Partitioning Axis-Parallel Lines in 3D

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Abstract

Let L be a set of n axis-parallel lines in $\mathbb{R}^3$. We are are interested in partitions of $\mathbb{R}^3$ by a set H of three planes such that each open cell in the arrangement $\mathcal{A}(H)$ is intersected by as few lines from L as possible. We study such partitions in three settings, depending on the type of splitting planes that we allow. We obtain the following results.

- There are sets L of n axis-parallel lines such that, for any set H of three splitting planes, there is an open cell in $\mathcal{A}(H)$ that intersects at least $\lfloor n/3 \rfloor - 1 \approx \frac{1}{3}n$ lines.
- If we require the splitting planes to be axis-parallel, then there are sets L of n axis-parallel lines such that, for any set H of three splitting planes, there is an open cell in $\mathcal{A}(H)$ that intersects at least $\frac{3}{2}\lfloor n/4 \rfloor - 1 \approx \left(\frac{1}{3} + \frac{1}{24}\right) n$ lines.
  Furthermore, for any set L of n axis-parallel lines, there exists a set H of three axis-parallel splitting planes such that each open cell in $\mathcal{A}(H)$ intersects at most $\frac{7}{18}n = \left(\frac{1}{3} + \frac{1}{18}\right) n$ lines.
- For any set L of n axis-parallel lines, there exists a set H of three axis-parallel and mutually orthogonal splitting planes, such that each open cell in $\mathcal{A}(H)$ intersects at most $\lceil \frac{5}{12}n \rceil \approx \left(\frac{1}{3} + \frac{1}{12}\right) n$ lines.

Keywords and phrases Space partitions, axis-parallel lines, equipartitions

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1 Introduction

Partitioning problems of point sets in $\mathbb{R}^d$ have been studied extensively. For instance, the famous Ham-Sandwich Theorem states that, given d finite point sets in $\mathbb{R}^d$, there exists a hyperplane that bisects each of the sets, in the sense of having at most half of the set in each of its two open halfspaces. Another well-known result is that for any set of n points in the plane, there are two lines that partition the plane into four open cells that each contain at most $\lfloor n/4 \rfloor$ points (a *4-partition*). The latter result has several stronger forms, where one can specify the orientation of one of the lines, or that the two lines be orthogonal to each other [6] (but not both). The 4-partition question naturally generalizes to the problem of 2^d-partitioning of a point set in $\mathbb{R}^d$ by d hyperplanes. Such a triple of planes indeed always exists in $\mathbb{R}^3$ [9, 16], in fact, with the orientation of one of the planes prespecified.

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Alternatively, one of the partitioning planes can be required to be perpendicular to the other two [4]. It is known that 2^d-partition does not always exist in $d > 4$ [1]. The case $d = 4$ remains stubbornly open. Results on other partitioning problems for finite point sets can be found in the surveys by Kaneko and Kano [10] and by Kano and Urrutia [11].

Similar theorems have been obtained for equipartitioning continuous measures. For example, the Ham-Sandwich theorem [15] is traditionally stated in continuous setting: Given d finite absolutely continuous measures in $\mathbb{R}^d$, there exists a hyperplane that bisects each measure, in the sense of having half of the mass of each measure on each side; see [2] for some early history of the theorem. The 2^d-partition problem mentioned above was originally asked by Grünbaum in 1960 [7] for measures, as well; for a survey, see [3]. For an overview of equipartitioning problems, see [15, 17].

In this paper we are interested in the following question: given a set L of n lines, partition the space into open cells such that each cell intersects only few lines. This problem has been studied extensively, in the context of cuttings and polynomial partitions, but these works typically focus on asymptotic results that use a (possibly constant but) large number of partitioning planes, or the zero set of a (constant but) large degree polynomial. For example, a classical result on *cuttings* in the plane [5] is that, for any choice of parameter r, $1 \leqslant r \leqslant n$, there exists a tiling of the plane by $O(r^2)$ trapezoids so that each open trapezoid meets n/r of the lines of L. In a similar spirit, a result of Guth [8] states that, for any degree $D > 1$, there exists a non-zero bivariate polynomial f of degree at most D, such that the removal of its zero set $Z(f)$ from the plane produces $O(D^2)$ open connected sets, each meeting at most n/D of the lines of L. Analogous results are known for higher dimensions.

Another variant that has been studied is to partition $\mathbb{R}^3$ recursively using planes, until each cell meets $O(1)$ of the input objects. This results in a so-called binary space partition (BSP) of the objects. It has been shown that any set of lines (or disjoint triangles) admits a BSP of size $O(n^2)$ [14], and any set of axis-parallel lines (or disjoint axis-parallel rectangles) admits a BSP of size $O(n^{3/2})$ [13]. Both bounds are tight in the worst-case.

In contrast to the above settings, we are interested in what can be achieved with a very small number of planes, similar to the results on Ham-Sandwich cuts and equipartitions. In particular, we are interested in partitions for a set L of n lines in $\mathbb{R}^3$ that use only three planes. More precisely, we want to partition $\mathbb{R}^3$ using a set H of three planes such that each open cell in the arrangement $\mathcal{A}(H)$ meets only few of the lines from L. Note that if the planes of H are in general position, each such cell is an "octant."

We are not aware of any work directly addressing this question in three or higher dimensions, for unrestricted sets of lines. The two-dimensional case was settled in [12, Lemma 7]: for any set of n lines in the plane in general position, there exists a pair of lines with the property that each open quadrant formed by them is met by at most $\lfloor n/4 \rfloor$ of the lines. In this initial study of the problem, we consider the case of axis-parallel lines in $\mathbb{R}^3$.

Problem statement and results. Let L be a set of n axis-parallel lines in $\mathbb{R}^3$. We would like to partition the space by three planes h_1, h_2, h_3, so that each of the open cells of $\mathcal{A}(H)$ meets as few lines of L as possible, where $H \coloneqq \{h_1, h_2, h_2\}$. We consider three variants, of increasing generality. The variants depend on the orientation of the planes used in the partitioning. To this end we define a plane to be *axis-parallel* if it is parallel to two of the main coordinate axes, we define it to be *semi-tilted* if it is parallel to exactly one of the main coordinate axis, and we define it to be *tilted* if not parallel to any coordinate axis. We will consider the following three types of partitionings.

Function	Lower bound	Reference	Upper bound	Reference
$g_\perp(n)$	$\frac{3}{2}\left\lfloor\frac{n}{4}\right\rfloor \approx \left(\frac{1}{3}+\frac{1}{24}\right)n$	Theorem 10	$\left\lceil\frac{5}{12}n\right\rceil \approx \left(\frac{1}{3}+\frac{1}{12}\right)n$	Theorem 16
$g_\parallel(n)$	$\frac{3}{2}\left\lfloor\frac{n}{4}\right\rfloor - 1 \approx \left(\frac{1}{3}+\frac{1}{24}\right)n$	Theorem 10	$\left\lfloor\frac{7}{18}n\right\rfloor \approx \left(\frac{1}{3}+\frac{1}{18}\right)n$	Theorem 15
$g(n)$	$\left\lfloor\frac{1}{3}n\right\rfloor - 1 \approx \frac{1}{3}n$	Corollary 2	$\left\lfloor\frac{7}{18}n\right\rfloor \approx \left(\frac{1}{3}+\frac{1}{18}\right)n$	Theorem 15

Table 1 Summary of general results

- The set H must consist of three mutually orthogonal axis-parallel planes; let $g_\perp$ be the corresponding function.
- The planes in H must all be axis-parallel ($g_\parallel$), but need not be pairwise orthogonal.
- The planes in H can be chosen arbitrarily (g), that is, H can also use semi-tilted or tilted planes.

For a set L of lines, we let $g(L)$ be the minimum integer k, such that there is a set H of three planes in $\mathbb{R}^3$ with every open cell of $\mathcal{A}(H)$ meeting at most k lines of L. Define $g_\parallel(L)$ and $g_\perp(L)$ analogously. Now define $g(n) := \max_{|L|=n} g(L)$, with the maximum taken over all sets of axis-parallel lines of size n. Define $g_\parallel(n)$ and $g_\perp(n)$ similarly. Clearly, $g(n) \leqslant g_\parallel(n) \leqslant g_\perp(n)$. Table 1 summarizes our results.

The rest of the paper is organized as follows. In Section 2 we introduce our simplest "three bundle" construction, which implies lower bounds on all three functions and illustrates some of the methods used in the rest of the paper. We prove stronger lower bounds on $g_\perp(n)$ and $g_\parallel(n)$ in Section 3. In Section 4 we present constructive upper bounds on $g_\parallel$ and $g_\perp$, and consequently also on g.

We will use the following convention throughout the paper: we say that a line ℓ *crosses* a plane h in $\mathbb{R}^3$, if ℓ meets h, but is not contained in it.

2 Three bundles

As a warm-up exercise we analyze a simple and natural configuration of lines. The analysis will give a first lower bound on the functions $g_\perp(n)$, $g_\parallel(n)$, and $g(n)$. In fact, the bound we obtain for $g(n)$ is the best bound we have for this case, where the splitting planes can be completely arbitrary. The configuration consists of three bundles of $n/3$ lines, where n is a multiple of 3; see Fig. 1 for an illustration. We call this configuration the *three-bundle configuration*, and denote it by $L^3(n)$, or just L^3 for short. It is defined as $L^3 := L^3_x \cup L^3_y \cup L^3_z$, where

$$L^3_x := \{(0,i,i) + \lambda(1,0,0) : 1 \leqslant i \leqslant n/3\},$$
$$L^3_y := \{(i,0,2n/3+1-i) + \lambda(0,1,0) : 1 \leqslant i \leqslant n/3\}, \text{ and}$$
$$L^3_z := \{(i,i,0) + \lambda(0,0,1) : n/3 < i \leqslant 2n/3\}.$$

To be able to handle the case of tilted splitting planes, we need to slightly shift the lines in L^3, so that no plane (titled or otherwise) contains three or more lines. For example, L^3_x is actually defined as $L^3_x := \{(0,i+\varepsilon_i,i) + \lambda(1,0,0) : 1 \leqslant i \leqslant n/3\}$, where the ε_i's are small, distinct real numbers that guarantee that no plane contains more than two of the lines from L^3_x. The sets L^3_y and L^3_z are shifted similarly. Thus no plane can contain more than two lines from the same bundle. Note that a plane cannot contain two lines from different

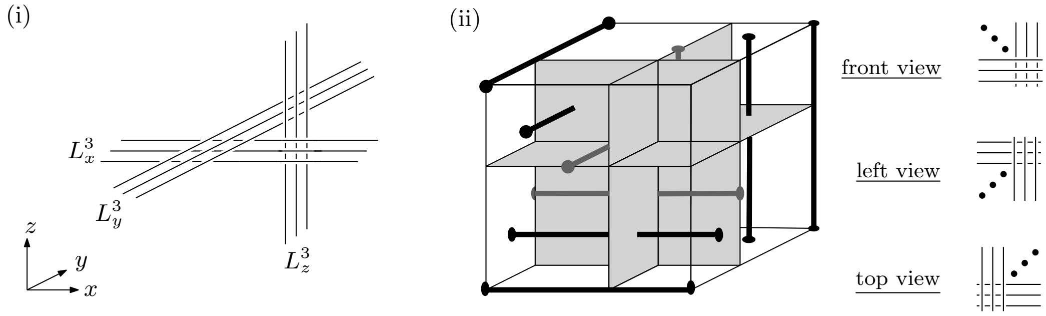


Figure 1 (i) Rough illustration of the set L^3 for the case $n = 9$. (ii) A more accurate illustration of L^3, which also shows three splitting planes that partition space into eight cells, each containing at most two lines. The dark grey lines are contained in a splitting plane. The front, left, and top views show that the construction is symmetric with respect to the axes, up to reversing the directions of some axes.

bundles either. With a slight abuse of notation we will still denote the shifted sets by L_x^3, L_y^3, and L_z^3. For simplicity, we will refrain from showing the perturbations in our figures.

It is straightforward to verify that $g_\perp(L^3) \leqslant n/3-1$. Indeed, if we take $h_1 := \{x = n/3+1\}$, and $h_2 := \{y = n/3\}$, and $h_3 := \{z = n/3 + 1\}$, then six of the eight cells in the resulting arrangement $\mathcal{A}(H)$ are intersected[1] by $n/3-1$ lines (all coming from a single bundle) and the remaining two cells are not intersected at all; see Fig. 1(ii). The main result of this section is that this is tight: even if one is allowed to use tilted splitting planes, it is not possible to ensure that all cells in the partitioning are intersected by strictly fewer than $n/3 - 1$ lines. This gives the following theorem.

▶ **Theorem 1.** *Let $L^3(n)$ be the three-bundle configuration on n lines, where n is divisible by 3, as defined above. Then*
- $g_\perp(L^3(n)) = g_{||}(L^3(n)) = g(L^3(n)) = n/3 - 1$ *for* $n \geqslant 9$,
- $g_\perp(L^3(n)) = g_{||}(L^3(n)) = 1$ *and* $g(L^3(n)) = 0$ *for* $n = 6$,
- $g_\perp(L^3(n)) = g_{||}(L^3(n)) = g(L^3(n)) = 0$ *for* $n = 3$.

Theorem 1 immediately gives the following corollary. The cases where $n = 3$ and $n = 6$ are easy to verify. In the remainder of this section we assume $n \geqslant 9$.

▶ **Corollary 2.** *For $n \geqslant 9$, $g(n) \geqslant \lfloor n/3 \rfloor - 1$.*

Theorem 1 also implies that $g_\perp(n) \geqslant \lfloor n/3 \rfloor - 1$ and $g_{||}(n) \geqslant \lfloor n/3 \rfloor - 1$, of course, but in later sections we will provide stronger lower bounds on $g_\perp(n)$ and $g_{||}(n)$; see Theorem 10.

Since we already argued that $g_\perp(L^3) \leqslant n/3 - 1$, and we have $g_\perp(L^3) \geqslant g_{||}(L^3) \geqslant g(L^3)$, it suffices to prove that $g(L^3) \geqslant n/3 - 1$. It will be convenient, however, to first prove this for $g_\perp(L^3)$, then extend the proof to $g_{||}(L^3)$, and finally extend it to $g(L^3)$.

▶ **Lemma 3.** *$g_\perp(L^3(n)) \geqslant n/3 - 1$.*

Proof. Suppose for a contradiction that there is a set $H = \{h_1, h_2, h_3\}$ of mutually orthogonal axis-parallel planes such that each cell in the arrangement $\mathcal{A}(H)$ meets fewer than $n/3-1$ lines from L^3. Let h_1, h_2, and h_3 be the planes orthogonal to the x-, y-, and z-axis, respectively.

[1] Recall that we consider the cells to be open, so lines fully contained in a splitting plane do not contribute to the counts.

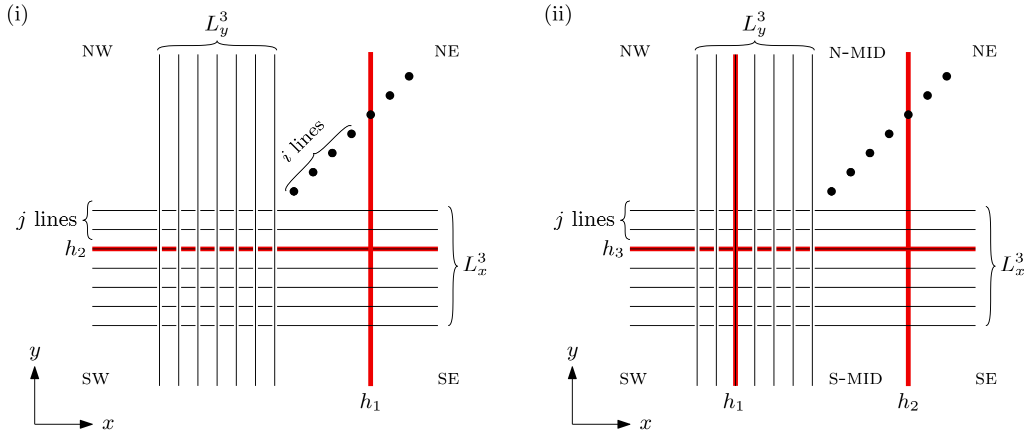


Figure 2 Illustration for the proof of Lemmas 3 and 4, showing the projection onto the xy-plane, where above/below crossings are drawn as seen from $z = +\infty$.

Consider the projection of L^3 and the planes h_1 and h_2 onto the xy-plane; refer to Fig. 2(i). The lines from L_x^3 and L_y^3 appear as lines in the projection. The planes h_1 and h_2 appear as lines as well, and they partition the xy-plane into four quadrants, which we label as NE, SE, SW, and NW in the natural way. The lines from L_z^3 appear as points in the projection. Below we will, with a slight abuse of terminology, sometimes speak of "points from L_z^3."

Note that the NE-quadrant contains at most $n/3 - 2$ points from L_z^3; otherwise there would be two octants in $\mathcal{A}(H)$ meeting at least $n/3 - 1$ lines. Hence, at least one of the following two cases occurs

(a) at least one point from L_z^3 lies to the left of h_1, or

(b) at least one point from L_z^3 lies below h_2.

Due to symmetry we can assume without loss of generality that case (a) holds, which implies that all lines from L_y^3 lie to the left of h_1. Now observe that the SW-quadrant meets at most $2n/3 - 3$ lines from $L_x^3 \cup L_y^3$; otherwise one of the two octants in $\mathcal{A}(H)$ that correspond to the SW-quadrant—these octants are generated when the z-vertical column corresponding to the SW-quadrant is split by the splitting plane h_3—will meet at least $n/3 - 1$ lines. (Note that the plane h_3 can contain at most one line from $L_x^3 \cup L_y^3$.) Since we already observed that all lines from L_y^3 lie to the left of h_1, this implies that at least two lines from L_x^3 lie above h_2. Hence, we can assume that the projection looks like the one depicted in Fig. 2(i).

Let i be the number of lines from L_z^3 lying (strictly) to the left of h_1, and let j be the number of lines from L_x^3 lying (strictly) above h_2; by the above arguments we have $i \geqslant 1$ and $j \geqslant 2$. Consider the remaining cutting plane h_3, which is parallel to the xy-plane, and let k be the number of lines from L_y^3 lying (strictly) below h_3. We have $k \geqslant 1$ because otherwise the NW-TOP octant of $\mathcal{A}(H)$—this is the cell that is above h_3 and whose projection onto the xy-plane is the NW quadrant—will meet i lines of L_z^3 and at least $n/3 - 1$ lines of L_x^3, which is a contradiction. Since $k \geqslant 1$ we also know that all j lines from L_x^3 must lie below h_3, since these lines lie below the lines from L_y^3. We can thus derive the following.

(i) The NE-BOTTOM octant meets $n/3 - i - 1$ lines of L_z^3 and j lines of L_x^3. Since we assumed all cells intersect fewer than $n/3 - 1$ lines, we thus have $n/3 - i - 1 + j < n/3 - 1$, which implies $i > j$.

(ii) The NW-TOP octant meets $n/3 - k - 1$ lines of L_y^3 and i lines of L_z^3. For this octant to intersect fewer than $n/3 - 1$ lines, we must thus have $k > i$.

(iii) The SW-BOTTOM octant meets k lines of L^3_y and $n/3 - j - 1$ lines of L^3_x. For this octant to intersect fewer than $n/3 - 1$ lines, we must thus have $k < j$.

But (i) and (ii) together imply $k > j$, which contradicts (iii), thus finishing the proof of the lemma. ◀

We now consider the case where the planes, though still axis-parallel, need not be mutually orthogonal.

▶ **Lemma 4.** $g_{\|}(L^3(n)) \geqslant n/3 - 1$.

Proof. Suppose for a contradiction that there is a set $H = \{h_1, h_2, h_3\}$ of three axis-parallel planes such that each cell of $\mathcal{A}(H)$ meets fewer than $n/3 - 1$ lines from L^3.

The case of three mutually orthogonal planes has been handled in Lemma 3. If h_1, h_2, and h_3 are all parallel to each other, then clearly there is a cell that is intersected by at least $n/3$ lines from L^3, since any line from the bundle orthogonal to the planes intersects all four cells. Thus the remaining case is when exactly two of the planes are parallel.

Assume without loss of generality that h_1 and h_2 are parallel to the yz-plane, with h_2 being to the right of h_1, and that h_3 is parallel to the xz-plane. Now consider the situation in the projection onto the xy-plane, as shown in Fig. 2(ii). There are six cells in the projection, which we label as NE, N-MID, NW, SE, S-MID, and SW. If each cell in the projection meets fewer than $n/3 - 1$ lines, then at least one line of L^3_y (in the projection) should be above h_3, otherwise the cells below h_3 intersect more than $n/3 - 1$ lines.

Let $j \geqslant 1$ be the number of lines from L^3_x above h_3. Then the number of lines from L^3_z—these are points in the figure—that are to the right of h_2 is at most $n/3 - j - 2$, otherwise the NE cell meets at least $n/3 - 1$ lines. Similarly, the number of lines from L^3_y to the left of h_1 must be at most $j - 1$, otherwise the SW cell meets at least $(n/3 - j - 1) + j = n/3 - 1$ lines. But then we have

- (# lines from L^3_x in N-MID) $= j$,
- (# lines from L^3_y in N-MID) $\geqslant n/3 - (j - 1) - 1 = n/3 - j$,
- (# lines from L^3_z in N-MID) $\geqslant n/3 - (n/3 - j - 2) - 1 = j + 1$.

This brings the total number of lines in the N-MID cell to at least $n/3 + j + 1$, thus contradicting our assumption. ◀

We now turn our attention to the general case, where we are also allowed to use splitting planes that are tilted or semi-tilted. (Recall that a semi-tilted plane is parallel to exactly one coordinate axis, and a tilted plane is not parallel to any coordinate axis.)

▶ **Lemma 5.** *For $n \geqslant 9$, we have $g(L^3(n)) \geqslant n/3 - 1$.*

Proof. Consider three splitting planes h_1, h_2 and h_3, and suppose for a contradiction that each cell in $\mathcal{A}(H)$ intersects fewer than $n/3 - 1$ lines. Define

- $a :=$ number of axis-parallel planes in H,
- $s :=$ number of semi-tilted planes in H, and
- $t :=$ number of tilted planes in H.

Note that $a + s + t = 3$. The case where all three planes are axis-parallel has already been handled in the previous two lemmas, so we can assume that $a < 3$. Any axis-parallel plane is crossed by $n/3$ lines from L^3, any semi-tilted plane is crossed by $2n/3$ lines from L^3, and any tilted plane is crossed by n lines from L^3. (Recall that, when we say that a line *crosses* a plane, or a plane *crosses* a line, we mean that there is a proper intersection; that is, the line is not contained in the plane.) Thus

$$(\text{\# fragments generated by the planes in } H) = \frac{n}{3} \cdot (3 + a + 2s + 3t).$$

Some fragments may be contained in one of the splitting planes and, hence, not appear inside any cell. Note that an axis-parallel splitting plane may contain at most one line from L^3, a semi-tilted splitting plane may contain at most two lines from L^3, and a tilted splitting plane cannot contain any line from L^3. Furthermore, a line contained in splitting plane consists of at most three fragments, arising due to the line crossing the other two splitting planes. Thus we have at least $(n/3)\cdot(3+a+2s+3t)-3(a+2s)$ fragments appearing inside the cells in $\mathcal{A}(H)$. Rewriting this, we conclude that

$$(\text{\# cell-line intersections}) \geqslant (n/3)\cdot(6+s+2t)-3(a+2s). \tag{1}$$

Since $\mathcal{A}(H)$ has at most eight cells, there must be a cell with $\left\lceil \frac{(n/3)\cdot(6+s+2t)-3(a+2s)}{8} \right\rceil$ intersections. We now make a case distinction, depending on the values of a, s and t. Recall that n is a multiple of 3 and that we are now dealing with the case $n \geqslant 9$. The first five cases we consider can be handled easily using Eq. (1), as follows.

- If $t \geqslant 2$, or $t=1$ and $s=2$, there is a cell with at least $\left\lceil \frac{(n/3)\cdot 10-6}{8} \right\rceil \geqslant n/3-1$ fragments.
- If $t=s=1$ (hence, $a=1$) there is a cell with at least $\left\lceil \frac{(n/3)\cdot 9-9}{8} \right\rceil = \left\lceil \frac{9}{8}(n/3-1) \right\rceil \geqslant n/3-1$ fragments.
- If $t=1$ and $s=0$ (hence, $a=2$) there is a cell with at least $\left\lceil \frac{(n/3)\cdot 8-6}{8} \right\rceil = \frac{n}{3}-1$ fragments.
- If $s=3$ (hence, $t=a=0$) there is a cell with at least $\left\lceil \frac{(n/3)\cdot 9-18}{8} \right\rceil = \left\lceil \frac{n}{3}-1+\frac{n-30}{24} \right\rceil$ fragments. For $n \geqslant 9$ this is at least $n/3-1$.
- If $t=0$ and $s=2$ (hence, $a=1$) there is a cell with at least $\left\lceil \frac{(n/3)\cdot 8-15}{8} \right\rceil = \lceil n/3-15/8 \rceil = n/3-1$ fragments.

Each of the cases above gives us the desired contradiction. The only remaining, and most difficult, case is when $t=0$, $s=1$, and $a=2$. In this case we need a more refined analysis, given next.

In the rest of the proof we assume there are two axis-parallel splitting planes, say h_1 and h_2, and one semi-tilted splitting plane h_3. If h_1 and h_2 are parallel to each other, the number of cells in $\mathcal{A}(H)$ is six. If all three planes are parallel to the same axis, then $\mathcal{A}(H)$ has seven cells. In either case, from Equation 1 it follows that there is a cell with

$$\left\lceil \frac{(n/3)\cdot(6+s+2t)-3(a+2s)}{7} \right\rceil = \left\lceil \frac{(n/3)\cdot 7-12}{7} \right\rceil = \lceil n/3-12/7 \rceil \geqslant n/3-1.$$

intersections giving the desired contradiction. Recall that the construction is symmetric with respect to the axes—see Fig. 1(ii)—and so without loss of generality we may now assume that h_1 is perpendicular to the x-axis, h_2 is perpendicular to the y-axis, and h_3 is parallel to the y-axis. We now proceed to derive a contradiction with our assumption that each cell in $\mathcal{A}(H)$ meets less than $n/3-1$ lines. To this end we first prove several properties that the splitting planes h_1, h_2, h_3 must satisfy, under our assumption. In the following, we consider the projection onto the xy-plane, and statements like "to the left of" or "above" refer to the situation in this projection. See Figure 3(i), and note that the lines of L_z^3 show up as points in the figure.

▷ Claim 6. At least one line of L_x^3 must lie above h_2. Hence, all lines of L_z^3 lie above h_2.

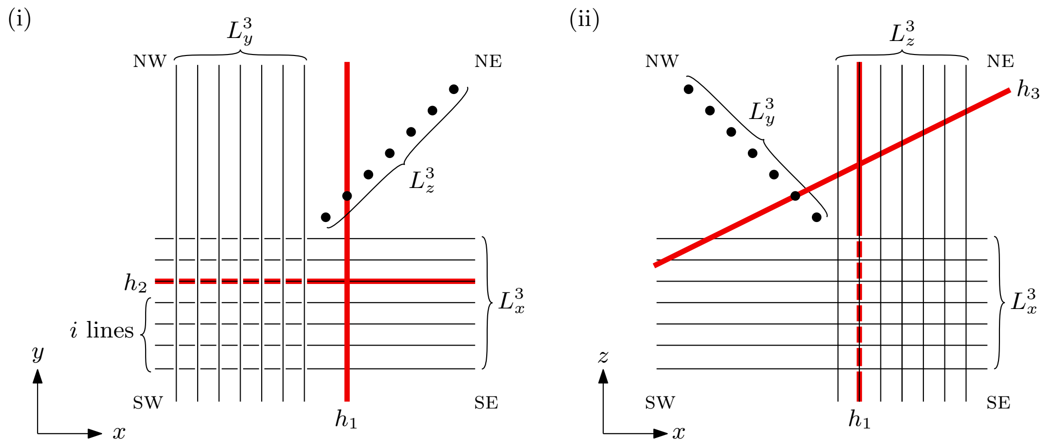


Figure 3 Illustration for the proof of Lemma 5. Note that in part (ii) of the figure, the semi-tilted plane h_3 can also show up as a line with negative slope. However, due to the small perturbation of the lines in L_y^3, h_3 can contain at most two lines from L_y^3 (here showing up as points).

Proof. For a contradiction, assume that no line of L_x^3 lies strictly above h_2. There is at most one line contained in h_2, so at least $n/3-1$ lines of L_x^3 lie below h_2. Each such line crosses h_1 and h_3—recall that the latter splitting plane is semi-tilted and parallel to the y-axis—and therefore crosses three of the four octants below h_2. Moreover, each line of L_y^3, except at most one line contained in h_1 and at most two lines contained in h_3, intersects one of these octants. Hence, the total count for the four octants below h_2 is at least $3 \cdot (n/3-1) + (n/3-3)$, and consequently, at least one of the four octants must intersect at least

$$\left\lceil \frac{3 \cdot (n/3-1) + (n/3-3)}{4} \right\rceil = \left\lceil \frac{4n/3-6}{4} \right\rceil = n/3-1$$

lines, contradicting our assumption. (Here we use our assumption that n is a multiple of 3 and that $n \geqslant 9$.) ◀

Claim 6 can be used to restrict the possible positions of h_1.

▷ Claim 7. h_1 must have at least one line of L_z^3 on either side of it. Hence, all lines of L_y^3 lie to the left of h_1.

Proof. By Claim 6, all lines from L_z^3 lie above h_2. If h_1 does not have lines from L_z^3 on either side of it, then one side has at least $n/3-1$ of these lines. Hence, one of the four quadrants defined by h_1, h_2 in the xy-projection—in particular, one of the quadrants above h_2—contains at least $n/3-1$ lines from L_z^3. Recall that the splitting plane h_3 is a semi-tilted plane parallel to the y-axis. Hence, h_3 is crossed by all lines from L_z^3, and so there would be an octant in $\mathcal{A}(H)$ that intersects at least $n/3-1$ lines, thus contradicting our assumptions. ◀

Next we restrict the possible positions for the semi-tilted splitting plane h_3.

▷ Claim 8. h_3 must have at least one line of L_y^3 on either side of it.

Proof. Observe that h_3 can contain at most two lines from L^3_y. Suppose for a contradiction that there are at least $n/3-2$ lines to one side of h_3, and consider the four octants of $\mathcal{A}(H)$ lying to this side. Every line from L^3_x and L^3_z intersects at least one of these octants, except at most one line from L^3_x contained in h_2 and at most one line from L^3_z contained in h_1. Moreover, each of the at least $n/3-2$ lines from L^3_y not contained in h_3, intersects h_2 and, hence, two of the octants. It follows that there is an octant that intersects

$$\left\lceil \frac{(2n/3-2)+2(n/3-2)}{4} \right\rceil = \left\lceil \frac{4n/3-6}{4} \right\rceil = n/3-1$$

lines, thus contradicting our assumptions. ◀

Let i be the number of lines of L^3_x below h_2.

▷ Claim 9. $i \geqslant \frac{2}{9}n - \frac{1}{3}$.

Proof. Consider the two quadrants above h_2 and the four octants of $\mathcal{A}(H)$ corresponding to those quadrants. Each line of L^3_x above h_2 intersects h_1 and h_3, and thus intersects three of these octants, for a total of $3(n/3-i-1)$ octant-line intersections. Each line of L^3_y, except for at most two lines contained in h_3, intersects one octant, thus contributing $n/3-2$ octant-line intersections. Finally, each line of L^3_z, except for at most one line contained in h_1, intersects two octants (since they intersect h_3), contributing $2(n/3-1)$ octant-line intersections. Hence, there is an octant intersecting

$$\left\lceil \frac{3(n/3-i-1)+(n/3-2)+2(n/3-1)}{4} \right\rceil = \left\lceil n/3-2+\frac{2n/3-3i+1}{4} \right\rceil$$

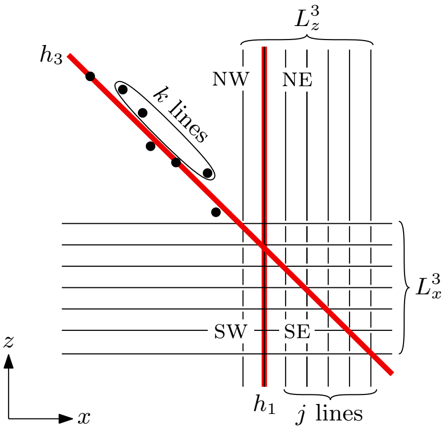
lines. In order not to contradict the assumption that no octant intersects more than $n/3-2$ lines, we must thus have $2n/3-3i+1 \leqslant 0$. Hence, $i \geqslant 2n/9-1/3$, as claimed. ◀

To finish the proof of Lemma 5, we switch views and consider the projection onto the xz-plane. Here h_3 shows up as a slanted line, as shown in Fig. 3(ii). The are two cases, depending on the slope of h_3 in the xz-projection.

Case 1: The projection of h_3 onto the xz-plane has positive slope.

Consider the SW-quadrant in Fig. 3(i), that is, the quadrant below h_2 and to the left of h_1. This quadrant corresponds to a vertical column in $\mathbb{R}^3$, which is cut into two octants by the semi-tilted plane h_3. Clearly, each line in L^3_y, except for at most two lines contained in h_3, intersects one of these two octants. Furthermore, each of the i lines of L^3_x below h_2 intersects both octants, because h_3 has positive slope in the xz-projection. Indeed, by Claim 8 the plane h_3 must have at least one line from L^3_y on either side of it—see Fig. 3 where these lines show up as points—and together with the fact that h_3 has positive slope this implies that all intersections of L^3_x with h_3 lie to the left of h_1. Thus, all i lines of L^3_x below h_2 intersects both octants, contributing $2i$ octant-line intersections. Hence, the total number of octant-line intersections in the two octants is at least $(n/3-2)+2i$. Since $i \geqslant \frac{2}{9}n - \frac{1}{3}$ by Claim 9 there is an octant that intersects at least

$$\left\lceil \frac{(n/3-2)+2i}{2} \right\rceil \geqslant \left\lceil \frac{n/3+(4n/9-2/3)-2}{2} \right\rceil = \left\lceil n/3-2+\frac{(n/9-2/3)}{2} \right\rceil$$



- The spliting plane h_2 is parallel to the xz-plane and, hence, not shown.
- All lines from L_z^3 lie above h_2 by Claim 7.
- By definition, i lines from L_x^3 are below h_2, and so at least $n/3 - i - 1$ lines from L_x^3 are above h_2

Figure 4 Illustration for Case 2 of the proof of Lemma 5. Recall that the lines in the construction were perturbed slightly. For the lines in L_y^3, which show up as points in the figure, these perturbations are shown larger than they actually are. Due to the perturbations, h_3 can contain at most two lines from L_y^3.

lines. Since $n \geqslant 9$ there is an octant intersecting more than $n/3 - 2$ lines, thus giving the desired contradiction for Case 1.

Case 2: The projection of h_3 onto the xz-plane has negative slope.

This case is illustrated in Fig. 4. Let j be the number of lines of L_z^3 to the right of h_1 and let k be the number of lines of L_y^3 above h_3; see Fig. 4. We label the "quadrants" induced by h_1, h_3 the xz-projection as NE, SE, SW, NW. Each of the these quadrants defines a column in $\mathbb{R}^3$, which is partitioned into two octants by h_2. We label the octants in the NE-column that are above and below h_2 by NE-TOP and NE-BOTTOM, respectively. The octants in the other columns are labeled similarly.

- Consider the NW-TOP octant. It meets at least $n/3 - j - 1$ lines of L_z^3, because all lines of L_z^3 lie above h_2 by Claim 6. Moreover, it meets k lines of L_y^3. (It may also meet lines from L_x^3 but we need not take them into account.) Since any octant is assumed to intersect less than $n/3 - 1$ lines, we must have $(n/3 - j + 1) + k < n/3 - 1$. Hence, $j > k + 2$.
- Now consider the SW-BOTTOM octant. It meets i lines from L_x^3 and at least $n/3 - k - 2$ lines of L_y^3, and so we must have $i + (n/3 - k - 2) < n/3 - 1$. Hence, $k > i - 1$.
- Finally, consider the NE-TOP octant. It meets j lines from L_z^3, because all lines of L_z^3 lie above h_2 by Claim 6. Note that all lines from L_x^3 intersect the NE quadrant, since h_3 has negative slope in the xz-projection. At least $n/3 - i - 1$ of these lines lie above h_2 and, hence, intersect the NE-TOP octant. We can conclude that we must have $j + (n/3 - i - 1) < n/3 - 1$, and so $i > j$.

Putting these three inequalities together we obtain

$$j > k + 2 > i + 1 > j + 1.$$

This is the desired contradiction and finishes the proof. ◀

3 Better lower bounds for axis-parallel splitting planes

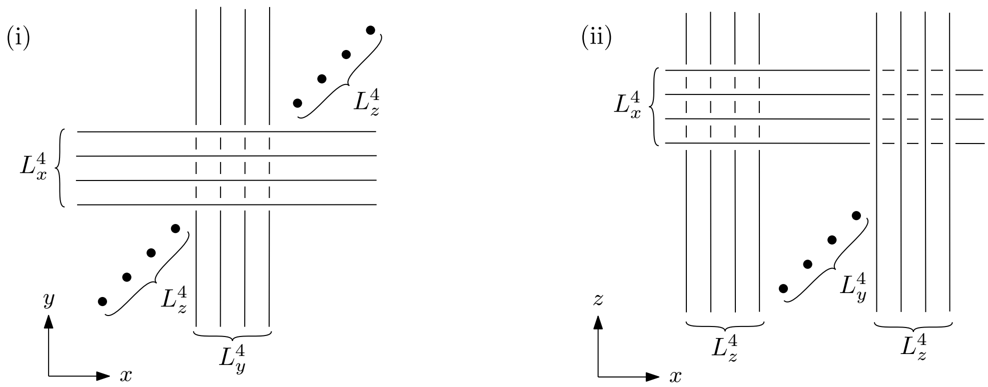
In this section we give improved lower bounds on $g_\perp(n)$ and $g_{||}(n)$. To this end we present a family of configurations $L^4(n)$ of n lines, with n divisible by 8, and prove that any decomposition of the type under consideration must have a cell intersected by at least $3n/8-1$ lines. Interestingly, for mutually orthogonal splitting planes the -1 term disappears.

▶ **Theorem 10.** *For every n divisible by 8, there exists a configuration $L^4 = L^4(n)$ such that $g_\perp(L^4(n)) = 3n/8$ and $g_{||}(L^4(n)) = 3n/8-1$. Hence, for any $n \geqslant 8$, we have $g_\perp(n) \geqslant \frac{3}{2}\lfloor n/4 \rfloor$ and $g_{||}(n) \geqslant \frac{3}{2}\lfloor n/4 \rfloor - 1$.*

The rest of this section is devoted to the proof of Theorem 10, where L^4 is defined as the union of the following sets:

$$L^4_x := \{(0, i, i+n/4) + \lambda(1,0,0) : i \in [n/4, n/2)\},$$
$$L^4_y := \{(i+n/4, 0, i) + \lambda(0,1,0) : i \in [0, n/4)\}, \text{ and}$$
$$L^4_z := \{(i, i, 0) + \lambda(0,0,1) : i \in [0, n/4) \cup [n/2, 3n/4)\}.$$

Note that $|L^4_x| = |L^4_y| = n/4$ and $|L^4_z| = n/2$, and that, up to symmetries, any projection to an axis-parallel plane looks like one of projections in Fig. 5. We will prove the theorem in



Projection onto xy-plane, view from $z = +\infty$ Projection onto xz-plane, view from $y = +\infty$

Figure 5 Line set L^4 from the proof of Theorem 10

two steps, by first considering the case of mutually orthogonal splitting planes and then the case of arbitrary axis-parallel splitting planes. In the remainder of this section, we assume that n is divisible by 8 and that $n \geqslant 8$.

▶ **Lemma 11.** *For any n divisible by 8, we have $g_\perp(L^4(n)) = 3n/8$.*

Proof. Consider three splitting planes h_1, h_2, and h_3, orthogonal to the x-, y-, and z-axis, respectively. Let $\ell_1 : x = i$ and $\ell_2 : y = j$ be the projections of h_1 and h_2 onto the xy-plane; see Fig. 6. Suppose that h_3 has k of the lines from $L^4_x \cup L^4_y$ lying strictly above it. (Note that for the purposes of lower-bound analysis we can assume that h_1, h_2, and h_3 each contain a line of L^4, as shifting them until they do can only decrease the number of lines of L^4 meeting each open octant. We will make this assumption hereafter in this proof, that is, we suppose that i, j, and k are integers in the relevant range.) We use NE to denote the open north-east quadrant defined by ℓ_1, ℓ_2 and define SE, SW, and NW similarly. For a quadrant

Q $\in \{$NW, NE, SW, SE$\}$, we define Q-TOP and Q-BOTTOM to be the open octants induced by h_3. With a slight abuse of notation, we also use NE (and, similarly, the other variables) to denote the total number of objects (lines and points) incident to the region NE.

We first observe that by setting $i = j = 3n/8$ and $k = n/4$ we obtain $g_\perp(L^4) \leqslant 3n/8$. It remains to argue that $g_\perp(L^4) \geqslant 3n/8$. Up to translation, rotation, and reflection, it suffices to consider the following cases.

Case 1: $0 \leqslant i < n/4$ and $0 \leqslant j < n/2$; see Fig. 6(i).

Observe that NE is incident to, at least, all $n/4$ lines in L^4_y, and $n/4$ lines of L^4_z. It follows that the number of incidences in NE-TOP and NE-BOTTOM together is at least $n/4 + 2(n/4) - 1 = 3n/4 - 1$, implying that at least one of NE-TOP, NE-BOTTOM is incident to

$$\left\lceil \frac{3n}{8} - \frac{1}{2} \right\rceil \geqslant \frac{3n}{8}$$

lines, where the inequality uses the fact that since n is divisible by 8.

Case 2: $0 \leqslant i < n/4$ and $n/2 \leqslant j < 3n/4$; see Fig. 6(ii).

We first additionally assume that $0 \leqslant k < n/4$. Thus, h_3 meets some line of L^4_x. NE-BOTTOM meets $n/4$ lines of L^4_y and $3n/4 - j - 1$ lines of L^4_z, so $n - j - 1$ lines in total. If $j < 5n/8$ then we have $n - j - 1 > 3n/8 - 1$ and, hence, $n - j - 1 \geqslant 3n/8$. So we may assume $j \geqslant 5n/8$. Observe that SE is incident to all $n/2$ lines in $L^4_x \cup L^4_y$, and at least $j - n/2 \geqslant n/8$ lines of L^4_z. Since h_3 contains a line of L^4_x, the number of incidences in SE-TOP and SE-BOTTOM is at least $2(n/8) + n/2 - 1 = 3n/4 - 1$, implying that at least one of SE-TOP and SE-BOTTOM is incident to at least

$$\left\lceil \frac{3n}{8} - \frac{1}{2} \right\rceil = \frac{3n}{8}$$

lines.

Now suppose $n/4 \leqslant k < n/2$, i.e., h_3 meets some line of L^4_y. SW-TOP meets i lines of L^4_z and all $n/4$ lines of L^4_x, i.e., SW-TOP $= i + n/4$. When $i \geqslant n/8$ (recall that n is a multiple of 8), we get SW-TOP $\geqslant 3n/8$, so we can assume $i < n/8$. Observe that SE is incident to all $n/2$ lines in $L^4_x \cup L^4_y$, and

$$\frac{n}{4} - i - 1 > \frac{n}{4} - \frac{n}{8} - 1 = \frac{n}{8} - 1,$$

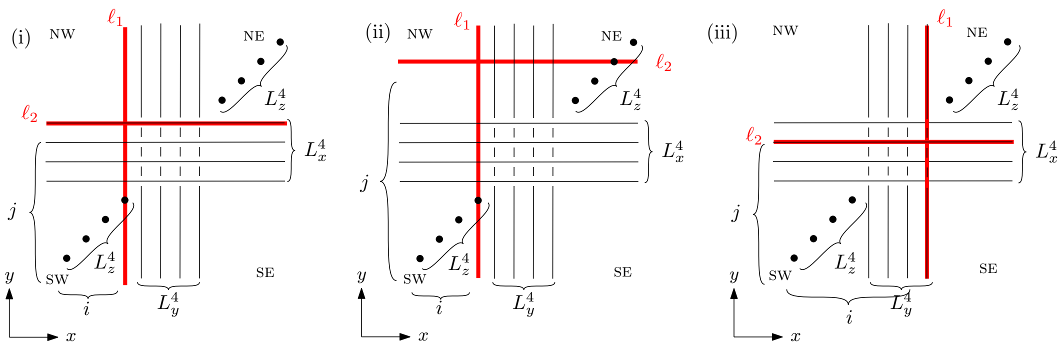


Figure 6 Different cases of partitioning with three orthogonal planes, projected to the xy-plane showing view from $z = +\infty$.

that is, at least $n/8$ lines of L_z^4. Thus the number of incidences in SE-TOP and SE-BOTTOM together is at least

$$2\left(\frac{n}{8}\right)+\frac{n}{2}-1=\frac{3n}{4}-1,$$

implying that at least one of SE-TOP and SE-BOTTOM meets at least $3n/8$ lines.

Case 3: $n/4 \leqslant i < n/2$ and $n/4 \leqslant j < n/2$; see Fig. 6(iii).

Suppose also that $0 \leqslant k < n/4$ (i.e., h_3 contains a line of L_x^4). Then, all but one of the lines of L_y^4 (so $n/4-1$) are incident to either NE-BOTTOM or SW-BOTTOM. Additionally, NE-BOTTOM and SW-BOTTOM are each incident to $n/2$ lines of L_z^4. That is, number of incidences in NE-BOTTOM and SW-BOTTOM is at least $n/4-1+n/2=3n/4-1$. It follows that at least one of these cells is incident to $3n/8$ lines.

Now suppose $n/4 \leqslant k < n/2$. i.e., h_3 contains a line of L_y^4. Then, all but one of the lines of L_x^4 (so $n/4-1$) are incident to either NE-TOP or SW-TOP. Additionally, NE-TOP and SW-TOP are each incident to $n/2$ lines of L_z^4. Then the number of incidences in NE-TOP and SW-TOP is at least $n/4-1+n/2=3n/4-1$. It follows that at least one of these cells is incident to $3n/8$ lines.

We conclude that all three cases give the desired number of incidences, which finishes the proof of the lemma. ◀

To finish the proof of Theorem 10 it remains to deal with axis-parallel splitting planes that need not be mutually orthogonal.

▶ **Lemma 12.** *For any n divisible by 8, we have $g_{\|}(L^4(n)) = 3n/8-1$.*

Proof. We first observe that the case of all three planes being orthogonal to the same axis is not interesting: Such planes would cut each line of one of the sets L_x^4, L_y^4, L_z^4 into four pieces, producing a total of at least

$$4\cdot\frac{n}{4}+\frac{n}{4}-1+\frac{n}{2}-1=\frac{7n}{4}-2$$

line-cell incidences, so at least one of the four cells would meet at least

$$\frac{1}{4}\left(\frac{7n}{4}-2\right)=\frac{7n}{16}-\frac{1}{2}>\frac{3n}{8}-1$$

lines. Hence, we will focus on partitions where two of the planes are parallel. Up to a permutation and reorientation of the coordinates, it is sufficient to consider the three cases illustrated in Figure 7. As before, it is safe to assume that each of the three planes contains a line of L^4.

We need prove that in each of the three cases, there is a cell that meets at least $3n/8-1$ lines, and that this bound can also be achieved as an upper bound in at least one of the cases.

Case (a): *h_1 and h_2 are parallel to the yz-plane and h_3 is parallel to the xz-plane.*

We label the cells as NW, N, NE, SW, S, and SE. Suppose there are i lines from $L_y^4 \cup L_z^4$ strictly to the left of h_1, and j lines from $L_y^4 \cup L_z^4$ strictly to the right of h_2, and k lines from $L_x^4 \cup L_z^4$ strictly above h_3. Due to the symmetry in the configuration, we may assume that the number of lines from L_x^4 below h_3 is at least the number of lines from L_x^4 above h_3. We can assume that h_3 contains a line of L_x^4 (and not a line of L_z^4), since otherwise

$$\text{SW}+\text{S}+\text{SE}=3\left(\frac{n}{4}\right)+\frac{n}{4}+\frac{n}{4}-2=\frac{5n}{4}-2,$$

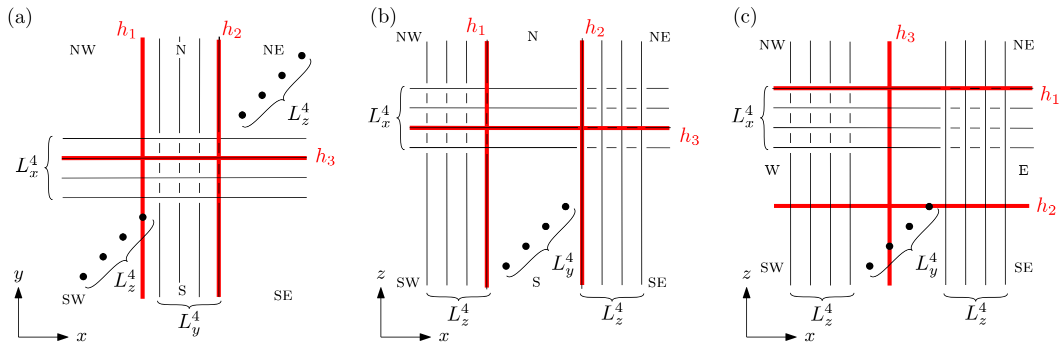


Figure 7 Different cases in the proof of Lemma12. For Case (a) the projection onto the xy-plane is shown, viewed from $z = +\infty$. For Cases (b) and (c) the projection onto the xz-plane is shown, viewed from $y = +\infty$.

implying that at least one of these cells contains

$$\left\lceil \frac{5n}{12} - \frac{2}{3} \right\rceil \geqslant \frac{3}{8}n - 1$$

lines. Hence, from now on we assume $0 \leqslant k < 3n/8$.

Note that SW is incident to $n/2 - k - 1 \geqslant n/8$ lines of L_x^4 and i lines of $L_y^4 \cup L_z^4$. If $i \geqslant n/4$, then this implies SW $\geqslant n/8 + i \geqslant 3n/8$ and so we are done. Similarly, if $j < n/4$, then S is incident to at least $n/8$ lines of L_x and $n/4$ lines of L_y, for a total of $3n.8$ incidences, and we are done as well. Hence, from here on we assume that $i < n/4$ (so h_1 contains a line of L_z^4), and that $j \geqslant n/4$ (so h_2 contains a line of L_y^4). Observe that SW, NE are incident to $i + j$ lines of $L_y^4 \cup L_z^4$, and $n/4 - 1$ lines of L_x^4, that is,

$$\text{SW} + \text{NE} = i + j + \frac{n}{4} - 1.$$

Moreover, S is incident to at least $n/8$ lines of L_x^4, to $n/2 - j - 1$ lines of L_y^4, and to $n/4 - i - 1$ lines of L_z^4 giving

$$\text{S} = \frac{n}{8} + \left(\frac{n}{2} - j - 1\right) + \left(\frac{n}{4} - i - 1\right) = \frac{7n}{8} - (i + j) - 2.$$

By combining these two inequalities we see that

$$\text{SW} + \text{NE} + \text{S} = \frac{9n}{8} - 3,$$

implying one of the three cells is incident to at least $3n/8 - 1$ lines.

This finishes the lower-bound proof for Case (a). Note that we can also achieve an upper bound of $3n/8 - 1$, by setting $k = 3n/8 - 1$, $i = n/4 - 1$, and $j = n/4$.

Case (b): h_1 and h_2 are parallel to the yz-plane and h_3 is parallel to the xy-plane.

As before, we label the cells as NW, N, NE, SW, S, and SE. Suppose there are i lines from $L_y^4 \cup L_z^4$ strictly to the left of h_1, and j lines from $L_y^4 \cup L_z^4$ strictly to the right of h_2, and k lines from $L_x^4 \cup L_y^4$ strictly above h_3.

Each of the lines in L_y^4 and L_z^4 are incident to exactly one of SW, S, SE, except for the two lines contained in h_1 and in h_2. If $k < n/8$, then at least $n/8$ of the lines in L_x^4 are incident to each of the cells SW, S, SE, implying

$$\text{SW} + \text{S} + \text{SE} = 3\left(\frac{n}{8}\right) + \frac{3n}{4} - 2 = \frac{9n}{8} - 2.$$

Hence, one of $\textsc{sw}, \textsc{s}, \textsc{se}$ is incident to at least $\lceil 3n/8 - 2/3 \rceil = 3n/8$ lines, and we are done. From here on, we therefore assume $k \geqslant n/8$.

Each line in L_x^4 is incident to either of $\textsc{sw}$ or $\textsc{nw}$, except for the line contained in h_3. If $i \geqslant n/4$ then $n/4$ lines in L_z^4 are incident to both $\textsc{sw}$ and $\textsc{nw}$, and so

$$\textsc{sw} + \textsc{nw} = 2\left(\frac{n}{4}\right) + \frac{n}{4} - 1 = \frac{3n}{4} - 1.$$

It follows that $\textsc{sw}$ or $\textsc{nw}$ must be incident to at least $3n/8$ lines, and we are done again. So we can assume $i < n/4$. By symmetry, we also can assume $j < n/4$.

Each of the k lines of L_x^4 above h_3 are incident to both $\textsc{nw}$ and $\textsc{ne}$. The cells $\textsc{nw}$ and $\textsc{ne}$ are also incident to i and j lines from L_z^4, respectively, and so $\textsc{nw} + \textsc{ne} = 2k + i + j$. Combining this with $\textsc{nw} + \textsc{ne} \leqslant 3n/4 - 2$, we get that $i + j \leqslant 3n/4 - 2k - 2$. On the other hand, $\textsc{s}$ is incident to $n/2 - k - 1$ lines of $L_x^4 \cup L_y^4$ and $n/2 - (i + j) - 2$ lines of L_z^4. Hence,

$$\begin{aligned}
\textsc{s} &= \left(\frac{n}{2} - k - 1\right) + \left(\frac{n}{2} - (i+j) - 2\right) \\
&= n - k - (i+j) - 3 \\
&\geqslant n - k - \left(\frac{6n}{8} - 2k - 2\right) - 3 \\
&= \frac{n}{4} + k - 1 \\
&\geqslant \frac{3n}{8} - 1.
\end{aligned}$$

This finishes the proof of the lower bound for Case (b). Note that also for this case we can achieve an upper bound of $3n/8 - 1$ lines, by setting $k = n/8$ and $i = j = n/4 - 1$.

Case (c): h_1 and h_2 are parallel to the xy-plane and h_3 is parallel to the yz-plane.

Label the cells as $\textsc{nw}, \textsc{w}, \textsc{sw}, \textsc{ne}, \textsc{e}$, and $\textsc{se}$. Let k be the number of lines in $L_y^4 \cup L_z^4$ that are strictly to the left of h_3. By symmetry, we may assume that $k \geqslant 3n/8$. Now each of the cells $\textsc{nw}, \textsc{w}, \textsc{sw}$ are incident to $n/4$ of the lines from L_z^4. All $n/4$ lines in L_x^4 and at least $n/8$ lines in L_y^4 are incident to exactly one of these cells, except for two that might be contained in h_1 and h_2. Hence,

$$\textsc{nw} + \textsc{w} + \textsc{sw} \geqslant 3\left(\frac{1}{4}n\right) + \frac{n}{8} + \frac{n}{4} - 2 = \frac{9n}{8} - 2,$$

implying one of these cells is incident to at least $\lceil 3n/8 - 2/3 \rceil = 3n/8$ lines.

This finishes the lower-bound proof for Case (c) and, hence, the proof of the lemma. ◀

4 Upper bounds

We now prove upper bounds on $g_\perp(n)$ and $g_\parallel(n)$. More precisely, we present algorithms that produce, for any set L of n lines, a decomposition of the type under consideration, with each cell intersected by at most a certain number of the lines.

Let $L := L_x \cup L_y \cup L_z$ be a set of axis-parallel lines in $\mathbb{R}^3$, where L_x, L_y, L_z denote the subsets parallel to the x-, y-, and z-axis, respectively. We say L_x is in *general position* if no pair of distinct lines in L_x share y- or z-coordinates; we define general position for L_y and L_z similarly. We say L is in general position if L_x, L_y, and L_z are in general position, and no two lines in L have a common point. We first argue that, for the purposes of upper bounds, it suffices to restrict our attention to sets in general position.

▶ **Lemma 13.** *Let $L = L_x \cup L_y \cup L_z$ be a set of axis-parallel lines not in general position. Then there exists a set $L' = L'_x \cup L'_y \cup L'_z$ of axis-parallel lines in general position such that*

$$g_{\|}(L) \leqslant g_{\|}(L') \quad \text{and} \quad g_{\perp}(L) \leqslant g_{\perp}(L').$$

Proof. Suppose that $L = L_x \cup L_y \cup L_z$ is a set of lines not in general position. Assume, without loss of generality, that the intersection points of lines in L_x (resp. L_y, and L_z) with the plane $x = 0$ (resp. $y = 0$, and $z = 0$) have integer coordinates. We obtain L' by a generic perturbation of the lines of L. More specifically, for each line $\ell \in L$, we let ℓ' be a generic line parallel to ℓ inside a tube of radius $1/3$ centered at ℓ.

Consider a triple $H' = (h'_1, h'_2, h'_3)$ of axis-parallel splitting planes (which need not be mutually orthogonal). Suppose, without loss of generality, that h'_1 is orthogonal to the x-axis and is given by $x = \alpha'$ for some $\alpha' \in \mathbb{R}$. Let h_1 be the plane given by $x = \alpha$ where α is α' rounded to the nearest integer with ties broken arbitrarily; and define h_2, h_3 similarly. Let $H = (h_1, h_2, h_3)$ be the resulting triple of axis-parallel splitting planes.

Let $\mathcal{C}' \in \mathcal{A}(H')$, and let $\mathcal{C} \in \mathcal{A}(H)$ be the corresponding cell. (Since two or more planes in H' may be "rounded" to the same plane in H, the cell $\mathcal{C}$ can be the empty set. In this case the following claim trivially holds.) We claim that the number of incidences of $\mathcal{C}$ with lines of L is at most the number of incidences of $\mathcal{C}'$ with lines of L'. This claim follows from the observation that, for each plane $h_i \in H$, the number of lines of L lying strictly on one side of h_i is upper bounded by the number of lines of L' lying on the same side of the corresponding $h'_i \in H$. To see this, note that if $\ell' \in L'$ lies on h_i, on h'_i, or between them, then the corresponding line $\ell \in L$ lies on h_i. ◀

4.1 Upper bounds on $g_{\|}$

We start with a simple observation.

▶ **Observation 14.** *If $\max(|L_x|, |L_y|, |L_z|) = m$ then there is a set H of three axis-parallel planes (two of which are parallel) such that any cell in $\mathcal{A}(H)$ meets at most $(5n - m)/12$ lines from L.*

Proof. Assume without loss of generality that L_z is the smallest of the three sets. By assumption, $|L_z| \leqslant \lfloor (n-m)/2 \rfloor$. Partition $L_x \cup L_y$ into three equal-size subsets using two planes h_1, h_2 parallel to the xy-plane, and partition L_z into two equal-size subsets using a plane h_3 parallel to the yz- or xz-pane. As in earlier arguments, we can always choose the planes so that they contain a line of L. Set $H := \{h_1, h_2, h_3\}$. Then the number of lines each of the six cells in $\mathcal{A}(H)$ meets is at most

$$\begin{aligned}
\left\lceil \frac{|L_x| + |L_y| - 2}{3} \right\rceil + \left\lceil \frac{|L_z| - 1}{2} \right\rceil &\leqslant \frac{|L_x| + |L_y|}{3} + \frac{|L_z|}{2} \\
&= \frac{n - |L_z|}{3} + \frac{|L_z|}{2} \\
&\leqslant \frac{n}{3} + \frac{\lfloor (n-m)/2) \rfloor}{6} \\
&\leqslant \frac{5n - m}{12}.
\end{aligned}$$ ◀

The following theorem gives an upper bound on $g_{\|}$.

▶ **Theorem 15.** *For any set L of n axis-parallel lines in $\mathbb{R}^3$, there is a set H of three axis-parallel planes (two of which are parallel) such that any cell in $\mathcal{A}(H)$ meets at most $7n/18$ lines from L. Hence, $g_{\|}(n) \leqslant \lfloor 7n/18 \rfloor$.*

Proof. Define L_x, L_y, and L_z as above. The largest of the groups has size at least $\lceil n/3 \rceil$. The theorem now follows from Observation 14 with $m = \lceil n/3 \rceil$. ◀

4.2 Upper bounds on $g_\perp$

▶ **Theorem 16.** *For any set L of n axis-parallel lines, there is a set H of three planes, one orthogonal to each axis direction, such that any cell in $\mathcal{A}(H)$ meets at most $\lceil 5n/12 \rceil$ of the lines; in other words, $g_\perp(n) \leqslant \lceil 5n/12 \rceil$.*

Proof. Assume without loss of generality that $|L_z| \geqslant \max(|L_x|, |L_y|)$, and consider the projection of L onto the xy-plane. In the projection the lines from L_z become points, the lines from the other sets are still lines. We denote the splitting planes orthogonal to the x-, y-, and z-axis by h_1, h_2, and h_3, respectively. We will first explain how we pick h_1 and h_2—in the projection these correspond to splitting lines, which we denote by ℓ_1 and ℓ_2, respectively—and then finish the construction by placing h_3. We distinguish two cases.

Case 1: $|L_z| \geqslant n/2$.

This is the easy case: we pick ℓ_1 and ℓ_2 such that each open quadrant contains at most $\lfloor |L_z|/3 \rfloor$ points, and we pick h_3 such that at most half the lines from $L_x \cup L_y$ are above h_1 and at most half are below. Thus each cell in the resulting decomposition intersects at most

$$\left\lfloor \frac{|L_z|}{3} \right\rfloor + \left\lfloor \frac{|L_x| + |L_y|}{2} \right\rfloor \leqslant \frac{|L_x| + |L_y| + |L_z|}{3} + \frac{|L_x| + |L_y|}{6} \leqslant \frac{5n}{12},$$

lines, since $|L_x| + |L_y| + |L_z| = n$ and $|L_x| + |L_y| \leqslant n/2$.

Case 2: $|L_z| < n/2$.

For two given splitting lines ℓ_1 and ℓ_2 in the xy-plane, we use $\textsc{ne}$ to denote the number of lines in L whose projection (which can be a line or a point) intersects the open north-east quadrant defined by ℓ_1, ℓ_2, and we define $\textsc{se}$, $\textsc{sw}$, and $\textsc{nw}$ similarly. Let $\textsc{n} = \textsc{nw} + \textsc{ne}$, and define $\textsc{e}, \textsc{s}, \textsc{w}$ similarly. Note that lines from L_x that lie in the northern part are counted twice in $\textsc{n}$, once for their intersection with the north-west quadrant and once for their intersection with the north-east quadrant. Finally, we use $\textsc{ne}_x$ to denote the number of lines from L_x intersecting the north-east quadrant, and we use $\textsc{ne}_y$, $\textsc{s}_z$, and so on, in a similar way. Finally, let $\textsc{t} = \textsc{ne} + \textsc{nw} + \textsc{se} + \textsc{sw}$ denote the total number of incidences.

Let $W(L) := 2(|L_x| + |L_y|) + |L_z| = n + |L_x| + |L_y|$, and note that

$$\frac{3n+1}{2} \leqslant W(L) \leqslant \lfloor 5n/3 \rfloor$$

which follows from the facts that $\max(|L_x|, |L_y|) \leqslant |L_z|$ and $|L_z| < n/2$. We will require ℓ_1 and ℓ_2 together each contain the projection of a line; note that such a line may subtract 1 from the total count $\textsc{t}$ (if its projection is a point), or 2 (if its projection is a line). Hence,

$$\frac{3n-7}{2} \leqslant W(L) - 4 \leqslant \textsc{t} \leqslant W(L) - 2 \leqslant \lfloor 5n/3 \rfloor - 2.$$

We now explain how to pick ℓ_2 (and, hence, h_2), the splitting line orthogonal to the y-axis. Place ℓ_2 at the highest y-coordinate where we still have $\textsc{s} \leqslant \frac{\lfloor 5n/3 \rfloor}{2}$. This is always possible since $\frac{\lfloor 5n/3 \rfloor}{2} < \frac{3n-7}{2}$ for all $n \geqslant 2$. Now, we have

$$\frac{\lfloor 5n/3 \rfloor}{2} - 1 \leqslant \textsc{s} \leqslant \frac{\lfloor 5n/3 \rfloor}{2}$$

where the lower bound comes from the fact that moving ℓ_2 could change the number of incidences by two (if ℓ_2 contains a line of L_y). Furthermore, since $\textsc{s} + \textsc{n} = \textsc{t} \leqslant \lfloor 5n/3 \rfloor - 2$, we have

$$\textsc{n} \leqslant \frac{\lfloor 5n/3 \rfloor}{2} - 1.$$

We also assume, without loss of generality, that $\textsc{s}_z \geqslant \textsc{n}_z$. Indeed, if this is not the case, we can interchange the roles of $\textsc{s}$ and $\textsc{n}$; simply enlarge $\textsc{n}$ (by shifting ℓ_2 down) until the above inequalities hold in interchanged form.

Now pick ℓ_1 (and, hence, h_1) such that $\lfloor \textsc{s}/2 \rfloor \leqslant \textsc{sw} \leqslant \lceil \textsc{s}/2 \rceil$ and $\lfloor \textsc{s}/2 \rfloor \leqslant \textsc{se} \leqslant \lceil \textsc{s}/2 \rceil$. From now on we assume, without loss of generality, that $\textsc{ne} \geqslant \textsc{nw}$. Note that this implies that $\textsc{nw} \leqslant \lfloor \textsc{n}/2 \rfloor$. Furthermore, we have

$$\textsc{sw}, \textsc{se} \leqslant \lceil \textsc{s}/2 \rceil \leqslant \lceil \lfloor 5n/3 \rfloor /4 \rceil \leqslant \lceil 5n/12 \rceil$$

and

$$\textsc{nw} \leqslant \lfloor \textsc{n}/2 \rfloor \leqslant \lceil (\lfloor 5n/3 \rfloor - 2)/4 \rceil < \lfloor 5n/12 \rfloor.$$

Thus the corresponding "columns" in $\mathbb{R}^3$ already have the desired number of incidences.

We choose the remaining splitting plane h_3 such that, within the north-eastern column, at most half the lines from $L_x \cup L_y$ are above h_3 and at most half are below (and one line is contained in h_3). We conclude that the number of lines intersected by each of the two cells resulting from splitting this column by h_3 is at most

$$\left\lfloor \frac{\textsc{ne}_x + \textsc{ne}_y}{2} \right\rfloor + \textsc{ne}_z \leqslant \frac{\textsc{ne} + \textsc{ne}_z}{2}. \tag{2}$$

To bound the expression in (2), we rely on the following.

Claim. With ℓ_1, ℓ_2 as above, we have $\textsc{ne} \leqslant n - \textsc{sw}$.

Proof. Since $\textsc{n}_x = 2\,\textsc{ne}_x$ we have

$$\textsc{n}_z = \textsc{n} - \textsc{n}_x - \textsc{n}_y = \textsc{n} - 2\,\textsc{ne}_x - \textsc{n}_y.$$

Trivially, we also have

$$\textsc{ne} = \textsc{ne}_x + \textsc{ne}_y + \textsc{ne}_z \leqslant \textsc{ne}_x + \textsc{ne}_y + \textsc{n}_z.$$

Combining these we get

$$\textsc{ne} \leqslant \textsc{ne}_x + \textsc{ne}_y + \big(\textsc{n} - 2\,\textsc{ne}_x - \textsc{n}_y\big) = \textsc{n} - \textsc{n}_y + \textsc{ne}_y - \textsc{ne}_x. \tag{3}$$

Moreover, since $\textsc{se}_x = \textsc{e}_x - \textsc{ne}_x$ and $\textsc{ne}_y = \textsc{se}_y$ we have

$$\textsc{se} = \textsc{se}_x + \textsc{se}_y + \textsc{se}_z = \big(\textsc{e}_x - \textsc{ne}_x\big) + \textsc{ne}_y + \textsc{se}_z,$$

which can be rewritten as

$$\textsc{ne}_y - \textsc{ne}_x = \textsc{se} - \textsc{e}_x - \textsc{se}_z. \tag{4}$$

Note also that

$$\textsc{n} = \textsc{t} - \textsc{s} = 2\,\textsc{n}_y + 2\,\textsc{e}_x + \textsc{t}_z - \textsc{s}. \tag{5}$$

Using that $\mathrm{E}_x \leqslant |L_x|$ and $\mathrm{N}_y \leqslant |L_y|$ and $\mathrm{T}_z \leqslant |L_z|$, we obtain

$$\begin{aligned}
\mathrm{NE} &\leqslant \mathrm{N} - \mathrm{N}_y + (\mathrm{NE}_y - \mathrm{NE}_x) && \text{by (3)}\\
&= \mathrm{N} - \mathrm{N}_y + (\mathrm{SE} - \mathrm{E}_x - \mathrm{SE}_z) && \text{by (4)}\\
&= (2\,\mathrm{N}_y + 2\,\mathrm{E}_x + \mathrm{T}_z - \mathrm{S}) - \mathrm{N}_y + (\mathrm{SE} - \mathrm{E}_x - \mathrm{SE}_z) && \text{by (5)}\\
&\leqslant |L_x| + |L_y| + |L_z| - \mathrm{S} + \mathrm{SE} - \mathrm{SE}_z\\
&= n - \mathrm{SW} - \mathrm{SE}_z\\
&\leqslant n - \mathrm{SW},
\end{aligned}$$

which finishes the proof of the claim. $\triangleleft$

Now recall that

$$\mathrm{SW} \geqslant \lfloor \mathrm{S}/2 \rfloor \geqslant \left\lfloor \frac{\lfloor 5n/3 \rfloor - 2}{4} \right\rfloor. \tag{6}$$

Moreover, since we assumed $\mathrm{N}_z \leqslant \mathrm{S}_z$ and we are in Case 2 (so $|L_z| \leqslant (n-1)/2$) we have

$$\mathrm{NE}_z \leqslant \mathrm{N}_z \leqslant \lfloor |L_z|/2 \rfloor \leqslant \lfloor (n-1)/4 \rfloor. \tag{7}$$

Finally, from (2), the number of incidences in each cell of the north-eastern column is at most

$$\begin{aligned}
\frac{\mathrm{NE} + \mathrm{NE}_z}{2} &\leqslant \frac{n - \mathrm{SW} + \mathrm{NE}_z}{2} && \text{by the Claim above}\\
&\leqslant \frac{n - \lfloor (\lfloor 5n/3 \rfloor - 2)/4 \rfloor + \mathrm{NE}_z}{2} && \text{by (6)}\\
&\leqslant \frac{n - \lfloor (\lfloor 5n/3 \rfloor - 2)/4 \rfloor + \lfloor (n-1)/4 \rfloor}{2} && \text{by (7)}\\
&\leqslant \lceil 5n/12 \rceil.
\end{aligned}$$

For the last inequality, define $f(n) := \frac{n - \left\lfloor (\lfloor 5n/3 \rfloor - 2)/4 \right\rfloor + \lfloor (n-1)/4 \rfloor}{2} - \lceil 5n/12 \rceil$. We need to show that $f(n) \leqslant 0$ for all $n \in \mathbb{N}$. Observe that $f(n) = f(n+12)$ for any n. Hence, it suffices to show that $f(n) \leqslant 0$ for all integer n from 0 to 11, which can be verified by a straightforward computation. ◀

5 Conclusions

Two obvious open problems remain:

- Close the gaps between the upper and the lower bounds for axis-parallel lines.
- Answer the same question for lines with arbitrary orientations in $\mathbb{R}^3$: given a set L of n lines, minimize the number of lines meeting each open cell of the arrangement $\mathcal{A}(H)$ formed by a set H of three planes. A simple calculation shows that if L is in general position, then at least one of the open cells of $\mathcal{A}(H)$ meets at least $4(n-3)/8 \approx n/2$ lines from L, since any plane can contain or be parallel to at most one of the lines of L. Can we prove larger lower bounds, and what upper bounds can we obtain?

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