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On Gilles Pisier's approach to Gaussian concentration, isoperimetry, and Poincaré-type inequalities*

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Abstract

We discuss a natural extension of Gilles Pisier's approach to the study of measure concentration, isoperimetry, and Poincaré-type inequalities. This approach allows one to explore counterparts of various results about Gaussian measures in the class of rotationally invariant probability distributions on Euclidean spaces, including multidimensional Cauchy measures.

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Contents

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1 G. Pisier's approach and its consequences

Let γ_n denote the standard Gaussian measure on $\mathbb{R}^n$, thus with density

$$\frac{d\gamma_n(x)}{dx} = (2\pi)^{-\frac{n}{2}}\, e^{-|x|^2/2}, \quad x \in \mathbb{R}^n,$$

with respect to the Lebesgue measure, where $|\cdot|$ stands for the canonical Euclidean norm. In the mid 1980's G. Pisier [18] proposed the following remarkable family of integro-differential inequalities involving this measure.

Theorem 1.1. *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a convex function. For any smooth function f on $\mathbb{R}^n$ with gradient ∇f,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi(f(y)-f(x))\, d\gamma_n(x)\, d\gamma_n(y) \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(x), y\rangle\Big)\, d\gamma_n(x)\, d\gamma_n(y). \tag{1.1}$$

In particular, if f has γ_n-mean zero, then

$$\int_{\mathbb{R}^n} \Psi(f)\, d\gamma_n \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(x), y\rangle\Big)\, d\gamma_n(x)\, d\gamma_n(y). \tag{1.2}$$

This result was actually stated in a more general setting of Gaussian measures on locally convex spaces, which is readily suggested by the dimension-free character of (1.1). What is also surprising, this inequality admits a rather simple proof. For reader's convenience, we include it below with simplications due to B. Maurey (according to [18]).

Proof. Given independent random vectors X and Y in $\mathbb{R}^n$ with distribution γ_n, put $X(t) = X\cos t + Y \sin t$ for $0 \le t \le \frac{\pi}{2}$. By the Leibniz formula,

$$\Delta \equiv f(Y) - f(X) = \int_0^{\pi/2} \frac{d}{dt} f(X(t))\, dt = \int_0^{\pi/2} \langle \nabla f(X(t)), X'(t)\rangle\, dt,$$

where $X'(t) = -X\sin t + Y\cos t$. Hence, by Jensen's inequality,

$$\Psi(\Delta) \le \frac{2}{\pi}\int_0^{\pi/2} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(X(t)), X'(t)\rangle\Big)\, dt.$$

Taking the expectation, we get

$$\mathbb{E}\,\Psi(\Delta) \le \frac{2}{\pi}\int_0^{\pi/2} \mathbb{E}\,\Psi\Big(\frac{\pi}{2}\,\langle \nabla f(X(t)), X'(t)\rangle\Big)\, dt. \tag{1.3}$$

Now, a crucial point is that, since the Gaussian measure $\gamma_{2n} = \gamma_n \otimes \gamma_n$ is rotationally invariant on $\mathbb{R}^{2n}$, the couple $(X(t), X'(t))$ represents an independent copy of (X, Y). In particular, the second expectation in (1.3) in the integrand does not depend on t. $\square$

The relations (1.1)-(1.2) have several important consequences. Choosing $\Psi(r) = |r|^p$ with $p \geq 1$, we get a Poincaré-type inequality for L^p-norms in Gauss space $(\mathbb{R}^n, \gamma_n)$. Namely,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p \, d\gamma_n(x)\, d\gamma_n(x) \, \leq \, c_p \int_{\mathbb{R}^n} |\nabla f|^p \, d\gamma_n, \tag{1.4}$$

and, by Jensen's inequality,

$$\int_{\mathbb{R}^n} \Big|f - \int_{\mathbb{R}^n} f\, d\gamma_n\Big|^p \, d\gamma_n \, \leq \, c_p \int_{\mathbb{R}^n} |\nabla f|^p \, d\gamma_n \tag{1.5}$$

with

$$c_p = \Big(\frac{\pi}{2}\Big)^p \, \mathbb{E}\,|\xi|^p = \Big(\frac{\pi}{2}\Big)^p \, \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\, \Gamma\Big(\frac{p+1}{2}\Big),$$

where ξ is a normal random variable with distribution γ_1. Such relations are known to hold for general measures as a consequence of the particular case $p = 1$ when the latter is true – however, with constants growing like $(cp)^p$ for large p (where $c > 0$ is an absolute constant). In the present formulation, c_p is of the order $(cp)^{p/2}$ which is asymptotically correct.

While the best constants c_p are unknown except for $p = 1, 2$, the constant $c_1 = \sqrt{\frac{\pi}{2}}$ is sharp. In this case, (1.4)-(1.5) are equivalent to the Cheeger-type isoperimetric inequality

$$\gamma_n^+(\partial A) \geq \sqrt{\frac{2}{\pi}}\, \min\{\gamma_n(A), 1 - \gamma_n(A)\} \quad (A \subset \mathbb{R}^n \text{ Borel}),$$

where γ_n^+ denotes the Gaussian perimeter. This relation becomes an equality for any half-space A in $\mathbb{R}^n$ whose boundary contains the origin.

Another choice $\Psi(r) = e^r$ leads in (1.2) to the exponential bound

$$\mathbb{E}\, e^{f(X)} \leq \mathbb{E}\, e^{\frac{\pi^2}{8}\, |\nabla f(X)|^2} \quad \big(\mathbb{E} f(X) = 0\big), \tag{1.6}$$

which is one of the ways to express the Gaussian dimension-free concentration phenomenon. Later in [6], the constant $\frac{\pi^2}{8}$ was removed from the exponent by applying the logarithmic Sobolev inequality. In particular, if the function f is 1-Lipschitz, (1.6) implies that the random variable $f(X)$ is subgaussian, i.e. $\mathbb{E}\exp\{cf(X)^2\} \leq 2$ for some constant $c > 0$. See also [15] for further refinements.

2 Extension to general measures. Cauchy distributions

The integration in (1.1) is carried out with respect to the Gaussian measure $\gamma_{2n} = \gamma_n \otimes \gamma_n$ on $\mathbb{R}^n \times \mathbb{R}^n$. This inspires a natural idea to explore the applicability of Pisier's approach to general positive measures μ on $\mathbb{R}^n \times \mathbb{R}^n$. Recently, motivated by the Enflo problem, Ivanisvili, van Handel and Volberg [14] have found a dimension-free analogue of Pisier's inequality (1.2) on the discrete cube $\{-1, 1\}^n$ for the Bernoulli distribution and for functions f with values in an arbitrary Banach space. Alternatively, we directly follow Pisier's argument and introduce the following transform, without requiring that the measures have a product structure.

Definition 2.1. *Given a (positive) measure μ on $\mathbb{R}^n \times \mathbb{R}^n$, denote by μ_t the image of μ under the orthogonal linear transformation*

$$U_t(x,y) = (x \cos t + y \sin t, -x \sin t + y \cos t), \quad x, y \in \mathbb{R}^n,$$

from $\mathbb{R}^n \times \mathbb{R}^n$ into itself (i.e. U_t pushes forward μ onto μ_t). We call the measure on $\mathbb{R}^n \times \mathbb{R}^n$

$$\widehat{\mu} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mu_t \, dt \tag{2.1}$$

a spherical cap for μ.

For example, a spherical cap for γ_{2n} is γ_{2n} itself. Moreover, $\widehat{\mu} = \mu$ as long as μ is rotationally invariant, although such measures lose the product structure in the non-Gaussian case. As for the general situation, since $\widehat{\mu}$ appears as an average over a family of orthogonal transforms of μ, it is "more round" and becomes closer to the class of spherically invariant measures. If μ is the Lebesgue measure restricted to a convex body $K \subset \mathbb{R}^n \times \mathbb{R}^n$, the measure $\widehat{\mu}$ will be supported on a larger set $\widehat{K}$ and will partially dominate μ, in particular, on the Euclidean ball inscribed in K.

Theorem 1.1 has the following extension. Without loss of generality (and in order to avoid integrability questions), we assume that the convex function Ψ is non-negative.

Theorem 2.2. *Given a positive measure* μ *on* $\mathbb{R}^n \times \mathbb{R}^n$ *with a spherical cap* $\widehat{\mu}$*, for any smooth function* f *on* $\mathbb{R}^n$*,*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi(f(y) - f(x))\, d\mu(x,y) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(u), v\rangle\Big)\, d\widehat{\mu}(u,v). \tag{2.2}$$

In particular, this holds true with $\widehat{\mu} = \mu$ if μ is rotationally invariant. We discuss the transform $\mu \to \widehat{\mu}$ and a similar transformation for densities in Section 3. Under certain isotropy-type conditions, the right integral in (2.2) can be simplified for the quadratic function $\Psi(r) = r^2$, which leads to weighted Poincaré-type inequalities (Section 4). Moreover, L^p-Poincaré-type inequalities with arbitrary $p \geq 1$ can be obtained from (2.2) for the class of rotationally invariant measures. The example of the uniform distribution on the sphere is discussed in Section 5.

We also illustrate this kind of applications on the example of Cauchy (also called Student) probability distributions. Recall that the n-dimensional Cauchy measure $\mathfrak{m}_{n,\alpha}$ on $\mathbb{R}^n$ of order $\alpha > \frac{n}{2}$ has density proportional to $(1+|x|^2)^{-\alpha}$, $x \in \mathbb{R}^n$. We will remind of basic properties and identities related to this class of probability distributions in Section 6. Here let us only mention that, similarly to the Gaussian case, if the couple of random vectors (X,Y) in $\mathbb{R}^{2n}$ has a Cauchy distribution of order $\alpha > n$, then both X and Y have a Cauchy distribution on $\mathbb{R}^n$ of order $\alpha - \frac{n}{2}$. As a Cauchy-type analog of (1.4), we have:

Theorem 2.3. *Let* $\alpha > n + \frac{1}{2}$ *and* $1 \leq p < 2\,(\alpha - n)$*. For any smooth function* f *on* $\mathbb{R}^n$*,*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|^p\, d\mathfrak{m}_{2n,\alpha}(x,y) \,\leq\, C\Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p\, d\mathfrak{m}_{n,\beta}(x), \tag{2.3}$$

where $\beta = \alpha - \frac{n+p}{2}$*, and where the constant depends on* (n,p,α) *and is given by*

$$C = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\frac{p+1}{2})\Gamma(\alpha - n - \frac{p}{2})}{\Gamma(\alpha - n)}.$$

In particular, for $p = 2$ and with $\beta = \alpha - \frac{n}{2} - 1$, (2.3) takes the form

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|^2\, d\mathfrak{m}_{2n,\alpha}(x,y) \,\leq\, \frac{\pi^2}{8\,(\alpha - n - 1)} \int_{\mathbb{R}^n} |\nabla f|^2\, d\mathfrak{m}_{n,\beta}. \tag{2.4}$$

The right integral in (2.3) may be written over the measure $\mathfrak{m}_{n,\alpha}$ with the weight function $(1+|x|^2)^{\frac{n+p}{2}}$ using a different constant in place of C. Hence, this relation may be viewed as a weighted Poincaré-type inequality for L^p-norms.

If α is sufficiently large, the constants in (2.3)-(2.4) do not exceed a multiple of $1/\alpha$, which is a correct growth rate. Moreover, consider the image $\widetilde{\mathfrak{m}}_{2n,\alpha}$ of $\mathfrak{m}_{2n,\alpha}$ under the linear map $(x,y) \to \sqrt{2\alpha}\,(x,y)$ and the image $\widetilde{\mathfrak{m}}_{n,\beta}$ of $\mathfrak{m}_{n,\beta}$ under the map $x \to \sqrt{2\alpha}\,x$, which represent the probability measures with densities

$$d\widetilde{\mathfrak{m}}_{2n,\alpha}(x,y) = \frac{(2\alpha)^{-n}}{c_{2n,\alpha}} \left(1 + \frac{|x|^2 + |y|^2}{2\alpha}\right)^{-\alpha} dx\,dy,$$

$$d\widetilde{\mathfrak{m}}_{n,\beta}(x) = \frac{(2\alpha)^{-n/2}}{c_{n,\beta}} \left(1 + \frac{|x|^2}{2\alpha}\right)^{-\beta} dx$$

for suitable normalization constants $c_{2n,\alpha}$, $c_{n,\beta}$ (see (6.3)). In terms of these measures, then the inequality (2.3) becomes

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)-f(y)|^p\, d\widetilde{\mathfrak{m}}_{2n,\alpha}(x,y) \,\le\, C\,(2\alpha)^{\frac{p}{2}} \left(\frac{\pi}{2}\right)^p \int_{\mathbb{R}^n} |\nabla f|^p\, d\widetilde{\mathfrak{m}}_{n,\beta}. \tag{2.5}$$

In the limit as $\alpha \to \infty$, $\widetilde{\mathfrak{m}}_{2n,\alpha} \to \gamma_{2n}$ and $\widetilde{\mathfrak{m}}_{n,\beta} \to \gamma_n$ in the weak topology (and actually in total variation distance), while $C\,(2\alpha)^{\frac{p}{2}} \to 2^{p/2}\pi^{-1/2}\,\Gamma((p+1)/2)$. Hence, (2.5) recovers Pisier's Poincare-type inequality (1.4). In this sense Cauchy's distributions after a proper normalization represent a pre-Gaussian model, which may be used to recover various relations for the Gaussian measure.

On the other hand, this class of probability distributions is of independent interest and has been intensively studied in the literature. In particular, inequalities such as (2.4), i.e. (1.4) for $p = 2$, were considered with respect to the product measure $\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}$ on the left-hand side, and with weight $1 + |x|^2$ over $\mathfrak{m}_{n,\alpha}$ on the right-hand side, cf. e.g. [7], [8]. For comparison, several results in this direction will be discussed at the end of this paper. One should mention here that the $2n$-dimensional Cauchy measure $\mathfrak{m}_{2n,\alpha}$ is rather close to the product $\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}$ for the growing parameter α. In particular,

$$\mathfrak{m}_{2n,\alpha} \ge d\,\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}, \quad d = d_{n,\alpha} = \frac{\Gamma(\alpha - \frac{n}{2})^2}{\Gamma(\alpha-n)\,\Gamma(\alpha)}. \tag{2.6}$$

For example, $d \ge \frac{1}{2}$ for $\alpha \ge n^2$.

Of a special interest in (2.3) is the case $p = 1$, when this inequality yields

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)-f(y)|\, d\mathfrak{m}_{2n,\alpha}(x,y) \,\le\, \frac{\sqrt{\pi}}{\sqrt{\alpha-n}} \int_{\mathbb{R}^n} |\nabla f|\, d\mathfrak{m}_{n,\beta} \tag{2.7}$$

with $\alpha \ge n+1$ and $\beta = \alpha - \frac{n+1}{2}$. Being applied to (nearly) indicator functions, (2.7) together with (2.6) lead to the Cheeger-type isoperimetric-type inequality for the Cauchy measures. Let us recall that the ν-perimeter, or the outer Minkowski content for a (Borel) probability measure ν on $\mathbb{R}^n$ is defined for all Borel sets A in $\mathbb{R}^n$ by

$$\nu^+(\partial A) = \liminf_{\varepsilon\to 0} \frac{\nu(A_\varepsilon) - \nu(A)}{\varepsilon}, \tag{2.8}$$

where A_ε ($\varepsilon > 0$) denotes an open ε-neighborhood of A for the Euclidean distance.

Corollary 2.4. *Let $\beta \ge \frac{n+1}{2}$ and $\beta^* = \beta + \frac{n+1}{2}$. For any Borel set A in $\mathbb{R}^n$,*

$$\mathfrak{m}_{n,\beta}^+(\partial A) \ge c\,\mathfrak{m}_{n,\beta^*}(A)(1 - \mathfrak{m}_{n,\beta^*}(A)), \tag{2.9}$$

where one may take $c = \frac{d}{\sqrt{\pi}}\sqrt{2\beta}$ with $d = d_{n,\beta^*}$. Here $c \ge \frac{1}{\sqrt{2\pi}}\sqrt{\beta}$, if $\beta \ge n^2$.

Usually, the isoperimetric problem is aimed to estimate the perimeter $\nu^+(\partial A)$ via the "size" $\nu(A)$. Here, however, for the lower bound on $\mathfrak{m}_{n,\beta}^+(\partial A)$ we use a different Cauchy measure $\mathfrak{m}_{n,\beta^*}$ with a good isoperimetric profile function. In particular, if $\mathfrak{m}_{n,\beta^*}(A) = \frac{1}{2}$, the $\mathfrak{m}_{n,\beta}$-perimeter of A is bounded from below by a large quantity for large values of β. This is consistent with the isoperimetric inequality (1.6) for the Gaussian measure.

Finally, let us emphasize a concentration aspect behind the Poincaré-type inequality (2.3) generalizing the dimension free concentration phenomenon for the Gaussian measure. The latter can be stated as the property that, for any function f on $\mathbb{R}^n$ with Lipschitz semi-norm $\|f\|_{\rm Lip} \leq 1$, the function $f(x) - f(y)$ is subgaussian under γ_{2n}, i.e.

$$\gamma_{2n}\big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f(y)| \geq t\big\} \leq 2\,e^{-ct^2}, \quad t > 0,$$

with some absolute constant $c > 0$. A similar assertion continues to hold on a bounded t-interval for the Cauchy measures after the natural normalization of the space variable. To be more precise, consider the probabilities

$$p_{n,\alpha}(t) = \mathfrak{m}_{2n,\alpha}\big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \sqrt{\alpha - n}\,|f(x) - f(y)| \geq t\big\}.$$

Corollary 2.5. *If $\alpha \geq n+1$, for any function f on $\mathbb{R}^n$ with $\|f\|_{\rm Lip} \leq 1$,*

$$p_{n,\alpha}(t) \leq \begin{cases} 2\,\exp\big\{-t^2/14\big\}, & 0 \leq t \leq t_0, \\ 2\,\exp\big\{-(t\log t)/5\big\}, & t_0 \leq t \leq t_1, \\ 2\left(\frac{2t_0}{t}\right)^{t_1}, & t \geq t_1, \end{cases} \tag{2.10}$$

where $t_0 = \sqrt{\alpha - n}$ and $t_1 = \alpha - n$.

Analogous relations for similar regions have been explored in [7] for the product measures $\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}$. In particular, a subgaussian deviation inequality was derived there by means of a certain weighted logarithmic Sobolev inequality over $m_{n,\alpha}$ like in (2.10) for the interval $0 \leq t \leq t_0$ (which cannot be extended to the whole half-axis). The intermediate range $t_0 \leq t \leq t_1$ was studied in [7] on the basis of the weighted Poincaré-type inequality for the Cauchy measure (with the canonical power $p = 2$). However, this led to a large deviation bound with an exponential decay, while in (2.10) we deal with Poissonian-type tails. Note also that, in view of (2.6), from (2.10) we obtain analogous bounds

$$\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}\big\{\sqrt{\alpha - n}\,|f(x) - f(y)| \geq t\big\} \leq \begin{cases} 4\,\exp\big\{-t^2/14\big\}, & 0 \leq t \leq t_0, \\ 4\,\exp\big\{-(t\log t)/5\big\}, & t_0 \leq t \leq t_1, \\ 4\left(\frac{2t_0}{t}\right)^{t_1}, & t \geq t_1, \end{cases} \tag{2.11}$$

provided that $\alpha \geq n^2$.

3 Spherical caps of measures and associated transforms

First, let us employ Pisier's argument and prove Theorem 2.2. Suppose that we are given a positive Borel measure μ on $\mathbb{R}^n \times \mathbb{R}^n$ (in general, not necessarily finite). Given a smooth function f on $\mathbb{R}^n$, we consider fluctuations of $\Delta = f(y) - f(x)$ under μ. Introduce the same path $x(t) = x\cos t + y\sin t$, $0 \leq t \leq \frac{\pi}{2}$, connecting x with y, i.e., with $x(0) = 0$, $x(\pi/2) = y$. Again,

$$\Delta = \int_0^{\pi/2} \frac{d}{dt} f(x(t))\,dt = \int_0^{\pi/2} \langle \nabla f(x(t)), x'(t)\rangle\,dt.$$

Hence, given a convex non-negative function Ψ, by Jensen's inequality,

$$\Psi(\Delta) \leq \frac{2}{\pi}\int_0^{\pi/2} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(x(t)), x'(t)\rangle\Big)\,dt,$$

and after integration, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi(\Delta)\,d\mu \leq \frac{2}{\pi}\int_0^{\pi/2}\int_{\mathbb{R}^{2n}} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(x(t)), x'(t)\rangle\Big)\,d\mu\,dt. \tag{3.1}$$

As in Definition 2.1, denote by μ_t the image of μ under the orthogonal linear transformation $U_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ given by

$$U_t(x,y) = (u,v) = (x(t), x'(t)) = (x\cos t + y\sin t, -x\sin t + y\cos t).$$

Then using the change of variables formula, (3.1) may be rewritten as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi(\Delta)\,d\mu \leq \frac{2}{\pi}\int_0^{\pi/2}\int_{\mathbb{R}^{2n}} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(u), v\rangle\Big)\,dt\,d\mu_t(u,v),$$

which is the desired relation (2.2) for the spherical cap $\widehat{\mu}$ defined in (2.1). □

Let us state (2.2) once more on a functional level, assuming that μ has density $w(x,y)$ with respect to the Lebesgue measure. From the definition of μ_t it follows that, for any bounded measurable function g on $\mathbb{R}^n \times \mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(u,v)\,d\mu_t(u,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(u,v)\,w(u\cos t - v\sin t, u\sin t + v\cos t)\,du\,dv.$$

Hence, according to Definition 2.1, the spherical cap $\widehat{\mu}$ has density

$$(Uw)(u,v) = \widehat{w}(u,v) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} w(u\cos t - v\sin t, u\sin t + v\cos t)\,dt. \tag{3.2}$$

As a result, we obtain the transform $Uw = \widehat{w}$, which may be extended by this formula as a linear positive operator on $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Theorem 2.2 has therefore the following functional counterpart.

Theorem 3.1. *Let $\Psi : \mathbb{R} \to [0,\infty)$ be a convex function. Given a smooth function f on $\mathbb{R}^n$, for any non-negative Borel measurable function $w(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi(f(y)-f(x))\,w(x,y)\,dx\,dy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\,\langle \nabla f(u), v\rangle\Big)\,\widehat{w}(u,v)\,du\,dv.$$

In particular, for any $p \geq 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)-f(x)|^p\,w(x,y)\,dx\,dy \leq \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\,\langle \nabla f(u), v\rangle\,|^p\,\widehat{w}(u,v)\,du\,dv.$$

Let us stress how the operator U defined in (3.2) is acting on the Lebesgue spaces $L^p = L^p(\mathbb{R}^n \times \mathbb{R}^n, dx\,dy)$ with standard norms $\|\cdot\|_p$.

Proposition 3.2. *The linear operator U defined in* (3.2) *is acting as a contraction from L^p to L^p for any $p \in [1,\infty]$. Moreover, it represents a unitary operator on L^2.*

Proof. For each t, the linear mapping

$$T_t(u,v) = (u\cos t - v\sin t, u\sin t + v\cos t), \quad u,v \in \mathbb{R}^n, \tag{3.3}$$

pushes forward the Lebesgue measure λ_{2n} on $\mathbb{R}^n \times \mathbb{R}^n$ to λ_{2n}. Hence, the first claim immediately follows from (3.3) and the triangle inequality:

$$\|Uw\|_p \leq \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \|w(T_t)\|_p\,dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \|w\|_p\,dt = \|w\|_p.$$

For the second one, let us recall that the equation $T_t(u,v) = (x,y)$ is solved as

$$\begin{aligned}(u,v) &= T_t^{-1}(x,y) = U_t(x,y)\\ &= (x\cos t + y\sin t, -x\sin t + y\cos t) = R(T_t(y,x)), \quad x,y \in \mathbb{R}^n,\end{aligned}$$

where $R(\xi,\eta) = (\eta,\xi)$, $\xi,\eta \in \mathbb{R}^n$, is the reflection around the main diagonal in the space $\mathbb{R}^n \times \mathbb{R}^n$. Hence, for all complex-valued functions $f, g \in L^2$,

$$\begin{aligned}\langle Uf, g\rangle_{L^2} &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \bigg[\int f(T_t(u,v))\,\bar{g}(u,v)\,du\,dv\bigg]\,dt\\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \bigg[\int f(x,y)\,\bar{g}(T_t^{-1}(x,y))\,dx\,dy\bigg]\,dt\\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \bigg[\int f(x,y)\,\bar{g}(R(T_t(y,x)))\,dx\,dy\bigg]\,dt\\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \bigg[\int f(R(x,y))\,\bar{g}(T_t(y,x))\,dx\,dy\bigg]\,dt\\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \bigg[\int f(y,x)\,\bar{g}(T_t(y,x))\,dx\,dy\bigg]\,dt = \langle f, Ug\rangle_{L^2}.\end{aligned}$$

The latter means that the operator U is unitary. □

Let us illustrate Definiton 2.1 on two examples for the dimension $n = 1$, i.e., when μ is supported on the plane $\mathbb{R}^2$.

Examples. 1. For the symmetric Bernoulli measure $\nu = \frac{1}{2}\,\delta_1 + \frac{1}{2}\,\delta_{-1}$ assigning the mass $1/2$ to the points ± 1, consider its product with itself

$$\mu = \nu \otimes \nu = \frac{1}{4}\left(\delta_{(1,1)} + \delta_{(1,-1)} + \delta_{(-1,1)} + \delta_{(-1,-1)}\right),$$

which is the Bernoulli measure on the discrete square $\{-1,1\} \times \{-1,1\}$. It should be clear from (2.1) that $\widehat{\mu}$ represents the normalized Lebesgue measure on the circle $\sqrt{2}\,S^1 \subset \mathbb{R}^2$ of radius $\sqrt{2}$.

2. For a continous counterpart of the previous example, let ν be the uniform distribution on the interval $[-1,1]$ (the normalized Lebegue measure) and let $\mu = \nu \otimes \nu$ be the uniform distribution on the square $[-1,1] \times [-1,1]$, thus with density $w(x,y) = 1/4$. Then the spherical cap $\widehat{\mu}$ will be supported on the disc $x^2 + y^2 < 2$. Moreover, tedious computations show that it has density

$$\widehat{w}(u,v) = \begin{cases} \frac{\pi}{4}, & \text{if} \quad u^2+v^2 \le 1,\\ \arcsin\frac{1}{\sqrt{u^2+v^2}} - \frac{\pi}{4}, & \text{if} \quad 1 \le u^2+v^2 \le 2,\\ 0, & \text{if} \quad u^2+v^2 \ge 2.\end{cases}$$

Note that in both examples, the spherical cap is a spherically invariant measure.

4 Isotropic measures

One important particular case in Theorem 2.2 is the bound

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)-f(x)|^2\,d\mu(x,y) \le \frac{\pi^2}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle \nabla f(u), v\rangle^2\,d\widehat{\mu}(u,v). \tag{4.1}$$

In order to simplify the right integral, some additional assumptions about the measures μ or $\widehat{\mu}$ are needed. A good possible property is isotropy. A finite Borel measure λ on $\mathbb{R}^n$

is called isotropic, if

$$\int \langle \theta, v\rangle^2 \, d\lambda(v) = \sigma^2 |\theta|^2 \quad \text{for all} \;\; \theta \in \mathbb{R}^n$$

with some finite σ^2 ($\sigma \geq 0$) independent of θ, called the isotropic constant of λ. In this case,

$$\sigma^2 = \frac{1}{n} \int |v|^2 \, d\lambda(v). \tag{4.2}$$

This definition is consistent with the one in Convex Geometry, when λ represents the Lebesgue measure restricted to a symmetric convex body. The body is then called isotropic.

Any rotationally invariant measure on $\mathbb{R}^n$ is isotropic. In particular, when λ represents the normalized Lebesgue measure on the sphere $rS^{n-1} \subset \mathbb{R}^n$ of radius $r > 0$ with center at the origin, then $\sigma^2 = r^2/n$ according to (4.2). Intuitively, the isotropy means that in some sense the measure λ is "round" and is not dilated in any direction. On the other hand, this is a normalization type condition. By simple algebra, any measure with finite second moment becomes isotropic after some linear transformation. Note that an orthogonal transformation of an isotropic measure is isotropic.

When λ is a probability measure on $\mathbb{R}^n$, and a random vector $X = (X_1, \dots, X_n)$ is distributed according to λ, the property of being isotropic is equivalent to the non-correlatedness of coordinates and the requirement that all coordinates have equal L^2-norms: $\mathbb{E} X_i X_j = \sigma^2 \delta_{ij}$.

Now, given a finite Borel measure $d\nu(u,v)$ on $\mathbb{R}^n \times \mathbb{R}^n$, let π denote its projection to the "first" coordinate u, which is the marginal of ν defined by

$$\pi(A) = \nu(A \times \mathbb{R}^n), \quad A \subset \mathbb{R}^n \text{ (Borel)}.$$

According to the general Measure Theory (cf. [19], [9]), for the partition $\mathbb{R}^n \times \mathbb{R}^n = \bigcup_{u \in \mathbb{R}^n} \{u\} \times \mathbb{R}^n$, there exists a (unique) family of finite measures ν_u defined for π-almost all u, called conditional measures, such that each ν_u is supported on $\{u\} \times \mathbb{R}^n$ and

$$\nu = \int_{\mathbb{R}^n} \nu_u \, d\pi(u). \tag{4.3}$$

Moreover, if ν is a probability measure, then π and ν_u are probability measures as well.

It will be more convenient to consider $\mathbb{R}^n$ as the space for the support of ν_u rather then the slice $\{u\} \times \mathbb{R}^n$, so that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(u,v) \, d\nu(u,v) = \int_{\mathbb{R}^n} \bigg[\int_{\mathbb{R}^n} g(u,v) \, d\nu_u(v) \bigg] \, d\pi(u) \tag{4.4}$$

for any ν-integrable function g on $\mathbb{R}^n \times \mathbb{R}^n$. Equality (4.3) thus provides a canonical representation for ν in the case of the partition of the $2n$-space along the first coordinate. With this convention, if ν has density $q(u,v)$, then the marginal measure π has density $u \mapsto c_u = \int q(u,v) \, dv$, and ν_u has density

$$q_u(v) = \frac{1}{c_u} \, q(u,v) \quad \text{for} \;\; c_u > 0.$$

Definition 4.1. *Let us say that a finite measure ν on $\mathbb{R}^n \times \mathbb{R}^n$ is isotropic along the first coordinate, if π-almost all conditional measures ν_u in the canonical representation* (4.3) *are isotropic on $\mathbb{R}^n$. Equivalently, for π-almost all u,*

$$\int_{\mathbb{R}^n} \langle \theta, v\rangle^2 \, d\nu_u(v) = \sigma^2(u) |\theta|^2 \quad (\theta \in \mathbb{R}^n)$$

with some finite $\sigma^2(u)$, which we call the isotropic function of ν along the first coordinate.

According to (4.2), in that case the isotropic function is given by

$$\sigma^2(u) = \frac{1}{n} \int_{\mathbb{R}^n} |v|^2 \, d\nu_u(v). \tag{4.5}$$

One can now return to Theorem 2.2, assuming that $\nu = \widehat{\mu}$ is isotropic along the first coordinate, with the isotropic function $\widehat{\sigma}^2(u)$. Using (4.4), on the right-hand side of (4.1) we then deal with the integral

$$\begin{aligned}\int_{\mathbb{R}^n \times \mathbb{R}^n} \langle \nabla f(u), v\rangle^2 \, d\widehat{\mu}(u,v) &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \langle \nabla f(u), v\rangle^2 \, d\widehat{\mu}_u(v)\right] d\pi(u) \\ &= \int_{\mathbb{R}^n} \widehat{\sigma}^2(u)\, |\nabla f(u)|^2 \, d\pi(u).\end{aligned}$$

As a result, we arrive in (4.1) at the weighted Poincaré-type inequality.

A similar conclusion can also be made in the case $\Psi(r) = |r|^p$ in Theorem 2.2 with $1 \le p \le 2$, which leads to the weighted Cheeger-type inequality. Assume that μ is a probability measure, so that $\nu = \widehat{\mu}$ and ν_u are probability measures as well. Then, using (4.4) and applying Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\, \langle \nabla f(u), v\rangle\Big)\, d\widehat{\mu}(u,v) &= \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |\langle \nabla f(u), v\rangle|^p \, d\widehat{\mu}_u(v)\right] d\pi(u) \\ &\le \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \langle \nabla f(u), v\rangle^2 \, d\widehat{\mu}_u(v)\right)^{p/2} d\pi(u) \\ &= \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n} \widehat{\sigma}^p(u)\, |\nabla f(u)|^p \, d\pi(u).\end{aligned}$$

Let us summarize.

Corollary 4.2. *Suppose that, for a probability measure μ on $\mathbb{R}^n \times \mathbb{R}^n$, the spherical cap $\widehat{\mu}$ is an isotropic measure on $\mathbb{R}^n \times \mathbb{R}^n$ along the first coordinate with the isotropic function $\widehat{\sigma}^2$. Let π be the first marginal of $\widehat{\mu}$. For any smooth function f on $\mathbb{R}^n$, and any $1 \le p \le 2$,*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(y) - f(x)|^p \, d\mu(x,y) \le \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n} \widehat{\sigma}^p(u)\, |\nabla f(u)|^p \, d\pi(u).$$

Corollary 4.2 might not be true for $p > 2$ if nothing is known about the measure beyond isotropy of the measure $\widehat{\mu}$. Indeed, isotropy only guarantees the finiteness of the 2nd moment for linear functionals, while their p-th moments might be infinite.

5 Spherically invariant measures

A natural generalization of Pisier's result is Theorem 2.2 with measures μ that are spherically (rotationally) invariant on $\mathbb{R}^n \times \mathbb{R}^n$. In the absolutely continuous case, this means that μ has a density

$$w(x,y) = w(r) = w\big(\sqrt{|x|^2 + |y|^2}\big),$$

depending only on the norm $r = |(x,y)|$. Since the transforms U_t and T_t are orthogonal, we have $|U_t(x,y)| = |T_t(x,y)| = |(x,y)|$. So, $\mu_t = \mu$ for all t and thus $\widehat{\mu} = \mu$ similarly to the Gaussian measures $\mu = \gamma_n \otimes \gamma_n$. Let us emphasize this particular case in (2.2) once more.

Theorem 5.1. *Let a positive Borel measure μ on $\mathbb{R}^n \times \mathbb{R}^n$ be rotationally invariant. Then, for any smooth function f on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi(f(y) - f(x))\, d\mu(x,y) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\, \langle \nabla f(u), v\rangle\Big)\, d\mu(u,v).$$

In particular,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(y) - f(x)|^2\, d\mu(x,y) \leq \frac{\pi^2}{4} \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle \nabla f(u), v\rangle^2\, d\mu(u,v). \tag{5.1}$$

All rotationally invariant measures are isotropic along the first coordinate, so Corollary 4.2 is applicable, and one can evaluate explicitly the involved marginal π and the isotropic function σ^2 which serves as a weight (since $\widehat{\mu} = \mu$, we may omit the hat-sign).

Let us illustrate these relations on the example of the uniform distribution on the sphere. In the sequel, we denote by

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\,\Gamma(\frac{n}{2})} \tag{5.2}$$

the volume of the unit ball $B_n = \{x \in \mathbb{R}^n : |x|^2 \leq 1\}$, and then $s_{n-1} = n\omega_n$ describes the surface measure of the unit sphere S^{n-1}. Thus, let $\mu = \sigma_{2n-1}$, $n \geq 2$, be the normalized Lebesgue measure on the $(2n-1)$-dimensional unit sphere, which may also be defined as the normalized restriction of the Hausdorff measure of dimension $2n-1$ to the sphere

$$S^{2n-1} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x|^2 + |y|^2 = 1\}.$$

Every section

$$S_x^{2n-1} = \big\{y \in \mathbb{R}^n : (x,y) \in S^{2n-1}\big\} = \sqrt{1 - |x|^2}\; S^{n-1} \quad (x \in \mathbb{R}^n,\ |x| < 1)$$

represents the sphere in $\mathbb{R}^n$ of radius $\sqrt{1-|x|^2}$ with center at the origin. In this case, the conditional measure ν_x in (4.3)-(4.4) for $\nu = \sigma_{2n-1}$ represents the uniform distribution on S_x^{2n-1}. Therefore, by (4.5), the isotropic function for μ is given by

$$\sigma^2(x) = \frac{1}{n} \int_{S_x^{2n-1}} |y|^2\, d\nu_x(y) = \frac{1 - |x|^2}{n}, \quad |x| < 1. \tag{5.3}$$

Now, let us describe the coresponding marginal distribution

$$\pi(A) = \sigma_{2n-1}\big(S^{2n-1} \cap (A \times \mathbb{R}^n)\big), \quad A \subset \mathbb{R}^n \text{ (Borel)}.$$

It is supported on the unit ball B_n, where it has a density of the form $q(x) = q(|x|)$, i.e. π is also rotationally invariant. To find q, let μ_ε, $\varepsilon > 0$, denote the uniform distribution in the region

$$D_\varepsilon = \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : 1 < \sqrt{|x|^2 + |y|^2} < 1 + \varepsilon\big\}.$$

Then $\mu_\varepsilon \to \sigma_{2n-1}$ weakly as $\varepsilon \to 0$, and a similar convergence holds true for the marginal distributions of these measures, π_ε and π. Moreover, the density $q_\varepsilon(x)$ of π_ε is convergent to $q(x)$ on the unit ball $|x| < 1$. Denoting by mes_n the Lebesgue measure on $\mathbb{R}^n$, for any Borel set $A \subset B_n$ we have by Fubini's theorem

$$\pi_\varepsilon(A) = \frac{1}{\mathrm{mes}_{2n}(D_\varepsilon)} \int_{D_\varepsilon \cap (A \times \mathbb{R}^n)} dx\, dy = \frac{1}{\mathrm{mes}_{2n}(D_\varepsilon)} \int_A dx \int_{\{y \in \mathbb{R}^n : (x,y) \in D_\varepsilon\}} dy.$$

Therefore, for all the points $|x| < 1$ we have

$$\begin{aligned} q_\varepsilon(x) &= \frac{1}{\mathrm{mes}_{2n}(D_\varepsilon)}\, \mathrm{mes}_n\big\{y \in \mathbb{R}^n : 1 < \sqrt{|x|^2 + |y|^2} < 1 + \varepsilon\big\} \\ &= \frac{\omega_n}{\omega_{2n}\,((1+\varepsilon)^{2n} - 1)} \Big(\big((1+\varepsilon)^2 - |x|^2\big)^{\frac{n}{2}} - \big(1 - |x|^2\big)^{\frac{n}{2}}\Big). \end{aligned}$$

Let $u(\varepsilon)$ denote the expression in the last brackets. We have $u(0) = 0$ and $u'(0) = n(1 - |x|^2)^{\frac{n}{2}-1}$. Since also $(1+\varepsilon)^{2n} - 1 = 2n\varepsilon + O(\varepsilon^2)$, it follows that

$$q_\varepsilon(x) = \frac{\omega_n}{2\omega_{2n}} \big(1 - |x|^2\big)^{\frac{n}{2}-1} + O(\varepsilon)$$

and therefore

$$q(x) = \frac{\omega_n}{2\omega_{2n}} \big(1 - |x|^2\big)^{\frac{n}{2}-1}, \quad |x| < 1.$$

To describe the coefficient explicitly, one may refer to the formula (5.2). As a consequence, we conclude that the marginal distribution of σ_{2n-1} is a probability measure on the unit ball B_n in $\mathbb{R}^n$ with density

$$\frac{d\pi(u)}{du} = \frac{\Gamma(n)}{\pi^{\frac{n}{2}}\,\Gamma(\frac{n}{2})} \big(1 - |u|^2\big)^{\frac{n}{2}-1}, \quad |u| < 1. \tag{5.4}$$

Hence, recalling formula (5.3) for the isotropic function, we can apply and state Corollary 4.2 in a more explicit form, choosing the value $p = 2$.

Corollary 5.2. *For any smooth function f on $\mathbb{R}^n$,*

$$\int_{S^{2n-1}} |f(x) - f(y)|^2\, d\sigma_{2n-1}(x,y) \ \le\ \frac{\pi^2}{4n} \int_{|u|<1} |\nabla f(u)|^2\, (1 - |u|^2)\, d\pi(u), \tag{5.5}$$

where the probability measure π is given in (5.4)*.*

Let us see what kind of concentration is hidden in (5.5). As easy to check,

$$\int_{|u|<1} (1 - |u|^2)^{p-1}\, du = \pi^{\frac{n}{2}}\, \frac{\Gamma(p)}{\Gamma(\frac{n}{2} + p)}, \quad p > 0.$$

We use this identity for $p = \frac{n}{2} + 1$. If $|\nabla f| \le 1$, the last integral in (5.5) does not exceed

$$\int_{|u|<1} (1 - |u|^2)\, d\pi(u) = \frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\, \frac{\Gamma(p)}{\Gamma(\frac{n}{2} + p)} = \frac{1}{2}.$$

Hence, by (5.5),

$$\int_{S^{2n-1}} |f(x) - f(y)|^2\, d\sigma_{2n-1}(x,y) \ \le\ \frac{\pi^2}{8n}.$$

A similar conclusion can be made on the basis of the Poincaré inequality on the sphere S^{2n-1}, when it is applied to the function $f(x) - f(y)$.

The relation (5.5) is rather similar to the Poincaré-type inequality for the measure π,

$$\int_{|x|<1} \int_{|y|<1} |f(x) - f(y)|^2\, d\pi(x)\, d\pi(y) \ \le\ \frac{c}{n} \int_{|u|<1} |\nabla f(u)|^2\, d\pi(u),$$

which was derived in [3] in a more general setting of log-concave, spherically symmetric probability measures on $\mathbb{R}^n$.

6 Background on Cauchy measures

Let us recall basic definitions and facts about the multidimensional Cauchy distributions.

Definition 6.1. *The n-dimensional Cauchy measure $\mathfrak{m}_{n,\alpha}$ on $\mathbb{R}^n$ of order $\alpha > \frac{n}{2}$ has density*

$$w_{n,\alpha}(x) = \frac{1}{c_{n,\alpha}}\,(1+|x|^2)^{-\alpha}, \quad x \in \mathbb{R}^n, \tag{6.1}$$

where $c_{n,\alpha}$ is a normalizing constant such that $\mathfrak{m}_{n,\alpha}(\mathbb{R}^n) = 1$.

To see that the condition $\alpha > \frac{n}{2}$ is necessary in order to obtain a probability measure, one may integrate in polar coordinates:

$$\int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^\alpha} = n\omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^\alpha}\,dr. \tag{6.2}$$

The latter integral is finite if and only if $\alpha > \frac{n}{2}$. This will be always assumed when speaking about Cauchy measures on $\mathbb{R}^n$.

Changing the variable $r = \sqrt{s}$ and then $s = \frac{1}{t} - 1$, the last integral in (6.2) is equal to

$$\begin{aligned} \frac{1}{2}\int_0^\infty \frac{s^{\frac{n}{2}-1}}{(1+s)^\alpha}\,ds &= \frac{1}{2}\int_0^1 t^{\alpha-\frac{n}{2}-1}\,(1-t)^{\frac{n}{2}-1}\,dt \\ &= \frac{1}{2}\,B\Big(\alpha-\frac{n}{2},\frac{n}{2}\Big) = \frac{1}{2}\,\frac{\Gamma(\alpha-\frac{n}{2})\,\Gamma(\frac{n}{2})}{\Gamma(\alpha)}. \end{aligned}$$

Hence, the normalizing constant in is given by

$$c_{n,\alpha} = \frac{n\,\omega_n}{2}\,\frac{\Gamma(\alpha-\frac{n}{2})\,\Gamma(\frac{n}{2})}{\Gamma(\alpha)} = \pi^{\frac{n}{2}}\,\frac{\Gamma(\alpha-\frac{n}{2})}{\Gamma(\alpha)}. \tag{6.3}$$

Characterization by means of χ squared distribution. Write $\alpha = \frac{n+d}{2}$ for

$$d = 2\alpha - n > 0.$$

The probability measure $\mathfrak{m}_{n,\alpha}$ may be characterized as the distribution of the random vector $X = Z/\eta$, where $Z = (Z_1,\dots,Z_n)$ is a random vector in $\mathbb{R}^n$ with the standard Gaussian distribution γ_n, and where $\eta > 0$ is a random variable independent of Z and having the χ_d −distribution with d degrees of freedom. That is, η has density

$$\chi_d(r) = \frac{1}{2^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}\,r^{d-1}\,e^{-r^2/2}, \quad r > 0.$$

Hence, $\mathfrak{m}_{n,\alpha}$ represents the image of the product measure $\gamma_n \otimes \chi_d$ on $\mathbb{R}^n \times (0,\infty)$ under the map $(z,r) \to z/r$ (with some abuse, we denote by χ_d the measure with density $\chi_d(r)$). When d is integer, χ_d represents the distribution of the Euclidean norm $\eta = |V| = (\xi_1^2 + \dots + \xi_d^2)^{1/2}$ of a standard normal random vector $V = (\xi_1,\dots,\xi_d)$ in $\mathbb{R}^d$.

Essential support of $\mathfrak{m}_{n,\alpha}$. Note that $\mathbb{E}\,|Z|^2 = n$, while $\mathbb{E}\,|V|^2 = d$. Moreover, for large n and d, with high probability we have $\frac{n}{2} < |Z|^2 < 2n$ and $\frac{d}{2} < |V|^2 < 2d$ (as a consequence of the Gaussian concentration phenomenon). Hence, with high probability

$$\frac{n}{4d} < |X|^2 < \frac{4n}{d}.$$

In other words, the measure $\mathfrak{m}_{n,\alpha}$ is almost concentrated on the Euclidean ball with center at zero and radius R of order $\sqrt{n/d}$. If α is essentailly greater than $\frac{n}{2}$, then d is of order 2α, so that R is approximately $\sqrt{n/(2\alpha)}$.

Projections. The χ_d-characterization has a useful consequence about the images $\mathfrak{m}_{n,\alpha}T^{-1}$ of $\mathfrak{m}_{n,\alpha}$ under linear projections $T : \mathbb{R}^n \to H$ to k-dimensional linear subspaces H in $\mathbb{R}^n$ ($1 \le k \le n$). Since $\mathfrak{m}_{n,\alpha}$ is spherically invariant, it is sufficient to consider canonical projections from $\mathbb{R}^n$ to $\mathbb{R}^k$, that is, the maps

$$T(x_1, \dots, x_n) = (x_1, \dots, x_k), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Let $Z = (Z_1, \dots, Z_n)$ be a random vector in $\mathbb{R}^n$ with the standard Gaussian distribution, and η be a random variable independent of Z and having the χ_d −distribution with $d = 2\alpha - n$ degrees of freedom. Then the random vector $X = Z/\eta$ has distribution $\mathfrak{m}_{n,\alpha}$. Moreover,

$$T(X) = \frac{1}{\eta}\, T(Z) = \frac{1}{\eta}\, (Z_1, \dots, Z_k)$$

is of a similar form as X itself. Since $(Z_1, \dots, Z_k)$ is a standard normal random vector in $\mathbb{R}^k$, we conclude that the projection $\mathfrak{m}_{n,\alpha}T^{-1}$ represents the k-dimensional Cauchy measure of the order β such that $d = 2\beta - k$. That is, the k-dimensional projection of $\mathfrak{m}_{n,\alpha}$ is a Cauchy measure $\mathfrak{m}_{k,\beta}$ of order

$$\beta = \frac{d+k}{2} = \alpha - \frac{n-k}{2}. \tag{6.4}$$

Gaussian limit. If a random vector X has a Cauchy distribution $\mathfrak{m}_{n,\alpha}$, then the random vector $X_\alpha = \sqrt{2\alpha}\, X$ has density

$$\widetilde{w}_{n,\alpha}(x) = (2\alpha)^{-n/2}\, w_{n,\alpha}\Big(\frac{1}{\sqrt{2\alpha}}\, x\Big) = \frac{1}{c'_{n,\alpha}(1 + \frac{1}{2\alpha}\,|x|^2)^\alpha}, \quad x \in \mathbb{R}^n,$$

where $c'_{n,\alpha} = (2\alpha)^{n/2}\, c_{n,\alpha}$. This normalization is consistent with the size of the essential support of the Cauchy measure for large values of α (in which case the values of $|X_\alpha|$ do not exceed a muptiple of $\sqrt{n}$ in a full analogy with standard Gaussian random vectors). Indeed,

$$\widetilde{w}_{n,\alpha}(x) \to (2\pi)^{-n/2}\, e^{-|x|^2/2} \quad \text{as} \ \ \alpha \to \infty.$$

Therefore, the linear image $\widetilde{\mathfrak{m}}_{n,\alpha}$ of $\mathfrak{m}_{n,\alpha}$ under the map $x \to \sqrt{2\alpha}\, x$ approaches for growing α the standard Gaussian measure γ_n on $\mathbb{R}^n$ in total variation norm. This property also follows from the above χ_d-characterization and the convergence in distribution

$$\frac{|V|^2}{d} = \frac{\xi_1^2 + \dots + \xi_d^2}{d} \to 1 \quad \text{as} \ \ d \to \infty,$$

which is the weak law of large numbers for independent normal random variables ξ_i.

Thus, the class of Cauchy measures may serve as a pre-Gaussian model, in the sense that many properties of the Gaussian measure may be potentially obtained in the limit from similar ones for the Cauchy measures.

Moments of the Euclidean norm. Since the density of the Cauchy measure $\mathfrak{m}_{n,\alpha}$ on $\mathbb{R}^n$ decay polynomially at infinity, the Euclidean norm has finite L^p-norms for a certain range $p \ge 0$ only, namely, for $p < 2\alpha - n$. Integrating in polar coordinates and repeating the previous computations, we have

$$\int_{\mathbb{R}^n} |x|^p\, dm_\alpha(x) = \frac{1}{c_{n,\alpha}} \int_{\mathbb{R}^n} \frac{|x|^p}{(1+|x|^2)^\alpha}\, dx = \frac{\Gamma(\alpha - \frac{n+p}{2})\, \Gamma(\frac{n+p}{2})}{\Gamma(\alpha - \frac{n}{2})\, \Gamma(\frac{n}{2})}.$$

In particular, the second moment is finite if and only if $\alpha > \frac{n+2}{2}$, in which case

$$\int_{\mathbb{R}^n} |x|^2\, d\mathfrak{m}_{n,\alpha}(x) = \frac{n}{2\alpha - n - 2}, \quad \int_{\mathbb{R}^n} \langle x, \theta\rangle^2\, d\mathfrak{m}_{n,\alpha}(x) = \frac{|\theta|^2}{2\alpha - n - 2}.$$

This also shows that the $\sqrt{2\alpha}$-normalization in the definition of $\widetilde{\mathfrak{m}}_{n,\alpha}$ is natural, when α is essentialy larger than n.

7 Cauchy measures on $\mathbb{R}^n \times \mathbb{R}^n$. Proof of Theorem 2.3

According to the general definition (6.1) and the formulas (5.2)–(6.3), the $2n$-dimensional Cauchy measures $\mathfrak{m}_{2n,\alpha}$ on $\mathbb{R}^n \times \mathbb{R}^n$ have densities

$$w_{2n,\alpha}(x,y) = \frac{1}{c_{2n,\alpha}} \big(1 + |x|^2 + |y|^2\big)^{-\alpha}, \quad x, y \in \mathbb{R}^n, \tag{7.1}$$

where $\alpha > n$ is a parameter and where the normalizing constant is given by

$$c_{2n,\alpha} = n\,\omega_{2n}\,\frac{\Gamma(\alpha - n)\,\Gamma(n)}{\Gamma(\alpha)} = \pi^n\,\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)}. \tag{7.2}$$

As we know from (6.4) with $k = n$ and with n being replaced with $2n$, the marginal of $\mathfrak{m}_{2n,\alpha}$ represents the n-dimensional Cauchy measure $\mathfrak{m}_{n,\beta}$ of order $\beta = \alpha - \frac{n}{2}$. Equivalently, if the couple of random vectors (X, Y) in $\mathbb{R}^{2n}$ has a Cauchy distribution of order α, then both X and Y have a Cauchy distribution in $\mathbb{R}^n$ of order $\alpha - \frac{n}{2}$. This fact may be obtained directly by computation of the marginal density.

By Theorem 5.1 applied with $\Psi(t) = |t|^p$, $p \geq 1$, for any smooth function f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p\,d\mathfrak{m}_{2n,\alpha}(x,y) \leq \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\left<\nabla f(x), y\right>|^p\,d\mathfrak{m}_{2n,\alpha}(x,y). \tag{7.3}$$

In order to simplify the last integral, let us fix a real number $p \geq 0$, a vector $v = r\theta$, $r \geq 0$, $\theta \in S^{n-1}$, and consider the integrals

$$I_p(x,v) = \int_{\mathbb{R}^n} \frac{|\left<v, y\right>|^p}{(1 + |x|^2 + |y|^2)^\alpha}\,dy = r^p \int_{\mathbb{R}^n} \frac{|\left<\theta, y\right>|^p}{(1 + |x|^2 + |y|^2)^\alpha}\,dy. \tag{7.4}$$

In particular, the quantity $I_0(x) = I_0(x,v)$ corresponding to $p = 0$ does not depend on v. In this case, according to (7.1), the equality

$$\frac{d\nu(x)}{dx} = \frac{1}{c_{2n,\alpha}}\,I_0(x), \quad x \in \mathbb{R}^n, \tag{7.5}$$

defines a probability measure on $\mathbb{R}^n$ which is the image ν of $\mathfrak{m}_{2n,\alpha}$ under the projection $T(x_1,\dots,x_{2n}) = (x_1,\dots,x_n)$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$. So, ν is a Cauchy distribution on $\mathbb{R}^n$ of order $\alpha - \frac{n}{2}$.

We will recover this fact as part of a more general case $p \geq 0$ in (7.3). Put

$$G(n,p) = \frac{\mathbb{E}\,|\xi|^p}{\mathbb{E}\,|Z|^p}, \tag{7.6}$$

where ξ and Z are standard normal random vectors in $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

Lemma 7.1. *For any $\alpha > \frac{n+p}{2}$, $p \geq 0$, we have*

$$I_p(x,v) = I_p(x)\,|v|^p, \quad \text{where } I_p(x) = A\,(1 + |x|^2)^{-\beta}, \quad \beta = \alpha - \frac{n+p}{2}, \tag{7.7}$$

and

$$A = G(n,p)\,\frac{n\omega_n}{2}\,B\Big(\alpha - \frac{n+p}{2}, \frac{n+p}{2}\Big). \tag{7.8}$$

In particular,

$$I_0(x) = \frac{A}{(1 + |x|^2)^{\alpha - \frac{n}{2}}}.$$

This shows that the measure ν defined in (7.5) is a Cauchy distribution of order $\alpha - \frac{n}{2}$.

Proof. By the rotational invariance of the Lebesgue measure, the last integral in (7.4) does not depend on θ, so that

$$I_p(x) \equiv \mathbb{E}_\theta I_p(x,\theta)$$

for all $x \in \mathbb{R}^n$, where $\mathbb{E}_\theta$ means the integral (the mean) with respect to σ_{n-1}. Thus, for all θ,

$$I_p(x,\theta) = \int_{\mathbb{R}^n} \frac{\mathbb{E}_\theta |\langle \theta, y\rangle|^p}{(1+|x|^2+|y|^2)^\alpha}\,dy.$$

This expectation is actually a multiple of $|y|^p$ by the rotational invariance of σ_{n-1}, that is,

$$\mathbb{E}_\theta |\langle \theta, y\rangle|^p = G(n,p)\,|y|^p$$

with some constant $G(n,p)$. Taking $y = (1,0,\dots,0)$, we get $G(n,p) = \mathbb{E}_\theta |\theta_1|^p$. Moreover, let $Z = (Z_1,\dots,Z_n)$ be a standard normal vector in $\mathbb{R}^n$, so that $Z/|Z|$ is uniformly distributed in the unite sphere S^{n-1} and is independent of $|Z|$. Hence, we have the product of two independent random variables $Z_1 = \frac{Z_1}{|Z|}\,|Z|$, which gives

$$\mathbb{E}\,|Z_1|^p = \mathbb{E}\Big|\frac{Z_1}{|Z|}\Big|^p\,\mathbb{E}\,|Z|^p = \mathbb{E}_\theta|\theta_1|^p\,\mathbb{E}\,|Z|^p = G(n,p)\,\mathbb{E}\,|Z|^p.$$

Thus, the constant $G(n,p)$ is described according to (7.6). As a consequence,

$$\begin{aligned} I_p(x,v) &= G(n,p)\,|v|^p \int_{\mathbb{R}^n} \frac{|y|^p}{(1+|x|^2+|y|^2)^\alpha}\,dy \\ &= \frac{G(n,p)}{(1+|x|^2)^{\alpha-\frac{n+p}{2}}}\,|v|^p \int_{\mathbb{R}^n} \frac{|z|^p}{(1+|z|^2)^\alpha}\,dz. \end{aligned}$$

The last integral is finite if and only if $\alpha > \frac{n+p}{2}$, which is assumed. In this case, the last integral can be evaluated in polar coordinates: It is equal to

$$\begin{aligned} n\omega_n \int_0^\infty \frac{r^{n+p-1}}{(1+r^2)^\alpha}\,dr &= \frac{n\omega_n}{2}\int_0^\infty \frac{s^{\frac{n+p}{2}-1}}{(1+s)^\alpha}\,ds \\ &= \frac{n\omega_n}{2}\int_0^\infty t^{\alpha-\frac{n+p}{2}-1}\,(1-t)^{\frac{n+p}{2}-1}\,dt \\ &= \frac{n\omega_n}{2}\,B\Big(\alpha-\frac{n+p}{2},\frac{n+p}{2}\Big), \end{aligned}$$

where we changed the variable $r = \sqrt{s}$ and then $s = \frac{1}{t}-1$, and where B denotes the classical beta-function. As a result, we arrive at (7.7) with the constant A as in (7.8). □

Proof. (of Theorem 2.3.) Using (7.1) and applying Lemma 7.1, the second integral in (7.3) is simplified to

$$\begin{aligned} \frac{1}{c_{2n,\alpha}}\int_{\mathbb{R}^n} I_p(x,\nabla f(x))\,dx &= \frac{A}{c_{2n,\alpha}}\int_{\mathbb{R}^n} |\nabla f(x))|^p(1+|x|^2)^{-\beta}\,dx \\ &= A\,\frac{c_{n,\beta}}{c_{2n,\alpha}}\int_{\mathbb{R}^n} |\nabla f(x))|^p\,d\mathfrak{m}_{n,\beta}. \end{aligned}$$

Note that the probability measure $\mathfrak{m}_{n,\beta}$ is defined for $\beta = \alpha - \frac{n+p}{2} > \frac{n}{2}$, which forces $p < 2(\alpha-n)$ given by assumption in the statement. It remains to simplify the constant in front of the last integral. Recall that, according to (6.3) and (7.2),

$$\begin{aligned} c_{n,\beta} &= \frac{n\omega_n}{2}\,\frac{\Gamma(\beta-\frac{n}{2})\,\Gamma(\frac{n}{2})}{\Gamma(\beta)} = \pi^{\frac{n}{2}}\,\frac{\Gamma(\alpha-\frac{2n+p}{2})}{\Gamma(\alpha-\frac{n+p}{2})}, \\ c_{2n,\alpha} &= n\,\omega_{2n}\,\frac{\Gamma(\alpha-n)\,\Gamma(n)}{\Gamma(\alpha)} = \pi^n\,\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}. \end{aligned}$$

Hence

$$\frac{c_{n,\beta}}{c_{2n,\alpha}} = \frac{1}{\pi^{\frac{n}{2}}} \frac{\Gamma(\alpha - \frac{2n+p}{2})}{\Gamma(\alpha - \frac{n+p}{2})} \frac{\Gamma(\alpha)}{\Gamma(\alpha - n)}. \tag{7.9}$$

Next, in order to compute the constant $G(n,p)$ in (7.6), we use the formula (5.2) for ω_n in terms of the Gamma function and integrate in polar coordinates. Changing the variable $r = \sqrt{2s}$, we get that

$$\begin{aligned} \mathbb{E}\,|Z|^p &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |x|^p \, e^{-|x|^2/2}\, dx \;=\; \frac{n\omega_n}{(2\pi)^{n/2}} \int_0^\infty r^{n+p-1}\, e^{-r^2/2}\, dr \\ &= \frac{n\omega_n}{\pi^{n/2}}\, 2^{p/2} \int_0^\infty s^{\frac{n+p-1}{2}} e^{-s}\, ds \;=\; 2^{\frac{p}{2}}\, \frac{\Gamma(\frac{n+p}{2})}{\Gamma(\frac{n}{2})}. \end{aligned}$$

In particular, in dimension one

$$\mathbb{E}\,|\xi|^p \;=\; \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\, \Gamma\Big(\frac{p+1}{2}\Big).$$

It follows that

$$G(n,p) = \frac{\mathbb{E}\,|\xi|^p}{\mathbb{E}\,|Z|^p} = \frac{1}{\sqrt{\pi}}\, \frac{\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{n+p}{2})}.$$

Hence, according to (7.8),

$$A = G(n,p)\, \frac{n\omega_n}{2}\, B\Big(\alpha - \frac{n+p}{2}, \frac{n+p}{2}\Big) = \frac{\pi^{\frac{n}{2}}}{\sqrt{\pi}}\, \frac{\Gamma(\alpha - \frac{n+p}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\alpha)}.$$

Thus, using (7.7),

$$A\, \frac{c_{n,\beta}}{c_{2n,\alpha}} = \frac{1}{\sqrt{\pi}}\, \frac{\Gamma(\frac{p+1}{2})\Gamma(\alpha - \frac{2n+p}{2})}{\Gamma(\alpha - n)}.$$

□

8 Poincaré-type inequalities for L^1-norm and isoperimetry

In the important particular case $p = 1$, Theorem 2.3 is reduced to the following assertion about Cauchy measures on $\mathbb{R}^{2n}$ and $\mathbb{R}^n$.

Corollary 8.1. *Let $\alpha > n + \frac{1}{2}$ and $\beta = \alpha - \frac{n+1}{2}$. For any smooth function f on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|\, d\mathfrak{m}_{2n,\alpha}(x,y) \le \frac{\sqrt{\pi}}{2}\, \frac{\Gamma(\alpha - n - \frac{1}{2})}{\Gamma(\alpha - n)} \int_{\mathbb{R}^n} |\nabla f|\, d\mathfrak{m}_{n,\beta}. \tag{8.1}$$

In particular, for $\alpha \ge n + 1$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|\, d\mathfrak{m}_{2n,\alpha}(x,y) \le \sqrt{\pi}\, \frac{1}{\sqrt{\alpha - n}} \int_{\mathbb{R}^n} |\nabla f|\, d\mathfrak{m}_{n,\beta}. \tag{8.2}$$

To bound from above the constant in (8.1) by a simpler expression, one may use Wendel's inequality $\Gamma(x + \frac{1}{2}) \le \Gamma(x)\sqrt{x}$ ($x > 0$), which for $x \ge 1$ gives

$$\Gamma\Big(x - \frac{1}{2}\Big) = \frac{1}{x - \frac{1}{2}}\, \Gamma\Big(x + \frac{1}{2}\Big) \le \frac{\sqrt{x}}{x - \frac{1}{2}}\, \Gamma(x) \le \frac{2}{\sqrt{x}}\, \Gamma(x).$$

Applying this with $x = \alpha - n$, we get (8.2).

If α is sufficiently large, for example, $\alpha \ge 2n$, the constants in these inequalities do not exceed a multiple of $1/\sqrt{\alpha}$. Moreover, as was already explained in Section 2, after rescaling of the space variable and in the limit as $\alpha \to \infty$, (8.1) yields the L^1-Poincare-type inequality over the Gaussian measure

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|\, d\gamma_n(x)\, d\gamma_n(y) \ \le\ \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}^n} |\nabla f|\, d\gamma_n. \tag{8.3}$$

Here the constant is optimal and is attained asymptotically, when f approaches the indicator function of a half-space whose boundary passes through the origin.

Let us comment on the geometric meaning of Sobolev-type inequalities such as (8.1)-(8.3). One can prove the following general isoperimetric-type characterization, using the notion of the ν-perimeter $\nu^+(\partial A)$ defined in (2.8). Denote by $A^c = \mathbb{R}^n \setminus A$ the complement of the set A.

Lemma 8.2. *Let ν be an absolutely continuous probability measure on $\mathbb{R}^n$ and let μ be a finite measure on $\mathbb{R}^n \times \mathbb{R}^n$ which is invariant under the map $(x, y) \to (y, x)$. The following two assertions are equivalent:* $a)$ *For any smooth function f on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)|\, d\mu(x, y) \le \int_{\mathbb{R}^n} |\nabla f|\, d\nu. \tag{8.4}$$

$b)$ *For any closed set A in $\mathbb{R}^n$,*

$$\nu^+(\partial A) \ge 2\mu(A \times A^c). \tag{8.5}$$

Proof. The argument is standard, and we give it here for completeness. The property $a)$ can be equivalently stated for different classes of functions:

$a')$ The relation (8.4) holds for all C^∞-smooth functions f on $\mathbb{R}^n$ with a compact support;

$a'')$ The relation (8.4) holds for all locally Lipschitz functions f on $\mathbb{R}^n$ with the generalized modulus of gradient

$$|\nabla f(x)| = \limsup_{y\to x} \frac{|f(x) - f(y)|}{|x - y|}, \quad x \in \mathbb{R}^n.$$

The property of being locally Lipschitz means that f has a finite Lipschitz semi-norm in some neighborhood of any point in $\mathbb{R}^n$. In particular, f has to be continuous and a.e. differentiable (by Rademacher's theorem). Moreover, $|\nabla f(x)|$ is finite, Borel measurable, and coincides with the modulus of the usual gradient at every point x where f is differentiable.

It shoud be clear that $a'') \Rightarrow a) \Rightarrow a')$. Using a smoothing argument, these implications can be reversed; cf. [5], Proposition 5.4.1, for the proof of a similar statement about Poincaré-type inequalities for L^2-norms (at this point, the absolute continuity of ν is required).

For the implication $a'') \Rightarrow b)$, given a non-empty closed set A in $\mathbb{R}^n$, one may pick up functions $f_k : \mathbb{R}^n \to [0, 1]$ with finite Lipschitz semi-norm such that $f_k \to 1_A$ pointwise and

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} |\nabla f_k|\, d\mu \le \nu^+(\partial A)$$

(cf. Proposition 5.2.2 in [5]). Since, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} |f_k(x) - f_k(y)|\, d\mu(x, y) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |1_A(x) - 1_A(y)|\, d\mu(x, y) \\ &= \mu(A \times A^c) + \mu(A^c \times A) = 2\mu(A \times A^c), \end{aligned}$$

the desired inequality (8.5) follows.

Finally, to derive (8.4) from (8.5), assume that f is smooth and bounded. We may assume that $f \geq 0$, since the inequality (8.4) does not change when adding any constant to f. The sets $A_t = \{x \in \mathbb{R}^n : f(x) \geq t\}$, $t > 0$, are closed, so that (8.5) is applicable. Applying the co-area inequality (cf. Proposition 5.2.2 in [5]) together with Fubini's theorem, we then get

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f| \, d\nu &\geq \int_0^\infty \nu^+(A_t) \, dt \, \geq \, 2 \int_0^\infty \mu(A_t \times A_t^c) \, dt \\ &= \int_0^\infty \bigg[\int_{\mathbb{R}^n \times \mathbb{R}^n} |1_{A_t}(x) - 1_{A_t}(y)| \, d\mu(x,y) \bigg] \, dt \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg[\int_0^\infty |1_{A_t}(x) - 1_{A_t}(y)| \, dt \bigg] d\mu(x,y). \end{aligned}$$

Using the triangle inequality, we see that the last expression is greater than or equal to

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \Big| \int_0^\infty (1_{A_t}(x) - 1_{A_t}(y)) \, dt \Big| \, d\mu(x,y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| \, d\mu(x,y).$$

As a result, we arrive at (8.4). □

When μ is a multiple of a product measure, Lemma 8.2 is reduced to the next well-known characterization.

Lemma 8.3. *Let ν be an absolutely continuous probability measure on $\mathbb{R}^n$. Given $h > 0$, the following two assertions are equivalent:*

$a)$ *For any smooth function f on $\mathbb{R}^n$,*

$$h \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)| \, d\nu(x) \, d\nu(y) \leq \int_{\mathbb{R}^n} |\nabla f| \, d\nu. \tag{8.6}$$

$b)$ *For any closed set A in $\mathbb{R}^n$,*

$$\nu^+(\partial A) \geq 2h \, \nu(A)(1 - \nu(A)). \tag{8.7}$$

As an equivalent functional form for the isoperimetric inequality (8.7), one may also consider the Sobolev-type inequality

$$h \int_{\mathbb{R}^n} |f - m| \, d\nu \leq \int_{\mathbb{R}^n} |\nabla f| \, d\nu, \tag{8.8}$$

where $m = \int f \, d\nu$ is the ν-mean of f, and where the indicator functions $f = 1_A$ still play an extremal role in the asymptotic sense.

Both (8.6) and (8.8) are particular cases of Sobolev-type inequalities of the form

$$Lf \leq \int_{\mathbb{R}^n} |\nabla f| \, d\nu, \quad Lf = \sup_{g \in G} \int_{\mathbb{R}^n} fg \, d\nu, \tag{8.9}$$

for arbitrary families G of functions g on $\mathbb{R}^n$ such that the functional Lf is well-defined (at least, for bounded f). Indeed, (8.8) corresponds to (8.9) for the class G of all Borel measurable g with ν-mean zero such that $|g(x)| \leq h$ for all $x \in \mathbb{R}^n$. Similarly, (8.6) corresponds to (8.9) for the class G of functions g representable as

$$g(x) = \int (u(x,y) - u(y,x)) \, d\nu(y)$$

with arbitrary Borel measurable functions u such that $|u(x,y)| \leq h$, $x,y \in \mathbb{R}^n$. In this case,

$$\begin{aligned}\int fg\,d\nu &= \int f(x)\bigg[\int u(x,y)\,d\nu(y)\bigg]\,d\nu(x) - \int f(x)\bigg[\int u(y,x)\,d\nu(y)\bigg]\,d\nu(x)\\ &= \int f(x)\bigg[\int u(x,y)\,d\nu(y)\bigg]\,d\nu(x) - \int f(y)\bigg[\int u(x,y)\,d\nu(x)\bigg]\,d\nu(y)\\ &= \int\!\!\int (f(x)-f(y))\,u(x,y)\,d\nu(x)\,d\nu(y),\end{aligned}$$

where in the second last step we changed notations by replacing x with y and y with x, using also the property that the product measure $\nu \otimes \nu$ on $\mathbb{R}^n \times \mathbb{R}^n$ is invariant under the mapping $(x,y) \to (y,x)$. From this we obtain that

$$\sup_{g\in G} \int fg\,d\nu = h \int\!\!\int |f(x)-f(y)|\,d\nu(x)\,d\nu(y).$$

By the Rothaus theorem (cf. [20], [5]), the relation (8.9) holds true in the class of all smooth bounded f on $\mathbb{R}^n$ if and only if the isoperimetric inequality

$$\max\{L(1_A), L(-1_A)\} \leq \nu^+(A)$$

holds true in the class of all Borel (or equivalently, closed) sets A in $\mathbb{R}^n$. Hence, the indicator functions play an extremal role in (8.9). This characterization shows that the functional forms (8.6) and (8.8) are equivalent to the same isoperimetric inequality (8.7).

As a closely related, let us also mention an isoperimetric inequality of Cheeger-type

$$\nu^+(\partial A) \geq h' \min\{\nu(A), 1-\nu(A)\}, \tag{8.10}$$

in which the optimal value h' is called Cheeger's isoperimetric constant associated to the measure ν. A functional form of (8.10) is a Sobolev-type inequality

$$h' \int_{\mathbb{R}^n} |f-m|\,d\nu \leq \int_{\mathbb{R}^n} |\nabla f|\,d\nu, \tag{8.11}$$

where now $m = m(f)$ is a median of f under ν, that is, a real number such that

$$\nu\{f \leq m\} \geq \frac{1}{2}, \quad \nu\{f \geq m\} \geq \frac{1}{2}$$

(in general the median is not unique, but the left integral in (8.11) does not depend on the choice of m). The equivalence of (8.10) and (8.11) is a standard fact.

Comparing (8.10) with (8.7), it is clear that

$$h \leq h' \leq 2h.$$

Often, however, $h' = h$, including the Gaussian measure $\nu = \gamma_n$. In this case, by the isoperimetric theorem in Gauss space ([13], [21]), the perimeter $\gamma_n^+(\partial A)$ subject to $\gamma_n(A) = 1/2$ is minimized for any half-space $A_\theta = \{x \in \mathbb{R}^n : \langle x,\theta\rangle \leq 0\}$. But for such sets $\gamma_n^+(\partial A_\theta) = 1/\sqrt{2\pi}$, hence $h = h' = \sqrt{2/\pi}$. Thus, the constant $\sqrt{\pi/2}$ is optimal in the Sobolev-type inequality (8.3), which may also be stated as the isoperimetric inequality

$$\gamma_n^+(\partial A) \geq 2\sqrt{\frac{2}{\pi}}\,\gamma_n(A)(1-\gamma_n(A))$$

with an equality for all $A = A_\theta$.

9 Isoperimetric inequalities for Cauchy measures

Using Lemma 8.2, the Sobolev-type inequality (8.2) in Corollary 8.1 may be equivalently stated as an isoperimetric inequality.

Corollary 9.1. *Let $\alpha \geq n+1$ and $\beta = \alpha - \frac{n+1}{2}$. For any closed set A in $\mathbb{R}^n$,*

$$\mathfrak{m}_{n,\beta}^{+}(\partial A) \geq \frac{2\sqrt{\alpha - n}}{\sqrt{\pi}}\, \mathfrak{m}_{2n,\alpha}(A \times A^c). \tag{9.1}$$

In order to bring this relation to the form (8.7) as in Lemma 8.3 and thus prove Corollary 2.4, we need to bound the Cauchy measure $\mathfrak{m}_{2n,\alpha}$ from below in terms of the product of two Cauchy measures on $\mathbb{R}^n$. To this aim, let us derive inequality (2.6).

Lemma 9.2. *For any $\alpha \geq n+1$,*

$$\mathfrak{m}_{2n,\alpha} \geq d\, \mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}, \quad d = d_{n,\alpha} = \frac{\Gamma(\alpha - \frac{n}{2})^2}{\Gamma(\alpha - n)\,\Gamma(\alpha)}. \tag{9.2}$$

In particular, $d \geq \frac{1}{2}$ for $\alpha \geq n^2$.

Thus, if α is sufficiently large, the constant d may be chosen to be universal.

Proof. Inequality (9.2) can be stated as a pointwise comparison relation for densities of the involved measures. According to the definitions (6.1) and (7.1), the densities of $\mathfrak{m}_{2n,\alpha}$ and $\mathfrak{m}_{n,\alpha}$ satisfy, for all $x, y \in \mathbb{R}^n$,

$$\begin{aligned} w_{2n,\alpha}(x,y) &= \frac{1}{c_{2n,\alpha}}\, \frac{1}{(1+|x|^2+|y|^2)^\alpha} \\ &\geq \frac{1}{c_{2n,\alpha}}\, \frac{1}{((1+|x|^2)(1+|y|^2))^\alpha} = \frac{c_{n,\alpha}^2}{c_{2n,\alpha}}\, w_{n,\alpha}(x) w_{n,\alpha}(y). \end{aligned}$$

The last product of w-functions represents the density of the product measure $\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}$. In addition, according to (6.3) and (7.2),

$$\frac{c_{n,\alpha}^2}{c_{2n,\alpha}} = \frac{\Gamma(\alpha - \frac{n}{2})^2}{\Gamma(\alpha - n)\,\Gamma(\alpha)}.$$

This proves (9.2).

Our next task is to bound the last fraction. To this end, one may use Stirling's formula. Alternatively, one may appeal to the following statement proved in [3]: If a random variable $\eta > 0$ has a log-concave density $q(t)$ on $(0,\infty)$, then the function $\log \mathbb{E}\,\eta^x - x \log x$ is concave in $x \geq 0$. For example, if η has a standard exponential distribution with density $q(t) = e^{-t}$, $t > 0$, we have $\mathbb{E}\,\eta^x = \Gamma(x+1)$, so that the function $\log \Gamma(x+1) - x \log x$ is concave. Therefore,

$$\log \Gamma(x) = \psi(x-1) + u(x), \quad \psi(x) = x \log x, \;\; x \geq 1, \tag{9.3}$$

for some concave function $u(x)$. Introduce the operators

$$\Delta_h U(x) = U(x) - \frac{1}{2}\, U(x-h) - \frac{1}{2}\, U(x+h), \quad h \geq 0.$$

By Jensen's inequality, if $x - h \geq 1$, then, by (9.3),

$$\Delta_h \log \Gamma(x) \geq \Delta_h \psi(x-1).$$

In the interval $0 \le h < x$, the function $\varphi_x(h) = -\Delta_h \psi(x)$ has the first two derivatives

$$\begin{aligned}\varphi_x'(h) &= \frac{1}{2}\,\log(x+h) - \frac{1}{2}\,\log(x-h),\\ \varphi_x''(h) &= \frac{1}{2\,(x+h)} + \frac{1}{2\,(x-h)} \,\le\, \frac{1}{x-h}.\end{aligned}$$

Since $\varphi_x(0) = \varphi_x'(0) = 0$, Taylor's formula implies $\varphi_x(h) \le \frac{h^2}{2(x-h)}$. Hence

$$\Delta_h \log\Gamma(x) \ge -\varphi_{x-1}(h) \ge -\frac{h^2}{2(x-h-1)}.$$

Applying this with $x = \alpha - \frac{n}{2}$ and $h = \frac{n}{2}$, we get

$$\begin{aligned}\log\Gamma\Big(\alpha - \frac{n}{2}\Big) - \frac{1}{2}\,\log\Gamma(\alpha-n) - \frac{1}{2}\,\log\Gamma(\alpha) &\ge -\frac{n^2}{8\,(\alpha-n-1)}\\ &\ge -\frac{n^2}{8\,(n^2-n-1)} \,\ge\, -\frac{1}{2},\end{aligned}$$

since $\frac{n^2}{n^2-n-1} \le 4$ for $n \ge 2$ (and where we used the assumption $\alpha \ge n^2$). It follows that

$$\frac{\Gamma(\alpha-\frac{n}{2})^2}{\Gamma(\alpha-n)\,\Gamma(\alpha)} \ge e^{-1/2} > \frac{1}{2}. \tag{9.4}$$

In the case $n = 1$, we use the original assumption $\alpha \ge n+1$, implying $\alpha \ge 2$. We employ Gautschi's inequality which asserts in particular that

$$\sqrt{x} < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x+1}, \quad x > 0.$$

Here the left inequality was already used (as Wendel's inequality). Applying the right inequality with $x = \alpha - 1$, we have

$$\Gamma\Big(\alpha - \frac{1}{2}\Big) > \frac{1}{\sqrt{\alpha}}\,\Gamma(\alpha).$$

Hence for the ratio on the left-hand side in (9.4) we get

$$\frac{\Gamma(\alpha-\frac{1}{2})^2}{\Gamma(\alpha-1)\,\Gamma(\alpha)} > \frac{\Gamma(\alpha)}{\alpha\Gamma(\alpha-1)} = \frac{\alpha-1}{\alpha} \ge \frac{1}{2}.$$

□

Proof. (of Corollary 2.4.) Recall that $\beta^* = \beta + \frac{n+1}{2}$ with $\beta \ge \frac{n+1}{2}$. We apply Lemma 9.2 and Corollary 9.1 with $\alpha = \beta^*$, so that the condition $\alpha \ge n+1$ is fulfilled. By (9.2), the right-hand side of (9.1) is bounded from below by

$$\frac{2d}{\sqrt{\pi}}\sqrt{\alpha-n}\;\mathfrak{m}_{n,\alpha}(A)\,\mathfrak{m}_{n,\alpha}(\bar{A}).$$

Since $\alpha - n = \beta - \frac{n-1}{2} \ge \frac{1}{2}\,\beta$, the desired inequality (2.9) follows.

Also, the condition $\alpha \ge n^2$ as in Lemma 9.2 is fulfilled as long as $\beta \ge n^2$. In this case,

$$\frac{2d}{\sqrt{\pi}}\,\sqrt{\alpha-n} \ge \frac{1}{\sqrt{\pi}}\sqrt{\beta/2}.$$

□

10 Moderate and large deviations. Proof of Corollary 2.5

Let us return to Theorem 2.3 for the general parameter p, where it was assumed that $\alpha > n$ and $1 \leq p < 2\,(\alpha - n)$. It follows from (2.3) that, for any function f on $\mathbb{R}^n$ with Lipschitz semi-norm $\|f\|_{\rm Lip} \leq 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p\,d\mathfrak{m}_{2n,\alpha}(x,y) \ \leq\ \frac{1}{\sqrt{\pi}}\,\Big(\frac{\pi}{2}\Big)^p\,\frac{\Gamma(\frac{p+1}{2})\,\Gamma(\alpha-n-\frac{p}{2})}{\Gamma(\alpha-n)}. \tag{10.1}$$

This bound can be used to explore probabilities of moderate and large deviations of $f(x)-f(y)$ under the Cauchy measure $\mathfrak{m}_{2n,\alpha}$. To this aim, it is worthwile to realize how the expression on the right-hand side of (10.1) grows with respect to the growing parameter p. We employ the following two-sided bound for the Gamma function proposed by Batir [1]: For any $x \geq \frac{1}{2}$,

$$\sqrt{2e}\,\Big(\frac{x}{e}\Big)^x \leq \Gamma\Big(x+\frac{1}{2}\Big) \leq \sqrt{2\pi}\,\Big(\frac{x}{e}\Big)^x. \tag{10.2}$$

Here the constants $\sqrt{2e}$ and $\sqrt{2\pi}$ are optimal for the indicated x-range and are attained for $x = \frac{1}{2}$ on the left and $x = \infty$ on the right-hand side. Let us add that the right inequality continues to hold for $0 \leq x \leq \frac{1}{2}$. Indeed, the Gamma function is convex, while $(\frac{x}{e})^x$ is decreasing in $0 \leq x \leq 1$. Hence, one only needs to check (10.2) for the points $x = 0$ and $x = \frac{1}{2}$, but then the right inequality in (10.2) is evident.

As a consequence, whenever $x \geq \frac{1}{2}$ and $x \geq h \geq 0$,

$$\begin{aligned}\frac{\Gamma(x-h+\frac{1}{2})}{\Gamma(x+\frac{1}{2})} \ &\leq\ \sqrt{\frac{\pi}{e}}\,e^h\,\frac{(x-h)^{x-h}}{x^x}\\ &=\ \sqrt{\frac{\pi}{e}}\,\Big(1-\frac{h}{x}\Big)^{x-h}\,\Big(\frac{e}{x}\Big)^h \ \leq\ \sqrt{\frac{\pi}{e}}\,\Big(\frac{e}{x}\Big)^h.\end{aligned}$$

Applying this with $x = \alpha-n-\frac{1}{2}$, $h = \frac{p}{2}$, and assuming that $\alpha-n \geq 1$ and $1 \leq p \leq 2(\alpha-n)-1$, we get

$$\frac{\Gamma(\alpha-n-\frac{p}{2})}{\Gamma(\alpha-n)} \ \leq\ \sqrt{\frac{\pi}{e}}\,\Big(\frac{e}{\alpha-n-\frac{1}{2}}\Big)^{p/2}.$$

In addition, by (10.2) once more,

$$\Gamma\Big(\frac{p+1}{2}\Big) \leq \sqrt{2\pi}\,\Big(\frac{p}{2e}\Big)^{p/2}.$$

Hence, the right-hand side of (10.1) can be bounded from above by

$$\frac{1}{\sqrt{\pi}}\,\Big(\frac{\pi}{2}\Big)^p\,\sqrt{2\pi}\,\Big(\frac{p}{2e}\Big)^{p/2}\,\sqrt{\frac{\pi}{e}}\,\Big(\frac{e}{\alpha-n-\frac{1}{2}}\Big)^{p/2} < 2\,\Big(\frac{\frac{\pi^2}{8}\,p}{\alpha-n-\frac{1}{2}}\Big)^{p/2}.$$

To simplify the last expression, one may use $\alpha-n-\frac{1}{2} \geq \frac{1}{2}\,(\alpha-n)$. We can summarize.

Corollary 10.1. *Let $\alpha \geq n+1$ and $1 \leq p \leq 2\,(\alpha-n)-1$. For any function f on $\mathbb{R}^n$ with $\|f\|_{\rm Lip} \leq 1$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p\,d\mathfrak{m}_{2n,\alpha}(x,y) \ \leq\ 2\,\Big(\frac{cp}{\alpha-n}\Big)^{p/2} \tag{10.3}$$

with $c = \pi^2/4$.

Proof. (of Corollary 2.5.) Recall the notations for the points $t_0 = \sqrt{\alpha - n}$ and $t_1 = \alpha - n$. By Markov's inequality, for any $t > 0$ and the range of (n, p) as in Corollary 10.1, from (10.3) we get

$$p_{n,\alpha}(t) = \mathfrak{m}_{2n,\alpha}\big\{\sqrt{\alpha - n}\,|f(x) - f(y)| \geq t\big\} \leq 2\left(\frac{cp}{t^2}\right)^{p/2}. \tag{10.4}$$

Choosing here $p = t^2/(ce)$, the right-hand side becomes $2\,e^{-p/2} = 2\,e^{-t^2/(2ce)} \leq 2\,e^{-t^2/14}$, which is applicable for $c_0^2 \leq t^2 < ce\,(2(\alpha - n) - 1)$ with $c_0 = \sqrt{ce}$. Thus, for this interval

$$p_{n,\alpha}(t) \leq 2\,e^{-t^2/14}. \tag{10.5}$$

This inequality continues to hold for $t \leq c_0$, since $2\,e^{-c_0^2/14} > 1$, while the left probability in (10.5) does not exceed 1. Also, the right endpoint of the interval may be simplified using $2(\alpha - n) - 1 \geq \alpha - n$ and $ce > 1$. As a result, the subgaussian deviation inequality (10.5), that is, (2.10) holds true in the interval $0 \leq t \leq t_0$.

Next, for the interval $t_0 \leq t \leq t_1$ we choose $p = t/c$ in (10.4) which yields

$$p_{n,\alpha}(t) \leq 2\,e^{-(t\log t)/2c}. \tag{10.6}$$

The requirement $p \leq 2\,(\alpha - n) - 1$ is then fulfilled, by the choice of t_1, while the condition $p \geq 1$ means that $t \geq c$. But in the case $0 \leq t \leq c$, (10.6) holds automatically, since the right-hand side is minimized for $t = c$, when it is equal to $2/\sqrt{c} > 1$. Since $2c > 5$, we arrive in (10.6) at the second (Poissonian) bound in (2.10).

Finally, for the range $t \geq t_1$, we choose $p = \alpha - n = t_1$ in (10.4) and use $\sqrt{c} < 2$. $\square$

If additionally $\alpha \geq n^2$, then, by Lemma 9.2, $\mathfrak{m}_{2n,\alpha} \geq \frac{1}{2}\,\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}$, which gives (2.11) as a consequence of (2.10).

11 Convexity of Cauchy measures

The class of Cauchy measures have been intensively studied in the literature in the framework of convex (hyperbolic) measures. Let us recall several results in this direction.

A probability measure ν on $\mathbb{R}^n$ is called κ-concave, where $-\infty \leq \kappa \leq \infty$, if it satisfies the Brunn-Minkowski-type inequality

$$\nu(tA + (1-t)B) \geq \big(t\nu(A)^\kappa + (1-t)\nu(B)^\kappa\big)^{1/\kappa} \tag{11.1}$$

for all $t \in (0,1)$ and for all Borel measurable sets $A, B \subset \mathbb{R}^n$ with positive measure. Here $tA + (1-t)B = \{tx + (1-t)y : x \in A, y \in B\}$ stands for the Minkowski sum of the two sets.

Since the right-hand side of (11.1) is non-decreasing in κ, the class of κ-concave measure is getting smaller for the growing parameter κ and contains only δ-measure in the limit case $\kappa = \infty$. Otherwise, necessarily $\kappa \leq 1$. The Lebesgue measure and its restriction to convex bodies in $\mathbb{R}^n$ are κ-concave with $\kappa = \frac{1}{n}$. This is the content of the Brunn-Minkowski-Lyusternik theorem. If $\kappa = 0$, we obtain the important class of log-concave measures, in which case (11.1) takes the form

$$\nu(tA + (1-t)B) \geq \nu(A)^t \nu(B)^{1-t}.$$

The largest class corresponding to the other limit value $\kappa = -\infty$ is described by the Brunn-Minkowski-type inequality

$$\nu(tA + (1-t)B) \geq \min\{\nu(A), \nu(B)\}.$$

Such measures are called convex (following Borell) or hyperbolic (according to V. Milman).

A characterization of κ-concave measures was given by Borell [12]. For simplicity suppose that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Then any convex measure should be supported on an open convex set $\Omega \subset \mathbb{R}^n$ where it has a continuous density. Moreover, the log-concavity of ν is equivalent to the property that its density is log-concave on Ω. In the case $\kappa < 0$, ν is κ-concave if and only if it has density of the form

$$w(x) = \frac{d\nu(x)}{dx} = V(x)^{-p}, \quad \kappa = -\frac{1}{p-n}\ (p \geq n),$$

for some convex function V on Ω.

The n-dimensional Cauchy measure $\nu = \mathfrak{m}_{n,\alpha}$, $\alpha > \frac{n}{2}$, corresponds to the convex function $V(x) = \sqrt{1+|x|^2}$. Hence this measure is κ-concave with the optimal parameter

$$\kappa = -\frac{1}{2\alpha - n}.$$

It was shown in [4] that any κ-concave probability measure ν on $\mathbb{R}^n$ with $\kappa \leq 1$ satisfies an isoperimetric inequality

$$\nu^+(\partial A) \geq \frac{c(\kappa)}{m}\,\big(\nu(A)(1-\nu(A))\big)^{1-\kappa}, \tag{11.2}$$

where m is the ν-median of the Euclidean norm $x \to |x|$, and where $c(\kappa)$ is a positive continuous function in the range $(-\infty, 1]$. A closely related weighted Sobolev-type inequality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|\,d\nu(x)\,d\nu(y) \,\leq\, C(\kappa)\int_{\mathbb{R}^n} |\nabla f(x)|\,(m - \kappa\,|x|)\,d\nu(x) \tag{11.3}$$

was derived in [8] for $\kappa \leq 0$, where $m = \exp\int \log|x|\,d\nu(x)$ is the geometric mean (or L^0-norm) of the Euclidean norm with respect to ν.

If ν is log-concave, these inequalities are equivalent within universal factors, and (11.3) is reduced to the non-weighted L^1-Poincaré-type inequality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|\,d\nu(x)\,d\nu(y) \,\leq\, Cm\int_{\mathbb{R}^n} |\nabla f(x)|\,d\nu(x). \tag{11.4}$$

When ν is a uniform distribution on a convex body, (11.4) corresponds to the result by Kannan, Lovász and Simonovits [16]; the general log-concave case $\kappa = 0$ was considered in [3]. Here the quantity m is equivalent to the L^1-norm $\int |x|\,d\mu(x)$. Hence, when ν is isotropic with $\int |x|^2\,d\nu(x) = 1$, (11.4) yields

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|\,d\nu(x)\,d\nu(y) \,\leq\, C\sqrt{n}\int_{\mathbb{R}^n} |\nabla f(x)|\,d\nu(x).$$

This is far from beeing optimal, since the factor $\sqrt{n}$ can be replaced with $\sqrt{\log n}$, due to a recent result of Klartag [17].

As for the general case $\kappa < 0$, the inequalities (11.2)-(11.3) cannot be used to recover the concentration of measure phenomenon, such as the one for the Gaussian measure. In this respect, known results about Cauchy measures are more accurate than (11.2)-(11.3). For example. it was shown in [7] that every $\mathfrak{m}_{n,\alpha}$, $\alpha \geq n$, admits a weighted Poincaré-type inequality

$$\mathrm{Var}_{\mathfrak{m}_{n,\alpha}}(f) \leq \frac{C_\alpha}{2(\alpha-1)}\int |\nabla f(x)|^2\,(1+|x|^2)\,d\mathfrak{m}_{n,\alpha}(x) \tag{11.5}$$

in the class of all smooth functions f on $\mathbb{R}^n$ with constant $C_\alpha = \big(\sqrt{1+\frac{2}{\alpha-1}} + \sqrt{\frac{2}{\alpha-1}}\,\big)^2$ (see also [11] and [10, Theorem 2] for the optimal values of C_α in the whole range of admissible α's). Since $C_\alpha \to 1$ as $\alpha \to \infty$, after the linear rescaling of the space variable, (11.5) yields in the limit the Gaussian Poincaré-type inequality

$$\mathrm{Var}_{\gamma_n}(f) \leq \int |\nabla f(x)|^2\, d\gamma_n(x).$$

See also [10] As was already mentioned, Theorem 2.3 with $p = 1$ also yields the L^1-Poincaré-type inequality for the Gaussian measure with a dimension free optimal constant in the limit as $\alpha \to \infty$.

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