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Title: We Are Wanderers: Abstract geometry reflects spatial navigation

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Author's Note:

All materials, methods, and analyses were preregistered on the Open Science Framework unless otherwise stated. These preregistrations as well as the stimuli, data, and analysis code are available at: https://osf.io/gvn6k/.

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Abstract

Philosophers throughout history have debated the relations between the abstract geometry of formal mathematics and the physical geometry of the natural world. We provide evidence that abstract geometry reflects the geometry humans and non-human animals use for spatial navigation. Across two preregistered experiments, educated adults watched short videos of two points and two line segments forming an open figure on an otherwise blank screen. These simple visuals were described with sparse and minimally different language, creating different spatial contexts. After watching each video, participants were asked to click: anywhere (*anywhere* condition); to complete the triangle (*triangle* condition); where the next corner of an object would be (*object* condition); where the next stop on an agent's path would be (*navigation* condition); or where the next point on an abstract plane would be (*abstract* condition). Across spatial contexts, participants produced responses that reflected strikingly different sets of geometric representations; in particular, preserving distance and direction for open paths in the *navigation* condition but preserving length and angle for closed shapes in the *object* condition. In the *navigation* and *abstract* contexts, however, the elicited geometry was remarkably similar. Human language may thus effectively isolate phylogenetically ancient geometric representations used for navigating the physical world and recognizing the objects in it. Moreover, the cognitive origins of uniquely human abstract geometry may lie in representations used for navigating the physical world.

Keywords: abstract thought; geometry; navigation; object recognition; spatial language

Public Significance Statement

Philosophers throughout history have debated the relation between the "abstract geometry" in our mind and the "physical geometry" in our world. Since antiquity, geometry has been the model of human abstract thought. But where do our ideas about geometry come from? Are our geometric concepts rooted in everyday experience? Our work informs debates over the origins of geometry. Using only minimally different language, our experiments were able to get adults to think differently about the same simple geometric forms. We humans are able to think about geometry in many different ways, through our experiences with places or with objects. Most surprisingly, however, when we think about abstract geometry, we wander the Euclidean plane like we wander the land. There is a connection between how we navigate the world and how we think about formal geometry. Perhaps, we suggest, we can harness this connection to inform the development of mathematics education.

"We are wanderers." —Carl Sagan (Sagan, 1985)

Our experience of the everyday spatial world seems rich and unitary, integrating the geometry of the places we navigate with our recognition of the objects and visual forms that populate those places. Nevertheless, decades of research in the psychological, cognitive, and neural sciences suggests that the representations underlying our understanding of places and objects are not as unitary as they might seem. For example, seminal work exploring the search behavior in simple rectangular environments of disoriented animals, from fish and chickens to rats and humans, suggests that when we navigate such environments, we tend to keep track of the allocentric distances and directions of the environment's boundaries (Cheng et al., 2013; Doeller et al., 2008; Doeller & Burgess, 2008; Julian et al., 2015; Spelke & Lee, 2012). In contrast, work exploring the sensitivities of humans and non-human animals to small-scale 2D and 3D visual forms suggests that when we recognize objects, it matters less how far away an object is or what direction it is facing, as long as it is the right shape (Blough & Blough, 1997; Kourtzi & Kanwisher, 2001; Lehrer & Campan, 2004; Logothetis et al., 1994; Tanaka, 1997; Zoccolan et al., 2009).

Previous research has thus found surprising dissociations in human's use of geometric information for representing the physical world, dissociations shared with non-human animals. Nevertheless, representing the physical world is not unique to humans, and shared environments and common phylogeny may lead to similar cognitive and neural processes for navigation and object recognition across humans and non-human animals. But what of the geometric intuitions that are *unique* to humans? At the foundation of formal geometry and cultural productions like science, technology, art, and architecture lies humans' intuitive *abstract geometry*, which affords humans the unique capacity to imagine and reason about a spatial world of zero-dimensional points, infinitely long lines, and endless planes. Indeed, since antiquity, this geometry has often been held up by philosophers and scientists as *the* model of human abstract thought (Hodge, 1978; Olson, 1995). Do uniquely human abstract geometric intuitions rely on the same foundational geometry humans and non-human animals use to represent the physical spatial world? What are the cognitive origins of human abstract geometry?

Current research in the cognitive and neural sciences provides several proposals for these origins. One proposal is that humans uniquely possess early emerging and abstract representations of Euclidean principles, which get composed in an algorithmic-like "language of thought" for geometry (Amalric et al., 2017; Dehaene et al., 2022; Sablé-Meyer et al., 2021). These principles include foundational concepts like linearity, parallelism, perpendicularity, and symmetry represented as symbolic concepts. Recent evidence for this proposal has come from cross-species, cross-cultural, and computational work using an odd-ball paradigm that presented participants with arrays of planar figures—like squares, parallelograms, and trapezoids—simply asking them to find the one figure in each array that did not belong with the rest. The more Euclidean that the principles exemplified by the context figures or deviant figure were, the easier the task was for human adults and children but not for baboons (Sablé-Meyer et al., 2021). Moreover, while state-of-the-art convolutional-neural-network models performed on this task like baboons, only a model that included a symbolic list of discrete geometric properties performed like humans.

Another proposal for the cognitive origins of abstract geometry is an appeal to phylogenetically ancient and dissociable geometric systems for everyday navigation and 2D and 3D form analysis (Dillon et al., 2013; Dillon & Spelke, 2018; Izard et al., 2011; Spelke et al., 2010). This proposal suggests that the system dedicated to navigation prioritizes distance and directional information for paths through space while the system dedicated to form analysis prioritizes length and angle information for closed shapes and objects. Through human development and supported by the combinatorial capacities of uniquely human and domain-general symbolic systems like language and pictures, the complementary geometries of the navigation and form-analysis systems merge to support an intuitive natural geometry that captures Euclidean geometry, including granting symbolic concepts of linearity, parallelism, perpendicularity, and symmetry.

A third proposal for the cognitive origins of abstract geometry is that noisy and dynamic mental simulations of moving though the physical world, akin to a correlated random walk of a navigating animal, approximate Euclidean principles and ground abstract geometric reasoning (Hart et al., 2018, 2022). Consistent with the second proposal, such navigation prioritizes distance and directional information, but this third proposal emphasizes the role of simulated navigation (Banino et al., 2018; Dragoi & Tonegawa, 2010; Gupta et al., 2010) in abstract reasoning. The proposal is built from a series of findings suggesting that older children and adults reason about the Euclidean properties of planar figures not with language-like rules or symbolic concepts, but rather with approximate and dynamic mental simulations of figures. The sides and corners of these figures are mentally simulated with local, smooth motion (and with it, local noise akin to the noise in a random walk) as well as a global correction process that reflects and preserves the basic Euclidean principle of scale-invariant angle/directional representations (making the random walk's steps correlated from one step to the next; Hart et al., 2018, 2022). This tradeoff between maintaining both smooth motion and motion in a certain direction is inherent in the navigational abilities of a variety of animal species (Cheung et al., 2007;

Papastamatiou et al., 2011; Peleg & Mahadevan, 2016; Wehner et al., 1996; Wiltschko & Wiltschko, 2005). This proposal then suggests that reasoning consistent with Euclidean geometry may emerge in human development when children start to reason about the general properties of planar figures using such mental simulation. In doing so, children develop a natural geometry within which are principles that allow for an intuitive but approximate grasp of Euclidean geometry and a capacity to learn formal Euclidean geometry (Huey et al., 2023), like concepts of linearity, parallelism, perpendicularity, and symmetry.

These three proposals differ in their commitments to foundational representations of physical geometry as a grounding for abstract geometry: the first proposal suggests no such grounding; the second suggests grounding in both navigation and object recognition; and the third suggests grounding primarily in navigation. The three proposals thus also make different predictions about whether and how adults' judgments about figures described as abstract points and lines should relate to figures described in the contexts of navigation and object recognition. The first proposal, because of its emphasis on the immediately symbolic nature of abstract geometry, might suggest that abstract figures would show no necessary geometric resemblance to figures contextualized in navigational or object-recognition contexts. Or, this proposal might suggest that all planar figures, regardless of context, are represented with the same abstract geometric principles. The second proposal, because of its emphasis on navigation and object recognition as two independent sources of different geometric representations, might suggest that figures contextualized in navigation would preserve distance and direction for open paths through space while those contextualized in object recognition would preserve length and angle information for closed shapes. Abstract figures would then reflect a combination of these two contexts, preserving distance, direction, angle, and length, and forming both open and closed figures with equal facility. The third proposal, because of its emphasis on navigation, might suggest that abstract figures would resemble figures contextualized in navigation and would preserve distance and direction for open paths through space.

With simple visual displays and short, minimally contrastive linguistic descriptions, the present work intervenes on this debate on the origins of human abstract geometry by investigating whether foundational representations of the geometry of navigation and object recognition, which are shared by humans and non-human animals alike, are reflected in educated adults' representations of abstract points and lines. In finding that a verbal description of a navigating agent uniquely elicits the same geometric information as a verbal description of abstract points and lines, the present study suggests that abstract geometry reflects spatial navigation—a unique prediction of the third proposal outlined above.

Finally, the present work also probes a novel question about geometric cognition and human language, not directly addressed by the proposals outlined above: i.e., whether language can consistently evoke foundational geometric representations for navigation and object recognition, effectively isolating the different geometries that may contribute to our sense of a unified spatial world. The second proposal outlined above is committed to the persistence of foundational geometric representations of navigation and object recognition being present and active throughout the lifespan of human and non-human animals, merging these different geometries only in humans via natural language. The present work reveals that such representations are not only still present and active in human adults but also that and are individually accessible through language used to describe the very same simple planar figures.

Methods

Transparency and Openness

All materials, methods, and analyses were preregistered on the Open Science Framework unless otherwise stated. These preregistrations as well as the stimuli, data, and analysis code are available at: https://osf.io/gvn6k/. Our report follows the guidelines of JARS-Quant (Applebaum et al., 2018).

Overview

In two preregistered experiments (**Experiment 1**: https://osf.io/5jvwn; **Experiment 2**: https://osf.io/dbe8h), educated adults from the United States watched short videos of two points and two line segments forming an open figure on an otherwise blank screen. At the end of each video, the lines disappeared, leaving the two points (**Figure 1**; **Movie S1**). Across five randomly assigned conditions, these simple visuals were described with sparse and minimally different language, creating different spatial contexts in which participants provided a mouse-click response. Participants were asked to click: anywhere (*anywhere* condition); to complete the triangle (*triangle* condition); where the next corner of the object would be (*object* condition); where the next stop on the agent's path would be (*navigation* condition); or where the next point on the abstract plane would be (*abstract* condition).

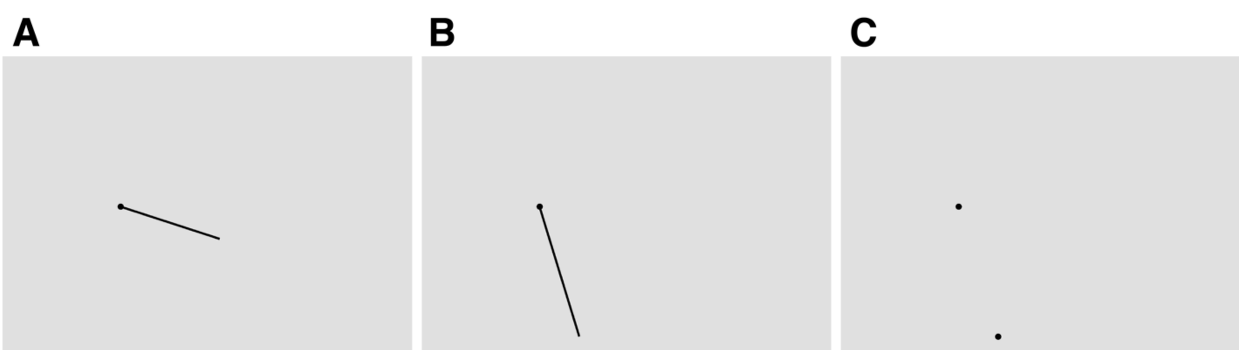

Figure 1. Example frames of a stimulus video showing a 55° angle. **A.** The first line is drawn from the figure's origin, which is not displayed as a point, and ends with a first visible point. **B.** The second line is drawn from the first visible point and ends with a second visible point. **C.** The two points remain visible at the end of the video.

Participants

The sample size was chosen after exploring participant responses in scatterplots from pilot experiments. Participants were recruited from our university's participant pool and received course credit. They reported their age and gender in a text box; no other participant demographic information was collected. The study was approved by the institutional review board for the use of human participants at our university.

In **Experiment 1**, 150 adults (111 women, 38 men, 1 gender non-binary; age: $M = 20$ years, $SD = 1.71$, range = 18–33) participated. Based on preregistered exclusion criteria, an additional 84 participants were excluded for not completing the experiment (34), taking longer than 60 minutes to complete the experiment (16), answering the comprehension question incorrectly after two attempts (20); or answering the catch question incorrectly (14; see the **SI**). In **Experiment 2**, 150 adults (107 women, 41 men, 2 gender non-binary; age: $M = 19.41$ years, $SD = 1.36$, range = 17–24) participated. An additional 64 participants were excluded for not completing the experiment (9), taking longer than 60 minutes to complete the experiment (6), answering the comprehension question incorrectly after two attempts (10); or answering the catch question incorrectly (39; see the **SI**). The exclusion rates are comparable to the exclusion rates reported in previous unmonitored online studies with adults (Ludwin-Peery et al., 2020, 2021).

Stimuli

Participants saw a total of 84 trials presenting stimuli videos generated in MATLAB at 1120 px X 840 px and forming angles of six different sizes (25°, 55°, 80°, 100°, 125°, and 155°), with 14 unique examples of each angle size, which varied in length, orientation, and location (see the **SI**). The individual lines forming the angles were between 20% and 45% the height of the full gray background area (840 px); neither line was oriented within 10° of the vertical or horizontal; both lines had at minimum a 4% distance to the gray background's boundary; and the line drawn second was to the right of the first line in a random half of the videos and to the left in the other half of the videos. The videos were presented at a faster speed in **Experiment 2** compared with **Experiment 1** to reduce participant fatigue (**Experiment 1**: M = 9649 ms, SD = 1459 ms, range = 6800–12700 ms; **Experiment 2**: M = 5789 ms, SD = 874 ms, range = 4080–7620 ms). Prior to seeing the test trial videos, participants practiced making mouse-click responses to three practice videos, which showed similar figures but with angle sizes not used in the test videos (13°, 72°, and 167°).

Sparse and minimally different language, which described five different spatial contexts, characterized these simple visuals across conditions (see the **SI** for full task instructions). In the *anywhere* condition, participants were told, "For each question, you will see one video. Please watch the video closely. After the video ends, click anywhere you'd like." In the *triangle* condition, participants were told, "For each question, you will see one video showing a partial triangle. The sides of the partial triangle will disappear, but the vertices will remain visible. Please watch the video closely. After viewing the video, click to complete the triangle." In the *object* condition, participants were told, "For each question, you will see one video of some of the edges and corners of an object. Please watch the video closely. The edges will disappear at the end of the video, but two corners of the object will remain visible. After viewing the edges and corners, click where you think the next corner of that object will be." In the *navigation* condition, participants were told, "For each question, you will see one video of some of the paths and stops that an agent has travelled on a land. Please watch the video closely. The paths will disappear at the end of the video, but the two stops the agent visited will remain visible. After viewing the paths and stops, click where you think the next stop of that agent will be." Finally, in the *abstract* condition, participants were told, "For each question, you will see one video of some points and lines on a plane. The lines will disappear, but two points will remain visible. Please watch the video closely. After viewing the points and lines, click where you think the next point will be."

Procedure

In both experiments, participants were randomly assigned to one of the five conditions and completed the experiment online and unmonitored through the survey platform Qualtrics. Participants first read the instructions for their assigned condition and answered comprehension questions that assessed their understanding of those instructions. They were given two chances to answer each comprehension question correctly; participants who answered incorrectly on both attempts were allowed to complete the experiment but were excluded from the analysis (see the **SI**). After completing the comprehension questions, participants completed three practice trials. The practice trials were presented in a random order for each participant. Before proceeding to the test trials, participants were reminded of the instructions. No feedback was given on practice or test trials.

In **Experiment 1**, participants were asked to make only one click at the end of the video, adding one point to the two points given in the video. In **Experiment 2**, participants were free to make at least one but up to ten clicks. Participants saw their click(s) appear as a point on the screen, but they did not see any lines drawn from the last given point to the click(s) they generated.

In both experiments, the same subset of test trials was randomized within the first, second, or third of three blocks, and there was an optional break at the end of each block. During the breaks, participants could watch a two-minute video of cute animals. The animal videos were embedded in the experiment so participants did not need to navigate away from the experiment.

After the first block of trials and before the first optional break, participants saw a catch question in which they were presented with four still pictures, one which most closely resembled a still from the stimuli videos (see the **SI**). Participants had only one opportunity to click on the picture that most closely resembled the stimuli videos, and they received no feedback. If they responded incorrectly, they were allowed to complete the experiment but were excluded from the analysis.

Analysis

In two preregistered analysis plans (**Experiment 1**: https://osf.io/d8ts9; **Experiment 2**: https://osf.io/xvwm4), we evaluated whether participants responded to the given figures differently depending on the spatial framing provided by the language of their assigned condition and whether participants responded in the abstract condition in a way similar to any other condition. To do so, we used multivariate (2D) kernel density estimations to estimate and compare the probability density functions of participants' mouse-click responses across conditions.

First, we preprocessed the data in MATLAB to align participants' responses to the different examples they saw of the same angle size. To conduct this preprocessing, we rotated, flipped, and translated the 14 examples of each angle to align the angles' vertices, keeping the line lengths unchanged. Then, we applied the same set of rotations, flips, and translations to the coordinates of participants' mouse-click responses to determine their new coordinates in relation to the now aligned angles. Aligning the stimuli and responses resulted in a square response space instead of the rectangle space in the stimulus videos.

To conduct the kernel density estimation analysis, we first defined a region within the square response space that included almost all of participants' responses (**Experiment 1**: 99.97% responses; **Experiment 2**: 99.94% responses), excluding a small number of outlier responses detected upon visual inspection by a researcher masked to the condition of those responses (**Experiment 1**: *anywhere* condition, 80° [3 responses]; *anywhere* condition 100° [1 response]; **Experiment 2**: *anywhere* condition, 80° [5 responses]; *object* condition 80° [1 response]; *navigation* condition 80° [1 response]; *abstract* condition 80° [1 response]). With the *kde* function in the *ks* R package, we then used the default 151 X 151 grid size to fit a separate kernel density estimation to each participant's 14 responses per angle and used cross validation to find the optimal bandwidth matrix (smoothing factor) for each of these kernel density estimations. We averaged these bandwidth matrices across all participants and angles to obtain a shared bandwidth matrix and fit a new set of kernel density estimations to each participant's responses to each angle using this shared bandwidth matrix. Next, we compared the kernel density estimations across participants at each cell of the grid for each angle across the five conditions using one-way ANOVAs with permutation tests (to control for the family-wise error), permuting all of the possible kernel density estimations, the dependent variable, while keeping condition, the independent variable, fixed. Finally, at each location with a significant effect of condition revealed by the ANOVA, we conducted pairwise contrasts across the conditions with permutation tests (to control for the family-wise error) and Holm correction on the resulting *p*-values to adjust for multiple comparisons.

In an addition to this preregistered analysis, we conducted an unplanned analysis focused on characterizing the differences between three target conditions—*object*, *navigation*, and *abstract*—using earth mover's distance, a method of image comparison that evaluates the minimum cost required to align one distribution of responses to another (Rubner et al., 2000).

To conduct the earth mover's distance analysis, we used the same optimal bandwidth matrix as the preregistered analysis, and first fit new kernel density estimations to each condition and each angle size, this time collapsing across participants. Because this analysis was too computationally demanding to conduct with the original kernel-density-estimation 151 X 151 grid, we down-sampled to a 51 X 51 grid (following Beller et al., 2022), visually inspecting the original and down-sampled kernel density estimations to ensure the down-sampling preserved the patterns observed at the original grid size. Then, we computed the earth mover's distance for each pair of conditions for each angle size using the *EMD* function of the *opencv* package in Python, with one distance measure for each pair. Finally, we performed a one-way ANOVA and pairwise contrasts across angles to compare the earth mover's distances of the *object*/*navigation* comparison, *object*/*abstract* comparison, and *navigation*/*abstract* comparison.

Results

Figure 2A presents scatter plots of all mouse-click responses to the 55° angle across the five conditions of **Experiment 1**. **Figure 2B** presents the results of the kernel density estimations, with the colored patches in each plot representing significantly different response densities ($p < .05$) across the two compared conditions.

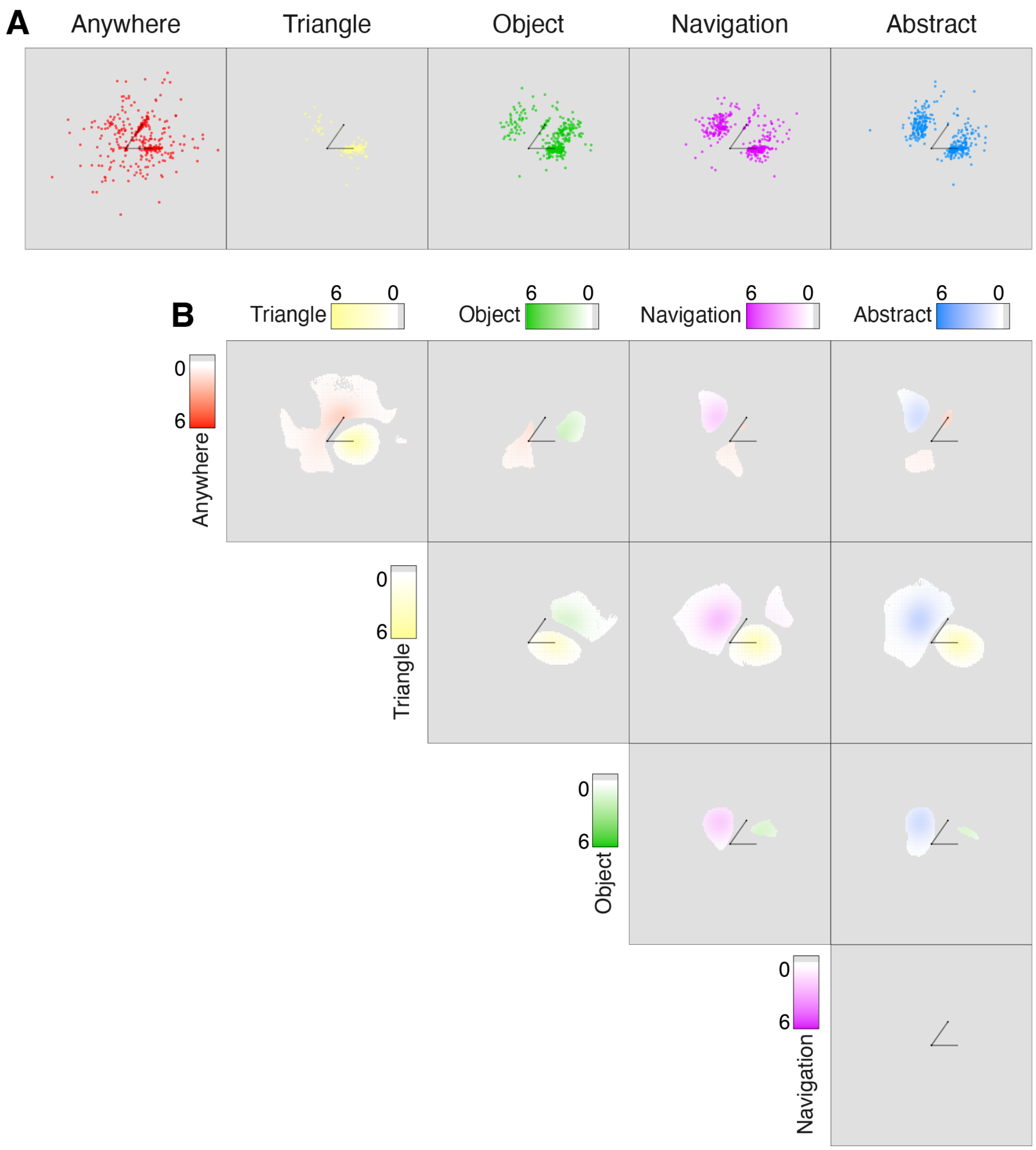

Figure 2. A Scatter plots of all mouse-click responses to the 55° angle in **Experiment 1**. **B** Results of the kernel density estimation analysis, with the colored patches in each plot representing significantly different response densities ($p < .05$, Holm-corrected) across the two compared conditions. The range of the kernel density estimations are provided on the figure's color scale. A magnitude of 6 (the scale's maximum) refers to a 0.15 percentage-point increase of responding at that location in the grid in the indicated condition.

All conditions had a cluster of clicks at the figure's origin (which was not represented as a point), suggesting some tendency to complete the given open figure to form a triangle. This tendency was nevertheless strongest in the *triangle* condition. The *anywhere* condition showed much more distributed responses compared with the responses in all the other conditions. Unique responses in the *object* condition clustered opposite the given figure's first visible point, as if participants were preserving the overall shape of a closed figure to form the third side of what would be a parallelogram. Unique responses in the *navigation* condition, in contrast, clustered past the end of the given figure's second visible point, as if participants were continuing the agent's path with the same distance and direction as the first line in an open zig-zag. Despite the identical visual stimuli and the minimal linguistic differences across these conditions, participants produced very different responses. Strikingly, however, participants' responses in the *abstract* condition closely resembled participants' responses in the *navigation* condition: There was a large cluster of responses extending beyond the given figure's second visible point and no difference in the response patterns across these two conditions. The results across the six angles tested (**Figure 3**) corroborate these results (for the scatter plots and complete kernel density estimations for the additional angles, see the **SI**).

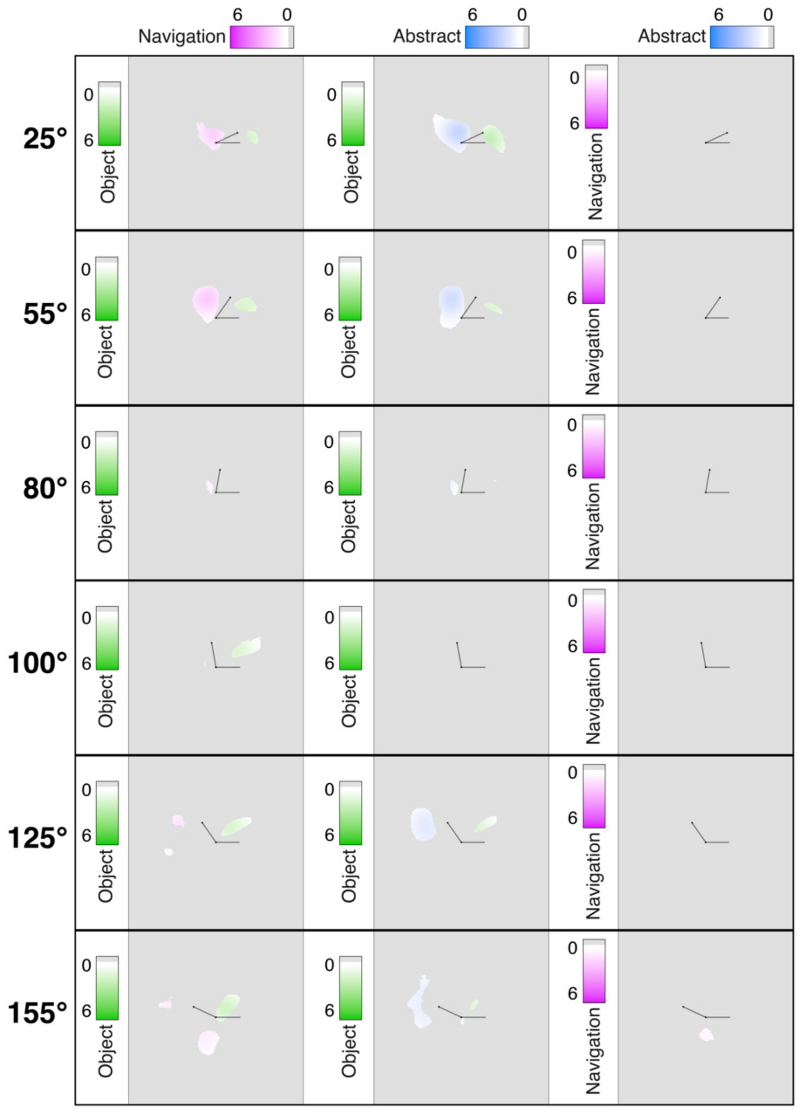

Figure 3. Results of the kernel density estimation analysis in the *object*, *navigation*, and *abstract* conditions across the six angles in **Experiment 1**, with the colored patches in each plot representing significantly different response densities ($p < .05$) across the two compared conditions.

The results of the unplanned earth mover's distance analysis, which focused on the *object*, *navigation*, and *abstract* conditions, were consistent with those of the preregistered kernel density estimation analysis. **Figure 4A** displays the earth mover's distance comparing each pair of conditions across the six angles in **Experiment 1**, with smaller values representing smaller differences between conditions. The *object/abstract* comparison had the largest earth mover's distance ($M = 8.09$, $SD = 1.40$), followed by the *object/navigation* comparison ($M = 7.90$, $SD = 1.26$). The *navigation/abstract* comparison had the smallest earth mover's distance ($M = 1.73$, $SD = 0.74$). The one-way ANOVA on earth mover's distance across conditions revealed a significant effect of condition ($F(1, 15) = 57.22$, $p < .001$), and pairwise contrasts revealed no significant difference between the *object/navigation* comparison and the *object/abstract* comparison ($p = .958$), but significant differences between the *object/navigation* comparison and *navigation/abstract* comparison ($p < .001$) as well as between the *object/abstract* comparison and *navigation/abstract* comparison ($p < .001$).

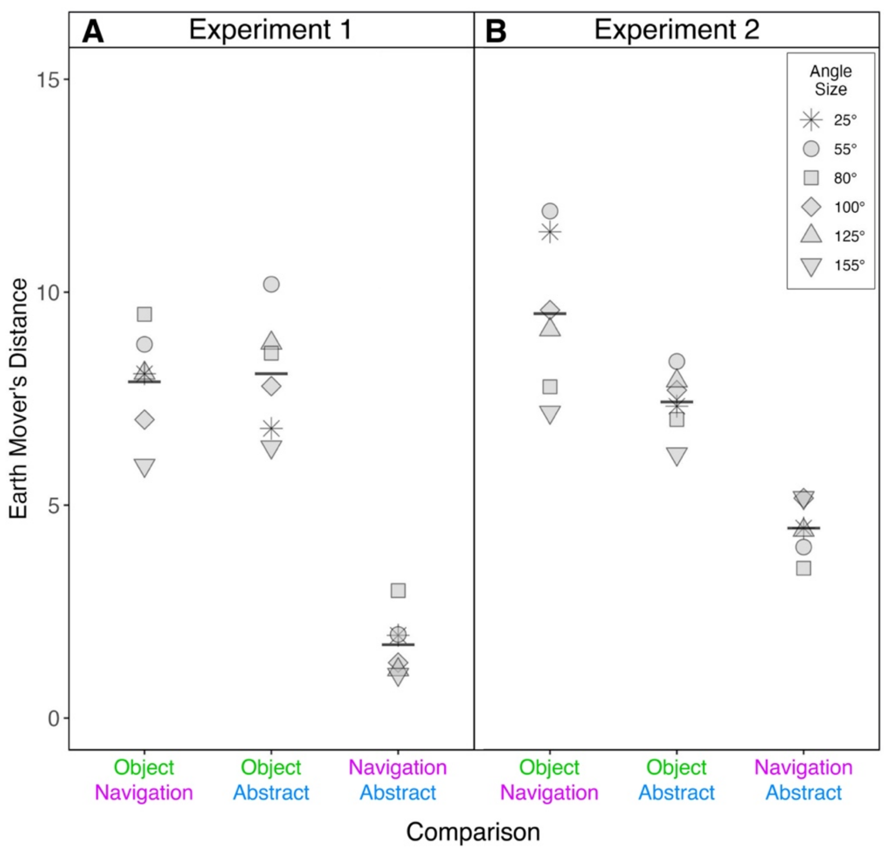

Figure 4. A The earth mover's distance comparing the *object*, *navigation*, and *abstract* conditions across the six angles in **Experiment 1**. **B** The earth mover's distance comparing the *object*, *navigation*, and *abstract* conditions across the six angles in **Experiment 2**. The black lines represent the mean earth mover's distance across the six angles of each comparison.

Experiment 2 replicated and extended **Experiment 1**'s findings to cases in which participants were not constrained to responding with only one click. Compared with **Experiment 1**, **Experiment 2** thus provided an even more open-ended procedure both to ensure that the findings from **Experiment 1** were generalizable (i.e., not idiosyncratic to a one-click response) and to provide more opportunities for the conditions to converge or diverge, contributing to a stronger test of our hypothesis. Across all conditions and angle sizes, participants differed in their number of clicks (mixed-model Poisson regression, Wald Test, $X^2(4) = 38.81$, $p < .001$). The number of clicks in the *anywhere* (mode = 3 clicks) and *triangle* (mode = 3 clicks) conditions did not differ from each other (Holm-corrected pairwise contrast, $p = 1.000$), but did differ from the other conditions (Holm-corrected pairwise contrasts, *anywhere* - *object*, $p < .001$; *anywhere* - *navigation*: $p < .001$; *anywhere* - *abstract*: $p = .015$; *triangle* - *object*: $p < .001$; *triangle* - *navigation*: $p < .001$; *triangle* - *abstract*: $p = .006$). The number of clicks in the *object*, *navigation*, and *abstract* conditions did not differ from each other (all modes = 1 click; *object* - *navigation*: $p =1.000$; *object* - *abstract*: $p = .598$; *navigation* - *abstract*: $p = 1.000$).

Figure 5 presents the scatter plots (**A**) and kernel density estimations (**B**) from for the responses to the 55° angle across the five conditions of **Experiment 2**, and **Figure 6** shows the results of the *object*, *navigation*, and *abstract* conditions across all six angles tested (for the scatter plots and complete kernel density estimations for these additional angles, see the **SI**). Consistent with the findings of **Experiment 1**, participants in all conditions showed some tendency to complete the triangle, but those in the *triangle* condition did so the most. Participants in the *anywhere* condition showed the most distributed responses, participants in the *object* condition uniquely responded opposite the first given point in what would be a closed parallelogram, and participants in the *navigation* condition uniquely responded by continuing the agent's path in an open zig-zag. As in **Experiment 1**, participants responded remarkably differently across conditions, except in the *navigation* and the *abstract* conditions, in which responses closely resembled one another.

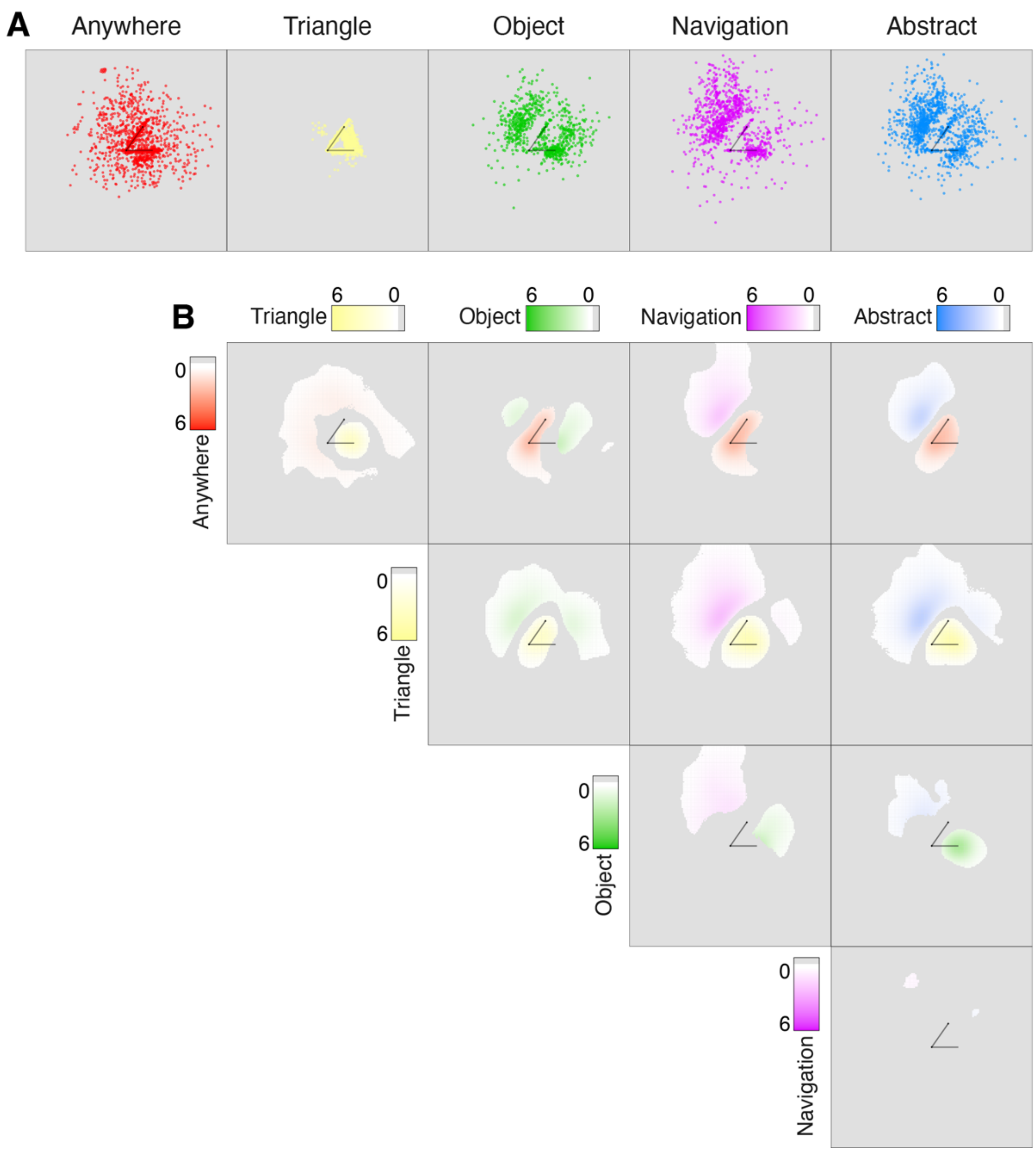

Figure 5. A Scatter plots of all mouse-click responses to the 55° angle in **Experiment 2**. **B** Results of the kernel density estimation analysis, with the colored patches in each plot representing significantly different response densities ($p < .05$, Holm-corrected) across the two compared conditions. The range of the kernel density estimations are provided on the figure's color scale. A magnitude of 6 (the scale's maximum) refers to a 0.15 percentage-point increase of responding at that location in the grid in the indicated condition.

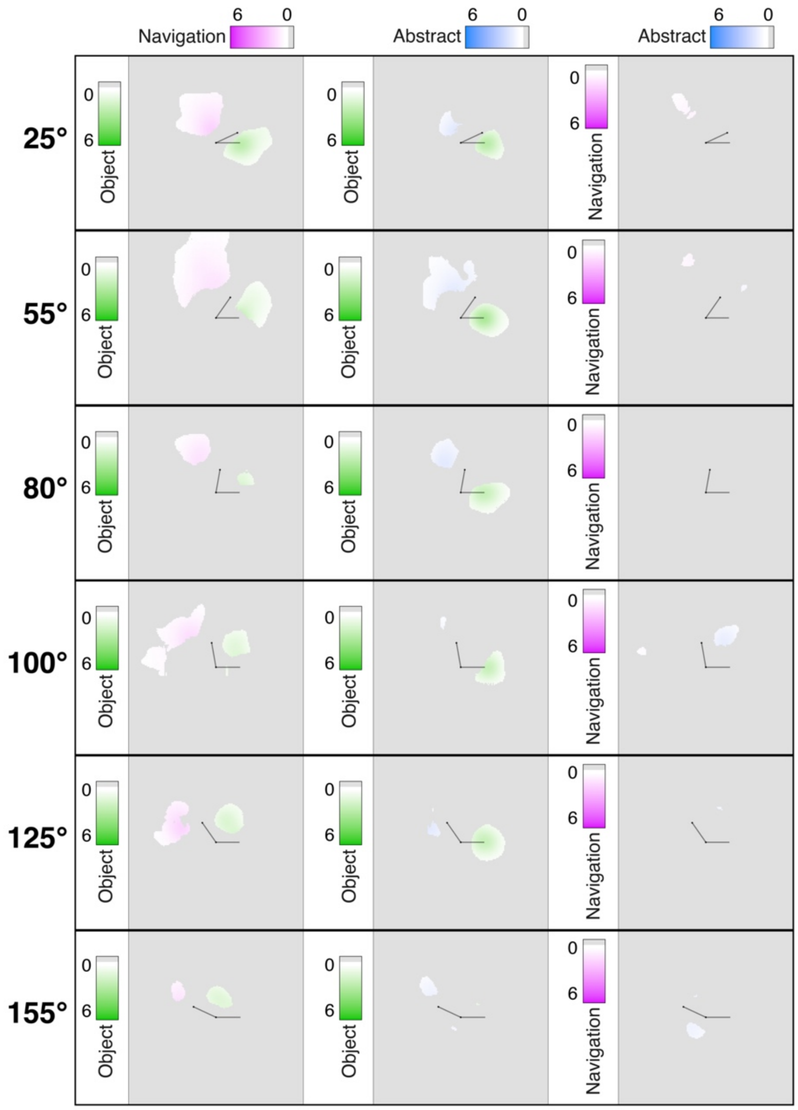

Figure 6. Results of the kernel density estimation analysis in the *object*, *navigation*, and *abstract* conditions across the six angles in **Experiment 2**, with the colored patches in each plot representing significantly different response densities ($p < .05$) across the two compared conditions.

As in **Experiment 1**, moreover, the results from the unplanned earth mover's distance analysis were consistent with those of the preregistered kernel density estimation analysis. **Figure 4B** displays the earth mover's distance comparing each pair of conditions across the six angles in **Experiment 2**. The *object*/*navigation* comparison had the largest earth mover's distance (M = 9.50, SD = 1.89), followed by the *object*/*abstract* comparison (M = 7.42, SD =

0.76). The *navigation*/*abstract* comparison had the smallest earth mover's distance (M = 4.46, SD = 0.65). The one-way ANOVA revealed a significant effect of condition ($F(1, 15) = 25.15$, $p < .001$), but unlike in **Experiment 1**, all pairwise contrasts were significant (*object*/*navigation* comparison - *object*/*abstract* comparison, $p = .028$; *object*/*navigation* comparison - *navigation*/*abstract* comparison, $p < .001$, *object*/*abstract* comparison - *navigation*/*abstract* comparison, $p = .002$).

Discussion

The results yield three main findings. First, the same simple visual displays, characterized only by sparse and minimally different linguistic descriptions, readily elicited in educated adults strikingly different sets of geometric representations. These geometric representations are inherent in phylogenetically ancient cognitive systems of geometry for navigating places by distance and direction and for recognizing objects by shape: Participants in the *navigation* condition preserved the distance and direction of the initial figure's first line segment, while those in the *object* condition instead preserved the global shape of the initial figure. The clarity and consistency with which participants' responses reflected the geometry of these cognitive systems are particularly surprising given the open-ended and subjective nature of the procedure and participants' education in formal geometry. Participants could have imagined a path or object with *any* geometry, especially considering that there was no actual navigation or object recognition involved in the task. Despite this freedom but consistent with other tasks' use of an open-ended tapping procedure to elicit consistent spatial representations (Firestone & Scholl, 2014; Boger & Ullman, 2023), participants more often imagined open paths with distance and direction preserved and objects with global convex shape preserved. Human language may thus both naturally invoke the particular geometric representations inherent to navigation and object recognition given a basic description of the spatial context and also allow us easily to mentally wander among our different domains of foundational spatial knowledge about places and objects.

Sensitivity to this foundational geometric information is shared across cultures and through human development (Dehaene et al., 2006; Izard et al., 2022; Izard, Pica, Dehaene, et al., 2011; Sablé-Meyer et al., 2021). Nevertheless, future studies might explore whether these same sets of geometric representations can be elicited using the same visual stimuli and similarly minimal linguistic descriptions as the present study in adults across cultures and who speak different languages. Future studies might also explore when and how language might elicit such geometric representations in children. If the present results rely on adults' capacity to combine place and object geometry for geometric tasks like reorientation (Hermer-Vazquez et al., 1999; Pyers et al., 2010; Shusterman et al., 2011) or their successful verbal reasoning about relations among the distance, direction, and shape properties of visual forms (Dillon & Spelke, 2018; Hart et al., 2022; Izard, Pica, Spelke, et al., 2011), then the same results may not be obtained until late childhood.

Second, while human language may enrich (e.g., Dessalegn & Landau, 2008; Landau et al., 2009) or combine disparate spatial representations (e.g., Hermer-Vazquez et al., 1999; Shusterman et al., 2011) to solve spatial tasks, the results from the present study show that language can also effectively *isolate* such foundational spatial representations, even in human adults who can use them together. Language's powerful capacity for selectivity has been revealed across languages, for example: in representations of motion verbs by directing a speaker's and listener's attention either to the manner by which a motion is taken or to the properties of that motion's path; and in choice of reference frames by specifying egocentric, geocentric, or object-centered reference frames (see Landau et al., 2009 for a review). Language's selectivity in the present study modulated spatial representations within a single natural language. Moreover, language here selected over content that in our experience of the everyday spatial world seems unitary, carving this experience at joints that fall along separable but complementary geometries, which are separate in human infancy and childhood. What seems like a unified system of spatial intuitions in the minds of human adults may thus actually be the merger of two systems of persisting and complementary geometries for representing places and objects, consistent with the second proposal outlined in the **Introduction**. The present work makes the novel suggestion that these merged geometries can nevertheless be later re-isolated through language. Future studies could thus explore whether and how language might isolate other foundational human knowledge in adult cognitive systems that seem unified, like a system of moral reasoning, which may rely on a merged representation of another person as both an intentional agent with goals and a phenomenal being with experiences and emotions (Gray et al., 2007; Knobe & Prinz, 2008; Spelke, 2022).

Third, the present findings demonstrate a striking resemblance between the geometry educated adults use to extrapolate the next point given points and lines on a plane, and the geometry they use to extrapolate the next stop of an agent, given the agent's previous stops and paths on a land. Consistent with the third proposal reviewed in the **Introduction**, these findings suggest a link between our basic abstract geometric intuitions about points and lines on the Euclidean plane and the specific geometric intuitions we call upon in contexts of navigation: Just as humans and other animals navigate places dynamically by distance and direction, participants in the *navigation* and *abstract* conditions preserved the distance and direction of the initial figure's first line segment to form open paths. Indeed, we humans may wander the abstract

Euclidean plane like we and other animals wander the physical world. Future studies might nevertheless explore the specificity of this link and the language required to elicit it. For example, since the visual displays themselves were dynamic, participants may have by default interpreted these visuals as displays of navigation, suggesting that other verbal prompts could elicit the same kinds of responses as the *navigation* and *abstract* conditions and/or that the associated "points" and "lines" verbal descriptions in the *abstract* condition may not have conveyed anything critical about abstract entities. While the *anywhere* condition served as a control for these possibilities, other control conditions using language like, "click what will come next," which highlight the dynamism of the visual displays, should be explored.

The link between navigating the physical world of everyday life and the abstract world of formal geometry suggested by the present study informs not only philosophical debates over the origins of geometry (Husserl, 1970/1954; Kant, 1998/1781), but also our interpretation of past empirical findings. For example, the present results are consistent with the navigationally grounded abstract inferences shown in a seminal study of a young blind child's ability to complete triangular paths in a novel environment (Landau et al., 1981). And, the present work suggests that describing abstract geometric concepts, such as zero-dimensional points and infinitely long lines, in a navigational context may be essential to eliciting accurate Euclidean judgements in adults across cultures (Izard, Pica, Spelke, et al., 2011). In particular, Izard et al. (2011; see also Dillon & Spelke, 2018; Huey et al., 2023) probed the intuitions of adults from the Amazon, who had received no formal education in geometry, about the general properties of points and lines, including probing concepts like linearity and parallelism, by asking them to imagine points as villages and lines as paths on a land. Future studies could examine whether the same accurate abstract intuitions could be elicited by instead grounding descriptions of abstract points and lines as the corners and sides of an object. An object context might be less likely to elicit abstract geometry. And, while such navigational descriptions successfully elicited adults' abstract intuitions, we might wonder whether previous studies like these are probing fully abstract geometry or merely eliciting intuitions about navigation.

A link between abstract geometry and navigation may also have implications for the development of geometry pedagogy and educational interventions. Future studies might thus explore the development of this link to determine whether children's responses to the *navigation* and *abstract* conditions resemble one another. This link might precede children's achievement of geometric reasoning consistent with Euclidean geometry or this link might emerge only after that achievement. In either case, teaching abstract geometry using navigational contexts might better serve math instruction (Dillon et al., 2017), which should also take into account how an individual's navigation experience (Coutrot et al., 2022; Newcombe et al., 2022) might affect their capacity for learning. Relatedly, future studies might explore whether and what specific properties of navigation—like straight-ahead motion without turning—might support the acquisition of foundational Euclidean concepts of linearity, parallelism, perpendicularity, and symmetry. Finally, in finding that a verbal description of a navigating agent uniquely elicits the same geometric information as a verbal description of abstract points and lines, the present study suggests that our explorations of abstract geometry often rely on our spatial navigation, suggesting, perhaps, that the origins of geometry may lie in dynamic mental simulations of moving through the physical world, akin to a correlated random walk of a navigating animal. Here, mathematical invention and education coincide, as we see on the pages of math textbooks that traditionally, since at least Gauss, imagine geometric manifolds as traversed by ants (Ault, 2018).

Constraints on Generality

Geometric representations for everyday navigation and object recognition are present across human cultures and languages, regardless of formal education, and with varied experiences of the physical world (Dehaene et al., 2006; Dillon et al., 2017; Heimler et al., 2021; Landau et al., 1981); they are also present in other animal species (Cheng et al., 2013; Spelke & Lee, 2012). Moreover, studies on human abstract geometry have shown striking consistency among the responses of formally educated adults from the United States and France, adults from the Amazon, where there is no formal education in geometry and no specialized geometric vocabulary, and even mathematics enthusiasts and professionals (Amalric & Dehaene, 2016; 2018; Butterworth, 2006; Hart et al., 2022; Izard et al., 2011). These previous findings suggest that our results with educated college-aged English-speaking adults may generalize to adults from other cultures, who speak different languages, and with varied educational and everyday experiences. Future studies should explore such differences to further inform the implementation of scalable mathematics education.

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