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Actuator Selection for Dynamical Networks with Multiplicative Noise ⋆

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Abstract: We propose a greedy algorithm for actuator selection considering multiplicative noise in the dynamics and actuator architecture of a discrete-time, linear system network model. We show that the resultant architecture achieves mean-square stability with lower control costs and for smaller actuator sets than the deterministic model, even in the case of modeling uncertainties. Networks with multiplicative noise may fail to be mean-square stabilizable by any small actuator set, leading to a failure of a cost-based greedy algorithm. To account for this, we propose a multi-metric greedy algorithm that allows actuator sets to be evaluated effectively even when none of them stabilize the system. We illustrate our results on networks with multiplicative noise in the open-loop dynamics and the actuator inputs, and we analyze control costs for random graphs of different network sizes and generation parameters.

Keywords: Control of Networks, Stochastic Control, Stochastic Optimal Control, Complex Dynamical Networks, Actuator Selection, Optimal Control, Multiplicative Noise

1. INTRODUCTION

Complex dynamical systems are critical foundations of modern technology and infrastructure with applications to power grids, transportation networks, biological system models, water delivery systems and many others. These networks have practical constraints on the number of actuators, and an important emerging problem is to design the network control architecture to optimize certain metrics of controllability and feedback control performance. To effectively solve network control architecture design problems, we need knowledge about the system dynamics and uncertainties, and also a suitable algorithm for actuator selection. We propose a multi-metric variation of the greedy algorithm for actuator selection for linear discrete-time network dynamics with multiplicative noise. We evaluate our algorithm through numerical experiments of noisy systems using control sets and feedback from models with and without multiplicative noise over variations of systems parameters such as network size and actuator set size.

Network control architecture and control design decisions are affected by the accuracy of the model. However, the accuracy of mathematical models for real-world dynamics is affected by unmodelled dynamics and disturbances. Additive noise models do not capture errors that depend on states and inputs. Multiplicative noise models capture this valuable information during control design, leading to robustness. Our study here focuses on this intersection of actuator selection in dynamic networks with multiplicative noise. Networks models encode dynamic interactions between states into edges of the adjacency matrix. Inputs and additive noises are modelled as inputs to nodes while multiplicative noises can be viewed as perturbations on the edge weights of the adjacency matrix.

In network control architecture design problems, we aim to find a subset of control inputs that optimizes a feedback control metric. The feasibility, selection and comparison of controllability of actuator sets depends on the choice of metric used to define it, such as the system graph structure, matrix rank, Gramian, cost functions and their combinations (Ruths and Ruths [2014], Liu et al. [2011], Pasqualetti et al. [2014], Ganapathy et al. [2021]). The greedy algorithm offers tractable heuristic to many selection problems. In some cases, submodularity of the cost metric allow for approximation guarantees, while others can even be hard to approximate (Jadbabaie et al. [2019], Zhang et al. [2017], Olshevsky [2014], Summers [2016], Summers et al. [2016], Cortesi et al. [2014], Chamon et al. [2021]). A difficulty with greedy actuator selection is when all available actuator subsets may fail to stabilize the system and give infinite cost. Improvements to the standard greedy algorithm such as the reverse greedy algorithm (Guo et al. [2021]) provide an alternative approach. However, for large networks with small actuator set size restrictions, this can take a significant number of computations. Further, most of these works restrict themselves to greedy selection of closed-loop stable actuators for deterministic dynamic models or consider additive disturbances.

We aim to extend the greedy algorithm and evaluate its performance on multiplicative noise models. Current research on actuator selection for the multiplicative noise problem proposes a gradient descent algorithm with high guarantees (Belabbas and Kirkoryan [2018]) and others

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focus on the design of controllers for fixed architectures (Gravell et al. [2021a,b]). We propose a multi-metric variation of the greedy algorithm for actuator selection using both control cost and time to overcome the limitations of the standard greedy algorithm. We then evaluate our algorithm through numerical experiments with discrete-time linear systems using models with and without multiplicative noise. We study variations of systems parameters such as network size and actuator set size to empirically demonstrate the benefits.

Our contributions in this paper are summarized as follows.

- We propose a greedy actuator selection algorithm for actuator selection while considering multiplicative noise in the dynamics and actuator set for mean-square stability. To account for limitations of the greedy selection when small actuator sets fail to stabilize the system, we propose a multi-metric greedy algorithm to take finite-time performance into consideration when selecting actuators (Section 3).
- We demonstrate the benefit of modelling multiplicative noise in open-loop system dynamics for dynamical networks with multiplicative noise. We show improved performance with lower control costs and smaller actuator set requirements compared to the deterministic model, even in cases of model errors (Section 4.1). We extend our analysis to mean-square stability statistics of different actuator set sizes, network sizes and graph generation parameters for both the Erdos-Renyi (ER) and Barabasi-Albert (BA) graph generators, reflecting similar results of lower costs and smaller actuator sets stabilizing a higher fraction of the generated networks (Section 4.2).
- We extend our experiments on models of multiplicative noise models of actuator inputs. Even with errors from mismatch of true and modelled noise in the actuators, we show the benefits from reduced control costs for all actuator set sizes (Section 4.3).

The code to generate, test and visualize these models can be found on Github at `https://github.com/TSummersLab/multiplicative_noise_greedy_control_architecture`.

1.1 Notations

The real-valued vectors in n-dimensional space is denoted by $\mathbf{R}^n$. The canonical basis vectors of $\mathbf{R}^n$ are denoted by $\mathbf{e}_i \, \forall i \in [1, n]$ where $\mathbf{e}_i$ is a vector of zeros with the i^{th} term as 1. The set of real-valued positive definite, symmetric matrices are denoted by $\mathbf{S}^n_{++}$ where $A \in \mathbf{S}^n_{++} \Leftrightarrow A \succ 0$. The set of positive semi-definite symmetric matrices are denoted by $\mathbf{S}^n_{+}$ where $A \in \mathbf{S}^n_{+} \Leftrightarrow A \succeq 0$. The n-dimensional identity matrix is denoted by $\mathbf{I}_n$. The vector of ones in $\mathbf{R}^n$ is denoted by $\mathbf{1}_n$, the vector of zeros in $\mathbf{R}^n$ is denoted by $\mathbf{0}_n$ and matrix of zeros in $\mathbf{R}^{n\times m}$ is denoted by $\mathbf{0}_{n\times m}$. The expectation of a random variable x is denoted by $\mathbf{E}\{x\}$. The Kronecker delta $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise.

2. DEFINING THE ACTUATOR SELECTION PROBLEM

2.1 Dynamic Network Model With Multiplicative Noise

The network dynamics are modelled by the discrete-time linear dynamical system evolving on the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as

$$x_{t+1} = \left(A + \sum_{i=1}^{N_\nu} \nu_{i,t} A_i\right) x_t + \left(B_S + \sum_{j=1}^{N_\eta} \eta_{j,t} B_j\right) u_t \quad (1)$$

where $x_t \in \mathbf{R}^n$ are the states of the system, $u_t \in \mathbf{R}^m$ are the control inputs. The non-zero entries of the open-loop network dynamics matrix, $A \in \mathbf{R}^{n\times n}$, model the edges $\mathcal{E}$ of the graph $\mathcal{G}$. The state-dependent multiplicative noise, $\nu_t = [\nu_{1,t} \ \dots \ \nu_{N_\nu,t}]^\top \in \mathbf{R}^{N_\nu}$, perturbs dynamics through the corresponding matrices $A_i \in \mathbf{R}^{n\times n} \, \forall i \in [1, N_\nu]$. The non-zero entries of the A_i indicate how each noise term affects certain edge weights in the graph. These noises are zero-mean identically and independently distributed (i.i.d) with covariance $\mathbf{E}\{\nu_{i,t_1}, \nu_{i,t_2}\} = \alpha_i \delta_{t_1,t_2}$ with $\alpha_i \geq 0$.

We define the set of all possible actuators in terms of the canonical basis vectors $\mathbf{e}_i \, \forall i \in [1, n]$. The set of available actuators is given by $\mathcal{B}$. The subset of active actuators, $S \subseteq \mathcal{B}$ form the actuator matrix $B_S \in \mathbf{R}^{n\times|S|}$ where the columns of the matrix are the corresponding basis vectors and the number of actuators is given by $|S|$. The input-dependent multiplicative noise, $\eta_t = [\eta_{1,t} \ \dots \ \eta_{N_\eta,t}]^\top \in \mathbf{R}^{N_\eta}$ perturb the actuator inputs at the corresponding nodes through the corresponding matrices $B_j \in \mathbf{R}^{n\times|S|} \, \forall j \in [1, N_\eta]$. We assume that input-dependent noises only affect the same node as the corresponding active actuator, so that the matrices B_j have 1 in the corresponding entry and zeros elsewhere. These noises are zero-mean i.i.d with covariance $\mathbf{E}\{\eta_{j,t_1}, \eta_{j,t_2}\} = \beta_j \delta_{t_1,t_2}$ with $\beta_j \geq 0$.

2.2 Optimal Control Problem

The control cost is defined over the T-step horizon as

$$V_{T,S}(x_0) \triangleq \mathbf{E}_{\nu,\eta}\left\{\sum_{t=0}^{T-1}\left(x_t^\top Q x_t + u_t^\top R u_t\right) + x_T^\top Q x_T\right\} \quad (2)$$

where $Q \in \mathbf{S}^n_{+}$ is the cost on the states and $R \in \mathbf{S}^{|S|}_{++}$ is the cost on control inputs. The control costs are a function of the initial states x_0 and the control architecture S. The optimal feedback control problem aims to find an optimal state feedback controller that minimizes the control cost:

$$V^*_{T,S}(x_0) = \min_{\mathbf{u}(\cdot)} V_{T,S}(x_0) \quad (3)$$

For linear systems with multiplicative noise and quadratic costs, the dynamic programming algorithm provides backwards recursion to compute the cost functions and state feedback control policies (Smith and Bertsekas [1996]). In this case, the optimal cost-to-go functions are quadratic with $V^*_{T,S}(x_0) = x_0^\top P_0 x_0$, the optimal state feedback policies are linear with $u_t = K_t x_t$, and the coefficients can be computed from the recursion

$$P_{t-1} = Q + A^\top P_t A + \sum_{i=1}^{N_\nu} \left(\alpha_i A_i^\top P_t A_i\right) + \left[A^\top P_t B_S K_t\right] \tag{4}$$

$$K_t = -\left(R + B_S^\top P_t B_S + \sum_{j=1}^{N_\eta} (\beta_i B_j^\top P_t B_j)\right)^{-1} B_S^\top P_t A \tag{5}$$

with the initialization $P_T = Q$. Note that the cost matrices P_t for $t = [0, T-1]$ are functions of the set of actuators S.

For the infinite horizon problem, if the system is stabilizable in the mean-square sense, we can calculate the fixed gain K and the cost matrix P via the generalized Riccati equation

$$P = Q + A^\top P A + \sum_{i=1}^{N_\nu} \left(\alpha_i A_i^\top P A_i\right) + \left[A^\top P B_S K_t\right], \tag{6}$$

$$K = -\left(R + B_S^\top P B_S + \sum_{j=1}^{N_\eta} (\beta_i B_j^\top P B_j)\right)^{-1} B_S^\top P A \tag{7}$$

and the corresponding cost $V_{\infty,S}(x_0) = x_0^\top P x_0$. Similar to (5), cost matrix P is a function of the actuator set S.

2.3 Actuator Selection Problem

The actuator selection problem is aimed at optimizing the control architecture to minimize the control cost. This problem is defined as

$$S^* = \underset{S \subseteq \mathcal{B}, |S| \leq \mathcal{S}}{\arg\min} \; J^*_{S,x_0} \tag{8}$$

where $|S| \leq \mathcal{S}$ is a bound on the maximum number of actuators and J^*_{S,x_0} is related to $V^*_S(x_0)$ in various ways depending on the initial state, e.g.,

$$J^*_{S,x_0} = V^*_{T,S}(x_0) = x_0^\top P_0 x_0, \qquad \textbf{fixed} \tag{9a}$$

$$J^*_S = tr(P_0 X_0), \qquad \textbf{average} \tag{9b}$$

$$J^*_S = \max_{|x_0|=1} x_0^\top P_0 x_0 = \lambda_{\max}(P_0), \qquad \textbf{worst-case}. \tag{9c}$$

With x_0 fixed, we can define the cost as (9a). Given the covariance X_0 of the initial states, we can define the average-cost metric as (9b). Alternatively, we can optimize over the worst case of initial states distributed over the unit circle as (9c). For $T \to \infty$, we would substitute P_0 with P from (7).

3. MULTI-METRIC GREEDY ACTUATOR SELECTION

Actuator selection is a combinatorial optimization problem which makes it computationally difficult, further scaling with network size. However, greedy selection algorithms perform well empirically in many practical problems. We propose using a greedy algorithm to optimize the control architecture for the problem posed in Section 2.

A complication for greedy actuator selection may occur when there does not exist any actuator subset of a certain (small) cardinality that stabilizes the system. This is especially challenging in systems with multiplicative noise, since the noise may cause mean square instability even when the system without noise is stabilizable. In such cases, the standard greedy algorithm fails to provide meaningful actuator selection in the first iterations when there are few or no actuators already selected. To address this, we propose a multi-metric algorithm.

In particular, for the primary metric, we use a finite T-step horizon cost $J^*(S)$ from (9). If this cost is larger than a user-specified upper bound $J_{\max}$, then for the secondary metric, we define the largest time for which this cost bound is satisfied

$$T^*(S) = \max\{T \mid J^*(S) < J_{\max}\}. \tag{10}$$

In this way, one of these two metrics is always finite for every possible actuator subset. This allows effective actuators to be selected even when every actuator subset in an iteration of the greedy algorithm fails to stabilize the system in the mean square sense.

Algorithm 1 shows the multi-metric greedy algorithm that we propose for actuator selection in dynamical networks with multiplicative noise.

Algorithm 1 Greedy actuator selection

Require: System model: $A, \alpha_i, A_i, \beta_j, B_j$, Set of available actuators: $\mathcal{B}$, Given actuator set: default $S = \emptyset$, actuator selection constraint: $\mathcal{S}$, cost parameters: Q, R, time horizon: T, cost bound: $J_{\max}$, primary metric: $J^*(S)$ from (9), secondary metric: $T^*(S)$ from (10)
Ensure: $\mathcal{B} \neq \emptyset$ and $\mathcal{B} \cap S = \emptyset$

```
while |S| ≤ 𝒮 and |ℬ| > 0 do
    if ∃b ∈ ℬ such that J*(S + b) < J_max then
        b* ← arg min_{b∈ℬ} J*(S + b)
    else
        b* ← arg max_{b∈ℬ} T*(S + b)
    end if
    S ← S + b*, ℬ ← ℬ − b*
end while
return S
```

4. NUMERICAL EXPERIMENTS

Our analysis focuses on two broad cases of multiplicative noise modelling, first in the network dynamics and second in the actuator inputs. Networks models, with each node representing a state variable and each actuator injecting a control input at a certain node, can be used to study how network properties influence actuator selection. We utilize realizations of Erdos-Renyi (ER) (Erdos and Rényi [2011]) and the Barabasi-Albert (BA) (Barabási and Albert [2011]) random graph generators to evaluate performance statistics. They reflect networks with varying degree distributions and allow us analyze the effect on actuator selection and control costs. We ensure that all graphs generated are undirected and connected (there is a path between any pair of nodes) to eliminate trivial cases involving isolated nodes. We assume $Q = \mathbf{I}_n$ and $R = \mathbf{I}_{|S|}$ for all the numerical tests in this paper.

4.1 Benefits of modelling multiplicative noise in dynamics

We first demonstrate the benefits of multiplicative noise modelling in the system dynamics. We show that even in the case of imperfect information of the network edges, multiplicative noise models give better controllability properties for smaller actuator sets and at lower costs.

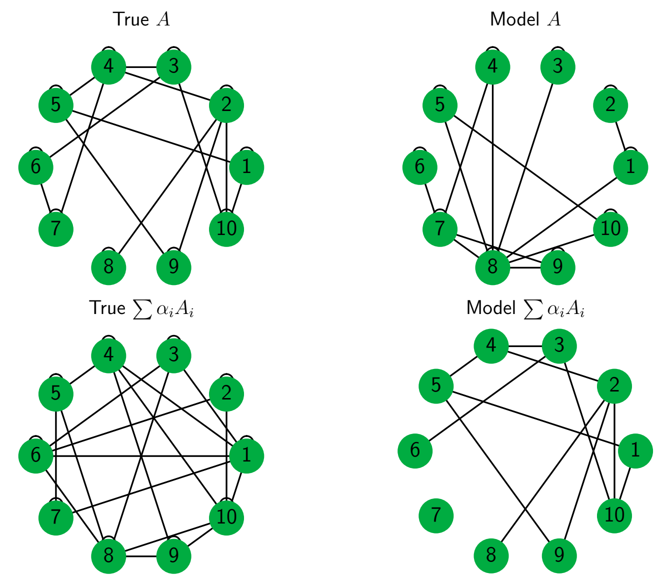
Model setup We consider a network size of $n = 10$ nodes. The true and model network parameters are described in Fig. 1a. The adjacency matrices are realizations ER networks with edge probability $p = 0.3$ with the edge weights scaled such that $|\lambda_{\max}(A)| = 0.8$ for the open-loop nominal dynamics of each network. The dynamics of the true system (True A) has multiplicative noise (True $\sum \alpha_i A_i$). Both system models Sys_{Nom} and Sys_{MPL} have imperfect, incorrect information on the dynamics of the true system (Model A). Sys_{MPL} assumes dynamic multiplicative noise (Model $\sum \alpha_i A_i$). This model multiplicative noise captures partial information of edges of the true dynamic network (True A - Model A) assuming ($\alpha_i = 0.1$) and does not completely capture the true multiplicative noise (True $\sum \alpha_i A_i$) which is assumed to be unknown to the model. The initial state is assumed to be $x_0 = 10 * \mathbf{1}_n$, and we use the corresponding metric (9a). We use (10) with an upper bound $J_{\max} = 10^8$ over a simulation time horizon of 200 steps.

Numerical results We run greedy actuator selection on Sys_{Nom} and Sys_{MPL} separately. The cost function (2) for Sys_{MPL} can be adapted for Sys_{Nom} by setting $\alpha_i = \beta_j = 0 \; \forall \; i, j$. The resultant actuator sets shown in the corresponding models in Fig. 1b are different, indicating that the multiplicative noise in the dynamics influences optimal actuator selection. We then simulate the true system with its modelled unknown disturbances using the gains and feedback from each model. The cost trajectories and difference in cost are plotted in Fig. 2 with the corresponding steady-state values in Table 1.

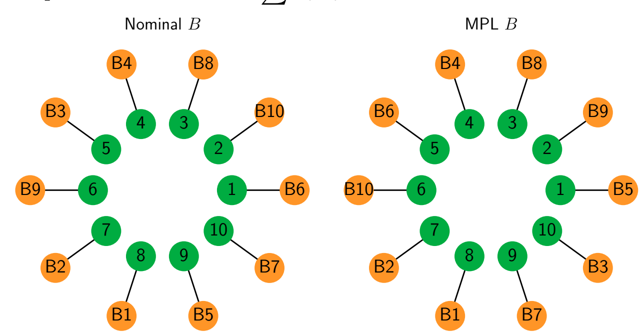
Table 1. Control cost comparison: Simulation of the True System using Sys_{Nom} vs Sys_{MPL} for multiplicative noise in dynamics. The last two columns shows the signed magnitude and % of cost improvement of actuator selection using Sys_{MPL} with reference to Sys_{Nom}.

|S|	SysNom	SysMPL	SysNom − SysMPL	% Cost Improvement
1	-	-	-	-
2	-	-	-	-
3	-	-	-	-
4	-	5.70e+06	-	-
5	-	8.76e+04	-	-
6	1.75e+05	3.42e+04	1.41e+05	80.4%
7	3.84e+04	2.25e+04	1.59e+04	41.5%
8	1.91e+04	1.15e+04	7.57e+03	39.7%
9	1.32e+04	5.85e+03	7.32e+03	55.6%
10	6.54e+03	4.85e+03	1.69e+03	25.9%

Discussion We see benefits of Sys_{MPL} over Sys_{Nom} in all three cases: (1) time to failure when both models fail, (2) size of the smallest actuator set for feasible control and (3) control cost for a fixed actuator set when both are feasible. We see that controllers designed for Sys_{Nom} fails over a significantly shorter time horizon than those from Sys_{MPL}. Disturbances in the current simulation are identical for both test models and we see that there is a significant difference in time to failure. This is an indicator of better resilience of control using Sys_{MPL} to uncertainties in the system dynamics.



(a) True system has dynamics True A and multiplicative noise True $\sum \alpha_i A_i$. Nominal model Sys_{Nom} assumes dynamics Model A. Multiplicative system Sys_{MPL} assumes dynamics Model A and multiplicative noise Model $\sum \alpha_i A_i$



(b) Greedy actuator selection order for Sys_{Nom} and Sys_{MPL}. Actuator $B1$ in orange refers to the first greedy actuator, $B2$ is the second and so on.

Fig. 1. True System, modelled Sys_{Nom} and Sys_{MPL} and their corresponding greedy actuator set selection

Note that control fails for $|S| \leq 3$ for both cases. The standard greedy algorithm would fail to select any actuators as cost metrics (9) fail, while the proposed algorithm uses the time metric (10) as the next priority to continue the selection algorithm.

For an actuator set constraint $|S| = \{4, 5\}$, we see that Sys_{MPL} succeeds in controlling the system despite the imperfect information in the modelling while Sys_{Nom} fails. This indicates better control guarantees for smaller actuator set constraints for multiplicative noise modelling.

When $6 \leq |S| \leq 10$, we see that both models have feasible controllers despite the differences in the actuator set until $|S| = 10 = n$ corresponding to full actuation. From the last column of Table 1, we see that there is a significant reduction in costs when using Sys_{MPL} compared to Sys_{Nom}. This difference can be quite substantial for smaller actuator sets and decreases as we approach full actuation. For fully actuated models, the only difference comes from the multiplicative noise model in the dynamics

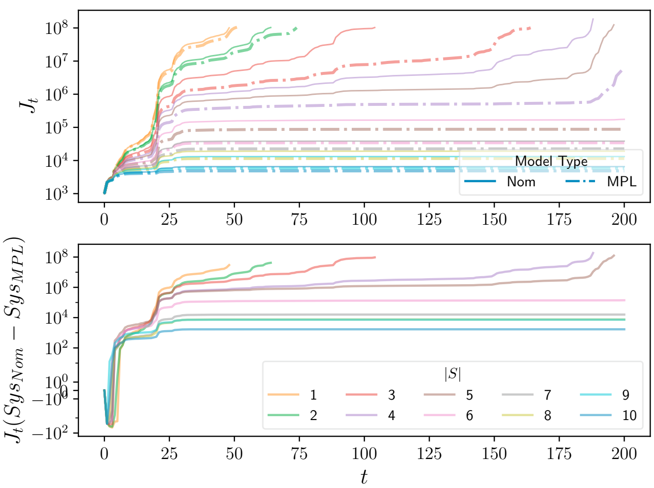


Fig. 2. The plots describe the benefits of multiplicative noise modelling in system dynamics. The top graph shows the cost over time for simulations of the true system over range of $|S|$ of Sys_{Nom} and Sys_{MPL} greedy actuators in solid and dotted lines respectively. The bottom graph shows the difference in cost over time for range of $|S|$ with final values in Table 1.

and its effect on the feedback gains from the Riccati equation. This emphasizes value of the multiplicative noise modelling in system dynamics even under full actuation.

4.2 Statistics of modelling multiplicative noise in dynamics

In this section, we expand on the benefits of modelling multiplicative noise in system dynamics by observing the statistics of actuator selection and control costs over realizations of random graphs under variations of both the network generation parameters and the network size. Our numerical experiments reported here focus on ER networks, though we also did experiments with BA graphs and observed qualitatively similar results.

Model Setup We use a similar modelling process as in Section 4.1 for a true system, a nominal system model (Sys_{Nom}) and a dynamics multiplicative noise model (Sys_{MPL}). The nominal dynamics of each model are scaled for open-loop stability. The vector of initial states are randomly generated from a zero-mean normal distribution with covariance $X_0 = 10 * \mathbf{I}_n$ where n is the number of nodes in the test network.

We start with ER network with $n = 10$ nodes and edge-probability of $p = 0.4$. We then expand our analysis to two more cases of networks: (1) $n = 20, 30$ nodes for fixed $p = 0.4$ and (2) $p = 0.2, 0.6$ with fixed $n = 10$ nodes.

Numerical results We generate 100 realizations of the test system. The parameters recorded are the feasibility of control for a particular actuator set size and the steady-state cost of control if feasible. The results for the reference graph of ER are plotted in Fig. 3. We then tested for variation in network size as plotted in Fig. 4. Additional test for variation in edge probability of ER graphs were conducted. The extensive test results are available in the code and discussed briefly here. For all plots, the bottom graph is the fraction of the realizations that had bounded control costs over the different sizes of the actuator sets. The top graph plots the mean and median of the control cost across all realizations. The median value is significant and finite only when at least half of the tested realizations have bounded cost and the mean value only when all realizations have bounded cost for a given actuator set size.

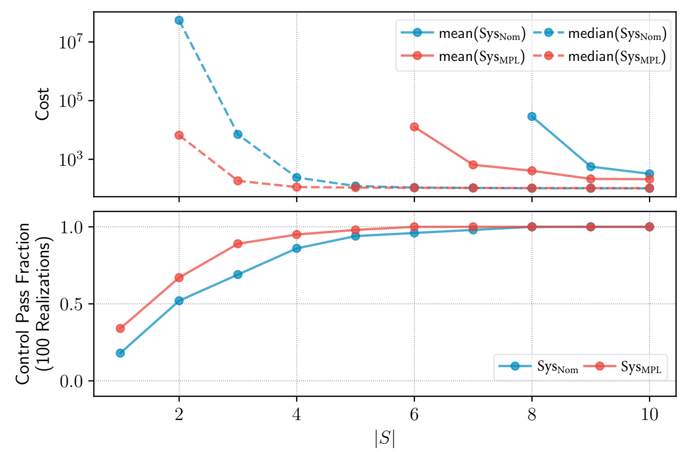


Fig. 3. The plots show the statistics of control cost and the fraction of $n = 10$ node ER graph networks ($p = 0.4$) with increasing actuator set size for the Sys_{Nom} and Sys_{MPL}. Both plots show the better performance of modelling multiplicative noise.

Discussion We begin our analysis from Fig. 3. The bottom plot shows us that Sys_{MPL} has better fraction of controllable systems than Sys_{Nom} across all sizes of actuator sets. There is a significant order of magnitude difference between median control costs and a difference in the size of the actuator sets before all the realizations are controllable and the mean cost becomes a significant metric of comparison.

When comparing the size of the networks for a constant edge probability Fig. 4 and the edge probability parameter for constant network size, we see a significant shift to the right. As expected, the number of actuators required for feasible control increases with increasing number of nodes and Sys_{MPL} outperforms Sys_{Nom} in both small actuator set requirements and lower control costs. There is a more significant increase of actuator requirements for network size than degree of connectivity. For networks of $n = 30$, Sys_{Nom} fails to control even half of the test realizations under full actuation as the optimal feedback fails to keep the multiplicative noise in check while Sys_{MPL} achieves control despite imperfect information.

Numerical experiments with BA graphs also reflect similar benefits of Sys_{MPL} over Sys_{Nom} as with ER graphs. Further variations on the modelling parameters and other random graph generators can provide further valuable insight into actuator selection for different network structures. These experiments demonstrate the benefit of modelling multiplicative noise in the dynamics for optimal control.

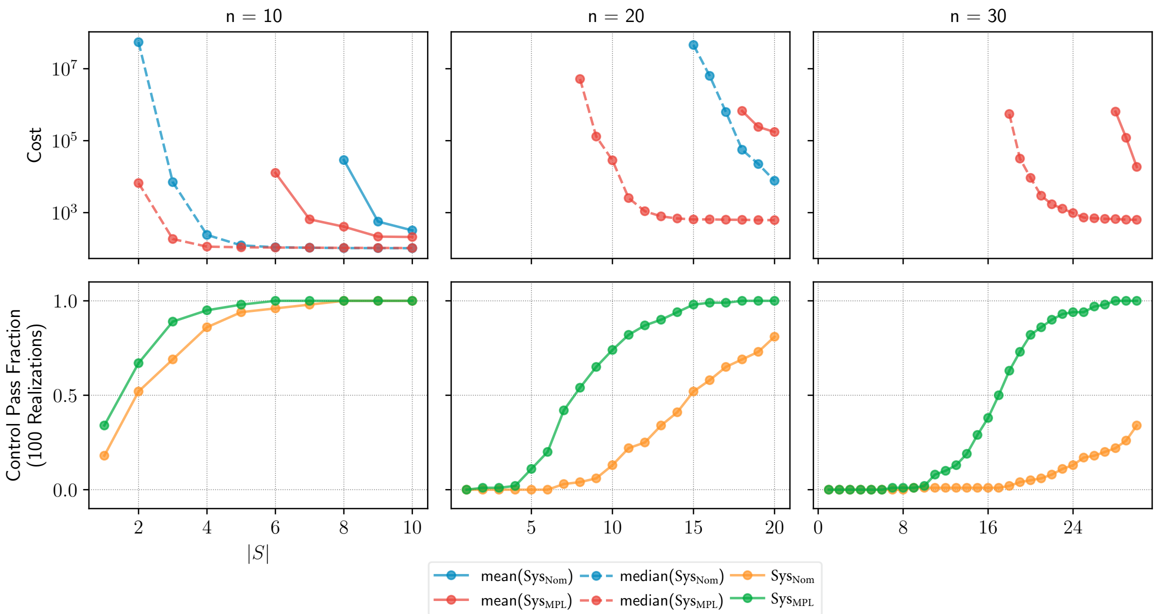


Fig. 4. The plots show the statistics of control cost and the fraction of $n = [10, 20, 30]$ node ER graph networks with $p = 0.4$ under control using the $\mathrm{Sys}_{\mathrm{Nom}}$ and $\mathrm{Sys}_{\mathrm{MPL}}$ showing the effect of change in network size and the size of the actuator set.

4.3 Benefits of modelling multiplicative noise in actuators

Multiplicative noise models of uncertainties scaling with inputs also have a significant effect on feedback control performance and actuator selection.

Model setup We consider a network size of $n = 10$ nodes. The dynamics matrix is fixed and deterministic with $(|\lambda_{\max}(A)| = 0.8)$. We assume a set of $|\mathcal{B}| = 10$ actuators are available with multiplicative disturbances β_j for $j \in [1, n]$. The nominal model ($\mathrm{Sys}_{\mathrm{Nom}}$) assumes actuators are deterministic while the multiplicative model ($\mathrm{Sys}_{\mathrm{MPL}}$) uses a multiplicative noise model for the actuators but with incorrect matching of multiplicative input noise to actuator. The initial state is sampled from the zero-mean distribution of $X_0 = 10 * \mathbf{I}_n$ and use the corresponding metric (9a). We use (10) with $J_{\max} = 10^8$ as the upper bound on (9c) to bound the most unstable cost mode over a simulation time horizon of 200 steps.

Numerical results The procedure is identical to Section 4.1. We run greedy actuator selection on both models separately, then use the actuator set and feedback from each to run simulations of the true system. The results are shown in Fig. 5 with the steady-state cost values in Table 2.

Discussion Our analysis from Fig. 5 shows that all non-empty actuator sets from both models provide feasible control for the system. We extend our comparison to the cost of control.

For simulations of the true system using $\mathrm{Sys}_{\mathrm{MPL}}$, we see a monotonic decrease in control costs with increasing actuator set size as in Table 2. The overall difference between control costs for $|S| = 1$ and $|S| = 10$ is quite small with minimal improvements with saturation of actuators.

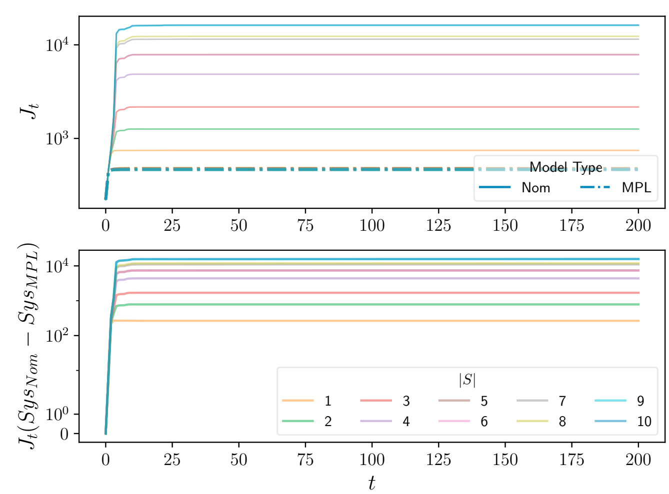


Fig. 5. The plots describe the benefits of multiplicative noise modelling in control inputs. The top graph shows the cost over time for simulations of the true system over range of $|S|$ of $\mathrm{Sys}_{\mathrm{Nom}}$ and $\mathrm{Sys}_{\mathrm{MPL}}$ greedy actuators in solid and dotted lines respectively. The bottom graph shows the difference in cost over time for range of $|S|$ with final values in Table 2.

Comparing these results to simulations using $\mathrm{Sys}_{\mathrm{Nom}}$, we see a wide fluctuation in control costs in Table 2. Simulations of the true system with the same models for different realizations of the noise resulted in inconsistent overall trends of control cost with increasing actuator set sizes. For small actuator set sizes, adding a new actuator increases the noise influencing the system. As the system is saturated with actuators, the noise influence is better controlled as illustrated by the reduced magnitude of change in costs as they converge. However, even at $|S| = 10 = n$, we see a significant difference in costs which

Table 2. Cost comparison: Simulation of the True System using $\mathrm{Sys}_{\mathrm{Nom}}$ vs $\mathrm{Sys}_{\mathrm{MPL}}$ for multiplicative noise in actuators. The last two columns shows the signed magnitude and % of cost improvement of actuator selection using $\mathrm{Sys}_{\mathrm{MPL}}$ with reference to $\mathrm{Sys}_{\mathrm{Nom}}$.

|S|	SysNom	SysMPL	SysNom − SysMPL	% Cost Improvement
1	7.45e+02	4.80e+02	2.66e+02	35.6%
2	1.26e+03	4.75e+02	7.85e+02	62.3%
3	2.17e+03	4.73e+02	1.69e+03	78.2%
4	4.85e+03	4.70e+02	4.38e+03	90.3%
5	7.87e+03	4.68e+02	7.41e+03	94.1%
6	7.82e+03	4.67e+02	7.35e+03	94.0%
7	1.15e+04	4.66e+02	1.10e+04	95.9%
8	1.23e+04	4.65e+02	1.19e+04	96.2%
9	1.61e+04	4.65e+02	1.56e+04	97.1%
10	1.63e+04	4.65e+02	1.58e+04	97.1%

is a result of the difference in feedback. The benefits of $\mathrm{Sys}_{\mathrm{MPL}}$ over $\mathrm{Sys}_{\mathrm{Nom}}$ for multiplicative noise models of the actuators despite the model errors compared to the true system are evident from the control costs.

5. CONCLUSION

Our illustrations and analysis have shown the benefits of modelling multiplicative noise in both dynamics and actuators for optimal control and the effectiveness of the multi-metric greedy selection of actuators. Analysis of different noise models can give us further insights to how network topology may influence actuator selection and control performance. Actuator selection for dynamic games with multiplicative noise, the dual problem of sensor selection, and the problem of simultaneous actuator and sensor selection for the generalized system framework would be interesting for future work.

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