Graphs and Combinatorics (2023) 39:86
https://doi.org/10.1007/s00373-023-02677-1

ORIGINAL PAPER

Isomorphisms and Properties of TAR Graphs for Zero Forcing and Other X-set Parameters

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Received: 16 October 2022 / Revised: 8 June 2023 / Accepted: 17 June 2023 /
Published online: 19 July 2023

Abstract

In a TAR (token addition/removal) reconfiguration graph, each vertex is a set of vertices, with an edge between two vertices if one can be obtained from the other by adding or removing one element. This paper considers the X-TAR graph of a base graph G where the vertices of the X-TAR graph of G are the X-sets of G, which are subsets of $V(G)$ that satisfy certain conditions. Dominating sets, power dominating sets, zero forcing sets, and positive semidefinite zero forcing sets are all examples of X-sets. For graphs G and G' with no isolated vertices, it is shown that G and G' have isomorphic X-TAR graphs if and only if there is a relabeling of the vertices of G' such that G and the relabeled G' have exactly the same X-sets. These results are applied to show certain zero forcing TAR graphs are unique up to isomorphism of the base graph. Furthermore, results related to the connectedness of the zero forcing TAR graph are presented.

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Keywords Reconfiguration · Token addition and removal · TAR · Vertex · X-set · Isomorphism · X-irrelevant · Zero forcing

Mathematics Subject Classification 05C69 · 68R10 · 05C50 · 05C57 · 05C60

1 Introduction

The study of reconfiguration examines relationships among solutions to a problem. These solutions are modeled as vertices in a graph called a *reconfiguration graph*. A *reconfiguration rule* describes the adjacency relationship in a reconfiguration graph. There is a growing body literature about reconfiguration graphs in various contexts. Typical questions addressed are structural (whether a reconfiguration graph is connected, Hamiltonian, etc.), existence/uniqueness (whether a given graph can be a reconfiguration graph of some base graph, and when a reconfiguration graph determines its base graph) and algorithmic. A recent survey paper by Mynhardt and Nasserasr [15] describes the current state of understanding of the graph properties of reconfiguration graphs, while a survey by Nishimura [17] discusses algorithmic and complexity questions in a wide variety of reconfiguration settings.

For many graph theory problems, the solutions to be reconfigured are sets of vertices of a *base graph* G defined by a property such as being a dominating set, power dominating set, or zero forcing set of G. In such cases one natural and widely studied reconfiguration rule is the *token addition or removal rule* in which two sets S and S' are adjacent in the *TAR reconfiguration graph* (or *TAR graph*) if S' can be obtained from S by adding or removing one element. We focus on TAR reconfiguration graphs. Properties of the TAR graph for dominating sets have been studied in [1, 2, 12, 13, 15, 16]. Different types of reconfiguration graphs for the parameters discussed here have also been studied: the token exchange (or token jumping) reconfiguration graph for dominating sets in [15, 18, 20] and for zero forcing sets in [11]; the token sliding reconfiguration graph for dominating sets in [10, 15]. Reconfiguration problems for irredundant sets and for graph colorings have also been studied (see, for example, the survey [15] and and the references therein) but these are less closely related to the work described here.

In [5], Bjorkman et al. introduced a universal framework for parameters that we call X-set parameters (see Definition 1.2), and we call the associated TAR graphs X-TAR graphs. Examples of X-set parameters include the domination number, the power domination number, the zero forcing number, and the positive semidefinite zero forcing number (definitions of these parameters are presented in Sect. 1.1), so results proved for X-set parameters apply to all these parameters. Various elementary consequences of the universal definition were established in [5] (see Theorem 1.9 for an application of such results to the zero forcing TAR-graph). Here we establish deeper results about X-TAR graphs, specifically about isomorphisms of X-TAR graphs.

For a given graph G and X-set parameter, the X-TAR graph of G is completely determined by the X-sets of G, which are the vertices of the X-TAR graph of G. Questions naturally arise regarding when X-TAR graphs of base graphs G and G' are isomorphic and what this says about G and G'. Obviously two base graphs with the same X-sets have isomorphic X-TAR graphs.[1] One of our main results is the converse of this statement (when isolated vertices are excluded):

Theorem 1.1 *Suppose base graphs G and G' have no isolated vertices and their X-TAR graphs are isomorphic. Then there is a relabeling of the vertices of G' such that G and G' have exactly the same X-sets.*

Theorem 1.1 is an immediate consequence of Theorems 2.8 and 2.9 proved below. The omission of isolated vertices does not significantly weaken the result, because the effect of isolated vertices in the base graph on the X-TAR graph is easy to describe (see Remark 1.3), and modulo that caveat this is the strongest possible result. Finding a relabeling to obtain the same X-sets for G and G' is likely not any easier than finding an isomorphism of X-TAR graphs. However, we did find this result a bit surprising. Of course, an isomorphism of X-TAR graphs can be viewed as a relabeling of the vertices of one of the *TAR graphs*, but there does not seem to be any a priori reason why the existence of a relabeling of the vertices of one of the *base graphs* so as to have exactly the same X-sets should be a necessary condition for an isomorphism of X-TAR graphs. Theorem 1.1 applies to the domination number, the power domination number, the zero forcing number, and the positive semidefinite zero forcing number.

To prove Theorems 2.8 and 2.9, we introduce the concept of an X-irrelevant vertex (or an X-irrelevant set of vertices) in Sect. 2.2. A vertex is *X-irrelevant* if it does not appear in any minimal X-set. In Theorem 2.13, X-irrelevant sets are used to characterize the automorphism group of an X-TAR graph. This characterization implies that the X-TAR graph of G has an automorphism that maps an X-set to an X-set of different cardinality if and only if G has at least one X-irrelevant vertex. The existence (or nonexistence) of X-irrelevant vertices highlights significant differences among the graph parameters discussed: There are graphs with X-irrelevant vertices for X-sets being zero forcing sets or power dominating sets, whereas domination and positive semidefinite zero forcing do not allow X-irrelevant vertices, because any vertex of a base graph G can be in a minimal dominating set of G, and similarly for positive semidefinite zero forcing (see Corollaries 2.18 and 2.20). This means that any automorphism of a dominating TAR graph or a positive semidefinite zero forcing TAR graph fixes the cardinalities of the vertices of the TAR graph.

Theorems 2.8 and 2.9 are useful in the study of which X-TAR graphs are unique (up to isomorphism of the base graph and with isolated vertices prohibited). The details are parameter-specific, and we examine this question for zero forcing TAR graphs in Sect. 3.1.

Many studies of reconfiguration for a specific parameter address connectedness, that is, whether any one feasible solution can be transformed into any other by the reconfiguration rule. Unfortunately, connectedness properties seem to be very parameter-specific, and thus unsuitable for the universal approach using X-TAR graphs. The main results in this paper relate to isomorphisms of X-TAR graphs, as discussed in Sect. 2. However, in Sect. 3.2 we present results related to connectedness for zero forcing TAR graphs. This includes exhibiting families of graphs that

[1] Note that nonisomorphic base graphs can have the same X-sets for each of the X-set parameters discussed here.

exceed known bounds for parameters describing connectedness of the zero forcing TAR graphs (see Propositions 3.4 and 3.9).

In the next section we present basic graph theory definitions and precise definitions of zero forcing number, positive semidefinite zero forcing number, domination number, and power domination number. Section 1.2 contains the definitions of an X-set parameter, its TAR graph, and related parameters. In Sect. 1.3 we apply basic results from [5] to the zero forcing TAR graph and determine the zero forcing TAR graphs for several graph families.

1.1 Definitions and Notation

All graphs are simple, undirected, finite, and have nonempty vertex sets. Let $G = (V(G), E(G))$ be a graph. The *neighborhood* of a vertex v of G is the set of all vertices adjacent to v and is denoted by $N_G(v)$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. If S is a set of vertices of G, then $N_G[S] = \cup_{x\in S} N_G[x]$. The *order* of a graph G is the cardinality of $V(G)$. The *maximum degree* and *minimum degree* of G are denoted $\Delta(G)$ and $\delta(G)$ respectively.

Let G be a graph and color all the vertices of G blue or white. Zero forcing is a process that changes the color of white vertices to blue by applying the *standard color change rule*, i.e., a blue vertex v can force a white vertex w to change color to blue if w is the only white vertex in the neighborhood of v. A *zero forcing set* is a set $S \subseteq V(G)$ that can result in all the vertices of G turning blue by repeated application of the standard color change rule starting with exactly the vertices in S blue. In the context of TAR reconfiguration, "solutions" to zero forcing on a graph G are zero forcing sets. The *zero forcing number* of G, denoted by $\operatorname{Z}(G)$, is the minimum cardinality of a zero forcing set. The zero forcing TAR graph allows us to consider all the zero forcing sets (of various sizes) and the relationships among them for a particular base graph. It is well known (and easy to see) that $\delta(G) \le \operatorname{Z}(G)$. The study of zero forcing reconfiguration was initiated in [11], where the token exchange zero forcing reconfiguration graph was studied. Note that for the zero forcing token exchange reconfiguration graph, only minimum zero forcing sets are used as vertices and two such sets are adjacent if one can be obtained from the other by exchanging one vertex.

Positive semidefinite zero forcing is a variant of zero forcing defined by the *PSD color change rule*: Let S be the set of blue vertices and let $W_1, \dots, W_k$ be the sets of vertices of the $k \ge 1$ components of $G - S$ (the graph obtained from G by deleting the vertices in S). If $u \in S$, $w \in W_i$, and w is the only white neighbor of u in $G[W_i \cup S]$, then change the color of w to blue (where $G[U]$ denotes the subgraph of G induced by the vertices of U). A *PSD zero forcing set* is a set $S \subseteq V(G)$ that can result in all the vertices of G turning blue by repeated application of the PSD color change rule starting with exactly the vertices in S blue. The *PSD zero forcing number* of G, denoted by $\operatorname{Z}_+(G)$, is the minimum cardinality of a PSD zero forcing set. More information on zero forcing and positive semidefinite zero forcing can be found in [14].

A set S of vertices of a graph G is a *dominating set* of G if $N_G[S] = V(G)$ and the *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A set S of vertices of a graph G is a *power dominating set* of G if $N_G[S]$ is a zero forcing set of G, and the *power domination number* $\gamma_P(G)$ of G is the minimum cardinality of a power dominating set of G. The γ-TAR graph $\mathscr{D}^{\mathrm{TAR}}(G)$ was studied in [1, 2, 12]. Bjorkman et al. introduced the study of the γ_P-TAR graph $\mathscr{P}\mathscr{D}^{\mathrm{TAR}}(G)$ in [5], where it was observed that some proofs for the domination TAR graph remained valid for power domination. This led to the rewriting of these proofs in universal notation similar to that used here.

1.2 *X*-Set Parameter Definitions and Initial Results

The next definition is adapted from [5] (the term X-*set parameter* is new but the list of conditions in Definition 1.2 is taken from [5, Definition 2.1]).

Definition 1.2 An X-*set parameter* is a graph parameter $X(G)$ defined to be the minimum cardinality of an X-set of G, where the X-sets of G are defined by a given property and satisfy the following conditions:

(1) If S is an X-set of G and $S \subseteq S'$, then S' is an X-set of G.

(2) The empty set is not an X-set of any graph.

(3) An X-set of a disconnected graph is the union of an X-set of each component.

(4) If G has no isolated vertices, then every set of $|V(G)| - 1$ vertices is an X-set.

Note that $\mathrm{Z}(G)$, $\mathrm{Z}_+(G)$, $\gamma(G)$, and $\gamma_P(G)$ are all X-set parameters. Let X be an X-set parameter. The following definitions are adapted from [5]. The X token addition and removal reconfiguration graph (X-*TAR graph)* of a base graph G, denoted by $\mathscr{X}^{\mathrm{TAR}}(G)$, has the set of all X-sets of G as the set of vertices, and there is an edge between two vertices S_1 and S_2 of $\mathscr{X}^{\mathrm{TAR}}(G)$ if and only if $|S_1 \ominus S_2| = 1$ where $A \ominus B = (A \cup B) \setminus (A \cap B)$ denotes the *symmetric difference* of sets A and B.

As noted in [5], it is sufficient to study X-TAR reconfiguration graph for base graphs with no isolated vertices. The disjoint union of graphs G and H is denoted by $G \sqcup H$ and rK_1 denotes r isolated vertices.

Remark 1.3 [5] Let X be a property satisfying the conditions in Definition 1.2. If $G' = G \sqcup rK_1$, G has no isolated vertices and the isolated vertices of G' are denoted by $z_1, \ldots, z_r$, and then the mapping $S \to S \cup \{z_1, \ldots, z_r\}$ is a bijection of X-sets of G to the X-sets of G' and $\mathscr{X}^{\mathrm{TAR}}(G') \cong \mathscr{X}^{\mathrm{TAR}}(G)$.

As mentioned above, determining when a TAR graph (or any reconfiguration graph) is connected is important because it corresponds to determining if all solutions can be reached from each other. Observe that $\mathscr{X}^{\mathrm{TAR}}(G)$ is connected for any graph G. The k-*token addition and removal reconfiguration graph for* X of G, denoted by $\mathscr{X}_k^{\mathrm{TAR}}(G)$, is the subgraph of $\mathscr{X}^{\mathrm{TAR}}(G)$ induced by the set of all X-sets of cardinality at most k. Thus determining for which k the graph $\mathscr{X}_k^{\mathrm{TAR}}(G)$ is connected is a main theme of the study of TAR graph reconfiguration. Connectedness is also a theme of other types of reconfiguration.

The least k such that $\mathscr{X}_k^{\mathrm{TAR}}(G)$ is connected is denoted by $\underline{x_0}(G)$. The least k such that $\mathscr{X}_i^{\mathrm{TAR}}(G)$ is connected for $k \le i \le |V(G)|$ is denoted by $x_0(G)$. The maximum cardinality of a minimal X-set of G is the *upper X-number* of G and is denoted by $\overline{X}(G)$.

Let G be a graph of order $n \ge 2$ with no isolated vertices. As noted in [5], $\Delta(\mathscr{X}^{\mathrm{TAR}}(G)) = n$. Observe that an X-set $S \neq V(G)$ of G has $\deg_{\mathscr{X}_{|S|+1}^{\mathrm{TAR}}(G)}(S) = n$ if and only if $S \setminus \{x\}$ is an X-set for every $x \in S$.

In order to reduce consideration of a problem to connected base graphs, we establish the form of the X-TAR graph of a of disconnected graph. If G_1 and G_2 are disjoint, the *Cartesian product* of G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with $V(G \square G') = V(G_1) \times V(G_2)$ such that (v_1, v_2) and (u_1, u_2) are adjacent if and only if (1) $v_1 = u_1$ and $v_2 u_2 \in E(G_2)$, or (2) $v_2 = u_2$ and $v_1 u_1 \in E(G_1)$.

Proposition 1.4 *Let X be an X-set parameter and let $G = G_1 \sqcup G_2$. Then $\mathscr{X}^{\mathrm{TAR}}(G) \cong \mathscr{X}^{\mathrm{TAR}}(G_1) \,\square\, \mathscr{X}^{\mathrm{TAR}}(G_2)$.*

Proof Let S and T be X-sets of G. Then $S = S_1 \sqcup S_2$ and $T = T_1 \sqcup T_2$, where S_1 and T_1 are zero forcing sets of G_1, and S_2 and T_2 are zero forcing sets of G_2. Observe that S and T are adjacent in $\mathscr{X}^{\mathrm{TAR}}(G)$ if and only if there exists a vertex $v \in V(G_1) \sqcup V(G_2)$ such that

- $S_1 = T_1$ and ($T_2 = S_2 \setminus \{v\}$ or $S_2 = T_2 \setminus \{v\}$), or
- $S_2 = T_2$ and ($T_1 = S_1 \setminus \{v\}$ or $S_1 = T_1 \setminus \{v\}$).

Further, $T_2 = S_2 \setminus \{v\}$ or $S_2 = T_2 \setminus \{v\}$ if and only if S_2 is adjacent to T_2 in $\mathscr{X}^{\mathrm{TAR}}(G_2)$, and $T_1 = S_1 \setminus \{v\}$ or $S_1 = T_1 \setminus \{v\}$ if and only S_1 is adjacent to T_1 in $\mathscr{X}^{\mathrm{TAR}}(G_1)$. $\square$

1.3 Initial Observations on Zero Forcing TAR Graphs

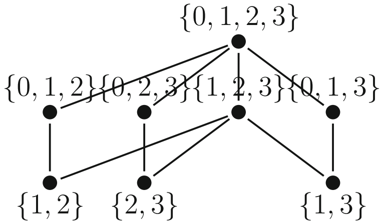
In this section, we implement definitions in Sect. 1.2 for zero forcing.

Definition 1.5 The *zero forcing token addition and removal* (Z*z-TAR*) *zgraph* of a base graph G, denoted by $\mathscr{Z}^{\mathrm{TAR}}(G)$, has the set of all zero forcing sets of G as the set of vertices, and there is an edge between two vertices S_1 and S_2 of $\mathscr{Z}^{\mathrm{TAR}}(G)$ if and only if $|S_1 \ominus S_2| = 1$. The *zero forcing k-token addition and removal reconfiguration graph* of G, denoted by $\mathscr{Z}_k^{\mathrm{TAR}}(G)$, is the subgraph of $\mathscr{Z}^{\mathrm{TAR}}(G)$ induced by the set of all zero forcing sets of cardinality at most k.

Definition 1.6 The maximum cardinality of a minimal zero forcing set of a graph G is the *upper zero forcing number* of G and is denoted by $\overline{\mathrm{Z}}(G)$. The least k such that $\mathscr{Z}_k^{\mathrm{TAR}}(G)$ is connected is denoted by $\underline{z_0}(G)$. The least k such that $\mathscr{Z}_i^{\mathrm{TAR}}(G)$ is connected for each $i = k, \dots, |V(G)|$ is denoted by $z_0(G)$.

It was shown in [4] that a graph with no isolated vertices has more than one minimum zero forcing set and no vertex is in every minimum zero forcing set. Let G be a graph that has no isolated vertices. Since a minimal zero forcing set of cardinality k has no neighbors in $\mathscr{Z}_k^{\mathrm{TAR}}(G)$, $\mathscr{Z}_k^{\mathrm{TAR}}(G)$ is not connected whenever there is a minimal zero forcing set of cardinality k. In particular, $\mathscr{Z}_{\mathrm{Z}(G)}^{\mathrm{TAR}}(G)$ and $\mathscr{Z}_{\overline{\mathrm{Z}}(G)}^{\mathrm{TAR}}(G)$ are not connected.

Fig. 1 $\mathscr{Z}^{\mathrm{TAR}}(K_{1,3})$



Brimkov and Carlson constructed an example of an infinite family of graphs G_n with $\mathrm{Z}(G_n) = 5$ and $\overline{\mathrm{Z}}(G_n) = n - 2$ in [7].

Next we determine the zero forcing TAR graphs and related parameters for two families of graphs (additional families are discussed in Sect. 3.) The complete graph is denoted by K_n and $K_{1,r}$ denotes the star with r leaves (where a *leaf* is a vertex of degree one); the *center vertex* of $K_{1,r}$ is the vertex of degree r.

Example 1.7 Let $n \geq 2$. It is well known (and immediate from $\delta(K_n) = n - 1$) that $\mathrm{Z}(K_n) = n - 1$ (implying $\overline{\mathrm{Z}}(K_n) = n - 1$) and the zero forcing sets of K_n consist of all subsets of vertices of K_n that contain at least $n - 1$ vertices. Thus $\mathscr{Z}^{\mathrm{TAR}}(K_n) = K_{1,n}$. Since $\mathscr{Z}^{\mathrm{TAR}}_{\mathrm{Z}(G)}(G)$ is disconnected for any connected graph of order at least two, $z_0(K_n) = \underline{z_0}(K_n) = n$.

Figure 1 illustrates the next result. Throughout this paper, we discuss both base graphs and their TAR graphs. To avoid confusion, we use open circles for vertices of a base graph (with the label written in the circle if a label is present) and black dots for vertices of TAR reconfiguration graphs, with the vertex labels (which are sets of base graph vertices) written nearby.

Proposition 1.8 *Let $r \geq 2$. Then $\mathscr{Z}^{\mathrm{TAR}}(K_{1,r}) \cong K_{1,r} \,\square\, K_2$. Furthermore, $\overline{\mathrm{Z}}(K_{1,r}) = \mathrm{Z}(K_{1,r}) = r - 1$ and $z_0(K_{1,r}) = \underline{z_0}(K_{1,r}) = r$.*

Proof Let $V(K_{1,r}) = \{0, 1, \ldots, r\}$ where 0 is the center vertex. The collection of zero forcing sets of $K_{1,r}$ can be partitioned as

$$T_1 = \{V(K_{1,r})\} \cup \{S \subseteq V(K_{1,r}) : |S| = r \text{ and } 0 \in S\}$$

and

$$T_2 = \{V(K_{1,r}) \setminus \{0\}\} \cup \{S \subseteq V(K_{1,r}) : |S| = r - 1 \text{ and } 0 \notin S\}.$$

Observe that the induced subgraph of $\mathscr{Z}^{\mathrm{TAR}}(K_{1,r})$ on either T_1 or T_2 is $K_{1,r}$. Further, zero forcing sets $S_1 \in T_1$ and $S_2 \in T_2$ are adjacent in $\mathscr{Z}^{\mathrm{TAR}}(G)$ if and only if $S_2 = S_1 \setminus \{0\}$. Thus $\mathscr{Z}^{\mathrm{TAR}}(K_{1,r}) = K_{1,r} \,\square\, K_2$.

It is immediate from the list of zero forcing sets that $\overline{\mathrm{Z}}(K_{1,r}) = \mathrm{Z}(K_{1,r}) = r - 1$ and $z_0(K_{1,r}) = \underline{z_0}(K_{1,r}) = r$. □

Next we apply results established for TAR graphs of X-set parameters and known properties of zero forcing to the zero forcing TAR graph and related parameters. The t

dimensional hypercube Q_t can be defined recursively in terms of the Cartesian product as $Q_0 = K_1$ and $Q_t = Q_{t-1} \square K_2$.

Proposition 1.9 [5] *Let G be a graph of order $n \geq 2$ with no isolated vertices. Then*

1. $\Delta(\mathscr{Z}^{\text{TAR}}(G)) = n$.
2. *If $n \geq 2$, then $\delta(\mathscr{Z}^{\text{TAR}}(G)) = n - \overline{\mathrm{Z}}(G)$.*
3. $\overline{\mathrm{Z}}(G) + 1 \leq z_0(G) \leq \min\{\overline{\mathrm{Z}}(G) + \mathrm{Z}(G), n\}$.
4. *If $\mathrm{Z}(G) = 1$, then $z_0(G) = \overline{\mathrm{Z}}(G) + 1$.*
5. *For every $k \geq \mathrm{Z}(G)$, $\mathscr{Z}_k^{\text{TAR}}(G)$ is a subgraph of Q_n. Thus $\mathscr{Z}_k^{\text{TAR}}(G)$ is bipartite.*
6. *No zero forcing TAR graph is isomorphic to a hypercube.*

Proposition 1.9 allows us to show that the values of certain parameters of the base graphs of isomorphic zero forcing TAR graphs must be equal.

Corollary 1.10 [5] *Let G and G' be graphs with no isolated vertices. If $\mathscr{Z}^{\text{TAR}}(G) \cong \mathscr{Z}^{\text{TAR}}(G')$, then*

1. $|V(G)| = |V(G')|$,
2. $\mathrm{Z}(G) = \mathrm{Z}(G')$, *and*
3. $\overline{\mathrm{Z}}(G) = \overline{\mathrm{Z}}(G')$.

Note that Proposition 1.4 also applies to the zero forcing TAR graph.

2 *X*-TAR Graph Isomorphisms and Irrelevant Sets

We show that for graphs G and G' with no isolated vertices, $\mathscr{X}^{\text{TAR}}(G) \cong \mathscr{X}^{\text{TAR}}(G')$ implies there is a relabeling of the vertices of G' such that G and G' have exactly the same X-sets (see Theorem 2.9). Irrelevant sets are introduced in Section 2.1 to prove that if $\mathscr{X}^{\text{TAR}}(G) \cong \mathscr{X}^{\text{TAR}}(G')$, then there is an isomorphism that preserves the cardinality of X-sets (see Theorem 2.8). The focus of Section 2.1 is proving Theorems 2.8 and 2.9, but irrelevant sets are interesting in their own right (and exhibit significant differences among various X-set parameters). These ideas are discussed in Sect. 2.2.

2.1 Isomorphisms of the *X*-TAR Graph

There is a natural poset structure on the vertices of an X-TAR graph whose partial ordering is determined by set inclusion. This poset structure can be used to better understand the isomorphisms between X-TAR graphs. We begin this subsection with a result showing that the vertices of induced hypercubes of X-TAR graphs are intervals in the aforementioned poset. To formally state this result, we recall some basic poset terminology found in [19].

Let Y be a set and $P = (Y, \leq)$ be a poset. An element $u \in Y$ is *maximal* (respectively, *minimal*) if $u \not< y$ (respectively, $u \not> y$) for each $y \in Y$. Elements $u, v \in Y$ are *comparable* provided $u \leq v$ or $v \leq u$. If $u \leq v$ in P, then the set $[u, v] = \{y : u \leq y \leq v\}$ is called an *interval*. A *chain* is a poset such that each pair of elements is comparable. The *length* of a chain C is $|C| - 1$ and the length of the interval $[u, v]$ is the maximum length of a chain in $[u, v]$ and is denoted by $\ell(u, v)$.

For a set T, let $\mathcal{P}(T)$ denote the power set of T. We remark that $\mathscr{X}^{\mathrm{TAR}}(G)$ is a join-semilattice contained in $(\mathcal{P}(V(G)), \subseteq)$ and $\ell(S, S') = \mathrm{dist}(S, S')$ for any interval $[S, S']$ in $\mathscr{X}^{\mathrm{TAR}}(G)$ since S and S' are comparable (where $\mathrm{dist}(S, S')$ is the distance in the graph $\mathscr{X}^{\mathrm{TAR}}(G)$).

Lemma 2.1 *Let X be an X-set parameter, let $t \geq 0$ be an integer, let G be a graph on n vertices, and let H be an induced subgraph of $\mathscr{X}^{\mathrm{TAR}}(G)$. If $H \cong Q_t$, then $V(H)$ is an interval of length t in the poset $(\mathcal{P}(V(G)), \subseteq)$.*

Proof Assume $H \cong Q_t$. The claim is obvious for $t = 0, 1$. Suppose that $t \geq 2$. We begin by showing that $(V(H), \subseteq)$ has exactly 1 maximal element.

Assume, to the contrary, that $(V(H), \subseteq)$ has at least 2 maximal elements u and v. Since H is connected, there exists a path from u to v in H. Every path P from u to v in H can be written in the form $uy_1 \cdots y_i v$. For each such P, let d_P to be the smallest index k such that $y_{k-1} \subseteq y_k$, where $y_0 = u$ and $y_{i+1} = v$. Note that since u and v are maximal in $(V(H), \subseteq)$, $d_P \geq 2$ is always defined.

Let d be the minimum d_P amongst all paths P from u to v in H. Pick a path P^* in H of the form $ux_1 \cdots x_i v$ such that $d_{P^*} = d$, and let $x_0 = u$ and $x_{i+1} = v$. Then $x_{d-1} \subseteq x_{d-2}$ because $x_{d-2} \nsubseteq x_{d-1}$ and x_{d-2} is adjacent to x_{d-1}. Since x_{d-2} and x_d have x_{d-1} as a common neighbor and each pair of vertices in a hypercube share exactly 0 or 2 common neighbors, there exists some vertex $w \neq x_{d-1}$ in H that is adjacent to x_{d-2} and x_d. By our choice of d, $w \subseteq x_{d-2}$ and $w \subseteq x_d$ or else $P' = ux_1 \ldots x_{d-2}wx_d \ldots x_i v$ would be a path with $d_{P'} < d_{P^*}$ since $w = x_{d-2} \cup x_d$. Hence $x_{d-2} \cap x_d = w$. This is absurd since $x_{d-2} \cap x_d = x_{d-1}$ and $w \neq x_{d-1}$.

Thus, H has exactly 1 maximal element T. A similar argument shows H has exactly 1 minimal element S in the subset partial ordering of $\mathscr{X}^{\mathrm{TAR}}(G)$. Thus, $S \subseteq R \subseteq T$ for every $R \in V(H)$. Since $H \cong Q_t$, $\mathrm{dist}(S, T) \leq \mathrm{diam}(Q_t) = t$. So there are at most 2^t elements in the interval $[S, T]$. But $|V(H)| = |V(Q_t)| = 2^t$. Therefore, $\mathrm{dist}(S, T) = t$ and $V(H) = [S, T]$. □

We remark that the proof of Lemma 2.1 is really an argument about cover graphs. For a poset $(Y, \leq)$ and $x, y \in Y$, we say that y covers x provided $x < y$ and there is no $z \in Y$ such that $x < z < y$. The *cover graph of* P is the simple graph with vertices in Y and an edge uv if and only if u covers v, or v covers u. Lemma 2.1 says that if $G \cong Q_t$, then $V(G)$ is an interval in $(\mathcal{P}(Y), \subseteq)$.

Lemma 2.2 *Let X be an X-set parameter, let G and G' be graphs. Suppose $\varphi : \mathscr{X}^{\mathrm{TAR}}(G) \to \mathscr{X}^{\mathrm{TAR}}(G')$ is an isomorphism. Let $S' = \varphi(V(G))$. If Y' is a minimal X-set of G', then $Y' \subseteq S'$.*

Proof Let Y' be a minimal X-set of G'. Define $Y = \varphi^{-1}(Y')$. The interval $[Y, V(G)]$ in $(\mathcal{P}(V(G), \subseteq)$ forms an induced Q_t in $\mathscr{X}^{\mathrm{TAR}}(G)$ for some integer $t > 0$. By Lemma 2.1, $\varphi([Y, V(G)])$ is an interval $[Z', W']$ in $(\mathcal{P}(V(G')), \subseteq)$. Since Y' is a minimal X-set of G', $Z' = Y'$. Thus, $Y' \subseteq S'$. □

Some zero forcing TAR graphs have automorphisms that do not preserve the cardinality of the zero forcing sets that are the vertices of the TAR graph (see Example 2.4). The next definition allows us to manage that issue.

Definition 2.3 Let G be a graph. We say that a vertex $v \in V(G)$ is X-*irrelevant* if $x \notin S$ for every minimal X-set S of G. A set $R \subseteq V(G)$ is an X-*irrelevant set* if every vertex of R is X-irrelevant.

Irrelevant vertices exist for zero forcing as seen in the next example, but not for domination as discussed in Sect. 2.2.

Example 2.4 The proof of Proposition 1.8 shows the center vertex of $K_{1,r}$ is a Z-irrelevant vertex and is in fact the only Z-irrelevant vertex of $K_{1,r}$.

Definition 2.5 Let G be a graph and $R \subseteq V(G)$. We define the map

$$\nu_R : V(\mathscr{X}^{\mathrm{TAR}}(G)) \to \mathcal{P}(V(G)) \quad \text{via} \quad \nu_R(S) = S \ominus R.$$

For an X-irrelevant set R, the map ν_R maps X-sets to X-sets, and is thus an automorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$. This is illustrated in the next example and shown in Theorem 2.7. Observe that ν_R maps S to S' with $|S'| \neq |S|$ if R is nonempty.

Example 2.6 Consider $K_{1,r}$ with center vertex 0. Recall that $\mathscr{Z}^{\mathrm{TAR}}(K_{1,r}) = K_{1,r} \square K_2$ (see Fig. 1 for $\mathscr{Z}^{\mathrm{TAR}}(K_{1,3})$). For $R = \{0\}$, ν_R is the automorphism of $K_{1,r} \square K_2$ obtained by reversing the two vertices of K_2.

Theorem 2.7 *Let X be an X-set parameter, let G be a graph, and let $R \subseteq V(G)$. Then ν_R is a graph automorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$ if and only if R is X-irrelevant.*

Proof Suppose that ν_R is an automorphism. By Lemma 2.2 every minimal X-set of G is a subset of $\nu_R(V(G)) = V(G) \setminus R$. Thus, R is X-irrelevant.

Suppose that R is X-irrelevant. Let S be an X-set of G. Then there exists some minimal X-set $T \subseteq S$. Since R is X-irrelevant, $T \subseteq S \setminus R \subseteq S$. Thus, $\nu_R(S) \supseteq S \setminus R$ is an X-set of G. By construction, adjacency is preserved by ν_R. Therefore, ν_R is an automorphism. □

We are now ready to state one of our main results, namely that if there is an isomorphism between X-TAR graphs that does not preserve the sizes of the X-sets, then the difference in size is due to an X-irrelevant set. Moreover, there is an isomorphism between these X-TAR graphs that preserves the size of the X-sets.

Theorem 2.8 *Let X be an X-set parameter, let G and G' be graphs with no isolated vertices, and let $\tilde{\varphi} : \mathscr{X}^{\mathrm{TAR}}(G) \to \mathscr{X}^{\mathrm{TAR}}(G')$ be an isomorphism. Then $R' = V(G') \setminus \tilde{\varphi}(V(G))$ is X-irrelevant and $\varphi = \nu_{R'} \circ \tilde{\varphi}$ is an isomorphism such that $|\varphi(S)| = |S|$ for every $S \in V(\mathscr{X}^{\mathrm{TAR}}(G))$.*

Proof Let $n = |V(G)|$. If $\tilde{\varphi}(V(G)) = V(G')$, then $R' = \emptyset$ is X-irrelevant. So suppose that $\tilde{\varphi}(V(G)) = S'$, where $S' \neq V(G')$. Since $\tilde{\varphi}$ is an isomorphism, Lemma 2.2 implies every minimal X-set of G' is a subset of S' and hence R' is X-irrelevant. By Theorem 2.7, $\nu_{R'}$ is an automorphism of $\mathscr{X}^{\mathrm{TAR}}(G')$. Thus, $\varphi = \nu_{R'} \circ \tilde{\varphi}$ is an isomorphism such that $\varphi(V(G)) = V(G')$.

Note that $|V(G')| = n$ and $X(G') = X(G)$. Let $S \in V(\mathscr{X}^{\mathrm{TAR}}(G))$. The interval $H = [S, V(G)] \in \mathscr{X}^{\mathrm{TAR}}(G)$ is an induced hypercube in $\mathscr{X}^{\mathrm{TAR}}(G)$, so $\varphi(H)$ is an induced hypercube in $\mathscr{X}^{\mathrm{TAR}}(G')$. By Lemma 2.1 and since $\varphi(V(G)) = V(G')$, $\varphi(H) = [S', V(G')]$ in $\mathscr{X}^{\mathrm{TAR}}(G')$ and $\mathrm{dist}(S', V(G')) = \mathrm{dist}(S, V(G))$. We show by induction on $|S|$ that $|\varphi(S)| = |S|$. We say S is a k-X-set if S is an X-set and $|S| = k$.

For the base case, assume $|S| = X(G)$, so $\mathrm{dist}(S', V(G')) = \mathrm{dist}(S, V(G)) = n - X(G) = |V(G')| - X(G')$. This implies $|S'| = X(G') = X(G) = |S|$. The same reasoning applies using φ^{-1}, since $\varphi^{-1}(V(G')) = V(G)$. Thus φ defines a bijection between minimum X-sets of G and minimum X-sets of G'.

Now assume φ defines a bijection between i-X-sets of G and i-X-sets of G' for $X(G) \leq i \leq k$ and let S be a $(k+1)$-X-set of G. This implies $|\varphi(W)| \geq k+1$ for $W \in [S, V(G)]$. By Lemma 2.1, $\varphi([S, V(G)]) = [S', V(G')]$ in $\mathscr{X}^{\mathrm{TAR}}(G')$ and $\mathrm{dist}(S', V(G')) = n - k - 1$. Thus S' is a $(k+1)$-X-set of G. □

In the next theorem, we show that if two graphs G and G' (with no isolated vertices) have isomorphic X-TAR graphs, then there is a bijection between the vertices of G and the vertices of G' that results in the correspondence of the X-sets. Note this bijection need not be a graph isomorphism between G and G'. A cycle and a cycle plus one edge provide an example of nonisomorphic graphs with the same zero forcing sets (see Example 3.3). We first give some useful notation. For any map $\psi : A \to A'$ and subset $B \subseteq A$ we write $\psi(B)$ to mean the image of B, i.e., $\psi(B) = \{\psi(b) : b \in B\}$. This is particularly useful when working with a map $\psi : V(G) \to V(G')$ that maps X-sets of G to X-sets of G', since this convention naturally induces a map $\psi : V(\mathscr{X}^{\mathrm{TAR}}(G)) \to V(\mathscr{X}^{\mathrm{TAR}}(G'))$.

Theorem 2.9 *Let X be an X-set parameter, let G and G' be graphs with no isolated vertices, and suppose $\varphi : \mathscr{X}^{\mathrm{TAR}}(G) \to \mathscr{X}^{\mathrm{TAR}}(G')$ is a graph isomorphism. Then $|\varphi(S)| = |S|$ for every X-set S if and only if there exists a bijection $\psi : V(G) \to V(G')$ such that $\psi(S) = \varphi(S)$ for every X-set S of G.*

Suppose $|\varphi(S)| = |S|$ for every X-set S and $\psi : V(G) \to V(G')$ is a bijection such that $\psi(S) = \varphi(S)$ for every X-set S of G. Then there is a relabeling of the vertices of G' such that the relabeled graph has the same X-sets as G and the same X-TAR graph as G.

Proof Let $n = |V(G)|$. For any graph H and $v \in V(H)$, define $S_v = V(H) \setminus \{v\}$. For $v \in V(G)$ and $v' \in V(G')$, note that S_v is an X-set of G and $S_{v'}$ is an X-set of G'.

Begin by assuming $|\varphi(S)| = |S|$ for every X-set S. Since $|\varphi(S_v)| = n - 1$ for every $v \in V(G)$, we may define $\psi(v) = v'$ where v' is the unique vertex such that $\varphi(S_v) = S_{v'}$. Note that $\psi : V(G) \to V(G')$ is a bijection.

The proof that $\psi(S) = \varphi(S)$ for every X-set S of G proceeds iteratively from $|S| = n$ to $|S| = X(G)$. By the choice of φ and the definition of ψ, we have $\varphi(S) = \psi(S)$ for $|S| = n, n-1$. Assume $\varphi(S) = \psi(S)$ for each X-set S of order k for some k with $n - 1 \geq k > X(G)$. Let S be an X-set of order $k - 1$. Since $k - 1 \leq n - 2$, there exist distinct vertices $a, b \in V(G) \setminus S$ such that $S \cup \{a\}$ and $S \cup \{b\}$ are X-sets of order k. Since S is adjacent to $S \cup \{a\}$ and $S \cup \{b\}$ in $\mathscr{X}^{\mathrm{TAR}}(G)$, and $|\varphi(S \cup \{a\})| = |\varphi(S \cup \{b\})| > |\varphi(S)|$, there exist distinct $a', b' \in V(G') \setminus \varphi(S)$ such that $\varphi(S \cup \{a\}) = \varphi(S) \cup \{a'\}$ and $\varphi(S \cup \{b\}) = \varphi(S) \cup \{b'\}$. Thus,

$$\varphi(S) = \varphi(S \cup \{a\}) \cap \varphi(S \cup \{b\}) = \psi(S \cup \{a\}) \cap \psi(S \cup \{b\}) = \psi(S),$$

where the last equality follows since ψ is a bijection.

Now assume there exists a bijection $\psi : V(G) \to V(G')$ such that $\psi(S) = \varphi(S)$ for every X-set S of G. Then $|S| = |\psi(S)| = |\varphi(S)|$ as desired.

Finally, suppose $|\varphi(S)| = |S|$ for every X-set S and $\psi : V(G) \to V(G')$ is a bijection such that $\psi(S) = \varphi(S)$ for every X-set S of G. Define G'' from G' by relabeling vertices of G' so that $v' \in V(G')$ is labeled by $\psi^{-1}(v')$. Then $G'' \cong G'$ and G'' and G have the same X-sets. □

One of the advantages of the universal approach is that universal results apply to many different kinds of TAR graphs. Theorem 2.9 shows that if G and G' have no isolated vertices and the zero forcing (respectively, domination, power domination, PSD zero forcing) TAR graphs of G and G' are isomorphic, then G' can be relabeled so that G and G' have exactly the same zero forcing (respectively, dominating, power dominating, positive semidefinite zero forcing) sets.

The next two results focus on mappings and minimal X-sets and are used in the proof of Theorem 2.13.

Proposition 2.10 *Let X be an X-set parameter and let G and G' be graphs with no isolated vertices. Suppose $\varphi : \mathscr{X}^{\mathrm{TAR}}(G) \to \mathscr{X}^{\mathrm{TAR}}(G')$ is a graph isomorphism such that $|\varphi(S)| = |S|$. Then φ maps minimal X-sets to minimal X-sets (of the same size).*

Proof If S is minimal X-set of G, then $\deg_{\mathscr{X}^{\mathrm{TAR}}(G)}(S) = n - |S|$. If T is an X-set of G that is not minimal in $\mathscr{X}^{\mathrm{TAR}}(G)$, then $\deg_{\mathscr{X}^{\mathrm{TAR}}(G)}(T) \geq n - |T| + 1$ (since T has a subset neighbor). Analogous statements are true for G'. □

Proposition 2.11 *Let X be an X-set parameter and let G and G' be graphs with no isolated vertices. Suppose $\psi : V(G) \to V(G')$ is a bijection.*

(1) *Suppose ψ maps X-sets of G to X-sets of G'. Then the induced mapping $\psi : V(\mathscr{X}^{\mathrm{TAR}}(G)) \to V(\mathscr{X}^{\mathrm{TAR}}(G'))$ is an isomorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$ and $\psi(\mathscr{X}^{\mathrm{TAR}}(G))$. If every X-set of G' is the image of an X-set of G, then $\psi : V(\mathscr{X}^{\mathrm{TAR}}(G)) \to V(\mathscr{X}^{\mathrm{TAR}}(G'))$ is an isomorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$ and $\mathscr{X}^{\mathrm{TAR}}(G')$.*
(2) *Suppose ψ maps minimal X-sets of G to minimal X-sets of G'. Then ψ maps X-sets of G to X-sets of G'. If every minimal X-set of G' is the image of a minimal X-set of G, then ψ is a bijection from X-sets of G to X-sets of G'.*
(3) *Suppose ψ maps minimal X-sets of G to minimal X-sets of G'. Then the induced mapping $\psi : V(\mathscr{X}^{\mathrm{TAR}}(G)) \to V(\mathscr{X}^{\mathrm{TAR}}(G'))$ is an isomorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$ and $\psi(\mathscr{X}^{\mathrm{TAR}}(G))$. If every minimal X-set of G' is the image of a minimal X-set of G, then $\psi : V(\mathscr{X}^{\mathrm{TAR}}(G)) \to V(\mathscr{X}^{\mathrm{TAR}}(G'))$ is an isomorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$ and $\mathscr{X}^{\mathrm{TAR}}(G')$.*

Proof (1): Since ψ maps X-sets of G to X-sets of G', ψ induces a bijection between the vertices of $\mathscr{X}^{\mathrm{TAR}}(G)$ and a subset of the vertices of $\mathscr{X}^{\mathrm{TAR}}(G')$ (X-sets of G' of the form $\psi(S)$ where S is an X-set of G). Assume that $S_1, S_2 \in V(\mathscr{X}^{\mathrm{TAR}}(G))$ are adjacent in $\mathscr{X}^{\mathrm{TAR}}(G)$. Without loss of generality, $|S_1 \setminus S_2| = 1$. Since ψ is a bijection, $|\psi(S_1) \setminus \psi(S_2)| = 1$. Thus, $\psi(S_1)$ and $\psi(S_2)$ are adjacent in $\mathscr{X}^{\mathrm{TAR}}(G')$. Hence ψ is an isomorphism from $\mathscr{X}^{\mathrm{TAR}}(G)$ to $\psi(\mathscr{X}^{\mathrm{TAR}}(G))$.

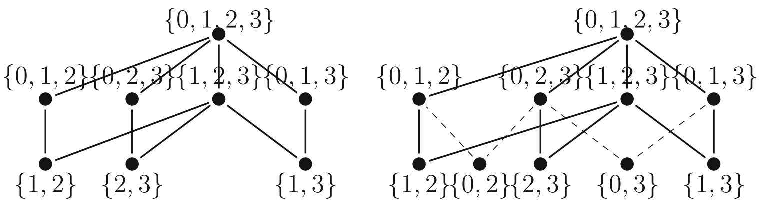


Fig. 2 $\mathscr{Z}^{\mathrm{TAR}}(K_{1,3})$ and $\mathscr{Z}^{\mathrm{TAR}}(P)$

(2): Let $S \in V(\mathscr{X}^{\mathrm{TAR}}(G))$ be an X-set. There is a minimal X-set $T \subseteq S$ of G and $\psi(T) \subseteq \psi(S)$. Since $\psi(T)$ is a minimal X-set of G', $\psi(S)$ is an X-set of G'.

Statement (3) is immediate from statements (2) and (1) □

It is possible to have a vertex bijection ψ that maps minimal zero forcing sets of G to minimal zero forcing sets of G' but not vice versa, as the next example shows (using the identity function as ψ).

Example 2.12 Consider the graph $K_{1,3}$ with vertices {0, 1, 2, 3} where 0 is the center vertex, and the paw graph P constructed from $K_{1,3}$ by adding the edge 23. The minimal zero forcing sets of $K_{1,3}$ are {1, 2}, {1, 3}, and {2, 3}. The minimal zero forcing sets of P are {0, 2}, {0, 3}, {1, 2}, {1, 3}, and {2, 3}. The zero forcing TAR graphs are shown in Fig. 2. The dotted lines in $\mathscr{Z}^{\mathrm{TAR}}(P)$ indicate edges of this graph that are not present in $\mathscr{Z}^{\mathrm{TAR}}(K_{1,3})$.

2.2 *X*-Irrelevant Vertices and Automorphisms of *X*-TAR Graphs

In this section we explore the existence (or non-existence) of X-irrelevant vertices for various X-set parameters and consequences for automorphisms of X-TAR graphs and we apply these results to zero forcing, power domination, domination, and PSD zero forcing. Let $M_X(G)$ denote the set of bijections $\psi : V(G) \to V(G)$ that send minimal X-sets of G to minimal X-sets of G of the same size.

Theorem 2.13 *Let G be a graph with no isolated vertices. Then the automorphism group of $\mathscr{X}^{\mathrm{TAR}}(G)$ is generated by*

$$\{\nu_R : R \textit{ is } X\textit{-irrelevant}\} \cup M_X(G).$$

Proof By Theorem 2.7 $\nu_R \in \mathrm{aut}(\mathscr{X}^{\mathrm{TAR}}(G))$ for every X-irrelevant set R. By Proposition 2.11(3), $\psi \in \mathrm{aut}(\mathscr{X}^{\mathrm{TAR}}(G))$ for every $\psi \in M_X(G)$ (since $G' = G$ here, having ψ map minimal X-sets to minimal X-sets is sufficient).

We now show that $\{\nu_R : R$ is X-irrelevant$\} \cup M_X(G)$ generates $\mathrm{aut}(\mathscr{X}^{\mathrm{TAR}}(G))$. Let φ by an automorphism of $\mathscr{X}^{\mathrm{TAR}}(G)$. Suppose first that $V(G)$ is fixed by φ. Then $|S| = n - \mathrm{dist}(V(G), S) = n - \mathrm{dist}(V(G), \varphi(S)) = |\varphi(S)|$ for each $S \in V(\mathscr{X}^{\mathrm{TAR}}(G))$. By Theorem 2.9 there exists a bijection $\psi : V(G) \to V(G)$ such that $\psi(S) = \varphi(S)$ for every X-set S of G. By Proposition 2.10, ψ maps minimal X-sets to minimal X-sets of the same size.

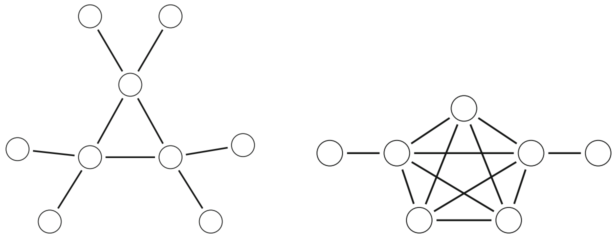


Fig. 3 The graphs $K_3 \circ 2K_1$ and $G(5, 2)$

Suppose that $V(G)$ is not fixed by φ. By Theorem 2.8 and the preceding argument, there exists a bijection $\psi \in M_X(G)$ such that $\psi = \nu_R \circ \varphi$, where $R = V(G) \setminus \varphi(G)$. Thus, $\varphi = \nu_R^{-1} \circ \psi$. □

Not every automorphism of $\mathscr{Z}^{\text{TAR}}(G)$ induced by a mapping on G comes from an automorphism of G itself, as seen in the next example.

Example 2.14 Consider the path P_4 with vertices 1, 2, 3, 4 and edges 12, 23, 34. Define $\psi : V(G) \to V(G)$ by $\psi(1) = 4$, $\psi(2) = 2$, $\psi(3) = 3$ and $\psi(1) = 4$. The minimal zero forcing sets are $\{1\}$, $\{4\}$, and $\{2, 3\}$, so ψ maps minimal forcing sets to minimal forcing sets. Thus ψ defines an automorphism of $\mathscr{Z}^{\text{TAR}}(P_4)$. However, ψ is *not* an automorphism of P_4.

The next result provides many examples of graphs with nonempty Z-irrelevant sets (and includes $K_{1,r}$ discussed in Example 2.4). Let H and G be graphs, and for each $v \in V(H)$, let G_v denote a copy of G such that H and G_v, $v \in V(H)$ are all disjoint graphs. The *corona* of H with G, denoted by $H \circ G$, has $V(H \circ G) = V(H) \cup \bigcup_{v \in V(H)} V(G_v)$ and $E(H \circ G) = E(H) \cup \bigcup_{v \in V(H)} E(G_v) \cup \bigcup_{v \in V(H)} \{vx : x \in V(G_v)\}$. The graph $K_3 \circ 2K_1$ is shown in Fig. 3.

Proposition 2.15 *Let H be a connected graph and $G = H \circ rK_1$ with $r \geq 2$. Then $T \subseteq V(G)$ is a Z-irrelevant set of G if and only if $T \subseteq V(H)$.*

Proof Let $V(H) = \{v_1, \ldots, v_k\}$ and denote the leaves adjacent to v_i by $x_{i,j}$ for $j = 1, \ldots, r$. Then $S \subseteq V(G)$ is a zero forcing set of G if and only if S contains at least $r - 1$ of the vertices $x_{i,1}, \ldots, x_{i,r}$ for all $i = 1, \ldots, k$. Thus $T \subseteq V(G)$ is Z-irrelevant set if and only if $T \subseteq V(H)$. □

Next we apply the results in Sect. 2.1 to other X-set parameters. The power domination TAR graph of a graph G was introduced in [5]. We define a family of graphs with nonempty γ_P-irrelevant sets. For $r \geq 3$ and $1 \leq \ell \leq r - 2$, construct $G(r, \ell)$ from $K_r \circ K_1$ by deleting $r - \ell$ leaves. The graph $G(5, 2)$ is shown in Fig. 3.

Proposition 2.16 *For $r \geq 3$ and $1 \leq \ell \leq r - 2$, let L denote the set of leaves of $G(r, \ell)$. Then $T \subseteq V(G(r, \ell))$ is a γ_P-irrelevant set of $G(r, \ell)$ if and only if $T \subseteq L$.*

Proof The set L is not a power dominating set of $G(r, \ell)$ because no vertex dominated by L is adjacent to exactly one vertex of $G(r, \ell)$ that is not dominated by L, and L does not dominate G. But any one vertex of K_r is a power dominating set. □

In contrast to zero forcing and power domination, neither domination nor PSD zero forcing has irrelevant vertices, as we now show. Let G be a graph and $v \in V(G)$. A minimal dominating set containing v can be constructed by starting with $S = \{v\}$ and repeatedly adding $w \notin N_G[S]$ until $N_G[S] = V(G)$. This immediately implies the next result.

Proposition 2.17 *If G is a graph and $S \subseteq V(G)$ is γ-irrelevant, then $S = \emptyset$.*

The next result is immediate from Proposition 2.17 and Theorems 2.8 and 2.9.

Corollary 2.18 *Let G and G' be graphs with no isolated vertices, and suppose $\varphi : \mathscr{D}^{\mathrm{TAR}}(G) \to \mathscr{D}^{\mathrm{TAR}}(G')$ is a graph isomorphism. For every dominating set S of G, $|\varphi(S)| = |S|$. Furthermore, there exists a bijection $\psi : V(G) \to V(G')$ such that $\psi(S) = \varphi(S)$ for every dominating set S of G.*

Let $\mathscr{Z}_+^{\mathrm{TAR}}(G)$ denote the positive semidefinite zero forcing TAR graph of G, which has positive semidefinite zero forcing sets as vertices. No work on this reconfiguration graph has appeared, but the positive semidefinite zero forcing number is an X-set parameter. Since the results stated for zero forcing in Proposition 1.9 and Corollary 1.10 were established for X-set parameters in [5], they remain true when $\mathscr{Z}^{\mathrm{TAR}}(G)$ and related zero forcing parameters are replaced by $\mathscr{Z}_+^{\mathrm{TAR}}(G)$ and related PSD zero forcing parameters. Known results about positive semidefinite zero forcing and Theorems 2.8 and 2.9 provide information about isomorphisms of $\mathscr{Z}_+^{\mathrm{TAR}}(G)$.

Proposition 2.19 [14, Theorem 9.36] *Let G be a graph. Then for any vertex $v \in V(G)$, there exists a minimum positive semidefinite zero forcing set S such that $v \in S$.*

Corollary 2.20 *Let G and G' be graphs with no isolated vertices, and suppose $\varphi : \mathscr{Z}_+^{\mathrm{TAR}}(G) \to \mathscr{Z}_+^{\mathrm{TAR}}(G')$ is a graph isomorphism. For every positive semidefinite zero forcing set S of G, $|\varphi(S)| = |S|$. Furthermore, there exists a bijection $\psi : V(G) \to V(G')$ such that $\psi(S) = \varphi(S)$ for every positive semidefinite zero forcing set S of G.*

3 The Zero Forcing TAR Graph

In this section we return to the study of zero forcing TAR graphs introduced in Sect. 1.3. In Sect. 3.1 we establish that certain zero forcing TAR graphs are unique up to isomorphism of the base graph (assuming the base graph has no isolated vertices) and also give an example of nonisomorphic base graphs with same zero forcing TAR graphs. We study the connectedness of $\mathscr{Z}_k^{\mathrm{TAR}}(G)$ and related parameters in Sect. 3.2, including presenting a family of graphs H where $z_0(H)$ exceeds the lower bound $\overline{\mathrm{Z}}(H) + 1$ by an arbitrary amount.

We begin with a simple application of our universal isomorphism results to zero forcing polynomials. Let G be a graph of order n and let $z(G; k)$ denote the number of zero forcing sets of cardinality k. Boyer et al. defined the *zero forcing polynomial* of G to be $Z(G; x) = \sum_{k=\mathrm{Z}(G)}^{n} z(G; k)x^k$ in [6]. Theorem 2.8 (or Theorem 2.9) applied to zero forcing implies that if $\mathscr{Z}^{\mathrm{TAR}}(G) \cong \mathscr{Z}^{\mathrm{TAR}}(G')$ for graphs G and G'

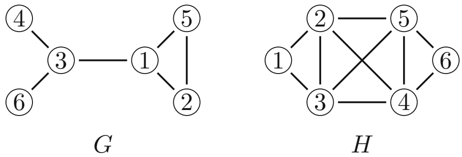


Fig. 4 Two graphs the the same zero forcing polynomial and nonisomorphic zero forcing TAR graphs

with no isolated vertices, then the zero forcing polynomials of G and G' are equal. The converse to this statement is false. Small examples of graphs with the same zero forcing polynomial and nonisomorphic zero forcing TAR graphs are easy to find with software, and one such example is presented next.

Example 3.1 The graphs G and H in Fig. 4 have

$$Z(G; x) = Z(H; x) = x^6 + 6x^5 + 13x^4 + 8x^3$$

(see [8] for the computations). The minimal zero forcing sets of G are $\{1, 2, 4\}$, $\{1, 2, 6\}$, $\{1, 4, 5\}$, $\{1, 5, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$, $\{2, 5, 6\}$, $\{4, 5, 6\}$ and the minimal zero forcing sets of H are $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$, $\{2, 5, 6\}$, $\{3, 4, 6\}$, $\{3, 5, 6\}$, $\{2, 3, 4, 5\}$. Thus $\overline{\mathrm{Z}}(G) = 3$, $\overline{\mathrm{Z}}(H) = 4$, and $\mathscr{Z}^{\mathrm{TAR}}(G) \not\cong \mathscr{Z}^{\mathrm{TAR}}(H)$.

3.1 Uniqueness and Nonuniqueness

For a graph H with no isolated vertices, we say its zero forcing TAR graph is *unique* if $\mathscr{Z}^{\mathrm{TAR}}(G) \cong \mathscr{Z}^{\mathrm{TAR}}(H)$ implies $G \cong H$ for any graph G with no isolated vertices. In this section we present examples of *unique* Z-TAR graphs. We also present examples of nonisomorphic graphs G and H with no isolated vertices such that $\mathscr{Z}^{\mathrm{TAR}}(G) \cong \mathscr{Z}^{\mathrm{TAR}}(H)$, or equivalently, the vertices of G can be labelled so that G and H have the same zero forcing sets.

It is immediate that the Z-TAR graphs of the path and the complete graph are unique, because the path is the only graph G with $\mathrm{Z}(G) = 1$ and K_n is the only graph G of order n with no isolated vertices having $\mathrm{Z}(G) = n - 1$. It is less obvious that $K_{1,r}$ has a unique Z-TAR graph, as we show in the next result. The proof uses some additional definitions and known properties of zero forcing. Let G be a graph. For a given zero forcing set S, carry out a forcing process to color all vertices of G blue, recording the forces; the set of these forces is denoted by $\mathcal{F}$. A set of forces $\mathcal{F}$ of S defines a *reversal* of S, namely the set of vertices that do not perform a force (using the set of forces $\mathcal{F}$). It is known that the reversal of a zero forcing set is also a zero forcing set [14, Theorem 9.10(1)]. The next process is sometimes called *neighbor trading* e.g., in [11]. Let S be a zero forcing set of G. Suppose $v \in S$, $v \to w$ can be the first force performed in a forcing process, and $\deg_G(v) \geq 2$. Let $u \in N_G(v)$ and $u \neq w$. Then $u \in S$ and $S \setminus \{u\} \cup \{w\}$ is a zero forcing set (of the same cardinality) with first force $v \to w$ replaced by $v \to u$ and the other forces remaining the same.

Proposition 3.2 *Let G be a graph on $n \geq 3$ vertices with no isolated vertices. If $\mathscr{Z}^{\mathrm{TAR}}(G) \cong K_{1,r} \square K_2$, then $G \cong K_{1,r}$.*

Table 1 Number of graphs with unique zero forcing TAR graph for small orders

# Vertices in G	2	3	4	5	6	7	8
# Graphs with unique $\mathscr{Z}^{\text{TAR}}(G)$	1	2	4	7	34	303	5318
# Graphs with no isolated vertices	1	2	7	23	122	888	11302
Ratio (# unique/# no isolated)	1	1	0.5714	0.3043	0.2787	0.3412	0.4705

Proof Recall that $\mathscr{Z}^{\text{TAR}}(K_{1,r}) \cong K_{1,r} \,\square\, K_2$ by Proposition 1.8. Now suppose $\mathscr{Z}^{\text{TAR}}(G) \cong K_{1,r} \,\square\, K_2$. By Proposition 1.4, $\mathscr{Z}^{\text{TAR}}(G_1 \sqcup G_2) \cong \mathscr{Z}^{\text{TAR}}(G_1) \,\square\, \mathscr{Z}^{\text{TAR}}(G_2)$. Since K_2 is a hypercube, there does not exist a graph G_2 with $\mathscr{Z}^{\text{TAR}}(G_2) = K_2$ by Proposition 1.9(6). Thus we may assume G is connected.

Since $\mathscr{Z}^{\text{TAR}}(G) \cong \mathscr{Z}^{\text{TAR}}(K_{1,r})$, the order of G is $r+1$ and we label the vertices of G so that the zero forcing sets of G are exactly the zero forcing sets of $K_{1,r}$ where the vertices of both graphs are $\{0, \dots, r\}$ and 0 is the center vertex of $K_{1,r}$. Note that every minimal zero forcing set of $K_{1,r}$, and thus of G, is a minimum zero forcing set. Recall from Example 2.4 that the center vertex 0 is Z-irrelevant, i.e., not in any minimal zero forcing set.

Let B be a minimum zero forcing set of G. Since $\mathrm{Z}(G) = |V(G)| - 2$, exactly two forces are performed to color all vertices blue starting with the vertices in B blue. Since a vertex that does not force is in the zero forcing set of the reversal of the set of forces, 0 must perform a force and the set of forces must be $\{i \to 0, 0 \to j\}$. If $\deg_G i \geq 2$, then by neighbor trading 0 would be in a minimum zero forcing set. Thus $\deg_G i = 1$. By considering the reversal, $\deg_G j = 1$. If $n = 3$, then $G \cong K_{1,2}$, so assume $n \geq 4$ (i.e., $r \geq 3$). Then there is another vertex k, which must be adjacent to 0 and not adjacent to i or j. If $r = 3$, then $G \cong K_{1,r}$. If $r \geq 4$ then there exists another vertex ℓ. Since $\ell \notin N_G(i)$ and $\ell \notin N_G(j)$, $\ell \in N_G(0)$ or $\ell \in N_G(k)$. Let $\ltimes$ denote the graph obtained from K_3 by adding adding two leaves to one vertex. If $\ell \in N_G(0) \cap N_G(k)$, then $G[\{0, i, j, k, \ell\}] \cong \ltimes$, which is a forbidden induced subgraph for $\mathrm{Z}(G) = |V(G)| - 2$ [14, Theorem 9.12(6)]. If $\ell \in N_G(k)$ and $\ell \notin N_G(0)$, then $G[\{0, i, k, \ell\}] \cong P_4$, which is a forbidden induced subgraph for $\mathrm{Z}(G) = |V(G)| - 2$. Thus $G[\{0, i, j, k, \ell\}] \cong K_{1,4}$. The argument for ℓ shows any additional vertices must also be leaves and $G \cong K_{1,r}$. $\square$

For a graph G and vertices u and v that are not adjacent in G, the graph $G + uv$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{uv\}$.

Example 3.3 Let $n \geq 4$. For any two nonadjacent vertices u and v of C_n, $\mathscr{Z}^{\text{TAR}}(C_n) = \mathscr{Z}^{\text{TAR}}(C_n + uv)$ because a set of vertices S is a zero forcing set if and only if S contains two vertices that are consecutive on the cycle for both C_n and $C_n + uv$. Note also that $\mathrm{Z}(C_n) = \overline{\mathrm{Z}}(C_n) = 2$ and $z_0(C_n) = \underline{z_0}(C_n) = 3$.

Table 1 shows the number of (nonismorphic) graphs without isolated vertices of order at most eight that have unique Z-TAR graphs (this data was computed in [8]).

3.2 Connectedness Properties of the Zero Forcing TAR Graph

The focus of this section is connectedness properties of the zero forcing TAR graph. We exhibit a family of graphs H where $z_0(H)$ exceeds the lower bound $\overline{\mathrm{Z}}(H)+1$ by an arbitrary amount and another family of graphs G where $\underline{z_0}(G)$ is strictly less than $z_0(G)$. These examples are interesting because the more common situation is $\underline{z_0}(G) = z_0(G) = \overline{\mathrm{Z}}(G) + 1$. The path graph illustrates this: Let $n \geq 4$ and consider the path P_n with vertices in path order. Then $S \subseteq V(P_n)$ is a zero forcing set if and only if S contains an endpoint or S contains two consecutive vertices of the path. The set $\{2, 3\}$ is a zero forcing set, but is not adjacent to any zero forcing set in $\mathscr{Z}_2^{\mathrm{TAR}}(P_n)$. Thus $\overline{\mathrm{Z}}(G) = 2$ and $\mathscr{Z}_2^{\mathrm{TAR}}(P_n)$ is not connected. Since adding an end vertex makes any set a zero forcing set, $\mathscr{Z}_k^{\mathrm{TAR}}(G)$ is connected for $k \geq 3$. Thus $z_0(P_n) = \underline{z_0}(P_n) = 3$.

For TAR reconfiguration of the X-set parameters domination and power domination, examples are known such that $x_0(G)$ exceeds the lower bound $\overline{X}(G) + 1$ (see [5, 12]). Naturally, such examples are specific to the X-set parameter being studied, and knowing examples for one X-set parameter does not generally help construct such examples for other X-set parameters. Next we construct a family of examples such that $\overline{\mathrm{Z}}(G)+r \leq z_0(G) < \min\{\overline{\mathrm{Z}}(G)+\mathrm{Z}(G), |V(G)|\}$ for $r \geq 2$. Software such as [8] is a useful tool for finding examples (and found $H(2)$, which we then generalized).

Define $H(r)$ to be the graph with $2r + 4$ vertices such that both sets $V_1 = \{1, \dots, r + 2\}$ and $V_2 = \{r + 3, \dots, 2r + 4\}$ form cliques and there is a matching between the vertices $\{1, \dots, r\}$ and $\{r + 3, \dots, 2r + 2\}$. The graphs $H(2)$ and $\mathscr{Z}_5^{\mathrm{TAR}}(H(2))$ are shown in Figure 5. Note that the diagrams in this figure do not respect the poset structure of $\mathscr{Z}_5^{\mathrm{TAR}}(H(2))$.

Proposition 3.4 *For $r \geq 2$, $\mathrm{Z}(H(r)) = \overline{\mathrm{Z}}(H(r)) = r + 2$ and $\underline{z_0}(H(r)) = z_0(H(r)) = 2r + 2 = \overline{\mathrm{Z}}(H(r)) + r$.*

Proof Observe that the vertices $r+1$, $r+2$, $2r+3$ and $2r+4$ have degree $r+1$ and all other vertices of $H(r)$ have degree $r+2$ so $\delta(H(r)) = r+1$. Let $U_1 = \{r+1, r+2\}$ and $U_2 = \{2r+3, 2r+4\}$. If $v \in V_i$, then $|N_{H(r)}(v) \cap V_i| = r+1$. Let $S \subset V(H(r))$ and let $S_i = S \cap V_i$ for $i = 1, 2$. We show that for a set S of vertices such that $|S_1| \geq |S_2|$, S is a zero forcing set of $H(r)$ if and only if $|S_1| \geq r + 1$ and $|S_2 \cap U_2| \geq 1$ (the argument where $|S_2| \geq r + 1$ is similar). Once this is established, $\mathrm{Z}(H(r)) = \overline{\mathrm{Z}}(H(r)) = r + 2$ with every minimal zero forcing set S having ($|S_1| = r + 1$ and $|S_2| = 1$) or ($|S_2| = r + 1$ and $|S_1| = 1$). Furthermore, $\mathscr{Z}_{2r+1}^{\mathrm{TAR}}(H(r))$ has 2 components, one containing zero forcing sets S with $|S_1| > |S_2|$ and the other with $|S_2| > |S_1|$, whereas $\mathscr{Z}_{2r+2}^{\mathrm{TAR}}(H(r))$ is connected.

Suppose first that $|S_1| = r + 1$, $|S_2| = 1$, and $S_2 \subset U_2$. There must be a vertex $u \in U_1 \cap S_1$, and u can force the one white vertex of S_1. Then $i \to r + 2 + i$ for $i = 1, \dots, r$. Finally any blue vertex of V_2 can force the one remaining white vertex in V_2. Thus S is a zero forcing set.

Now suppose S is a zero forcing set. Let v be the vertex that performs the first force, and without loss of generality $v \in V_1$. Since v and all but one of its neighbors must be in S, S contains at least $r + 1$ of the vertices in S. Since at most r vertices of V_2

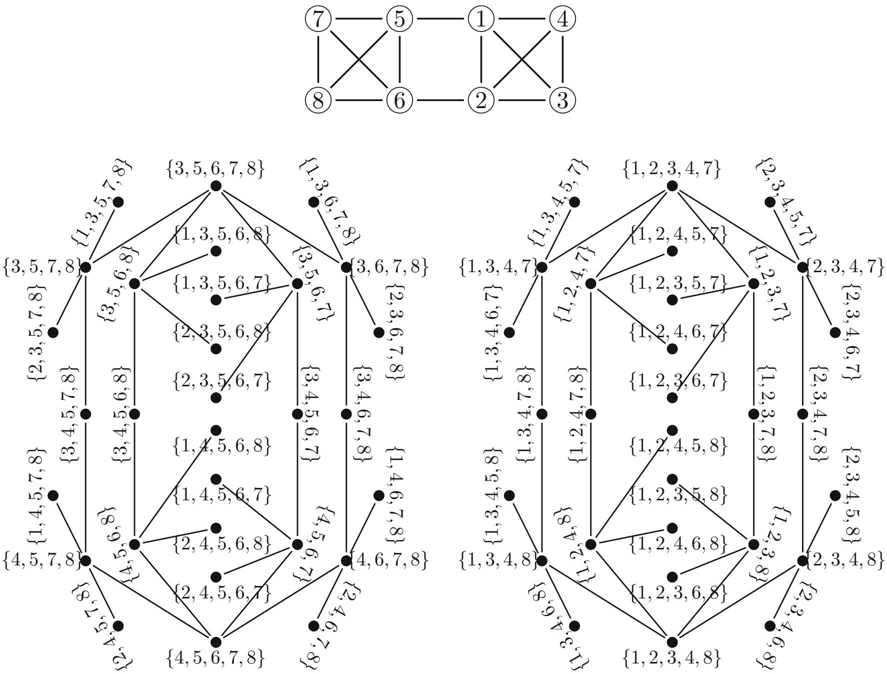
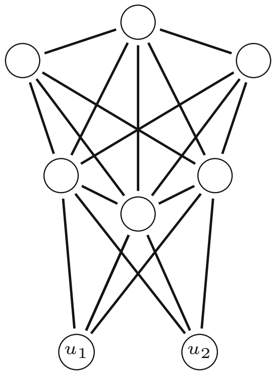


Fig. 5 The graphs $H(2)$ (top) and $\mathscr{Z}_5^{\mathrm{TAR}}(H(2))$ (bottom)

Fig. 6 A graph H_2 satisfying $z_0(H_2) > \underline{z_0}(H_2)$



can be forced by vertices in V_1, S must contain a vertex of V_2 that has no neighbor in V_1, i.e., S must contain at least one vertex of U_2. $\square$

A computer search on graphs up to 8 vertices (with no isolated vertices) found exactly 2 graphs G such that $z_0(G) > \underline{z_0}(G)$. For example, the graph H_2 in Fig. 6 has $z_0(H_2) = 7$ and $\underline{z_0}(H_2) = 5$ because there are minimal zero forcing sets of orders 4 and 6 but not 5, and $\mathscr{Z}_5^{\mathrm{TAR}}(H_2)$ and $\mathscr{Z}_7^{\mathrm{TAR}}(H_2)$ are connected (and $|V(H_2)| = 8$); see [8].

Next we show that the graph H_2 can be used to create an infinite family of graphs H_r such that $z_0(H_r) > \underline{z_0}(H_r)$. Observe that $N_{H_2}(u_2) = N_{H_2}(u_1)$. Construct H_r from H_2 by adding vertices $u_3, \ldots, u_r$ with $N_{H_r}(u_k) = N_{H_r}(u_1)$ for $k = 3, \ldots, r$. Note that $|V(H_r)| = r + 6$.

In general, vertices v_1 and v_2 in a graph G are called *twins* if $N_G(v_1) = N_G(v_2)$ (what we define as twins are often called *independent twins*). A *set of twins* is a set $\{v_1, \ldots, v_k\} \subseteq V(G)$ such that v_i and v_j are twins for all $1 \leq i < j \leq k$.

Observation 3.5 *If G is a graph with a set of twins $\{u_1, \ldots, u_r\}$, then any zero forcing set of G must contain at least $r - 1$ of the vertices $\{u_1, \ldots, u_r\}$.*

Proposition 3.6 *Let G be a graph that has a set of twins $T = \{u_1, \ldots, u_r\}$ with $r \geq 3$, and let $G_i = G - u_i$. If S_i is a zero forcing set of G_i, then $S = S_i \cup \{u_i\}$ is a zero forcing set of G. If S is a zero forcing set of G and $u_i \in S$, then $S_i = S \setminus \{u_i\}$ is a zero forcing set of G_i. Thus there is a bijection between zero forcing sets of G_i and zero forcing sets of G that contain u_i, and a zero forcing set S_i of G_i is minimal if and only if $S = S_i \cup \{u_i\}$ is a minimal zero forcing set of G.*

Proof It is immediate that S_i being a zero forcing set of G_i implies $S = S_i \cup \{u_i\}$ is a zero forcing set of G (without any assumption about twins). Let S be a zero forcing set of G. Since $r \geq 3$, S must contain at least two vertices in T, say u_i and u_j. Choose a set of forces $\mathscr{F}$ that color every vertex of G blue. We show that at least one of u_i and u_j does not perform a force. Neither u_i nor u_j can perform a force until all but one of their common set of neighbors are blue. So if u_i forces the one white neighbor blue, then u_j cannot perform a force and if u_j forces, then u_i can't (it is possible neither forces). Thus we may assume u_i does not perform a force. Then $S_i = S \setminus \{u_i\}$ is a zero forcing set of G_i using the set of forces $\mathscr{F}$. The last sentence is immediate. $\square$

The technique of removing a nonforcing vertex v from a zero forcing set S of G to obtain a zero forcing set $S \setminus \{v\}$ for $G - v$ is well known (see, for example, the proof of Theorem 2.7 in [9]). When there are only two twins, it is not true that $S_i = S \setminus \{u_i\}$ must be a zero forcing set of G_i, because it is possible that only one of u_1 and u_2 is in S and that vertex is required to perform a force.

This is illustrated in the next example.

Example 3.7 Consider the *double star* $DS(2, 2)$ constructed by adding an edge between the centers of two disjoint copies of $K_{1,2}$. If the leaves of one star are u_1 and u_2, then $\mathrm{Z}(DS(2, 2)) = 2 = \mathrm{Z}(DS(2, 2) - u_1)$ (see [8]).

Lemma 3.8 *Let G be a graph that has a set of twins $T = \{u_1, \ldots, u_r\}$ with $r \geq 3$, and let $G_i = G - u_i$. If $\mathscr{Z}_k^{\mathrm{TAR}}(G_i)$ is connected for some i, then $\mathscr{Z}_{k+1}^{\mathrm{TAR}}(G)$ is connected.*

Proof The graphs G_j are isomorphic and hence the $\mathscr{Z}_k^{\mathrm{TAR}}(G_j)$ are isomorphic as well. Thus if $\mathscr{Z}_k^{\mathrm{TAR}}(G_i)$ is connected for some i, then $\mathscr{Z}_k^{\mathrm{TAR}}(G_j)$ is connected for all j.

For $S \subseteq V(G)$ with $u_i \in S$, let $S_i = S \setminus \{u_i\}$. Since $r \geq 3$, any zero forcing set S of G must contain at least two vertices in T, say u_i and u_j. By Proposition 3.6, a set

$S \subseteq V(G)$ that contains u_i, u_j is a zero forcing set of G if and only if $S \setminus u_i$ is a zero forcing set of G_i and $S \setminus u_j$ is a zero forcing set of G_j.

Let S, S' be two zero forcing sets of G of size $k+1$ or less. Since $r \geq 3$ and each can omit at most one vertex in T, their intersection must contain at least one u_i. Then $S \setminus u_i, S' \setminus u_i$ are zero forcing sets for G_i. Since by assumption $\mathscr{Z}_k^{\mathrm{TAR}}(G_i)$ is connected, that means that there is a path between $S \setminus u_i$ and $S' \setminus u_i$ in $\mathscr{Z}_k^{\mathrm{TAR}}(G_i)$ and hence a path between S, S' in $\mathscr{Z}_{k+1}^{\mathrm{TAR}}(G)$. □

Proposition 3.9 *For* $r \geq 2$, $\mathrm{Z}(H_r) = r+2$, $\overline{\mathrm{Z}}(H_r) = r+4$, $\underline{z_0}(H_r) = r+3$, *and* $z_0(H_r) = r+5$.

Proof The proof is by induction. The base case is $H = H_2$, which has minimal zero forcing sets of sixes 4 and 6 and has $\mathscr{Z}_5^{\mathrm{TAR}}(H_2)$ is connected. Apply Proposition 3.6 to show that H_r has minimal zero forcing sets of sizes $r+2$ and $r+4$. Apply Lemma 3.8 to show $\mathscr{Z}_{r+3}^{\mathrm{TAR}}(H_r)$ and $\mathscr{Z}_{r+5}^{\mathrm{TAR}}(H_r)$ are connected. Thus $\underline{z_0}(H_r) = r+3$ (because $\mathscr{Z}_{\mathrm{Z}(G)}^{\mathrm{TAR}}(G)$ is disconnected for every G). Since $|V(H_r)| = r+6$, $\overline{\mathrm{Z}}(H_r) = r+4$ and $z_0(H_r) = r+5$. □

4 Concluding Remarks

We have established that the X-TAR graphs of G and G' are isomorphic if and only if there is a relabeling of the vertices of G' such that G and G' have exactly the same X-sets. However, we have not investigated the question of how to determine whether two X-TAR graphs are isomorphic (beyond providing obvious necessary conditions). This question would be interesting to investigate.

We have only begun the study of zero forcing TAR graphs. One question we have not addressed is determining whether a graph can be realized as a zero forcing TAR graph (for a graph that satisfies the necessary condition of being a subgraph of a hypercube). We have presented examples of $\mathscr{Z}^{\mathrm{TAR}}(G)$ for various G, e.g., $K_{1,r} = \mathscr{Z}^{\mathrm{TAR}}(K_n)$. As noted in [5], $\mathscr{Z}^{\mathrm{TAR}}(G)$ is not isomorphic to any hypercube for any graph of order $n \geq 2$. But these are just examples of graphs that can and cannot be realized as zero forcing TAR graphs, and a more systematic study would be of interest. Additional structural study, such as properties of the base graph that ensure the existence of Hamilton paths in $\mathscr{Z}^{\mathrm{TAR}}(G)$, would also be of interest.

Acknowledgements The authors thank the referees for helpful comments that have improved the paper. The authors thank the American Institute of Mathematics (AIM) and the National Science Foundation (NSF) for support of this research. The research of B. Curtis was also partially supported by NSF Grant 1839918.

Author Contributions All authors contributed to the study conception and design. All authors read and approved the final manuscript.

Funding This work was supported by the National Science Foundation (Grant Numbers 1839918, 1929334).

Data Availibility Statement The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose.

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