hand, a linear time algorithm for computing the determining number and associated minimum determining set of trees was provided in [35]. The determining number of some Cartesian products of graphs was also determined in [35]. The determining number of Levenshtein graphs (see section 5.2) was determined in [180].

8.8. Metric Space Connections. Graphs are special instances of metric spaces, and the notion of metric dimension extends naturally to abstract metric spaces. Given a metric space (X, d) , a subset $R \subset X$ is called *resolving* if for all $u, v \in X$, with $u \neq v$, there is an $r \in R$ such that $d(r, u) \neq d(r, v)$. The metric dimension of (X, d) , denoted as $\beta(X, d)$, corresponds to the cardinality of a smallest resolving set. When the metric d is implicit from the context, we write $\beta(X)$ instead of $\beta(X, d)$.

If (X, d) is complete, convex, and every pair of different points resolves it, then (X, d) is isometric to $\mathbb{R}$ [165]. If (X, d) is connected and compact, and $\beta(X) = 1$, then X is homeomorphic to the interval $[0, 1]$. Instead, if (X, d) is connected and noncompact, every closed ball is compact, and $\beta(X) = 1$, then X is homeomorphic to the interval $[0, +\infty)$ [21]. If V is an affine subspace of $\mathbb{R}^n$, then $\beta(V) = \dim(V) + 1$ [21]; in particular, $\beta(\mathbb{R}^n) = n + 1$.

Let $x = (x_1, \dots, x_n)$ denote a generic element in $\mathbb{R}^n$. The *hyperbolic space* $\mathbb{H}^n := \{x \text{ such that } x_i > 0 \text{ for } i = 1, \dots, n\}$ endowed with the path metric derived from dx/x_n , the *spherical space* $\mathbb{S}^n := \{x \text{ such that } \|x\|_2 = 1\}$ endowed with the path metric induced by the Euclidean distance in $\mathbb{R}^{n+1}$, and the *unit ball* $\mathbb{B}^n := \{x \text{ such that } \|x\|_2 \leq 1\}$ endowed with the path metric induced by $2|dx|/(1 - \|x\|_2^2)$, have metric dimension $(n + 1)$ each. The same applies to the metric dimension of any open subset of these metric spaces or $\mathbb{R}^n$ [21]. Next, consider $X \subset \mathbb{R}^n$ with the metric induced by the Euclidean distance. Then, any metric basis $R = \{r_1, \dots, r_k\}$ of X must be *affinely independent*, i.e., the vectors $(r_2 - r_1), \dots, (r_k - r_1)$ must be linearly independent. Furthermore, if $\mathcal{A}(X)$ is the smallest affine subspace of $\mathbb{R}^n$ containing X , then $\beta(X) \leq \beta(\mathcal{A}(X))$. If in addition X has a nonempty interior, then $\beta(X) \in \{n, n + 1\}$. Furthermore, if X is a convex subset with a nonempty interior, then $\beta(X) = n$ if and only if X has a supporting hyperplane H such that $\mathcal{A}(X \cap H) = H$ [21].

We briefly mention some other connections. A metric space generalization of corona product graphs (see section 5.3) is introduced and studied in [177]; in particular, a formula is given for their metric dimension. Results about the metric dimension of so-called wreath products of two metric spaces can be found in [172], and about Riemann surfaces and Riemannian manifolds in [106, 21]. Finally, the notion of k -metric dimension (see section 8.1) has been generalized to metric spaces [22, 176, 54].

9. Conclusion and Open Questions. Metric dimension is a straightforward idea. Its close relation to GPS and trilateration makes applications concerning locating graphs' nodes immediately apparent. While determining the exact metric dimension

of general graphs is an NP-complete problem, several approximation methods exist, including the ICH algorithm. The ICH gives resolving sets guaranteed to be close to optimal cardinality but has cubic run time, which can be impractical for real world networks. Nevertheless, a great deal is known concerning exact formulae, asymptotic behavior, and bounds for an array of graph families. Recently, the metric dimension of random trees and forests and Erdős–Rényi random graphs have also been characterized. This understanding is a critical step toward practical application in different settings.

Potential applications of metric dimension include identifying the source of a spread in a network, detecting network motifs, and embedding symbolic data into comparatively low-dimensional Euclidean spaces. A better theoretical understanding of metric dimension on random graph models, coupled with new approximate or exact algorithms tailored to specific families of graphs, will have a large impact on these applications. Further, many of these settings may accommodate relaxations or variations of metric dimension, such as doubly resolving sets and multilateration.

Possible directions for future work abound, both in deepening our theoretical understanding of metric dimension and related concepts and in applying these concepts in practice. For instance, the precise treewidth, between 2 and 24, at which the decision version of metric dimension becomes NP-hard is unknown. Moreover, the existence of additional types of randomly k -dimensional graphs remains open [47], and asymptotic or explicit characterizations of the metric dimension of graph families with comparatively small automorphism groups, such as Levenshtein graphs, and that of discrete (finite or infinite) metric spaces could prove helpful for generating Euclidean embeddings in different contexts.

Acknowledgments. The authors thank the reviewers for their very insightful comments on the original version of this article and their suggestions for important papers to include in the survey. The authors also thank E. W. Weisstein for valuable comments about the arXiv.org version of this article.

REFERENCES

- [1] M. ABAS AND T. VETRÍK, *Metric dimension of Cayley digraphs of split metacyclic groups*, Theoret. Comput. Sci., 809 (2020), pp. 61–72. (Cited on p. 938)
- [2] R. ADAR AND L. EPSTEIN, *The k -metric dimension*, J. Combin. Optim., 34 (2017), pp. 1–30. (Cited on p. 950)
- [3] R. ADAR AND L. EPSTEIN, *The metric dimension of two-dimensional extended meshes*, Acta Cybernet., 23 (2018), pp. 761–772. (Cited on p. 937)
- [4] S. AHMAD, M. A. CHAUDHRY, I. JAVAID, AND M. SALMAN, *On the metric dimension of generalized Petersen graphs*, Quaest. Math., 36 (2013), pp. 421–435. (Cited on pp. 931, 940)
- [5] M. R. ALFURAIDAN AND J. HALL, *Smith’s theorem and a characterization of the 6-cube as distance-transitive graph*, J. Algebraic Combin., 24 (2006), pp. 195–207. (Cited on p. 939)
- [6] Y. ALIZADEH, E. ESTAJI, S. KLAVŽAR, AND M. PETKOVŠEK, *Metric properties of generalized Sierpiński graphs over stars*, Discrete Appl. Math., 266 (2019), pp. 48–55. (Cited on p. 931)
- [7] R. ARRATIA, L. GORDON, AND M. S. WATERMAN, *The Erdős–Rényi law in distribution, for coin tossing and sequence matching*, Ann. Statist., 18 (1990), pp. 539–570. (Cited on p. 947)
- [8] L. BABAI, *On the complexity of canonical labeling of strongly regular graphs*, SIAM J. Comput., 9 (1980), pp. 212–216, <https://doi.org/10.1137/0209018>. (Cited on p. 939)
- [9] L. BABAI, *On the order of uniprimitive permutation groups*, Ann. of Math. (2), 113 (1981), pp. 553–568. (Cited on p. 939)
- [10] L. BABAI, *Graph isomorphism in quasipolynomial time*, in Proceedings of the 48th Annual ACM Symposium on Theory of Computing, 2016, pp. 684–697. (Cited on p. 946)

- [11] L. BABAI, P. ERDŐS, AND S. M. SELKOW, *Random graph isomorphism*, SIAM J. Comput., 9 (1980), pp. 628–635, <https://doi.org/10.1137/0209047>. (Cited on pp. 941, 944, 946, 947)
- [12] M. BAČA, E. T. BASKORO, A. SALMAN, S. SAPUTRO, AND D. SUPRIJANTO, *The metric dimension of regular bipartite graphs*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 54 (2011), pp. 15–28. (Cited on p. 932)
- [13] B. BAGHERI, M. JANNESARI, AND B. OMOOMI, *Uniquely Dimensional Graphs*, preprint, <https://arxiv.org/abs/1205.0327>, 2012. (Cited on p. 952)
- [14] R. F. BAILEY, *The metric dimension of small distance-regular and strongly regular graphs*, Australas. J. Combin., 62 (2015), pp. 18–34. (Cited on pp. 939, 941)
- [15] R. F. BAILEY, *On the metric dimension of imprimitive distance-regular graphs*, Ann. Comb., 20 (2016), pp. 641–659. (Cited on p. 939)
- [16] R. F. BAILEY, *On the metric dimension of incidence graphs*, Discrete Math., 341 (2018), pp. 1613–1619. (Cited on p. 939)
- [17] R. F. BAILEY, J. CÁCERES, D. GARIJO, A. GONZÁLEZ, A. MÁRQUEZ, K. MEAGHER, AND M. L. PUERTAS, *Resolving sets for Johnson and Kneser graphs*, European J. Combin., 34 (2013), pp. 736–751. (Cited on p. 932)
- [18] R. F. BAILEY AND P. J. CAMERON, *Base size, metric dimension and other invariants of groups and graphs*, Bull. London Math. Soc., 43 (2011), pp. 209–242. (Cited on p. 939)
- [19] R. F. BAILEY AND P. SPIGA, *Metric dimension of dual polar graphs*, Arch. Math., 120 (2023), pp. 467–478, <https://doi.org/10.1007/s00013-023-01829-2>. (Cited on p. 932)
- [20] D. BARTOLI, T. HÉGER, G. KISS, AND M. TAKÁTS, *On the metric dimension of affine planes, biaffine planes and generalized quadrangles*, Australas. J. Combin., 72 (2018), pp. 226–248. (Cited on p. 941)
- [21] S. BAU AND A. BEARDON, *The metric dimension of metric spaces*, Comput. Methods Funct. Theory, 13 (2013), pp. 295–305. (Cited on p. 953)
- [22] A. F. BEARDON AND J. A. RODRÍGUEZ-VELÁZQUEZ, *On the k -metric dimension of metric spaces*, Ars Math. Contemp., 16 (2019), pp. 25–38. (Cited on pp. 950, 953)
- [23] L. BEAUDOU, P. DANKELMANN, F. FOUCAUD, M. A. HENNING, A. MARY, AND A. PARREAU, *Bounding the order of a graph using its diameter and metric dimension: A study through tree decompositions and VC dimension*, SIAM J. Discrete Math., 32 (2018), pp. 902–918, <https://doi.org/10.1137/16M1097833>. (Cited on p. 929)
- [24] Z. BEERLIOVA, F. EBERHARD, T. ERLEBACH, A. HALL, M. HOFFMANN, M. MIHAL'AK, AND L. S. RAM, *Network discovery and verification*, IEEE J. Selected Areas Commun., 24 (2006), pp. 2168–2181. (Cited on p. 927)
- [25] R. BELMONTE, F. V. FOMIN, P. A. GOLOVACH, AND M. S. RAMANUJAN, *Metric dimension of bounded tree-length graphs*, SIAM J. Discrete Math., 31 (2017), pp. 1217–1243, <https://doi.org/10.1137/16M1057383>. (Cited on p. 927)
- [26] J. BENSMAIL, F. MC INERNEY, AND N. NISSE, *Metric dimension: From graphs to oriented graphs*, Electron. Notes Theoret. Comput. Sci., 346 (2019), pp. 111–123. (Cited on p. 952)
- [27] P. BERMAN, B. DASGUPTA, AND M.-Y. KAO, *Tight approximability results for test set problems in bioinformatics*, J. Comput. Syst. Sci., 71 (2005), pp. 145–162. (Cited on p. 927)
- [28] L. BLUMENTHAL, *Theory and Applications of Distance Geometry*, Oxford University Press, 1953. (Cited on p. 921)
- [29] B. BOLLOBÁS, D. MITSCHKE, AND P. PRALAT, *Metric dimension for random graphs*, Electron. J. Combin., 20 (2013), art. 1. (Cited on pp. 941, 942)
- [30] É. BONNET AND N. PUROHIT, *Metric dimension parameterized by treewidth*, Algorithmica, 83 (2021), pp. 2606–2633. (Cited on p. 927)
- [31] N. BOOTH AND J. A. MATIC, *Mapping and leveraging influencers in social media to shape corporate brand perceptions*, Corporate Commun., 16 (2011), pp. 184–191. (Cited on p. 945)
- [32] D. L. BOUTIN, *Identifying graph automorphisms using determining sets*, Electron. J. Combin., 13 (2006), art. 78. (Cited on p. 953)
- [33] B. BREŠAR, S. KLAVŽAR, AND D. F. RALL, *Domination game and an imagination strategy*, SIAM J. Discrete Math., 24 (2010), pp. 979–991, <https://doi.org/10.1137/100786800>. (Cited on p. 945)
- [34] P. BUCZKOWSKI, G. CHARTRAND, C. POISSON, AND P. ZHANG, *On k -dimensional graphs and their bases*, Period. Math. Hungar., 46 (2003), pp. 9–15. (Cited on pp. 930, 931)
- [35] J. CÁCERES, D. GARIJO, M. L. PUERTAS, AND C. SEARA, *On the determining number and the metric dimension of graphs*, Electron. J. Combin., 17 (2010), art. 63. (Cited on p. 953)
- [36] J. CÁCERES, C. HERNANDO, M. MORA, I. M. PELAYO, AND M. L. PUERTAS, *On the metric dimension of infinite graphs*, Electron. Notes Discrete Math., 35 (2009), pp. 15–20. (Cited on pp. 933, 937, 940)

- [37] J. CÁCERES, C. HERNANDO, M. MORA, I. M. PELAYO, M. L. PUERTAS, C. SEARA, AND D. R. WOOD, *On the metric dimension of some families of graphs*, Electron. Notes Discrete Math., 22 (2005), pp. 129–133. (Cited on p. 937)
- [38] J. CÁCERES, C. HERNANDO, M. MORA, I. M. PELAYO, M. L. PUERTAS, C. SEARA, AND D. R. WOOD, *On the metric dimension of Cartesian products of graphs*, SIAM J. Discrete Math., 21 (2007), pp. 423–441, <https://doi.org/10.1137/050641867>. (Cited on pp. 935, 945, 949)
- [39] Y.-D. CAI, K.-Y. FENG, Y.-X. LI, AND K.-C. CHOU, *Support vector machine for predicting α -turn types*, Peptides, 24 (2003), pp. 629–630. (Cited on p. 948)
- [40] M. CHA, H. HADDADI, F. BENEVENUTO, AND K. P. GUMMADI, *Measuring user influence in Twitter: The million follower fallacy*, in the Fourth International AAAI Conference on Weblogs and Social Media, 2010, pp. 10–17. (Cited on p. 931)
- [41] G. G. CHAPPELL, J. GIMBEL, AND C. HARTMAN, *Bounds on the metric and partition dimensions of a graph*, Ars Combin., 88 (2008), pp. 349–366. (Cited on p. 952)
- [42] G. CHARTRAND, L. EROH, M. A. JOHNSON, AND O. R. OELLERMANN, *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math., 105 (2000), pp. 99–113. (Cited on pp. 922, 929, 930, 931, 932, 933, 934, 935, 937, 944)
- [43] G. CHARTRAND, C. POISSON, AND P. ZHANG, *Resolvability and the upper dimension of graphs*, Comput. Math. Appl., 39 (2000), pp. 19–28. (Cited on pp. 922, 931, 951)
- [44] G. CHARTRAND, M. RAINES, AND P. ZHANG, *The directed distance dimension of oriented graphs*, Math. Bohemica, 125 (2000), pp. 155–168. (Cited on p. 951)
- [45] G. CHARTRAND, E. SALEHI, AND P. ZHANG, *On the partition dimension of a graph*, Congr. Numer., 130 (1998), pp. 157–168. (Cited on p. 952)
- [46] G. CHARTRAND, E. SALEHI, AND P. ZHANG, *The partition dimension of a graph*, Aequationes Math., 59 (2000), pp. 45–54. (Cited on p. 952)
- [47] G. CHARTRAND AND P. ZHANG, *On the chromatic dimension of a graph*, Congr. Numer., 145 (2000), pp. 97–108. (Cited on p. 954)
- [48] G. CHARTRAND AND P. ZHANG, *The theory and applications of resolvability in graphs*, Congr. Numer., 160 (2003), pp. 47–68. (Cited on p. 948)
- [49] K. CHAU AND S. GOSSELIN, *The metric dimension of circulant graphs and their Cartesian products*, Opuscula Math., 37 (2017), pp. 509–534. (Cited on p. 932)
- [50] V. CHVÁTAL, *Mastermind*, Combinatorica, 3 (1983), pp. 325–329. (Cited on pp. 944, 945)
- [51] M. CLAVEROL, A. GARCÍA, G. HERNÁNDEZ, C. HERNANDO, M. MAURESO, M. MORA, AND J. TEJEL, *Metric dimension of maximal outerplanar graphs*, Bull. Malaysian Math. Sci. Soc., 44 (2021), pp. 2603–2630. (Cited on p. 940)
- [52] D. CONTE, P. FOGGIA, C. SANSONE, AND M. VENTO, *Thirty years of graph matching in pattern recognition*, Internat. J. Pattern Recognition Artificial Intell., 18 (2004), pp. 265–298. (Cited on p. 946)
- [53] S. A. COOK, *The complexity of theorem-proving procedures*, in Proceedings of the Third Annual ACM Symposium on Theory of Computing, ACM, 1971, pp. 151–158. (Cited on p. 946)
- [54] S. CORREGIDOR AND Á. MARTÍNEZ-PÉREZ, *Finite metric and k -metric bases on ultrametric spaces*, Proc. Amer. Math. Soc., 149 (2021), pp. 4487–4499. (Cited on p. 953)
- [55] H. CRAMÉR, *Sur un nouveau théorème-limite de la théorie des probabilités*, Actual. Sci. Ind., 736 (1938), pp. 5–23. (Cited on p. 947)
- [56] M. CRASMARU, C. GLASSER, K. W. REGAN, AND S. SENGUPTA, *A protocol for serializing unique strategies*, in International Symposium on Mathematical Foundations of Computer Science, Springer, 2004, pp. 660–672. (Cited on p. 946)
- [57] J. D. CURRIE AND O. R. OELLERMAN, *The metric dimension and metric independence of a graph*, J. Combin. Math. Combin. Comput., 39 (2001), pp. 157–167. (Cited on p. 939)
- [58] L. DAVIS, *Handbook of Genetic Algorithms*, Van Nostrand Reinhold, New York, 1991. (Cited on p. 928)
- [59] J. DÍAZ, O. POTTONEN, M. SERNA, AND E. J. VAN LEEUWEN, *On the complexity of metric dimension*, in European Symposium on Algorithms, Springer, 2012, pp. 419–430. (Cited on pp. 927, 932)
- [60] J. DIAZ, O. POTTONEN, M. SERNA, AND E. J. VAN LEEUWEN, *Complexity of metric dimension on planar graphs*, J. Comput. System Sci., 83 (2017), pp. 132–158. (Cited on pp. 932, 940)
- [61] D. DOLŽAN, *The metric dimension of the total graph of a finite commutative ring*, Canad. Math. Bull., 59 (2016), pp. 748–759. (Cited on p. 939)
- [62] E. DUCHENE, V. GLEDEL, A. PARREAU, AND G. RENAULT, *Maker-breaker domination game*, Discrete Math., 343 (2020), art. 111955. (Cited on p. 945)

- [63] P. EICHENBERGER, M. FUJITA, S. T. JENSEN, E. M. CONLON, D. Z. RUDNER, S. T. WANG, C. FERGUSON, K. HAGA, T. SATO, J. S. LIU, AND R. LOSICK, *The program of gene transcription for a single differentiating cell type during sporulation in Bacillus subtilis*, PLoS Biol., 2 (2004), art. e328. (Cited on p. 946)
- [64] D. EPPSTEIN, *Metric dimension parameterized by max leaf number*, J. Graph Algorithms Appl., 19 (2015), pp. 313–323. (Cited on p. 927)
- [65] L. EPSTEIN, A. LEVIN, AND G. J. WOEGINGER, *The (weighted) metric dimension of graphs: Hard and easy cases*, Algorithmica, 72 (2015), pp. 1130–1171. (Cited on pp. 927, 932)
- [66] P. ERDŐS, F. HARARY, AND W. T. TUTTE, *On the dimension of a graph*, Mathematika, 12 (1965), pp. 118–122. (Cited on p. 921)
- [67] P. ERDŐS AND A. RÉNYI, *On two problems of information theory*, Magyar Tud. Akad. Mat. Kutató Int. Közl., 8 (1963), pp. 229–243. (Cited on p. 935)
- [68] P. ERDŐS AND J. L. SELFRIDGE, *On a combinatorial game*, J. Combin. Theory Ser. A, 14 (1973), pp. 298–301. (Cited on p. 945)
- [69] L. EROH, P. FEIT, C. X. KANG, AND E. YI, *The effect of vertex or edge deletion on the metric dimension of graphs*, J. Combin., 6 (2015), pp. 433–444. (Cited on p. 930)
- [70] L. EROH, C. KANG, AND E. YI, *A comparison between the metric dimension and zero forcing number of trees and unicyclic graphs*, Acta. Math. Sin. (English Ser.), 33 (2017), pp. 731–747. (Cited on pp. 930, 933, 934)
- [71] L. EROH, C. X. KANG, AND E. YI, *The connected metric dimension at a vertex of a graph*, Theoret. Comput. Sci., 806 (2020), pp. 53–69. (Cited on p. 952)
- [72] A. ESTRADA-MORENO, J. A. RODRIGUEZ-VELÁZQUEZ, AND I. G. YERO, *The k -metric dimension of a graph*, Appl. Math. Inform. Sci., 9 (2015), pp. 2829–2840. (Cited on p. 950)
- [73] A. ESTRADA-MORENO, I. G. YERO, AND J. A. RODRÍGUEZ-VELÁZQUEZ, *On the (k, t) -metric dimension of graphs*, Comput. J., 64 (2021), pp. 707–720. (Cited on p. 950)
- [74] M. FEHR, S. GOSSELIN, AND O. R. OELLERMANN, *The metric dimension of Cayley digraphs*, Discrete Math., 306 (2006), pp. 31–41. (Cited on pp. 932, 938)
- [75] M. FENG, X. MA, AND K. WANG, *The structure and metric dimension of the power graph of a finite group*, European J. Combin., 43 (2015), pp. 82–97. (Cited on p. 938)
- [76] M. FENG AND K. WANG, *On the metric dimension of bilinear forms graphs*, Discrete Math., 312 (2012), pp. 1266–1268. (Cited on p. 932)
- [77] M. FENG, M. XU, AND K. WANG, *On the metric dimension of line graphs*, Discrete Appl. Math., 161 (2013), pp. 802–805. (Cited on pp. 931, 932)
- [78] H. FERNAU, P. HEGGERNES, P. VAN’T HOF, D. MEISTER, AND R. SAEI, *Computing the metric dimension for chain graphs*, Inform. Process. Lett., 115 (2015), pp. 671–676. (Cited on pp. 927, 932)
- [79] G. FIJAVŽ AND B. MOHAR, *Rigidity and separation indices of Paley graphs*, Discrete Math., 289 (2004), pp. 157–161. (Cited on p. 932)
- [80] F. FOUCAUD, G. B. MERTZIOS, R. NASERASR, A. PARREAU, AND P. VALICOV, *Identification, location-domination and metric dimension on interval and permutation graphs. I. Bounds*, Theoret. Comput. Sci., 668 (2017), pp. 43–58. (Cited on p. 929)
- [81] F. FOUCAUD, G. B. MERTZIOS, R. NASERASR, A. PARREAU, AND P. VALICOV, *Identification, location-domination and metric dimension on interval and permutation graphs. II. Algorithms and complexity*, Algorithmica, 78 (2017), pp. 914–944. (Cited on p. 927)
- [82] M. R. GAREY AND D. S. JOHNSON, *Computers and Intractability: A Guide to the Theory of NP-completeness*, W. H. Freeman and Company, New York, 1979. (Cited on p. 925)
- [83] D. GARIJO, A. GONZÁLEZ, AND A. MÁRQUEZ, *On the metric dimension, the upper dimension and the resolving number of graphs*, Discrete Appl. Math., 161 (2013), pp. 1440–1447. (Cited on p. 951)
- [84] D. GARIJO, A. GONZÁLEZ, AND A. MÁRQUEZ, *The difference between the metric dimension and the determining number of a graph*, Appl. Math. Comput., 249 (2014), pp. 487–501. (Cited on p. 953)
- [85] J. GENESON, *Metric dimension and pattern avoidance in graphs*, Discrete Appl. Math., 284 (2020), pp. 1–7. (Cited on pp. 930, 952)
- [86] J. GENESON AND E. YI, *The Distance- k Dimension of Graphs*, preprint, <https://arxiv.org/abs/2106.08303>, 2021. (Cited on pp. 950, 951)
- [87] W. GODDARD, *Mastermind revisited*, J. Combin. Math. Combin. Comput., 51 (2004), pp. 215–220. (Cited on p. 944)
- [88] O. GOLDBREICH, *Computational Complexity: A Conceptual Perspective*, Cambridge University Press, 2008. (Cited on p. 925)
- [89] M. T. GOODRICH, *On the algorithmic complexity of the Mastermind game with black-peg results*, Inform. Process. Lett., 109 (2009), pp. 675–678. (Cited on p. 944)

- [90] M. GOSAK, R. MARKOVIČ, J. DOLENŠEK, M. S. RUPNIK, M. MARHL, A. STOŽER, AND M. PERC, *Network science of biological systems at different scales: A review*, Phys. Life Rev., 24 (2018), pp. 118–135. (Cited on p. 946)
- [91] C. GRIGORIOUS, P. MANUEL, M. MILLER, B. RAJAN, AND S. STEPHEN, *On the metric dimension of circulant and Harary graphs*, Appl. Math. Comput., 248 (2014), pp. 47–54. (Cited on p. 932)
- [92] A. GROVER AND J. LESKOVEC, *node2vec: Scalable feature learning for networks*, in Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, ACM, 2016, pp. 855–864. (Cited on p. 948)
- [93] J. GUO, K. WANG, AND F. LI, *Metric dimension of some distance-regular graphs*, J. Combin. Optim., 26 (2013), pp. 190–197. (Cited on p. 932)
- [94] G. GUTIN, M. RAMANUJAN, F. REIDL, AND M. WAHLSTRÖM, *Alternative parameterizations of metric dimension*, Theoret. Comput. Sci., 806 (2020), pp. 133–143. (Cited on p. 927)
- [95] V. HÄHNKE, M. RUPP, M. KRIER, F. RIPPMANN, AND G. SCHNEIDER, *Pharmacophore alignment search tool: Influence of canonical atom labeling on similarity searching*, J. Comput. Chem., 31 (2010), pp. 2810–2826. (Cited on p. 946)
- [96] A. HAKANEN, V. JUNNILA, T. LAIHONEN, AND I. G. YERO, *On vertices contained in all or in no metric basis*, Discrete Appl. Math., 319 (2022), pp. 407–423. (Cited on p. 952)
- [97] M. HALLAWAY, C. X. KANG, AND E. YI, *On metric dimension of permutation graphs*, J. Combin. Optim., 28 (2014), pp. 814–826. (Cited on p. 941)
- [98] R. W. HAMMING, *Error detecting and error correcting codes*, Bell Labs Tech. J., 29 (1950), pp. 147–160. (Cited on p. 934)
- [99] F. HARARY AND R. A. MELTER, *On the metric dimension of a graph*, Ars Combin., 2 (1976), pp. 191–195. (Cited on pp. 921, 927, 931)
- [100] S. HARTUNG AND A. NICHTERLEIN, *On the parameterized and approximation hardness of metric dimension*, in the 2013 IEEE Conference on Computational Complexity, IEEE, 2013, pp. 266–276. (Cited on p. 927)
- [101] M. HAUPTMANN, R. SCHMIED, AND C. VIEHMANN, *Approximation complexity of metric dimension problem*, J. Discrete Algorithms, 14 (2012), pp. 214–222. (Cited on p. 927)
- [102] T. HÉGER, P. SZILÁRD, AND M. TAKÁTS, *The metric dimension of the incidence graphs of projective and affine planes of small order*, Australas. J. Combin., 78 (2020), pp. 352–375. (Cited on p. 941)
- [103] T. HÉGER AND M. TAKÁTS, *Resolving sets and semi-resolving sets in finite projective planes*, Electron. J. Combin., 19 (2012), art. P30. (Cited on p. 941)
- [104] C. HERNANDO, M. MORA, I. M. PELAYO, C. SEARA, J. CÁCERES, AND M. L. PUERTAS, *On the metric dimension of some families of graphs*, Electron. Notes Discrete Math., 22 (2005), pp. 129–133. (Cited on pp. 931, 932, 937)
- [105] C. HERNANDO, M. MORA, I. M. PELAYO, C. SEARA, AND D. R. WOOD, *Extremal graph theory for metric dimension and diameter*, Electron. J. Combin., 17 (2010), art. R30. (Cited on p. 929)
- [106] M. S. HEYDARPOUR AND S. MAGHSOUDI, *The metric dimension of geometric spaces*, Topology Appl., 178 (2014), pp. 230–235. (Cited on p. 953)
- [107] S. HOFFMANN, A. ELTERMAN, AND E. WANKE, *A linear time algorithm for metric dimension of cactus block graphs*, Theoret. Comput. Sci., 630 (2016), pp. 43–62. (Cited on pp. 927, 932)
- [108] J. HOLLAND, *Adaptation in Natural and Artificial Systems: An Introductory Analysis with Applications to Biology, Control, and Artificial Intelligence*, University of Michigan Press, 1975. (Cited on p. 946)
- [109] P. W. HOLLAND AND S. LEINHARDT, *The Statistical Analysis of Local Structure in Social Networks*, Tech. Report, National Bureau of Economic Research, 1974. (Cited on p. 946)
- [110] J. HU AND X. SHANG, *Detection of network motif based on a novel graph canonization algorithm from transcriptional regulation networks*, Molecules, 22 (2017), art. 2194. (Cited on pp. 944, 946, 947)
- [111] *ILOG CPLEX Documentation*, Ver. 9.0, ILOG CPLEX Optimization Inc., 2005. (Cited on pp. 928, 939)
- [112] M. IMRAN, A. BAIG, AND A. AHMAD, *Families of plane graphs with constant metric dimension*, Utilitas Math., 88 (2012), pp. 43–57. (Cited on p. 940)
- [113] M. IMRAN, A. Q. BAIG, S. A. U. H. BOKHARY, AND I. JAVAID, *On the metric dimension of circulant graphs*, Appl. Math. Lett., 25 (2012), pp. 320–325. (Cited on p. 932)
- [114] M. IMRAN, A. Q. BAIG, M. K. SHAFIQ, AND I. TOMESCU, *On metric dimension of generalized Petersen graphs $P(n, 3)$* , Ars Combin., 117 (2014), pp. 113–130. (Cited on p. 932)
- [115] M. IMRAN, S. A. U. H. BOKHARY, AND A. Q. BAIG, *On families of convex polytopes with*

- constant metric dimension, *Comput. Math. Appl.*, 60 (2010), pp. 2629–2638. (Cited on p. 940)
- [116] M. IMRAN AND H. SIDDIQUI, *Computing the metric dimension of convex polytopes generated by wheel related graphs*, *Acta Math. Hungar.*, 149 (2016), pp. 10–30. (Cited on p. 940)
 - [117] S. IMRAN, M. K. SIDDIQUI, M. IMRAN, AND M. HUSSAIN, *On metric dimensions of symmetric graphs obtained by rooted product*, *Mathematics*, 6 (2018), art. 191. (Cited on p. 936)
 - [118] H. ISWADI, E. T. BASKORO, A. SALMAN, AND R. SIMANJUNTAK, *The metric dimension of amalgamation of cycles*, *Far East J. Math. Sci.*, 41 (2010), pp. 19–31. (Cited on p. 931)
 - [119] H. ISWADI, E. T. BASKORO, R. SIMANJUNTAK, AND A. SALMAN, *The metric dimension of graph with pendant edges*, *J. Combin. Math. Combin. Comput.*, 65 (2008), pp. 139–145. (Cited on p. 936)
 - [120] G. JÄGER, *An optimal strategy for static black-peg Mastermind with two pegs*, in *International Conference on Combinatorial Optimization and Applications*, Springer, 2016, pp. 670–682. (Cited on pp. 944, 945)
 - [121] G. JÄGER AND F. DREWES, *The metric dimension of $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ is $\lfloor 3n/2 \rfloor$* , *Theoret. Comput. Sci.*, 806 (2020), pp. 344–362. (Cited on pp. 944, 945)
 - [122] M. JANNESARI AND B. OMOOMI, *The metric dimension of the lexicographic product of graphs*, *Discrete Math.*, 312 (2012), pp. 3349–3356. (Cited on pp. 938, 943, 950)
 - [123] M. JANNESARI AND B. OMOOMI, *Characterization of n -vertex graphs with metric dimension $n - 3$* , *Math. Bohem.*, 139 (2014), pp. 1–23. (Cited on p. 930)
 - [124] I. JAVAID, M. T. RAHIM, AND K. ALI, *Families of regular graphs with constant metric dimension*, *Utilitas Math.*, 75 (2008), pp. 21–34. (Cited on pp. 931, 932)
 - [125] Z. JIANG AND N. POLYANSKII, *On the metric dimension of Cartesian powers of a graph*, *J. Combin. Theory Ser. A*, 165 (2019), pp. 1–14. (Cited on p. 935)
 - [126] M. JOHNSON, *Structure-activity maps for visualizing the graph variables arising in drug design*, *J. Biopharmaceutical Statist.*, 3 (1993), pp. 203–236. (Cited on p. 944)
 - [127] M. JOHNSON, *Browsable structure-activity datasets*, *Adv. Molecular Similarity*, 2 (1998), pp. 153–170. (Cited on p. 944)
 - [128] C. X. KANG, S. KLAVŽAR, I. G. YERO, AND E. YI, *Maker-breaker resolving game*, *Bull. Malaysian Math. Sci. Soc.*, 44 (2021), pp. 2081–2099. (Cited on pp. 944, 945)
 - [129] R. M. KARP, *Reducibility among combinatorial problems*, in *Complexity of Computer Computations*, Springer, 1972, pp. 85–103. (Cited on p. 925)
 - [130] M. G. KARPOVSKY, K. CHAKRABARTY, AND L. B. LEVITIN, *On a new class of codes for identifying vertices in graphs*, *IEEE Trans. Inform. Theory*, 44 (1998), pp. 599–611. (Cited on p. 950)
 - [131] Z. KATONA, P. P. ZUBCSEK, AND M. SARVARY, *Network effects and personal influences: The diffusion of an online social network*, *J. Marketing Res.*, 48 (2011), pp. 425–443. (Cited on p. 945)
 - [132] A. KELENC, A. T. M. TOSHI, R. SKREKOVSKI, AND I. G. YERO, *On Metric Dimensions of Hypercubes*, preprint, <https://arxiv.org/abs/2102.10916>, 2021. (Cited on p. 935)
 - [133] A. KELENC, N. TRATNIK, AND I. YERO, *Uniquely identifying the edges of a graph: The edge metric dimension*, *Discrete Appl. Math.*, 251 (2018), pp. 204–220. (Cited on p. 952)
 - [134] S. KHULLER, B. RAGHAVACHARI, AND A. ROSENFELD, *Landmarks in graphs*, *Discrete Appl. Math.*, 70 (1996), pp. 217–229. (Cited on pp. 923, 925, 927, 929, 930, 932, 937, 940, 944, 952)
 - [135] S. KLAVŽAR AND S. S. ZEMLJIĆ, *On distances in Sierpiński graphs: Almost-extreme vertices and metric dimension*, *Appl. Anal. Discrete Math.*, 7 (2013), pp. 72–82. (Cited on p. 931)
 - [136] D. J. KLEIN AND E. YI, *A comparison on metric dimension of graphs, line graphs, and line graphs of the subdivision graphs*, *European J. Pure Appl. Math.*, 5 (2012), pp. 302–316. (Cited on p. 932)
 - [137] D. E. KNUTH, *The computer as master mind*, *J. Recreational Math.*, 9 (1976), pp. 1–6. (Cited on p. 944)
 - [138] J. KRATICA, *Strong metric dimension: A survey*, *Yugoslav J. Oper. Res.*, 24 (2016), pp. 187–198. (Cited on p. 949)
 - [139] J. KRATICA, D. CVETKOVIC, M. CANGALOVIC, V. KOVACEVIC-VUJČIC, AND J. KOJIC, *The metric dimension of strongly regular graphs*, in *Proceedings of SYMOPIS*, 2008, pp. 341–344. (Cited on p. 939)
 - [140] J. KRATICA, V. KOVAČEVIĆ-VUJČIĆ, AND M. ČANGALOVIĆ, *Computing the metric dimension of graphs by genetic algorithms*, *Comput. Optim. Appl.*, 44 (2009), pp. 343–361. (Cited on p. 928)
 - [141] W. J. KRZANOWSKI, *Principles of Multivariate Analysis: A User's Perspective*, Oxford University Press, Oxford, 2000. (Cited on p. 948)

- [142] V. KUMAR AND R. MIRCHANDANI, *Increasing the ROI of social media marketing*, MIT Sloan Management Rev., 54 (2012), p. 55. (Cited on p. 945)
- [143] D. KUZIAK, I. PETERIN, AND I. G. YERO, *Resolvability and strong resolvability in the direct product of graphs*, Results Math., 71 (2017), pp. 509–526. (Cited on p. 937)
- [144] D. KUZIAK, J. A. RODRÍGUEZ-VELÁZQUEZ, AND I. G. YERO, *Corrections to the article “The metric dimension of graph with pendant edges”* [Journal of Combinatorial Mathematics and Combinatorial Computing, 65 (2008) 139–145], J. Combin. Math. Combin. Comput., 87 (2013), pp. 43–50. (Cited on pp. 931, 932, 936)
- [145] D. KUZIAK, J. A. RODRÍGUEZ-VELÁZQUEZ, AND I. G. YERO, *Computing the metric dimension of a graph from primary subgraphs*, Discussiones Math. Graph Theory, 37 (2017), pp. 273–293. (Cited on pp. 936, 937)
- [146] D. KUZIAK AND I. G. YERO, *Metric Dimension Related Parameters in Graphs: A Survey on Combinatorial, Computational and Applied Results*, preprint, <https://arxiv.org/abs/2107.04877>, 2021. (Cited on pp. 948, 949)
- [147] L. LAIRD, R. C. TILLQUIST, S. BECKER, AND M. E. LLADSER, *Resolvability of Hamming graphs*, SIAM J. Discrete Math., 34 (2020), pp. 2063–2081, <https://doi.org/10.1137/19M1274511>. (Cited on p. 935)
- [148] T. I. LEE, N. J. RINALDI, F. ROBERT, D. T. ODOM, Z. BAR-JOSEPH, G. K. GERBER, N. M. HANNETT, C. T. HARBISON, C. M. THOMPSON, I. SIMON, ET AL., *Transcriptional regulatory networks in Saccharomyces cerevisiae*, Science, 298 (2002), pp. 799–804. (Cited on p. 946)
- [149] C. S. LESLIE, E. ESKIN, AND W. S. NOBLE, *The spectrum kernel: A string kernel for SVM protein classification*, in Pacific Symposium on Biocomputing, Vol. 7, 2002, pp. 566–575. (Cited on p. 948)
- [150] V. I. LEVENSHTIN, *Binary codes capable of correcting deletions, insertions, and reversals*, Soviet Phys. Dokl., 10 (1966), pp. 707–710. (Cited on p. 935)
- [151] S. LI AND M. PILIPCZUK, *Hardness of metric dimension in graphs of constant treewidth*, in 16th International Symposium on Parameterized and Exact Computation (IPEC 2021), Dagstuhl Publishing, 2021, pp. 24:1–24:13. (Cited on p. 927)
- [152] B. LINDSTRÖM, *On a combinatorial detection problem I*, I. Magyar Tud. Akad. Mat. Kutató Int. Közl., 9 (1964), pp. 195–207. (Cited on p. 935)
- [153] A. Y. LOKHOV, M. MÉZARD, H. OHTA, AND L. ZDEBOROVÁ, *Inferring the origin of an epidemic with a dynamic message-passing algorithm*, Phys. Rev. E, 90 (2014), art. 012801. (Cited on p. 945)
- [154] P. MANUEL, R. BHARATI, I. RAJASINGH, AND M. M. CHRIS, *On minimum metric dimension of honeycomb networks*, J. Discrete Algorithms, 6 (2008), pp. 20–27. (Cited on pp. 931, 940)
- [155] P. MANUEL, B. RAJAN, I. RAJASINGH, AND M. C. MONICA, *Landmarks in torus networks*, J. Discrete Math. Sci. Cryptography, 9 (2006), pp. 263–271. (Cited on p. 932)
- [156] P. D. MANUEL, M. I. ABD-EL-BARR, I. RAJASINGH, AND B. RAJAN, *An efficient representation of Benes networks and its applications*, J. Discrete Algorithms, 6 (2008), pp. 11–19. (Cited on p. 932)
- [157] B. D. MCKAY AND A. PIPERNO, *Practical graph isomorphism*, II, J. Symb. Comput., 60 (2014), pp. 94–112. (Cited on p. 947)
- [158] K. MEAGHER AND R. F. BAILEY, *On the metric dimension of Grassmann graphs*, Discrete Math. Theoret. Comput. Sci., 13 (2011), pp. 97–104. (Cited on p. 932)
- [159] R. A. MELTER AND I. TOMESCU, *Metric bases in digital geometry*, Comput. Vision Graph. Image Process., 25 (1984), pp. 113–121. (Cited on pp. 923, 932)
- [160] R. MEUSEL, S. VIGNA, O. LEHMBERG, AND C. BIZER, *Graph structure in the web—revisited: A trick of the heavy tail*, in Proceedings of the 23rd International Conference on World Wide Web, ACM, 2014, pp. 427–432. (Cited on p. 931)
- [161] D. MITSCHKE AND J. RUÉ, *On the limiting distribution of the metric dimension for random forests*, European J. Combin., 49 (2015), pp. 68–89. (Cited on pp. 941, 942, 943)
- [162] N. MLADENović, J. KRATICA, V. KOVAČEVIĆ-VUJČIĆ, AND M. ČANGALOVIĆ, *Variable neighborhood search for metric dimension and minimal doubly resolving set problems*, European J. Oper. Res., 220 (2012), pp. 328–337. (Cited on pp. 928, 935)
- [163] C. MOORE AND M. E. NEWMAN, *Epidemics and percolation in small-world networks*, Phys. Rev. E, 61 (2000), pp. 5678–5682. (Cited on p. 945)
- [164] M. MOSCARINI, *Computing a metric basis of a 2-connected bipartite distance-hereditary graph*, Theoret. Comput. Sci., 804 (2020), pp. 186–206. (Cited on p. 932)
- [165] G. MURPHY, *A metric basis characterization of Euclidean space*, Pacific J. Math., 60 (1975), pp. 159–163. (Cited on p. 953)

- [166] M. E. NEWMAN, *Spread of epidemic disease on networks*, Phys. Rev. E, 66 (2002), art. 016128. (Cited on p. 945)
- [167] O. R. OELLERMANN, C. D. PAWLICK, AND A. STOKKE, *The metric dimension of Cayley digraphs of abelian groups*, Ars Combin., 81 (2006), pp. 97–111. (Cited on p. 938)
- [168] R. PASTOR-SATORRAS AND A. VESPIGNANI, *Epidemic spreading in scale-free networks*, Phys. Rev. Lett., 86 (2001), pp. 3200–3203. (Cited on p. 945)
- [169] P. C. PINTO, P. THIRAN, AND M. VETTERLI, *Locating the source of diffusion in large-scale networks*, Phys. Rev. Lett., 109 (2012), art. 068702. (Cited on p. 945)
- [170] N. PISANTI, *Recent Duplications in Genomes: A Graph Theory Approach*, Memoire de Diplôme d'Études Approfondies, 1998. (Cited on p. 935)
- [171] C. POISSON AND P. ZHANG, *The metric dimension of unicyclic graphs*, J. Combin. Math. Combin. Comput., 40 (2002), pp. 17–32. (Cited on p. 933)
- [172] B. S. PONOMARCHUK, *Metric dimension of metric transform and wreath product*, Carpathian Math. Publ., 11 (2019), pp. 418–421. (Cited on p. 953)
- [173] J. R. QUINLAN, *Induction of decision trees*, Machine Learning, 1 (1986), pp. 81–106. (Cited on p. 927)
- [174] B. RAJAN, I. RAJASINGH, J. A. CYNTHIA, AND P. MANUEL, *Metric dimension of directed graphs*, Internat. J. Comput. Math., 91 (2014), pp. 1397–1406. (Cited on p. 931)
- [175] B. RAJAN, I. RAJASINGH, AND P. MANUEL, *On minimum metric dimension of circulant networks*, J. Comput. Math. Sci., 1 (2010), pp. 155–162. (Cited on p. 932)
- [176] J. A. RODRÍGUEZ-VELÁZQUEZ, *Lexicographic metric spaces*, Appl. Anal. Discrete Math., 14 (2020), pp. 20–32. (Cited on p. 953)
- [177] J. A. RODRÍGUEZ-VELÁZQUEZ, *Corona metric spaces: Basic properties, universal lines, and the metric dimension*, AIMS Math., 7 (2022), pp. 13763–13776. (Cited on p. 953)
- [178] J. A. RODRÍGUEZ-VELÁZQUEZ, D. KUZIAK, I. G. YERO, AND J. M. SIGARRETA, *The metric dimension of strong product graphs*, Carpathian J. Math., 31 (2015), pp. 261–268. (Cited on p. 937)
- [179] P. RUTH, *Numerical Encoding of Symbolic Data: Standard, State of the Art, and New Techniques*, Undergraduate Honors Thesis, University of Colorado, 2021. (Cited on p. 935)
- [180] P. E. RUTH AND M. E. LADSER, *Levenshtein graphs: Resolvability, automorphisms & determining sets*, Discret. Math., 346 (2023), art. 113310. (Cited on pp. 935, 936, 953)
- [181] V. SAENPHOLPHAT AND P. ZHANG, *Conditional resolvability in graphs: A survey*, Int. J. Math. Math. Sci., 2004 (2004), pp. 1997–2017. (Cited on p. 948)
- [182] F. SALA, R. GABRYS, C. SCHOENY, AND L. DOLECEK, *Three novel combinatorial theorems for the insertion/deletion channel*, in 2015 IEEE International Symposium on Information Theory (ISIT), IEEE, 2015, pp. 2702–2706. (Cited on p. 935)
- [183] S. SAPUTRO, E. T. BASKORO, A. SALMAN, AND D. SUPRIJANTO, *The metric dimensions of a complete n -partite graph and its Cartesian product with a path*, J. Combin. Math. Combin. Comput., 71 (2009), pp. 283–293. (Cited on p. 932)
- [184] S. W. SAPUTRO, R. SIMANJUNTAK, S. UTTUNGGADEWA, H. ASSIYATUN, E. T. BASKORO, A. SALMAN, AND M. BAČA, *The metric dimension of the lexicographic product of graphs*, Discrete Math., 313 (2013), pp. 1045–1051. (Cited on p. 938)
- [185] U. SCHÖNING, *Graph isomorphism is in the low hierarchy*, J. Comput. System Sci., 37 (1988), pp. 312–323. (Cited on p. 946)
- [186] A. SEBÓ AND E. TANNIER, *On metric generators of graphs*, Math. Oper. Res., 29 (2004), pp. 383–393. (Cited on pp. 921, 949)
- [187] J. SEDLAR AND R. ŠKREKOVSKI, *Bounds on metric dimensions of graphs with edge disjoint cycles*, Appl. Math. Comput., 396 (2021), art. 125908. (Cited on p. 952)
- [188] D. SHAH AND T. ZAMAN, *Rumors in a network: Who's the culprit?*, IEEE Trans. Inform. Theory, 57 (2011), pp. 5163–5181. (Cited on p. 945)
- [189] B. SHANMUKHA, B. SOORYANARAYANA, AND K. HARINATH, *Metric dimension of wheels*, Far East J. Appl. Math., 8 (2002), pp. 217–229. (Cited on pp. 923, 924, 931)
- [190] Z. SHEN, S. CAO, W.-X. WANG, Z. DI, AND H. E. STANLEY, *Locating the source of diffusion in complex networks by time-reversal backward spreading*, Phys. Rev. E, 93 (2016), art. 032301. (Cited on p. 945)
- [191] S. S. SHEN-ORR, R. MILO, S. MANGAN, AND U. ALON, *Network motifs in the transcriptional regulation network of Escherichia coli*, Nature Genetics, 31 (2002), pp. 64–68. (Cited on p. 946)
- [192] H. M. A. SIDDIQUI AND M. IMRAN, *Computing the metric dimension of wheel related graphs*, Appl. Math. Comput., 242 (2014), pp. 624–632. (Cited on p. 940)
- [193] P. J. SLATER, *Leaves of trees*, in Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory and Computing, Congress. Numer. XIV, Utilitas Mathematica,

- Winnipeg, 1975, pp. 549–559. (Cited on pp. 921, 927, 931, 944)
- [194] P. J. SLATER, *Domination and location in acyclic graphs*, Networks, 17 (1987), pp. 55–64. (Cited on p. 950)
- [195] P. J. SLATER, *Dominating and reference sets in a graph*, J. Math. Phys. Sci., 22 (1988), pp. 445–455. (Cited on p. 921)
- [196] S. SÖDERBERG AND H. S. SHAPIRO, *A combinatorial detection problem*, Amer. Math. Monthly, 70 (1963), pp. 1066–1070. (Cited on p. 934)
- [197] B. SOORYANARAYANA, S. KUNIKULLAYA, AND N. N. SWAMY, *Metric dimension of generalized wheels*, Arab J. Math. Sci., 25 (2019), pp. 131–144. (Cited on pp. 932, 940)
- [198] B. M. SPINELLI, E. CELIS, AND P. THIRAN, *Observer placement for source localization: The effect of budgets and transmission variance*, in 54th Annual Allerton Conference on Communication, Control, and Computing, IEEE, 2016, pp. 743–751. (Cited on pp. 944, 945, 946, 949)
- [199] F. STAHLBERG, *Discovering Vocabulary of a Language through Cross-Lingual Alignment*, Ph.D. thesis, Karlsruhe Institute of Technology, 2011. (Cited on p. 935)
- [200] G. SUDHAKARA AND H. A. KUMAR, *Graphs with metric dimension two—a characterization*, Adv. Appl. Discrete Math., 4 (2009), pp. 169–186. (Cited on p. 930)
- [201] R. C. TILLQUIST, *Low-Dimensional Embeddings for Symbolic Data Science*, Ph.D. thesis, University of Colorado, 2020. (Cited on pp. 941, 943, 950)
- [202] R. C. TILLQUIST, R. M. FRONGILLO, AND M. E. LLADSER, *Truncated Metric Dimension for Finite Graphs*, preprint, <https://arxiv.org/abs/2106.14314>, 2021. (Cited on pp. 950, 951)
- [203] R. C. TILLQUIST AND M. E. LLADSER, *Low-dimensional representation of genomic sequences*, J. Math. Biol., 79 (2019), pp. 1–29. (Cited on pp. 931, 932, 935, 944, 947, 948, 950)
- [204] R. C. TILLQUIST AND M. E. LLADSER, *Multilateration of Random Networks with Community Structure*, preprint, <https://arxiv.org/abs/1911.01521>, 2019. (Cited on pp. 941, 943, 950)
- [205] I. TOMESCU AND M. IMRAN, *Metric dimension and R-sets of connected graphs*, Graphs Combin., 27 (2011), pp. 585–591. (Cited on p. 940)
- [206] I. TOMESCU AND I. JAVAID, *On the metric dimension of the Jahangir graph*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 50 (2007), pp. 371–376. (Cited on p. 931)
- [207] M. TRUSOV, A. V. BODAPATI, AND R. E. BUCKLIN, *Determining influential users in internet social networks*, J. Marketing Res., 47 (2010), pp. 643–658. (Cited on p. 945)
- [208] L. R. VARSHNEY, J. KUSUMA, AND V. K. GOYAL, *On palimpsests in neural memory: An information theory viewpoint*, IEEE Trans. Molecular Biol. Multi-Scale Commun., 2 (2016), pp. 143–153. (Cited on p. 935)
- [209] T. VETRÍK, *The metric dimension of circulant graphs*, Canad. Math. Bull., 60 (2017), pp. 206–216. (Cited on p. 932)
- [210] C. VON DER MALSBURG, *Pattern recognition by labeled graph matching*, Neural Networks, 1 (1988), pp. 141–148. (Cited on p. 946)
- [211] K. WAGNER, *Über eine Eigenschaft der ebenen Komplexe*, Math. Ann., 114 (1937), pp. 570–590. (Cited on p. 930)
- [212] T. WASHIO AND H. MOTODA, *State of the art of graph-based data mining*, ACM SIGKDD Explorations Newsletter, 5 (2003), pp. 59–68. (Cited on p. 946)
- [213] I. WEGENER, *Complexity Theory: Exploring the Limits of Efficient Algorithms*, Springer Science & Business Media, 2005. (Cited on p. 946)
- [214] D. WEININGER, *Smiles, a chemical language and information system. 1. Introduction to methodology and encoding rules*, J. Chem. Inform. Comput. Sci., 28 (1988), pp. 31–36. (Cited on p. 946)
- [215] I. YERO, J. RODRIGUEZ-VELÁZQUEZ, AND D. KUZIAK, *Closed Formulae for the Metric Dimension of Rooted Product Graphs*, preprint, <https://arxiv.org/abs/1309.0641>, 2013. (Cited on p. 936)
- [216] I. G. YERO, D. KUZIAK, AND J. A. RODRÍGUEZ-VELÁZQUEZ, *On the metric dimension of corona product graphs*, Comput. Math. Appl., 61 (2011), pp. 2793–2798. (Cited on p. 936)
- [217] S. ZEJNLOVIĆ, D. MITSCHKE, J. GOMES, AND B. SINOPOLI, *Extending the metric dimension to graphs with missing edges*, Theoret. Comput. Sci., 609 (2016), pp. 384–394. (Cited on p. 946)
- [218] X. ZHANG, Y. ZHANG, T. LV, AND Y. YIN, *Identification of efficient observers for locating spreading source in complex networks*, Phys. A, 442 (2016), pp. 100–109. (Cited on p. 945)
- [219] X. ZHONG, F. HEINICKE, AND S. RAYNER, *miRBaseMiner, a tool for investigating miRBase content*, RNA Biol., 16 (2019), pp. 1534–1546. (Cited on p. 935)
- [220] N. ZUBRILINA, *On the edge dimension of a graph*, Discrete Math., 341 (2018), pp. 2083–2088. (Cited on p. 952)