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SIMPLE FORM CONTROL POLICIES FOR RESOURCE SHARING NETWORKS WITH HGI PERFORMANCE

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We consider a family of resource sharing networks, known as bandwidth sharing models, in heavy traffic with general service and interarrival times. These networks, introduced in Massoulié and Roberts (*Telecommun. Syst.* **15** (2000) 185–201) as models for internet flows, have the feature that a typical job may require simultaneous processing by multiple resources in the network. We construct simple form threshold policies that asymptotically achieve the Hierarchical Greedy Ideal (HGI) performance. This performance benchmark, which was introduced in Harrison et al. (*Stoch. Syst.* **4** (2014) 524–555), is characterized by the following two features: every resource works at full capacity whenever there is work for that resource in the system; total holding cost of jobs of each type at any instant is the minimum cost possible for the associated vector of workloads. The control policy we provide is explicit in terms of a finite collection of vectors, which can be computed offline by solving a system of linear inequalities. Proof of convergence is based on path large deviation estimates for renewal processes, Lyapunov function constructions and analyses of suitable sample path excursions.

CONTENTS

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1. Introduction. In Harrison et al. [11], the authors formulated a challenging open problem of constructing simple form control policies for Massoulié and Roberts [17] Resource Sharing Networks (RSN) in heavy traffic that asymptotically achieve the so-called Hierarchical Greedy Ideal performance. In a recent work [6], we had made partial progress toward this goal. In the current work, we give a much more general treatment that removes some of the main restrictive assumptions of [6] and also provides a substantially more intuitive and easier to implement control policy.

We begin by describing the network model and the basic problem from [11]. Resource sharing networks of the form considered in this work, which were originally proposed as models for internet flows [17] (the terminology is due to [19]), are quite general processing systems that have the distinguishing feature that a typical job may require simultaneous processing by multiple resources. We remark that they are a subset of the collection of stochastic networks introduced in [10] and have the key feature that they are "one pass" systems with no feedback. A RSN considered in this work consists of I resources (labeled $1, \dots, I$) where the ith resource has processing rate capacity C_i, $i = 1, \dots, I$. Jobs of type $1, \dots, J$ arrive according to independent renewal processes with distributions depending on the job type. The job sizes of different job types are i.i.d. with distribution (depending on the type) supported on the positive real line. We make usual assumptions on mutual independence. In general, a job will require simultaneous processing by several network resources. This is made precise through a $I \times J$ incidence matrix K for which $K_{i,j} = 1$ if jth job type requires processing by resource i and $K_{i,j} = 0$ otherwise. The processing of a job is accomplished by allocating to it the same instantaneous *flow rate* by all the resources responsible for its simultaneous processing, and a job departs from the system when the integrated flow rate equals the size of the job. Throughout this work, the term Resource Sharing Network (RSN) will refer to a network of the above form. If at any time instant $\mathfrak{b} = (\mathfrak{b}_1, \dots, \mathfrak{b}_J)'$ is the nonnegative vector of flow rates allocated to various job types, then $\mathfrak{b}$ must satisfy the capacity constraint $K\mathfrak{b} \le C$, where $C = (C_1, \dots, C_I)'$. There is a holding cost per unit time, which is a linear function of the queue length. Specifically, we let $h_j > 0$ denote the cost per unit time per job for the jth job type.

Optimal resource allocation problems for such networks are in general hard and intractable. A common approach when the system is critically loaded is through certain diffusion approximations that replace the original allocation problem by a certain Brownian control problem (BCP) (cf. [9]). Although in special cases such BCP can be explicitly solved, in general, closed-form solutions are not available and using this approach for constructing asymptotically optimal simple-form allocation policies for general RSN becomes challenging. To address this, in [11], authors propose to focus on a less demanding goal than constructing asymptotically optimal policies, which is to construct control policies that achieve

Hierarchical Greedy Ideal (HGI) performance in the heavy traffic limit. Roughly speaking, HGI performance is the (in general suboptimal) cost for a control in the BCP, which has the following two features: (a) *No idleness*: Every resource works at full capacity whenever there is work for that resource in the system; (b) *Instantaneous holding cost minimization*: Total holding cost of jobs of each type at any instant is the minimum cost possible for the associated vector of workloads. Good performance of HGI control policies and comparison with other types of allocation policies, for example, proportional fairness policies [15, 16, 18] has been discussed in detail in [11] through simulations and numerical examples. This paper also put forward the challenging open problem of constructing simple-form control policies for broad families of RSN that achieve HGI performance in the heavy traffic limit.

In our recent work [6], we made partial progress toward this open problem. Under a set of conditions on the topology of the RSN and system parameters (arrival and service rates, holding costs), the paper [6] constructed a sequence of threshold-type control policies, which when the interarrival and service times are *exponentially distributed*, was proved to asymptotically achieve HGI. One of the key assumptions imposed in [6] is the existence of a certain *ranking map*, which identified a suitable form of priority among the various job types. This paper also provided some tractable sufficient conditions under which such a ranking map exists for a given network; nevertheless, this is a restrictive condition and it is easy to produce examples of networks where this condition fails (see, for instance, [6], Example 6.6). Furthermore, in general constructing an explicit ranking map can be challenging.

In the current paper, we revisit [11] and give a fairly complete solution to the open problem formulated there. In particular, we remove the key restrictive condition on the existence of a ranking map that was imposed in [6] and also allow for nonexponential interarrival and job sizes. We impose a standard heavy traffic and a stability condition (Condition 2), and a local traffic condition (Condition 3). Both of these conditions were also assumed in [11] and [6]. The latter condition, first introduced in [13] is needed in order to ensure that the state space of the *workload process* is all of the positive orthant. Finally, although interarrival and job sizes are not required to be exponential, we require them to satisfy a suitable exponential integrability condition (Condition 1). Under these three conditions, we introduce in Definition 2.3 a sequence of control policies and prove that these policies achieve the HGI in the heavy traffic limit. Our two main results are Theorem 2.4 and Theorem 2.5. The first proves the convergence to the HGI when the underlying cost is an infinite horizon discounted cost while the second shows the same result for the long-time cost per unit time (ergodic cost). Specifically, HGI performance in the discounted cost setting is defined as the expectation of a functional of a reflected Brownian motion (with drift) in $\mathbb{R}^I_+$ with normal reflections (namely the first expression in (11)) while in the ergodic cost setting it is given as the expectation under the unique stationary distribution of the same reflected Brownian motion (namely the second expression in (11)). The fact that these functionals involve the minimizer $\hat{h}$ defined in (4) captures the feature of instantaneous holding cost minimization, and the fact that the limit process is a reflected Brownian motion in $\mathbb{R}^I_+$, which has the feature that the reflection (which roughly corresponds to the asymptotic idleness) occurs only when the process hits the faces of the orthant, and captures the no-idleness property of HGI.

In addition to removing undesirable and restrictive assumptions, the other main contribution of this work is to the form of the control policy, which we believe is significantly more intuitive and easy to implement than the policy presented in [6]. The control policy that we introduce in this work is given in terms of explicit thresholds determined from system parameters, and in addition requires the evaluation, for each $z \in \{0, 1\}^J$, of J-dimensional vectors $v^b(z)$ and $v^c(z)$ (see below Proposition 2.3). The evaluation of these vectors can be done offline. Specifically, determining $v^b(z)$ for each fixed z requires solving the inequalities $Av = 0$ and $Bv > 0$ where A and B are given matrices (depending on z) with dimensions $r \times J$ and

$s \times J$ where $r \leq I$, $s \leq J$. Similarly, determining $v^c(z)$ for each fixed z, requires solving the inequalities $Av = 0$, $Bv \geq 0$, $v \cdot f > 0$ (or determining that no such v exists) where A and B are given matrices of similar form and f is a given vector in $\mathbb{R}^J$. Once the vectors $v^c(z)$, $v^b(z)$ and the thresholds are determined, the policy takes a simple explicit form, which can be described by a single line (see (7)) as opposed to the half-page description of the policy constructed in [6]. Roughly speaking, the policy works as follows. Consider a vector $z \in \{0, 1\}^J$ representing the state of the system at some given time instant. An entry of 0 in the vector z means that the corresponding job type's queue is "far from empty" and an entry of 1 means that the queue is "close to empty." The vectors $v^c(z)$ and $v^b(z)$ will be used to make adjustments, depending on the state z, to the nominal instantaneous flow rate at the given instant to give the overall flow rate. The role of $v^c(z)$ will be to reduce the holding cost while keeping the net workload to be the same and ensuring that the queues close to empty do not get more than the nominal allocation. This vector helps with achieving the second feature of the HGI performance. The vector $v^b(z)$ is instrumental in achieving the first feature of the HGI performance by pushing the queues that are close to empty, away from 0, so that idleness due to "blocking" is prevented. A detailed discussion of the policy can be found in Remark 1.

We now make some comments on the main ingredients in the proofs (additional discussion can be found in Remark 1). As noted previously, the two key characteristic properties of HGI performance are: *no idleness* and *instantaneous holding cost minimization*. The main effort in the proof is to show that the sequence of control policies constructed here asymptotically have these two features. The main results that enable the verification of the first property are Propositions 3.2 and 3.3 whereas the second feature emerges as a consequence of Proposition 3.5. Furthermore, as we need to consider an ergodic cost criterion, one needs to obtain suitable stability estimates that are uniform in the traffic parameter. This is done in Proposition 3.4. Proofs of these three results are the technical heart of this work. Some recurring tools in the various proofs are path large deviation estimate for renewal processes and excursion analyses of strong Markov processes. We were not able to find suitable references for the path of large deviation estimates that we need, and so we provide self-contained proofs of these results for the reader's convenience (see Theorem 6.2). The excursion analyses that are used in the various proofs are guided by the form of our control policy, which is described in terms of thresholds determined by a sequence of stopping times. The general approach of considering suitable excursions together with appropriate large deviation estimates in the analysis of control policies for networks in heavy traffic originates from the work of Bell and Williams [2, 3] (see also [1, 5] and [7]). Our recent paper [6] also used an analogous approach; however, in general, the precise forms of excursions to consider, and study of their properties is problem specific and indeed such analyses constitute the major effort in the proofs.

The control policy we construct is in general not asymptotically optimal. However, it can be argued that when the function $\hat{h}$ introduced in (4) is a nondecreasing function then the minimality properties of the Skorohod map imply that our policy is indeed asymptotically optimal. This is a consequence of a more general result on asymptotic lower bounds on costs of arbitrary control policies for resource sharing networks, which will be reported elsewhere.

The remaining paper is organized as follows. We close this section with the notation and conventions used in this work. Section 2 presents the model description, our main assumptions, the sequence of control policies that we study and our main results. The rest of the paper is devoted to the proofs of the main results, namely Theorems 2.4 and 2.5. Details of proof organization can be found at the end of Section 2.5.

1.1. *Notation and conventions.* For $j \in \mathbb{N}$, let $\mathcal{D}^j = \mathcal{D}([0, \infty) : \mathbb{R}^j)$ (resp., $\mathcal{D}^j_+ = \mathcal{D}([0, \infty) : \mathbb{R}^j_+)$) denote the space of functions that are right continuous with left limits (RCLL) from $[0, \infty)$ to $\mathbb{R}^j$ (resp., $\mathbb{R}^j_+$) equipped with the usual Skorohod topology. Also, let $\mathcal{C}^j = \mathcal{C}([0,\infty):\mathbb{R}^j)$ (resp., $\mathcal{C}_+^j = \mathcal{C}_+([0,\infty):\mathbb{R}^j)$) denote the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^j$ (resp., $\mathbb{R}_+^j$) equipped with the local uniform topology. For $T<\infty$, the spaces $\mathcal{D}([0,T]:\mathbb{R}^j)$, $\mathcal{C}([0,T]:\mathbb{R}^j)$, $\mathcal{D}([0,T]:\mathbb{R}_+^j)$, $\mathcal{C}([0,T]:\mathbb{R}_+^j)$ are defined similarly. All stochastic processes in this work will have sample paths that are RCLL unless noted explicitly. For $m\in\mathbb{N}$, we denote by $\mathbb{A}_m$ the set $\{1,2,\ldots,m\}$ and χ_m the finite set of all vectors in $\mathbb{R}^m$ with entries 0 or 1. For $r\in\mathbb{N}$, $\mathbb{N}_{1/r}\doteq\frac{1}{r}\mathbb{N}_0$ is the scaled (nonnegative) integer lattice. We will frequently do componentwise operations on vectors. For instance, given two vectors $v^1,v^2\in\mathbb{R}^d$, we will use v^1v^2 to denote componentwise multiplication. The vector v^1/v^2 is defined similarly. For a vector $v\in\mathbb{R}^d$ and a constant $c\in\mathbb{R}$, we will interpret $v\vee c$, $v\wedge c$, and $v-c$ componentwise, for example, $v\vee c=(v_1\vee c,v_2\vee c,\ldots,v_J\vee c)$. We will also treat inputs to vectors of functions componentwise, so for a vector $v\in\mathbb{R}^d$, a constant $c\in\mathbb{R}$ and a vector of functions $f=(f_1,f_2,\ldots,f_d)$, from $\mathbb{R}$ to itself, we write

$$f(v)=\bigl(f_1(v_1),f_2(v_2),\ldots,f_d(v_d)\bigr),\qquad \mathbb{I}_{\{v\geq c\}}=(\mathbb{I}_{\{v_1\geq c\}},\mathbb{I}_{\{v_2\geq c\}},\ldots,\mathbb{I}_{\{v_d\geq c\}}).$$

We will use coordinatewise inequalities on vectors, for example, for $v^1,v^2\in\mathbb{R}^d$ and $c\in\mathbb{R}$ the statements $v^1\geq v^2$ and $v^1\geq c$ mean $v_j^1\geq v_j^2$ and $v_j^1\geq c$ for all $j\in\mathbb{A}_d$, respectively. Inequalities for vector-valued functions are interpreted pointwise and coordinatewise. For $v\in\mathbb{R}^d$, $|v|_1=\sum_{j=1}^d|v_j|$ and $|v|_2=(\sum_{j=1}^d|v_j|^2)^{\frac{1}{2}}$. As a convention, for a real sequence $\{a(l)\}_{l\in\mathbb{N}}$, $\sum_{l=1}^n a(l)$ is taken to be 0 if $n=0$.

2. Main results. A RSN in heavy traffic is described through a sequence of models, indexed by a traffic parameter $r\in\mathbb{N}$, each of which has the same underlying network topology. Each network in the sequence consists of J types of jobs and I resources and the network topology is described through an $I\times J$ matrix K for which $K_{i,j}=1$ if the jth job requires service from the ith resource and $K_{i,j}=0$ otherwise. A job of type j will be processed simultaneously by all resources in the set $\{i:K_{i,j}=1\}$ and so each resource in this set will allocate the same amount of processing capacity to the job at any instant. As r goes to ∞, the networks approach criticality in that the traffic intensity converges to 1. Specifically, as made precise in Condition 2, with the asymptotic load vector ρ as defined in that condition, $C_i=\sum_{j=1}^J K_{i,j}\rho_j$ for every resource $i=1,\ldots,I$, which says that the capacity of each resource equals the asymptotic load on the resource. In the rth system, $\{u_j^r(l)\}_{l=1}^\infty$ and $\{v_j^r(l)\}_{l=1}^\infty$ are the i.i.d., mutually independent, interarrival times and job sizes for job type j, given on some probability space $(\Omega,\mathcal{F},P)$, with means $E[u_j^r(l)]=\frac{1}{\alpha_j^r}$ and $E[v_j^r(l)]=\frac{1}{\beta_j^r}$ and finite standard deviations $\sigma_j^{u,r}$ and $\sigma_j^{v,r}$. We assume a first-in, first-out (FIFO) policy meaning that for each job type the oldest job in the queue is processed before another one is started. In the case where the job sizes are exponentially distributed, the "memoryless" property implies that the precise manner of the allocation of the flow rate among jobs in the queue of a particular job type does not impact the distribution of the queue-length process. Consequently, if the job sizes are exponentially distributed we can drop the FIFO assumption in favor of something else, such as a processor sharing policy that is common in the literature, without altering the results.

We now introduce our main assumptions.

2.1. *Assumptions.* We will assume that

$$P\bigl(u_j^r(1)>0\bigr)=P\bigl(v_j^r(1)>0\bigr)=1\quad\text{for all } r \text{ and } j.$$

In fact, for notational convenience we will simply assume (without loss of generality) that $v_j^r(l)>0$ and $u_j^r(l)>0$ for all $j\in\mathbb{A}_J$ and $l\in\mathbb{N}$.

The following assumption on finite moment-generating functions in the neighborhood of the origin will be used to obtain certain large deviation estimates. Recall that for $m\in\mathbb{N}$, $\mathbb{A}_m=\{1,\ldots,m\}$.

CONDITION 1. *There exists $\delta > 0$ such that for all $y < \delta$ and $j \in \mathbb{A}_J$.*

$$\sup_{r>0} E\big[e^{y v_j^r(1)}\big] < \infty, \qquad \sup_{r>0} E\big[e^{y u_j^r(1)}\big] < \infty.$$

The following is our main *heavy traffic condition*.

CONDITION 2. *For each $j \in \mathbb{A}_J$, there exist $\bar{\alpha}_j, \bar{\beta}_j, \alpha_j, \beta_j, \sigma_j^u, \sigma_j^v \in (0, \infty)$ such that*

$$\lim_{r\to\infty} r\big(\alpha_j^r - \alpha_j\big) = \bar{\alpha}_j, \qquad \lim_{r\to\infty} r\big(\beta_j^r - \beta_j\big) = \bar{\beta}_j, \qquad \lim_{r\to\infty} \sigma_j^{u,r} = \sigma_j^u, \qquad \lim_{r\to\infty} \sigma_j^{v,r} = \sigma_j^v.$$

Furthermore, with $\rho = \alpha/\beta$ we have $C = K\rho$, and with $\eta = \beta^{-2}(\bar{\alpha}\beta - \alpha\bar{\beta})$ and $\theta \doteq K\eta$, we have $\theta < 0$. Note that this implies $\lim_{r\to\infty} r(\rho^r - \rho) = \eta$ where $\rho^r = \alpha^r/\beta^r$.

The $\theta < 0$ part of the condition is a key stability assumption, which will be crucially used in obtaining various types of uniform in time moment estimates (see, e.g., Section 8).

Finally, we make the following local traffic assumption. It says that every resource has at least one associated job type that requires service from only that particular resource. This assumption was also made in [11, 13] and [6].

CONDITION 3. *For every $i \in \mathbb{A}_I$, there exists $j \in \mathbb{A}_J$ such that $K_{i,j} = 1$ and $K_{l,j} = 0$ for $l \neq i$.*

When the above condition does not hold, the situation is quite different in that the HGI performance is not given by a reflected Brownian motion (RBM) in an orthant since the workload process may not achieve all vectors in $\mathbb{R}_+^I$. Consider, for example, the case where $I = 2$, $J = 2$ and the job-resource matrix K is $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. In this case, if $\beta = [1, 1]'$, the limiting Brownian motion will live in the wedge $\{(x_1, x_2) : 0 \le x_2 \le x_1 < \infty\}$. Treating such settings will require additional new ideas and is beyond the scope of the current work.

2.2. *State processes.* The arrival and service processes for job type $j \in \mathbb{A}_J$ are, respectively,

$$A_j^r(s) = \max\left\{n \ge 0 : \sum_{l=1}^n u_j^r(l) \le s\right\}, \qquad S_j^r(s) = \max\left\{n \ge 0 : \sum_{l=1}^n v_j^r(l) \le s\right\}.$$

The initial state of the system is described by the J-dimensional queue length vector $q^r \in \mathbb{N}^J$ and residual arrival and service time vectors in $\mathbb{R}_+^J$, defined as

$$\Upsilon^{A,r} = \big(\Upsilon_1^{A,r}, \Upsilon_2^{A,r}, \dots, \Upsilon_J^{A,r}\big), \qquad \Upsilon^{S,r} = \big(\Upsilon_1^{S,r}, \Upsilon_2^{S,r}, \dots, \Upsilon_J^{S,r}\big).$$

Here, $\Upsilon_j^{A,r}$ [resp., $\Upsilon_j^{S,r}$] represent the deterministic times after which the arrivals [resp., processing times] are governed by the renewal processes $\{A_j^r(s)\}_{s\ge 0}$ [resp., $\{S_j^r(s)\}_{s\ge 0}$]. These quantities capture initial state configurations when the evolution is viewed onwards from a time instant at which the system has been in operation for some time.

A key ingredient in the state evolution of the queue-length process is a capacity allocation control policy, which is described as a continuous $\mathbb{R}_+^J$ valued stochastic process $B^r(\cdot)$ with appropriate measurability and feasibility properties that will be specified later in the section. Roughly speaking, for $j \in \mathbb{A}_J$, $B_j^r(t)$ specifies the cumulative amount of capacity used by type-j jobs, in the rth system, by time instant t. Given such initial conditions and a control policy, the J-dimensional queue-length process is given by the equation

$$(1)\quad Q^r(t) = q^r + A^r\big((t - \Upsilon^{A,r})^+\big) + \mathcal{I}_{\{t \ge \Upsilon^{A,r} > 0\}} - S^r\big((B^r(t) - \Upsilon^{S,r})^+\big) - \mathcal{I}_{\{B^r(t) \ge \Upsilon^{S,r} > 0\}}.$$

This evolution captures the fact that the queue length, of say the jth job type, corresponds to the initial queue length q^r_j, plus all the arrivals that have occurred according to the renewal process A^r_j, with an additional arrival at time instant $\Upsilon^{A,r}$ if this quantity is positive, minus all the jobs that have been been completed according to the renewal process S^r_j, with an additional departure at instant t where $B^r(t) = \Upsilon^{S,r}$ if the latter quantity is nonzero.

Let M^r be the $J \times J$ diagonal matrix with entries $\{\frac{1}{\beta^r_j}\}_{j=1}^J$ and let M be the $J \times J$ diagonal matrix with entries $\{\frac{1}{\beta_j}\}_{j=1}^J$. The I-dimensional workload process $W^r(t)$ associated with the queue-length process $Q^r(t)$ is then given by the equation

$$\begin{aligned} W^r(t) &\doteq KM^rQ^r(t) \\ &= KM^rq^r + KM^r\big(A^r((t-\Upsilon^{A,r})^+) - S^r((B^r(t)-\Upsilon^{S,r})^+)\big) \\ &\quad + KM^r\mathcal{I}_{\{t\ge\Upsilon^{A,r}>0\}} - KM^r\mathcal{I}_{\{B^r(t)\ge\Upsilon^{S,r}>0\}}. \end{aligned}$$

Note that this is a I-dimensional process, which captures the amount of work in the system at any instant for each of the I resources in the network.

Scaled processes. In order to study the behavior as the systems approach criticality, we will consider two types of scaling: diffusion scaling and fluid scaling. In both of these scalings, time is accelerated by a factor of r^2, but in the first type of scaling the magnitude is scaled down by a factor of r, while in the second scaling the magnitude is scaled down by factor of r^2. Processes obtained using diffusion scaling will typically be denoted using a "hat" symbol while the processes with fluid scaling will be denoted using a "bar" symbol. In particular, we define

$$\hat\Upsilon^{A,r} = \frac{1}{r}\Upsilon^{A,r}, \qquad \hat\Upsilon^{S,r} = \frac{1}{r}\Upsilon^{S,r}, \qquad \bar\Upsilon^{A,r} = \frac{1}{r^2}\Upsilon^{A,r}, \qquad \bar\Upsilon^{S,r} = \frac{1}{r^2}\Upsilon^{S,r}.$$

Similarly, we define $\hat Q^r(t) \doteq Q^r(r^2t)/r$, $\hat W^r(t) \doteq W^r(r^2t)/r$ and $\bar B^r(t) \doteq B^r(r^2t)/r^2$. Letting

$$\begin{aligned} \hat A^r_j(s) &= \frac{1}{r}\max\left\{n\ge 0 : \sum_{l=1}^n u^r_j(l) \le r^2 s\right\} - rs\alpha^r_j, \\ \hat S^r_j(s) &= \frac{1}{r}\max\left\{n\ge 0 : \sum_{l=1}^n v^r_j(l) \le r^2 s\right\} - rs\beta^r_j \end{aligned}$$

we see that, with $\hat q^r = q^r/r$,

$$\begin{aligned} \hat Q^r(t) &= \hat q^r + \hat A^r((t-\bar\Upsilon^{A,r})^+) - \hat S^r((\bar B^r(t)-\bar\Upsilon^{S,r})^+) + \frac{1}{r}\mathcal{I}_{\{t\ge\bar\Upsilon^{A,r}>0\}} \\ &\quad - \frac{1}{r}\mathcal{I}_{\{\bar B^r(t)\ge\bar\Upsilon^{S,r}>0\}} + r\big(\alpha^r t - \beta^r\bar B^r(t)\big) - r\alpha^r\big(t\wedge\bar\Upsilon^{A,r}\big) + r\beta^r\big(\bar B^r(t)\wedge\bar\Upsilon^{S,r}\big) \end{aligned}$$

and the corresponding diffusion-scaled workload process, using the identities $M^r\alpha^r = \rho^r$ and $K\rho = C$, is

$$\begin{aligned} \hat W^r(t) &= KM^r\hat q^r + KM^r\hat A^r((t-\bar\Upsilon^{A,r})^+) - KM^r\hat S^r((\bar B^r(t)-\bar\Upsilon^{S,r})^+) \\ &\quad + \frac{1}{r}KM^r\mathcal{I}_{\{t\ge\bar\Upsilon^{A,r}>0\}} - \frac{1}{r}KM^r\mathcal{I}_{\{\bar B^r(t)\ge\bar\Upsilon^{S,r}>0\}} + rKM^r\big(\alpha^r t - \beta^r\bar B^r(t)\big) \\ &\quad - rKM^r\alpha^r\big(t\wedge\bar\Upsilon^{A,r}\big) + rKM^r\beta^r\big(\bar B^r(t)\wedge\bar\Upsilon^{S,r}\big) \\ &= KM^r\hat q^r + KM^r\hat A^r((t-\bar\Upsilon^{A,r})^+) - KM^r\hat S^r((\bar B^r(t)-\bar\Upsilon^{S,r})^+) \end{aligned}$$

$$+\frac{1}{r}KM^r\mathcal{I}_{\{t\geq\bar{\Upsilon}^{A,r}>0\}}-\frac{1}{r}KM^r\mathcal{I}_{\{\bar{B}^r(t)\geq\bar{\Upsilon}^{S,r}>0\}}+rtK(\rho^r-\rho)$$
$$+r(Ct-K\bar{B}^r(t))-rK\rho^r(t\wedge\bar{\Upsilon}^{A,r})+rK(\bar{B}^r(t)\wedge\bar{\Upsilon}^{S,r}).$$

It will be convenient to introduce the processes

$$\begin{aligned}\hat{X}^r(t)&\doteq KM^r\hat{A}^r((t-\bar{\Upsilon}^{A,r})^+)+\frac{1}{r}KM^r\mathcal{I}_{\{t\geq\bar{\Upsilon}^{A,r}>0\}}\\&\quad-KM^r\hat{S}^r((\bar{B}^r(t)-\bar{\Upsilon}^{S,r})^+)-\frac{1}{r}KM^r\mathcal{I}_{\{\bar{B}^r(t)\geq\bar{\Upsilon}^{S,r}>0\}}+rtK(\rho^r-\rho)\\&\quad-rK\rho^r(t\wedge\bar{\Upsilon}^{A,r})+rK(\bar{B}^r(t)\wedge\bar{\Upsilon}^{S,r})\end{aligned}\tag{2}$$

and

$$\hat{U}^r(t)=r(Ct-K\bar{B}^r(t))$$

so that, with $\hat{w}^r\doteq KM^r\hat{q}^r$, we have

$$\hat{W}^r(t)=\hat{w}^r+\hat{X}^r(t)+\hat{U}^r(t).\tag{3}$$

2.3. *Admissible control policies.* We now specify what types of allocation policies are admissible. Roughly speaking, an admissible control policy should not look into the future. This is made precise by introducing certain multiparameter filtrations. Recall the probability space $(\Omega,\mathcal{F},P)$ on which the sequences $\{u_j^r(l)\}_{l=1}^\infty$ and $\{v_j^r(l)\}_{l=1}^\infty$ are defined.

DEFINITION 2.1. For $n=(n_1,\dots,n_J)\in\mathbb{N}_0^J$ and $m=(m_1,\dots,m_J)\in\mathbb{N}_0^J$, let

$$\mathcal{F}^r(n,m)=\sigma\{u_j^r(\tilde{n}_j),v_j^r(\tilde{m}_j):0\leq\tilde{n}_j\leq n_j,0\leq\tilde{m}_j\leq m_j,j\in\mathbb{A}_J\},$$

where by convention $u_j^r(0)=v_j^r(0)=0$. Let

$$\mathcal{F}^r=\sigma\Big\{\bigcup_{(n,m)\in\mathbb{N}^{2J}}\mathcal{F}^r(n,m)\Big\}.$$

Note that $\{\mathcal{F}^r(n,m),n,m\in\mathbb{N}^J\}$ is a multiparameter filtration generated by the arrival and service times with the following partial ordering:

$$(n,m)\leq(\tilde{n},\tilde{m})\quad\text{if and only if}\quad n_j\leq\tilde{n}_j\text{ and }m_j\leq\tilde{m}_j\quad\text{for all }j\in\mathbb{A}_J.$$

For some basic definitions of multiparameter filtrations and multiparameter stopping times, see [8], Chapter 2, Section 8. An admissible control policy is defined as follows.

DEFINITION 2.2. For $r\in\mathbb{N}$, $B^r(\cdot)$ is an admissible policy (for the rth system) for the initial condition $(q^r,\Upsilon^r=(\Upsilon^{A,r},\Upsilon^{S,r}))\in\mathbb{N}_0^J\times\mathbb{R}_+^{2J}$ if the following hold:

(a) The stochastic process $B^r(\cdot)$ has sample paths that are absolutely continuous, nonnegative, nondecreasing functions from $[0,\infty)\to\mathbb{R}^J$ with $B^r(0)=0$.
(b) $C\geq K\frac{d}{dt}B^r(t)$ for a.e. $t\geq 0$, a.s.
(c) The process $Q^r(\cdot)$ defined by the right-hand side of (1) satisfies $Q^r(t)\geq 0$ for all $t\geq 0$.
(d) For each $r\in\mathbb{N}$ and $t\geq 0$, consider the $\mathbb{N}^{2J}$ valued random variable

$$\tau^r(t)\doteq(\tau_1^{r,A}(t),\dots,\tau_J^{r,A}(t),\tau_1^{r,S}(t),\dots,\tau_J^{r,S}(t)),$$

where for $j\in\mathbb{N}^J$,

$$\tau_j^{r,A}(t)=\min\Big\{n\geq 0:\sum_{l=1}^n u_j^r(l)\geq r^2(t-\bar{\Upsilon}_j^A)^+\Big\}$$

and

$$\tau_j^{r,S}(t) = \min\left\{n \geq 0 : \sum_{l=1}^{n} v_j^r(l) \geq r^2\big(\bar{B}_j^r(t) - \bar{\Upsilon}_j^S\big)^+\right\},$$

where by convention, $\tau_j^{r,A}(t)$ [resp., $\tau_j^{r,S}(t)$] is defined to be 0 if $t \leq \bar{\Upsilon}_j^A$ [resp., $\bar{B}_j^r(t) \leq \bar{\Upsilon}_j^S$]. Then $\tau^r(t)$ is an $\{\mathcal{F}^r(n,m)\}$-stopping time for all $t \geq 0$.

(e) Consider the filtration

$$\mathcal{G}^r(t) \doteq \mathcal{F}^r\big(\tau^r(t)\big) = \sigma\big\{A \in \mathcal{F}^r : A \cap \big\{\tau^r(t) \leq (n,m)\big\} \in \mathcal{F}^r(n,m) \text{ for all } (n,m) \in \mathbb{N}^{2J}\big\}.$$

Then $B^r(r^2 t)$ is $\{\mathcal{G}^r(t)\}$-adapted.

Define $\mathcal{A}$ to be the set of admissible controls.

Parts (b) and (c) give the feasibility requirements on the control policy whereas parts (d) and (e) make precise the nonanticipativity property of an admissible policy.

2.4. *Proposed control policy.* In this section, we introduce our proposed control policy, which will be shown to achieve the HGI performance asymptotically. Recall that K is an $I \times J$ matrix and M is a $J \times J$ diagonal matrix with diagonal entries $\{\frac{1}{\beta_j}\}_{j=1}^J$. Let $h \in \mathbb{R}^J$ such that $h > 0$ be a given holding cost vector. This vector will determine the discounted and ergodic cost functions that will be introduced later.

For $w \in \mathbb{R}_+^I$, define

$$\Lambda(w) = \big\{q \in \mathbb{R}_+^J : KMq = w\big\}, \qquad \hat{h}(w) = \inf_{q \in \Lambda(w)} (h \cdot q). \tag{4}$$

Let v_j be the jth column of K and note that due to the local traffic condition (Condition 3) the span of $\{v_j\}_{j=1}^J$ is $\mathbb{R}^I$. As a consequence, for any $w \in \mathbb{R}^I$, $\Lambda(w)$ is a nonempty compact subset of $\mathbb{R}^J$. In particular, there exists $q^* = q^*(w) \in \Lambda(w)$ such that $h \cdot q^* = \hat{h}(w)$.

Define

$$\mathcal{C}_K^h = \big\{u \in \mathbb{R}^J : \mathbf{0} = Ku \text{ and } (h\beta) \cdot u = 0\big\}$$

and note that $\mathcal{C}_K^h$ is a linear subspace of $\ker(K)$. Note that either $\mathcal{C}_K^h = \ker(K)$ or $\dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1 = J - I - 1$. If $\mathcal{C}_K^h = \ker(K)$, then for any $w \in \mathbb{R}_+^I$ all $q \in \mathbb{R}_+^J$, which satisfy $w = KMq$ have the same cost, namely if $q, \tilde{q} \in \mathbb{R}_+^J$ satisfy $KMq = KM\tilde{q} = w$ then $h \cdot q = h \cdot \tilde{q}$. Note that this situation, where all queue length vectors, which give the same workload have the same holding cost, is trivial in the sense that instantaneous holding cost minimization is no longer a concern and the focus is entirely on preventing idleness. The policy that we will present below (see Definition 2.3) takes a simpler form in this setting in that the vector $v^c(z)$ used for rate allocation adjustment in (7) is simply 0. However, as was pointed out to us by Mike Harrison in a private communication, one can treat this trivial case in a more elementary manner by simply giving lowest priority to the local traffic jobs so that the nonlocal traffic class form a subcritical RSN. As a result in the diffusion-scaled heavy traffic limit the nonlocal traffic queue-lengths go to 0 and the local traffic queue-lengths give the desired reflected Brownian motion workload. We have chosen to present the alternative policy for this case as in Definition 2.3 since it allows us to cover the two cases (namely $\dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1$ and $\dim(\mathcal{C}_K^h) = \dim(\ker(K))$) in a unified approach.

We select an orthonormal basis of $\mathbb{R}^J$, $(u_1, \ldots, u_J)$, such that $(u_1, \ldots, u_{J-I})$ is an orthonormal basis of $\ker(K)$, and in the case $\dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1 = J - I - 1$, $\operatorname{span}(u_1, \ldots, u_{J-I-1}) = \mathcal{C}_K^h$ and u_{J-I} is orthogonal to $\mathcal{C}_K^h$ and satisfies $h\beta \cdot u_{J-I} < 0$.

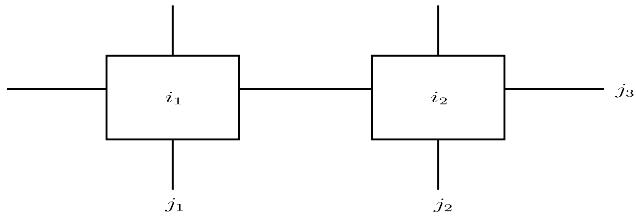

FIG. 1. *2LLN network.*

The latter quantity will play an important role and we define

$$\lambda \doteq h\beta \cdot u_{J-I}.$$

Note that since $\lambda < 0$, adjusting the queue length by adding βu_{J-I} reduces the cost while maintaining the same workload. For $q \in \mathbb{R}^J$, define

$$\Xi(q) \doteq \{v \in \ker(K) : q + v\beta \geq 0\}.$$

Note that $\Xi(q)$ is a compact set. Let

$$\tilde{d}(q) \doteq \begin{cases} \sup_{v \in \Xi(q)} \{v \cdot u_{J-I}\} & \text{if } \dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1 = J - I - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition gives a precise measure for how far away q is from the cost minimizing queue length for the workload KMq.

PROPOSITION 2.1 (Proof in Section 10.1). *For all $q \in \mathbb{R}_+^J$, we have $h \cdot q - \hat{h}(KMq) = |\lambda|\tilde{d}(q)$.*

To illustrate the distinction between the trivial case where $\mathcal{C}_K^h = \ker(K)$ and the nontrivial case where $\dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1$, we consider two examples of networks with $I = 2$ resources and $J = 3$ job types (referred to as 2LLN in [11]) pictured in Figure 1.

EXAMPLE 2.1. Consider a 2LLN network (see Figure 1) where $\alpha = \beta = (1, 1, 1)'$ so $\rho = (1, 1, 1)'$ and $C = (2, 2)'$. Let the holding cost vector be $h = (1, 1, 2)'$.

EXAMPLE 2.2. Just like Example 2.1, we consider a 2LLN network (see Figure 1) where $\alpha = \beta = (1, 1, 1)'$ so $\rho = (1, 1, 1)'$ and $C = (2, 2)'$. However, the holding cost vector is now $h = (1, 1, 1)'$.

For these 2LLN networks, the incidence matrix is

$$K = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

so $\ker(K)$ is one-dimensional and an orthonormal basis of $\ker(K)$ is the vector $u_1 = \frac{1}{\sqrt{3}}(-1, -1, 1)'$. Consequently, for both Example 2.1 and Example 2.2, all changes to the queue length that maintain the same workload involve adding a constant multiple of βu_1. In Example 2.1, we have $h\beta \cdot u_1 = 0$ so $\mathcal{C}_K^h = \ker(K)$ and we say this network is trivial (in terms of instantaneous holding cost minimization) because all queue lengths, which give the same workload have identical cost. In Example 2.2, we have $\lambda = h\beta \cdot u_1 = h \cdot u_1 = -\frac{1}{\sqrt{3}}$ so $\dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1$ and in this case adding a positive multiple of βu_1 will maintain the same workload but reduce the cost. This is because the queue lengths $(1, 1, 0)'$ and

$(0,0,1)'$ result in the same workload but $(0,0,1)'$ has a lower holding cost. In contrast to Example 2.1, where all queue lengths are cost minimizing for their corresponding workloads, the only cost minimizing queue lengths in Example 2.2 are on the boundary where either $q_1 = 0$ or $q_2 = 0$ meaning we want as much workload as possible to come from type 3 jobs.

Recall that χ_J denotes the finite set of all vectors in $\mathbb{R}^J$ with entries 0 or 1. For $z \in \chi_J$, define

$$\mathcal{A}_z = \{i \in \mathbb{A}_I : \text{ there exists } j \in \mathbb{A}_J \text{ such that } K_{i,j} = 1 \text{ and } z_j = 0\}.$$

Also, for $q \in \mathbb{R}^J_+$ define $z^q \in \chi_J$ by $z^q \doteq \mathbb{I}_{\{q=0\}}$. Here, for a given queue-length state $q \in \mathbb{R}^J_+$, $z^q \in \chi_J$ is a vector of indicators, that will tell us which queues are "empty" (corresponding to coordinates that equal 1) and the set $\mathcal{A}_{z^q}$ will tell us which resources are used by at least one job type whose queue length is not empty. In fact, in describing the scheme we will use approximate versions of such indicator vectors given as χ_J valued processes $\mathcal{Z}^r(\cdot)$ (see Definition 2.3), which will tell us which queues are *near empty* so we can push them away from the boundary to avoid "blocking" and ensure the servers can work at full capacity. In the policy, we describe the nominal allocation for each job type j is ρ_j (essentially the amount needed to keep up with the arrival rate). This nominal allocation is modified through two types of vectors, v^b and v^c in $\mathbb{R}^J$, that represent the amount we subtract (amounts can be positive or negative) from the nominal capacity allocation while maintaining capacity constraints. A positive entry in these vectors indicates an underallocation that should result in queue length growth and a negative entry results in queue length decline.

We now introduce a subset $\mathcal{M}$ of χ_J that will play an important role. In the case when $\dim(\mathcal{C}^h_K) = \dim(\ker(K)) - 1 = J - I - 1$, we define

$$\mathcal{M} \doteq \{z \in \chi_J : \text{ there exists } v \in \ker(K) \text{ such that } v_j \geq 0 \text{ if } z_j = 1 \text{ and } v \cdot u_{J-I} > 0\}.$$

We define $\mathcal{M}$ to be the empty set when $\dim(\mathcal{C}^h_K) = \dim(\ker(K)) = J - I$. The set $\mathcal{M}$ lists the configurations of empty and nonempty queues, as indicated by z, which allow us to reduce cost while maintaining the same workload. In particular, the following proposition demonstrates that if $z^q \notin \mathcal{M}$, then q is a cost minimizing queue length for the workload KMq and so for such configurations q, we cannot reduce cost while maintaining the same workload.

PROPOSITION 2.2. *Let $q \in \mathbb{R}^J_+$ be such that $z^q \notin \mathcal{M}$. Then $\tilde{d}(q) = 0$.*

PROOF. The result is immediate when $\dim(\mathcal{C}^h_K) = \dim(\ker(K)) = J - I$. Consider now the case $\dim(\mathcal{C}^h_K) = \dim(\ker(K)) - 1 = J - I - 1$. Suppose $\tilde{d}(q) > 0$. Then there exists $v \in \Xi(q)$ such that $v \cdot u_{J-I} > 0$. However, $v \in \Xi(q)$ implies that $v \in \ker(K)$ and $q + \beta v \geq 0$. For any $j \in \mathbb{A}_J$ such that $z^q_j = 1$, we have $0 \leq q_j + \beta_j v_j = \beta_j v_j$ and since $\beta_j > 0$ we must have $v_j \geq 0$. Consequently, there exists $v \in \ker(K)$ such that $v_j \geq 0$ for all j with $z^q_j = 1$. Also, $v \cdot u_{J-I} > 0$, which says that $z^q \in \mathcal{M}$. Thus, we have a contradiction and the result follows. □

The following proposition will allow us to construct an allocation policy, which underallocates resource capacity to queues that are close to empty to increase their queue lengths while simultaneously ensuring each resource utilizes its full capacity unless all of the job-types that use this resource have queue lengths near 0 (in other words, all resources in $\mathcal{A}_{\mathcal{Z}^r(t)}$, where $\mathcal{Z}^r(t)$ is as in Definition 2.3, allocate their full capacity).

PROPOSITION 2.3 (Proof in Section 10.2). *For any $z \in \chi_J$, there exists $v \in \mathbb{R}^J$ such that for all $i \in \mathcal{A}_z$ we have $(Kv)_i = 0$ and for all $j \in \mathbb{A}_J$ such that $z_j = 1$, we have $v_j > 0$.*

Vectors $v^c(z)$ *and* $v^b(z)$. We now introduce our main control policy. It will be defined in terms of two vector functions v^c on the finite set $\mathcal{M}$ and v^b on the finite set χ_J. Define $\rho^* \doteq \min_{j\in\mathbb{A}_J}\{\rho_j\}$. Recall that for any $z\in\mathcal{M}$ we can find $v^c(z)\in\ker(K)$ such that $v^c_j(z)\geq 0$ if $z_j=1$, and $v^c(z)\cdot u_{J-I}>0$. Without loss of generality, we can assume that $\rho - v^c(z) > \frac{\rho^*}{2}$. Also, note that

$$(h\beta)\cdot v^c(z) = \sum_{m=1}^{J-I}\big(v^c(z)\cdot u_m\big)\big((h\beta)\cdot u_m\big) = \big(v^c(z)\cdot u_{J-I}\big)\big((h\beta)\cdot u_{J-I}\big)$$
$$= \lambda\big(v^c(z)\cdot u_{J-I}\big) \doteq \lambda_c(z) < 0.$$

The c superscript refers to the fact that these vectors will be used to reduce the cost while being workload neutral. In particular, the vectors $v^c(z)$ determine changes to bandwidth allocation (from nominal allocation) that reduces cost in a manner that the rate allocation to queues that are close to empty is not increased. Next, we introduce the vector function v^b. Let

$$\tilde{\lambda} \doteq \max\{\lambda_c(z) : z\in\mathcal{M}\}. \tag{5}$$

From Proposition 2.3, for any $z\in\chi_J$ we can find $v^b(z)\in\mathbb{R}^J$ such that

$$\begin{aligned}&\big|v^b(z)\big| \leq \min\left\{\frac{\rho^*}{4}, \frac{|\tilde{\lambda}|}{4|\beta||h|}\right\},\\ &\big(Kv^b(z)\big)_i = 0 \text{ for all } i\in\mathcal{A}_z, \text{ and } v^b_j(z) > 0 \text{ for all } j\in\mathbb{A}_J \text{ such that } z_j = 1.\end{aligned} \tag{6}$$

The b superscript refers to the fact that these vectors are used to keep the queue lengths away from the boundary.

DEFINITION 2.3 (Control policy). Let $c_1 < c_2$ and $0<\kappa<\frac{1}{4}$ be arbitrary and let $\tilde{c}_1 = \min_j\{\beta_j\}c_1$ and $\tilde{c}_2 = \min_j\{\beta_j\}c_2$. Define the χ_J valued process $\mathcal{Z}^r(t) \doteq \mathbb{I}_{\{Q^r(t)<\tilde{c}_2 r^\kappa\}}$ and for $j\in\mathbb{A}_J$, consider the stopping times: $\tilde{\tau}_1^{r,j} = \inf\{t\geq 0 : Q^r_j(t) < \tilde{c}_1 r^\kappa\}$, and for $l\geq 1$,

$$\tilde{\tau}_{2l}^{r,j} = \inf\{t\geq \tilde{\tau}_{2l-1}^{r,j} : Q^r_j(t)\geq \tilde{c}_2 r^\kappa\}, \qquad \tilde{\tau}_{2l+1}^{r,j} = \inf\{t\geq \tilde{\tau}_{2l}^{r,j} : Q^r_j(t) < \tilde{c}_1 r^\kappa\}.$$

Define $\mathcal{E}^r_j(t) \doteq \sum_{l=1}^\infty \mathbb{I}_{[\tilde{\tau}_{2l-1}^{r,j},\tilde{\tau}_{2l}^{r,j})}(t)$ for $j\in\mathbb{N}_j$ and let

$$x^r(t) \doteq \rho - v^c\big(\mathcal{Z}^r(t)\big)\mathbb{I}_{\{\mathcal{Z}^r(t)\in\mathcal{M}\}} - v^b\big(\mathcal{Z}^r(t)\big), \qquad \mathfrak{b}^r(t) \doteq x^r(t)\mathbb{I}_{\{\mathcal{E}^r(t)=0\}}. \tag{7}$$

The control policy is then given as $B^r(t) = \int_0^t \mathfrak{b}^r(s)\,ds$, $t\geq 0$.

Going forward, we will assume that r is sufficiently large that $|\rho^r - \rho| < \frac{\rho^*}{4}$ componentwise and $(\tilde{c}_2 - \tilde{c}_1)r^\kappa > 1$.

To illustrate the scheme, we now apply it to the 2LLN network in Example 2.2. Recall that in this simple example $\ker(K)$ is the one-dimensional space spanned by $u_1 = \frac{1}{\sqrt{3}}(-1,-1,1)'$ and u_1 satisfies $\lambda = h\beta\cdot u_1 = -\frac{1}{\sqrt{3}}$. Note that since the first two components of u_1 are negative we have $\mathcal{M} = \{(0,0,0)', (0,0,1)'\}$, and we can define

$$v^c\big((0,0,0)'\big) = v^c\big((0,0,1)'\big) = \frac{1}{\sqrt{3}}u_1.$$

This says that if both q_1 and q_2 are positive we can reduce the cost while maintaining the same workload by moving the queue length in the $\frac{1}{\sqrt{3}}\beta u_1 = \frac{1}{3}(-1,-1,1)'$ direction, and if not, the queue length provides the minimum cost for its corresponding workload. For $z \in \chi_J$, define

$$v_1^b(z) = \frac{1}{36}\mathcal{I}_{\{z_1=1\}\cup(\{z_2=1\}\cap\{z_3=0\})} - \frac{1}{36}\mathcal{I}_{\{z_1=0\}\cap\{z_3=1\}},$$
$$v_2^b(z) = \frac{1}{36}\mathcal{I}_{\{z_2=1\}\cup(\{z_1=1\}\cap\{z_3=0\})} - \frac{1}{36}\mathcal{I}_{\{z_2=0\}\cap\{z_3=1\}},$$
$$v_3^b(z) = \frac{1}{36}\mathcal{I}_{\{z_3=1\}} - \frac{1}{36}\mathcal{I}_{\{z_3=0\}\cap(\{z_1=1\}\cup\{z_2=1\})}$$

and note that this definition of $v^b(z)$ satisfies (6). Observe that the magnitudes of $v^b(z)$ and $v^c(z)$ have been chosen so that while $v^b(z)$ is attempting to keep the queue lengths away from the boundary and to avoid any unnecessary idle time it does not overwhelm the cost reduction efforts of $v^c(z)$ that is trying to shift the workload to the less expensive type 3 jobs.

REMARK 1. The basic idea underlying the proposed scheme is as follows. We want $\tilde{d}(\hat{Q}^r(t))$ close to zero so that (from Proposition 2.1) our queue lengths are near cost-minimizing for the given workload. When $\mathcal{Z}^r(t) \in \mathcal{M}$, the policy uses the vector $v^c(\mathcal{Z}^r(t))$ to reduce $\tilde{d}(\hat{Q}^r(t))$ and when $\mathcal{Z}^r(t) \notin \mathcal{M}$, due to Lemma 7.4 and Proposition 7.7, $\tilde{d}(\hat{Q}^r(t))$ is already close to 0 so the cost associated with that configuration of queue lengths is close to optimal for the corresponding workload. We also want the resources to be utilized at full capacity when their workloads are not near the origin so that asymptotically these workloads behave like reflected Brownian motions. In order to ensure this, we want to prevent all queues from being completely empty so that idleness, when there is work present, is avoided. To achieve this behavior, when $\hat{Q}_j^r(t)$ falls below $\tilde{c}_2 r^{\kappa-1}$ the allocation scheme attempts to increase the corresponding queue. This is because, due to the property $v_j^c(z) \geq 0$ when $z_j = 1$, the vectors $v^c(\mathcal{Z}^r(t))$ will not attempt to decrease queue lengths of job types that fall below this threshold while the vectors $v^b(z)$ will attempt to increase queue lengths of job types below this threshold. If $\hat{Q}_j^r(t)$ continues to decline past $\tilde{c}_1 r^{\kappa-1}$, so that $\mathcal{E}_j^r(t) = 1$, then we stop processing this job type altogether until the corresponding scaled queue length exceeds $\tilde{c}_2 r^{\kappa-1}$ again. The magnitudes of the vectors $v^c(z)$ and $v^b(z)$ are chosen so that while the vector $v^b(z)$ is keeping queue lengths nonempty and ensuring resources can operate at full capacity it does not overwhelm $v^c(z)$ and prevent it from reducing $\tilde{d}(\hat{Q}^r(t))$ to make the queue length configuration to be near cost-minimizing for the associated workload. Note that as long as at least one job type that uses resource i has a queue length greater than $\tilde{c}_2 r^{\kappa-1}$ and all of the job types that use resource i are still being processed (meaning $\mathcal{E}_j^r(t) = 0$ for all $j \in \mathbb{A}_J$ such that $K_{i,j} = 1$) then from the properties $(Kv^b(z))_i = 0$ for $i \in \mathcal{A}_z$, $v^c(z) \in \ker(K)$, and $\mathfrak{b}_j^r(t) = x_j^r(t)$ for all j with $\mathcal{E}_j^r(t) = 0$, we see that under the policy resource i is working at full capacity. In particular, recalling the relation between c_2 and $\tilde{c}_2$, it follows that for any $i \in \mathbb{A}_I$, and large r, if

$$(8)\qquad \hat{W}_i^r(t) \geq 2Jc_2 r^{\kappa-1} \quad \text{and} \quad \sum_{j\in\mathbb{A}_J} K_{i,j}\mathcal{I}_{\{\mathcal{E}_j^r(t)=1\}} = 0, \quad \text{then} \sum_{j\in\mathbb{A}_J} K_{i,j}\mathfrak{b}_j^r(t) = C_i.$$

To argue that the workload process asymptotically behaves like a reflected Brownian motion, we want a resource to (almost always) work at full capacity when its workload exceeds (the asymptotically 0) level $2Jc_2 r^{\kappa-1}$, namely in view of (8), we want to show that $\sum_{j\in\mathbb{A}_J} K_{i,j}\mathcal{I}_{\{\mathcal{E}_j^r(t)=1\}} > 0$ does not happen too frequently. This key property is established in Proposition 3.3, which provides an exponential decay bound on the probability of this happening frequently. In addition, Proposition 3.5 shows that under this scheme the difference between the holding cost of the queue-length process and the optimal cost for the corresponding workload is arbitrarily small for large values of r. These two propositions, which capture the two main ingredients of the HGI performance, are crucial to the proofs of our main results, namely Theorems 2.4 and 2.5, which say that the cost associated with this scheme achieves the hierarchical greedy ideal performance.

2.5. *Main results.* We now introduce some additional notation used in the rest of the paper. Some of this notation will be used to expand the state space of the process under our scheme defined above so that it becomes a Markov process.

DEFINITION 2.4. For $x \in \mathbb{R}_+$ and $j \in \mathbb{A}_J$, define

$$A_j^{r,x}(s) = \max\left\{n \geq 0 : \sum_{l=\tau_j^{r,A}(x)+1}^{\tau_j^{r,A}(x)+n} u_j^r(l) \leq s\right\},$$

$$S_j^{r,x}(s) = \max\left\{n \geq 0 : \sum_{l=\tau_j^{r,S}(x)+1}^{\tau_j^{r,S}(x)+n} v_j^r(l) \leq s\right\}$$

along with their diffusion-scaled versions

$$\hat{A}_j^{r,x}(s) = \frac{1}{r} A_j^{r,x}(r^2 s) - rs\alpha_j^r, \qquad \hat{S}_j^{r,x}(s) = \frac{1}{r} S_j^{r,x}(r^2 s) - rs\beta_j^r.$$

We next introduce the following processes.

DEFINITION 2.5. For $t \in [0, \infty)$ and $j \in \mathbb{A}_J$, define

$$\bar{\xi}_j^{A,r}(t) = \frac{1}{r^2} \sum_{l=1}^{\tau_j^{r,A}(t)} u_j^r(l), \qquad \bar{\xi}_j^{S,r}(t) = \frac{1}{r^2} \sum_{l=1}^{\tau_j^{r,S}(t)} v_j^r(l).$$

We will also need the following fluid-scaled residual arrival and service times at an arbitrary instant t.

DEFINITION 2.6. Let, for $t \geq 0$, $r \in \mathbb{N}$ and $j \in \mathbb{A}_J$,

$$\bar{\Upsilon}_j^{A,r}(t) \doteq \bar{\xi}_j^{A,r}(t) - \big(t - \bar{\Upsilon}_j^{A,r}\big), \qquad \bar{\Upsilon}_j^{S,r}(t) \doteq \bar{\xi}_j^{S,r}(t) - \big(\bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}\big).$$

In addition, define

$$\hat{\Upsilon}_j^{A,r}(t) = r\bar{\Upsilon}_j^{A,r}(t), \qquad \hat{\Upsilon}_j^{S,r}(t) = r\bar{\Upsilon}_j^{S,r}(t) \quad \text{and} \quad \hat{\Upsilon}^r(t) = \big(\hat{\Upsilon}^{A,r}(t), \hat{\Upsilon}^{S,r}(t)\big).$$

In addition, we will use the following fluid-scaled version of the indicator vector $\mathcal{E}^r$, which tells us which jobs are currently not being processed due to their queue lengths being close to 0.

DEFINITION 2.7. For $j \in \mathbb{N}_j$, $t \geq 0$ and $r \in \mathbb{N}$, define $\tilde{\mathcal{E}}_j^r(t) \doteq \mathcal{E}_j^r(r^2 t)$.

From (2), for all $j \in \mathbb{A}_J$ and $0 \leq s < t$, we have

$$\begin{aligned}\hat{Q}_j^r(t) &= \hat{Q}_j^r(s) + r^{-1}\mathcal{I}_{\{t-s \geq \bar{\Upsilon}_j^{A,r}(s) > 0\}} + r^{-1} A_j^{r,s}\big(r^2\big(t - s - \bar{\Upsilon}_j^{A,r}(s)\big)^+\big) \\ &\quad - r^{-1}\mathcal{I}_{\{\bar{B}_j^r(t) - \bar{B}_j^r(s) \geq \bar{\Upsilon}_j^{S,r}(s) > 0\}} - r^{-1} S_j^{r,s}\big(r^2\big(\bar{B}_j^r(t) - \bar{B}_j^r(s) - \bar{\Upsilon}_j^{S,r}(s)\big)^+\big).\end{aligned} \tag{9}$$

It then follows that $\hat{Y}^r(t) \doteq (\hat{Q}^r(t), \hat{\Upsilon}^r(t), \tilde{\mathcal{E}}^r(t))$ is a strong Markov process with values in $\mathcal{Y}^r \doteq \mathbb{N}^J_{1/r} \times \mathbb{R}^{2J}_+ \times \chi_J$, with respect to the filtration $\{\mathcal{G}^r(t)\}_{t\ge 0}$. For $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$ and bounded measurable $f : \mathcal{Y}^r \to \mathbb{R}$, we use the notation

$$\begin{aligned} &E_y\big[f\big((\hat{Q}^r(t), \hat{\Upsilon}^r(t), \tilde{\mathcal{E}}^r(t))\big)\big] \\ &\quad = E\big[f\big((\hat{Q}^r(t), \hat{\Upsilon}^r(t), \tilde{\mathcal{E}}^r(t))\big)|\big(\hat{Q}^r(0), \hat{\Upsilon}^r(0), \tilde{\mathcal{E}}^r(0)\big) = y\big]. \end{aligned}$$

When $f = 1_A$, we will write $E_y[f(\hat{Y}^r(t))]$ as $P_y(\hat{Y}^r(t) \in A)$.

Recall the holding cost vector h from Section 2.4. We consider two types of costs. The first is the infinite horizon discounted cost. Fix a discount factor $\varsigma \in (0, \infty)$. For $r \in \mathbb{N}$ and $y^r \in \mathcal{Y}^r$, the infinite horizon discounted cost associated with the control policy B^r in Definition 2.3 is

$$J^r_D(B^r, y^r) \doteq \int_0^\infty e^{-\varsigma t} E_{y^r}\big[h \cdot \hat{Q}^r(t)\big]\,dt.$$

The second cost we consider is the long-term cost per unit time (also referred to as the ergodic cost). Define, for $r \in \mathbb{N}$, $y^r \in \mathcal{Y}^r$, and $T > 0$,

$$J^{r,T}_E(B^r, y^r) \doteq E_{y^r}\left[\frac{1}{T}\int_0^T h \cdot \hat{Q}^r(t)\,dt\right].$$

Then the long-term cost per unit time for B^r and $y^r \in \mathcal{Y}^r$ is

$$J^r_E(B^r, y^r) \doteq \limsup_{T\to\infty} J^{r,T}_E(B^r, y^r).$$

In order to describe the limit model under the heavy traffic scaling, we now recall the definition of a Skorokhod map on the positive orthant $\mathbb{R}^d_+$ associated with normal reflections at the boundary. The reason such a Skorohod map emerges in our analysis is that (i) due to the local traffic condition and the fact that (under HGI performance) every resource utilizes its full capacity whenever there is work for that resource in the system, the state space of the workload process is all of the positive orthant; and (ii) since this is a "one pass" system, idleness of one resource does not (directly) impact the flow of work to any other resource, and as a consequence, one does not get oblique reflections from idleness.

DEFINITION 2.8. Let $T \in (0, \infty)$ and $f \in \mathcal{D}([0, T] : \mathbb{R}^d)$. We say that $(\phi, h) \in \mathcal{D}([0, T] : \mathbb{R}^d) \times \mathcal{D}([0, T] : \mathbb{R}^d)$ solves the Skorohod problem for f if (a) $\phi(t) = f(t) + h(t)$ for all $t \in [0, T]$, (b) $h(\cdot)$ is nondecreasing and $h(0) = -f(0) \vee 0$, (c) $\phi(\cdot) \ge 0$, (d) $\int_{[0,\infty)} \mathcal{I}_{\{\phi_i(t)>0\}}\,dh_i(t) = 0$ for all $i \in \mathbb{N}_d$.

It is known that there is a unique solution to the above Skorohod problem with normal reflections (which can be essentially regarded as d-one dimensional Skorohod problems) for every $f \in D([0, T] : \mathbb{R}^d)$ and denoting the unique ϕ associated with f as $\Gamma_d(f)$, the Skorohod map $\Gamma_d : \mathcal{D}([0, T] : \mathbb{R}^d) \to \mathcal{D}([0, T] : \mathbb{R}^d)$ has the following Lipschitz property: There exists $K_{\Gamma_d} \in (0, \infty)$ such that for all $T > 0$ and $f_1, f_2 \in D([0, T] : \mathbb{R}^d)$,

$$\sup_{0\le t\le T} \big|\Gamma_d(f_1)(t) - \Gamma_d(f_2)(t)\big| \le K_{\Gamma_d} \sup_{0\le t\le T} \big|f_1(t) - f_2(t)\big|.$$

Note that for $f \in D([0, T] : \mathbb{R}^d)$, $\Gamma_d(f)_i = \Gamma_1(f_i)$ for all $i = 1, \dots d$. When $d = I$, we will write $\Gamma_d = \Gamma_I$ as simply Γ. Also, it is easily verified that K_{Γ_1} can be taken to be 2. We refer the reader to [14], Section 3.6.C, for a discussion of the one-dimensional Skorohod problem.

Consider diagonal matrices Σ^u and Σ^v given with diagonal entries $\{\sigma^u_j\}^J_{j=1}$ and $\{\sigma^v_j\}^J_{j=1}$, respectively, and define $\Sigma \doteq KM(\Sigma^u + \Sigma^v R)M^\top K^\top$. Let $(\check{\Omega}, \check{\mathcal{F}}, \{\check{\mathcal{F}}_t\}, \check{P})$ be a filtered probability space on which is given an I-dimensional $\{\check{\mathcal{F}}_t\}$-Brownian motion $\{\hat{X}(t)\}$ with drift

θ and covariance matrix Σ. For $w_0 \in \mathbb{R}_+^I$, let $\check{W}^{w_0}$ be a $\mathbb{R}_+^I$ valued continuous stochastic process defined as

$$\check{W}^{w_0}(t) = \Gamma\big(w_0 + \hat{X}(\cdot)\big)(t), \quad t \geq 0. \tag{10}$$

The process $\check{W}^{w_0}$ is referred to as a I-dimensional reflected Brownian motion with initial value w_0, drift θ and covariance matrix Σ. It is well known [12] that, under our conditions (specifically the property $\theta < 0$), $\{\check{W}^{w_0}\}_{w_0 \in \mathbb{R}_+^I}$ defines a Markov process that has a unique invariant probability distribution, which we denote as π.

The *hierarchical greedy ideal* (HGI) associated with the discounted and the ergodic cost, for $w_0 \in \mathbb{R}_+^I$, are given respectively as

$$\mathrm{HGI}_D(w_0) \doteq \int_0^\infty e^{-\varsigma t} E\big[\hat{h}\big(\check{W}^{w_0}(t)\big)\big]\,dt, \qquad \mathrm{HGI}_E \doteq \int_{\mathbb{R}_I^+} \hat{h}(w)\pi(dw). \tag{11}$$

The following theorem says that our scheme achieves the hierarchical greedy ideal infinite horizon discounted cost.

THEOREM 2.4 (Proof in Section 3.2). *Let $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$ be an arbitrary sequence satisfying* $\lim_{r\to\infty} \hat{q}^r = \tilde{q}$ *for some* $\tilde{q} \in \mathbb{R}_+^J$ *and* $\sup_r r\hat{\Upsilon}^r < \infty$. *Define* $w_0 \doteq KM\tilde{q}$. *Then* $\lim_{r\to\infty} J_D^r(B^r, y^r) = \mathrm{HGI}_D(w_0)$.

The next theorem gives a similar result for the long-term cost per unit time.

THEOREM 2.5 (Proof in Section 3.3). *Let $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$ be an arbitrary sequence satisfying* $\sup_r \hat{q}^r < \infty$ *and* $\sup_r r\hat{\Upsilon}^r < \infty$. *Then* $\lim_{r\to\infty} J_E^r(B^r, y^r) = \mathrm{HGI}_E$.

For some background and rationale for the terminology of HGI for the costs in (11), we refer the reader to [11]. Roughly speaking, the fact that the asymptotic cost for our sequence of policies is given in terms of the function $\hat{h}$ corresponds to the feature of instantaneous holding cost minimization for the given workload (recall the definition of $\hat{h}$), and the fact that the limit is determined by a reflected Brownian motion, which has the feature that the reflection (which roughly corresponds to the asymptotic idleness) occurs only when the process hits the faces of the orthant (namely one of the coordinates is zero), captures the no-idleness property of HGI.

In the rest of the paper, we simplify the notation by leaving out the subscript y on the expected value that specifies the initial condition unless it is particularly relevant in that situation.

The following result is immediate from the definition of the control policy $B^r(\cdot)$. For part (d), see Remark 1 and for part (e) recall that $v_j^b(z) > 0$ if $z_j = 1$.

PROPOSITION 2.6. *The scheme given in Definition 2.3 has following properties*:

(a) *For any $j \in \mathbb{A}_J$ and $t \geq 0$ if $\tilde{\mathcal{E}}_j^r(t) = 1$, then $\frac{d}{dt}\bar{B}_j^r(t) = 0$.*

(b) *For any $j \in \mathbb{A}_J$ and $t \geq 0$ if $\tilde{\mathcal{E}}_j^r(t) = 0$, then $\tilde{\mathcal{E}}_j^r(s) = 0$ for all $s \geq t$ such that $\bar{B}_j^r(s) - \bar{B}_j^r(t) < \bar{\Upsilon}_j^{S,r}(t)$).*

(c) *For r sufficiently large for all $j \in \mathbb{A}_J$ and $t \geq 0$, we have $x_j^r(t) \geq \frac{\rho^*}{4}$, and consequently, if $\mathcal{E}_j^r(t) = 0$, then $\mathfrak{b}_j^r(t) \geq \frac{\rho^*}{4}$.*

(d) *For all $i \in \mathbb{A}_I$ and $t \geq 0$ if $\hat{W}_i^r(t) \geq 2Jc_2 r^{\kappa-1}$ and $\sum_{j=1}^J K_{i,j}\mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(t)=1\}} = 0$, then $\frac{d}{dt}(K\bar{B}^r(t))_i = (K\mathfrak{b}^r)_i(r^2 t) = C_i$.*

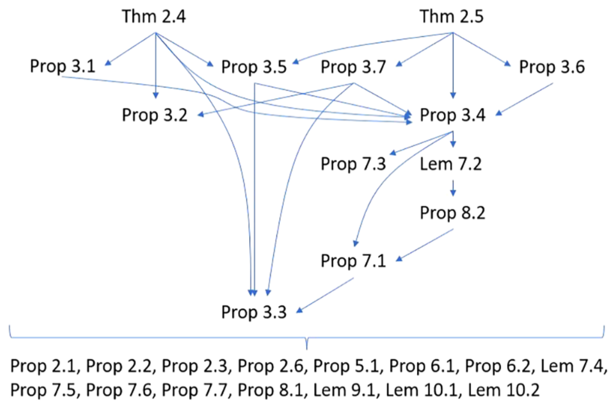

FIG. 2. *The interdependence of results.*

(e) *For r sufficiently large there exists $\Delta > 0$, such that for all $j \in \mathbb{A}_J$ and $t \geq 0$ if $Q^r_j(t) < \tilde{c}_2 r^\kappa$ then $\beta^r_j \mathfrak{b}^r_j(t) \leq \alpha^r_j - \Delta$.*

The remainder of the paper is organized as follows. Section 3 proves the main results of this work, namely Theorems 2.4 and 2.5, by introducing a set of seven propositions. The rest of the paper is devoted to the proof of these propositions. Propositions 3.6 and 3.7 give the tightness of certain path occupation measures and characterize the weak limit points. These results are needed only in the treatment of the ergodic cost. Analogous results when the interarrival times and service times are exponential were established in [6] and so we only provide a sketch of the proofs. These are in Section 11. Proposition 3.1 gives a functional central limit theorem, which proceeds by standard methods using the heavy traffic condition and central limit theorem for renewal processes. Proof is sketched in Section 4. Propositions 3.2 and 3.3 are key ingredients in establishing the no-idleness feature of the HGI performance. The first is proved in Section 5 while the second is proved in Section 6 using certain large deviation estimates for renewal processes (Proposition 6.2) the proofs of which is deferred to Section 9.2. Proposition 3.4 gives a key uniform in time exponential moment estimate. The proof of this result is in Section 7.1 based on several Lyapunov function lemmas, the proofs of which are relegated to Section 8. Finally, Proposition 3.5 is the key ingredient in showing that our policy achieves the instantaneous cost minimization feature of the HGI performance. Proof of this proposition is in Section 7.2, based on some auxiliary lemmas that are proved in Sections 9 and 10.

To describe the interdependence of results of the paper, we provide Figure 2 with the interpretation that the results can only depend on other results located lower in the figure. To simplify the figure and to make it easier to read, we have placed results, which are not directly used in the proofs of HGI performance (Theorems 2.4 and 2.5) and are not heavily dependent on other results into a "foundation" group at the bottom. The proofs of results in this foundation group depend on at most one other result in the paper, which is also in the foundation group, but these foundation results may be used in the proofs of results higher up the figure. The interdependence of results outside of the foundation group is indicated using arrows, where for instance, an arrow going from Theorem 2.4 to Proposition 3.1 indicates that Proposition 3.1 is used in the proof of Theorem 2.4.

3. Proof of main theorems.

3.1. *Some auxiliary results.* Proof of the following proposition follows by standard methods. A sketch is given in Section 4.

PROPOSITION 3.1. *Let $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$ be an arbitrary sequence satisfying $\sup_r \hat{q}^r < \infty$ and $\lim_{r\to\infty} \hat{\Upsilon}^r = 0$. Then $\hat{X}^r(\cdot) \to \hat{X}(\cdot)$ in distribution in $\mathcal{D}^I$ where $\hat{X}(\cdot)$ is as introduced above* (10).

Next, two propositions are needed to establish the nonidleness feature of the HGI performance.

PROPOSITION 3.2 (Proof in Section 5). *For all $r \in \mathbb{N}$, $y^r \in \mathcal{Y}^r$, $t \geq 0$ and $T \in (0, \infty)$, we have P_{y^r} a.s.,*

$$\sup_{s\in[0,T]} \{ |\Gamma(\hat{W}_i^r(t) + \hat{X}_i^r(t+\cdot) - \hat{X}_i^r(t))(s) - \hat{W}_i^r(t+s)| \}$$

$$\leq 3Jc_2 r^{\kappa-1} + 2r^{-1} C_i \sum_{j=1}^{J} \int_{r^2 t}^{r^2(t+T)} \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}} \, ds.$$

PROPOSITION 3.3 (Proof in Section 6). *For each $\epsilon > 0$, there exists $B, R \in (0, \infty)$ such that for all $r \geq R$, $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$, $T \geq 1$ and $j \in \mathbb{A}_J$ we have*

$$P_{y^r}\left(\int_0^{r^2 T} \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}} \, ds \geq r \hat{\Upsilon}_j^{A,r} + \epsilon T r^{\frac{7}{4}\kappa} \right) \leq e^{-B \mathrm{Tr}^{\frac{1}{8}\kappa}}.$$

The proofs of the next two propositions are in Section 7. The first gives a key uniform in time exponential moment estimate. The second is key in showing that our policy achieves the instantaneous cost minimization feature of the HGI performance.

PROPOSITION 3.4 (Proof in Section 7). *There exist constants $\tilde{R}, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{\delta}, \delta > 0$ such that for all $r \geq \tilde{R}$, $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$, $c \in (0, \delta]$ and $t \geq 0$ we have*

$$E_{y^r}[e^{c|\hat{W}^r(t)|}] \leq \tilde{B}_1 e^{-\tilde{\delta} t + \tilde{B}_2(|\hat{q}^r| + |\hat{\Upsilon}^r|)} + \tilde{B}_3.$$

PROPOSITION 3.5 (Proof in Section 7). *For any $\epsilon \in (0, 1)$ and $M < \infty$, there exist constants $T^*, R \in (0, \infty)$ such that for all $r \geq R$, $T \geq T^*$, $t \geq 0$ and $y^r \in \mathcal{Y}^r$ satisfying $\hat{q}^r \leq M$ and $r\hat{\Upsilon}^r \leq M$ we have*

$$E_{y^r}\left[\frac{1}{T} \int_0^T |h \cdot \hat{Q}^r(t+s) - \hat{h}(\hat{W}^r(t+s))| \, ds \right] \leq \epsilon$$

and

$$E_{y^r}\left[\int_0^\infty e^{-\varsigma s} |h \cdot \hat{Q}^r(t+s) - \hat{h}(\hat{W}^r(t+s))| \, ds \right] \leq \epsilon.$$

3.2. *Proof of Theorem* 2.4. Fix $T \in (0, \infty)$. From Proposition 3.1, $\hat{X}^r(\cdot) \to \hat{X}(\cdot)$ in distribution on $D([0, T] : \mathbb{R}^I)$ where $\hat{X}(\cdot)$ is as introduced above (10). Since $KM^r \hat{q}^r \to w_0$, by the continuity of the Skorokhod map, we have $\Gamma(KM^r\hat{q}^r + \hat{X}^r(\cdot)) \to \check{W}^{w_0}$ in distribution on $D([0, T] : \mathbb{R}^d)$, where $\check{W}^{w_0}(t)$ is the reflected Brownian motion given by (10). In addition, using Propositions 3.2 and 3.3 (and recalling that $\sup_r r\hat{\Upsilon}^r < \infty$ and $\kappa < 1/4$), we see that

$$\sup_{t\in[0,T]} |\Gamma(KM^r\hat{q}^r + \hat{X}^r(\cdot))(t) - \hat{W}^r(t)| \to 0$$

in probability. Combining this gives $\hat{W}^r \to \check{W}^{w_0}$ in distribution on $D([0, T] : \mathbb{R}^d)$. For $k \in (0, \infty)$, define $\hat{h}_k(w) = k \wedge \hat{h}(w)$ and note that $\hat{h}_k$ is a bounded, continuous function on $\mathbb{R}_+^I$. Consequently, it follows that for any $T < \infty$,

$$\lim_{r\to\infty} E\left[\int_0^T e^{-\varsigma t} \hat{h}_k(\hat{W}^r(t)) \, dt \right] = E\left[\int_0^T e^{-\varsigma t} \hat{h}_k(\check{W}^{w_0}(t)) \, dt \right].$$

Let $\epsilon > 0$ be arbitrary. Proposition 3.5 (recall that $\hat{q}^r \to \tilde{q}$ and $\sup_r r\hat{\Upsilon}^r < \infty$) tells us that there exists $R_1 \in (0, \infty)$ such that for all $r \geq R_1$ we have

$$\begin{aligned}\bigg|E\bigg[\int_0^\infty e^{-\varsigma t} h \cdot \hat{Q}^r(t)\,dt\bigg] - E\bigg[\int_0^\infty e^{-\varsigma t} \hat{h}(\hat{W}^r(t))\,dt\bigg]\bigg| \\ \leq E\bigg[\int_0^\infty e^{-\varsigma t} |\hat{h}(\hat{W}^r(t)) - h \cdot \hat{Q}^r(t)|\,dt\bigg] \leq \frac{\epsilon}{6}.\end{aligned}$$

Due to Proposition 3.4 (and once more using $\hat{q}^r \to \tilde{q}$ and $\sup_r r\hat{\Upsilon}^r < \infty$), there exists $B_1 \in (0, \infty)$ and $R_2 \in [R_1, \infty)$ such that for all $r \geq R_2$ and $t \geq 0$ we have $E[|\hat{W}^r(t)|] \leq B_1$. In addition, the definition of $\hat{h}$ implies that there exists $B_2 < \infty$ such that for all $w \in \mathbb{R}_+^I$ we have $\hat{h}(w) \leq B_2|w|$. Choose $T_1 \in (0, \infty)$ sufficiently large that $e^{-\varsigma T_1} \frac{B_2 B_1}{\varsigma} \leq \frac{\epsilon}{3}$. Consequently, for all $T \geq T_1$ and $r \geq R_2$, we have

$$\bigg|\int_0^\infty e^{-\varsigma t} E[\hat{h}(\hat{W}^r(t))]\,dt - \int_0^T e^{-\varsigma t} E[\hat{h}(\hat{W}^r(t))]\,dt\bigg| \leq \frac{B_2 B_1}{\varsigma} e^{-\varsigma T_1} \leq \frac{\epsilon}{3}.$$

Proposition 3.4 also implies that there exist constants $K_1 \in (0, \infty)$ and $R_3 \in [R_2, \infty)$ such that for all $r \geq R_3$ and $t \geq 0$ we have $E[\mathcal{I}_{\{|\hat{W}^r(t)| \geq K_1\}} |\hat{W}^r(t)|] \leq \varsigma\epsilon/(3B_2)$. Consequently, for $r \geq R_3$, $t \geq 0$ and $k \geq K_1 B_2$ we have

$$\begin{aligned}&E[|\hat{h}(\hat{W}^r(t)) - \hat{h}_k(\hat{W}^r(t))|] \\ &\quad \leq E[\mathcal{I}_{\{\hat{h}(\hat{W}^r(t)) \geq k\}} \hat{h}(\hat{W}^r(t))] \leq B_2 E[\mathcal{I}_{\{|\hat{W}^r(t)| \geq K_1\}} |\hat{W}^r(t)|] \leq \frac{\varsigma\epsilon}{3}.\end{aligned}$$

Then for all $T < \infty$ we have

$$\bigg|E\bigg[\int_0^T e^{-\varsigma t} \hat{h}(\hat{W}^r(t))\,dt\bigg] - E\bigg[\int_0^T e^{-\varsigma t} \hat{h}_k(\hat{W}^r(t))\,dt\bigg]\bigg| \leq \int_0^\infty e^{-\varsigma t} \frac{\varsigma\epsilon}{3}\,dt \leq \frac{\epsilon}{3}.$$

In addition, due to the monotone convergence theorem there exists $T_2 \in [T_1, \infty)$ and $K_2 \in [K_1, \infty)$ such that for all $T \geq T_2$ and $k \geq K_2$ we have

$$\bigg|E\bigg[\int_0^T e^{-\varsigma t} \hat{h}_k(\check{W}^{w_0}(t))\,dt\bigg] - E\bigg[\int_0^\infty e^{-\varsigma t} \hat{h}(\check{W}^{w_0}(t))\,dt\bigg]\bigg| \leq \frac{\epsilon}{6}.$$

Finally, for $k \geq K_2$ and $T \geq T_2$ we have

$$\begin{aligned}&\limsup_{r\to\infty} J_D^r(B^r, y_0^r) \\ &\quad = \limsup_{r\to\infty} E\bigg[\int_0^\infty e^{-\varsigma t} h \cdot \hat{Q}^r(t)\,dt\bigg] \leq \limsup_{r\to\infty} E\bigg[\int_0^\infty e^{-\varsigma t} \hat{h}(\hat{W}^r(t))\,dt\bigg] + \frac{\epsilon}{6} \\ &\quad \leq \limsup_{r\to\infty} E\bigg[\int_0^T e^{-\varsigma t} \hat{h}(\hat{W}^r(t))\,dt\bigg] + \frac{3\epsilon}{6} \leq \limsup_{r\to\infty} E\bigg[\int_0^T e^{-\varsigma t} \hat{h}_k(\hat{W}^r(t))\,dt\bigg] + \frac{5\epsilon}{6} \\ &\quad = E\bigg[\int_0^T e^{-\varsigma t} \hat{h}_k(\check{W}^{w_0}(t))\,dt\bigg] + \frac{5\epsilon}{6} \leq E\bigg[\int_0^\infty e^{-\varsigma t} \hat{h}(\check{W}^{w_0}(t))\,dt\bigg] + \epsilon.\end{aligned}$$

Similarly,

$$\liminf_{r\to\infty} J_D^r(B^r, y_0^r) \geq E\bigg[\int_0^\infty e^{-\varsigma t} \hat{h}(\check{W}^{w_0}(t))\,dt\bigg] - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows.

3.3. *Proof of Theorem* 2.5. We begin with some auxiliary results. From Proposition 3.4 it follows that, with y^r as in Theorem 2.5, there exist constants $R, B_1 \in (0,\infty)$ and $c > 0$ such that for all $r \geq R$ and $t \geq 0$ we have $E_{y^r}[e^{c|\hat{W}^r(t)|}] \leq B_1$. As a result, $\sup_{r\geq R}\{J_E^r(B^r, y^r)\} < \infty$.

DEFINITION 3.1. For $r \in \mathbb{N}$ and $y^r \in \mathcal{Y}^r$ such that $J_E^r(B^r, y^r) < \infty$ choose $T_r \in [r,\infty)$ such that $|J_E^{r,T_r}(B^r) - J_E^r(B^r)| < \frac{1}{r}$. If $J_E^r(B^r, y^r) = \infty$, set $T_r = 1$. Define the random variable ν^r with values in $\mathcal{P}(\mathbb{R}_+^I \times D([0,1]:\mathbb{R}^I))$ by

$$\nu^r \doteq \frac{1}{T_r}\int_0^{T_r} \delta_{(\hat{W}^r(t), \hat{X}^r(t+\cdot)-\hat{X}^r(t))}\, dt.$$

The next two propositions give tightness of the above occupation measure and characterize its weak limit points. Since analogous results for exponential primitives were studied in [6], we only give proof sketches in Section 11.

PROPOSITION 3.6. *Let* $\{\nu^r\}$ *be as in Definition* 3.1 *associated with a* $\{y^r\} \in \mathcal{Y}^r$ *and assume* $\sup_r \hat{q}^r < \infty$ *and* $\sup_r r\hat{\Upsilon}^r < \infty$. *Then* $\{\nu^r\}$ *is tight as a sequence of random variables with values in* $\mathcal{P}(\mathbb{R}_+^I \times D([0,1]:\mathbb{R}^I))$.

PROPOSITION 3.7. *Assume* $\{y^r\} \in \mathcal{Y}^r$ *satisfies* $\sup_r \hat{q}^r < \infty$ *and* $\sup_r r\hat{\Upsilon}^r < \infty$ *and consider a subsequence* $\{\nu^{r_m}\}_{m=1}^\infty$ *of the tight sequence in Proposition* 3.6 *that converges in distribution to a random variable* ν^* *with values in* $\mathcal{P}(\mathbb{R}_+^I \times D([0,1]:\mathbb{R}^I))$. *Then the coordinate maps* (w,x) *on* $\mathbb{R}_+^I \times D([0,1]:\mathbb{R}^I)$ *satisfy, under* $\nu^*(\omega)$, *for a.e.* ω,

(a) x *is a Brownian motion with drift* θ *and covariance* Σ *with respect to the filtration* $\mathcal{F}(t) = \sigma(w, x(s)) : s \leq t)$,
(b) $\Gamma_I(w + x(\cdot))(s) \overset{d}{=} w$ *for every* $s \in [0,1]$.

We now prove Theorem 2.5. It suffices to show that for any subsequence of $\{J_E^r(B^r, y^r)\}_r$ there is a further subsequence that converges to HGI_E. Let ν^r be the random variables given by Definition 3.1 associated with a $y^r \in \mathcal{Y}^r$ as in the statement of Theorem 2.5. For an arbitrary subsequence, Propositions 3.6 and 3.7 show that there is a further subsequence (which we will index by r_m), which satisfies $\nu^{r_m} \to \nu^*$ where ν^* is such that the coordinate variables (x, w) under $\nu^*(\omega)$ for a.e. ω satisfy $w \overset{d}{=} \Gamma_I(w + x(\cdot))$ and x is a Brownian motion with drift θ and covariance Σ with respect to the filtration $\mathcal{F}(t) = \sigma(w, x(s)) : s \leq t)$. Since the invariant distribution π of the reflected Brownian motion in equation (10) is unique, it follows that $\nu_{(1)}^*(\omega) \overset{d}{=} \pi$ for a.e. ω, where $\nu_{(1)}^*$ is the first marginal of ν^*. Consequently,

$$\int_{\mathbb{R}_+^I} \hat{h}(w)\pi(dw) = E\left[\int_{\mathbb{R}_+^I} \hat{h}(w)\nu_{(1)}^*(dw)\right].$$

For $k < \infty$, let $\hat{h}_k(w) \doteq k \wedge \hat{h}(w)$. Then since $\hat{h}_k(\cdot)$ is a bounded, continuous function

$$\lim_{m\to\infty}\int_{\mathbb{R}_+^I} E[\hat{h}_k(w)\nu_{(1)}^{r_m}(dw)] = E\left[\int_{\mathbb{R}_+^I} \hat{h}_k(w)\nu_{(1)}^*(dw)\right] = \int_{\mathbb{R}_+^I} \hat{h}_k(w)\pi(dw).$$

Let $\epsilon > 0$ be arbitrary. As in the proof of Theorem 2.4, using Proposition 3.4 (recall that $\sup_r \hat{q}^r < \infty$ and $\sup_r r\hat{\Upsilon}^r < \infty$), we see that there exist constants $K_1, R_1 \in (0,\infty)$ such that for all $r \geq R$, $t \geq 0$ and $k \geq K_1$,

$$E[\hat{h}(\hat{W}^r(t)) - \hat{h}_k(\hat{W}^r(t))] \leq \frac{\epsilon}{4}.$$

Using the monotone convergence theorem, we can choose $K_2 \in [K_1, \infty)$ such that for all $k \geq K_2$

$$\left| E\left[\int_{\mathbb{R}_+^I} \hat{h}(w)\nu_{(1)}^*(dw)\right] - E\left[\int_{\mathbb{R}_+^I} \hat{h}_k(w)\nu_{(1)}^*(dw)\right]\right| \leq \frac{\epsilon}{2}.$$

In addition, from Proposition 3.5 (recall that $\sup_r \hat{q}^r < \infty$ and $\sup_r r\hat{\Upsilon}^r < \infty$), there exists a constant $R_2 \in [R_1, \infty)$ such that for all $r \geq R_2$,

$$E\left[\frac{1}{T_r}\int_0^{T_r} \left|h \cdot \hat{Q}^r(t) - \hat{h}(\hat{W}^r(t))\right| dt\right] \leq \frac{\epsilon}{4}.$$

Consequently, for all $k \geq K_2$, we have

$$\begin{aligned}
&\limsup_{m\to\infty} J_E^{r_m}(B^r) \\
&\quad\leq \limsup_{m\to\infty} E\left[\frac{1}{T_{r_m}}\int_0^{T_{r_m}} h \cdot \hat{Q}^r(t)\,dt\right] \leq \limsup_{m\to\infty} E\left[\frac{1}{T_{r_m}}\int_0^{T_{r_m}} \hat{h}(\hat{W}^r(t))\,dt\right] + \frac{\epsilon}{4} \\
&\quad\leq \limsup_{m\to\infty} E\left[\frac{1}{T_{r_m}}\int_0^{T_{r_m}} \hat{h}_k(\hat{W}^r(t))\,dt\right] + \frac{\epsilon}{2} = \limsup_{m\to\infty} E\left[\int_{\mathbb{R}_+^I} \hat{h}_k(w)\nu_{(1)}^{r_m}(dw)\right] + \frac{\epsilon}{2} \\
&\quad= E\left[\int_{\mathbb{R}_+^I} \hat{h}_k(w)\nu_{(1)}^*(dw)\right] + \frac{\epsilon}{2} \leq E\left[\int_{\mathbb{R}_+^I} \hat{h}(w)\nu_{(1)}^*(dw)\right] + \epsilon = \int_{\mathbb{R}_+^I} \hat{h}(w)\pi(dw) + \epsilon.
\end{aligned}$$

Similarly, $\liminf_{m\to\infty} J_E^{r_m}(B^r) \geq \int_{\mathbb{R}_+^I} \hat{h}(w)\pi(dw) - \epsilon$. Since $\epsilon > 0$ was arbitrary, the result follows.

4. Proof sketch of Proposition 3.1. Recall the definition of $\hat{X}^r(\cdot)$ in (2). The central limit theorem for renewal processes (see, e.g., [4], Theorem 14.6) and the fact that $M^r \to M$) implies that $\theta\,\mathrm{id}(\cdot) + KM^r\hat{A}^r(\cdot) - KM^r\hat{S}^r(\rho\,\mathrm{id}(\cdot)) \to \hat{X}(\cdot)$ in distribution in $\mathcal{D}^I$, where id is the identity map. Clearly, $\frac{1}{r}KM^r\mathcal{I}_{\{\mathrm{id}(\cdot)\geq\bar{\Upsilon}^{A,r}>0\}} \to 0$, $\frac{1}{r}KM^r\mathcal{I}_{\{\bar{B}^r(\cdot)\geq\bar{\Upsilon}^{S,r}>0\}} \to 0$, and (due to Condition 2 and the paragraph that follows) $rK(\rho^r - \rho)\,\mathrm{id}(\cdot) \to \theta\,\mathrm{id}(\cdot)$ in $\mathcal{D}^I$. Since $\lim_{r\to\infty}\hat{\Upsilon}^r = 0$, it follows that $(\mathrm{id}(\cdot) - \bar{\Upsilon}^{A,r})^+ \to \mathrm{id}(\cdot)$, $rK\rho^r(\mathrm{id}(\cdot)\wedge\bar{\Upsilon}^{A,r}) \to 0$ and $rK(\bar{B}^r(\cdot)\wedge\bar{\Upsilon}^{S,r}) \to 0$ in $\mathcal{D}^I$. In addition, it can be shown using Proposition 3.4, Proposition 6.2 and the assumptions that $\sup_r \hat{q}^r < \infty$ and $\sup_r r\hat{\Upsilon}^r < \infty$ that $(\bar{B}^r(\cdot) - \bar{\Upsilon}^{S,r})^+ \to \rho\,\mathrm{id}(\cdot)$ in $\mathcal{D}^I$. The proof of this is very similar to the proof of [6], Theorem 15, part 2, and is therefore omitted. Putting all of this together implies that $\hat{X}^r(\cdot) \to \hat{X}(\cdot)$ in distribution in $\mathcal{D}^I$ and completes the proof.

5. Proof of Proposition 3.2. We begin with an auxiliary result.

PROPOSITION 5.1. *Let $T \in (0,\infty)$ and $f \in \mathcal{D}([0,T]:\mathbb{R})$ be arbitrary and let $\phi_1 = \Gamma_1(f)$. Assume $\phi_2 = f + h_2$ where $h_2 \in \mathcal{D}([0,T]:\mathbb{R})$ is a nondecreasing function satisfying $0 \leq h_2(0) \leq (-f(0))^+$ and $\int_0^T \mathcal{I}_{\{\phi_2(t)>0\}}\,dh_2(t) = 0$. Then $\phi_2 \leq \phi_1$. Let $\phi_3 = f + h_3$ where $h_3 \in \mathcal{D}([0,T]:\mathbb{R})$ is a nondecreasing function satisfying $h_3(0) \geq 0$ and $\phi_3 \geq 0$. Then $\phi_1 \leq \phi_3$.*

PROOF. Define $h_1(t) = \sup_{s\in[0,t]}\{(-f(s))^+\}$ and note that $\phi_1 = f + h_1$. We will first prove $\phi_2 \leq \phi_1$ and to do so it is sufficient to prove $h_2 \leq h_1$. Note that $0 \leq h_2(0) \leq (-f(0))^+$ implies $h_2(0) \leq h_1(0)$. Arguing via contradiction, assume there exists $t_2^* \in (0,T]$ such that $h_2(t_2^*) > h_1(t_2^*) = \sup_{s\in[0,t_2^*]}\{(-f(s))^+\}$. For notational convenience, let $a \doteq \sup_{s\in[0,t_2^*]}\{(-f(s))^+\}$ and define $t_1^* = \sup\{s \in [0,T] : h_2(s) \leq a\}$. Note that since

$h_2(t_2^*) > a$ we have $t_1^* \le t_2^*$. If $h_2(t_1^*) > a$ then, since $h_2(s) \le a$ for all $s < t_1^*$, we must have $\int_{\{t_1^*\}} dh_2(u) > 0$. However,

$$\begin{aligned}\phi_2(t_1^*) &= f(t_1^*) + h_2(t_1^*) > f(t_1^*) + a = f(t_1^*) + \sup_{s\in[0,t_2^*]} \{(-f(s))^+\} \\ &\ge f(t_1^*) + \sup_{s\in[0,t_1^*]} \{(-f(s))^+\} \ge 0\end{aligned}$$

which contradicts the fact that $\int_0^T \mathcal{I}_{\{\phi_2(t)>0\}} \, dh_2(t) = 0$. Therefore, we must have $h_2(t_1^*) \le a$ and so $t_1^* < t_2^*$. By the definition of t_1^*, we have $h_2(t) > a$ for all $t \in (t_1^*, t_2^*]$. Therefore, for any $t \in (t_1^*, t_2^*]$, we have

$$\begin{aligned}\phi_2(t) &= f(t) + h_2(t) > f(t) + a = f(t) + \sup_{s\in[0,t_2^*]} \{(-f(s))^+\} \\ &\ge f(t) + \sup_{s\in[0,t]} \{(-f(s))^+\} \ge 0.\end{aligned}$$

However, since $h_2(t_2^*) > a \ge h_2(t_1^*)$ we have $\int_{(t_1^*,t_2^*]} dh_2(t) > 0$, which since $\phi_2(t) > 0$ for all $t \in (t_1^*, t_2^*]$ contradicts the fact that $\int_0^T \mathcal{I}_{\{\phi_2(t)>0\}} \, dh_2(t) = 0$. Thus, we have a contradiction and so we must have $h_2(t) \le h_1(t)$ for all $t \in [0, T]$, which implies $\phi_2(t) \le \phi_1(t)$ for all $t \in [0, T]$.

Now we will prove that $\phi_1 \le \phi_3$. It is sufficient to show that $h_3 \ge h_1$. Once again arguing via contradiction, assume there exists $t_2^* \in [0, T]$ such that $h_3(t_2^*) < h_1(t_2^*) = \sup_{s\in[0,t_2^*]}\{(-f(s))^+\}$. Then there exists $t_1^* \in [0, t_2^*]$ such that $(-f(t_1^*))^+ > h_3(t_2^*) \ge h_3(t_1^*)$, which implies $(-f(t_1^*))^+ = f(t_1^*$ and $\phi_3(t_1^*) = f(t_1^*) + h_3(t_1^*) < f(t_1^*) - f(t_1^*)$ meaning $\phi_3(t_1^*) < 0$. However, this contradicts the fact that $\phi_3 \ge 0$, which proves that $h_3 \ge h_1$. □

We now proceed to the proof of Proposition 3.2. Fix $t \ge 0$. Define

$$\hat{I}_i^r(s) \doteq \int_{(t,t+s]} \mathcal{I}_{\{\hat{W}_i^r(u) - Jc_2 r^{\kappa-1} > 0\}} \, d\hat{U}_i^r(u), \quad s \ge 0.$$

Then, from (3), for $s, t \ge 0$,

$$\begin{aligned}\hat{W}_i^r(t+s) - Jc_2 r^{\kappa-1} &= \hat{W}_i^r(t) - Jc_2 r^{\kappa-1} + \hat{X}_i^r(t+s) - \hat{X}_i^r(t) \\ &\quad + \hat{I}_i^r(s) + \int_{(t,t+s]} \mathcal{I}_{\{\hat{W}_i^r(u) - Jc_2 r^{\kappa-1} \le 0\}} \, d\hat{U}_i^r(u),\end{aligned}$$

and consequently, from the first part of Proposition 5.1 we have

$$\hat{W}_i^r(t+s) - Jc_2 r^{\kappa-1} \le \Gamma_1\big(\hat{W}_i^r(t) - Jc_2 r^{\kappa-1} + \hat{X}_i^r(t+\cdot) - \hat{X}_i^r(t) + \hat{I}_i^r(\cdot)\big)(s)$$

for all $s \in [0, T]$. In addition,

$$\hat{W}_i^r(t+s) = \hat{W}_i^r(t) + \hat{X}_i^r(t+s) - \hat{X}_i^r(t) + \hat{U}_i^r(t+s) - \hat{U}_i^r(t)$$

where $\hat{U}_i^r(t+\cdot) - \hat{U}_i^r(t)$ is nondecreasing and nonnegative, and $\hat{W}_i^r(t+\cdot) \ge 0$ so using the second part of Proposition 5.1 once more, we have

$$\Gamma_1\big(\hat{W}_i^r(t) + \hat{X}_i^r(t+\cdot) - \hat{X}_i^r(t)\big)(s) \le \hat{W}_i^r(t+s) \tag{12}$$

for all $s \in [0, T]$. Since

$$\begin{aligned}\sup_{s\in[0,T]} &\big|\Gamma_1\big(\hat{W}_i^r(t) + \hat{X}_i^r(t+\cdot) - \hat{X}_i^r(t)\big)(s) \\ &\quad - \Gamma_1\big(\hat{W}_i^r(t) - Jc_2 r^{\kappa-1} + \hat{X}_i^r(t+\cdot) - \hat{X}_i^r(t) + \hat{I}_i^r(\cdot)\big)(s)\big|\end{aligned}$$

$$
\begin{aligned}
&\leq 2 \sup_{s\in[0,T]} \big|\big(\hat{W}_i^r(t)+\hat{X}_i^r(t+s)-\hat{X}_i^r(t)\big) \\
&\qquad - \big(\hat{W}_i^r(t)-Jc_2r^{\kappa-1}+\hat{X}_i^r(t+s)-\hat{X}_i^r(t)+\hat{I}_i^r(s)\big)\big| \\
&\leq 2Jc_2r^{\kappa-1}+2\int_{(t,t+T]} \mathcal{I}_{\{\hat{W}_i^r(u)-Jc_2r^{\kappa-1}>0\}}\,d\hat{U}_i^r(u)
\end{aligned}
$$

we have, for all $s \in [0, T]$,

$$
\begin{aligned}
&\Gamma_1\big(\hat{W}_i^r(t)+\hat{X}_i^r(t+\cdot)-\hat{X}_i^r(t)\big)(s) \\
&\quad \geq \Gamma_1\big(\hat{W}_i^r(t)-Jc_2r^{\kappa-1}+\hat{X}_i^r(t+\cdot)-\hat{X}_i^r(t)+\hat{I}_i^r(\cdot)\big)(s)-2Jc_2r^{\kappa-1} \\
&\qquad -2\int_{(t,t+T]} \mathcal{I}_{\{\hat{W}_i^r(u)-Jc_2r^{\kappa-1}>0\}}\,d\hat{U}_i^r(u) \qquad (13)\\
&\quad \geq \hat{W}_i^r(t+s)-3Jc_2r^{\kappa-1}-2\int_{(t,t+T]} \mathcal{I}_{\{\hat{W}_i^r(u)-Jc_2r^{\kappa-1}>0\}}\,d\hat{U}_i^r(u).
\end{aligned}
$$

Combining (12) and (13), we have

$$
\begin{aligned}
&\sup_{s\in[0,T]} \big\{\big|\Gamma_1\big(\hat{W}_i^r(t)+\hat{X}_i^r(t+\cdot)-\hat{X}_i^r(t)\big)(s)-\hat{W}_i^r(t+s)\big|\big\} \\
&\qquad \leq 3Jc_2r^{\kappa-1}+2\int_{(t,t+T]} \mathcal{I}_{\{\hat{W}_i^r(u)-Jc_2r^{\kappa-1}>0\}}\,d\hat{U}_i^r(u).
\end{aligned}
$$

In addition due to Proposition 2.6, part (d), we have

$$
\int_{(t,t+T]} \mathcal{I}_{\{\hat{W}_i^r(u)-Jc_2r^{\kappa-1}>0\}}\,d\hat{U}_i^r(u) \leq r^{-1}C_i \sum_{j=1}^{J} \int_{r^2t}^{r^2(t+T)} \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds.
$$

The result follows.

6. Proof of Proposition 3.3. We will use the following two propositions in the proof of Proposition 3.3. The proof of the first proposition is a simpler version of that of Proposition 7.5, which gives a similar estimate for service times. Since the proof of the latter result is given in full detail in Section 9.1, we omit the proof of Proposition 6.1. We make the convention that $u_j^r(0)=0$ for $r\in\mathbb{N}$ and $j\in\mathbb{A}_J$.

PROPOSITION 6.1. *Let δ be as in Condition 1. There exists $R<\infty$ and for any $c<\delta$ a corresponding $K(c)<\infty$ such that for any $j\in\mathbb{A}_J$, $y^r\in\mathcal{Y}^r$ and $t\geq 0$ we have*

$$
\sup_{r\geq R} E\big[e^{cu_j^r(\tau_j^{r,A}(t))}\big] < K(c).
$$

The proof of the next proposition is in Section 9.2.

PROPOSITION 6.2 (Proof in Section 9.2). *Let $j\in\mathbb{A}_J$, $c_1,c_2\geq 0$ and $\epsilon>0$ be arbitrary. Then there exists $B_1,B_2,R\in(0,\infty)$ such that for all $T\in[0,\infty)$ and $r\geq R$ we have*

$$
P\Big(\sup_{0\leq t\leq r^{2c_1+c_2}T} \big|A_j^r(t)-t\alpha_j^r\big| \geq \epsilon r^{c_1+c_2}T\Big) \leq B_1e^{-r^{c_2}TB_2} \tag{14}
$$

and

$$
P\Big(\sup_{0\leq t\leq r^{2c_1+c_2}\max_i\{C_i\}T} \big|S_j^r(t)-t\beta_j^r\big| \geq \epsilon r^{c_1+c_2}T\Big) \leq B_1e^{-r^{c_2}TB_2}. \tag{15}
$$

In particular, for $\kappa \geq 0$,

$$P\Big(\sup_{0\leq t\leq r^{\frac{3}{2}\kappa}T} \big|A_j^r(t) - t\alpha_j^r\big| \geq \epsilon r^\kappa T\Big) \leq B_1 e^{-\mathrm{Tr}^{\frac{1}{2}\kappa} B_2} \tag{16}$$

and

$$P\Big(\sup_{0\leq t\leq r^{\frac{3}{2}\kappa}\max_i\{C_i\}T} \big|S_j^r(t) - t\beta_j^r\big| \geq \epsilon r^\kappa T\Big) \leq B_1 e^{-\mathrm{Tr}^{\frac{1}{2}\kappa} B_2}. \tag{17}$$

We note that equations (16) and (17) are immediate from (14) and (15) on taking $c_1 = c_2 = \kappa/2$. The former set of equations are essential to understanding the behavior of $Q_j^r(t)$ in the region $[0, \tilde{c}_2 r^\kappa]$ in the proof of Proposition 3.3; see, for example, the proof of (28) completed below (30), and the estimates in (34) and (38).

We now proceed to the proof of Proposition 3.3.

Overall proof idea. Recall, as described in Definition 2.3, that $\mathcal{E}_j^r(t)$ changes from 0 to 1 when $Q_j^r(t)$ drops below $\tilde{c}_1 r^\kappa$. This says that we stop processing type j jobs until $Q_j^r(t)$ reaches $\tilde{c}_2 r^\kappa$ at which point $\mathcal{E}_j^r(t)$ changes back to 0 and the processing of type j jobs resumes. The key to this proof is that if $Q_j^r(t) < \tilde{c}_2 r^\kappa$ then type j jobs are processed at a rate lower than their arrival rate (see Proposition 2.6, part (e)) in an attempt to boost $Q_j^r(t)$ back up to $\tilde{c}_2 r^\kappa$. To prove this result, we first get an upper bound on $P(\mathcal{C}^r)$ where $\mathcal{C}^r$, defined in (19), is the event that $Q_j^r(r^2 t)$ fails to exceed $\tilde{c}_1 r^\kappa - 1$ by time $t = \bar{\Upsilon}_j^{A,r} + \mathrm{Tr}^{\frac{3}{2}\kappa-2}$. Note that due to Proposition 2.6, part (a) once $Q_j^r(r^2 t)$ enters the region $[\tilde{c}_1 r^\kappa - 1, \infty)$ it never leaves, and it is convenient for the remainder of this proof to focus on the event $(\mathcal{C}^r)^c$ where this entrance has occurred by time $t = \bar{\Upsilon}_j^{A,r} + \mathrm{Tr}^{\frac{3}{2}\kappa-2}$. The bulk of the proof is devoted to providing an upper bound on

$$P\bigg(\int_{r\hat{\Upsilon}_j^{A,r} + \mathrm{Tr}^{\frac{3}{2}\kappa}}^{r^2 T} \mathcal{I}_{\{\mathcal{E}_j^r(t)=1\}}\,dt \geq \epsilon T r^{\frac{7}{4}\kappa}\bigg).$$

This is accomplished by dividing the time interval $[\bar{\Upsilon}_j^{A,r} + \mathrm{Tr}^{\frac{3}{2}\kappa-2}, T]$ into subintervals of length $r^{\frac{3}{2}\kappa-2}$ and showing that bad behavior on the nth subinterval, as defined by the set $\mathcal{U}_n^r$ in (20), is unlikely. On the set $(\mathcal{C}^r)^c$, we can use the frequency of the events $\{\mathcal{U}_n^r\}_{n\geq 1}$ to bound the amount of time the process is in the state $\mathcal{E}_j^r(t) = 1$ using (27). Demonstrating that with high probability $\mathcal{U}_n^r$ only occurs for a small percentage of subintervals completes the proof.

Let $j \in \mathbb{A}_J$ be arbitrary. Let $R_1 < \infty$ and $\Delta > 0$ be such that for all $r \geq R_1$ we have

$$\begin{aligned} &2 < r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16}, r^\kappa \frac{5(\tilde{c}_2 - \tilde{c}_1)}{\Delta 4} < r^{\frac{3}{2}\kappa} \quad \text{and if} \quad Q_j^r(t) < \tilde{c}_2 r^\kappa, \\ &\text{then } \beta_j^r \mathfrak{b}_j^r(t) \leq \alpha_j^r - \Delta. \end{aligned} \tag{18}$$

Existence of such a R_1 and Δ follows from Proposition 2.6, part (e).

For $n \in \mathbb{N}_0$, define $\hat{t}_n = \bar{\Upsilon}_j^{A,r} + \mathrm{Tr}^{\frac{3}{2}\kappa-2} + (n-1) r^{\frac{3}{2}\kappa-2}$. In addition, for $n \in \mathbb{N}_0$ define the sets

$$\mathcal{A}_n^{r,1} = \bigg\{ r^2 \bar{\Upsilon}_j^{A,r}(\hat{t}_n) \geq r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha_j^r 16} \bigg\}, \qquad \mathcal{B}_n^{r,1} = \{\tau_j^{r,S}(\hat{t}_n) < \tau_j^{r,S}(\hat{t}_{n+1})\},$$

and

$$\mathcal{A}_n^{r,2} = \Big\{ \sup_{\bar{\xi}_j^{A,r}(\hat{t}_n) \le s \le \bar{\xi}_j^{A,r}(\hat{t}_n) + r^{\frac{3}{2}\kappa - 2} - \bar{\Upsilon}_j^{A,r}(\hat{t}_n)} \big| A_j^r(r^2 s) - A_j^r\big(r^2 \bar{\xi}_j^{A,r}(\hat{t}_n)\big) - r^2\big(s - \bar{\xi}_j^{A,r}(\hat{t}_n)\big)\alpha_j^r \big| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16} \Big\}$$

$$= \Big\{ \sup_{0 \le s \le r^{\frac{3}{2}\kappa - 2} - \bar{\Upsilon}_j^{A,r}(\hat{t}_n)} \big| A_j^{r,\hat{t}_n}(r^2 s) - r^2 s \alpha_j^r \big| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16} \Big\},$$

$$\mathcal{B}_n^{r,2} = \Big\{ \sup_{\bar{\xi}_j^{S,r}(\hat{t}_n) \le s \le \bar{B}_j^r(\hat{t}_{n+1}) - \bar{B}_j^r(\hat{t}_n) + \bar{\xi}_j^{S,r}(\hat{t}_n) - \bar{\Upsilon}_j^{S,r}(\hat{t}_n)} \big| S_j^r(r^2 s) - S_j^r\big(r^2 \bar{\xi}_j^{S,r}(\hat{t}_n)\big) - r^2\big(s - \bar{\xi}_j^{S,r}(\hat{t}_n)\big)\beta_j^r \big| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16} \Big\}$$

$$= \Big\{ \sup_{0 \le s \le \bar{B}_j^r(\hat{t}_{n+1}) - \bar{B}_j^r(\hat{t}_n) - \bar{\Upsilon}_j^{S,r}(\hat{t}_n)} \big| S_j^{r,\hat{t}_n}(r^2 s) - r^2 s \beta_j^r \big| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16} \Big\}.$$

Also, define

$$\mathcal{C}^r = \{ Q_j^r(r^2 \hat{t}_1) < r^\kappa \tilde{c}_1 - 1 \}. \tag{19}$$

From (1), we have, for all $s \ge 0$ and $j \in \mathbb{A}_J$,

$$Q^r(r^2 s) = q_j^r + A_j^r\big(r^2(s - \bar{\Upsilon}_j^{A,r})^+\big) + \mathcal{I}_{\{s \ge \bar{\Upsilon}_j^{A,r} > 0\}} - S_j^r\big(r^2(\bar{B}_j^r(s) - \bar{\Upsilon}_j^{S,r})^+\big) - \mathcal{I}_{\{\bar{B}_j^r(s) \ge \bar{\Upsilon}_j^{S,r} > 0\}}$$

and for any $n \ge 1$ and $\hat{t}_n \le s \le \hat{t}_{n+1}$ we have

$$A_j^r\big(r^2(s - \bar{\Upsilon}_j^{A,r})^+\big) + \mathcal{I}_{\{s \ge \bar{\Upsilon}_j^{A,r} > 0\}} = A_j^r\big(r^2(\hat{t}_n - \bar{\Upsilon}_j^{A,r})^+\big) + \mathcal{I}_{\{\hat{t}_n \ge \bar{\Upsilon}_j^{A,r} > 0\}} + \mathcal{I}_{\{s - \hat{t}_n \ge \bar{\Upsilon}_j^{A,r}(\hat{t}_n) > 0\}} + A_j^{r,\hat{t}_n}\big(r^2(s - \hat{t}_n - \bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+\big)$$

and

$$S_j^r\big(r^2(\bar{B}_j^r(s) - \bar{\Upsilon}_j^{S,r})^+\big) + \mathcal{I}_{\{\bar{B}_j^r(s) \ge \bar{\Upsilon}_j^{S,r} > 0\}}$$
$$= S_j^r\big(r^2(\bar{B}_j^r(\hat{t}_n) - \bar{\Upsilon}_j^{S,r})^+\big) + \mathcal{I}_{\{\bar{B}_j^r(\hat{t}_n) \ge \bar{\Upsilon}_j^{S,r} > 0\}} + \mathcal{I}_{\{\bar{B}_j^r(s) - \bar{B}_j^r(\hat{t}_n) \ge \bar{\Upsilon}_j^{S,r}(\hat{t}_n) > 0\}}$$
$$+ S_j^{r,\hat{t}_n}\big(r^2(\bar{B}_j^r(s) - \bar{B}_j^r(\hat{t}_n) - \bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+\big).$$

Let $\zeta_0^{r,n} \doteq \inf\{s \ge \hat{t}_n : Q_j^r(r^2 s) \ge r^\kappa \tilde{c}_2\}$ and for $l \ge 0$ define

$$\zeta_{2l+1}^{r,n} \doteq \inf\{s \ge \zeta_{2l}^{r,n} : Q_j^r(r^2 s) < r^\kappa \tilde{c}_2\}, \qquad \zeta_{2l+2}^{r,n} \doteq \inf\{s \ge \zeta_{2l+1}^{r,n} : Q_j^r(r^2 s) \ge r^\kappa \tilde{c}_2\}.$$

If $\zeta_0^{r,n} > \hat{t}_n$, then for all $s \in [\hat{t}_n, \zeta_0^{r,n})$ we have $Q_j^r(r^2 s) < r^\kappa \tilde{c}_2$. Consequently, for all $s \in [\hat{t}_n, \zeta_0^{r,n})$ we have $\beta_j^r(\bar{B}_j^r(s) - \bar{B}_j^r(\hat{t}_n)) \le (s - \hat{t}_n)(\alpha_j^r - \Delta)$. In addition, on $(\mathcal{C}^r)^c$, $Q_j^r(r^2 s) \ge r^\kappa \tilde{c}_1 - 1$ for all $s \ge \hat{t}_1$ because if $Q_j^r(r^2 s) < r^\kappa \tilde{c}_1$ then $\tilde{\mathcal{E}}_j^r(s) = 1$ so $\frac{d}{ds}\bar{B}_j^r(s) = 0$ due to Proposition 2.6, part (a). Consequently, with

$$\mathcal{U}_n^r \doteq \mathcal{A}_n^{r,1} \cup \mathcal{A}_n^{r,2} \cup (\mathcal{B}_n^{r,1} \mathcal{B}_n^{r,2}), \tag{20}$$

on the set $(\mathcal{U}_n^r)^c \cap (\mathcal{C}^r)^c$ for all $s \in [\hat{t}_n, \zeta_0^{r,n}) \cap [\hat{t}_n, \hat{t}_{n+1}]$ we have

$$\begin{aligned}
Q^r(r^2 s) &= Q^r(r^2\hat{t}_n) + \mathcal{I}_{\{s-\hat{t}_n \geq \bar{\Upsilon}_j^{A,r}(\hat{t}_n)>0\}} - \mathcal{I}_{\{\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)\geq \bar{\Upsilon}_j^{S,r}(\hat{t}_n)>0\}} \\
&\quad + A_j^{r,\hat{t}_n}(r^2(s-\hat{t}_n-\bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+) - S_j^{r,\hat{t}_n}(r^2(\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)-\bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+) \\
&\geq r^\kappa \tilde{c}_1 - 2 + r^2\alpha_j^r(s-\hat{t}_n-\bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+ \\
&\quad - r^2\beta_j^r(\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)-\bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+ - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{8} \\
&\geq r^\kappa \tilde{c}_1 - 2 + r^2\alpha_j^r(s-\hat{t}_n) - r^2\alpha_j^r\bar{\Upsilon}_j^{A,r}(\hat{t}_n) - r^2\beta_j^r(\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)) \qquad (21)\\
&\quad - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{8} \\
&\geq r^\kappa\tilde{c}_1 - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{16} + r^2\alpha_j^r(s-\hat{t}_n) - \alpha_j^r\left(r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{16\alpha_j^r}\right) \\
&\quad - r^2\beta_j^r(\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)) - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{8} \\
&\geq r^\kappa\tilde{c}_1 - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{4} + \Delta r^2(s-\hat{t}_n),
\end{aligned}$$

where the last two lines use (18). Recall that $\Delta > 0$ and from (18) $r^2(\hat{t}_{n+1}-\hat{t}_n) > r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{4\Delta}$ so if $\zeta_0^{r,n} - \hat{t}_n > r^{\kappa-2}\frac{5(\tilde{c}_2-\tilde{c}_1)}{4\Delta}$ then, on $(\mathcal{U}_n^r)^c \cap (\mathcal{C}^r)^c$,

$$Q^r\left(r^2\left(\hat{t}_n + r^{\kappa-2}\frac{5(\tilde{c}_2-\tilde{c}_1)}{4\Delta}\right)\right) \geq r^\kappa\tilde{c}_1 - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{4} + \Delta r^2\left(r^{\kappa-2}\frac{5(\tilde{c}_2-\tilde{c}_1)}{4\Delta}\right) = r^\kappa\tilde{c}_2,$$

which contradicts the definition of $\zeta_0^{r,n}$ so on the set $(\mathcal{U}_n^r)^c \cap (\mathcal{C}^r)^c$ we have $\zeta_0^{r,n} - \hat{t}_n \leq r^{\kappa-2}\frac{5(\tilde{c}_2-\tilde{c}_1)}{4\Delta}$, and consequently, $\zeta_0^{r,n} < \hat{t}_{n+1}$. Therefore, on the set $(\mathcal{U}_n^r)^c \cap (\mathcal{C}^r)^c$ we have

$$r^2(\zeta_0^{r,n} - \hat{t}_n) \leq r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{4}. \qquad (22)$$

By definition, for all $l \geq 0$, $Q_j^r(r^2 s) \geq r^\kappa\tilde{c}_2$ for all $s \in [\zeta_{2l}^{r,n}, \zeta_{2l+1}^{r,n}]$ so $\mathcal{E}_j^r(r^2 s) = 0$ for all $s \in [\zeta_{2l}^{r,n}, \zeta_{2l+1}^{r,n})$. On the set $(\mathcal{U}_n^r)^c \cap (\mathcal{C}^r)^c$ for any $l \geq 0$ satisfying $\zeta_{2l+1}^{r,n} < \hat{t}_{n+1}$ and $s \in [\zeta_{2l+1}^{r,n}, \zeta_{2l+2}^{r,n}) \cap [\zeta_{2l+1}^{r,n}, \hat{t}_{n+1}]$, we have

$$\begin{aligned}
&Q^r(r^2 s) \\
&= Q^r(r^2\zeta_{2l+1}^{r,n}) + \mathcal{I}_{\{s-\hat{t}_n \geq \bar{\Upsilon}_j^{A,r}(\hat{t}_n) > \zeta_{2l+1}^{r,n}-\hat{t}_n\}} - \mathcal{I}_{\{\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)\geq\bar{\Upsilon}_j^{S,r}(\hat{t}_n)>\bar{B}_j^r(\zeta_{2l+1}^{r,n})-\bar{B}_j^r(\hat{t}_n)\}} \\
&\quad + (A_j^{r,\hat{t}_n}(r^2(s-\hat{t}_n-\bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+) - A_j^{r,\hat{t}_n}(r^2(\zeta_{2l+1}^{r,n}-\hat{t}_n-\bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+)) \\
&\quad - (S_j^{r,\hat{t}_n}(r^2(\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)-\bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+) \\
&\quad - S_j^{r,\hat{t}_n}(r^2(\bar{B}_j^r(\zeta_{2l+1}^{r,n})-\bar{B}_j^r(\hat{t}_n)-\bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+)) \\
&\geq r^\kappa\tilde{c}_2 - 2 - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{8} + r^2\alpha_j^r((s-\hat{t}_n-\bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+ - (\zeta_{2l+1}^{r,n}-\hat{t}_n-\bar{\Upsilon}_j^{A,r}(\hat{t}_n))^+) \\
&\quad - r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{8} - r^2\beta_j^r((\bar{B}_j^r(s)-\bar{B}_j^r(\hat{t}_n)-\bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+ \\
&\quad - (\bar{B}_j^r(\zeta_{2l+1}^{r,n})-\bar{B}_j^r(\hat{t}_n)-\bar{\Upsilon}_j^{S,r}(\hat{t}_n))^+)
\end{aligned}$$

since $Q^r_j(r^2\zeta^{r,n}_{2l+1}) \ge r^\kappa\tilde{c}_2 - 1$. Consequently,

$$\begin{aligned}
&Q^r(r^2 s)\\
&\quad\ge r^\kappa\tilde{c}_2 - r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{16} + r^2\alpha^r_j(s-\zeta^{r,n}_{2l+1}) - r^2\alpha^r_j\bar{\Upsilon}^{A,r}_j(\hat{t}_n) - r^2\beta^r_j(\bar{B}^r_j(s)-\bar{B}^r_j(\zeta^{r,n}_{2l+1}))\\
&\quad\ge r^\kappa\tilde{c}_2 - r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{16} + r^2\Delta(s-\zeta^{r,n}_{2l+1}) - \alpha^r_j\left(r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{\alpha^r_j 16}\right)\\
&\quad\ge r^\kappa\left(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1\right) + r^2\Delta(s-\zeta^{r,n}_{2l+1}),
\end{aligned}$$

where once more we have used (18). Because $\Delta > 0$, we have $Q^r_j(r^2 s) \ge r^\kappa(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1)$ for all $s \in [\zeta^{r,n}_{2l+1}, \zeta^{r,n}_{2l+2}\wedge\hat{t}_{n+1}]$ and because by definition $\mathcal{E}^r_j(r^2\zeta^{r,n}_{2l+1}) = 0$ we have $\mathcal{E}^r_j(r^2 s) = 0$ for all $s \in [\zeta^{r,n}_{2l+1}, \zeta^{r,n}_{2l+2}) \cap [\zeta^{r,n}_{2l+1}, \hat{t}_{n+1}]$. Since $l \ge 0$, such that $\zeta^{r,n}_{2l+1} < \hat{t}_{n+1}$ was arbitrary, we have $\mathcal{E}^r_j(r^2 s) = 0$ for all $s \in [\zeta^{r,n}_0, \hat{t}_{n+1}]$ on the set $(\mathcal{U}^r_n)^c \cap (\mathcal{C}^r)^c$. In addition, we have shown that on the set $(\mathcal{U}^r_n)^c \cap (\mathcal{C}^r)^c$, we have $Q^r_j(r^2 s) \ge r^\kappa(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1)$ for all $s \in [\zeta^{r,n}_0, \hat{t}_{n+1}]$ so $Q^r_j(r^2\hat{t}_{n+1}) \ge r^\kappa(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1)$. Consequently, from (22), on the set $(\mathcal{U}^r_n)^c \cap (\mathcal{C}^r)^c$ we have

$$\int_{r^2\hat{t}_n}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds = \int_{r^2\hat{t}_n}^{r^2\zeta^{r,n}_0} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds + \int_{r^2\zeta^{r,n}_0}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds \le r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{4}, \tag{23}$$

$$\mathcal{E}^r_j(r^2\hat{t}_{n+1}) = 0, \quad\text{and,}\quad Q^r_j(r^2\hat{t}_{n+1}) \ge r^\kappa\left(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1\right). \tag{24}$$

In particular, with $H^r_n = \{\mathcal{E}^r_j(r^2\hat{t}_n) = 0\} \cap \{Q^r_j(r^2\hat{t}_n) \ge r^\kappa(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1)\}$, for $n > 2$ we have

$$(H^r_n)^c \cap (\mathcal{U}^r_{n-1})^c \cap (\mathcal{C}^r)^c = \varnothing. \tag{25}$$

Next, on the set $H^r_n \cap (\mathcal{U}^r_n)^c \cap (\mathcal{C}^r)^c$ for $s \in [\hat{t}_n, \zeta^{r,n}_0)$ (recall that $\zeta^{r,n}_0 < \hat{t}_{n+1}$ on $(\mathcal{U}^r_n)^c \cap (\mathcal{C}^r)^c$) we have, from similar calculations as in (22),

$$\begin{aligned}
Q^r(r^2 s) &\ge r^\kappa\left(\frac{5}{8}\tilde{c}_2+\frac{3}{8}\tilde{c}_1\right) + r^2\Delta(s-\hat{t}_n) - \alpha^r_j\left(r^\kappa\frac{\tilde{c}_2-\tilde{c}_1}{\alpha^r_j 16}\right) - r^\kappa\frac{3(\tilde{c}_2-\tilde{c}_1)}{16}\\
&\ge r^\kappa\left(\frac{3}{8}\tilde{c}_2+\frac{5}{8}\tilde{c}_1\right) + \Delta r^2(s-\hat{t}_n).
\end{aligned}$$

Because $\Delta > 0$, this implies $Q^r_j(r^2 s) > r^\kappa\tilde{c}_1$ for all $s \in [\hat{t}_n, \zeta^{r,n}_0)$ and since $\mathcal{E}^r_j(r^2\hat{t}_n) = 0$ this implies $\mathcal{E}^r_j(r^2 s) = 0$ for all $s \in [\hat{t}_n, \zeta^{r,n}_0)$. Consequently, on $H^r_n \cap (\mathcal{U}^r_n)^c \cap (\mathcal{C}^r)^c$ we have that (24) holds and

$$\int_{r^2\hat{t}_n}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds = \int_{r^2\hat{t}_n}^{r^2\zeta^{r,n}_0} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds + \int_{r^2\zeta^{r,n}_0}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds = 0. \tag{26}$$

Therefore, for any $N \ge 1$ we have, on $(\mathcal{C}^r)^c$,

$$\begin{aligned}
\int_{r^2\hat{t}_1}^{r^2\hat{t}_N} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds &= \sum_{n=1}^{N-1}\int_{r^2\hat{t}_n}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds\\
&= \sum_{n=1}^{N-1}\mathcal{I}_{\mathcal{U}^r_n}\int_{r^2\hat{t}_n}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds + \sum_{n=1}^{N-1}\mathcal{I}_{H^r_n\cap(\mathcal{U}^r_n)^c}\int_{r^2\hat{t}_n}^{r^2\hat{t}_{n+1}} \mathcal{I}_{\{\mathcal{E}^r_j(s)=1\}}\,ds
\end{aligned}$$

$$+\sum_{n=1}^{N-1}\mathcal{I}_{(H_n^r)^c\cap(\mathcal{U}_n^r)^c}\int_{r^2\hat{t}_n}^{r^2\hat{t}_{n+1}}\mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds$$

$$\le r^{\frac{3}{2}\kappa}\sum_{n=1}^{N-1}\mathcal{I}_{\mathcal{U}_n^r}+r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{4}\sum_{n=1}^{N-1}\mathcal{I}_{(H_n^r)^c\cap(\mathcal{U}_n^r)^c},$$

where we have used (24) and (26) in obtaining the last inequality.

From (25), we have, on $(\mathcal{C}^r)^c$,

$$\sum_{n=1}^{N-1}\mathcal{I}_{(H_n^r)^c\cap(\mathcal{U}_n^r)^c}\le 1+\sum_{n=1}^{N-1}\mathcal{I}_{\mathcal{U}_n^r}.$$

Therefore, on $(\mathcal{C}^r)^c$,

$$\int_0^{r^2\hat{t}_N}\mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds\le r\hat{\Upsilon}_j^{A,r}+\mathrm{Tr}^{\frac{3}{2}\kappa}+r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{4}+\left(r^{\frac{3}{2}\kappa}+r^\kappa\frac{5(\tilde{c}_2-\tilde{c}_1)}{4}\right)\sum_{n=1}^{N-1}\mathcal{I}_{\mathcal{U}_n^r}. \tag{27}$$

From the above estimate, in order to prove the result, it now suffices to show that there exists $R, B<\infty$ such that for all $r\ge R$ and $T\ge 1$, we have

$$P(\mathcal{C}^r)\le e^{-BTr^{\frac{1}{8}\kappa}}, \tag{28}$$

and

$$P\left(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil-1}\mathcal{I}_{\mathcal{U}_n^r}\ge 3r^{\frac{1}{8}\kappa}T\right)\le e^{-BTr^{\frac{1}{8}\kappa}}. \tag{29}$$

For (28), note that since from Proposition 2.6, part (a), if $Q_j^r(r^2s)<r^\kappa\tilde{c}_1$ then $\frac{d}{ds}\bar{B}_j^r(s)=0$, we have that

$$P(\mathcal{C}^r)\le P\bigl(A_j^r(\mathrm{Tr}^{3\kappa/2})\le r^\kappa\tilde{c}_1\bigr). \tag{30}$$

The estimate in (28) now follows readily from (16).

The estimate in (29) follows if we can show that there exists $R, B<\infty$ such that for all $r\ge R$ and $T\ge 1$, we have

$$P\left(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil-1}\mathcal{I}_{(\mathcal{B}_n^{r,1}\mathcal{B}_n^{r,2})}\ge r^{\frac{1}{8}\kappa}T\right)\le e^{-BTr^{\frac{1}{8}\kappa}}, \tag{31}$$

$$P\left(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil-1}\mathcal{I}_{\mathcal{A}_n^{r,1}}\ge r^{\frac{1}{8}\kappa}T\right)\le e^{-BTr^{\frac{1}{8}\kappa}}, \tag{32}$$

and

$$P\left(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil-1}\mathcal{I}_{(\mathcal{A}_n^{r,1})^c\cap\mathcal{A}_n^{r,2}}\ge r^{\frac{1}{8}\kappa}T\right)\le e^{-BTr^{\frac{1}{8}\kappa}}. \tag{33}$$

First we show (31), with $B=\frac{1}{4}$, for r sufficiently large. Define

$$\mathcal{F}_j^{r,S}(k)=\sigma\bigl\{u_l^r(m_l^u), v_{l'}^r(m_{l'}^v), v_j^r(m_j^v): m_{l'}^v\ge 0, m_l^u\ge 0, l\in\mathbb{A}_J, l'\in\mathbb{A}_J\setminus\{j\}, m_j^v\le k\bigr\},$$

which is the filtration that contains the information on all interarrival times, all service times from queues other than the jth queue, and the first k service times from queue j. Note that

$\tau_j^{r,S}(\hat{t}_n)$ is a $\mathcal{F}_j^{r,S}(k)$ stopping time, and thus with $\mathcal{H}_n^j \doteq \mathcal{F}_j^{r,S}(\tau_j^{r,S}(\hat{t}_n))$ for $n \geq 0$, $\{\mathcal{H}_n^j\}_{n\geq 0}$ is a filtration. Note that for $k < n$ $\mathcal{B}_k^{r,1} \cap \mathcal{B}_k^{r,2}$ is $\mathcal{H}_n^j$-measurable. In addition, $\bar{\xi}_j^{S,r}(\hat{t}_n)$ is $\mathcal{H}_n^j$-measurable and $S_j^{r,\hat{t}_n}$ is independent of $\mathcal{H}_n^j$. Write

$$\mathcal{U}_n = \exp\left\{\frac{1}{2}\mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}}\right\} = e^{1/2}\mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}} + \mathcal{I}_{(\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2})^c}.$$

Then we have

$$\begin{aligned} E\big[e^{\frac{1}{2}\sum_{n=1}^N \mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}}}\big] &= E\left[\prod_{n=1}^N \mathcal{U}_n\right] = E\left[E(\mathcal{U}_N \mid \mathcal{H}_{N-1}^j)\prod_{n=1}^{N-1}\mathcal{U}_n\right] \\ &\leq E\left[\big[(1 - P(\mathcal{B}_N^{r,2}|\mathcal{H}_{N-1}^j)) + e^{\frac{1}{2}}P(\mathcal{B}_N^{r,2}|\mathcal{H}_{N-1}^j)\big]\prod_{n=1}^{N-1}\mathcal{U}_n\right]. \end{aligned}$$

For $n \geq 0$, since $\bar{\Upsilon}_j^{S,r}(\hat{t}_n) \geq 0$ and $\bar{B}_j^r(\hat{t}_{n+1}) - \bar{B}_j^r(\hat{t}_n) \leq \max_i\{C_i\}r^{\frac{3}{2}\kappa-2}$, we have

$$P(\mathcal{B}_n^{r,2}|\mathcal{H}_n^j) \leq P\left(\sup_{0\leq s\leq \max_i\{C_i\}r^{\frac{3}{2}\kappa-2}} \big|S_j^{r,\hat{t}_n}(r^2 s) - r^2 s\beta_j^r\big| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16}\right)$$

and due to Proposition 6.2 (17), there exists $R_2 \in [R_1, \infty)$ and $B_1 < \infty$ such that for all $r \geq R_2$ and $n \geq 0$ we have

$$P\left(\sup_{0\leq s\leq \max_i\{C_i\}r^{\frac{3}{2}\kappa-2}} \big|S_j^{r,\hat{t}_n}(r^2 s) - r^2 s\alpha_j^r\big| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16}\right) \leq e^{-r^{\frac{\kappa}{2}}B_1}. \tag{34}$$

Since $(1 - p + pe^{\frac{1}{2}})$ is an increasing function of p, we have

$$E\big[e^{\frac{1}{2}\sum_{n=1}^N \mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}}}\big] \leq E\big[e^{\frac{1}{2}\sum_{n=1}^{N-1} \mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}}}\big]\big(1 + e^{-r^{\frac{\kappa}{2}}B_1}(e^{\frac{1}{2}} - 1)\big).$$

Now by a standard recursive argument, we see

$$E\big[e^{\frac{1}{2}\sum_{n=1}^N \mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}}}\big] \leq \big(1 + e^{-r^{\frac{\kappa}{2}}B_1}(e^{1/2} - 1)\big)^{N+1}.$$

Since $e^{\frac{1}{2}} - 1 \leq 1$, we have $1 + e^{-r^{\frac{\kappa}{2}}B_1}(e^{\frac{1}{2}} - 1) \leq e^{e^{-r^{\frac{\kappa}{2}}B_1}}$, which gives

$$E\big[e^{\frac{1}{2}\sum_{n=1}^N \mathcal{I}_{\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2}}}\big] \leq e^{(N+1)e^{-r^{\frac{\kappa}{2}}B_1}}. \tag{35}$$

Choose $R_3 \in [R_2, \infty)$ such that for all $r \geq R_3$ and $T \geq 1$, we have

$$-\frac{1}{2}r^{\frac{1}{8}\kappa}T + (\mathrm{Tr}^{2-\frac{3}{2}\kappa} + 1)e^{-r^{\frac{\kappa}{2}}B_1} \leq -\frac{1}{4}r^{\frac{1}{8}\kappa}T.$$

Then for all $r \geq R_3$ and $T \geq 1$, we have from (35) that

$$P\left(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil - 1} \mathcal{I}_{(\mathcal{B}_n^{r,1}\cap\mathcal{B}_n^{r,2})} \geq r^{\frac{1}{8}\kappa}T\right) \leq e^{-\frac{1}{2}r^{\frac{1}{8}\kappa}T + \lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil e^{-r^{\frac{\kappa}{2}}B_1}} \leq e^{-\frac{1}{4}\mathrm{Tr}^{\frac{1}{8}\kappa}}.$$

This completes the proof of (31). Let $\delta > 0$ be as in Condition 1 and let $\upsilon \in (0, \delta)$ be arbitrary. Now we will show that (32) holds with $B = \frac{\upsilon}{2}$ for r sufficiently large. For $x \in \mathbb{R}_+$, define

$$\tilde{\tau}_j^{r,A}(x) = \min\left\{k \geq 0 : \sum_{l=1}^k u_j^r(l) \geq x\right\}.$$

Let $\check{u}^r_j(x) \doteq \mathcal{I}_{\{x>0\}} u^r_j(\tilde{\tau}^{r,A}_j(x))$ and define

$$\Phi^r_\upsilon(x) \doteq E \exp\left\{ \upsilon \mathcal{I}_{\{\check{u}^r_j(x) > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}\}} + \upsilon \sum_{l=1}^{\infty} \mathcal{I}_{\{\check{u}^r_j(x) > l r^{\frac{3}{2}\kappa}\}} \right\}.$$

Let

$$\mathcal{F}^{r,A}_j(k) = \sigma\{u^r_{l'}(m^u_{l'}), v^r_l(m^v_l), u^r_j(m^u_j) : m^u_{l'} \geq 0, m^v_l \geq 0, l \in \mathbb{A}_J, l' \in \mathbb{A}_J \setminus \{j\}, m^u_j \leq k\},$$

which is the filtration that contains the information on all all service times, all interarrival times for queues other than the jth queue, and the first k arrival times from queue j. Note that $\tau^{r,A}_j(\hat{t}_n)$ is a $\mathcal{F}^{r,A}_j(k)$-stopping time, and thus with $\tilde{\mathcal{H}}^j_n \doteq \mathcal{F}^{r,A}_j(\tau^{r,A}_j(\hat{t}_n))$ for $n \geq 0$, $\{\tilde{\mathcal{H}}^j_n\}_{n \geq 0}$ is a filtration. For $n \geq 0$, and a $\tilde{\mathcal{H}}^j_n$-measurable positive random variable X, define

$$\tilde{\tau}^{r,A,n}_j(X) = \min\left\{ k \geq 0 : \sum_{l=\tau^{r,A}_j(\hat{t}_n)+1}^{l=\tau^{r,A}_j(\hat{t}_n)+k} u^r_j(l) \geq X \right\}$$

and

$$\psi^{r,n}_\upsilon(X) \doteq \exp\Bigg[\upsilon \mathcal{I}_{\{X>0, u^r_j(\tilde{\tau}^{r,A}_j(X) + \tau^{r,A}_j(\hat{t}_n)) > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}\}} + \upsilon \sum_{l=1}^{\infty} \mathcal{I}_{\{X>0, u^r_j(\tilde{\tau}^{r,A}_j(X) + \tau^{r,A}_j(\hat{t}_n)) > l r^{\frac{3}{2}\kappa}\}} \Bigg].$$

Note that

$$E[\psi^{r,n}_\upsilon(X) | \tilde{\mathcal{H}}^j_n] = \Phi^r_\upsilon(X). \tag{36}$$

Due to Proposition 6.1, there exists $R_4 \in [R_3, \infty)$ and $B_2 < \infty$ such that

$$\sup_{r \geq R_4, x \in \mathbb{R}_+} E[e^{\upsilon \check{u}^r_j(x)}] \leq B_2$$

and for all $r \geq R_4$ we have $e^{-\upsilon(r^{\frac{3}{2}\kappa} - 1)} < \frac{1}{2}$, $r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16} \leq r^{\frac{3}{2}\kappa}$, $2\alpha^r_j \geq \alpha_j$ and $r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 32} > 1$. Consequently, for all $x \in \mathbb{R}$ and $l \geq 1$, we have

$$P\left(\check{u}^r_j(x) \geq r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16} \right) \leq B_2 e^{-\upsilon r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}}, \qquad P(\check{u}^r_j(x) \geq l r^{\frac{3}{2}\kappa}) \leq B_2 e^{-\upsilon l r^{\frac{3}{2}\kappa}}.$$

Therefore, for $r \geq R_4$, we have

$$\begin{aligned}
\Phi^r_\upsilon(x) &\leq 1 + e^\upsilon B_2 e^{-\upsilon r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}} + \sum_{l=1}^{\infty} e^{\upsilon(l+1)} B_2 e^{-\upsilon l r^{\frac{3}{2}\kappa}} \\
&\leq 1 + e^\upsilon B_2 e^{-\upsilon r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}} + e^{2\upsilon} B_2 e^{-\upsilon r^{\frac{3}{2}\kappa}} \sum_{l=0}^{\infty} e^{-\upsilon l (r^{\frac{3}{2}\kappa} - 1)} \\
&\leq 1 + e^\upsilon B_2 e^{-\upsilon r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}} + 2 e^{2\upsilon} B_2 e^{-\upsilon r^{\frac{3}{2}\kappa}} \\
&\leq 1 + 3 B_2 e^{2\upsilon} e^{-\upsilon r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}} \leq e^{3 B_2 e^{2\upsilon} e^{-\upsilon r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}}}.
\end{aligned}$$

Letting $B_3 = 3B_2 e^{2\upsilon}$ and $B_4 = \upsilon \frac{2(\tilde{c}_2 - \tilde{c}_1)}{\alpha_j 16}$ we have that, for $r \geq R_4$, $\sup_{r \geq R_4, x \in \mathbb{R}} \{\Phi^r_\upsilon(x)\} \leq e^{B_3 e^{-B_4 r^\kappa}}$, which combined with equation (36) implies that for any $n \geq 0$ and $\tilde{\mathcal{H}}^j_n$-measurable real valued random variable X, we have a.e.

$$E\big[\psi^{r,n}_\upsilon(X) | \tilde{\mathcal{H}}^j_n\big] = \Phi^r_\upsilon(X) \leq e^{B_3 e^{-B_4 r^\kappa}}. \tag{37}$$

Next,

$$\mathcal{I}_{\mathcal{A}^{r,1}_N} \leq \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) \geq r^{\frac{3}{2}\kappa}\}} + \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) < r^{\frac{3}{2}\kappa}\}} \mathcal{I}_{\mathcal{A}^{r,1}_N}.$$

Also,

$$\begin{aligned}
&\mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) < r^{\frac{3}{2}\kappa}\}} \mathcal{I}_{\mathcal{A}^{r,1}_N} \\
&\quad \leq \mathcal{I}_{\{r^{\frac{3}{2}\kappa} - r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) > 0, u^r_j(\tau^{r,A}_j(\hat{t}_{N-1}) + \tilde{\tau}^{r,A,N-1}_j(r^{\frac{3}{2}\kappa} - r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}))) > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{\alpha^r_j 16}\}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{\upsilon \mathcal{I}_{\mathcal{A}^{r,1}_N}} &\leq e^{\upsilon \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) \geq r^{\frac{3}{2}\kappa}\}} + \upsilon \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) < r^{\frac{3}{2}\kappa}\}} \mathcal{I}_{\mathcal{A}^{r,1}_N}} \\
&\leq e^{\upsilon \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) \geq r^{\frac{3}{2}\kappa}\}}} \psi^{r,N-1}_\upsilon\big(r^{\frac{3}{2}\kappa} - r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1})\big).
\end{aligned}$$

By conditioning, using (37), and the fact that $r^{\frac{3}{2}\kappa} - r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1})$ is $\tilde{\mathcal{H}}^j_{N-1}$-measurable it then follows:

$$E\big[e^{\upsilon \sum_{n=1}^N \mathcal{I}_{\mathcal{A}^{r,1}_n}}\big] \leq e^{B_3 e^{-B_4 r^\kappa}} E e^{\upsilon \sum_{n=1}^{N-1} \mathcal{I}_{\mathcal{A}^{r,1}_n} + \upsilon \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_{N-1}) \geq r^{\frac{3}{2}\kappa}\}}}.$$

By a successive conditioning argument, we now have that

$$E e^{\upsilon \sum_{n=1}^N \mathcal{I}_{\mathcal{A}^{r,1}_n}} \leq e^{N B_3 e^{-B_4 r^\kappa}} E e^{\upsilon \mathcal{I}_{\mathcal{A}^{r,1}_0} + \upsilon \sum_{l=1}^N \mathcal{I}_{\{r^2 \bar{\Upsilon}^{A,r}_j(\hat{t}_0) \geq l r^{\frac{3}{2}\kappa}\}}}.$$

By definition $\bar{\Upsilon}^{A,r}_j(\hat{t}_0) = 0$, so the expectation on the right-hand side equals 1, which gives

$$E\big[e^{\upsilon \sum_{n=1}^N \mathcal{I}_{\mathcal{A}^{r,1}_n}}\big] \leq e^{N B_3 e^{-B_4 r^\kappa}}.$$

Choose $R_5 \in [R_4, \infty)$ such that for all $r \geq R_5$ and $T \geq 1$ we have

$$-T \upsilon r^{\frac{1}{8}\kappa} + \mathrm{Tr}^{2-\frac{3}{2}\kappa} B_3 e^{-B_4 r^\kappa} \leq -T \frac{\upsilon}{2} r^{\frac{1}{8}\kappa}.$$

Then for all $r \geq R_5$ and $T \geq 1$, we have

$$\begin{aligned}
P\left(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa} \rceil - 1} \mathcal{I}_{\mathcal{A}^{r,1}_n} \geq r^{\frac{1}{8}\kappa} T\right) &\leq e^{-\upsilon r^{\frac{1}{8}\kappa} T} E\big[e^{\upsilon \sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa} \rceil - 1} \mathcal{I}_{\mathcal{A}^{r,1}_n}}\big] \\
&\leq e^{-\upsilon r^{\frac{1}{8}\kappa} T + (\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa} \rceil - 1) B_3 e^{-B_4 r^\kappa}} \\
&\leq e^{-\upsilon r^{\frac{1}{8}\kappa} T + \mathrm{Tr}^{2-\frac{3}{2}\kappa} B_3 e^{-B_4 r^\kappa}} \leq e^{-\frac{\upsilon}{2} r^{\frac{1}{8}\kappa} T}.
\end{aligned}$$

This proves (32).

Finally, we will show (33), with $B = 1/4$ and for r sufficiently large. Note that for $n \geq 0$ since $A_j^{r,\hat{t}_n}(\cdot)$ is independent of $\mathcal{F}^r(\tau^r(\hat{t}_n))$, $\bar{\Upsilon}_j^{A,r}(\hat{t}_n)$ is $\mathcal{F}^r(\tau^r(\hat{t}_n))$-measurable, and $\bar{\Upsilon}_j^{A,r}(\hat{t}_n) \geq 0$ we have

$$P(\mathcal{A}_n^{r,2}|\mathcal{F}^r(\tau^r(\hat{t}_n))) \leq P\Bigl(\sup_{0\leq s\leq r^{\frac{3}{2}\kappa-2}} |A_j^{r,\hat{t}_n}(r^2 s) - r^2 s\alpha_j^r| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16} |\mathcal{F}^r(\tau^r(\hat{t}_n))\Bigr)$$

$$\leq P\Bigl(\sup_{0\leq s\leq r^{\frac{3}{2}\kappa-2}} |A_j^{r,\hat{t}_n}(r^2 s) - r^2 s\alpha_j^r| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16}\Bigr).$$

From Proposition 6.2 (16), there exists $R_6 \in [R_5, \infty)$ and $B_5 < \infty$ such that for all $r \geq R_6$ we have

$$P\Bigl(\sup_{0\leq s\leq r^{\frac{3}{2}\kappa-2}} |A_j^{r,\hat{t}_n}(r^2 s) - r^2 s\alpha_j^r| > r^\kappa \frac{\tilde{c}_2 - \tilde{c}_1}{16}\Bigr) \leq e^{-2B_5 r^{\frac{\kappa}{2}}} \tag{38}$$

and $e^{\frac{1}{2}} \leq B_5 r^{\frac{\kappa}{2}}$. Therefore, for all $r \geq R_6$ and $n \geq 0$, we have

$$P(\mathcal{A}_n^{r,2}|\mathcal{F}^r(\tau^r(\hat{t}_n))) \leq e^{-2B_5 r^{\frac{\kappa}{2}}}$$

and so

$$E[e^{\frac{1}{2}\mathcal{I}_{(\mathcal{A}_n^{r,1})^c\cap\mathcal{A}_n^{r,2}}}|\mathcal{F}^r(\tau^r(\hat{t}_n))] \leq (1 + e^{\frac{1}{2}} P(\mathcal{A}_n^{r,2}|\mathcal{F}^r(\tau^r(\hat{t}_n))))$$

$$\leq (1 + e^{\frac{1}{2}} e^{-2B_5 r^{\frac{\kappa}{2}}}) \leq (1 + e^{-B_5 r^{\frac{\kappa}{2}}}).$$

Since for $0 \leq m < n$, the set $(\mathcal{A}_m^{r,1})^c \cap \mathcal{A}_m^{r,2}$ is $\mathcal{F}^r(\tau^r(\hat{t}_n))$-measurable we have by a successive conditioning argument

$$E[e^{\frac{1}{2}\sum_{n=1}^N \mathcal{I}_{(\mathcal{A}_n^{r,1})^c\cap\mathcal{A}_n^{r,2}}}] \leq (1 + e^{-B_5 r^{\frac{\kappa}{2}}})^N E[e^{\frac{1}{2}\mathcal{I}_{(\mathcal{A}_0^{r,1})^c\cap\mathcal{A}_0^{r,2}}}]$$

$$\leq (1 + e^{-B_5 r^{\frac{\kappa}{2}}})^N e^{1/2} \leq e^{Ne^{-B_5 r^{\frac{\kappa}{2}}}} e^{1/2}.$$

Choose $R_7 \in [R_6, \infty)$ such that for all $r \geq R_7$ and $T \geq 1$, we have $-\frac{1}{2} r^{\frac{1}{8}\kappa} T + \mathrm{Tr}^{2-\frac{3}{2}\kappa} \times e^{-B_5 r^{\frac{\kappa}{2}}} + \frac{1}{2} \leq -\frac{1}{4} r^{\frac{1}{8}\kappa} T$. Then for all $r \geq R_7$, we have

$$P\Biggl(\sum_{n=1}^{\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil -1} \mathcal{I}_{(\mathcal{A}_n^{r,1})^c\cap\mathcal{A}_n^{r,2}} \geq r^{\frac{1}{8}\kappa} T\Biggr) \leq e^{-\frac{1}{2} r^{\frac{1}{8}\kappa} T + (\lceil \mathrm{Tr}^{2-\frac{3}{2}\kappa}\rceil - 1) e^{-B_5 r^{\frac{\kappa}{2}}} + \frac{1}{2}}$$

$$\leq e^{-\frac{1}{2} r^{\frac{1}{8}\kappa} T + \mathrm{Tr}^{2-\frac{3}{2}\kappa} e^{-B_5 r^{\frac{\kappa}{2}}} + \frac{1}{2}} \leq e^{-\frac{1}{4} r^{\frac{1}{8}\kappa} T}.$$

Consequently, for all $r \geq R_7$, we have that (31)–(33) hold, which proves (29). As noted previously, this completes the proof.

7. Proof of Propositions 3.4 and 3.5.

In this section, we provide proofs of Propositions 3.4 and 3.5.

7.1. *Proof of Proposition* 3.4. We begin with some preliminary stability results.

DEFINITION 7.1. Let $\tilde{\upsilon} > 0$ and $\xi \geq 0$ be arbitrary and define

$$\tilde{\gamma}^{r,\tilde{\upsilon}}_{i,\xi} \doteq \inf\left\{t \geq \xi : \hat{W}^r_i(t) + \sum_{j=1}^{J} \hat{\Upsilon}^{S,r}_j(t) + \sum_{j=1}^{J} \hat{\Upsilon}^{A,r}_j(t) < \tilde{\upsilon}\right\}.$$

Proof of the following proposition is in Section 8.1.

PROPOSITION 7.1 (Proof in Section 8.1). *There exist constants* $\tilde{\delta}, B_1, B_2, B_3, R \in (0,\infty)$ *such that for all* $c \in (0, \tilde{\delta}]$, $r \geq R$, $\tilde{\upsilon} > 0$, $\xi \geq 0$, $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$ *and* $i \in \mathbb{A}_I$ *we have*

$$E_{y^r}\big[e^{c\tilde{\gamma}^{r,\tilde{\upsilon}}_{i,\xi}}\big] \leq B_1 e^{B_2(\xi+\tilde{\upsilon}+\hat{w}^r_i+\sum_{j=1}^{J}\hat{\Upsilon}^{S,r}_j+\sum_{j=1}^{J}\hat{\Upsilon}^{A,r}_j)} + B_3,$$

where $\hat{w}^r = KM^r\hat{q}^r$.

DEFINITION 7.2. For $\tilde{\upsilon} > 0$, let $\tilde{\delta}$ be as in Proposition 7.1 and for all $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$ and $i \in \mathbb{A}_I$ define

$$V^{r,\tilde{\upsilon}}_i(y) = E_y\big[e^{\tilde{\delta}\tilde{\gamma}^{r,\tilde{\upsilon}}_{i,0}}\big].$$

$\{\hat{Y}^r(t)\}$ from Section 2.5.

LEMMA 7.2 (Proof in Section 8.2). *For* $\tilde{\upsilon} > 0$, *let* $\tilde{\delta}$ *be as in Proposition* 7.1. *There exist constants* $B, R \in (0,\infty)$ *such that for any* $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$, $t \geq 0$, $i \in \mathbb{A}_I$ *and* $r \geq R$, *we have*

$$E_y\big[V^{r,\tilde{\upsilon}}_i\big(\hat{Y}^r(t)\big)\big] \leq e^{-\tilde{\delta}t} V^{r,\tilde{\upsilon}}_i(y) + B.$$

PROPOSITION 7.3 (Proof in Section 8.3). *For any* $\tilde{\upsilon} > 0$, *there exist constants* $B_1, B_2, R \in (0,\infty)$ *such that for all* $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$, $i \in \mathbb{A}_I$ *and* $r \geq R$, *we have with* $\hat{w}^r = KM^r\hat{q}$,

$$V^{r,\tilde{\upsilon}}_i(y) \geq B_1 e^{B_2(\hat{w}^r_i - C_i \max_j\{\hat{\Upsilon}^A_j\})^+}.$$

We now proceed to the proof of Theorem 3.4.

Fix $\tilde{\upsilon} > 0$ and note that from Proposition 7.3 there exist constants $B_1, B_2, R_1 \in (0,\infty)$ such that for all $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$, $i \in \mathbb{A}_I$ and $r \geq R_1$, we have

$$B_1 e^{B_2(\hat{w}^r_i - C_i \max_j\{\hat{\Upsilon}^A_j\})^+} \leq V^{r,\tilde{\upsilon}}_i(y),$$

where $\hat{w}^r = KM^r\hat{q}$. From Lemma 7.2, there exist constants $B_3, \tilde{\delta} \in (0,\infty)$ and $R_2 \in [R_1,\infty)$ such that for any $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$, $t \geq 0$, $i \in \mathbb{A}_I$ and $r \geq R_2$ we have

$$E_y\big[V^{r,\tilde{\upsilon}}_i\big(\hat{Y}^r(t)\big)\big] \leq e^{-\tilde{\delta}t} V^{r,\tilde{\upsilon}}_i(y) + B_3.$$

In addition, Proposition 7.1 implies that there exist constants $B_4, B_5, B_6 \in (0,\infty)$ and $R_3 \in [R_2,\infty)$ such that for all $i \in \mathbb{A}_I$, $r \geq R_3$ and $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$, we have

$$V^{r,\tilde{\upsilon}}_i(y) \leq B_4 e^{B_5(|\hat{q}|+|\hat{\Upsilon}|)} + B_6.$$

Combining these three inequalities, we have, for all $t \geq 0$,

$$
\begin{aligned}
E_y\big[e^{B_2(\hat{W}_i^r(t)-C_i \max_j\{\hat{\Upsilon}_j^{A,r}(t)\})^+}\big] &\leq \frac{1}{B_1} E_y\big[V_i^{r,\tilde{\upsilon}}\big(\hat{Y}^r(t)\big)\big] \\
&\leq \frac{e^{-\tilde{\delta}t}}{B_1} V_i^{r,\tilde{\upsilon}}(y) + \frac{B_3}{B_1} \\
&\leq \frac{B_4}{B_1} e^{-\tilde{\delta}t + B_5(|\hat{q}|+|\hat{\Upsilon}|)} + \frac{B_6+B_3}{B_1}.
\end{aligned} \tag{39}
$$

Next, from Proposition 6.1, there exist constants $B_7 \in (0,\infty)$ and $R_4 \in [R_3 \vee 1, \infty)$, such that, with $\delta > 0$ as in Condition 1, for all $j \in \mathbb{A}_J$, and $c \in (0,\delta)$ we have

$$
\sup_{r \geq R_4, y \in \mathcal{Y}^r, t \in [0,\infty)} E_y\big[e^{c u_j^r(\tau_j^{A,r}(t))}\big] \leq B_7.
$$

Then for all $j \in \mathbb{A}_J$, $t \geq 0$, $c \in (0,\delta)$ and $r \geq R_4$, we have

$$
\begin{aligned}
E_y\big[e^{c\hat{\Upsilon}_j^{A,r}(t)}\big] &= E_y\big[e^{c\mathcal{I}_{\{\tau_j^{r,A}(t)=0\}}\hat{\Upsilon}_j^A + c\mathcal{I}_{\{\tau_j^{r,A}(t)>0\}}\hat{\Upsilon}_j^{A,r}(t)}\big] \\
&\leq e^{c\mathcal{I}_{\{\tau_j^{r,A}(t)=0\}}\hat{\Upsilon}_j^A} E_y\big[e^{\frac{c}{r}\mathcal{I}_{\{\tau_j^{r,A}(t)>0\}} u_j^r(\tau_j^{r,A}(t))}\big] \\
&\leq e^{c\hat{\Upsilon}_j^A} E_y\big[e^{c\mathcal{I}_{\{\tau_j^{r,A}(t)>0\}} u_j^r(\tau_j^{r,A}(t))}\big] \leq e^{c\hat{\Upsilon}_j^A} B_7.
\end{aligned} \tag{40}
$$

For all $c \in (0,\delta)$, $t \geq 0$ and $r \geq R_4$ we then have

$$
E_y\big[e^{c\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}}\big] \leq \sum_{j=1}^J E_y\big[e^{c\hat{\Upsilon}_j^{A,r}(t)}\big] \leq e^{c\max_j\{\hat{\Upsilon}_j^A\}} J B_7. \tag{41}
$$

Note that if $t > \max_j\{\hat{\Upsilon}_j^A\}$ and $r \geq R_4 \geq 1$, then $\tau_j^{r,A}(t) > 0$ for all $j \in \mathbb{A}_J$. Consequently, a similar estimate as for (41) shows that, for all $t > \max_j\{\hat{\Upsilon}_j^A\}$, $c \in (0,\delta)$ and $r \geq R_4$, we have

$$
E_y\big[e^{c\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}}\big] \leq \sum_{j=1}^J E_y\big[e^{c\hat{\Upsilon}_j^{A,r}(t)}\big] \leq J B_7. \tag{42}
$$

Let $\delta_1 \doteq \min\{B_2/2, \delta/(2C_i)\}$ and note that for all $c \in (0,\delta_1)$, $t \geq 0$, $r \geq R_4$ and $i \in \mathbb{A}_I$, we have

$$
\begin{aligned}
E_y\big[e^{c\hat{W}_i^r(t)}\big] &\leq E_y\big[\mathcal{I}_{\{\hat{W}_i^r(t)>2C_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}\}} e^{2c(\hat{W}_i^r(t)-C_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\})} \\
&\qquad + \mathcal{I}_{\{\hat{W}_i^r(t)\leq 2C_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}\}} e^{2cC_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}}\big] \\
&\leq E_y\big[e^{2c(\hat{W}_i^r(t)-C_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\})^+}\big] + E_y\big[e^{2cC_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}}\big] \\
&\leq E_y\big[e^{B_2(\hat{W}_i^r(t)-C_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\})^+}\big] + E_y\big[e^{2cC_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}}\big] \\
&\leq \frac{B_4}{B_1} e^{-\tilde{\delta}t + B_5(|\hat{q}|+|\hat{\Upsilon}|)} + \frac{B_6+B_3}{B_1} + E_y\big[e^{2cC_i\max_j\{\hat{\Upsilon}_j^{A,r}(t)\}}\big]
\end{aligned}
$$

where the last line used equation (39). Combining this with equations (41) and (42) implies that for any $c \in (0,\delta_1)$, $r \geq R_4$ and $i \in \mathbb{A}_I$ if $t \in [0, \max_j\{\hat{\Upsilon}_j^A\}]$ we have

$$
E_y\big[e^{c\hat{W}_i^r(t)}\big] \leq \frac{B_4}{B_1} e^{-\tilde{\delta}t + B_5(|\hat{q}|+|\hat{\Upsilon}|)} + \frac{B_6+B_3}{B_1} + e^{\delta_1\max_j\{\hat{\Upsilon}_j^A\}} J B_7
$$

and if $t > \max_j\{\hat{\Upsilon}_j^A\}$ we have

$$E_y\big[e^{c\hat{W}_i^r(t)}\big] \le \frac{B_4}{B_1}e^{-\tilde{\delta}t+B_5(|\hat{q}|+|\hat{\Upsilon}|)} + \frac{B_6+B_3}{B_1} + JB_7.$$

Note that since $|\hat{\Upsilon}| \ge \max_j\{\hat{\Upsilon}_j^A\}$ this implies that for any $c \in (0,\delta_1)$, $r \ge R_4$, $i \in \mathbb{A}_I$ and $t \ge 0$, we have

$$E_y\big[e^{c\hat{W}_i^r(t)}\big] \le \left(\frac{B_4}{B_1} + JB_7\right)e^{-\tilde{\delta}t+(B_5+\delta_1+\tilde{\delta})(|\hat{q}|+|\hat{\Upsilon}|)} + \frac{B_6+B_3}{B_1} + JB_7.$$

Define $\delta_2 \doteq I^{-\frac{1}{2}}\delta_1$ and note that for all $c \in (0,\delta_2)$, $t \ge 0$ and $r \ge R_4$, we have

$$\begin{aligned}
E_y\big[e^{c|\hat{W}^r(t)|_2}\big] &\le E_y\big[e^{cI^{\frac{1}{2}}\max_i\{\hat{W}_i^r(t)\}}\big] \le \sum_{i=1}^I E_y\big[e^{cI^{\frac{1}{2}}\hat{W}_i^r(t)}\big] \\
&\le \sum_{i=1}^I\left(\left(\frac{B_4}{B_1}+JB_7\right)e^{-\tilde{\delta}t+(B_5+\delta_1+\tilde{\delta})(|\hat{q}|+|\hat{\Upsilon}|)} + \frac{B_6+B_3}{B_1} + JB_7\right) \\
&\le I\left(\frac{B_4}{B_1}+JB_7\right)e^{-\tilde{\delta}t+(B_5+\delta_1+\tilde{\delta})(|\hat{q}|+|\hat{\Upsilon}|)} + I\left(\frac{B_6+B_3}{B_1} + JB_7\right).
\end{aligned}$$

Since $y \in \mathcal{Y}^r$ was arbitrary, this completes the proof.

7.2. *Proof of Proposition* 3.5. We begin with some auxiliary results.

LEMMA 7.4. *For all $s \ge 0$ if $\mathcal{Z}^r(s) \notin \mathcal{M}$, then $\tilde{d}((\hat{Q}^r(s) - \tilde{c}_2 r^{\kappa-1}) \vee 0) = 0$.*

PROOF. Let $s \ge 0$ be arbitrary and assume $\mathcal{Z}^r(s) \notin \mathcal{M}$. Note that for all $j \in \mathbb{A}_J$ we have $(\hat{Q}_j^r(s) - \tilde{c}_2 r^{\kappa-1})^+ = 0$ if $\mathcal{Z}_j^r(s) = 1$. Consequently, the result follows on applying Proposition 2.2 with $q = (\hat{Q}^r(s) - \tilde{c}_2 r^{\kappa-1})^+$ and noting that $z^q = \mathcal{Z}^r(s)$. □

The next result, which is analogous to Proposition 6.1, is proved in Section 9.1. Recall that, by convention, we take $v_j^r(0) = 0$ for $r > 0$ and $j \in \mathbb{A}_J$.

PROPOSITION 7.5 (Proof in Section 9.1). *Let δ be as in Condition 1. There exists $R < \infty$ and for any $c < \delta$ a corresponding $K(c) < \infty$ such that for any $j \in \mathbb{A}_J$, $y^r \in \mathcal{Y}^r$ and $t \ge 0$, we have*

$$\sup_{r \ge R} E_{y^r}\big[e^{cv_j^r(\tau_j^{r,S}(t))}\big] < K(c).$$

The next two results are proved in Section 10.3. Recall the constant λ from Section 2.4.

PROPOSITION 7.6 (Proof in Section 10.3). *There exists a constant $B_{\hat{h}} \in (0,\infty)$ and $R \in (0,\infty)$ such that for all $q^1, q^2 \in \mathbb{R}_+^J$ and $r \ge R$, we have*

$$|\lambda|\big(\tilde{d}(q^2) - \tilde{d}(q^1)\big) \le h \cdot (q^2 - q^1) + B_{\hat{h}}|KM^r q^2 - KM^r q^1|_2.$$

PROPOSITION 7.7 (Proof in Section 10.3). *There exists a constant $B_{\tilde{d}} < \infty$ such that for all $q^1, q^2 \in \mathbb{R}_+^J$, we have*

$$\big|\tilde{d}(q^1) - \tilde{d}(q^2)\big| \le B_{\tilde{d}}|q^1 - q^2|_2.$$

We now proceed to the proof of Proposition 3.5.

PROOF OF PROPOSITION 3.5. By adding and subtracting $\hat{h}(KM\hat{Q}^r)$, and using Proposition 2.1, Proposition 3.4 and the Lipschitz property of the map $\hat{h}$, it is sufficient to show that for any $\epsilon_0 \in (0,1)$ and $M < \infty$ there exists $T^*, R \in (0,\infty)$ such that for all $r \geq R$, $T \geq T^*$, $y^r \in \mathcal{Y}^r$ satisfying $\hat{q}^r \leq M$ and $\hat{\Upsilon}^r \leq r^{-1}M$, and $t \geq 0$ we have

$$E_{y^r}\left[\frac{1}{T}\int_0^T \tilde{d}(\hat{Q}^r(t+s))\,ds\right] \leq \epsilon_0, \qquad E_{y^r}\left[\int_0^\infty e^{-\varsigma s}\tilde{d}(\hat{Q}^r(t+s))\,ds\right] \leq \epsilon_0.$$

We now fix $\epsilon_0 \in (0,1)$ and let $\epsilon > 0$ be arbitrary, which will be chosen suitably later depending on ϵ_0.

The main idea behind the proof is that, on average, the vectors $v^c(\mathcal{Z}^r(t))$ reduce $\tilde{d}(\hat{Q}^r(t))$ faster than other factors like randomness and the vectors $v^b(\mathcal{Z}^r(t))$ (which are used to keep the queues nonempty) increase $\tilde{d}(\hat{Q}^r(t))$. This is used to show that for a given $\epsilon > 0$ there exists $k < \infty$ such that for all sufficiently large r and all $t \geq 0$ we have

$$E_{y^r}\left[\int_t^{t+kr^{-1}} \tilde{d}(\hat{Q}^r(s))\,ds\right] \leq \epsilon 3kr^{-1}, \tag{43}$$

and from this the proposition follows readily. Fix $\epsilon \in (0,1)$ and $M < \infty$. Proposition 3.4 implies that there exist constants $B_1 < \infty$ and $c > 0$ such that for sufficiently large r, all $t \geq 0$, and any $y^r \in \mathcal{Y}^r$ satisfying $\hat{q}^r \leq M$ and $\hat{\Upsilon}^r \leq r^{-1}M$ we have $E_{y^r}[e^{c|\hat{W}^r(t)|}] \leq B_1$. From the definition of $\tilde{d}$, there exists a constant $B_2 > 0$ such that for all $q \in \mathbb{R}^J_+$ and r we have $\tilde{d}(q) \leq B_2 e^{\frac{c}{2}|KM^r q|}$ and $\tilde{d}(q) \leq B_2|KM^r q|$. For notational convenience, let $\xi = \frac{\epsilon}{(1+4|\lambda||\tilde{\lambda}|^{-1})B_1 B_2} \wedge 1$, where $\tilde{\lambda}$ was defined in (5), and define for $n \in \mathbb{N}$,

$$\begin{aligned}\mathcal{A}^{r,n,t,j} = {}&\left\{\sup_{0\leq s\leq rn}|A_j^{r,t}(s) - s\alpha_j^r| \geq \frac{1}{5}nr^{\frac{1}{2}}\right\} \cup \left\{\sup_{0\leq s\leq r\max_i\{C_i\}n}|S_j^{r,t}(s) - s\beta_j^r| \geq \frac{1}{5}nr^{\frac{1}{2}}\right\}\\ &\cup \{r\bar{\Upsilon}_j^{A,r}(t) \geq \xi n\} \cup \left\{\beta_j^r r^{\frac{3}{2}}\bar{\Upsilon}_j^{S,r}(t) \geq \frac{1}{5}\xi n\right\},\end{aligned}$$

$$\mathcal{A}^{r,n,t} = \bigcup_{j=1}^J \mathcal{A}^{r,n,t,j},$$

and

$$\mathcal{B}^{r,n,t} \doteq \mathcal{A}^{r,n,t} \cup \{\tilde{d}(\hat{Q}^r(t+\xi nr^{-1})) > \xi n\} \cup \left\{\int_{r^2t}^{r^2t+rn}\sum_{j=1}^J \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds \geq \xi nr^{\frac{7}{4}\kappa}\right\}.$$

Main proof idea. We will choose k large enough that $P(\mathcal{B}^{r,k,t})$ is small and we will show that on $(\mathcal{B}^{r,k,t})^c$, because $\hat{\Upsilon}^r(t)$ has a small impact after $t+\xi kr^{-1}$, $\hat{Q}^r(s)$ has favorable behavior for $s \in [t+\xi kr^{-1}, t+kr^{-1}]$. When proving the key estimate (43), we break up the interval $[t, t+kr^{-1}]$ into the subintervals $[t, t+\xi k(1+4|\lambda||\tilde{\lambda}|^{-1})]$ and $[t+\xi k(1+4|\lambda||\tilde{\lambda}|^{-1}), t+kr^{-1}]$. We can think of ξ as determining the size of the first subinterval over which we rely on a crude bound on $E[\tilde{d}(\hat{Q}^r(s))]$ based on Proposition 3.4, but because ξ is small (recall its definition in terms of ϵ) we can use the length of the subinterval to show that

$$E_{y^r}\left[\int_t^{t+\xi k(1+4|\lambda||\tilde{\lambda}|^{-1})} \tilde{d}(\hat{Q}^r(s))\,ds\right] \leq \epsilon kr^{-1}.$$

For the second subinterval, we demonstrate that on the set $(\mathcal{B}^{r,k,t})^c$ $\tilde{d}(\hat{Q}^r(s))$ is small for all $s \in [t+\xi k(1+4|\lambda||\tilde{\lambda}|^{-1}), t+kr^{-1}]$, which combined with the fact that $(\mathcal{B}^{r,k,t})^c$ occurs with high probability, allows us to show

$$E_{y^r}\left[\int_{t+\xi k(1+4|\lambda||\tilde{\lambda}|^{-1})}^{t+kr^{-1}} \tilde{d}(\hat{Q}^r(s))\,ds\right] \le 2\epsilon k r^{-1}$$

and complete the proof of (43) from which the Proposition follows easily by appealing to Proposition 2.1.

Note that for any $j \in \mathbb{A}_J$ we have

$$\begin{aligned} r\bar{\Upsilon}_j^{A,r}(t) &\le \mathcal{I}_{\{\tau_j^{r,A}(t)=0\}}\hat{\Upsilon}_j^{A,r} + r^{-1}\mathcal{I}_{\{\tau_j^{r,A}(t)>0\}}u_j^r(\tau_j^{r,A}(t)) \\ &\le \mathcal{I}_{\{\tau_j^{r,A}(t)=0\}}r^{-1}M + r^{-1}\mathcal{I}_{\{\tau_j^{r,A}(t)>0\}}u_j^r(\tau_j^{r,A}(t)) \end{aligned}$$

and, similarly

$$\beta_j^r r^{\frac{3}{2}}\bar{\Upsilon}_j^{S,r}(t) \le \mathcal{I}_{\{\tau_j^{r,S}(t)=0\}}\beta_j^r r^{-\frac{1}{2}}M + \beta_j^r r^{-\frac{1}{2}}\mathcal{I}_{\{\tau_j^{r,S}(t)>0\}}v_j^r(\tau_j^{r,S}(t)).$$

This (recall that $\sup_j \beta_j^r < \infty$), combined with Propositions 6.2 ((14), (15), with $c_1 = 1/2$, $c_2 = 0$), 7.5 and 6.1 implies that there exist constants $\tilde{B}_1, \tilde{B}_2 < \infty$ such that for all $t \ge 0$ and all sufficiently large r, we have $P_{y^r}(\mathcal{A}^{r,n,t}) \le \tilde{B}_1 e^{-n\tilde{B}_2}$. In addition, for $r \ge \frac{1}{\xi}$ we have

$$\begin{aligned} &P_{y^r}\left(\int_{r^2t}^{r^2t+rn}\sum_{j=1}^J \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds \ge \xi n r^{\frac{7}{4}\kappa}\right) \\ &\quad\le P_{y^r}\left(\int_{r^2t}^{r^2(t+\xi n)}\sum_{j=1}^J \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds \ge \xi n r^{\frac{7}{4}\kappa}\right) \\ &\quad\le \sum_{j=1}^J P_{y^r}\left(\int_{r^2t}^{r^2(t+\xi n)} \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds \ge r\hat{\Upsilon}_j^{A,r}(t) + \frac{\xi n}{2J}r^{\frac{7}{4}\kappa}\right) \\ &\qquad+ \sum_{j=1}^J P_{y^r}\left(r\hat{\Upsilon}_j^{A,r}(t) \ge \frac{n\xi}{2J}r^{\frac{7}{4}\kappa}\right). \end{aligned}$$

This, combined with Proposition 3.3, Proposition 6.1 and the fact that for any $j \in \mathbb{A}_J$,

$$\begin{aligned} r\hat{\Upsilon}_j^{A,r}(t) &\le \mathcal{I}_{\{\tau_j^{r,A}(t)=0\}}r\hat{\Upsilon}_j^{A,r} + \mathcal{I}_{\{\tau_j^{r,A}(t)>0\}}u_j^r(\tau_j^{r,A}(t)) \\ &\le \mathcal{I}_{\{\tau_j^{r,A}(t)=0\}}M + \mathcal{I}_{\{\tau_j^{r,A}(t)>0\}}u_j^r(\tau_j^{r,A}(t)) \end{aligned}$$

implies that there exist constant $\tilde{B}_3, \tilde{B}_4 < \infty$ such that for sufficiently large r and for all $t \ge 0$ and $n \ge 0$, we have

$$P_{y^r}\left(\int_{r^2t}^{r^2t+rn}\sum_{j=1}^J \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds \ge \xi n r^{\frac{7}{4}\kappa}\right) \le \tilde{B}_3 e^{-n\tilde{B}_4}.$$

Furthermore, since $\tilde{d}(\hat{Q}^r(t+\xi nr^{-1})) \le B_2|\hat{W}^r(t+\xi nr^{-1})|$ and $E_{y^r}[e^{c|\hat{W}^r(t+\xi nr^{-1})|}] \le B_1$ for sufficiently large r, we have

$$P_{y^r}(\tilde{d}(\hat{Q}^r(t+\xi nr^{-1})) > \xi n) \le P_{y^r}\left(|\hat{W}^r(t+\xi nr^{-1})| > \frac{\xi n}{B_2}\right) \le B_1 e^{-\frac{c\xi}{B_2}n}.$$

Consequently, we know there exist constants $B_3, B_4 < \infty$ such that for sufficiently large r, we have for all $t \geq 0$ and $n \geq 0$ $P_{y^r}(\mathcal{B}^{r,n,t}) \leq B_3 e^{-nB_4}$.

Our goal is to show that for any $\epsilon > 0$ we can choose corresponding $k, R < \infty$ sufficiently large such that for all $r \geq R$ (43) holds. Our choices of $k, R < \infty$ will have to satisfy a large number of inequalities used throughout the remainder of the proof, so for the ease of the reader we list them all in one place below and reference them later when needed.

SUMMARY 1. *Recall the definitions of $\mathcal{B}^{r,n,t}$, B_1, B_2, B_3, B_4, ξ and c. We choose the constants $k \geq \frac{1}{\xi}$ and $R < \infty$ such that $B_3^{\frac{1}{2}} e^{-k\frac{B_4}{2}} B_2 B_1^{\frac{1}{2}} \leq \epsilon$ and, for all $r \geq R$, the following hold*:

(a) *For all $t \geq 0$, we have $E_{y^r}[e^{c|\hat{W}^r(t)|}] \leq B_1$.*
(b) *For all $q \in \mathbb{R}_+^J$, we have $\tilde{d}(q) \leq B_2 e^{\frac{c}{2}|KM^r q|}$ and $\tilde{d}(q) \leq B_2|KM^r q|$.*
(c) *For all $t \geq 0$, we have $P_{y^r}(\mathcal{B}^{r,k,t}) \leq B_3 e^{-kB_4}$.*
(d) $\epsilon r^{\frac{1}{8}} \geq k$, $r \geq \frac{1}{\xi}$, $r \geq 25$ *and* $\epsilon r^{\frac{1}{16}} \geq \xi k$.
(e) $2|\theta| + |\frac{1}{\beta^r}|_1 \leq r^{\frac{1}{8}}$
(f) *For all $i \in \mathbb{A}_I$, we have $r(\sum_{j=1}^J K_{i,j}\rho_j^r - C_i) \geq 2\theta_i$.*
(g) $J(c_2+1)r^{\kappa-1} + C_i \epsilon r^{\frac{7}{4}\kappa - \frac{15}{16}} + |\frac{1}{\beta^r}|_1 \epsilon r^{-\frac{3}{8}} \leq \epsilon r^{-\frac{1}{4}}$.
(h) *For all $i \in \mathbb{A}_I$, we have $(\sum_{j=1}^J K_{i,j}\rho_j^r - C_i) < 0$.*
(i) $|h||\beta^r||\rho - \rho^r| + |h||\beta^r - \beta|(\max_{z\in\mathcal{M}}|v^c(z)| + \max_{z\in\{0,1\}^J}|v^b(z)|) \leq \frac{|\tilde{\lambda}|}{4}$.
(j) $|\alpha^r| \leq 2|\alpha|$
(k) $(\frac{|\tilde{\lambda}|}{2} + 2|h||\alpha|)\epsilon r^{\frac{7}{4}\kappa - \frac{15}{16}} + |h|\epsilon r^{-\frac{3}{8}} \leq \epsilon r^{-\frac{1}{4}}$
(l) $\frac{\epsilon}{|\lambda|} r^{-\frac{1}{4}} + \frac{B_{\hat{h}}}{|\lambda|} I^{\frac{1}{2}} \epsilon r^{-\frac{1}{4}} \leq \xi k$
(m) $B_{\tilde{d}} J^{\frac{1}{2}} \tilde{c}_2 r^{\kappa-1} + J^{\frac{1}{2}} B_{\tilde{d}} r^{-1} + \frac{\epsilon}{|\lambda|} r^{-\frac{1}{4}} + 2B_{\hat{h}} I^{\frac{1}{2}} \frac{\epsilon}{|\lambda|} r^{-\frac{1}{4}} \leq \epsilon r^{-\frac{1}{8}}$.

We now bound $\tilde{d}(\hat{Q}^r(\cdot))$ from above on the set $(\mathcal{B}^{r,k,t})^c$. Using (9), on the set $(\mathcal{B}^{r,k,t})^c$ for all $r \geq R$, any $t + \xi k r^{-1} \leq s_1 < s_2 \leq t + kr^{-1}$, and any $j \in \mathbb{A}_J$, we have

$$
\begin{aligned}
&\hat{Q}_j^r(s_2) - \hat{Q}_j^r(s_1) \\
&\quad = \frac{1}{r}\mathcal{I}_{\{s_2 - t \geq \bar{\Upsilon}_j^{A,r}(t) > s_1 - t\}} - \frac{1}{r}\mathcal{I}_{\{\bar{B}_j^r(s_2) - \bar{B}_j^r(t) \geq \bar{\Upsilon}_j^{S,r}(t) > \bar{B}_j^r(s_1) - \bar{B}_j^r(t)\}} \\
&\qquad + \frac{1}{r}A_j^{r,t}\big(r^2\big(s_2 - t - \bar{\Upsilon}_j^{A,r}(t)\big)^+\big) - \frac{1}{r}A_j^{r,t}\big(r^2\big(s_1 - t - \bar{\Upsilon}_j^{A,r}(t)\big)^+\big) \\
&\qquad - \frac{1}{r}S_j^{r,t}\big(r^2\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+\big) + \frac{1}{r}S_j^{r,t}\big(r^2\big(\bar{B}_j^r(s_1) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+\big) \\
&\quad \leq \frac{1}{r}A_j^{r,t}\big(r^2\big(s_2 - t - \bar{\Upsilon}_j^{A,r}(t)\big)\big) - \frac{1}{r}A_j^{r,t}\big(r^2\big(s_1 - t - \bar{\Upsilon}_j^{A,r}(t)\big)\big) \\
&\qquad - \frac{1}{r}S_j^{r,t}\big(r^2\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+\big) + \frac{1}{r}S_j^{r,t}\big(r^2\big(\bar{B}_j^r(s_1) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+\big),
\end{aligned}
$$

since on the set $(\mathcal{B}^{r,k,t})^c$ we have $\bar{\Upsilon}_j^{A,r}(t) \leq \xi k r^{-1}$ meaning $\mathcal{I}_{\{s_2 - t \geq \bar{\Upsilon}_j^{A,r}(t) > s_1 - t\}} = 0$, and so

$$
\begin{aligned}
\hat{Q}_j^r(s_2) - \hat{Q}_j^r(s_1) \leq \alpha_j^r r(s_2 - s_1) + \frac{4}{5}kr^{-\frac{1}{2}} - \beta_j^r r\big(\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+ \\
- \big(\bar{B}_j^r(s_1) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+\big)
\end{aligned}
$$

$$\leq \alpha_j^r r(s_2 - s_1) - \beta_j^r r\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(s_1)\big) + \beta_j^r r \bar{\Upsilon}_j^{S,r}(t) + \frac{4}{5} k r^{-\frac{1}{2}}$$
$$\leq \alpha_j^r r(s_2 - s_1) - \beta_j^r r\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(s_1)\big) + k r^{-\frac{1}{2}}$$
$$\leq \alpha_j^r r(s_2 - s_1) - \beta_j^r r\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(s_1)\big) + \epsilon r^{-\frac{3}{8}}$$

due to Summary 1, part (d) and the fact that on the set $(\mathcal{B}^{r,k,t})^c$, we have $\beta_j^r r^{\frac{3}{2}} \bar{\Upsilon}_j^{S,r}(t) \leq \frac{1}{5}\xi k \leq \frac{1}{5}k$.

A similar calculation, using the inequalities,

$$\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+ - \big(\bar{B}_j^r(s_1) - \bar{B}_j^r(t) - \bar{\Upsilon}_j^{S,r}(t)\big)^+ \leq \bar{B}_j^r(s_2) - \bar{B}_j^r(s_1),$$

$-\frac{1}{r}\mathcal{I}_{\{\bar{B}_j^r(s_2)-\bar{B}_j^r(t)\geq \bar{\Upsilon}_j^{S,r}(t) > \bar{B}_j^r(s_1)-\bar{B}_j^r(t)\}} \geq -\frac{1}{r}$, properties in Summary 1, part (d) and the fact that on the set $(\mathcal{B}^{r,k,t})^c$ we have $\bar{\Upsilon}_j^{A,r}(t) \leq \xi k r^{-1}$ shows that on the set $(\mathcal{B}^{r,k,t})^c$ for any $t + \xi k r^{-1} \leq s_1 < s_2 \leq t + k r^{-1}$, $r \geq R$ and $j \in \mathbb{A}_J$, we have

$$\hat{Q}_j^r(s_2) - \hat{Q}_j^r(s_1) \geq \alpha_j^r r(s_2 - s_1) - \beta_j^r r\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(s_1)\big) - \epsilon r^{-\frac{3}{8}}.$$

Consequently, on the set $(\mathcal{B}^{r,k,t})^c$ for any $t + \xi k r^{-1} \leq s_1 < s_2 \leq t + k r^{-1}$, $r \geq R$ and $j \in \mathbb{A}_J$, we have

$$\big|\hat{Q}_j^r(s_2) - \hat{Q}_j^r(s_1) - r\big(\alpha_j^r(s_2 - s_1) - \beta_j^r\big(\bar{B}_j^r(s_2) - \bar{B}_j^r(s_1)\big)\big)\big| \leq \epsilon r^{-\frac{3}{8}}. \tag{44}$$

We will next use Proposition 7.6 to bound $\tilde{d}(\hat{Q}^r(\cdot))$ from above on the interval $[t + \xi k r^{-1}, t + k r^{-1}]$. For this, we will need to bound $\sup_{s\in[t+\xi k r^{-1}, t+k r^{-1}]} |\hat{W}_i^r(s) - \hat{W}_i^r(t + \xi k r^{-1})|$ for all $i \in \mathbb{A}_I$. Let $i \in \mathbb{A}_I$ be arbitrary. We will first bound $\inf_{s\in[t+\xi k r^{-1}, t+k r^{-1}]}(\hat{W}_i^r(s) - \hat{W}_i^r(t + \xi k r^{-1}))$ from below. On the set $(\mathcal{B}^{r,k,t})^c$ for all $r \geq R$ and $s \in [t + \xi k r^{-1}, t + k r^{-1}]$, we have from (44),

$$\begin{aligned}
\hat{W}_i^r(s) &\geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) - \left|\frac{1}{\beta^r}\right|_1 \epsilon r^{-\frac{3}{8}} \\
&\quad + r \sum_{j=1}^{J} K_{i,j} \frac{1}{\beta_j^r}\big(\alpha_j^r\big(s - t - \xi k r^{-1}\big) - \beta_j^r\big(\bar{B}_j^r(s) - \bar{B}_j^r\big(t + \xi k r^{-1}\big)\big)\big) \\
&\geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) + r\big(s - t - \xi k r^{-1}\big)\left(\sum_{j=1}^{J} K_{i,j}\rho_j^r - C_i\right) - \left|\frac{1}{\beta^r}\right|_1 \epsilon r^{-\frac{3}{8}} \\
&\geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) + 2\theta_i\big(s - t - \xi k r^{-1}\big) - \left|\frac{1}{\beta^r}\right|_1 \epsilon r^{-\frac{3}{8}} \\
&\geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) + 2\theta_i k r^{-1} - \left|\frac{1}{\beta^r}\right|_1 \epsilon r^{-\frac{3}{8}} \\
&\geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) + 2\theta_i \epsilon r^{-\frac{7}{8}} - \left|\frac{1}{\beta^r}\right|_1 \epsilon r^{-\frac{3}{8}} \\
&\geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) - \left(2|\theta_i| + \left|\frac{1}{\beta^r}\right|_1\right)\epsilon r^{-\frac{3}{8}} \geq \hat{W}_i^r\big(t + \xi k r^{-1}\big) - \epsilon r^{-\frac{1}{4}}
\end{aligned}$$

due to equation (44) and Summary 1, parts (d), (e) and (f). Therefore, on the set $(\mathcal{B}^{r,k,t})^c$ for all $r \geq R$, we have

$$\inf_{s\in[t+\xi k r^{-1}, t+k r^{-1}]}\big(\hat{W}_i^r(s) - \hat{W}_i^r\big(t + \xi k r^{-1}\big)\big) \geq -\epsilon r^{-\frac{1}{4}}.$$

Now we bound $\sup_{s\in[t+\xi kr^{-1},t+kr^{-1}]}(\hat{W}_i^r(s)-\hat{W}_i^r(t+\xi kr^{-1}))$ from above. For all $i\in\mathbb{A}_I$, define

$$\gamma_0^{r,i}\doteq\inf\{s\geq t+\xi kr^{-1}:\hat{W}_i^r(s)\geq Jc_2r^{\kappa-1}\}$$

and for $l\geq 0$ define

$$\gamma_{2l+1}^{r,i}\doteq\inf\{s\geq\gamma_{2l}^{r,i}:\hat{W}_i^r(s)<Jc_2r^{\kappa-1}\},\qquad \gamma_{2l+2}^{r,i}\doteq\inf\{s\geq\gamma_{2l+1}^{r,i}:\hat{W}_i^r(s)\geq Jc_2r^{\kappa-1}\}.$$

Using Proposition 2.6, part (d) and (44), for $r\geq R$, on the set $(\mathcal{B}^{r,k,t})^c$, for any $l\geq 0$ and all $s\in[\gamma_{2l}^{r,i},\gamma_{2l+1}^{r,i})\cap[\gamma_{2l}^{r,i},t+kr^{-1}]$,

$$\begin{aligned}
\hat{W}_i^r(s)&\leq\hat{W}_i^r(\gamma_{2l}^{r,i})+r\sum_{j=1}^J K_{i,j}\frac{1}{\beta_j^r}\big(\alpha_j^r(s-\gamma_{2l}^{r,i})-\beta_j^r(\bar{B}^r(s)-\bar{B}^r(\gamma_{2l}^{r,i}))\big)+\left|\frac{1}{\beta^r}\right|_1\epsilon r^{-\frac{3}{8}}\\
&\leq\hat{W}_i^r(t+\xi kr^{-1})+J(c_2+1)r^{\kappa-1}+r(s-\gamma_{2l}^{r,i})\left(\sum_{j=1}^J K_{i,j}\rho_j^r-C_i\right)\\
&\quad+r^{-1}C_i\int_{r^2t}^{r^2t+rk}\sum_{j=1}^J\mathcal{I}_{\{\mathcal{E}_j^r(u)=1\}}\,du+\left|\frac{1}{\beta^r}\right|_1\epsilon r^{-\frac{3}{8}}\\
&\leq\hat{W}_i^r(t+\xi kr^{-1})+J(c_2+1)r^{\kappa-1}+r(s-\gamma_{2l}^{r,i})\left(\sum_{j=1}^J K_{i,j}\rho_j^r-C_i\right)\\
&\quad+C_i\xi kr^{\frac{7}{4}\kappa-1}+\left|\frac{1}{\beta^r}\right|_1\epsilon r^{-\frac{3}{8}}\\
&\leq\hat{W}_i^r(t+\xi kr^{-1})+J(c_2+1)r^{\kappa-1}+r(s-\gamma_{2l}^{r,i})\left(\sum_{j=1}^J K_{i,j}\rho_j^r-C_i\right)\\
&\quad+C_i\epsilon r^{\frac{7}{4}\kappa-\frac{15}{16}}+\left|\frac{1}{\beta^r}\right|_1\epsilon r^{-\frac{3}{8}}\\
&\leq\hat{W}_i^r(t+\xi kr^{-1})+r(s-\gamma_{2l}^{r,i})\left(\sum_{j=1}^J K_{i,j}\rho_j^r-C_i\right)+\epsilon r^{-\frac{1}{4}}\\
&\leq\hat{W}_i^r(t+\xi kr^{-1})+\epsilon r^{-\frac{1}{4}}
\end{aligned}$$

due to Summary 1, parts (d), (g) and (h), and the fact that on the set $(\mathcal{B}^{r,k,t})^c$, we have $r^{-1}\int_{r^2t}^{r^2t+rk}\sum_{j=1}^J\mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds\leq\xi kr^{\frac{7}{4}\kappa-1}$. In addition, for $r\geq R$ on the set $(\mathcal{B}^{r,k,t})^c$, for all $s\in[t+\xi kr^{-1},\gamma_0^{r,i})\cap[t+\xi kr^{-1},t+kr^{-1}]$ and $s\in[\gamma_{2l+1}^{r,i},\gamma_{2l+2}^{r,i})\cap[\gamma_{2l+1}^{r,i},t+kr^{-1}]$ for any $l\geq 0$, we have

$$\hat{W}_i^r(s)\leq J(c_2+1)r^{\kappa-1}\leq J(c_2+1)r^{\kappa-1}+\hat{W}_i^r(t+\xi kr^{-1})\leq\hat{W}_i^r(t+\xi kr^{-1})+\epsilon r^{-\frac{1}{4}}$$

due to Summary 1, part (g). Since $i\in\mathbb{A}_I$ was arbitrary, we have shown that on the set $(\mathcal{B}^{r,k,t})^c$ for all $r\geq R$ and $i\in\mathbb{A}_I$, we have

$$\sup_{s\in[t+\xi kr^{-1},t+kr^{-1}]}\big|\hat{W}_i^r(s)-\hat{W}_i^r(t+\xi kr^{-1})\big|\leq\epsilon r^{-\frac{1}{4}}.\tag{45}$$

Now we will apply Proposition 7.6 to bound $\tilde{d}(\hat{Q}^r(\cdot))$ from above on the interval $[t+\xi kr^{-1},t+kr^{-1}]$. From Proposition 7.7, for any $q\in\mathbb{R}_+^J$,

$$\big|\tilde{d}(q)-\tilde{d}\big((q-\tilde{c}_2r^\kappa)\vee 0\big)\big|\leq B_{\tilde{d}}\big|q-(q-\tilde{c}_2r^\kappa)\vee 0\big|_2\leq B_{\tilde{d}}J^{\frac{1}{2}}\tilde{c}_2r^\kappa.\tag{46}$$

Define

$$\zeta_0^r \doteq \inf\{s \geq t + \xi k r^{-1} : \tilde{d}(Q^r(r^2 s)) \leq B_{\tilde{d}} J^{\frac{1}{2}} \tilde{c}_2 r^{\kappa}\}$$

and for $l \geq 0$ define

$$\zeta_{2l+1}^r \doteq \inf\{s \geq \zeta_{2l}^r : \tilde{d}(Q^r(r^2 s)) > B_{\tilde{d}} J^{\frac{1}{2}} \tilde{c}_2 r^{\kappa}\},$$

$$\zeta_{2l+2}^r \doteq \inf\{s \geq \zeta_{2l+1}^r : \tilde{d}(Q^r(r^2 s)) \leq B_{\tilde{d}} J^{\frac{1}{2}} \tilde{c}_2 r^{\kappa}\}.$$

Note that if $\mathcal{Z}^r(s) \notin \mathcal{M}$ then due to Lemma 7.4 we have $\tilde{d}((Q^r(r^2 s) - \tilde{c}_2 r^{\kappa}) \vee 0) = 0$ so, from (46),

$$\begin{aligned}\tilde{d}(Q^r(r^2 s)) &\leq \tilde{d}((Q^r(r^2 s) - \tilde{c}_2 r^{\kappa}) \vee 0) + |\tilde{d}(Q^r(r^2 s)) - \tilde{d}((Q^r(r^2 s) - \tilde{c}_2 r^{\kappa}) \vee 0)| \\ &\leq B_{\tilde{d}} J^{\frac{1}{2}} \tilde{c}_2 r^{\kappa}.\end{aligned}$$

Consequently, if $\tilde{d}(\hat{Q}^r(s)) > B_{\tilde{d}} J^{\frac{1}{2}} \tilde{c}_2 r^{\kappa-1}$ then $\mathcal{Z}^r(s) \in \mathcal{M}$ (here we have used the fact that for any $r > 0$ and $q \in \mathbb{R}_+^J$ we have $\tilde{d}(\frac{1}{r} q) = \frac{1}{r}\tilde{d}(q)$). In particular, $\mathcal{Z}^r(s) \in \mathcal{M}$ for all $s \in [t + \xi k r^{-1}, \zeta_0^r) \cap [t + \xi k r^{-1}, t + k r^{-1}]$ and all $s \in [\zeta_{2l+1}^r, \zeta_{2l+2}^r) \cap [\zeta_{2l+1}^r, t + k r^{-1}]$ for any $l \geq 0$.

Next, note that from Definition 2.3 for all $r \geq R$ if $\mathcal{Z}^r(s) \in \mathcal{M}$ and $\sum_{j=1}^J \mathcal{E}_j^r(s) = 0$ we have $\mathfrak{b}^r(s) = \rho - v^c(\mathcal{Z}^r(s)) - v^b(\mathcal{Z}^r(s))$, and so

$$\begin{aligned} h \cdot (\alpha^r - \beta^r \mathfrak{b}^r(s)) &= h \cdot (\alpha^r - \beta^r(\rho - v^c(\mathcal{Z}^r(s)) - v^b(\mathcal{Z}^r(s)))) \\ &= h \cdot (-\beta^r(\rho - \rho^r) + \beta v^c(\mathcal{Z}^r(s)) + \beta v^b(\mathcal{Z}^r(s)) \\ &\quad + (\beta^r - \beta) v^c(\mathcal{Z}^r(s)) + (\beta^r - \beta) v^b(\mathcal{Z}^r(s))) \\ &\leq |h||\beta^r||\rho - \rho^r| + |h||\beta^r - \beta|\Big(\max_{z \in \mathcal{M}} |v^c(z)| + \max_{z \in \{0,1\}^J} |v^b(z)|\Big) \\ &\quad + h\beta \cdot v^c(\mathcal{Z}^r(s)) + h\beta \cdot v^b(\mathcal{Z}^r(s)) \\ &\leq \frac{|\tilde{\lambda}|}{4} - |\tilde{\lambda}| + |h||\beta| \frac{|\tilde{\lambda}|}{4|h|\beta|} \leq -\frac{|\tilde{\lambda}|}{2}, \end{aligned} \tag{47}$$

where we have used Summary 1, part (i) and the fact that due to Definition 2.3, we have $\max_{z \in \mathcal{Z}} \{h\beta \cdot v^c(z)\} \leq -|\tilde{\lambda}|$ and $\max_{z \in \{0,1\}} |v^b(z)| \leq \frac{|\tilde{\lambda}|}{4|\beta||h|}$. Next, on the set $(\mathcal{B}^{r,k,t})^c$ for all $r \geq R$ and $s \in [t + \xi k r^{-1}, \zeta_0^r) \cap [t + \xi k r^{-1}, t + k r^{-1}]$, we have from (44),

$$\begin{aligned} & h \cdot (\hat{Q}^r(s) - \hat{Q}^r(t + \xi k r^{-1})) - |h|\epsilon r^{-\frac{3}{8}} \\ &\leq r h \cdot (\alpha^r(s - t - \xi k r^{-1}) - \beta^r(\bar{B}^r(s) - \bar{B}^r(t + \xi k r^{-1}))) \\ &\leq \int_{t+\xi k r^{-1}}^s \mathcal{I}_{\{\sum_{j=1}^J \mathcal{E}_j^r(r^2 z) = 0\}} r h \cdot (\alpha^r - \beta^r \mathfrak{b}^r(r^2 z))\, dz \\ &\quad + \int_{t+\xi k r^{-1}}^s \mathcal{I}_{\{\sum_{j=1}^J \mathcal{E}_j^r(r^2 z) > 0\}} r h \cdot (\alpha^r - \beta^r \mathfrak{b}^r(r^2 z))\, dz \\ &\leq -r\bigg(s - t - \xi k r^{-1} - \int_{t+\xi k r^{-1}}^s \mathcal{I}_{\{\sum_{j=1}^J \mathcal{E}_j^r(r^2 z) > 0\}}\, dz\bigg) \frac{|\tilde{\lambda}|}{2} \\ &\quad + r(h \cdot \alpha^r) \int_{t+\xi k r^{-1}}^s \mathcal{I}_{\{\sum_{j=1}^J \mathcal{E}_j^r(r^2 z) > 0\}}\, dz, \end{aligned}$$

where the last line is from (47). Thus,

$$
\begin{aligned}
&h \cdot (\hat{Q}^r(s) - \hat{Q}^r(t + \xi k r^{-1})) \\
&\le -r(s - t - \xi k r^{-1})\frac{|\tilde{\lambda}|}{2} + \left(\frac{|\tilde{\lambda}|}{2} + 2|h||\alpha|\right)\xi k r^{\frac{7}{4}\kappa - 1} + |h|\epsilon r^{-\frac{3}{8}} \\
&\le -r(s - t - \xi k r^{-1})\frac{|\tilde{\lambda}|}{2} + \left(\frac{|\tilde{\lambda}|}{2} + 2|h||\alpha|\right)\epsilon r^{\frac{7}{4}\kappa - \frac{15}{16}} + |h|\epsilon r^{-\frac{3}{8}} \\
&\le -r(s - t - \xi k r^{-1})\frac{|\tilde{\lambda}|}{2} + \epsilon r^{-\frac{1}{4}}
\end{aligned}
\tag{48}
$$

due to Summary 1, parts (d), (j) and (k), and the fact that on the set $(\mathcal{B}^{r,k,t})^c$ we have $\int_t^{t+kr^{-1}} \mathcal{I}_{\{\sum_{j=1}^J \mathcal{E}_j^r(r^2 z)>0\}}\,dz \le \xi k r^{\frac{7}{4}\kappa - 2}$. An almost identical argument shows that on the set $(\mathcal{B}^{r,k,t})^c$ for all $r \ge R$, any $l \ge 0$, and all $s \in [\zeta_{2l+1}^r, \zeta_{2l+2}^r) \cap [\zeta_{2l+1}^r, t + kr^{-1}]$ we have

$$
h \cdot (\hat{Q}^r(s) - \hat{Q}^r(\zeta_{2l+1}^r)) \le -r(s - \zeta_{2l+1}^r)\frac{|\tilde{\lambda}|}{2} + \epsilon r^{-\frac{1}{4}}. \tag{49}
$$

From Proposition 7.6, for $r \ge R$, on the set $(\mathcal{B}^{r,k,t})^c$ for all $s \in [t + \xi k r^{-1}, \zeta_0^r) \cap [t + \xi k r^{-1}, t + kr^{-1}]$, we have

$$
\begin{aligned}
\tilde{d}(\hat{Q}^r(s)) &\le \tilde{d}(\hat{Q}^r(t + \xi k r^{-1})) + \frac{1}{|\lambda|} h \cdot (\hat{Q}^r(s) - \hat{Q}^r(t + \xi k r^{-1})) \\
&\quad + \frac{B_{\hat{h}}}{|\lambda|}|\hat{W}^r(s) - \hat{W}^r(t + \xi k r^{-1})|_2 \\
&\le \xi k - r(s - t - \xi k r^{-1})\frac{|\tilde{\lambda}|}{2|\lambda|} + \frac{\epsilon}{|\lambda|} r^{-\frac{1}{4}} + \frac{B_{\hat{h}}}{|\lambda|} I^{\frac{1}{2}} \epsilon r^{-\frac{1}{4}} \\
&\le 2\xi k - r(s - t - \xi k r^{-1})\frac{|\tilde{\lambda}|}{2|\lambda|},
\end{aligned}
$$

where we have used (48), (45), Summary 1, part (l), and the fact that on the set $(\mathcal{B}^{r,k,t})^c$ we have $\tilde{d}(\hat{Q}^r(t + \xi k r^{-1})) \le \xi k$. This implies that on the set $(\mathcal{B}^{r,k,t})^c$ for $r \ge R$ if $\zeta_0^r - t - \xi k r^{-1} > \frac{4\xi k|\lambda|}{|\tilde{\lambda}|} r^{-1}$ we have

$$
\tilde{d}\left(\hat{Q}^r\left(t + \xi k r^{-1} + \frac{4\xi k|\lambda|}{|\tilde{\lambda}|} r^{-1}\right)\right) \le 2\xi k - r\left(\frac{4\xi k|\lambda|}{|\tilde{\lambda}|} r^{-1}\right)\frac{|\tilde{\lambda}|}{2|\lambda|} \le 0,
$$

which contradicts the definition of ζ_0^r. Consequently, on the set $(\mathcal{B}^{r,k,t})^c$ for $r \ge R$, we have

$$
\zeta_0^r \le t + \xi k r^{-1} + \frac{4\xi k|\lambda|}{|\tilde{\lambda}|} r^{-1}. \tag{50}
$$

Similarly, for $r \ge R$ on the set $(\mathcal{B}^{r,k,t})^c$ for all $s \in [\zeta_{2l+1}^r, \zeta_{2l+2}^r) \cap [\zeta_{2l+1}^r, t + kr^{-1}]$ for any $l \ge 0$, we have

$$
\begin{aligned}
\tilde{d}(\hat{Q}^r(s)) &\le \tilde{d}(\hat{Q}^r(\zeta_{2l+1}^r)) + \frac{1}{|\lambda|} h \cdot (\hat{Q}^r(s) - \hat{Q}^r(\zeta_{2l+1}^r)) + \frac{B_{\hat{h}}}{|\lambda|}|\hat{W}^r(s) - \hat{W}^r(\zeta_{2l+1}^r)|_2 \\
&\le \tilde{d}(\hat{Q}^r(\zeta_{2l+1}^r)) + \frac{1}{|\lambda|} h \cdot (\hat{Q}^r(s) - \hat{Q}^r(\zeta_{2l+1}^r))
\end{aligned}
$$

$$+\frac{B_{\hat{h}}}{|\lambda|}|\hat{W}^r(s)-\hat{W}^r(t+\xi kr^{-1})|_2+\frac{B_{\hat{h}}}{|\lambda|}|\hat{W}^r(\zeta^r_{2l+1})-\hat{W}^r(t+\xi kr^{-1})|_2$$
$$\le B_{\tilde{d}}J^{\frac{1}{2}}\tilde{c}_2r^{\kappa-1}+J^{\frac{1}{2}}B_{\tilde{d}}r^{-1}-r(s-\zeta^r_{2l+1})\frac{|\tilde{\lambda}|}{2|\lambda|}+\frac{\epsilon}{|\lambda|}r^{-\frac{1}{4}}+2B_{\hat{h}}I^{\frac{1}{2}}\frac{\epsilon}{|\lambda|}r^{-\frac{1}{4}}$$
$$\le \epsilon r^{-\frac{1}{8}}-r(s-\zeta^{r,\epsilon}_{2l+1})\frac{|\tilde{\lambda}|}{2|\lambda|},$$

where we have used (45), (49), Summary 1, part (m) and the fact that by the definition of ζ^r_{2l+1} and Proposition 7.7 we have $\tilde{d}(\hat{Q}^r(\zeta^r_{2l+1}))\le B_{\tilde{d}}J^{\frac{1}{2}}\tilde{c}_2r^{\kappa-1}+J^{\frac{1}{2}}B_{\tilde{d}}r^{-1}$. This, combined with the fact that for $r\ge R$, any $l\ge 0$ and all $s\in[\zeta^r_{2l},\zeta^r_{2l+1})$ we have $\tilde{d}(\hat{Q}^r(s))\le B_{\tilde{d}}J^{\frac{1}{2}}\tilde{c}_2r^{\kappa-1}\le \epsilon r^{-\frac{1}{8}}$ due to the definition of ζ^r_{2l+1} and Summary 1, part (m) implies that on the set $(\mathcal{B}^{r,k,t})^c$ for $r\ge R$ and all $s\in[\zeta^r_0,t+kr^{-1}]$ we have $\tilde{d}(\hat{Q}^r(s))\le\epsilon r^{-\frac{1}{8}}$. Therefore, from (50), on the set $(\mathcal{B}^{r,k,t})^c$ for $r\ge R$ we have

$$\tilde{d}(\hat{Q}^r(s))\le\epsilon r^{-\frac{1}{8}}\quad\text{for all } s\in\left[t+(\xi k+4\xi k|\lambda||\tilde{\lambda}|^{-1})r^{-1},t+kr^{-1}\right].\tag{51}$$

Note that for $r\ge R$ we have

$$E_{y^r}\int_t^{t+kr^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds\le E_{y^r}\int_t^{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds+E_{y^r}\int_{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}^{t+kr^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds\tag{52}$$

and

$$E_{y^r}\int_t^{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds\le\int_t^{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}B_2E_{y^r}\left[e^{c|\hat{W}^r(s)|}\right]ds\tag{53}$$
$$\le r^{-1}\left(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|}\right)B_2B_1\le r^{-1}k\epsilon\tag{54}$$

due to our choice of ξ and Summary 1, parts (a) and (b).

In addition, due to Summary 1, parts (a), (b), (c) and our choice of k, for all $r\ge R$, $t\ge 0$, and $s\ge 0$ we have

$$E_{y^r}\left[\mathcal{I}_{\mathcal{B}^{r,k,t}}\tilde{d}(\hat{Q}^r(s))\right]\le E_{y^r}\left[\mathcal{I}_{\mathcal{B}^{r,k,t}}B_2e^{\frac{c}{2}|\hat{W}^r(s)|}\right]$$
$$\le P_{y^r}(\mathcal{B}^{r,k,t})^{\frac{1}{2}}B_2E_{y^r}\left[e^{c|\hat{W}^r(s)|}\right]^{\frac{1}{2}}\le B_3^{\frac{1}{2}}e^{-k\frac{B_4}{2}}B_2B_1^{\frac{1}{2}}\le\epsilon.$$

So, for $r\ge R$, we have

$$\begin{aligned}&E_{y^r}\left[\int_{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}^{t+kr^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds\right]\\&\quad\le E_{y^r}\left[\mathcal{I}_{(\mathcal{B}^{r,k,t})^c}\int_{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}^{t+kr^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds\right]\\&\qquad+E_{y^r}\left[\mathcal{I}_{\mathcal{B}^{r,k,t}}\int_{t+(\xi k+\frac{4\xi k|\lambda|}{|\tilde{\lambda}|})r^{-1}}^{t+kr^{-1}}\tilde{d}(\hat{Q}^r(s))\,ds\right]\\&\quad\le r^{-1}\left((1-\xi)k-\frac{4\xi k|\lambda|}{|\tilde{\lambda}|}\right)\epsilon r^{-\frac{1}{8}}+r^{-1}\left((1-\xi)k-\frac{4\xi k|\lambda|}{|\tilde{\lambda}|}\right)\epsilon\le r^{-1}k2\epsilon.\end{aligned}\tag{55}$$

Combining (52), (54) and (55), we have that for $r \geq R$ and any $t \geq 0$,

$$E_{y^r}\left[\int_t^{t+kr^{-1}} \tilde{d}(\hat{Q}^r(s))\, ds\right] \leq r^{-1} 3k\epsilon.$$

We can now complete the proof. Consider first the discounted case. For $r \geq R$, we have

$$\begin{aligned}
E_{y^r}\left[\int_0^{\infty} e^{-s\varsigma} \tilde{d}(\hat{Q}^r(s))\, ds\right] &\leq \sum_{l=0}^{\infty} e^{-l\varsigma(kr^{-1})} E_{y^r}\left[\int_{lkr^{-1}}^{(l+1)kr^{-1}} \tilde{d}(\hat{Q}^r(s))\, ds\right] \\
&\leq \sum_{l=0}^{\infty} e^{-l\varsigma(kr^{-1})} r^{-1} 3k\epsilon \leq \frac{1}{1 - e^{-\varsigma(kr^{-1})}} r^{-1} 3k\epsilon \\
&\leq \frac{1}{e^{-\varsigma(kr^{-1})}\varsigma(kr^{-1})} 3r^{-1}k\epsilon \leq \frac{3\epsilon e^{\varsigma(kr^{-1})}}{\varsigma} \\
&\leq \frac{3\epsilon e^{\varsigma\epsilon r^{-\frac{7}{8}}}}{\varsigma} \leq \frac{3\epsilon e^{\varsigma\epsilon}}{\varsigma}
\end{aligned}$$

due to Summary 1, part (d). Taking $\epsilon = \frac{\varsigma e^{-\varsigma}}{3}\epsilon_0 \wedge 1$ proves the first statement in the proposition.

Consider now the ergodic cost. Let $T^* = kR^{-1}$. For any $t \geq 0$, $T \geq T^*$ and $r \geq R$, we have

$$\begin{aligned}
E_{y^r}\left[\frac{1}{T}\int_t^{t+T} \tilde{d}(\hat{Q}^r(s))\, ds\right] &\leq \frac{1}{T} \sum_{l=1}^{\lceil \frac{T}{kr^{-1}} \rceil} E_{y^r}\left[\int_{t+(l-1)kr^{-1}}^{t+lkr^{-1}} \tilde{d}(\hat{Q}^r(s))\, ds\right] \\
&\leq \frac{1}{T} \sum_{l=1}^{\lceil \frac{T}{kr^{-1}} \rceil} r^{-1} 3k\epsilon \leq \frac{1}{T}\left(\frac{T}{kr^{-1}} + 1\right) r^{-1} 3k\epsilon \\
&\leq 3\epsilon + \frac{3kR^{-1}\epsilon}{T^*} \leq 6\epsilon.
\end{aligned}$$

Taking $\epsilon = \epsilon_0/6$ completes the proof of the second statement in the proposition. □

8. Proofs of some stability results. In this section, we prove the key stability results, namely Proposition 7.1, Lemma 7.2 and Proposition 7.3, that were used in the proof of Proposition 3.4 in Section 7.1.

8.1. *Proof of Proposition* 7.1. We begin with the following auxiliary result, which is proved in Section 9.3.

PROPOSITION 8.1. *For any* $c > 0$ *and* $\epsilon > 0$, *there exist constants* $B, R \in (0, \infty)$ *such that for all* $T \geq 1$, $j \in \mathbb{A}_J$ *and* $r \geq R$ *we have*

$$P\left(\sum_{l=1}^{\lceil r^2 T \rceil} \mathcal{I}_{\{v_j^r(l) \geq rc\}} v_j^r(l) \geq \epsilon T\right) \leq e^{-BT} \tag{56}$$

and

$$P\left(\sum_{l=1}^{\lceil r^2 T \rceil} \mathcal{I}_{\{u_j^r(l) \geq rc\}} u_j^r(l) \geq \epsilon T\right) \leq e^{-BT}. \tag{57}$$

We now complete the proof of Proposition 7.1.

PROOF OF PROPOSITION 7.1. Recall ρ^* introduced below Proposition 2.3 and θ from Condition 2. Fix $i \in \mathbb{A}_I$ and $y^r = (\hat{q}^r, \hat{\Upsilon}^r, \tilde{\mathcal{E}}^r) \in \mathcal{Y}^r$. Let $\tilde{\upsilon} > 0$, $\xi \geq 0$ be given. Let $\epsilon_1 = \frac{\frac{|\theta_i|}{12}}{JC_i(1+\frac{4}{\rho^*})}$, $\epsilon_2 = \frac{|\theta_i|}{24JC_i}$ and $\epsilon_3 = \frac{|\theta_i| \min_j\{\beta_j\}}{96J}$. Define the sets

$$\begin{aligned}\mathcal{A}_j^{r,T} &= \Big\{\sup_{0\le t\le r^2T}\big|A_j^r(t) - t\alpha_j^r\big| \le \epsilon_3 rT\Big\} \cap \Big\{\sup_{0\le t\le r^2\max_i\{C_i\}T}\big|S_j^r(t) - t\beta_j^r\big| \le \epsilon_3 rT\Big\}\\ &\quad\cap \left\{\int_0^{r^2T}\mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\,ds \le r\hat{\Upsilon}_j^{A,r} + \epsilon_2 Tr\right\}\\ &\quad\cap \left\{\sum_{l=1}^{\lceil 2r^2\max_i\{C_i\}\beta_j^r T\rceil}\mathcal{I}_{\{v_j^r(l) > r\frac{\tilde{\upsilon}}{2J+1}\}}v_j^r(l) \le r\epsilon_1 T\right\}\\ &\quad\cap \left\{\sum_{l=1}^{\lceil 2r^2\alpha_j^r T\rceil}\mathcal{I}_{\{u_j^r(l) > r\frac{\tilde{\upsilon}}{2J+1}\}}u_j^r(l) \le r\epsilon_1 T\right\}\end{aligned}$$

and $\mathcal{A}^{r,T} = \bigcap_{j=1}^J \mathcal{A}_j^{r,T}$. Due to Propositions 6.2 (equations (14) and (15) with $c_1 = 1$ and $c_2 = 0$), 8.1 and 3.3 (and since $\kappa < 1/4$) we know there exist constants $B_1, B_2, R \in (0,\infty)$ such that for all $r \geq R$, $j \in \mathbb{A}_J$, $y^r \in \mathcal{Y}^r$ and $T \geq 1$ we have

$$P_{y^r}\big((\mathcal{A}^{r,T})^c\big) \le B_1 e^{-B_2 T}, \qquad \epsilon_3 T \le r\max_i\{C_i\}\beta_j^r T \quad\text{and}\quad \epsilon_3 T \le r\alpha_j^r T. \tag{58}$$

In addition, we assume R is sufficiently large that for all $r \geq R$ we have $\frac{3\theta}{4} \geq r(K\rho^r - C)$, $\tilde{\upsilon} > Jr^{-1}$, $\frac{\tilde{\upsilon}}{2J+1} \geq Jr^{\kappa-1}c_2$, $2\min_j\{\beta_j^r\} \geq \min_j\{\beta_j\}$ and for all $j \in \mathbb{A}_J$ and $t \geq 0$ we have $x_j^r(t) \geq \frac{\rho^*}{4}$ (see Proposition 2.6, part (c)). For the rest of the proof, we will restrict ourselves to values of r satisfying $r \geq R$. Note that if $\hat{W}_i^r(s) \geq \frac{\tilde{\upsilon}}{2J+1} \geq Jc_2 r^{\kappa-1}$ and $\sum_{j=1}^J \mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(s)=1\}} = 0$ then due to Proposition 2.6, part (d) we have $(K\mathfrak{b}^r)_i(r^2 s) = C_i$. This implies that if $\sum_{j=1}^J \mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(s)=1\}} = 0$,

$$\sum_{j=1}^J \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}} + \sum_{j=1}^J \mathcal{I}_{\{\hat{\Upsilon}_j^{A,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}} = 0, \quad\text{and}$$

$$\hat{W}_i^r(s) + \sum_{j=1}^J \hat{\Upsilon}_j^{S,r}(s) + \sum_{j=1}^J \hat{\Upsilon}_j^{A,r}(s) \geq \tilde{\upsilon}$$

then $(K\mathfrak{b}^r)_i(r^2 s) = C_i$. Define

$$\hat{t} \doteq \sup\left\{t \in [0,\xi] : \hat{W}_i^r(t) + \sum_{j=1}^J \hat{\Upsilon}_j^{S,r}(t) + \sum_{j=1}^J \hat{\Upsilon}_j^{A,r}(t) < \tilde{\upsilon}\right\}$$

with the convention that if

$$\hat{W}_i^r(t) + \sum_{j=1}^J \hat{\Upsilon}_j^{S,r}(t) + \sum_{j=1}^J \hat{\Upsilon}_j^{A,r}(s) \geq \tilde{\upsilon} \quad \text{for all } t \in [0,\xi] \text{ then } \hat{t} \doteq 0.$$

Define

$$T^* \doteq 1 + 2\xi + \frac{4}{|\theta_i|}\left(\hat{w}_i^r + 3\tilde{\upsilon} + (C_i+1)\sum_{j=1}^J \hat{\Upsilon}_j^{S,r} + 3C_i\sum_{j=1}^J \hat{\Upsilon}_j^{A,r}\right)$$

and let $T > T^*$. Then on the set $\{\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T\} \cap \mathcal{A}^{r,T}$, for all $s \in [\hat{t}, T]$, we have

$$\hat{W}_i^r(s) + \sum_{j=1}^J \hat{\Upsilon}_j^{S,r}(s) + \sum_{j=1}^J \hat{\Upsilon}_j^{A,r}(s) \geq \tilde{\upsilon}$$

so for any $s \in [\hat{t}, T]$ satisfying $\sum_{j=1}^J \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}} = 0$, and

$$\sum_{j=1}^J \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}} + \sum_{j=1}^J \mathcal{I}_{\{\hat{\Upsilon}_j^{A,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}} = 0,$$

we have $(K\mathfrak{b}^r)_i(s) = C_i$. Consequently, on the set $\{\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T\} \cap \mathcal{A}^{r,T}$,

$$\begin{aligned}
&\sum_{j=1}^J K_{i,j}\big(\bar{B}_j^r(T) - \bar{B}_j^r(\hat{t})\big) \\
&\quad = \int_{\hat{t}}^T \sum_{j=1}^J K_{i,j}\mathfrak{b}_j^r\big(r^2 s\big)\,ds \\
&\quad \geq C_i(T - \hat{t}) - C_i \sum_{j=1}^J \int_{\hat{t}}^T \mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(s)=1\}}\,ds \\
&\qquad - C_i \sum_{j=1}^J \int_{\hat{t}}^T \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}}\,ds - C_i \sum_{j=1}^J \int_{\hat{t}}^T \mathcal{I}_{\{\hat{\Upsilon}_j^{A,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}}\,ds \qquad (59)\\
&\quad \geq C_i(T - \hat{t}) - C_i \sum_{j=1}^J \int_0^T \mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(s)=1\}}\,ds \\
&\qquad - C_i \sum_{j=1}^J \int_0^T \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}}\,ds - C_i \sum_{j=1}^J \int_0^T \mathcal{I}_{\{\hat{\Upsilon}_j^{A,r}(s) > \frac{\tilde{\upsilon}}{2J+1}\}}\,ds.
\end{aligned}$$

For all $j \in \mathbb{A}_J$, we have

$$\begin{aligned}
&\int_0^T \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s) > \frac{\tilde{\upsilon}}{2J+1} r\}}\,ds \\
&\quad \leq r^{-1}\hat{\Upsilon}_j^{S,r} + \sum_{l=1}^{\tau_j^{r,S}(T)} \mathcal{I}_{\{v_j^r(l) > \frac{\tilde{\upsilon}}{2J+1} r\}} \frac{1}{r^2} \int_0^{r^2 T} \mathcal{I}_{\{\sum_{k=1}^{l-1} v_j^r(k) \leq B_j^r(s) \leq \sum_{k=1}^l v_j^r(k)\}}\,ds.
\end{aligned}$$

Recall that due to Proposition 2.6, part (c) and our assumption on the size of r, we have $\mathfrak{b}_j^r(s) \geq \frac{\rho^*}{4}$ unless $\mathcal{E}_j^r(s) = 1$, so for all $l \in \{1, \ldots, \tau_j^{r,S}(T)\}$ we have

$$\begin{aligned}
&\int_0^{r^2 T} \mathcal{I}_{\{\sum_{k=1}^{l-1} v_j^r(k) \leq B_j^r(s) \leq \sum_{k=1}^l v_j^r(k)\}}\,ds \\
&\quad \leq \int_0^{r^2 T} \mathcal{I}_{\{\mathcal{E}_j^r(s)=0\}} \mathcal{I}_{\{\sum_{k=1}^{l-1} v_j^r(k) \leq B_j^r(s) \leq \sum_{k=1}^l v_j^r(k)\}}\,ds \\
&\qquad + \int_0^{r^2 T} \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}} \mathcal{I}_{\{\sum_{k=1}^{l-1} v_j^r(k) \leq B_j^r(s) \leq \sum_{k=1}^l v_j^r(k)\}}\,ds \\
&\quad \leq \frac{4}{\rho^*} v_j^r(l) + \int_0^{r^2 T} \mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}} \mathcal{I}_{\{\sum_{k=1}^{l-1} v_j^r(k) \leq B_j^r(s) \leq \sum_{k=1}^l v_j^r(k)\}}\,ds.
\end{aligned}$$

Combining these last two inequalities give

$$\begin{aligned}
&\int_0^T \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s)>\frac{\tilde{\upsilon}}{2J+1}r\}}\,ds \\
&\quad\le r^{-1}\hat{\Upsilon}_j^{S,r} + \sum_{l=1}^{\tau_j^{r,S}(T)} \mathcal{I}_{\{v_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}\frac{1}{r^2}\frac{4}{\rho^*}v_j^r(l) \\
&\qquad + \sum_{l=1}^{\tau_j^{r,S}(T)} \mathcal{I}_{\{v_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}\frac{1}{r^2}\int_0^{r^2T}\mathcal{I}_{\{\mathcal{E}_j^r(s)=1\}}\mathcal{I}_{\{\sum_{k=1}^{l-1}v_j^r(k)\le B_j^r(s)\le\sum_{k=1}^{l}v_j^r(k)\}}\,ds \\
&\quad\le r^{-1}\hat{\Upsilon}_j^{S,r} + \frac{4}{\rho^*}\frac{1}{r^2}\sum_{l=1}^{\tau_j^{r,S}(T)} \mathcal{I}_{\{v_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}v_j^r(l) + \int_0^T \mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(s)=1\}}\,ds.
\end{aligned}$$

Note that on the set $\mathcal{A}^{r,T}$ we have $\tau_j^{r,S}(T)\le\lceil 2r^2\max_i\{C_i\}\beta_j^rT\rceil$ and

$$\frac{1}{r^2}\sum_{l=1}^{\lceil 2r^2\max_i\{C_i\}\beta_j^rT\rceil}\mathcal{I}_{\{v_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}v_j^r(l)\le r^{-1}\epsilon_1T,$$

which gives

$$\int_0^T \mathcal{I}_{\{\hat{\Upsilon}_j^{S,r}(s)\ge\frac{\tilde{\upsilon}}{2J+1}r\}}\,ds = r^{-1}\hat{\Upsilon}_j^{S,r} + r^{-1}\frac{4\epsilon_1}{\rho^*}T + \int_0^T \mathcal{I}_{\{\tilde{\mathcal{E}}_j^r(s)=1\}}\,ds.$$

Also, on the set $\mathcal{A}^{r,T}$, we have $\tau_j^{r,A}(T)\le\lceil 2r^2\alpha_j^rT\rceil$ and

$$\frac{1}{r^2}\sum_{l=1}^{\lceil 2r^2\alpha_j^rT\rceil}\mathcal{I}_{\{u_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}u_j^r(l)\le r^{-1}\epsilon_1T$$

so

$$\begin{aligned}
&\int_0^T \mathcal{I}_{\{\hat{\Upsilon}_j^{A,r}(s)>\frac{\tilde{\upsilon}}{2J+1}r\}}\,ds \\
&\quad\le r^{-1}\hat{\Upsilon}_j^{A,r} + \sum_{l=1}^{\tau_j^{r,A}(T)} \mathcal{I}_{\{u_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}\frac{1}{r^2}\int_0^{r^2T}\mathcal{I}_{\{\sum_{k=1}^{l-1}u_j^r(k)\le s\le\sum_{k=1}^{l}u_j^r(k)\}}\,ds \\
&\quad\le r^{-1}\hat{\Upsilon}_j^{A,r} + \frac{1}{r^2}\sum_{l=1}^{\tau_j^{r,A}(T)} \mathcal{I}_{\{u_j^r(l)>\frac{\tilde{\upsilon}}{2J+1}r\}}u_j^r(l) \le r^{-1}\hat{\Upsilon}_j^{A,r} + r^{-1}\epsilon_1T.
\end{aligned}$$

Thus, from (59), on $\{\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T\} \cap \mathcal{A}^{r,T}$,

$$\begin{aligned}
&\sum_{j=1}^{J} K_{i,j}\big(\bar{B}_j^r(T) - \bar{B}_j^r(\hat{t})\big) \\
&\quad \geq C_i(T-\hat{t}) - 2C_i \sum_{j=1}^{J} \int_0^T \mathcal{I}_{\{\mathcal{E}_j^r(r^2 s)=1\}}\, ds - r^{-1} J C_i \epsilon_1 T\left(1 + \frac{4}{\rho^*}\right) \\
&\qquad - r^{-1} C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{S,r} - r^{-1} C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{A,r} \qquad (60)\\
&\quad \geq C_i(T-\hat{t}) - r^{-1} C_i J 2\epsilon_2 T - r^{-1} J C_i \epsilon_1 T\left(1 + \frac{4}{\rho^*}\right) \\
&\qquad - r^{-1} C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{S,r} - 3r^{-1} C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{A,r}.
\end{aligned}$$

Consequently, on $\{\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T\} \cap \mathcal{A}^{r,T}$ we have

$$\begin{aligned}
&\hat{W}_i^r(T) \\
&\quad \leq \hat{W}_i^r(\hat{t}) + J r^{-1} + \sum_{j=1}^{J} K_{i,j} \frac{1}{\beta_j^r}\big(\hat{A}_j^r\big((T - \bar{\Upsilon}_j^{A,r})^+\big) - \hat{A}_j^r\big((\hat{t} - \bar{\Upsilon}_j^{A,r})^+\big)\big) \\
&\qquad - \sum_{j=1}^{J} K_{i,j} \frac{1}{\beta_j^r}\big(\hat{S}_j^r\big((\bar{B}_j^r(T) - \bar{\Upsilon}_j^{S,r})^+\big) - \hat{S}_j^r\big((\bar{B}_j^r(\hat{t}) - \bar{\Upsilon}_j^{S,r})^+\big)\big) + \sum_{j=1}^{J} K_{i,j} \hat{\Upsilon}^{S,r} \\
&\qquad + (T-\hat{t}) r\big((K\rho^r)_i - C_i\big) + r\left(C_i(T-\hat{t}) - \sum_{j=1}^{J} K_{i,j}\big(\bar{B}^r(T) - \bar{B}^r(\hat{t})\big)\right) \\
&\quad \leq \hat{W}_i^r(\hat{t}) + \tilde{\upsilon} + \frac{4J}{\min_j\{\beta_j^r\}} \epsilon_3 T + \frac{3\theta_i}{4}(T-\hat{t}) + C_i J 2\epsilon_2 T \\
&\qquad + J C_i \epsilon_1 T\left(1 + \frac{4}{\rho^*}\right) + (C_i + 1)\sum_{j=1}^{J} \hat{\Upsilon}_j^{S,r} + 3C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{A,r}.
\end{aligned}$$

Note that $\hat{W}_i^r(\hat{t}) \leq \max\{\hat{w}_i^r, \tilde{\upsilon} + r^{-1}J\} \leq \hat{w}_i^r + 2\tilde{\upsilon}$ which, combined with our choices of ϵ_i, gives

$$\begin{aligned}
\hat{W}_i^r(T) &\leq \hat{w}_i^r + 3\tilde{\upsilon} - \frac{\theta_i}{12}T + \frac{3\theta_i}{4}(T-\hat{t}) - \frac{\theta_i}{12}T - \frac{\theta_i}{12}T + (C_i+1)\sum_{j=1}^{J} \hat{\Upsilon}_j^{S,r} + 3C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{A,r} \\
&\leq \hat{w}_i^r + 3\tilde{\upsilon} + \frac{3\theta_i}{4}(T-\xi) - \frac{\theta_i}{4}T + (C_i+1)\sum_{j=1}^{J} \hat{\Upsilon}_j^{S,r} + 3C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{A,r} \\
&\leq \hat{w}_i^r + 3\tilde{\upsilon} + (C_i+1)\sum_{j=1}^{J} \hat{\Upsilon}_j^{S,r} + 3C_i \sum_{j=1}^{J} \hat{\Upsilon}_j^{A,r} + \frac{\theta_i}{4}(T-\xi),
\end{aligned}$$

where in the last inequality we have used $T < 2(T-\xi)$, which follows from our choice of T. Since $T > \xi + \frac{4}{|\theta_i|}(\hat{w}_i^r + 3\tilde{\upsilon} + (C_i+1)\sum_{j=1}^J \hat{\Upsilon}_j^S + 3C_i \sum_{j=1}^J \hat{\Upsilon}_j^A)$ and $\theta_i < 0$, this implies

$$\hat{W}_i^r(T) < \hat{w}_i^r + 3\tilde{\upsilon} + (C_i+1)\sum_{j=1}^J \hat{\Upsilon}_j^S + 3C_i \sum_{j=1}^J \hat{\Upsilon}_j^A + \frac{1}{4}\theta_i\left(-\frac{4}{\theta_i}\left(w_i^r + 3\tilde{\upsilon} + (C_i+1)\sum_{j=1}^J \hat{\Upsilon}_j^S + 3C_i \sum_{j=1}^J \hat{\Upsilon}_j^A\right)\right) = 0,$$

which contradicts the fact that $\hat{W}_i^r(t) \geq 0$ for all $t \geq 0$. Therefore, for all $T > T^*$ we have $\{\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T\} \cap \mathcal{A}^{r,T} = \varnothing$, which says that $\{\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T\} \subset (\mathcal{A}^{r,T})^c$, and consequently, from (58) $P_{y^r}(\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > T) \leq P_{y^r}((\mathcal{A}^{r,T})^c) \leq B_1 e^{-B_2 T}$. Finally, for any $c \in (0, \frac{1}{2}B_2]$ and $r \geq R$, we have

$$\begin{aligned} E_{y^r}[e^{c\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}}}] &\leq e^{cT^*} + \int_{(e^{cT^*},\infty)} P_{y^r}\left(\tilde{\gamma}_{i,\xi}^{r,\tilde{\upsilon}} > \frac{\ln(x)}{c}\right) dx \\ &\leq e^{cT^*} + \int_{(e^{cT^*},\infty)} B_1 e^{-\frac{B_2 \ln(x)}{c}}\, dx \\ &\leq e^{\frac{1}{2}B_2 T^*} + B_1. \end{aligned}$$

The result follows on recalling the definition of T^*. □

8.2. *Proof of Lemma* 7.2. Since $\hat{Y}^r(\cdot)$ is a $\mathcal{G}^r(t)$ Markov process, we have, for $0 \leq s < t$,

$$\begin{aligned} E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(t))] &= E_y[e^{\tilde{\delta}(\tilde{\gamma}_{i,t}^{r,\tilde{\upsilon}} - t)}] = E_y[E_y[e^{\tilde{\delta}(\tilde{\gamma}_{i,t}^{r,\tilde{\upsilon}} - t)} | \mathcal{G}^r(s)]] \\ &= E_y[E_{\hat{Y}^r(s)}[e^{\tilde{\delta}(\tilde{\gamma}_{i,t-s}^{r,\tilde{\upsilon}} - (t-s))}]] = E_y[E_{\hat{Y}^r(s)}[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(t-s))]]. \end{aligned}$$

PROPOSITION 8.2. *For* $\tilde{\upsilon} > 0$, *let* $\tilde{\delta}$ *be as in Proposition* 7.1. *There exist constants* $\tilde{R}, \tilde{B} \in (0,\infty)$ *such that for any* $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$, $i \in \mathbb{A}_I$, $s \in [0,1]$ *and* $r \geq \tilde{R}$, *we have*

$$E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(s))] \leq e^{-\tilde{\delta}s} V_i^{r,\tilde{\upsilon}}(y) + \tilde{B}.$$

PROOF. Let $\tilde{\upsilon} > 0$ be fixed, let B_1, B_2, B_3, R be as in Theorem 7.1 and let $y \in \mathcal{Y}^r$, $i \in \mathbb{A}_I$ and $s \in [0,1]$ be arbitrary. Due to the fact that $\hat{Y}^r(t)$ is a $\mathcal{G}^r(t)$ strong Markov process and $\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} \wedge s$ is a bounded $\mathcal{G}^r(t)$ stopping time, for all $r \geq R$ we have

$$\begin{aligned} E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(s))] &\leq E_y[e^{\tilde{\delta}(\tilde{\gamma}_{i,s}^{r,\tilde{\upsilon}} - s)}] \\ &= E_y[\mathcal{I}_{\{\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} \geq s\}} e^{\tilde{\delta}(\tilde{\gamma}_{i,s}^{r,\tilde{\upsilon}} - s)}] + E_y[\mathcal{I}_{\{\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} < s\}} e^{\tilde{\delta}(\tilde{\gamma}_{i,s}^{r,\tilde{\upsilon}} - s)}] \\ &\leq E_y[e^{\tilde{\delta}(\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} - s)}] + E_y[E_y[\mathcal{I}_{\{\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} < s\}} e^{\tilde{\delta}(\tilde{\gamma}_{i,s}^{r,\tilde{\upsilon}} - s)} | \mathcal{G}^r(\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} \wedge s)]] \\ &\leq e^{-\tilde{\delta}s} E_y[e^{\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}}}] + \sup_{(y,z): w_i + |\hat{\Upsilon}| \leq \tilde{\upsilon}, z \in [0,s]} \{E_y[e^{\tilde{\delta}\tilde{\gamma}_{i,z}^{r,\tilde{\upsilon}}}]\} \\ &\leq e^{-\tilde{\delta}s} V_i^{r,\tilde{\upsilon}}(y) + B_1 e^{B_2(1+2\tilde{\upsilon})} + B_3 \end{aligned}$$

where the last line uses Proposition 7.1. This completes the proof. □

We now complete the proof of Lemma 7.2.

PROOF OF LEMMA 7.2. Let $\tilde{\upsilon} > 0$ be fixed and note that, with $\tilde{\delta}$ as in Proposition 7.1, due to Proposition 8.2, there exist constants $B_1, R \in (0, \infty)$ such that for all $y \in \mathcal{Y}^r$, $r \geq R$, $i \in \mathbb{A}_I$ and $t \in [0, 1]$ we have

$$E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(t))] \leq e^{-\tilde{\delta}t} V_i^{r,\tilde{\upsilon}}(y) + B_1.$$

Consequently, for $T \in [0, 1]$, we already have the result (with $B = B_1$). Let $T > 1$ be arbitrary and note that there exists $t \in [\frac{1}{2}, 1]$ and $n \in \mathbb{N}$ such that $T = nt$. Using the Markov property, we have

$$\begin{aligned} &E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(T))] \\ &\quad = E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(nt))] = E_y[E[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(nt))|\mathcal{G}^r((n-1)t)]] \\ &\quad = E_y[E_{\hat{Y}^r((n-1)t)}[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(t))]] \leq e^{-\tilde{\delta}t} E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r((n-1)t))] + B_1. \end{aligned}$$

Now a standard recursive argument shows that

$$\begin{aligned} E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(T))] &\leq e^{-\tilde{\delta}nt} E_y[V_i^{r,\tilde{\upsilon}}(\hat{Y}^r(0))] + B_1 \sum_{l=0}^{n-1} e^{-\tilde{\delta}lt} \\ &\leq e^{-\tilde{\delta}T} V_i^{r,\tilde{\upsilon}}(y) + B_1 \sum_{l=0}^{\infty} e^{-\tilde{\delta}l\frac{1}{2}} \leq e^{-\tilde{\delta}T} V_i^{r,\tilde{\upsilon}}(y) + \frac{B_1}{1 - e^{-\frac{1}{2}\tilde{\delta}}}. \end{aligned}$$

This completes the proof on taking $B = B_1/(1 - e^{-\frac{1}{2}\tilde{\delta}})$. □

8.3. *Proof of Proposition* 7.3. Let $\tilde{\upsilon} > 0$ be fixed and let $y = (\hat{q}, \hat{\Upsilon}, \tilde{\mathcal{E}}) \in \mathcal{Y}^r$ and $i \in \mathbb{A}_I$ be arbitrary. For $T \geq 1$, define

$$\mathcal{A}_j^{r,T} = \left\{ \sup_{0 \leq t \leq r^2 T} |A_j^r(t) - t\alpha_j^r| \leq \frac{|\theta_i|}{4J} rT \right\} \cap \left\{ \sup_{0 \leq t \leq r^2 \max_i\{C_i\}T} |S_j^r(t) - t\beta_j^r| \leq \frac{|\theta_i|}{4J} rT \right\}$$

and $\mathcal{A}^{r,T} = \bigcap_{j=1}^J \mathcal{A}_j^{r,T}$. From Proposition 6.2 (equations (14) and (15) with $c_1 = 1$, $c_2 = 0$), we know there exist constants $B_1, B_2, R \in (0, \infty)$ such that for all $r \geq R$ and $T \geq 1$ we have

$$P(\mathcal{A}^{r,T}) \geq 1 - B_1 e^{-B_2 T}, \qquad r^{-1} \sum_{j=1}^J \frac{1}{\beta_j^r} \leq \tilde{\upsilon}, \quad \text{and,} \quad r\left(\sum_{j=1}^J K_{i,j}\rho_j^r - C_i\right) \geq \frac{3\theta_i}{2}.$$

Choose $L \in [1, \infty)$ such that $1 - B_1 e^{-B_2 L} \geq \frac{1}{2}$. If

$$\hat{w}_i - C_i \max_j\{\hat{\Upsilon}_j^A\} \geq 2|\theta_i|L + 2\tilde{\upsilon}, \quad \text{let } \tilde{T} \doteq \frac{1}{2|\theta_i|}\left(\hat{w}_i - C_i \max_j\{\hat{\Upsilon}_j^A\} - 2\tilde{\upsilon}\right)$$

and note that $\tilde{T} \geq L$. For all $r \geq R$ and $s \in [0, \tilde{T})$ on the set $\mathcal{A}^{r,\tilde{T}}$, we have

$$\begin{aligned} \hat{W}_i^r(s) &= \hat{w}_i + \sum_{j=1}^J K_{i,j} \frac{1}{r\beta_j^r} A_j^r(r^2(s - \bar{\Upsilon}_j^A)^+) + \sum_{j=1}^J K_{i,j} \frac{1}{r\beta_j^r} \mathcal{I}_{\{s \geq \bar{\Upsilon}_j^A > 0\}} \\ &\quad - \sum_{j=1}^J K_{i,j} \frac{1}{r\beta_j^r} S_j^r(r^2(\bar{B}^r(s) - \bar{\Upsilon}_j^S)^+) - \sum_{j=1}^J K_{i,j} \frac{1}{r\beta_j^r} \mathcal{I}_{\{\bar{B}^r(s) \geq \bar{\Upsilon}_j^S > 0\}} \\ &\geq \hat{w}_i + \sum_{j=1}^J K_{i,j} \frac{1}{r\beta_j^r} A_j^r(r^2(s - \bar{\Upsilon}_j^A)^+) - \sum_{j=1}^J K_{i,j} \frac{1}{r\beta_j^r} S_j^r(r^2 \bar{B}_j^r(s)) - r^{-1} \sum_{j=1}^J \frac{1}{\beta_j^r} \end{aligned}$$

$$
\begin{aligned}
&\geq \hat{w}_i + r\sum_{j=1}^{J} K_{i,j}\rho_j^r(s-\bar{\Upsilon}_j^A)^+ - r\sum_{j=1}^{J} K_{i,j}\bar{B}_j^r(s) + \frac{\theta_i}{2}\tilde{T} - \tilde{\upsilon} \\
&\geq \hat{w}_i + r\sum_{j=1}^{J} K_{i,j}\rho_j^r\Big(s-\max_j\{\bar{\Upsilon}_j^A\}\Big)^+ - rsC_i + \frac{\theta_i}{2}\tilde{T} - \tilde{\upsilon} \\
&\geq \hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} + sr\left(\sum_{j=1}^{J} K_{i,j}\rho_j^r - C_i\right) + \frac{\theta_i}{2}\tilde{T} - \tilde{\upsilon} \\
&\geq \hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} + \frac{3\theta_i}{2}s + \frac{\theta_i}{2}\tilde{T} - \tilde{\upsilon} > \hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} - \tilde{\upsilon} + 2\theta_i\tilde{T}.
\end{aligned}
$$

Since $\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} - \tilde{\upsilon} + 2\theta_i\tilde{T} = \tilde{\upsilon}$, this means that for $r \geq R$ and $s \in [0,\tilde{T})$ on the set $\mathcal{A}^{r,\tilde{T}}$ we have $\hat{W}_i^r(s) > \tilde{\upsilon}$, and consequently, $\{\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}} \geq \tilde{T}\} \supset \mathcal{A}^{r,\tilde{T}}$. Therefore, for $r \geq R$, if $\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} \geq 2|\theta_i|L + 2\tilde{\upsilon}$, we have

$$
\begin{aligned}
V_i^{r,\tilde{\upsilon}}(y) &= E_y\big[e^{\tilde{\delta}\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}}}\big] \geq P\big(\mathcal{A}^{r,\tilde{T}}\big)e^{\tilde{\delta}\tilde{T}} \\
&\geq \big(1-B_1e^{-B_2L}\big)e^{\tilde{\delta}\tilde{T}} \geq \frac{e^{-\frac{\tilde{\delta}\tilde{\upsilon}}{|\theta_i|}}}{2}e^{\frac{\tilde{\delta}}{2|\theta_i|}(\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\})}.
\end{aligned} \tag{61}
$$

If $\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} < 2|\theta_i|L + 2\tilde{\upsilon}$, by definition, we have $V_i^{r,\tilde{\upsilon}}(y) = E_y[e^{\tilde{\delta}\tilde{\gamma}_{i,0}^{r,\tilde{\upsilon}}}] \geq 1$. Let

$$
\tilde{B}_1 \doteq \min\left\{e^{-\frac{\tilde{\delta}}{2|\theta_i|}(2|\theta_i|L+2\tilde{\upsilon})}, \frac{e^{-\frac{\tilde{\delta}\tilde{\upsilon}}{|\theta_i|}}}{2}\right\}, \quad \text{and} \quad \tilde{B}_2 \doteq \frac{\tilde{\delta}}{2|\theta_i|}.
$$

Then if $\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} \leq 2|\theta_i|L + 2\tilde{\upsilon}$, we have $\tilde{B}_1 e^{\tilde{B}_2(\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\})^+} \leq 1 \leq V_i^{r,\tilde{\upsilon}}(y)$ and if $\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\} \geq 2|\theta_i|L + 2\tilde{\upsilon}$ we have from (61),

$$
\tilde{B}_1 e^{\tilde{B}_2(\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\})^+} \leq \frac{e^{-\frac{\tilde{\delta}\tilde{\upsilon}}{|\theta_i|}}}{2}e^{\frac{\tilde{\delta}}{2|\theta_i|}(\hat{w}_i - C_i\max_j\{\hat{\Upsilon}_j^A\})} \leq V_i^{r,\tilde{\upsilon}}(y).
$$

The result follows.

9. Proofs of some exponential estimates.

The following lemma gives a key independence property.

LEMMA 9.1. *Let $t \in \mathbb{R}_+$ and $r \in \mathbb{N}$. Then $(\hat{A}^{r,t}(\cdot), \hat{S}^{r,t}(\cdot))$ is independent of $\mathcal{G}^r(t)$ and $(\hat{A}^{r,t}(\cdot), \hat{S}^{r,t}(\cdot))$ has the same distribution as $(\hat{A}^r(\cdot), \hat{S}^r(\cdot))$.*

PROOF. For $a, b \in \mathbb{N}^J$, define

$$
\begin{aligned}
\tilde{u}_a^r(b) = \big(&u_1^r(a_1+1)\dots, u_1^r(a_1+b_1), \\
&u_2^r(a_2+1)\dots, u_2^r(a_2+b_2)\dots, u_J^r(a_J+1)\dots, u_J^r(a_J+b_J)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{v}_a^r(b) = \big(&v_1^r(a_1+1)\dots, v_1^r(a_1+b_1), \\
&v_2^r(a_2+1)\dots, v_2^r(a_2+b_2), \dots, v_J^r(a_J+1)\dots, v_J^r(a_J+b_J)\big).
\end{aligned}
$$

It suffices to show that for all $(\tilde{n},\tilde{m}) \geq (0,0)$ $(\tilde{u}^r_{\tau^{r,A}(t)}(\tilde{n}), \tilde{v}^r_{\tau^{r,S}(t)}(\tilde{m}))$ is independent of $\mathcal{G}^r(t) = \mathcal{F}^r(\tau^r(t))$ and $(\tilde{u}^r_{\tau^{r,A}(t)}(\tilde{n}), \tilde{v}^r_{\tau^{r,S}(t)}(\tilde{m}))$ has the same distribution as $(\tilde{u}^r_0(\tilde{n}), \tilde{v}^r_0(\tilde{m}))$. Let $(n,m) \geq (0,0)$, $f : \mathbb{R}_+^{|\tilde{n}|_1+|\tilde{m}|_1} \to \mathbb{R}$ and $G \in \mathcal{F}^r(\tau^r(t))$ be arbitrary. Then

$$
\begin{aligned}
&E\big[\mathcal{I}_G f\big(\tilde{u}^r_{\tau^{r,A}(t)}(\tilde{n}), \tilde{v}^r_{\tau^{r,S}(t)}(\tilde{m})\big)\big] \\
&\quad = \sum_{(n,m)\geq(0,0)} E\big[\mathcal{I}_{\{\tau^r(t)=(n,m)\}}\mathcal{I}_G f\big(\tilde{u}^r_{\tau^{r,A}(t)}(\tilde{n}), \tilde{v}^r_{\tau^{r,S}(t)}(\tilde{m})\big)\big] \\
&\quad = \sum_{(n,m)\geq(0,0)} E\big[\mathcal{I}_{G\cap\{\tau^r(t)=(n,m)\}} f\big(\tilde{u}^r_n(\tilde{n}), \tilde{v}^r_m(\tilde{m})\big)\big] \\
&\quad = \sum_{(n,m)\geq(0,0)} E[\mathcal{I}_{G\cap\{\tau^r(t)=(n,m)\}}]E\big[f\big(\tilde{u}^r_n(\tilde{n}), \tilde{v}^r_m(\tilde{m})\big)\big] \\
&\quad = E\big[f\big(\tilde{u}^r_0(\tilde{n}), \tilde{v}^r_0(\tilde{m})\big)\big] \sum_{(n,m)\geq(0,0)} E[\mathcal{I}_{G\cap\{\tau^r(t)=(n,m)\}}] = P(G)\big[f\big(\tilde{u}^r_0(\tilde{n}), \tilde{v}^r_0(\tilde{m})\big)\big],
\end{aligned}
$$

where the third equality is from the fact that $G \cap \{\tau^r(t) = (n,m)\}$ is $\mathcal{F}^r(n,m)$-measurable and $(\tilde{u}^r_n(\tilde{n}), \tilde{v}^r_m(\tilde{m}))$ is independent of $\mathcal{F}^r(n,m)$ and the fourth equality uses the fact that $(\tilde{u}^r_n(\tilde{n}), \tilde{v}^r_m(\tilde{m}))$ has the same distribution as $(\tilde{u}^r_0(\tilde{n}), \tilde{v}^r_0(\tilde{m}))$. The result follows. □

9.1. *Proof of Proposition* 7.5. Fix $j \in \mathbb{A}_J$. Define

$$\mathcal{F}^{r,S}_j(k) \doteq \sigma\{u^r_l(m^u_l), v^r_{l'}(m^v_{l'}) : m^u_l \in \mathbb{N}, l \in \mathbb{A}_J, m^v_{l'} \in \mathbb{N}, l' \in \mathbb{A}_J \setminus \{j\} \text{ and } m^v_j \leq k\},$$

which is the filtration that contains the information from all interarrival times, all service times from queues other than the jth queue and the first k service times from queue j. Note that $\tau^{r,S}_j(t)$ is a $\mathcal{F}^{r,S}_j(k)$ stopping time. For all $n \geq 1$, define $\tilde{L}^r_j(n) = \sup\{s \geq 0 : B^r_j(s) < \sum_{l=1}^n v^r_j(l)\}$ and since $B^r_j(\cdot)$ is continuous, we have $B^r_j(\tilde{L}^r_j(n)) = \sum_{l=1}^n v^r_j(l)$.

Define

$$o^r_j(n) = \inf\{s \geq \tilde{L}^r_j(n) : \mathcal{E}^r_j(s) = 0\}$$

and note that due to property (a) of Proposition 2.6 for all $s \in [\tilde{L}^r_j(n), o^r_j(n)]$ we have $B^r_j(s) = \sum_{l=1}^n v^r_j(l)$. Define $\tilde{a} = \frac{\rho^*}{4}$ and note that due to properties (b) and (c) of Proposition 2.6 for all r sufficiently large, we have $\mathfrak{b}^r_j(s) \geq \tilde{a} > 0$ for all $s \in [o^r_j(n), \tilde{L}^r_j(n+1)]$.

Let $n \geq 2$ be arbitrary and assume that $\tau^{r,S}_j((t - \frac{v^r_j(n)}{\tilde{a}r^2})^+) > n-1$ and $\tau^{r,S}_j(t) = n$. Note that in this case we must have $t > \frac{v^r_j(n)}{\tilde{a}r^2}$. Since $\tau^{r,S}_j(t - \frac{v^r_j(n)}{\tilde{a}r^2}) > n-1$ implies $B^r_j(r^2(t - \frac{v^r_j(n)}{\tilde{a}r^2})) > \sum_{l=1}^{n-1} v^r_j(l)$, we have $o^r_j(n-1) < r^2(t - \frac{v^r_j(n)}{\tilde{a}r^2})$ because $o^r_j(n-1) \geq r^2(t - \frac{v^r_j(n)}{\tilde{a}r^2})$ implies $B^r_j(r^2(t - \frac{v^r_j(n)}{\tilde{a}r^2})) = \sum_{l=1}^{n-1} v^r_j(l)$. In addition, $\tau^{r,S}_j(t) = n$ implies $\tilde{L}^r_j(n) \geq tr^2$ so $[r^2(t - \frac{v^r_j(n)}{r^2\tilde{a}}), r^2t] \subset [o^r_j(n-1), \tilde{L}^r_j(n)]$, and consequently, $\mathfrak{b}^r_j(s) \geq \tilde{a}$ for all $s \in [r^2(t - \frac{v^r_j(n)}{r^2\tilde{a}}), r^2t]$, and thus in particular $B^r_j(r^2t) - B^r_j(r^2(t - \frac{v^r_j(n)}{r^2\tilde{a}})) \geq v^r_j(n)$. Therefore,

$$\sum_{l=1}^n v^r_j(l) = \sum_{l=1}^{n-1} v^r_j(l) + v^r_j(n) < B^r_j\bigg(r^2\bigg(t - \frac{v^r_j(n)}{r^2\tilde{a}}\bigg)\bigg) + v^r_j(n) \leq B^r_j(tr^2),$$

which contradicts the assumption that $\tau^{r,S}_j(t) = n$. Consequently,

$$\bigg\{\tau^{r,S}_j\bigg(\bigg(t - \frac{v^r_j(n)}{\tilde{a}r^2}\bigg)^+\bigg) > n-1\bigg\} \cap \{\tau^{r,S}_j(t) = n\} = \varnothing$$

and so

$$\{\tau_j^{r,S}(t)=n\}=\left\{\tau_j^{r,S}\left(\left(t-\frac{v_j^r(n)}{\tilde{a}r^2}\right)^+\right)\le n-1\right\}\cap\{\tau_j^{r,S}(t)=n\}.$$

Thus, we have

$$\begin{aligned}\{\tau_j^{r,S}(t)=n\}&=\{\tau_j^{r,S}(t)>n-1\}\cap\{\tau_j^{r,S}(t)=n\}\\&=\{\tau_j^{r,S}(t)>n-1\}\cap\left\{\tau_j^{r,S}\left(\left(t-\frac{v_j^r(n)}{\tilde{a}r^2}\right)^+\right)\le n-1\right\}\cap\{\tau_j^{r,S}(t)=n\}\\&\subset\{\tau_j^{r,S}(t)>n-1\}\cap\left\{\tau_j^{r,S}\left(\left(t-\frac{v_j^r(n)}{\tilde{a}r^2}\right)^+\right)\le n-1\right\}.\end{aligned}$$

Note that for any $z\ge 0$, $t\ge 0$ and $n\ge 2$ both $\{\tau_j^{r,S}(t)>n-1\}$ and $\{\tau_j^{r,S}((t-\frac{z}{\tilde{a}r^2})^+)\le n-1\}$ are $\mathcal{F}_j^{r,S}(n-1)$-measurable and $v_j^r(n)$ is independent of $\mathcal{F}_j^{r,S}(n-1)$.

Let $\gamma_j^r(dz)$ denote the pdf of $v_j^r(1)$ (recall that the $\{v_j^r(l)\}_{l=1}^\infty$ are i.i.d.) and let $c\in(0,\delta)$ be arbitrary where δ is as in Condition 1. Since $P(\tau_j^{r,S}(t)<\infty)=1$, we have, recalling the convention $v_j^r(0)=0$,

$$\begin{aligned}E[e^{cv_j^r(\tau_j^{r,S}(t))}]&\le 1+E[e^{cv_j^r(1)}]+\sum_{n=2}^\infty E[e^{cv_j^r(n)}\mathbb{I}_{\{\tau_j^{r,S}(t)=n\}}]\\&\le 1+E[e^{cv_j^r(1)}]+\sum_{n=2}^\infty E[e^{cv_j^r(n)}\mathbb{I}_{\{\tau_j^{r,S}(t)>n-1\}\cap\{\tau_j^{r,S}((t-\frac{v_j^r(n)}{r^2\tilde{a}})^+)\le n-1\}}]\\&=1+\int_{(0,\infty)}e^{cz}E\left[1+\sum_{n=1}^\infty\mathbb{I}_{\{\tau_j^{r,S}((t-\frac{z}{\tilde{a}r^2})^+)\le n\}\cap\{\tau_j^{r,S}(t)>n\}}\right]\gamma_j^r(dz)\\&=1+\int_{(0,\infty)}e^{cz}\left(1+E\left[\tau_j^{r,S}(t)-\tau_j^{r,S}\left(\left(t-\frac{z}{\tilde{a}r^2}\right)^+\right)\right]\right)\gamma_j^r(dz).\end{aligned}\tag{62}$$

Let

$$\hat{\tau}_j^{r,S,t}(z)=\min\left\{n\ge 1:\sum_{l=\tau_j^{r,S}((t-\frac{z}{\tilde{a}r^2})^+)+1}^{\tau_j^{r,S}((t-\frac{z}{\tilde{a}r^2})^+)+n}v_j^r(l)\ge\frac{\max_i\{C_i\}z}{\tilde{a}}\right\}$$

and note that since $B_j^r(r^2t)-B_j^r(r^2(t-\frac{z}{\tilde{a}r^2})^+)\le\frac{\max_i\{C_i\}z}{\tilde{a}}$ and $B_j^r(r^2(t-\frac{z}{\tilde{a}r^2})^+)\le\sum_{l=1}^{\tau_j^{r,S}((t-\frac{z}{\tilde{a}r^2})^+)}v_j^r(l)$, we have $\hat{\tau}_j^{r,S,t}(z)\ge\tau_j^{r,S}(t)-\tau_j^{r,S}((t-\frac{z}{\tilde{a}r^2})^+)$. In addition, if we define $\hat{\tau}_j^{r,S}(z)=\min\{n\ge 1:\sum_{l=1}^n v_j^r(l)\ge\frac{\max_i\{C_i\}z}{\tilde{a}}\}$, then we have $\hat{\tau}_j^{r,S,t}(z)\overset{d}{=}\hat{\tau}_j^{r,S}(z)$ due to Lemma 9.1. Consequently, from (62),

$$E[e^{cv_j^r(\tau_j^{r,S}(t))}]\le 1+\int_{(0,\infty)}e^{cz}(1+E[\hat{\tau}_j^{r,S}(z)])\gamma_j^r(dz).\tag{63}$$

We will now bound $E[\hat{\tau}_j^{r,S}(z)]$ from above. For all r sufficiently large, we have $\frac{2}{\beta_j}\ge\frac{1}{\beta_j^r}\ge\frac{3}{4\beta_j}$ and $2\sigma_j^v\ge\sigma_j^{v,r}\ge\frac{1}{2}\sigma_j^v$. Due to Condition 1, there exists $1\le K<\infty$ such that $\sup_r\{E[v_j^r(1)\mathbb{I}_{\{v_j^r(1)>K\}}]\}\le\frac{1}{4\beta_j}$. Therefore, for all r sufficiently large, we have

$$\frac{3}{4\beta_j}\le\frac{1}{\beta_j^r}\le E[v_j^r(1)\mathbb{I}_{\{v_j^r(1)\le K\}}]+E[v_j^r(1)\mathbb{I}_{\{v_j^r(1)>K\}}]$$

and so

$$E\big[v_j^r(1)\mathbb{I}_{\{v_j^r(1)\le K\}}\big] \ge \frac{3}{4\beta_j} - E\big[v_j^r(1)\mathbb{I}_{\{v_j^r(1)>K\}}\big] \ge \frac{1}{2\beta_j}.$$

In addition,

$$\begin{aligned} E\big[v_j^r(1)\mathbb{I}_{\{v_j^r(1)\le K\}}\big] &\le E\big[v_j^r(1)\mathbb{I}_{\{v_j^r(1)<\frac{1}{4\beta_j}\}}\big] + E\big[v_j^r(1)\mathbb{I}_{\{\frac{1}{4\beta_j}\le v_j^r(1)\le K\}}\big] \\ &\le \frac{1}{4\beta_j}P\bigg(v_j^r(1) < \frac{1}{4\beta_j}\bigg) + KP\bigg(\frac{1}{4\beta_i} \le v_j^r(1) \le K\bigg) \\ &\le \frac{1}{4\beta_j} + KP\bigg(\frac{1}{4\beta_j} \le v_j^r(1)\bigg) \end{aligned}$$

so for all r sufficiently large we have

$$P\bigg(\frac{1}{4\beta_j} \le v_j^r(1)\bigg) \ge \frac{1}{K}E\big[v_j^r(1)\mathbb{I}_{\{v_j^r(1)\le K\}}\big] - \frac{1}{4K\beta_j} \ge \frac{1}{2K\beta_j} - \frac{1}{4K\beta_j} \ge \frac{1}{4K\beta_j}. \tag{64}$$

Define $C_j^r(n) = \sum_{l=1}^n \mathbb{I}_{\{v_j^r(l)\ge \frac{1}{4K\beta_j}\}}$ and $\zeta_j^r(z) = \min\{n \ge 0 : C_j^r(n) = \lceil 4\frac{z}{\tilde a}K\max_i\{C_i\}\beta_j\rceil\}$ and note that $E[\hat\tau_j^{r,S}(z)] \le E[\zeta_j^r(z)]$. However, because the $\{v_j^r(l)\}_{l=1}^\infty$ are i.i.d. it follows that $\zeta_j^r(z)$ is just the sum of $\lceil 4\frac{z}{\tilde a}K\max_i\{C_i\}\beta_j\rceil$ independent geometric distributions with probability of success $p \ge \frac{1}{4K\beta_j}$ for all r sufficiently large, which gives

$$E\big[\hat\tau_j^{r,S}(z)\big] \le E\big[\zeta_j^r(z)\big] \le \bigg(1 + 4\frac{z}{\tilde a}K\max_i\{C_i\}\beta_j\bigg)4K\beta_j \le 4K\beta_j + 16\frac{z}{\tilde a}K^2\max_i\{C_i\}\beta_j^2.$$

Thus, from (63) we have, for all r sufficiently large,

$$\begin{aligned} E\big[e^{cv_j^r(\tau_j^{r,S}(t))}\big] &\le 1 + \int_{(0,\infty)} e^{cz}\big(1 + E\big[\hat\tau_j^{r,S}(z)\big]\big)\gamma_j^r(dz) \\ &\le 1 + \int_{(0,\infty)} e^{cz}\bigg(1 + 4K\beta_j + 16\frac{z}{\tilde a}K^2\max_i\{C_i\}\beta_j^2\bigg)\gamma_j^r(dz) \\ &\le 1 + (1 + 4K\beta_j)\int_{(0,\infty)} e^{cz}\gamma_j^r(dz) + \frac{16K^2\beta_j^2}{\tilde a}\max_i\{C_i\}\int_{(0,\infty)} ze^{cz}\gamma_j^r(dz). \end{aligned}$$

The result follows on using Condition 1.

9.2. *Proof of Proposition* 6.2. In this section, we prove the key large deviation estimates given in Proposition 6.2, which have been used on several occasions.

Fix $j \in \mathbb{A}_J$. Since the proof is identical for A_j^r and S_j^r, we will only present it for A_j^r. Throughout in the proof, we suppress the subscript j. Define $C^r(n) \doteq \sum_{l=1}^n u^r(1)$ and $\Lambda^r(y) = \log(E[e^{y(u^r(1)-\frac{1}{\alpha^r})}])$. Note that $y \mapsto \Lambda^r(y)$ is infinitely differentiable for $|y| < \delta$ and due to Jensen's inequality $\Lambda^r(y) \ge 0$ for all $|y| < \delta$. Due to Condition 1, there exists $K_\Lambda < \infty$ such that

$$\sup_{r,|y|\le\frac{\delta}{2}}\bigg|\frac{d^3\Lambda^r}{dy^3}(y)\bigg| \le K_\Lambda.$$

Since $\Lambda^r(0) = 0$, $\frac{d\Lambda^r}{dy}(0) = 0$ and $\frac{d^2\Lambda^r}{dy^2}(0) = (\sigma^{u,r})^2$, we have for $|y| \le \delta/2$,

$$\bigg|\Lambda^r(y) - \frac{y^2}{2}(\sigma^{u,r})^2\bigg| \le \frac{|y|^3}{6}K_\Lambda. \tag{65}$$

We will first prove that, for any $c \geq 0$ there exist $B_1, B_2, R \in (0, \infty)$ such that for all $r \geq R$ and $T \geq 0$, we have

$$\begin{aligned}P\Big(\sup_{0\le t\le r^{2c}T}\big|A^r(t)-t\alpha^r\big|\ge \epsilon r^c T\Big) &= P\Big(\sup_{0\le t\le T}\big|A^r(r^{2c}t)-r^{2c}t\alpha^r\big|\ge \epsilon r^c T\Big)\\ &\le B_1 e^{-TB_2}.\end{aligned}\tag{66}$$

Due to the fact that A^r is integer-valued and the definition of C^r, for all $t \in [0, T]$ we have

$$\begin{aligned}\{A^r(r^{2c}t)-\alpha^r r^{2c}t \ge r^c T\epsilon\} &= \{A^r(r^{2c}t)\ge \lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil\}\\ &= \{C^r(\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil)\le r^{2c}t\}\end{aligned}$$

and

$$\begin{aligned}\{A^r(r^{2c}t)-\alpha^r r^{2c}t \le -r^c T\epsilon\} &= \{A^r(r^{2c}t)\le \lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor\}\\ &= \{C^r(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) > r^{2c}t\}.\end{aligned}$$

In addition, since for all $t \in [0, T]$ we have

$$\frac{1}{\alpha^r}\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil - \frac{1}{\alpha^r}(r^c T\epsilon) \ge r^{2c}t$$

and

$$\frac{1}{\alpha^r}(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) + \frac{1}{\alpha^r}(r^c T\epsilon - 1) \le r^{2c}t$$

it follows that for all $t \in [0, T]$ we have

$$\begin{aligned}&\{C^r(\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil)\le r^{2c}t\}\\ &\quad\subset \left\{\frac{1}{\alpha^r}\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil - C^r(\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil) \ge \frac{1}{\alpha^r}(r^c T\epsilon)\right\}\\ &\quad\subset \left\{\frac{1}{\alpha^r}\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil - C^r(\lceil \alpha^r r^{2c}t + r^c T\epsilon\rceil) > \frac{1}{\alpha^r}(r^c T\epsilon - 1)\right\}\end{aligned}$$

and

$$\begin{aligned}&\{C^r(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) > r^{2c}t\}\\ &\quad\subset \left\{\frac{1}{\alpha^r}(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) - C^r(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) < -\frac{1}{\alpha^r}(r^c T\epsilon - 1)\right\}\\ &\quad= \left\{C^r(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) - \frac{1}{\alpha^r}(\lfloor \alpha^r r^{2c}t - r^c T\epsilon\rfloor+1) > \frac{1}{\alpha^r}(r^c T\epsilon - 1)\right\}.\end{aligned}$$

Using these observations, we have with $p_c^r \doteq \lceil \alpha^r r^{2c}T + r^c\epsilon T\rceil$,

$$\left\{\sup_{0\le t\le T}(A^r(r^{2c}t)-\alpha^r r^{2c}t) \ge r^c T\epsilon\right\} \subset \left\{\sup_{0\le n\le p_c^r}\left(\frac{1}{\alpha^r}n - C^r(n)\right) > \frac{1}{\alpha^r}(r^c\epsilon T - 1)\right\}\tag{67}$$

and with $q_c^r = \lfloor \alpha^r r^{2c}T - r^c\epsilon T\rfloor$,

$$\begin{aligned}&\left\{\inf_{0\le t\le T}(A^r(r^{2c}t)-\alpha^r r^{2c}t) \le -r^c T\epsilon\right\}\\ &\quad\subset \left\{\sup_{0\le n\le q_c^r+1}\left(C^r(n) - \frac{1}{\alpha^r}n\right) > \frac{1}{\alpha^r}(r^c\epsilon T - 1)\right\}.\end{aligned}\tag{68}$$

For $0 < y < \frac{\delta}{4}$,

$$P\Big(\sup_{0\le n\le p_c^r}\Big(\frac{1}{\alpha^r}n - C^r(n)\Big) > \frac{1}{\alpha^r}\big(r^c\epsilon T - 1\big)\Big)e^{\frac{y}{\alpha^r}(r^c\epsilon T-1)} \le E\Big[\sup_{0\le n\le p_c^r} e^{y(\frac{1}{\alpha^r}n - C^r(n))}\Big]. \tag{69}$$

Since $\Lambda^r(-\frac{y}{2}) \ge 0$, we have

$$\begin{aligned} E\Big[\sup_{0\le n\le p_c^r}\big(e^{y(\frac{1}{\alpha^r}n - C^r(n))}\big)\Big] &= E\Big[\sup_{0\le n\le p_c^r}\big(e^{y(\frac{1}{\alpha^r}n-C^r(n))-n2\Lambda^r(-\frac{y}{2})}e^{n2\Lambda^r(-\frac{y}{2})}\big)\Big] \\ &\le e^{p_c^r 2\Lambda^r(-\frac{y}{2})}E\Big[\sup_{0\le n\le p_c^r}\big(e^{y(\frac{1}{\alpha^r}n-C^r(n))-n2\Lambda^r(-\frac{y}{2})}\big)\Big]. \end{aligned} \tag{70}$$

Since $e^{\frac{y}{2}(\frac{1}{\alpha^r}n-C^r(n))-n\Lambda^r(-\frac{y}{2})}$ is a nonnegative martingale and $Ee^{\frac{y}{2}(\frac{1}{\alpha^r}n-C^r(n))-n\Lambda^r(-\frac{y}{2})} < \infty$ for all $n \in \mathbb{N}$, the Doob–Kolmogorov inequality gives

$$\begin{aligned} E\Big[\sup_{0\le n\le p_c^r}\big(e^{y(\frac{1}{\alpha^r}n-C^r(n))-2n\Lambda^r(-\frac{y}{2})}\big)\Big] &\le 4E\big[e^{y(\frac{1}{\alpha^r}p_c^r - C^r(p_c^r))-2p_c^r\Lambda^r(-\frac{y}{2})}\big] \\ &\le 4e^{-2p_c^r\Lambda^r(-\frac{y}{2})}e^{p_c^r\Lambda^r(-y)}. \end{aligned}$$

Consequently, from (67), (69) and (70),

$$\begin{aligned} &P\Big(\sup_{0\le t\le T}\big(A^r(r^{2c}t) - \alpha^r r^{2c}t\big) \ge r^c T\epsilon\Big) \\ &\quad\le 4e^{-\frac{y}{\alpha^r}(r^c\epsilon T-1)+p_c^r\Lambda^r(-y)} \le 4e^{-\frac{y}{\alpha^r}(r^c\epsilon T-1)+p_c^r(\frac{y^2}{2}(\sigma^{u,r})^2+\frac{|y|^3}{6}K_\Lambda)} \\ &\quad\le 4e^{-\frac{y}{\alpha^r}(r^c\epsilon T-1)+(\alpha^r r^{2c}T+r^c\epsilon T+1)(\frac{y^2}{2}(\sigma^{u,r})^2+\frac{|y|^3}{6}K_\Lambda)}, \end{aligned} \tag{71}$$

where the second inequality is from (65).

Choose $R < \infty$ sufficiently large that for all $r \ge R$, we have $\alpha/2 \le \alpha^r \le 2\alpha$, $(\sigma^{u,r})^2 \le 2(\sigma^u)^2$.

Consider first the case where $c > 0$. Define $k = \epsilon/(16(\alpha\sigma^u)^2)$, and assume without loss that R is large enough that for all $r \ge R$,

$$\begin{gathered} r^{-c}\frac{1}{6}\alpha^r k^3 K_\Lambda + r^{-c}\frac{1}{2}\epsilon k^2(\sigma^{u,r})^2 + r^{-2c}\frac{1}{6}\epsilon k^3 K_\Lambda \le \frac{k\epsilon}{8\alpha}, \\ r^{-c}\frac{k}{\alpha^r} + r^{-2c}\frac{1}{2}k^2(\sigma^{u,r})^2 + r^{-3c}\frac{1}{6}k^3K_\Lambda \le 1 \quad \text{and} \quad kr^{-c} < \frac{\delta}{4}. \end{gathered}$$

Then for all $r \ge R$, the exponent in (71) with $y = kr^{-c}$ satisfies

$$\begin{aligned} &-\frac{kr^{-c}}{\alpha^r}(r^c\epsilon T - 1) + (\alpha^r r^{2c}T + r^c\epsilon T)\Big(\frac{(kr^{-c})^2}{2}(\sigma^{u,r})^2 + \frac{(kr^{-c})^3}{6}K_\Lambda\Big) \\ &\quad + \Big(\frac{(kr^{-c})^2}{2}(\sigma^{u,r})^2 + \frac{(kr^{-c})^3}{6}K_\Lambda\Big) \\ &= -\frac{k\epsilon T}{\alpha^r} + \frac{k}{r^c\alpha^r} + \frac{1}{2}\alpha^r Tk^2(\sigma^{u,r})^2 + \frac{1}{6r^c}\alpha^r Tk^3K_\Lambda + \frac{1}{2r^c}\epsilon Tk^2(\sigma^{u,r})^2 \\ &\quad + \frac{1}{6r^{2c}}\epsilon Tk^3K_\Lambda + \frac{1}{2r^{2c}}k^2(\sigma^{u,r})^2 + \frac{1}{6r^{3c}}k^3K_\Lambda \\ &= T\Big(\frac{1}{2}\alpha^r k^2(\sigma^{u,r})^2 - \frac{k\epsilon}{\alpha^r}\Big) + T\Big(r^{-c}\frac{1}{6}\alpha^r k^3K_\Lambda + r^{-c}\frac{1}{2}\epsilon k^2(\sigma^{u,r})^2 + r^{-2c}\frac{1}{6}\epsilon k^3K_\Lambda\Big) \end{aligned}$$

$$+r^{-c}\frac{k}{\alpha^r}+r^{-2c}\frac{1}{2}k^2(\sigma^{u,r})^2+r^{-3c}\frac{1}{6}k^3K_\Lambda$$
$$\le T\left(2\alpha k^2(\sigma^u)^2-\frac{k\epsilon}{2\alpha}\right)+T\left(\frac{k\epsilon}{8\alpha}\right)+1=T\left(\frac{k\epsilon}{8\alpha}-\frac{k\epsilon}{2\alpha}\right)+T\left(\frac{k\epsilon}{8\alpha}\right)+1=-T\frac{k\epsilon}{4\alpha}+1.$$

Consequently, for $r\ge R$, substituting $y=kr^{-c}$ in (71) gives

$$P\Big(\sup_{0\le t\le T}\big(A^r(r^{2c}t)-\alpha^r r^{2c}t\big)\ge r^cT\epsilon\Big)\le 4e^{-T\frac{k\epsilon}{4\alpha}+1}\le 4ee^{-T\frac{k\epsilon}{4\alpha}}.$$

Similarly, using the inequality (68), an almost identical argument (which we omit for the sake of brevity) provides a similar bound on $P(\inf_{0\le t\le T}(A^r(r^{2c}t)-\alpha^r r^{2c}t)\le -r^cT\epsilon)$ and combining the two bounds we have the estimate in (66) for $c>0$.

Now consider the case $c=0$ and define

$$q=\min\left\{1,\frac{\delta}{4},\frac{\epsilon}{8\alpha(\alpha(\sigma^u)^2+\frac{\alpha}{6}K_\Lambda+\epsilon(\sigma^u)^2+\frac{\epsilon}{6}K_\Lambda)}\right\}.$$

Then for all $r\ge R$ we have, for the exponent in (71) with $y=q$,

$$-\frac{q}{\alpha^r}(\epsilon T-1)+(\alpha^rT+\epsilon T)\left(\frac{q^2}{2}(\sigma^{u,r})^2+\frac{q^3}{6}K_\Lambda\right)+\left(\frac{q^2}{2}(\sigma^{u,r})^2+\frac{q^3}{6}K_\Lambda\right)$$
$$\le -\frac{\epsilon Tq}{2\alpha}+\frac{2q}{\alpha}+\alpha^rTq^2(\sigma^u)^2+\alpha^rT\frac{q^3}{6}K_\Lambda+\epsilon Tq^2(\sigma^u)^2+\epsilon T\frac{q^3}{6}K_\Lambda$$
$$+q^2(\sigma^u)^2+\frac{q^3}{6}K_\Lambda$$
$$\le -\frac{\epsilon Tq}{2\alpha}+2q^2T\left(\alpha(\sigma^u)^2+\frac{\alpha}{6}K_\Lambda+\epsilon(\sigma^u)^2+\frac{\epsilon}{6}K_\Lambda\right)+q^2(\sigma^u)^2+\frac{2q}{\alpha}+\frac{q^3}{6}K_\Lambda$$
$$\le -\frac{\epsilon Tq}{2\alpha}+\frac{\epsilon Tq}{4\alpha}+q^2(\sigma^u)^2+\frac{2q}{\alpha}+\frac{q^3}{6}K_\Lambda=-T\frac{\epsilon q}{4\alpha}+q^2(\sigma^u)^2+\frac{2q}{\alpha}+\frac{q^3}{6}K_\Lambda.$$

Consequently, for $r\ge R$, substituting $y=q$ in (71) with $c=0$, we have

$$P\Big(\sup_{0\le t\le T}\big(A^r(t)-\alpha^rt\big)\ge T\epsilon\Big)\le\big(4e^{q^2(\sigma^u)^2+\frac{2q}{\alpha}+\frac{q^3}{6}K_\Lambda}\big)e^{-T\frac{\epsilon q}{4\alpha}}.$$

Finally, using the inclusion in (68) with $c=0$, we have by an almost identical argument (which is omitted), a similar upper bound on $P(\inf_{0\le t\le T}(A^r(t)-\alpha^rt)\le -T\epsilon)$. Combining the two bounds, we now have the estimate in (66) for $c=0$. We have thus proved (14) with $c_2=0$. Now (14) for general $c_1,c_2\ge 0$ (and T replaced by $S\ge 0$) follows on taking $c=c_1$ and $T=Sr^{c_2}$ in (66). Finally, the inequality in (16) follows on taking $c_1=c_2=\frac{\kappa}{2}$ in (14).

9.3. *Proof of Proposition* 8.1. The proof for (57) is the same as that for (56) and so we will only show the latter. Let $c>0$, $\epsilon>0$ and $j\in\mathbb{A}_J$ be arbitrary. Once again, we suppress j from the notation. Due to Condition 1, we have $\sup_{r>0}\{Ee^{\frac{3\delta}{4}v^r(1)}\}=\tilde{K}<\infty$ where δ is as in Condition 1. Choose $R<\infty$ such that for all $r\ge R$, we have $(r^2+1)\tilde{K}e^{-r\frac{\delta}{4}c}\le\frac{\epsilon\delta}{4}$. Consequently,

$$E\big[e^{\frac{\delta}{2}\mathcal{I}_{\{v^r(1)\ge rc\}}v^r(1)}\big]=1+E\big[\mathcal{I}_{\{v^r(1)\ge rc\}}e^{\frac{\delta}{2}v^r(1)}\big]\le 1+E\big[\mathcal{I}_{\{v^r(1)\ge rc\}}e^{-r\frac{\delta}{4}c}e^{\frac{3\delta}{4}v^r(1)}\big]$$
$$\le 1+e^{-r\frac{\delta}{4}c}E\big[e^{\frac{3\delta}{4}v^r(1)}\big]\le 1+\tilde{K}e^{-r\frac{\delta}{4}c}\le e^{\tilde{K}e^{-r\frac{\delta}{4}c}}.$$

So, for all $T \geq 1$ and $r \geq R$, we have

$$E\big[e^{\frac{\delta}{2}\sum_{l=1}^{\lceil r^2T\rceil}\mathcal{I}_{\{v^r(l)\geq rc\}}v^r(l)}\big] \leq e^{T(r^2+1)\tilde{K}e^{-r\frac{\delta}{4}c}} \leq e^{T\frac{\epsilon\delta}{4}},$$

which implies

$$P\left(\sum_{l=1}^{\lceil r^2T\rceil}\mathcal{I}_{\{v^r(l)\geq rc\}}v^r(l) \geq \epsilon T\right) \leq e^{-\frac{\epsilon\delta}{2}T}e^{T\frac{\epsilon\delta}{4}} \leq e^{-\frac{\epsilon\delta}{4}T}.$$

10. General network and cost properties.

10.1. *Proof of Proposition* 2.1. From the discussion above, Proposition 2.1, the result is clearly true when $\mathcal{C}_K^h = \ker(K)$. Consider now the complementary case, namely $\dim(\mathcal{C}_K^h) = \dim(\ker(K)) - 1 = J - I - 1$. Let $q \in \mathbb{R}_+^J$ be arbitrary. For $\tilde{q} \in \Lambda(KMq)$, define $\tilde{v} = (\tilde{q} - q)/\beta$. Note that $0 \leq \tilde{q}_j = q_j + (\tilde{q}_j - q_j) = q_j + \tilde{v}_j\beta_j$ for all $j \in \mathbb{A}_J$ and $K\tilde{v} = KM(\tilde{q} - q) = KM\tilde{q} - KMq = 0$ and so $\tilde{v} \in \Xi(q)$. In addition, for any $\hat{v} \in \Xi(q)$ define $\hat{q} = q + \beta\hat{v}$ and note that by the definition of $\Xi(q)$, we have $\hat{q} \in \mathbb{R}_+^J$ and $KM\hat{q} = KM(q + \beta\hat{v}) = KMq + K\hat{v} = KMq$ so $\hat{q} \in \Lambda(KMq)$. Consequently, $\inf_{\tilde{q}\in\Lambda(KMq)}(h\cdot\tilde{q}) = \inf_{\tilde{v}\in\Xi(q)}(h\cdot(q+\beta\tilde{v}))$. Therefore,

$$\begin{aligned}
h\cdot q - \hat{h}(KMq) &= h\cdot q - \inf_{\tilde{q}\in\Lambda(KMq)}(h\cdot\tilde{q}) = h\cdot q - \inf_{\tilde{v}\in\Xi(q)}\big(h\cdot(q+\beta\tilde{v})\big)\\
&= \sup_{\tilde{v}\in\Xi(q)}\big(-h\cdot(\beta\tilde{v})\big) = \sup_{\tilde{v}\in\Xi(q)}\left(-\beta h\cdot\sum_{j=1}^{J-I}u_j(u_j\cdot\tilde{v})\right)\\
&= \sup_{\tilde{v}\in\Xi(q)}\big(-\beta h\cdot u_{J-I}(u_{J-I}\cdot\tilde{v})\big)\\
&= |\lambda|\sup_{\tilde{v}\in\Xi(q)}(u_{J-I}\cdot\tilde{v}) = |\lambda|\tilde{d}(q),
\end{aligned}$$

where the last equality on the second line uses $\beta h\cdot u_j = 0$ for $j = 1,\dots J - I - 1$, and the last line follows on recalling the definition of λ and $\tilde{d}$. The result follows.

10.2. *Proof of Proposition* 2.3. Fix $z \in \chi_J$. Due to the local traffic assumption (Condition 3) for each $l \in \mathcal{A}_z$, we can choose $s_l \in \mathbb{A}_J$ such that $K_{l,s_l} = 1$ and $\sum_{i\in\mathbb{A}_I}K_{i,s_l} = 1$. Define $S = \{s_l : l \in \mathcal{A}_z\}$, $P = \{j \in \mathbb{A}_J : z_j = 1\} \setminus S$ and $N = \{j \in \mathbb{A}_J : z_j = 0\} \setminus S$. For $j \in N$ define $v_j = -J$, for $j \in P$ define $v_j = 1$, and for all $l \in \mathcal{A}_z$ define $v_{s_l} = -\sum_{j\neq s_l}K_{l,j}v_j$. For any $l \in \mathcal{A}_z$, we have

$$\sum_{j\in\mathbb{A}_J}K_{l,j}v_j = \sum_{j\neq s_l}K_{l,j}v_j + v_{s_l} = \sum_{j\neq s_l}K_{l,j}v_j - \sum_{j\neq s_l}K_{l,j}v_j = 0.$$

This verifies the first statement in the proposition with the above choice of v. Next, consider $j \in \mathbb{N}_J$ such that $z_j = 1$. If $j \in P$, then by definition $v_j > 0$. To complete the proof, consider now $l \in \mathcal{A}_z$ with $z_{s_l} = 1$. Due to the definition of $\mathcal{A}_z$, we have $\sum_{j:z_j=0}K_{l,j} \geq 1$ and we also know that $K_{l,s_i} = 0$ for all $i \in \mathcal{A}_z \setminus \{l\}$. This implies that there exists $j^* \in N$ such that $K_{l,j^*} = 1$ and for all $j \in \mathbb{A}_{\mathbb{J}} \setminus \{j^*, s_l\}$ such that $K_{l,j} = 1$, either $v_j = 1$ or $v_j = -J$. Since $v_{j^*} = -J$, we have

$$v_{s_l} = -\sum_{j\neq s_l}K_{l,j}v_j = J - \sum_{j\in\mathbb{A}_J\setminus\{j^*,s_l\}}K_{l,j}v_j \geq J - (J-2) \geq 2.$$

Thus, we have shown that for all $j \in \mathbb{A}_J$ such that $z_j = 1$ we have $v_j > 0$. This proves the second statement in the proposition.

10.3. *Proofs of Propositions 7.6 and 7.7*. We begin with two auxiliary results.

LEMMA 10.1. *There exists a $\hat{B}^-_{\hat{h}} \in (0, \infty)$ such that for all $w^1, w^2 \in \mathbb{R}^I_+$ satisfying $w^1 \geq w^2$, we have*

$$\hat{h}(w^2) \leq \hat{h}(w^1) + \hat{B}^-_{\hat{h}} |w^1 - w^2|_2.$$

PROOF. For $i \in \mathbb{A}_I$, let $\mathcal{C}_i = \{j \in \mathbb{A}_J : K_{i,j} = 1\}$. For $j \in \mathcal{C}_i$, define $\mathcal{O}^j_i = \{l \in \mathbb{A}_I : K_{l,j} = 1 \text{ and } l \neq i\}$. Thus, the set $\mathcal{O}^j_i$ consists of all the resources that job type j impacts aside from resource i. For each $i \in \mathbb{A}_I$, let $s_i \in \mathbb{A}_J$ be such that $\sum_{l=1}^I K_{l,s_i} = K_{i,s_i} = 1$. For any $j \in \mathbb{A}_J$, let $e^j \in \mathbb{R}^J_+$ be the unit vector with one on the jth coordinate, so $e^j_j = 1$ and $e^j_l = 0$ for $l \neq j$. Note that for any $c > 0$ and $j \in \mathbb{A}_J$ if we define $\tilde{w}^1 = KMc\beta_j e^j = cK_{\cdot,j}$ (here $K_{\cdot,j}$ is the jth column vector of the matrix K) and $\tilde{w}^2 = KM(\sum_{l\in\mathcal{O}^j_i} c\beta_{s_l} e^{s_l})$ then $\tilde{w}^2_l = \tilde{w}^1_l$ for $l \neq i$ but $\tilde{w}^2_i = 0$ and $\tilde{w}^1_i = c$. In other words, by replacing $c\beta_j e^j$ with $\sum_{l\in\mathcal{O}^i_j} c\beta_{s_l} e^{s_l}$ in the queue length vector we have reduced the workload for server i by c and we have changed the cost by $c(\sum_{l\in\mathcal{O}^j_i} h_{s_l}\beta_{s_l} - h_j\beta_j)$. This is the key idea in the proof.

Define

$$\tilde{R}_i = \max_{j\in\mathcal{C}_i}\biggl(\sum_{l\in\mathcal{O}^j_i} h_{s_l}\beta_{s_l} - h_j\beta_j\biggr)^+ \quad \text{and} \quad \tilde{R} = \max_{i\in\mathbb{A}_I}\{\tilde{R}_i\}.$$

Now let $w^1, w^2 \in \mathbb{R}^I_+$ be such that $w^1 \geq w^2$ and let $\tilde{q}^0 \in \mathbb{R}^J_+$ satisfy $w^1 = KM\tilde{q}^0$ and $\hat{h}(w^1) = h \cdot \tilde{q}^0$. Since $\sum_{j\in\mathcal{C}_1} \frac{1}{\beta_j}\tilde{q}^0_j = \sum_{j\in\mathbb{A}_J} K_{1,j}\frac{1}{\beta_j}\tilde{q}^0_j = w^1_1 \geq w^1_1 - w^2_1$, we can choose $\tilde{c}^1_j \in [0, \tilde{q}^0_j/\beta_j]$ for all $j \in \mathcal{C}_1$ such that $\sum_{j\in\mathcal{C}_1} \tilde{c}^1_j = w^1_1 - w^2_1$. Define

$$\tilde{q}^1 = \tilde{q}^0 + \sum_{j\in\mathcal{C}_1}\biggl(\sum_{l\in\mathcal{O}^j_1} \tilde{c}^1_j\beta_{s_l}e^{s_l} - \tilde{c}^1_j\beta_j e^j\biggr)$$

and note that $\tilde{q}^1 \in \mathbb{R}^J_+$,

$$KM\tilde{q}^1 = \biggl(w^1_1 - \sum_{j\in\mathcal{C}_1}\tilde{c}^1_j, w^1_2, w^1_3, \ldots, w^1_I\biggr) = (w^2_1, w^1_2, w^1_3, \ldots, w^1_I),$$

and

$$\begin{aligned} h \cdot \tilde{q}^1 &= h \cdot \tilde{q}^0 + \sum_{j\in\mathcal{C}_1}\biggl(\sum_{l\in\mathcal{O}^j_1} \tilde{c}^1_j h_{s_l}\beta_{s_l} - \tilde{c}^1_j h_j\beta_j\biggr) \\ &\leq \hat{h}(w^1) + \sum_{j\in\mathcal{C}_1}\tilde{c}^1_j\biggl(\sum_{l\in\mathcal{O}^j_1} h_{s_l}\beta_{s_l} - h_j\beta_j\biggr)^+ \leq \hat{h}(w^1) + \tilde{R}|w^1_1 - w^2_1|. \end{aligned}$$

Now assume that for $k \in \{1, \ldots, I-1\}$ there exists $\tilde{q}^k \in \mathbb{R}^J_+$ such that

$$\begin{aligned} &KM\tilde{q}^k = (w^2_1, w^2_2, \ldots, w^2_k, w^1_{k+1}, w^1_{k+2}, \ldots, w^1_I), \quad \text{and} \\ &h \cdot \tilde{q}^k \leq \hat{h}(w^1) + \tilde{R}\sum_{i=1}^k |w^1_i - w^2_i|. \end{aligned} \tag{72}$$

Since $\sum_{j\in\mathcal{C}_{k+1}} \tilde{q}^k_j/\beta_j = \sum_{j\in\mathbb{A}_J} K_{k+1,j}\tilde{q}^k_j/\beta_j = w^1_{k+1} \geq w^2_{k+1}$, we can choose $\tilde{c}^{k+1}_j \in [0, \tilde{q}^k_j/\beta_j]$ for all $j \in \mathcal{C}_{k+1}$ such that $\sum_{j\in\mathcal{C}_{k+1}} \tilde{c}^{k+1}_j = w^1_{k+1} - w^2_{k+1}$. Define

$$\tilde{q}^{k+1} = \tilde{q}^k + \sum_{j\in\mathcal{C}_{k+1}} \biggl(\sum_{l\in\mathcal{O}^j_{k+1}} \tilde{c}^{k+1}_j \beta_{s_l} e^{s_l} - \tilde{c}^{k+1}_j \beta_j e^j \biggr).$$

Then, as before $\tilde{q}^{k+1} \in \mathbb{R}^J_+$, and from (72),

$$\begin{aligned} KM\tilde{q}^{k+1} &= \biggl(w^2_1, w^2_2, \dots, w^2_k, w^1_{k+1} - \sum_{j\in\mathcal{C}_{k+1}} \tilde{c}^{k+1}_j, w^1_{k+2}, \dots, w^1_I\biggr) \\ &= (w^2_1, w^2_2, \dots, w^2_k, w^2_{k+1}, w^1_{k+2}, \dots, w^1_I) \end{aligned}$$

and

$$\begin{aligned} h\cdot\tilde{q}^{k+1} &= h\cdot\tilde{q}^k + \sum_{j\in\mathcal{C}_{k+1}} \biggl(\sum_{l\in\mathcal{O}^j_{k+1}} \tilde{c}^{k+1}_j h_{s_l}\beta_{s_l} - \tilde{c}^{k+1}_j h_j\beta_j \biggr) \\ &\leq \hat{h}(w^1) + \tilde{R}\sum_{i=1}^k |w^1_i - w^2_i| + \sum_{j\in\mathcal{C}_{k+1}} \tilde{c}^{k+1}_j \biggl(\sum_{l\in\mathcal{O}^j_{k+1}} h_{s_l}\beta_{s_l} - h_j\beta_j \biggr)^+ \\ &\leq \hat{h}(w^1) + \tilde{R}\sum_{i=1}^k |w^1_i - w^2_i| + \tilde{R}|w^1_{k+1} - w^2_{k+1}| = \hat{h}(w^1) + \tilde{R}\sum_{i=1}^{k+1} |w^1_i - w^2_i|. \end{aligned}$$

By induction, this implies that there exists $\tilde{q}^I \in \mathbb{R}^J_+$ such that

$$KM\tilde{q}^I = w^2, \quad \text{and,} \quad h\cdot\tilde{q}^I \leq \hat{h}(w^1) + \tilde{R}\sum_{i=1}^I |w^1_i - w^2_i|.$$

Since $\sum_{i=1}^I |w^1_i - w^2_i| \leq \sqrt{J}|w^1 - w^2|_2$ and $\hat{h}(w^2) \leq h\cdot\tilde{q}^I$ due to the fact that $KM\tilde{q}^I = w^2$, we have

$$\hat{h}(w^2) \leq \hat{h}(w^1) + \tilde{R}\sqrt{J}|w^1 - w^2|_2.$$

This completes the proof. □

LEMMA 10.2. *There exists* $\hat{B}_{\hat{h}} \in (0,\infty)$ *such that for all* $w^1, w^2 \in \mathbb{R}^I_+$ *we have*

$$|\hat{h}(w^1) - \hat{h}(w^2)| \leq \hat{B}_{\hat{h}}|w^1 - w^2|_2.$$

PROOF. Let $w^1, w^2 \in \mathbb{R}^I_+$ be arbitrary. Let $q^* \in \mathbb{R}^J_+$ satisfy $KMq^* = w^1 \wedge w^2$ and $h\cdot q^* = \hat{h}(w^1 \wedge w^2)$. Note that for every $i \in \mathbb{A}_I$ we have $|w^1_i - w^2_i| = |w^1_i - w^1_i \wedge w^2_i| + |w^2_i - w^1_i \wedge w^2_i|$ so

$$|w^1 - w^1\wedge w^2|_2 \leq |w^1 - w^2|_2, \quad \text{and,} \quad |w^2 - w^1\wedge w^2|_2 \leq |w^1 - w^2|_2,$$

and

$$|w^1 - w^2|_2 \leq |w^1 - w^1\wedge w^2|_2 + |w^2 - w^1\wedge w^2|_2.$$

Due to Lemma 10.1, we have

$$\hat{h}(w^1\wedge w^2) \leq \hat{h}(w^1) + \hat{B}^-_{\hat{h}}|w^1 - w^1\wedge w^2|_2 \leq \hat{h}(w^1) + \hat{B}^-_{\hat{h}}|w^1 - w^2|_2$$

and

$$\hat{h}(w^1 \wedge w^2) \leq \hat{h}(w^2) + \hat{B}_{\hat{h}}^- |w^2 - w^1 \wedge w^2|_2 \leq \hat{h}(w^2) + \hat{B}_{\hat{h}}^- |w^1 - w^2|_2.$$

Let $\hat{B}_{\hat{h}}^+ \in (0, \infty)$ be such that for all $w \in \mathbb{R}_+^I$ we have $\hat{h}(w) \leq \hat{B}_{\hat{h}}^+ |w|_2$. Then, using the definition of $\hat{h}$,

$$\begin{aligned}\hat{h}(w^1) &\leq \hat{h}(w^1 \wedge w^2) + \hat{h}(w^1 - w^1 \wedge w^2) \leq \hat{h}(w^1 \wedge w^2) + \hat{B}_{\hat{h}}^+ |w^1 - w^1 \wedge w^2|_2 \\ &\leq \hat{h}(w^1 \wedge w^2) + \hat{B}_{\hat{h}}^+ |w^1 - w^2|_2\end{aligned}$$

and similarly,

$$\hat{h}(w^2) \leq \hat{h}(w^1 \wedge w^2) + \hat{B}_{\hat{h}}^+ |w^1 - w^2|_2.$$

Consequently,

$$|\hat{h}(w^1) - \hat{h}(w^1 \wedge w^2)| \leq \max\{\hat{B}_{\hat{h}}^+, \hat{B}_{\hat{h}}^-\} |w^1 - w^2|_2$$

and

$$|\hat{h}(w^2) - \hat{h}(w^1 \wedge w^2)| \leq \max\{\hat{B}_{\hat{h}}^+, \hat{B}_{\hat{h}}^-\} |w^1 - w^2|_2$$

and so

$$\begin{aligned}|\hat{h}(w^1) - \hat{h}(w^2)| &\leq |\hat{h}(w^1) - \hat{h}(w^1 \wedge w^2)| + |\hat{h}(w^2) - \hat{h}(w^1 \wedge w^2)| \\ &\leq 2 \max\{\hat{B}_{\hat{h}}^+, \hat{B}_{\hat{h}}^-\} |w^1 - w^2|_2.\end{aligned}$$

This completes the proof. □

We can now complete the proof of Proposition 7.6.

PROOF OF PROPOSITION 7.6. For any $q^1, q^2 \in \mathbb{R}_+^J$, from Proposition 2.1, we have

$$\begin{aligned}|\lambda|(\tilde{d}(q^2) - \tilde{d}(q^1)) &= h \cdot q^2 - \hat{h}(KMq^2) - (h \cdot q^1 - \hat{h}(KMq^1)) \\ &= h \cdot q^2 - h \cdot q^1 - (\hat{h}(KMq^2) - \hat{h}(KMq^1)) \\ &\leq h \cdot (q^2 - q^1) + |\hat{h}(KMq^2) - \hat{h}(KMq^1)|.\end{aligned}$$

The result now follows from Lemma 10.2 on observing that we can find $R \in (0, \infty)$ such that for all $x \in \mathbb{R}^J$ and $r \geq R$, $|KMx|_2 \leq 2|KM^r x|_2$. □

Finally, we complete the proof of Proposition 7.7.

PROOF OF PROPOSITION 7.7. Let $q^1, q^2 \in \mathbb{R}_+^J$ be arbitrary. Due to Proposition 7.6, we have

$$\begin{aligned}|\lambda||\tilde{d}(q^2) - \tilde{d}(q^1)| &\leq |h \cdot (q^2 - q^1)| + B_{\hat{h}} |KMq^2 - KMq^1|_2 \\ &\leq |h|_2 |q^2 - q^1|_2 + B_{\hat{h}} |KM| |q^2 - q^1|_2 \\ &\leq (|h|_2 + B_{\hat{h}} |KM|) |q^2 - q^1|_2,\end{aligned}$$

which completes the proof. □

11. Proofs of Propositions 3.6 and 3.7. Results analogous to Propositions 3.6 and 3.7 for exponential primitives were studied in [6] and, therefore, we only give proof sketches here.

11.1. *Proof of Proposition* 3.6. The fact that $J_E^r(B^r, y^r) < \infty$ for r sufficiently large follows from Proposition 3.4 and the fact that there exists a constant $B < \infty$ such that $h \cdot q \leq B|w|$ for all $q \in \Lambda(w)$ and $w \in \mathbb{R}_+^I$. To prove tightness of $\{\nu^r\}$, it is sufficient to show that the two marginals are tight. The tightness of $\nu_{(1)}^r$ follows immediately from Proposition 3.4. To show the tightness of the second marginal, $\nu_{(2)}^r(dx)$, since $x(0) = 0$ $\nu_{(2)}^r$-a.s. it is sufficient to show that for any $\epsilon_1, \epsilon_2 > 0$ there exists $\delta > 0$ and $R < \infty$ such that for all $r \geq R$ we have

$$E\Big[\nu_{(2)}^r\Big(\Big\{\sup_{s,t\in[0,1],|s-t|<\delta}\|x(s)-x(t)\| > \epsilon_1\Big\}\Big)\Big] < \epsilon_2. \tag{73}$$

Note that the left-hand side above equals

$$\frac{1}{T_r}\int_0^{T_r} P\Big(\sup_{s,t\in[u,u+1],|s-t|<\delta}\|\hat{X}^r(s)-\hat{X}^r(t)\| > \epsilon_1\Big)\,du$$

and for any $t, u \geq 0$ we have

$$\begin{aligned}\hat{X}^r(u+t) = {}& \hat{X}^r(u) + KM^r\hat{A}^{r,u}\big((t-\bar{\Upsilon}^{A,r}(u))^+\big) + \frac{1}{r}KM^r\mathcal{I}_{\{t\geq\bar{\Upsilon}^{A,r}(u)>0\}}\\ &- KM^r\hat{S}^{r,u}\big((\bar{B}^r(t+u)-\bar{B}^r(u)-\bar{\Upsilon}^{S,r}(u))^+\big)\\ &- \frac{1}{r}KM^r\mathcal{I}_{\{\bar{B}^r(t+u)-\bar{B}^r(u)\geq\bar{\Upsilon}^{S,r}(u)>0\}} + rtK(\rho^r-\rho)\\ &- rK\rho^r\big(t\wedge\bar{\Upsilon}^{A,r}(u)\big) + rK\big((\bar{B}^r(t+u)-\bar{B}^r(u))\wedge\bar{\Upsilon}^{S,r}(u)\big).\end{aligned} \tag{74}$$

From Proposition 6.2 and Lemma 9.1, it follows that for any $\epsilon_1, \epsilon_2 > 0$ there exists $\delta > 0$ and $R < \infty$ such that for all $r \geq R$ and $j \in \mathbb{A}_J$ we have

$$\frac{1}{T_r}\int_0^{T_r} P\Big(\sup_{s,t\in[u,u+1],|s-t|<\delta}\|\hat{A}_j^{r,u}(s)-\hat{A}_j^{r,u}(t)\| > \epsilon_1\Big)\,du < \epsilon_2$$

and

$$\frac{1}{T_r}\int_0^{T_r} P\Big(\sup_{s,t\in[u,u+\max_i\{C_i\}],|s-t|<\max_i\{C_i\}\delta}\|\hat{S}_j^{r,u}(s)-\hat{S}_j^{r,u}(t)\| > \epsilon_1\Big)\,du < \epsilon_2.$$

In addition, due to Proposition 6.1, Proposition 7.5 and the assumption that $\sup_r r\hat{\Upsilon}^r < \infty$ it follows that for any $\epsilon_1, \epsilon_2 > 0$, there exists $\delta > 0$ and $R < \infty$ such that for all $r \geq R$ we have

$$\frac{1}{T_r}\int_0^{T_r} P\big(\|rK\rho^r\bar{\Upsilon}^{A,r}(u)\| > \epsilon_1\big)\,du < \epsilon_2$$

and

$$\frac{1}{T_r}\int_0^{T_r} P\big(\|rK\bar{\Upsilon}^{S,r}(u)\| > \epsilon_1\big)\,du < \epsilon_2.$$

Also, note that $rK(\rho^r - \rho) \to \theta$ (due to Condition 2 and the paragraph that follows) and for all $s, t \geq 0$ and $j \in \mathbb{A}_J$ we have $|\bar{B}_j^r(s) - \bar{B}_j^r(t)| \leq \max_i\{C_i\}(t-s)$. These observations, together with the form of $\|\hat{X}^r(s) - \hat{X}^r(t)\|$ for $s, t \geq u$ given by (74), give (73) and complete the proof.

11.2. *Proof of Proposition* 3.7. Let ν^* and (w, x) be as in the statement of the proposition. In what follows, we let $P_{\nu^*(\omega)}$ and $E_{\nu^*(\omega)}$ denote probability and expectation under $\nu^*(\omega)$. The proof that for a.e. ω, $P_{\nu^*(\omega)}(x \in \mathcal{C}([0,1] : \mathbb{R}^I) = 1$ is the same as the proof of [6], Theorem 15, part 1. To complete the proof of part (a), it suffices to show that for any $f \in \mathcal{C}_c^2(\mathbb{R}^I)$ (which is the space of continuous, real-valued functions on $\mathbb{R}^I$ with compact support and continuous first and second derivatives) $f(x(t)) - \int_0^t \mathcal{L}f(x(s))\,ds$ is a $\sigma(w, x(s) : s \le t)$-martingale under $\nu^*(\omega)$ for a.e ω where

$$\mathcal{L}f(y) = \sum_{i=1}^{I} \theta_i \frac{\partial f}{\partial y_i}(y) + \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I} \Sigma_{i,j} \frac{\partial^2 f}{\partial y_i \partial y_j}(y), \quad y \in \mathbb{R}^I.$$

Let $0 \le s < t \le 1$ be fixed and let $g_s \in \mathcal{C}_b(\mathbb{R}_+^I \times \mathcal{D}([0,s]))$ and $f \in \mathcal{C}_c^2(\mathbb{R}^I)$ be arbitrary (here $\mathcal{C}_b(\mathbb{R}_+^I \times \mathcal{D}([0,s])$ denotes the space of continuous, bounded real-valued functions on $\mathbb{R}_+^I \times \mathcal{D}([0,s]))$. Denoting, for $s \in [0,1]$, by x_s the restriction of x on $[0,s]$, we see that

$$g_s(w, x_s)\Bigl(f\bigl(x(t)\bigr) - f\bigl(x(s)\bigr) - \int_s^t \mathcal{L}f\bigl(x(z)\bigr)\,dz\Bigr)$$

is a bounded, continuous function on $\mathbb{R}_+^I \times \mathcal{D}([0,1]:\mathbb{R}^I)$ so

$$\begin{aligned} &E\Bigl[E_{\nu^*(\omega)}\Bigl[g_s(w,x_s)\Bigl(f\bigl(x(t)\bigr) - f\bigl(x(s)\bigr) - \int_s^t \mathcal{L}f\bigl(x(u)\bigr)\,du\Bigr)\Bigr]^2\Bigr] \\ &\quad = \lim_{m\to\infty} E\Bigl[E_{\nu^{r_m}(\omega)}\Bigl[g_s(w,x_s)\Bigl(f\bigl(x(t)\bigr) - f\bigl(x(s)\bigr) - \int_s^t \mathcal{L}f\bigl(x(u)\bigr)\,du\Bigr)\Bigr]^2\Bigr]. \end{aligned} \tag{75}$$

For $(w,y) \in \mathbb{R}_+^I \times \mathcal{D}^I$ and $0 \le s \le 1$, define

$$G_s^u(w,y) = g_s\bigl(w, [y(u+\cdot) - y(u)]_s\bigr),$$

where $[y(u+\cdot) - y(u)]_s$ is the restriction of $y(u+\cdot) - y(u)$ on $[0,s]$ and for $0 \le s \le t \le 1$,

$$F_{s,t}^u(y) = f\bigl(y(u+t) - y(u)\bigr) - f\bigl(y(u+s) - y(u)\bigr) - \int_s^t \mathcal{L}f\bigl(y(u+z) - y(u)\bigr)\,dz.$$

Then the expectation in (75) can be written as

$$\begin{aligned} &E\Bigl[\Bigl(\frac{1}{T_{r_m}}\int_0^{T_{r_m}} G_s^u\bigl(\hat{W}^{r_m}(u), \hat{X}^{r_m}\bigr)F_{s,t}^u\bigl(\hat{X}^{r_m}\bigr)\,du\Bigr)^2\Bigr] \\ &\quad = E\Bigl[\frac{2}{T_{r_m}^2}\int_0^{T_{r_m}}\int_0^u G_s^u\bigl(\hat{W}^{r_m}(u), \hat{X}^{r_m}\bigr)F_{s,t}^u\bigl(\hat{X}^{r_m}\bigr)G_s^v\bigl(\hat{W}^{r_m}(v), \hat{X}^{r_m}\bigr)F_{s,t}^v\bigl(\hat{X}^{r_m}\bigr)\,dv\,du\Bigr]. \end{aligned}$$

It can be shown using Proposition 3.4, Proposition 6.2, Lemma 9.1, Propositions 6.1 and 7.5 and the assumptions that $\sup_r \hat{q}^r < \infty$ and $\sup_r r\hat{\Upsilon}^r < \infty$, that

$$\lim_{m\to\infty} \sup_{u\ge 0} P\Bigl(\sup_{t\in[0,1]} \|\bar{B}^r(t+u) - \bar{B}^r(u) - t\rho\| > \epsilon\Bigr) = 0. \tag{76}$$

The proof of the above assertion is very similar to that of [6], Theorem 15, part 2, and is therefore omitted.

For any $u \ge 0$, define

$$\tilde{X}^{r,u}(t) = KM^r\hat{A}^{r,u}(t) - KM^r\hat{S}^{r,u}(\rho t) + rtK\bigl(\rho^r - \rho\bigr)$$

and

$$\breve{X}^{r_m,u}(t) \doteq \begin{cases} \hat{X}^{r_m}(t) & \text{if } u \ge t, \\ \hat{X}^{r_m}(u) + \tilde{X}^{r,u}(t-u) & \text{otherwise.} \end{cases}$$

Note that due to Lemma 9.1 the distribution of $\tilde{X}^{r_m,u}$ does not depend on u, meaning $\tilde{X}^{r_m,u} \overset{d}{=} \tilde{X}^{r_m,0}$ for all $u \ge 0$, and from the central limit theorem for renewal processes (see, e.g., [4], Theorem 14.6) $\tilde{X}^{r,0} \to \hat{X}$ in distribution on $\mathcal{D}^I$ where $\hat{X}(\cdot)$ is as introduced above (10). Since $\hat{X} \in \mathcal{C}^I$ a.s., it follows from (76), Propositions 6.1 and 7.5 and the assumption that $\sup_r \hat{\Upsilon}^r < \infty$ that, for any $\epsilon > 0$,

$$\lim_{m\to\infty} \sup_{u\ge 0} P\Big(\sup_{z\in[0,1]} \|\hat{X}^{r_m}(u+z) - \breve{X}^{r_m,u}(u+z)\| > \epsilon\Big) = 0.$$

Since f and its first and second derivatives are bounded and uniformly continuous, it follows that

$$\lim_{m\to\infty} \sup_{u\ge 0} E\big[\|F^u_{s,t}(\hat{X}^{r_m}) - F^u_{s,t}(\breve{X}^{r_m,u+s})\|\big] = 0. \tag{77}$$

and using the fact that the distribution of $\tilde{X}^{r_m,u}$ does not depend on u, that $f \in \mathcal{C}^2_c(\mathbb{R}^I)$ (in particular it has compact support), and that

$$E\bigg[f\big(\hat{X}(t-s)+x\big) - f(x) - \int_s^t \mathcal{L}f\big(\hat{X}(z-s)+x\big)\,dz\bigg] = 0$$

for all $x \in \mathbb{R}^I$ we have

$$\begin{aligned}\lim_{m\to\infty} \sup_{x\in\mathbb{R}^I, u\ge 0} \bigg\{E\bigg[&f\big(\tilde{X}^{r_m,u+s}(t-s)+x\big) - f(x)\\ &- \int_s^t \mathcal{L}f\big(\tilde{X}^{r_m,u+s}(z-s)+x\big)\,dz\bigg]\bigg\} = 0.\end{aligned} \tag{78}$$

Recall that

$$\begin{aligned}F^u_{s,t}(\breve{X}^{r_m,u+s}) &= f\big(\tilde{X}^{r_m,u+s}(t-s) + \hat{X}^{r_m}(u+s) - \hat{X}^{r_m}(u)\big) - f\big(\hat{X}^{r_m}(u+s) - \hat{X}^{r_m}(u)\big)\\ &\quad - \int_s^t \mathcal{L}f\big(\tilde{X}^{r_m,u+s}(z-s) + \hat{X}^{r_m}(u+s) - \hat{X}^{r_m}(u)\big)\,dz\end{aligned}$$

and note that since $\tilde{X}^{r_m,u+s}$ is independent of $\mathcal{G}^{r_m}(u+s)$ and $\hat{X}^{r_m}(u+s) - \hat{X}^{r_m}(u)$ is $\mathcal{G}^{r_m}(u+s)$-measurable we have

$$\begin{aligned}&\sup_{u\ge 0} E\big[F^u_{s,t}(\breve{X}^{r_m,u+s})|\mathcal{G}^{r_m}(u)\big]\\ &\quad\le \sup_{x\in\mathbb{R}^I, u\ge 0} E\bigg[f\big(\tilde{X}^{r_m,u+s}(t-s)+x\big) - f(x) - \int_s^t \mathcal{L}f\big(\tilde{X}^{r_m,u+s}(z-s)+x\big)\,dz\bigg].\end{aligned} \tag{79}$$

The fact that $F^u_{s,t}$ and G^u_s are uniformly bounded in u gives

$$\begin{aligned}&\lim_{m\to\infty} E\frac{2}{T^2_{r_m}} \int_0^{T_{r_m}} \int_0^u G^u_s(\hat{W}^{r_m}(u), \hat{X}^{r_m}) F^u_{s,t}(\hat{X}^{r_m}) G^v_s(\hat{W}^{r_m}(v), \hat{X}^{r_m}) F^v_{s,t}(\hat{X}^{r_m})\,dv\,du\\ &= \lim_{m\to\infty} E\frac{2}{T^2_{r_m}} \int_0^{T_{r_m}} \int_0^{u-1} G^u_s(\hat{W}^{r_m}(u), \hat{X}^{r_m}) F^u_{s,t}(\breve{X}^{r_m,u+s})\\ &\quad G^v_s(\hat{W}^{r_m}(v), \hat{X}^{r_m}) F^v_{s,t}(\hat{X}^{r_m})\,dv\,du\\ &= \lim_{m\to\infty} E\frac{2}{T^2_{r_m}} \int_0^{T_{r_m}} \int_0^{u-1} E\big[F^u_{s,t}(\breve{X}^{r_m,u+s})|\mathcal{G}^{r_m}(u+s)\big] G^u_s(\hat{W}^{r_m}(u), \hat{X}^{r_m})\\ &\quad G^v_s(\hat{W}^{r_m}(v), \hat{X}^{r_m}) F^v_{s,t}(\hat{X}^{r_m})\,dv\,du = 0,\end{aligned}$$

where the first equality comes from (77), and the second comes from the fact that for $u-1 \geq v$ $G_s^u(\hat{W}^{r_m}(u), \hat{X}^{r_m}) G_s^v(\hat{W}^{r_m}(v), \hat{X}^{r_m}) F_{s,t}^v(\hat{X}^{r_m})$ and $\hat{X}^{r_m}(u+s) - \hat{X}^{r_m}(u)$ are $\mathcal{G}^{r_m}(u+s)$-measurable, and the third comes from (79) and (78). Putting this all together gives

$$E\left[E_{\nu^*(\omega)}\left[g_s(w, x(\cdot))\left(f(x(t)) - f(x(s)) - \int_s^t \mathcal{L}f(x(u))\,du\right)\right]^2\right] = 0.$$

Proof of (a) now follows by a standard separability argument.

We will now prove part (b). Let $f \in \mathcal{C}_c(\mathbb{R}^I)$ (the space of continuous functions on $\mathbb{R}^I$ with compact support)) and $s \in [0,1]$ be arbitrary From part (a), $P_{\nu^*(\omega)}(x \in \mathcal{C}([0,1]:\mathbb{R}^I)) = 1$ for a.e. ω, which implies

$$E\left|E_{\nu^*(\omega)}\left[f(w) - f\left(\Gamma\left(w + x(\cdot)\right)(s)\right)\right]\right| = \lim_{m\to\infty} E\left|E_{\nu^{r_m}(\omega)}\left[f(w) - f\left(\Gamma\left(w + x(\cdot)\right)(s)\right)\right]\right|.$$

The expectation on the right-hand side is bounded above by

$$E\left[\left|\frac{1}{T_{r_m}}\int_0^{T_{r_m}} f\left(\hat{W}^{r_m}(u+s)\right) - f\left(\Gamma\left(\hat{W}^{r_m}(u) + \hat{X}^{r_m}(u+\cdot) - \hat{X}^{r_m}(u)\right)(s)\right)du\right|\right]$$
$$+ E\left[\left|\frac{1}{T_{r_m}}\int_0^{T_{r_m}} f\left(\hat{W}^{r_m}(u)\right) - f\left(\hat{W}^{r_m}(u+s)\right)du\right|\right].$$

Due to Proposition 3.2, Proposition 3.3, Propositions 6.1 and 7.5, the assumption that $\sup_r \hat{\Upsilon}^r < \infty$ and the fact that f is uniformly continuous and bounded, we have

$$\lim_{m\to\infty} E\left|\frac{1}{T_{r_m}}\int_0^{T_{r_m}} f\left(\hat{W}^{r_m}(u+s)\right) - f\left(\Gamma\left(\hat{W}^{r_m}(u) + \hat{X}^{r_m}(u+\cdot) - \hat{X}^{r_m}(u)\right)(s)\right)du\right| = 0.$$

In addition, since f is bounded and $T_{r_m} \to \infty$, we have

$$\lim_{m\to\infty} E\left[\left|\frac{1}{T_{r_m}}\int_0^{T_{r_m}} \left[f\left(\hat{W}^{r_m}(u)\right) - f\left(\hat{W}^{r_m}(u+s)\right)\right]du\right|\right] = 0.$$

Together these observations show $E[|E_{\nu^*(\omega)}[f(w) - f(\Gamma(w + x(\cdot))(s))]|] = 0$. By standard separability arguments, it now follows that $w \stackrel{d}{=} \Gamma(w + x(\cdot))(s)$ for all $s \in [0,1]$ under $\nu^*(\omega)$ for a.e. ω, which proves part (b).

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