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ORIGINAL PAPER

A Necessary and Sufficient Condition for $(2d-2)$-Transversals in $\mathbb{R}^{2d}$

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Abstract

A *k-transversal* to a family of sets in $\mathbb{R}^d$ is a k-dimensional affine subspace that intersects each set of the family. In 1957 Hadwiger provided a necessary and sufficient condition for a family of pairwise disjoint, planar convex sets to have a 1-transversal. After a series of three papers among the authors Goodman, Pollack, and Wenger from 1988 to 1990, Hadwiger's Theorem was extended to necessary and sufficient conditions for $(d-1)$-transversals to finite families of convex sets in $\mathbb{R}^d$ with no disjointness condition on the family of sets. However, no such conditions for a finite family of convex sets in $\mathbb{R}^d$ to have a k-transversal for $0 < k < d-1$ have previously been proven or conjectured. We make progress in this direction by providing necessary and sufficient conditions for a finite family of convex sets in $\mathbb{R}^{2d}$ to have a $(2d-2)$-transversal.

Keywords Geometric Transversal Theory · Discrete Geometry · Convex Sets

Mathematics Subject Classification 52A35

1 Introduction

The well-known Helly's theorem [9] states that if a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ has the property that any choice of $d+1$ or fewer sets in $\mathcal{F}$ have a non-empty intersection, then there is a point in common to all the sets in $\mathcal{F}$ (see [3, 5] for surveys on Helly's theorem and related results). A *k-transversal* is a k-dimensional affine space that intersects each set of $\mathcal{F}$, so Helly's theorem provides a necessary and sufficient

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condition for $\mathcal{F}$ to have a 0-transversal. In 1935, Vincensini was interested in the natural extension of Helly's theorem of finding necessary and sufficient conditions for a finite family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ to have a k-transversal for $k > 0$ [15]. In particular, Vincensini asked if there exists some constant $r = r(k, d)$ such that if every choice of r or fewer sets in $\mathcal{F}$ has a k-transversal, then $\mathcal{F}$ has a k-transversal. However, Santaló provided examples showing that such a constant r does not exist for any $k > 0$ [14]. Many other related problems in geometric transversal theory have also been considered. For more information we refer the reader to the surveys [5, 7].

In 1957, Hadwiger made the first positive progress toward this extension of Helly's theorem considered by Vincensini by proving the following theorem.

Theorem 1.1 (Hadwiger [8]) *A finite family of pairwise disjoint convex sets in $\mathbb{R}^2$ has a 1-transversal if and only if the sets in the family can be linearly ordered such that any three sets have a 1-transversal consistent with the ordering.*

Hadwiger's theorem has been generalized in different ways, eventually resulting in an encompassing result for $(d-1)$-transversals in $\mathbb{R}^d$. The first significant result in this direction was made by Goodman and Pollack who showed that in $\mathbb{R}^d$, the linear ordering in Hadwiger's theorem can be replaced with the notion of an order type of points in $\mathbb{R}^{d-1}$ given the additional condition that the family is $(d-2)$-separable, which generalizes the disjointness condition in Hadwiger's theorem (see [6] for the precise statement of the theorem and definitions of these notions). Soon after, Wenger showed that the disjointness condition of Hadwiger's theorem can be dropped [16]. Finally, Pollack and Wenger completed the picture by proving the necessary and sufficient conditions required to have a $(d-1)$-transversal in $\mathbb{R}^d$ with no additional separability conditions on the family of sets [13]. We note that several extensions of this result, including colorful generalizations, have been studied for instance in [1, 2, 4, 10, 11].

Despite the previous work on the existence of $(d-1)$-transversals, no necessary and sufficient conditions for the existence of k-transversals in $\mathbb{R}^d$ for $0 < k < d-1$ have been proven or conjectured. In this paper, we make progress in this direction by providing necessary and sufficient conditions for $(2d-2)$-transversals in $\mathbb{R}^{2d}$.

2 Hyperplane Transversals Revisited

Here we will describe the result of Pollack and Wenger on $(d-1)$-transversals in $\mathbb{R}^d$ as presented in [5], then we will discuss an equivalent rephrasing of this theorem to put our main result in Sect. 3 into context.

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ and let P be a subset of points in $\mathbb{R}^k$ for some k. We say that $\mathcal{F}$ *separates consistently* with P if there exists a map $\phi : \mathcal{F} \to P$ such that for any two subfamilies $\mathcal{F}_1,\ \mathcal{F}_2 \subset \mathcal{F}$, we have that

$$\operatorname{conv}(\mathcal{F}_1) \cap \operatorname{conv}(\mathcal{F}_2) = \emptyset \implies \operatorname{conv}(\phi(\mathcal{F}_1)) \cap \operatorname{conv}(\phi(\mathcal{F}_2)) = \emptyset.$$

Here we mean $\operatorname{conv}(\mathcal{F}_i)$ to be $\operatorname{conv}(\cup_{F \in \mathcal{F}_i} F)$. Another way to think about this condition is that if the sets of $\mathcal{F}_1$ can be separated from the sets of $\mathcal{F}_2$ by a hyperplane in

$\mathbb{R}^d$, then the sets of points $\phi(\mathcal{F}_1)$ and $\phi(\mathcal{F}_2)$ can be separated by a hyperplane in $\mathbb{R}^k$. We also note that $\mathcal{F}$ separates consistently with P if and only if

$$\mathrm{conv}(\mathcal{F}_1) \cap \mathrm{conv}(\mathcal{F}_2) = \emptyset \implies \mathrm{conv}(\phi(\mathcal{F}_1)) \cap \mathrm{conv}(\phi(\mathcal{F}_2)) = \emptyset.$$

whenever $|\mathcal{F}_1| + |\mathcal{F}_2| \leq k+2$. This is a consequence of the well-known Kirchberger's theorem [12], which states that if U and V are finite point sets in $\mathbb{R}^k$ such that for every set of $k+2$ points $S \subset U \cup V$, we have that $\mathrm{conv}(S \cap U) \cap \mathrm{conv}(S \cap V) = \emptyset$, then $\mathrm{conv}(U) \cap \mathrm{conv}(V) = \emptyset$.

We now have the terminology to state the Goodman–Pollack–Wenger theorem.

Theorem 2.1 (Goodman–Pollack–Wenger theorem [13]) *A finite family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ has a $(d-1)$-transversal if and only if $\mathcal{F}$ separates consistently with a set $P \subset \mathbb{R}^{d-1}$.*

The condition in our main result of Sect. 3 is quite similar to the condition in Theorem 2.1, and we will first provide a slight rephrasing of the definition for $\mathcal{F}$ to separate consistently with P in order to make this similarity more apparent. By taking the contrapositive of the implication in the definition of separating consistently, we may equivalently say that $\mathcal{F}$ separates consistently with P if there exists a map $\phi : \mathcal{F} \to P \subset \mathbb{R}^k$ such that

$$\mathrm{conv}(\phi(\mathcal{F}_1)) \cap \mathrm{conv}(\phi(\mathcal{F}_2)) \neq \emptyset \implies \mathrm{conv}(\mathcal{F}_1) \cap \mathrm{conv}(\mathcal{F}_2) \neq \emptyset.$$

In other words, the existence of an affine dependence

$$\sum_{F \in \mathcal{F}_1 \cup \mathcal{F}_2} a_F = 0, \quad \sum_{F \in \mathcal{F}_1 \cup \mathcal{F}_2} a_F \phi(F) = 0$$

where $a_F \geq 0$ for all $F \in \mathcal{F}_1$ (not all 0) and $a_F \leq 0$ for all $F \in \mathcal{F}_2$ implies the existence of points $p_F \in F$ and real numbers $r_F \geq 0$ such that

$$\sum_{F \in \mathcal{F}_1 \cup \mathcal{F}_2} r_F a_F = 0, \quad \sum_{F \in \mathcal{F}_1 \cup \mathcal{F}_2} (r_F a_F) p_F = 0$$

is an affine dependence of the points p_F and the numbers $r_F a_F$ are not all 0.

3 Main Result

In order to simplify the statement of our main result, we will state our theorem as a necessary and sufficient condition for a finite family $\mathcal{F}$ of convex sets in $\mathbb{C}^d$ to have a *complex* $(d-1)$-transversal, where a complex $(d-1)$-transversal here is a complex $(d-1)$-dimensional affine subspace of $\mathbb{C}^d$ that intersects each set in $\mathcal{F}$. Since $\mathbb{C}^d$ can be identified with $\mathbb{R}^{2d}$ and a complex $(d-1)$-transversal can subsequently be identified with a $(2d-2)$-transversal in $\mathbb{R}^{2d}$, our result will equivalently provide a necessary and sufficient condition for a finite family of convex sets $\mathcal{F}$ in $\mathbb{R}^{2d}$ to have a $(2d-2)$-transversal.

First, following our discussion from Sect. 2, we make the following definition in order to articulate our main theorem, Theorem 3.4.

Definition 3.1 Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{C}^d$, and let $P \subset \mathbb{C}^k$. We say that $\mathcal{F}$ is *dependency-consistent* with P if there exists a map $\phi : \mathcal{F} \to P$ such that for every subfamily $\mathcal{F}' \subset \mathcal{F}$ and every affine dependence

$$\sum_{F \in \mathcal{F}'} a_F = 0, \ \sum_{F \in \mathcal{F}'} a_F \phi(F) = 0$$

for complex numbers a_F, there exist real numbers $r_F \geq 0$ and points $p_F \in F$ for $F \in \mathcal{F}'$ such that

$$\sum_{F \in \mathcal{F}'} r_F a_F = 0, \ \sum_{F \in \mathcal{F}'} (r_F a_F) p_F = 0$$

where not all of the values $r_F a_F$ are 0.

Remark 3.2 Note that we could add the additional restriction that $|\mathcal{F}'| \leq 2k+3$ in Definition 3.1, and we would get an equivalent definition. Indeed, by associating the points $(a_F \phi(F), a_F)$ with points in $\mathbb{R}^{2k+2}$, we have that the set of points $\{(a_F \phi(F), a_F)\}_{F \in \mathcal{F}'}$ contains $0 \in \mathbb{R}^{2k+2}$ in its convex hull. Therefore, by Carathéodory's Theorem, there exist $m \leq 2k+3$ sets $F_1, \ldots, F_m \in \mathcal{F}'$ and real numbers $s_i > 0$ such that $\sum_{i=1}^m s_i (a_{F_i} \phi(F_i), a_{F_i}) = 0$. In other words, there is the complex affine dependence

$$\sum_{i=1}^m s_i a_{F_i} = 0, \ \sum_{i=1}^m (s_i a_{F_i}) \phi(F_i) = 0$$

among the points $\phi(F_1), \ldots, \phi(F_m)$.

In our proof of Theorem 3.4 will make use of the Borsuk–Ulam theorem below. We note that the Borsuk–Ulam theorem was also employed in the proof of the Goodman–Pollack–Wenger theorem in [13], and in fact our proof of Theorem 3.4 takes significant inspiration from this proof.

Theorem 3.3 (Borsuk–Ulam theorem) *If $n \geq m$ and $f : S^n \to \mathbb{R}^m$ is an odd, continuous map, i.e. $f(-x) = -f(x)$ for all $x \in S^n$, then $0 \in Im(f)$.*

We are now ready to state and prove the main result. The non-trivial direction, and the contents of the following proof, is the "if" part of the statement. The "only if" direction is straightforward since a complex $(d-1)$-transversal can be associated with $\mathbb{C}^{d-1}$, and P can be constructed by choosing a point in the intersection of a set from $\mathcal{F}$ and the complex $(d-1)$-transversal for each set in $\mathcal{F}$.

Theorem 3.4 (Main theorem) *A finite family of convex sets $\mathcal{F}$ in $\mathbb{C}^d$ has a complex $(d-1)$-transversal if and only if $\mathcal{F}$ is dependency-consistent with a set $P \subset \mathbb{C}^{d-1}$.*

Proof We associate $\mathbb{C}^d$ with the set of points

$$H = \{(z_1, \ldots, z_{d+1}) \in \mathbb{C}^{d+1} \mid z_{d+1} = 1\}.$$

Thus, we think of the convex sets from the family $\mathcal{F}$ as lying in H as well.

We will now define a continuous odd map f from $\{\mathbf{z} \in \mathbb{C}^{d+1} \mid ||\mathbf{z}|| = 1\}$, which can be identified with the $(2d+1)$-dimensional sphere, S^{2d+1}, to $\mathbb{R}^{2d}$. Such a map will have a zero by Theorem 3.3, and we will show that for the map that we define, this zero will correspond to a complex $(d-1)$-transversal of $\mathcal{F}$.

Let $\mathbf{x} \in \{\mathbf{z} \in \mathbb{C}^{d+1} \mid ||\mathbf{z}|| = 1\}$, and let $P_\mathbf{x}$ be the complex subspace spanned by $\mathbf{x}$. Additionally, for a set $F \in \mathcal{F}$ we denote the orthogonal projection of F onto $P_\mathbf{x}$ by $F_\mathbf{x}$; note that $F_\mathbf{x}$ is a convex set.

For each set $F \in \mathcal{F}$, let $p_{\mathbf{x},F}$ be the complex number c such that $c\mathbf{x}$ is the element of $F_\mathbf{x}$ that is closest in distance to $\mathbf{0}$, the origin in $\mathbb{C}^{d+1}$. Note that the convexity of $F_\mathbf{x}$ implies that c is unique. Furthermore, for a fixed set F, the point $p_{\mathbf{x},F}$ varies continuously with $\mathbf{x}$. Let ϕ be the map witnessing the fact that $\mathcal{F}$ is dependency-consistent with P; we define $f(\mathbf{x}) \in \mathbb{C} \times \mathbb{C}^{d-1}$, which can be identified with $\mathbb{R}^{2d}$, as follows

$$f(\mathbf{x}) = \sum_{F \in \mathcal{F}} \left(p_{\mathbf{x},F},\ \overline{p_{\mathbf{x},F}}\phi(F)\right),$$

where $\overline{p_{\mathbf{x},F}}$ denotes the complex conjugate of $p_{\mathbf{x},F}$. Because $p_{\mathbf{x},F} = -p_{-\mathbf{x},F}$ and $p_{\mathbf{x},F}$ varies continuously with $\mathbf{x}$, we have that f is indeed a continuous odd map and thus has a zero, say $\mathbf{x}_0$, by Theorem 3.3.

It cannot be the case that $\mathbf{x}_0 = (0, \ldots, 0, z_{d+1})$ for some $z_{d+1} \neq 0$, since in that case, each F_x is the point $(0, \ldots, 0, 1)$ and thus $p_{x_0,F} = 1/z_{d+1}$ for every $F \in \mathcal{F}$. If $p_{x_0,F} = 1/z_{d+1}$ for each F, then clearly $\sum_{F \in \mathcal{F}} p_{x_0,F} \neq 0$. Thus, $\mathbf{x}_0 \neq (0, \ldots, 0, z_{d+1})$, and in particular, this implies that the orthogonal complement of $\mathbf{x}_0$ intersects H in a complex $(d-1)$-dimensional affine space, say T. If T intersects each set in $\mathcal{F}$, then $\mathcal{F}$ has a complex $(d-1)$-transversal, and we are done. Therefore, we assume that T does not intersect each set in $\mathcal{F}$, so we have that some of the values $p_{\mathbf{x}_0,F}$ must be nonzero. We show that this assumption leads to a contradiction.

Let $\mathcal{F}' \subset \mathcal{F}$ be the family of sets F such that $p_{\mathbf{x}_0,F} \neq 0$. Since $f(\mathbf{x}_0) = 0$, we have the affine dependence

$$\sum_{F \in \mathcal{F}'} p_{\mathbf{x}_0,F} = 0,\ \sum_{F \in \mathcal{F}'} \overline{p_{\mathbf{x}_0,F}}\phi(F) = 0.$$

Since $\mathcal{F}$ is dependency-consistent with P, we have that there exist points $p_F \in F$ for all $F \in \mathcal{F}'$ and real numbers $r_F \geq 0$ such that

$$\sum_{F \in \mathcal{F}'} r_F p_{\mathbf{x}_0,F} = 0,\ \sum_{F \in \mathcal{F}'} \overline{r_F p_{\mathbf{x}_0,F}} p_F = 0.$$

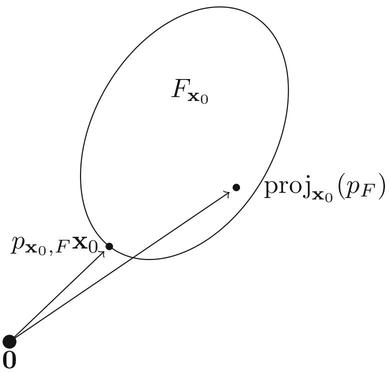

Fig. 1 Depiction of $F_{\mathbf{x}_0}$, $p_{\mathbf{x}_0,F}\mathbf{x}_0$, and $\mathrm{proj}_{\mathbf{x}_0}(p_F)$ in the complex plane $P_{\mathbf{x}_0}$

Now, consider the $\mathbb{C}$-linear map $\mathrm{proj}_{\mathbf{x}_0} : \mathbb{C}^{d+1} \to P_{\mathbf{x}_0}$ given by orthogonal projection onto $P_{\mathbf{x}_0}$. By linearity, we have that

$$\mathrm{proj}_{\mathbf{x}_0}\left(\sum_{F \in \mathcal{F}'} \overline{r_F\, p_{\mathbf{x}_0,F}}\, p_F\right) = \sum_{F \in \mathcal{F}'} \overline{r_F\, p_{\mathbf{x}_0,F}}\mathrm{proj}_{\mathbf{x}_0}(p_F) = 0.$$

However, this immediately leads to a contradiction. Indeed, recall that $p_{\mathbf{x}_0,F}\mathbf{x}_0$ is the point of $F_{\mathbf{x}_0}$ closest to $\mathbf{0}$ and $F_{\mathbf{x}_0} \ni \mathrm{proj}_{\mathbf{x}_0}(p_F)$ is a convex set. This implies that the angle between $p_{\mathbf{x}_0,F}\mathbf{x}_0$ and $\mathrm{proj}_{\mathbf{x}_0}(p_F)$ is less than 90° (see Fig. 1). Therefore, writing $\mathrm{proj}_{\mathbf{x}_0}(p_F) = c_F\mathbf{x}_0$, we have that the absolute difference in argument between the complex numbers $p_{\mathbf{x}_0,F}$ and c_F is less than 90°, which implies that $\mathrm{Re}(\overline{p_{\mathbf{x}_0,F}}c_F) > 0$. This is in contradiction to the fact that $\sum_{F\in\mathcal{F}'} \overline{r_F\, p_{\mathbf{x}_0,F}}\mathrm{proj}_{\mathbf{x}_0}(p_F) = 0$, and hence completes the proof. $\square$

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