



Nets in $\mathbb{P}^2$ and Alexander Duality

Nancy Abdallah¹ · Hal Schenck²

Received: 3 October 2021 / Revised: 21 August 2022 / Accepted: 3 September 2022 /

Published online: 24 April 2023

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Abstract

A net in $\mathbb{P}^2$ is a configuration of lines $\mathcal{A}$ and points X satisfying certain incidence properties. Nets appear in a variety of settings, ranging from quasigroups to combinatorial design to classification of Kac–Moody algebras to cohomology jump loci of hyperplane arrangements. For a matroid M and rank r , we associate a monomial ideal (a monomial variant of the Orlik–Solomon ideal) to the set of flats of M of rank $\leq r$. In the context of line arrangements in $\mathbb{P}^2$, applying Alexander duality to the resulting ideal yields insight into the combinatorial structure of nets.

Keywords Line arrangement · Net · Alexander duality · Free resolution

Mathematics Subject Classification 05B35 · 52C35

1 Introduction

The investigation of point-line incidence relations in $\mathbb{P}^2$ reaches back into the mists of time; for a comprehensive treatment see Grünbaum [23].

Definition 1.1 For a configuration of lines $\mathcal{A} \subset \mathbb{P}_{\mathbb{C}}^2$, if p is an intersection point of two or more lines, define $\mu(p) = |\text{lines through } p| - 1$, and let $L_2(\mathcal{A})$ be the set of all intersection points. A (k, d) net is a partition Π of the lines of $\mathcal{A}$ into $k \geq 3$ blocks,

Editor in Charge: Kenneth Clarkson

Schenck supported by NSF 2006410.

Nancy Abdallah
nancy.abdallah@hb.se

Hal Schenck
hks0015@auburn.edu

¹ Department of Mathematics, University of Borås, Borås, Sweden

² Department of Mathematics, Auburn University, Auburn, AL 36849, USA

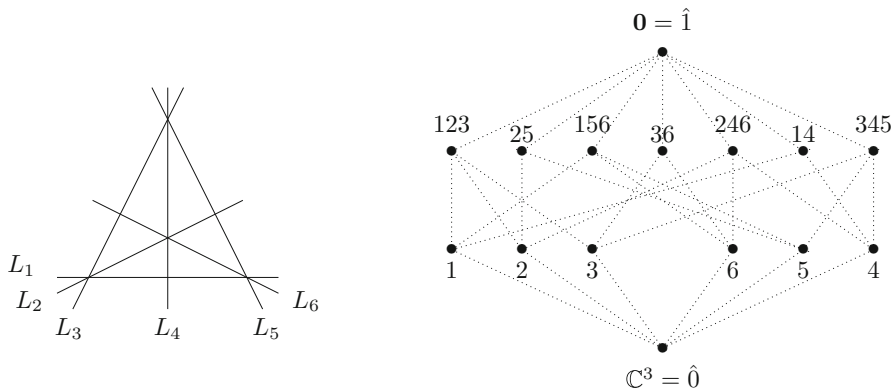


Fig. 1 The line configuration $A_3 \subseteq \mathbb{P}^2$ and $L(A_3)$ for $A_3 \subseteq \mathbb{C}^3$

each containing $d \geq 2$ lines, and a subset $X \subseteq \{p \in L_2(\mathcal{A}) \mid \mu(p) \geq 2\}$ of multiple points such that

- (i) every pair of lines from distinct blocks meet in some $p \in X$;
- (ii) there is exactly one line from each block of Π passing through a point $p \in X$.

A *potential* (k, d) net is a partition Π and subset X as above, but without the requirement that conditions (i) and (ii) hold.

If (Π, X) is a (k, d) net, then it is easy to show that every line meets X in d points, and $|X| = d^2$. In [43] Yuzvinsky proves that a (k, d) net must have $k \in \{3, 4\}$.

Example 1.2 For $1 \leq i < j \leq 4$ the equations $x_i - x_j = 0$ define a set of hyperplanes in $\mathbb{C}^4$ which all contain the subspace $W = \text{span}(1, \dots, 1)$. Projecting to $W^\perp$ yields a configuration of planes $A_3 \subseteq \mathbb{C}^3$ with common intersection at the origin, so A_3 also defines a line configuration in $\mathbb{P}^2$. The matroid defined by the lattice of intersections (in $\mathbb{C}^3$) is depicted on the right of Fig. 1. The partition $|14|25|36|$ and set of triple points $X = \{123, 156, 246, 345\}$ define a net.

It follows immediately from Definition 1.1 that if (Π, X) is a net, then every $\bigcap_i H_i = p \in L_2(\mathcal{A})$ is either an element of X , or has all H_i in the same block of Π . If $H_1 \cap H_2 = p \in L_2(\mathcal{A})$ and $\mu(p) = 1$, then H_1 and H_2 must be in the same block of Π .

The set of flats of a matroid, partially ordered by inclusion, form a lattice, so it is natural to ask:

Question 1.3 Is there a monomial ideal associated to a matroid that captures existence of a net?

Definition 1.4 For a matroid on ground set $\{1, \dots, n\}$ and choice of rank r and field $\mathbb{k}$, let $S = \mathbb{k}[x_1, \dots, x_n]$, and let J denote the ideal generated by monomials corresponding to the flats of rank $\leq r$. So a monomial $m = x_{i_1} \cdots x_{i_k} \in J \leftrightarrow [i_1, \dots, i_k]$ is a flat of rank at most r .

Example 1.5 For $r = 2$, the ideal J in Example 1.2 is generated by

$$\langle x_1x_4, x_2x_5, x_3x_6, x_1x_2x_3, x_1x_5x_6, x_2x_4x_6, x_3x_4x_5 \rangle.$$

Definition 1.4 works for any matroid; our interest stems from the study of complex projective hyperplane arrangements. In that setting, a flat of rank r corresponds to a (maximal) collection of hyperplanes meeting in codimension r . Work of Falk–Yuzvinsky in [20] shows that nets play a fundamental role in the study of the resonance variety of a hyperplane arrangement.

The resonance variety is defined in terms of the Orlik–Solomon algebra, and has attracted considerable attention: see for example work of Aomoto [1], Esnault–Schechtman–Viehweg [16], Schechtman–Terao–Varchenko [35], Yuzvinsky [40], Falk [18], Cohen–Suciu [9], Libgober–Yuzvinsky [26], and Falk–Yuzvinsky [20]. The Orlik–Solomon algebra is not needed to describe nets, but is used to define resonance varieties. For completeness we include in Sect. 5 an appendix on the Orlik–Solomon algebra and resonance varieties.

The generalization of Example 1.2 will serve as a running example. The *braid arrangement* A_n is defined by equations $x_i - x_j = 0$ for $1 \leq i < j \leq n + 1$. It plays a central role in many areas of mathematics: in mathematical physics, the complement of A_n is the configuration space for $n + 1$ non-colliding points. In combinatorics, the lattice of intersections $L(A_n)$ is isomorphic to the partition lattice Π_{n+1} , and in representation theory, A_n consists of fixed points of reflections in the Weyl group of $SL(n + 1)$.

Remark 1.6 By [2, Prop. 2.1], for a squarefree monomial ideal, results over the symmetric algebra may be translated to results over the exterior algebra, and vice versa. In this paper, we work over the symmetric algebra.

For hyperplane arrangements, a natural first guess at answering Question 1.3 is the initial ideal of the Orlik–Solomon ideal, which has been used to good effect in a number of settings, e.g. Björner–Ziegler [6]. It turns out that the initial ideal loses too much combinatorial information to be useful in identifying nets and resonance. The ideal J appearing in Definition 1.4 is our proposed answer to Question 1.3. As our main interest is in nets, we focus on the rank two case, and in this setting call the ideal J the *monomial OS ideal*.

A main tool in our investigation is Alexander duality. Alexander duality is a staple of both algebraic topology, commutative algebra, and combinatorics. A combinatorial proof of Alexander duality appears in [5]. In commutative algebra Alexander duality plays a key role in the study of squarefree monomial ideals. Applications of Alexander duality related to matroids and arrangements appear in works of Björner–Ziegler [7], Falk [17], and Eisenbud–Popescu–Yuzvinsky [15], but none address Question 1.3.

In Sect. 2 we review some necessary concepts from homological and commutative algebra: free resolutions, Castelnuovo–Mumford regularity, Betti numbers, and algebraic Alexander duality. In Sect. 3 we use Alexander duality to make a connection between nets and the ideal J , and in Sect. 4 we use Alexander duality to study the braid arrangement. As noted above, the appendix of Sect. 5 is a quick primer on the Orlik–Solomon algebra and resonance varieties.

1.1 Approach and Main Results

We state all results in terms of the symmetric algebra, which by Remark 1.6 can be translated to the exterior setting if desired.

Theorem A *If $\mathcal{A} = (\Pi, X)$ is a potential (k, d) net, with $\Pi = (\pi_1, \dots, \pi_k)$, let J_Π denote the ideal generated by the k monomials of degree d given by products of the variables within each block π_i , and let J_X denote the ideal generated by monomials corresponding to the points of X . Hence $J = J_X + J_Y$, where J_Y is generated by monomials corresponding to elements of $L_2(\mathcal{A})$ not in X . As noted earlier the degree two component of J satisfies $J_2 \subseteq J_Y$.*

(i) $\mathcal{A} = (\Pi, X)$ is a (k, d) net iff

$$(J_X^\vee)_d = (J_\Pi)_d,$$

where $I^\vee$ denotes the Alexander dual of I , defined in §2.

(ii) *If $\mathcal{A} = (\Pi, X)$ is a (k, d) net, then all intersections of lines within a block π_i of Π are normal crossing iff J_2 is a direct sum of k determinantal ideals $J_{\pi_1}, \dots, J_{\pi_k}$, with each J_{π_i} generated by all squarefree quadratic monomials in a block of d -variables, and the blocks are disjoint.*

In the setting of (ii), a corollary is that each J_{π_i} is the ideal of the 2×2 minors of a $2 \times d$ matrix, and the quadratic quotient S/J_2 factors as

$$S/J_2 \simeq \bigotimes_{i=1}^k S_{\pi_i}/J_{\pi_i}, \quad \text{with } S_{\pi_i} \text{ a polynomial ring in the variables of } \pi_i.$$

This in turn means that the free resolution of S/J_2 as an S -module is the tensor product of the free resolutions of the S/J_{π_i} . Hence the free resolution of S/J_2 is a tensor product of k Eagon–Northcott complexes, so S/J_2 is arithmetically Cohen–Macaulay (which means that $\text{codim}(J_2) = \text{pdim}(S/J_2)$), and the Alexander dual $S/J_2^\vee$ has a linear free resolution (which means that all differentials appearing in the free resolution of $S/J^\vee$ after the first step are matrices of linear forms).

Theorem B is an analysis of the monomial OS algebra for the type A reflection arrangements generalizing Example 1.2. An arrangement of type A_n is defined by the vanishing of the linear forms $x_i - x_j$ for $1 \leq i < j \leq n+1$, which is also the graphic arrangement (see [29, §2.4]) corresponding to the complete graph K_{n+1} . The monomial OS ideal for K_{n+1} is generated by cubic monomials corresponding to triangles in the graph, and quadratic monomials corresponding to pairs of disjoint edges.

Theorem B *For the braid arrangement A_{n-1} , let $J(K_n)$ denote the ideal J . Then the Hilbert series is $P(S/J(K_n), t)/(1-t)^{n-1}$, where the numerator is*

$$P(S/J(K_n), t) = n + (1-n)(1-t)^{n-1} - \binom{n}{2} t(1-t)^{n-2}.$$

The ring $S/J(K_n)$ has Castelnuovo–Mumford regularity two. For $S/J(K_n)^\vee$, the Hilbert series is $P(S/J(K_n)^\vee, t)/(1-t)^{n-1}$, where the numerator is

$$P(S/J(K_n)^\vee, t) = 1 - nt^{\binom{n-1}{2}} + \binom{n}{2}t^{\binom{n}{2}-1} - \binom{n-1}{2}t^{\binom{n}{2}},$$

and $S/J(K_n)^\vee$ has projective dimension three, with $\mathbb{Z}^n$ -graded Betti numbers

$$\begin{aligned} b_{0,\mathbf{m}}(S/J(K_n)^\vee) &= 1 & \text{if } |\mathbf{m}| &= 0, \\ b_{1,\mathbf{m}}(S/J(K_n)^\vee) &= 1 & \text{if } \mathbf{m} \leftrightarrow K_{n-1} \subseteq K_n, \\ b_{2,\mathbf{m}}(S/J(K_n)^\vee) &= 1 & \text{if } |\mathbf{m}| &= \binom{n}{2} - 1, \\ b_{3,\mathbf{m}}(S/J(K_n)^\vee) &= \binom{n-1}{2} & \text{if } |\mathbf{m}| &= \binom{n}{2}. \end{aligned} \quad (1.1)$$

Theorem 4.5 describes the entire minimal free resolution for $S/J(K_n)^\vee$. We discuss Alexander duality in Sect. 2, prove Theorem A in Sect. 3, and prove Theorem B in Sect. 4.

2 Alexander Duality

We will make use of two fundamental results involving Alexander duality: Primary decomposition, and Hochster's theorem. An excellent reference for both is [27].

2.1 Alexander Duality and Free Resolutions

Definition 2.1 Fix a field $\mathbb{k}$, and let Δ be a simplicial complex on vertex set V . If $|V| = n$, let $S = \mathbb{k}[x_1, \dots, x_n]$. The Stanley–Reisner ring of Δ is S/I_Δ , where

$$I_\Delta = \langle x_{i_1} \cdots x_{i_j} \mid [i_1, \dots, i_j] \text{ is not a face of } \Delta \rangle.$$

The ideal I_Δ encodes all the non-faces of Δ , so in particular the simplicial complex Δ and the ideal I_Δ carry the same information.

Example 2.2 Let Δ be a simplicial complex on four vertices, and edges $\{[12], [23], [34], [14], [13]\}$. The missing faces are the edge $[24]$, and all triangles. The missing triangles $[124]$ and $[234]$ and the missing full 3-simplex $[1234]$ are consequences of the missing edge, so

$$I_\Delta = \langle x_2x_4, x_1x_2x_3, x_1x_3x_4 \rangle.$$

The complement of a face of Δ is a *coface*; let $\text{CF}(\Delta)$ denote the set of minimal cofaces of Δ .

Theorem 2.3 [36, 5.3.3] *For a simplicial complex Δ , the primary decomposition of I_Δ is*

$$I_\Delta = \bigcap_{[v_{i_1} \cdots v_{i_k}] \in \text{CF}(\Delta)} \langle x_{i_1}, \dots, x_{i_k} \rangle.$$

Example 2.4 In Example 2.2 the minimal cofaces of Δ are $\{[34], [14], [12], [23], [24]\}$, so the primary decomposition is

$$I_\Delta = \langle x_1, x_2 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle. \quad (2.1)$$

The fact that the minimal generators of a primary component can be chosen as variables is special to squarefree monomial (= Stanley–Reisner) ideals. Notice that by choosing variables as minimal generators of a primary component, the product of the generators of a primary component of a Stanley–Reisner ideal is a monomial. The ideal generated by such monomials (one for each primary component) is called the *monomialization* of the primary decomposition of I_Δ .

Definition 2.5 The combinatorial Alexander dual of Δ is

$$\Delta^\vee = \{\sigma \subset V \mid \bar{\sigma} \notin \Delta\}.$$

The condition that $\bar{\sigma} \notin \Delta$ means that σ is the complement of a non-face of Δ .

Theorem 2.6 (see [12] or [24]) *Monomializing the primary decomposition of I_Δ yields $I_{\Delta^\vee}$.*

Example 2.7 The nonfaces of Δ of Example 2.2 are

$$\{[123], [124], [134], [234], [24]\},$$

so the complements of the nonfaces are

$$\{[4], [3], [2], [1], [13]\}.$$

Hence the maximal faces of $\Delta^\vee$ are the vertices $[2]$, $[4]$ and the edge $[13]$. In particular, all edges are missing from $\Delta^\vee$ except $[13]$. Monomializing the primary decomposition in (2.1) yields

$$I_{\Delta^\vee} = \langle x_1x_2, x_1x_4, x_2x_3, x_2x_4, x_3x_4 \rangle,$$

which is indeed the Stanley–Reisner ideal of $\Delta^\vee$.

2.2 Free Resolutions and Betti Tables

The Hilbert Syzygy Theorem [13] guarantees that any finitely generated $\mathbb{Z}$ -graded $S = \mathbb{k}[x_1, \dots, x_n]$ -module M has a *minimal graded finite free resolution*: an exact sequence of free modules $F_i \simeq \bigoplus_j S(-j)^{b_{i,j}}$ with $b_{i,j} \in \mathbb{Z}$:

$$0 \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \quad (2.2)$$

where $i \leq n$ and the entries of the d_i matrices are homogeneous of positive degree.

Definition 2.8 For M as above, the *regularity* and *projective dimension* are

$$\operatorname{reg}(M) = \sup \{j \mid b_{i,i+j} \neq 0\} \quad \text{and} \quad \operatorname{pdim}(M) = \sup \{i \mid b_{i,\bullet} \neq 0\}.$$

The *graded Betti numbers* are

$$b_{i,j} = \dim_{\mathbb{k}} \operatorname{Tor}_i(M, \mathbb{k})_j.$$

This data is compactly encoded in the *Betti table* [14]: an array whose entry in position (i, j) (reading over and down) is $b_{i,i+j}$. This indexing seems odd, but it is set up so that $\operatorname{reg}(M)$ is given by the index of the bottom row of the Betti table.

Example 2.9 The minimal free resolution for I_Δ from Example 2.2 is given below.

$$0 \longleftarrow I_\Delta \xleftarrow{[x_2x_4 \ x_1x_2x_3 \ x_1x_3x_4]} S(-2) \oplus S(-3)^2 \xleftarrow{\begin{bmatrix} x_1x_3 & 0 \\ -x_4 & x_4 \\ 0 & -x_2 \end{bmatrix}} S(-4)^2 \longleftarrow 0$$

The corresponding Betti table is

$j \backslash i$	0	1
2	1	—
3	2	2

The first column of the table reflects that I_Δ has one quadratic generator and two cubic generators. The second column shows there are two syzygies on the three generators. In the same fashion it is easy to write down the minimal free resolution for $S/I_{\Delta^\vee}$, which has Betti table

$j \backslash i$	0	1	2	3
0	1	—	—	—
1	—	5	6	2

[27, Cor. 5.59] shows that $\operatorname{reg}(I_\Delta) = \operatorname{pdim}(S/I_{\Delta^\vee})$; for the example above we have that $\operatorname{pdim}(S/I_{\Delta^\vee}) = 3 = \operatorname{reg}(I_\Delta)$.

2.3 Hochster's Theorem

(see [24] or [27]) As in Sect. 2.2, Δ is a simplicial complex on n vertices and $S = \mathbb{k}[x_1, \dots, x_n]$. However, we now endow S with the $\mathbb{Z}^n$ grading, with $\deg(x_i) = \mathbf{e}_i$.

Theorem 2.10 *For a simplicial complex Δ on n vertices and S graded by $\mathbb{Z}^n$, let $\mathbf{m} \in \mathbb{Z}^n$ be a multidegree, and $|\mathbf{m}| = \sum_i m_i$. Then*

$$b_{i,\mathbf{m}}(I_\Delta) = \dim(\tilde{H}_{|\mathbf{m}|-i-2}(\Delta_{\mathbf{m}}, \mathbb{k}))$$

where $\Delta_{\mathbf{m}}$ is the subcomplex of Δ consisting of the faces of Δ of weight $\mathbf{n}$ such that for all $i \in \{1, \dots, n\}$, $n_i \leq m_i$ (hence $\mathbf{n} \leq \mathbf{m}$ pointwise).

Example 2.11 For Example 2.2, we compute the homology groups. The generators of I_Δ occur in multidegrees $\{(0101), (1110), (1011)\}$. We now compute

$$\dim(\tilde{H}_{|\mathbf{m}|-3}(\Delta_{\mathbf{m}}, \mathbb{k}))$$

for $\mathbf{m} = (1111)$. For this multidegree, $\Delta_{\mathbf{m}}$ is clearly the entire complex Δ , so consists of two (hollow) triangles, sharing the common edge [13], hence

$$\dim(\tilde{H}_1(\Delta_{(1111)}, \mathbb{k})) = 2,$$

yielding two first syzygies on I_Δ . The syzygies themselves are

$$[x_1x_3, -x_4, 0], \quad [x_1x_3, 0, -x_2].$$

3 Proof of Theorem A

Alexander duality led us to Theorem A: it was computations with the quadratic component J_2 and the Alexander dual $J_2^\vee$ which indicated that $J_2^\vee$ had a linear resolution. Duality also is central in understanding nets.

Proof For the proof of (i), the key is Theorem 2.6: the Alexander dual $I^\vee$ of a squarefree monomial ideal I is obtained by monomializing the primary decomposition of I , as in Example 2.7, combined with the description of the primary decomposition in Theorem 2.3.

- We first show

$$(\Pi, X) \text{ is a } (k, d) \text{ net} \implies (J_\Pi)_d = (J_X^\vee)_d.$$

A component of the primary decomposition of the monomial ideal J_Π will contain exactly one variable from each block π_i of Π . Since (Π, X) is a net, this means that $J_X \subseteq J_\Pi^\vee$, so dualizing yields

$$J_\Pi \subseteq J_X^\vee.$$

Therefore $(J_\Pi)_d \subseteq (J_X^\vee)_d$; note that it is not true that $J_\Pi = J_X^\vee$. From the definition of J_Π , $\dim_{\mathbb{K}}(J_\Pi)_d = k$, so to prove equality it suffices to show that $\dim_{\mathbb{K}}(J_X^\vee)_d = k$. Because J_X is a squarefree monomial ideal, it is the Stanley–Reisner ideal of a simplicial complex Δ . By Theorems 2.3 and 2.6, to find the minimal degree generators of $J_X^\vee$, we need to find the biggest faces of Δ . As the monomials of J_X correspond to the nonfaces of Δ , the biggest faces of Δ correspond to monomials which are not divisible by any monomial in J_X . As soon as a monomial is divisible by at least one variable from each of the k blocks of Π , the net condition means it might be in J_X . However, the primary decomposition of J_Π will have d^k components, whereas J_X has d^2 .

We now argue that the maximal faces of Δ are exactly the complements of single blocks π_i ; to illustrate, in Example 1.2, the maximal faces of Δ are $\{[1245] = [36], [1346] = [25], [2356] = [14]\}$. To see this, notice that a set of lines σ corresponds to a non-face of Δ exactly when

$$\sigma \cap \pi_i \neq \emptyset \quad \text{for all blocks } \pi_i.$$

The maximal sets which fail to have this property are the complements of a single block of Π , and the result follows.

- For the other direction,

$$(J_\Pi)_d = (J_X^\vee)_d \implies (\Pi, X) \text{ is a } (k, d) \text{ net}.$$

For (Π, X) a potential (k, d) net, since $(J_\Pi)_d = (J_X^\vee)_d$ and J_Π is generated in degree d , we have

$$J_\Pi \subseteq J_X^\vee,$$

hence $J_X \subseteq J_\Pi^\vee$. As noted above, $J_\Pi^\vee$ is generated by monomials obtained by taking exactly one element from every block, so this means every generator of J_X satisfies this property, hence so also does X .

For the proof of (ii), the assumption on the net (Π, X) means that if (after a change of variables) $\pi = |1, \dots, d|$ is a block of the partition Π , then for $1 \leq i < j \leq d$, H_i and H_j meet in a normal crossing point p . Thus, for each block π as above, we have a subideal of J ,

$$J_\pi = \langle x_i x_j \mid 1 \leq i < j \leq d \rangle.$$

For generic $\{a_1, \dots, a_d\}$, J_π can be written as the ideal generated by the 2×2 minors of the $2 \times d$ matrix

$$M = \begin{bmatrix} x_1 & x_2 & \cdots & x_d \\ a_1 x_1 & a_2 x_2 & \cdots & a_d x_d \end{bmatrix},$$

which by [13, Thm. A2.10] has an Eagon–Northcott resolution

$$\cdots \longrightarrow \operatorname{Sym}^1(S^2)^* \otimes \Lambda^3(S^d) \longrightarrow \Lambda^2(S^d) \xrightarrow{\Lambda^2(M)} \Lambda^2(S^2) \longrightarrow S/I_2(M) \longrightarrow 0.$$

As the variables in the blocks of Π are distinct, we see that

$$J_2 \simeq \bigoplus_{\pi_i \in \Pi} J_{\pi_i},$$

The hypothesis on the quadratic component J_2 means that there is a partition Π of the $d \cdot k$ hyperplanes into k -blocks of size d , and that all hyperplanes within a block have normal crossing intersection. Therefore, any point of intersection with multiplicity greater than two cannot be contained in a block of Π , so must lie in X . The condition on the primary decomposition from (i) ensures that every multiple point in X meets exactly one line from each block of Π .

Remark 3.1 We thank an anonymous referee for suggesting a simplification in the proof of (i).

Example 3.2 An infinite family of $(3, d)$ nets is the Ceva family, given by the arrangement defined by the vanishing of the polynomial $(x^d - y^d)(x^d - z^d)(y^d - z^d)$. Note that the vanishing set of $x^d - y^d$ defines d lines passing through the point $(0:0:1)$, and similarly for the vanishing sets of $x^d - z^d$ and $y^d - z^d$. This yields a $(3, d)$ net

$$|1, \dots, d \mid d+1, \dots, 2d \mid 2d+1, \dots, 3d|.$$

Therefore J is generated by the three polynomials of degree d (which define J_Π)

$$\{x_1 \cdots x_d, x_{d+1} \cdots x_{2d}, x_{2d+1} \cdots x_{3d}\},$$

and d^2 cubics corresponding to the triple intersections. As noted in [4], for complex line arrangements in $\mathbb{P}^2$, this is the only infinite family known to have no normal crossing points. For $d = 3$ the matroid is depicted in Fig. 2, where points denote the lines of the configuration in $\mathbb{P}^2$. The set X consists of $\{147\}, \{258\}, \{369\}, \{168\}, \{249\}, \{267\}, \{348\}, \{159\}, \{357\}$.

In [3], Bartz gives a classification of complete 3-nets. As noted in [4], there are also two isolated examples of configurations in $\mathbb{P}_{\mathbb{C}}^2$ with no normal crossings: Klein's configuration has 21 lines, and Wiman's configuration has 45 lines. It would be interesting to check if these configurations support nets.

3.1 An Aside on the Singular Fibers of a Net or Pencil

In classical algebraic geometry, the term *net* refers to a three dimensional subspace of the space of sections of some line bundle on an algebraic variety Z , while a *pencil* is a two-dimensional subspace. Therefore a net gives a rational map ϕ from Z to $\mathbb{P}^2$, and a pencil gives a rational map ϕ from Z to $\mathbb{P}^1$.

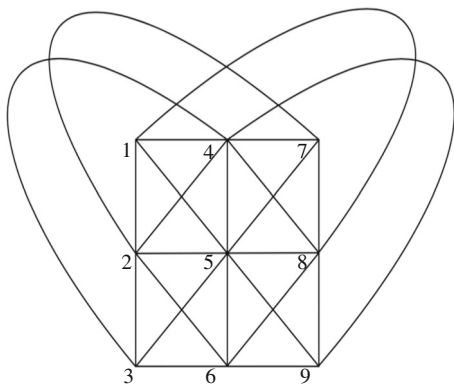


Fig. 2 The Ceva net for $d = 3$

Definition 3.3 [20, 3.3] If $F_1, F_2 \in \mathbb{C}[x, y, z]_d$ have no common factor, then the pencil $a_1 F_1 + a_2 F_2$ with $[a_1 : a_2] \in \mathbb{P}^1$ is *Ceva type* if there are three or more fibers that factor as products of linear forms, and after blowing up the base locus, the proper transforms of every fiber of ϕ are connected.

Proposition 3.4 If $\mathcal{A} = (\Pi, X)$ is a net satisfying part (ii) of Theorem A (all intersections of lines within a block are normal crossing), then the net has singular fibers beyond the singular fibers coming from the blocks of Π , unless $\mathcal{A}$ is a $(3, 2)$ net or a $(4, 3)$ net.

Proof In [20, Thm. 4.2], Falk–Yuzvinsky prove a result on the numerics of the Euler characteristic of a multinet (Π, X) , which for a (k, d) net takes the form

$$3 + |X| \geq (2 - k)(3d - d^2) + 2kd - \sum_{p \in \bar{X}} \mu(p), \quad (3.1)$$

where $\bar{X}$ is the set of points of intersection of $\mathcal{A}$ of the lines *within* the blocks π_i . They prove that equality holds in (3.1) iff the only singular fibers of the net are the blocks. Since $|X| = d^2$ for a net, if all intersections within the blocks have $\mu(p) = 1$, then

$$\sum_{p \in \bar{X}} \mu(p) = k \binom{d}{2}.$$

The equality in (3.1) is only possible if $k = 6 - 6/d$, which implies $(k, d) \in \{(3, 2), (4, 3), (5, 6)\}$, and by [43] only $(3, 2)$ and $(4, 3)$ can arise for a net.

The next example illustrates both Theorem A and Proposition 3.4.

Example 3.5 The Pappus and non-Pappus arrangements appear as Examples 9 and 10 in Suciu’s survey paper [39]. Both are arrangements of nine lines in $\mathbb{P}^2$; each has nine double points and nine triple points. There is a ninth line (not pictured) at infinity (see

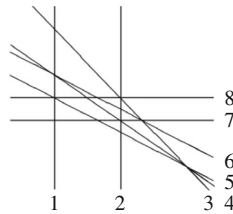


Fig. 3 non-Pappus arrangement

Fig. 3). The non-Pappus arrangement does not support a net. The Betti table of S/J_2 is

$j \backslash i$	0	1	2	3	4	5	6
0	1	—	—	—	—	—	—
1	—	9	9	—	—	—	—
2	—	—	18	18	—	—	—
3	—	—	—	3	9	9	2

The Pappus configuration in Fig. 4 supports a (3, 3) net, with blocks $|169|258|347|$. For the Pappus arrangement, the cubic generators of the monomial OS ideal are

$$\{x_1x_2x_7, x_1x_3x_5, x_1x_4x_8, x_2x_3x_9, x_2x_4x_6, x_3x_6x_8, x_4x_5x_9, x_5x_6x_7, x_7x_8x_9\}.$$

The primary decomposition of $J_\Pi = \langle x_1x_6x_9, x_2x_5x_8, x_3x_4x_7 \rangle$ has components

$$\begin{aligned} &(x_1, x_2, x_3), \quad (x_1, x_2, x_4), \quad (x_1, x_2, x_7), \quad (x_1, x_3, x_5), \quad (x_1, x_3, x_8), \quad (x_1, x_4, x_5), \\ &(x_1, x_4, x_8), \quad (x_1, x_5, x_7), \quad (x_1, x_7, x_8), \quad (x_2, x_3, x_6), \quad (x_2, x_3, x_9), \quad (x_2, x_4, x_6), \\ &(x_2, x_4, x_9), \quad (x_2, x_6, x_7), \quad (x_2, x_7, x_9), \quad (x_3, x_5, x_6), \quad (x_3, x_5, x_9), \quad (x_3, x_6, x_8), \\ &(x_3, x_8, x_9), \quad (x_4, x_5, x_6), \quad (x_4, x_5, x_9), \quad (x_4, x_6, x_8), \quad (x_4, x_8, x_9), \quad (x_5, x_6, x_7), \\ &(x_5, x_7, x_9), \quad (x_6, x_7, x_8), \quad (x_7, x_8, x_9). \end{aligned}$$

So the cubic minimal generators of J are elements of $J_\Pi^\vee$. The Betti table of S/J_2 is given by

$j \backslash i$	0	1	2	3	4	5	6
0	1	—	—	—	—	—	—
1	—	9	6	—	—	—	—
2	—	—	27	27	12	—	—
3	—	—	—	27	54	36	8

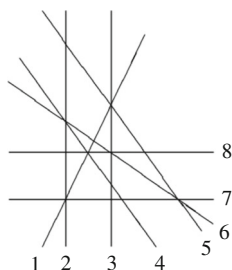


Fig. 4 Pappus arrangement

and the Betti table of the Alexander dual $S/J_2^\vee$ is

$j \backslash i$	0	1	2	3	4
0	1	—	—	—	—
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
4	—	—	—	—	—
5	—	27	54	36	8

In contrast, for the Alexander dual $S/J_2^\vee$ of the non-Pappus arrangement the Betti table is

$j \backslash i$	0	1	2	3	4
0	1	—	—	—	—
1	—	—	—	—	—
2	—	—	—	—	—
3	—	—	—	—	—
4	—	9	9	—	—
5	—	3	9	9	2

As the ideal $J_2^\vee$ for the Pappus arrangement has a linear resolution, by the Eagon–Reiner theorem [12], S/J_2 is Cohen–Macaulay, while the dual ideal $J_2^\vee$ of the non-Pappus arrangement does not have a linear resolution, so for the non-Pappus arrangement, S/J_2 is not Cohen–Macaulay.

Example 3.6 By Proposition 3.4, the Pappus arrangement has singular fibers in addition to the three normal crossing sets of lines of the blocks of Π . The unique (3, 2) net is that of Example 1.2, and the only known (4, 3) net is the Hessian; both satisfy the hypotheses of part (ii) of Theorem A.

4 The Monomial Orlik–Solomon Algebra for A_{n-1}

Let $J(K_n)$ denote the ideal J for the braid arrangement A_{n-1} (equivalently, the graphic arrangement K_n). Our focus in this section is on the algebraic behavior of the ideals

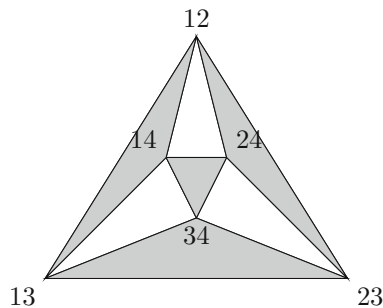


Fig. 5 The simplicial complex $\Delta_{J(K_4)}$

$J(K_n)$ and $J(K_n)^\vee$. To prove Theorem B, we first prove the projective dimension of $S/J(K_n)^\vee$ is three, which follows from an analysis of the corresponding ideal. With the bound on projective dimension of $S/J(K_n)^\vee$ in hand, an analysis using Theorem 2.10 yields the Betti numbers for $S/J(K_n)^\vee$.

Applying [27, Cor. 5.59] shows that $J(K_n)$ has regularity two. As there are only two rows in the Betti table of $J(K_n)$, to determine the Betti numbers, it suffices to determine the Hilbert series of $J(K_n)$ and one row of the Betti table.

The top row of the Betti table of $J(K_n)$ corresponds to a squarefree ideal generated by quadrics, so is an edge ideal of a graph Γ . The linear strand of the resolution is interesting in its own right, as it depends on the cut polynomial of Γ .

4.1 Hilbert Series of $J(K_n)$

We begin by describing the generators of the monomial ideals $J(K_n)$ and $J(K_n)^\vee$. For the complete graph K_n , the rank one elements of $L(A_{n-1})$ are the $\binom{n}{2}$ hyperplanes $V(x_i - x_j)$, so the rank two elements correspond to

- (a) triangles in K_n ,
- (b) pairs of disjoint edges in K_n .

We study the Stanley–Reisner ring in $\binom{n}{2}$ variables $S = \mathbb{k}[x_{ij} \mid 1 \leq i < j \leq n]$, modulo the ideal

$$J(K_n) = \langle x_{ij}x_{kl}, x_{ij}x_{ik}x_{jk} \mid \text{distinct } i, j, k, l \in \{1, \dots, n\} \rangle.$$

Example 4.1 For K_4 , the minimal non faces of $\Delta_{J(K_4)}$ are

$\{[12][34], [13][24], [14][23], [12][14][24], [23][24][34], [13][14][34], [12][13][23]\}.$

The complex in Fig. 5 corresponds to the braid arrangement in Fig. 1, where the vertices 12, 23, 13, 34, 14, and 24 in Fig. 5 correspond, respectively, to the lines $L_1, \dots, L_6$.

Lemma 4.2 *The simplicial complex $\Delta_{J(K_n)}$ corresponding to the ideal $J(K_n)$ consists of n simplices of dimension $n - 2$, each meeting the others in $n - 1$ points, and the face vector of $\Delta_{J(K_n)}$ is given by (with notation as in [44, §8.3])*

$$f(\Delta_{J(K_n)}, x) = x^{n-1} + n \left((n-1)x^{n-2} + \binom{n-1}{2}x^{n-3} + \binom{n-1}{3}x^{n-4} + \dots + (n-1)x + 1 \right) - \binom{n}{2}x^{n-2}.$$

Proof Vertices in $\Delta_{J(K_n)}$ correspond to edges in K_n . The maximal faces of $\Delta_{J(K_n)}$ correspond to simple graphs with vertex set $[n]$ that have no pairs of the types (a) or (b) above. The maximal such graphs are clearly the n star graphs with $n - 1$ edges,

$$\Delta_i = \{[i, j_1], \dots, [i, j_{n-1}]\}$$

for a fixed i . Every pair of such graphs share a common edge, and every edge lies in exactly two such graphs. This explains the term $\binom{n}{2}x^{n-2}$.

Remark 4.3 As pointed out by the referee, this complex is the nerve of the cover of K_n by closed edges, homotopy equivalent to K_n .

Example 4.4 For K_7 , the f -vector is

$$\begin{aligned} f(\Delta_{J(K_n)}) &= (1, 0, 0, 0, 0, 0, 0) + 7(0, 6, 15, 20, 15, 6, 1) - (0, 21, 0, 0, 0, 0, 0), \\ f(\Delta_{J(K_n)}, x) &= x^6 + 21x^5 + 105x^4 + 140x^3 + 105x^2 + 42x + 7. \end{aligned}$$

By [44, §8.3],

$$h(\Delta, x) = f(\Delta, x - 1),$$

so for K_n we have

$$\begin{aligned} h(\Delta_{J(K_n)}, x) &= (x-1)^{n-1} + n \left((n-1)(x-1)^{n-2} + \binom{n-1}{2}(x-1)^{n-3} + \dots + 1 \right) \\ &\quad - \binom{n}{2}(x-1)^{n-2}, \end{aligned}$$

which simplifies to

$$(x-1)^{n-1} + n(x^{n-1} - (x-1)^{n-1}) - \binom{n}{2}(x-1)^{n-2}.$$

In [25, §8.3], $h_i(\Delta)$ is the coefficient of $x^{\dim \Delta + 1 - i}$ in $h(\Delta, x)$, so since

$$P(S/I_\Delta, t) = (h_0 + h_1t + h_2t^2 + \dots)/(1-t)^{\dim \Delta + 1},$$

we need to reverse the order of the h -vector, and the result for the Hilbert series in Theorem B follows.

4.2 Resolution of the Alexander Dual

We now turn to the Alexander dual ideal $J(K_n)^\vee$. As noted in §2, $J(K_n)^\vee$ is the monomialization of the primary decomposition of $J(K_n)$. Letting $\text{CF}(\Delta)$ denote the set of minimal cofaces of Δ , Theorem 2.3 yields

$$J(K_n) = \bigcap_{[x_{i_1}, \dots, x_{i_k}] \in \text{CF}(\Delta)} \langle x_{i_1}, \dots, x_{i_k} \rangle.$$

By Lemma 4.2, Δ consists of n copies of the $n-2$ simplex Δ_{n-2} , glued at a total of $\binom{n}{2}$ vertices, so a maximal face of Δ is one of the Δ_{n-2} 's, whose complement coface consists of the remaining

$$\binom{n}{2} - (n-1) = \binom{n-1}{2}$$

vertices. Therefore $J(K_n)^\vee$ is generated by the corresponding n monomials of degree $\binom{n-1}{2}$.

Theorem 4.5 *The projective dimension of $S/J(K_n)^\vee$ is three, and the $\mathbb{Z}^{\binom{n}{2}}$ -graded Betti numbers are*

$$\begin{aligned} b_{0,\mathbf{m}}(S/J(K_n)^\vee) &= 1 & \text{if } |\mathbf{m}| &= 0, \\ b_{1,\mathbf{m}}(S/J(K_n)^\vee) &= 1 & \text{if } \mathbf{m} \leftrightarrow K_{n-1} \subseteq K_n, \\ b_{2,\mathbf{m}}(S/J(K_n)^\vee) &= 1 & \text{if } |\mathbf{m}| = \binom{n}{2} - 1, \\ b_{3,\mathbf{m}}(S/J(K_n)^\vee) &= \binom{n-1}{2} & \text{if } |\mathbf{m}| = \binom{n}{2}. \end{aligned}$$

Proof The proof follows from Hochster's formula. For each K_{n-1} in K_n , there is a generator with weight vector having entry 1 in the positions corresponding to edges of the K_{n-1} , hence n generators of weight $\binom{n-1}{2}$.

We now make use of the LCM lattice resolution, as described in [21]. The first syzygies of a monomial ideal correspond to the LCM of two generators. The two corresponding K_{n-1} subgraphs Δ_i and Δ_j intersect everywhere except along the missing edge $[ij]$ so the LCM corresponds to a weight vector which is one in all but the single entry $[ij]$. This yields $\binom{n}{2}$ first syzygies, with weight as above.

For the second syzygies, by Hochster's formula they must have weight $(1, 1, \dots, 1)$, and correspond to the LCM of triples of monomials, of which there are $\binom{n}{3}$. However, these choices are not independent, since every $\binom{n}{i}$ set with $i \geq 4$ has LCM of weight $(1, 1, \dots, 1)$. So accounting for dependencies, dependencies on dependencies, and so

on, we find that the number of minimal second syzygies is

$$\binom{n}{3} - \binom{n}{4} + \binom{n}{5} - \binom{n}{6} + \cdots = 1 - n + \binom{n}{2} = \binom{n-1}{2},$$

which concludes the proof. $\square$

Combining Lemma 4.2, Theorem 4.5, and [27, Cor. 5.59] proves Theorem B.

Example 4.6 The $\mathbb{Z}$ -graded Betti table for $S/J_7^\vee$ is

$j \backslash i$	0	1	2	3
0	1	—	—	—
1	—	—	—	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
13	—	—	—	—
14	—	7	—	—
15	—	—	—	—
16	—	—	—	—
17	—	—	—	—
18	—	—	21	15

The Hilbert series of $P(R/J_7^\vee, t)$ is therefore

$$\begin{aligned} & \frac{1 - 7t^{15} + 21t^{20} - 15t^{21}}{(1-t)^6} \\ &= \frac{1}{(1-t)^4} \left(\sum_{i=0}^{14} (i+1)t^i + 9t^{15} + 3t^{16} - 3t^{17} - 9t^{18} - 15t^{19} \right). \end{aligned}$$

4.3 The Linear Strand of $J(K_n)$ and the Cut Polynomial

A squarefree quadratic monomial ideal encodes the edges of a graph Γ , with a generator $x_i x_j$ corresponding to the edge $[ij]$; such ideals are often called *edge ideals*, and there is a wide literature on the topic; see [34]. It follows from Hochster's theorem that the Betti numbers $b_{i,i+1}(S/I_\Gamma)$ are determined by the *cut polynomial* of the graph Γ ; this is used by Papadima–Suciu in [30] to establish a formula for the Chen ranks of right angled Artin groups; their result is over E , but it translates to S .

Definition 4.7 For a simple (no loops or multiple edges) graph Γ , the cut polynomial is defined via

$$Q_\Gamma(t) = \sum_{j \geq 2} c_j(\Gamma) t^j, \quad \text{where } c_j(\Gamma) = \sum_{\substack{W \subseteq V \\ |W|=j}} (|\text{connected components of } \Gamma_W| - 1).$$

In [24], Hochster proves the Betti numbers $b_{j,j+1}$ of S/I_Γ satisfy

$$b_{j,j+1}(S/I_\Gamma) = c_{j+1}(\Gamma).$$

Since the regularity of $S/J(K_n)$ is two and we know the Hilbert series, to determine the Betti table, it suffices to determine the top row, hence to finding the coefficients of the cut polynomial of the graph Γ_n corresponding to the quadratic generators of $J(K_n)$. Let I_{Γ_n} denote the corresponding edge ideal. For small n the c_k are

n	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
4	3	0	0	0	0	0	0	0	0	0	0
5	15	30	10	0	0	0	0	0	0	0	0
6	45	210	390	285	100	15	0	0	0	0	0
7	105	840	3150	6510	7497	5565	2835	980	210	21	0

As any vertex $v \in \Delta_{J(K_n)}$ lies on a pair of $n - 2$ simplices, to disconnect v requires removing the $2(n - 2)$ vertices adjacent to v , leaving a total of $\binom{n}{2} - (2n - 3)$ vertices, hence $c_j(J(K_n))$ vanishes when $j \geq \binom{n-2}{2} + 2$.

Problem 4.8 Determine the cut polynomial for Γ_n . It is not hard to show that the first two c_i are

$$c_2 = b_{12}(S/I_{\Gamma_n}) = 3\binom{n}{4},$$

$$c_3 = b_{23}(S/I_{\Gamma_n}) = 3\binom{n}{2, 3, n-5} + \frac{1}{3}\binom{n}{2, 2, 2, n-6}.$$

5 Appendix: The Orlik–Solomon Algebra and Resonance Varieties

For a hyperplane arrangement

$$\mathcal{A} = \bigcup_{i=1}^m H_i \subseteq V \simeq \mathbb{K}^{\ell+1},$$

we write M for the complement $V \setminus \mathcal{A}$; unless otherwise noted V is a $\mathbb{K} = \mathbb{C}$ vector space. We focus on the case where $\mathcal{A} \subset V$ is *central* and *essential*, where central means the linear forms defining the H_i are homogeneous, and essential means the common intersection of the H_i is $0 \in V$. Note that $\mathcal{A}$ defines both an *affine* arrangement in V , as well as a projective arrangement in $\mathbb{P}(V)$.

Orlik and Solomon prove in [28] that the cohomology ring $H^*(M, \mathbb{Z})$ has a purely combinatorial description: it is determined by the intersection lattice $L(\mathcal{A})$. This lattice (in the graded poset sense) consists of the intersections of elements of $\mathcal{A}$, ordered by

reverse inclusion. The ambient vector space $V = \hat{0}$, rank one elements of $L_{\mathcal{A}}$ are hyperplanes, and the origin is $\hat{1}$.

For the complement M of a hyperplane arrangement $\mathcal{A}$ with fundamental group $\pi_1(M) = G$, the first resonance variety $R^1(G)$ is the jump locus for the cohomology of G . We work over a field $\mathbb{k}$ of characteristic zero; by Orlik–Solomon’s result $H^*(M, \mathbb{k}) \simeq E/I_{\mathcal{A}}$ (see Definition 5.4). By convention we write A for $H^*(M, \mathbb{k})$ and $R^1(A)$ for $R^1(G)$. A is $\mathbb{Z}$ -graded; as it is a cohomology ring we denote the i^{th} graded component by A^i . By Falk [18], the points $(a_1, \dots, a_m)$ of $R^1(A) \subseteq \mathbb{k}^m$ correspond to one-forms $a = \sum_1^m a_i e_i$ where the map $\lambda \mapsto \lambda \wedge a$ from A^1 to A^2 has rank $\leq m - 2$.

5.1 Combinatorics of Arrangements

Two important combinatorial players are the Möbius function and Poincaré polynomial:

Definition 5.1 The Möbius function $\mu: L(\mathcal{A}) \rightarrow \mathbb{Z}$ is given by

$$\mu(\hat{0}) = 1, \quad \mu(b) = - \sum_{a < b} \mu(a) \quad \text{if } \hat{0} < b.$$

The Poincaré polynomial $\pi(\mathcal{A}, t)$, defined as $\sum_{x \in L(\mathcal{A})} (-t)^{\text{rank}(x)} \mu(x)$, is equal to $\sum_{i=1}^{\ell+1} (\dim_{\mathbb{k}} A^i) t^i$.

In [18] Falk introduced the concept of a *neighborly partition*:

Definition 5.2 A partition Π of a subset U of $\mathcal{A}$ is neighborly if for every codimension two intersection $H_{i_1} \cap \dots \cap H_{i_k}$ with $\{i_1, \dots, i_k\} \subseteq U$ if all but one of the i_j are contained in a block of Π , then $\{i_1, \dots, i_k\}$ is contained in the block.

Example 5.3 For the arrangement A_3 in Example 1.2, the partition $\{14|25|36\}$ is a neighborly partition of the set $\{1, 2, 3, 4, 5, 6\}$ of all lines of the arrangement.

5.2 Algebra of Arrangements

The central algebraic object in the study of hyperplane arrangements is the cohomology ring of the arrangement complement, which was described by Orlik–Solomon in their landmark paper [28].

Definition 5.4 The Orlik–Solomon algebra A with coefficients in $\mathbb{k}$ of an arrangement

$$\mathcal{A} = \bigcup_{i=1}^m H_i \subseteq \mathbb{P}^{\ell}$$

is the quotient of the exterior algebra $E = \bigwedge(\mathbb{k}^m)$ on generators $e_1, \dots, e_m$ in degree 1 by the ideal $I_{\mathcal{A}}$ generated by all elements of the form

$$\partial(e_{i_1} \cdots e_{i_r}) = \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r} \quad \text{such that} \quad \text{codim}(H_{i_1} \cap \cdots \cap H_{i_r}) < r.$$

The ideal $I_{\mathcal{A}}$ is generated in degree ≥ 2 , so $A^0 = \mathbb{k}$ and $A^1 = \mathbb{k}^m$.

Example 5.5 For Example 1.2, $I_{\mathcal{A}} = \langle \partial(e_1 e_2 e_3), \partial(e_1 e_5 e_6), \partial(e_2 e_4 e_6), \partial(e_3 e_4 e_5) \rangle$.

It is clear from the definition that the Orlik–Solomon algebra A depends purely on the combinatorics of $L_{\mathcal{A}}$, as do the neighborly partitions appearing in Definition 5.2. Nets are connected to the resonance variety $R^1(A)$ via the results of [20]. As noted earlier, for each $a \in A^1$, we have $a \wedge a = 0$, so exterior right-multiplication by a defines a cochain complex of $\mathbb{k}$ -vector spaces

$$(A, a): \quad 0 \longrightarrow A^0 \xrightarrow{\wedge a} A^1 \xrightarrow{\wedge a} A^2 \xrightarrow{\wedge a} \cdots \xrightarrow{\wedge a} A^k \xrightarrow{\wedge a} \cdots \quad (5.1)$$

The complex (A, a) was introduced by Aomoto [1], and used by Esnault–Schechtman–Viehweg [16] and Schechtman–Terao–Varchenko [35] to study local system cohomology. For $a = \sum_{i=1}^n a_i e_i \in A^1$, we can define the loci where there complex is not exact:

$$R^j(A) = \{a \in A^1 \mid \dim H^j(A, a) \neq 0\},$$

which are homogeneous algebraic subvarieties of $\mathbb{P}^{m-1}$, introduced by Falk in [18]. Yuzvinsky shows in [40] that the complex is exact as long as $\sum_{i=1}^n a_i \neq 0$. Falk shows that each component of $R^1(A)$ is associated to a neighborly partition of a subset U of the hyperplanes of $\mathcal{A}$. In [20], Falk–Yuzvinsky give an interpretation of this in terms of the geometry of *multinets* in $\mathbb{P}^2$, which are collections of lines with multiplicity.

Example 5.6 For Example 1.2, since $e_i e_j - e_i e_k + e_j e_k = (e_i - e_j)(e_j - e_k)$, the generators of $I_{\mathcal{A}}$ can be written as

$$\begin{aligned} (e_1 - e_2) \wedge (e_2 - e_3), & \quad (e_1 - e_5) \wedge (e_5 - e_6), \\ (e_2 - e_4) \wedge (e_4 - e_6), & \quad (e_1 - e_3) \wedge (e_3 - e_5). \end{aligned}$$

Hence for exterior multiplication with the one form a

$$A^1 \xrightarrow{\wedge a} A^2,$$

the $\mathbb{P}^1$ spanned by $\{e_1 - e_2, e_2 - e_3\} \subseteq \mathbb{P}^5 \simeq \mathbb{P}(A^1)$ is a component of $R^1(A)$, as are the three $\mathbb{P}^1$'s associated to the other three quadrics in $I_{\mathcal{A}}$. A computation shows that

$$\begin{aligned} (e_1 - e_2 + e_4 - e_5) \wedge (e_2 - e_3 + e_5 - e_6) \\ = \partial(e_1 e_2 e_3) + \partial(e_1 e_5 e_6) - \partial(e_2 e_4 e_6) + \partial(e_3 e_4 e_5). \end{aligned}$$

This means there is a fifth $\mathbb{P}^1$ in $R^1(\mathcal{A})$. As shown by Falk in [18], $R^1(\mathcal{A})$ is the union of these five lines. The fifth component comes from a neighborly partition $\Pi = |14|25|36|$. As noted in Example 1.2 the partition Π is a $(3, 2)$ net.

For a maximal subset $U = \{H_{i_1}, \dots, H_{i_k}\}$ having codimension two intersection, the partition into singleton blocks $|i_1|i_2|\dots|i_k|$ is neighborly, and yields components of $R^1(\mathcal{A})$ which are called *local*. These components correspond to generators of $I_{\mathcal{A}}$, as in the first four $\mathbb{P}^1$'s in Example 5.6. In [9, 26] it is shown that $R^1(\mathcal{A})$ is a union of projectively disjoint projective subspaces.

Acknowledgements Our collaboration began at the CIRM conference on “Lefschetz Properties in Algebra, Geometry and Combinatorics”; we thank CIRM and the organizers. Computations in Macaulay2 [22] were essential to our work. We thank two referees for detailed and useful suggestions, especially a nice simplification in the proof of Theorem A.

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