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Dirichlet Sub-Laplacians on Homogeneous Carnot Groups: Spectral Properties, Asymptotics, and Heat Content

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We consider sub-Laplacians in open bounded sets in a homogeneous Carnot group and study their spectral properties. We prove that these operators have a pure point spectrum and prove the existence of the spectral gap. In addition, we give applications to the small ball problem for a hypoelliptic Brownian motion and the large time behavior of the heat content in a regular domain.

1 Introduction

This paper is focused on spectral properties of sub-Laplacians in subsets of homogeneous Carnot groups. The main difficulties include hypoellipticity of these operators and lack of smoothness of natural sets in such groups such as metric balls. Thus we cannot rely on standard partial differential equations' techniques. In particular, we are interested in domains that are balls with respect to a homogeneous distance on such groups. These are classical questions at the intersection of potential theory, spectral analysis, and probability in the setting of degenerate second-order differential operators.

The mathematical literature on the subject is vast, and while our goal is to prove results of a classical flavor, it seems that they are not readily available. We rely on the theory of Dirichlet forms to show that a sub-Laplacian with Dirichlet boundary

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conditions on sets with no restrictions on the boundary are infinitesimal generators of a Hunt process, namely, a killed process. To further study boundary behavior of eigenfunctions we address the issue of regularity of boundary points being defined differently in analysis, potential theory, and probability.

Our main results include Theorem 3.3 where we collect spectral properties of the Dirichlet sub-Laplacian $-\mathcal{L}_\Omega$ restricted to a bounded open connected subset Ω of a Carnot group $\mathbb{G}$. We prove that the spectrum of $-\mathcal{L}_\Omega$ is discrete, and then we show that the 1st Dirichlet eigenvalue is strictly positive and simple, that is, $-\mathcal{L}_\Omega$ has a spectral gap. Moreover, in Proposition 3.6, we prove uniform L^p-bounds and smoothness of the eigenfunctions of $-\mathcal{L}_\Omega$. These are well-known results for uniformly elliptic operators on domains with a smooth boundary; see [12, Chapter 6], [38, Corollary 5.1.2], and [45, Equation (3.3), p. 39]. Our results hold without assuming regularity of the boundary $\partial\Omega$ and in particular apply to uniform elliptic operators on domains with a nonsmooth boundary. If the boundary is regular, then we can show that the eigenfunctions are zero on the boundary. As standard PDEs' techniques are of limited use for hypoelliptic operators and for domains with a nonsmooth boundary, we rely on the Dirichlet form theory, the Krein–Rutman theorem and irreducibility of the corresponding semigroup.

The spectral analysis in Section 3 relies on the Dirichlet form theory on $L^2(\mathbb{G}, dx)$, where $\mathbb{G}$ is a homogeneous Carnot group and dx is a (bi-invariant) Haar measure. A natural question is if these techniques are applicable to more general sub-Riemannian manifolds, but in such a setting, we might not have a canonical choice of a measure m, which is needed to define a Dirichlet form on the corresponding L^2 space. We also indirectly rely on the fact that the Haar measure on $\mathbb{G}$ satisfies a volume doubling property. Finally, our approach uses the fact that sub-Laplacians on $L^2(\mathbb{G}, dx)$ are essentially self-adjoint on $C_c^\infty(\mathbb{G})$ in $L^2(\mathbb{G}, dx)$ by [10, Section 3].

Note that [36] proves a number of related results for sub-Laplacians on domains in homogeneous Carnot groups. The domains considered there are bounded open with a piecewise smooth and simple boundary. Lower bounds on the spectral gap for Dirichlet sub-Laplacians on compact domains with smooth boundary in sub-Riemannian manifolds have been studied recently in [33]. Their methods are different, and in particular, the Dirichlet form theory allows us to consider domains with a nonsmooth boundary. Small time asymptotic expansions for hypoelliptic heat kernels can be found in [7]. As one of the applications is to the heat content asymptotics, we mention that one of the technical difficulties in the sub-Riemannian setting is potential presence of characteristic points on $\partial\Omega$. These are points $x \in \partial\Omega$ where the horizontal distribution is tangent to $\partial\Omega$. We refer to [35, 43] for a more detailed analysis of such points.

We remark that Dirichlet forms in the context of free nilpotent groups have been used in [15, 16] in connection with the theory of rough paths. Moreover, they relied on the general Dirichlet forms such as [40–42] in the context of free nilpotent groups. In this setting, they derived small ball estimates in the context of support theorem for Markovian rough paths.

The paper is organized as follows. In Section 2, we describe homogeneous Carnot groups, sub-Laplacians, and Dirichlet forms on such groups. In Section 3, we describe spectral properties of Dirichlet sub-Laplacians that are collected in Theorem 3.3. In Section 4, we prove that analytic and probabilistic notions of boundary regular points are equivalent. We conclude Section 5 with two applications: the small ball problem for a hypoelliptic Brownian motion and the large time behavior of the heat content in a regular domain Ω. The latter is done with a natural assumption on the boundary $\partial\Omega$ being regular; therefore, we allow $\partial\Omega$ to contain characteristic points.

2 Preliminaries

2.1 Carnot groups

In this paper, we concentrate on a particular class of nilpotent groups, namely Carnot groups. We begin by recalling basic facts about Carnot (stratified) groups. A more detailed description of these spaces can be found in a number of references; see for example [3, 44].

Definition 2.1 (Carnot groups). We say that $\mathbb{G}$ is a Carnot group of step r if $\mathbb{G}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is *stratified*, that is, it can be written as

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r, \tag{2.1}$$

where

$$\begin{aligned} &\left[V_1, V_{i-1}\right] = V_i, \quad 2 \leqslant i \leqslant r, \\ &[V_1, V_r] = \{0\}. \end{aligned}$$

The stratification (2.1) is not unique as pointed out in [3, Section 2.2.1]. Moreover, if $(V_1, \ldots, V_r)$ and $(\tilde{V}_1, \ldots, \tilde{V}_s)$ are two stratification of a Carnot group $\mathbb{G}$, then $r = s$, which is referred to as the step and $\dim V_i = \dim \tilde{V}_i$ for every $i = 1, \ldots, r$ [3, Proposition 2.28]. If $\mathbb{G}$ is a Carnot group with stratification $(V_1, \ldots, V_r)$, by [3, Proposition 2.2.8], it follows that the step r and the number $m := \dim V_1$ of generators of $\mathbb{G}$ do not depend on the chosen stratification.

For the rest of the paper, we fix a stratification $(V_1, \ldots, V_s)$ of $\mathbb{G}$. This stratification determines the space $\mathcal{H} := V_1$ of horizontal vectors that generate the rest of the Lie algebra, nothing that $V_2 = [\mathcal{H}, \mathcal{H}], \ldots, V_r = \mathcal{H}^{(r)}$. The horizontal space $\mathcal{H}$ is used to construct a sub-Laplacian on $\mathbb{G}$ for which we refer to [3, Section 2.2].

To avoid degenerate cases, we assume that the dimension of the Lie algebra $\mathfrak{g}$ is at least 3, which implies $\dim V_1 \geqslant 2$. We generally assume that $r \geqslant 2$ to exclude the case when the corresponding Laplacian is elliptic.

In particular, Carnot groups are nilpotent. We will use $\mathcal{H} := V_1$ to denote the space of *horizontal* vectors that generate the rest of the Lie algebra, noting that $V_2 = [\mathcal{H}, \mathcal{H}], \ldots, V_r = \mathcal{H}^{(r)}$. As usual, we let

$$\exp : \mathfrak{g} \longrightarrow \mathbb{G},$$

$$\log : \mathbb{G} \longrightarrow \mathfrak{g}$$

denote the exponential and logarithmic maps, which are global diffeomorphisms for connected nilpotent groups; see for example [8, Theorem 1.2.1].

Finally, by [3, Propositions 2.2.17 and 2.2.18], we can assume without loss of generality that a Carnot group can be identified with a *homogeneous Carnot group*. For $i = 1, \ldots, r$, let $d_i = \dim V_i$ and $d_0 = 0$. The Euclidean space underlying $\mathbb{G}$ has dimension

$$N := \sum_{i=1}^{r} d_i$$

and the homogeneous dimension of $\mathbb{G}$ is given by

$$Q := \sum_{i=1}^{r} i \cdot d_i. \tag{2.2}$$

The identification of $\mathbb{G}$ with $\mathbb{R}^N$ allows us to define exponential coordinates as follows.

Definition 2.2. We say that a collection of left-invariant vector fields $\{X_1, \ldots, X_N\}$ on a Carnot group $\mathbb{G}$ is a *basis for* $\mathfrak{g}$ *adapted to the stratification* if the set $\left\{X_{d_{i-1}}, \ldots, X_{d_{i-1}+d_i}\right\}$ is a basis of V_i for each $i = 1, \ldots, r$.

The fact that exp is a global diffeomorphism can be used to parameterize $\mathbb{G}$ by its Lie algebra $\mathfrak{g}$. First, we recall the Baker–Campbell–Dynkin–Hausdorff formula.

Notation 2.3. For any $X, Y \in \mathfrak{g}$, we define the *Baker–Campbell–Dynkin–Hausdorff formula* by

$$\begin{aligned} \mathrm{BCDH}(X,Y) &:= \log(\exp X \exp Y) \\ &= \sum_{m=1}^{\infty} \sum_{p_i+q_i \geqslant 1} \frac{(-1)^{m-1}}{m(\sum_{i=1}^{m} p_i + q_i)} \\ &\times \frac{[\overbrace{[\cdots[X,X],\cdots],X]}^{p_1},\overbrace{Y],\cdots,Y]}^{q_1},\cdots\overbrace{\cdots,X],\cdots X}^{p_m}\overbrace{\cdots,Y],\cdots Y]}^{q_m}}{p_1!\,q_1!\cdots p_m!\,q_m!}. \end{aligned}$$

Definition 2.4. Suppose $\{X_1,\ldots,X_N\}$ is a basis of $\mathfrak{g}$ adapted to the stratification. For a point $g \in \mathbb{G} \cong \mathbb{R}^N$, we say that $(x_1,\ldots,x_N) \in \mathbb{R}^N$ are *exponential coordinates of the 1st kind* relative to the basis $\{X_1,\ldots,X_N\}$ if

$$g = \exp\left(\sum_{i=1}^{N} x_i X_i\right).$$

We equip $\mathbb{R}^N$ with the group operation pulled back from $\mathbb{G}$ by

$$\begin{aligned} z &:= x \star y, \\ \sum_{i=1}^{N} z_i X_i &= \mathrm{BCDH}\left(\sum_{i=1}^{N} x_i X_i, \sum_{i=1}^{N} y_j X_j\right). \end{aligned}$$

In particular, in this identification $x^{-1} = -x$. Note that $\mathbb{R}^N$ with this group law is a Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Both $\mathbb{G}$ and $(\mathbb{R}^N, \star)$ are nilpotent connected and simply connected; therefore, the exponential coordinates give a diffeomorphism between $\mathbb{G}$ and $\mathbb{R}^N$.

A stratified Lie algebra is equipped with a natural family of *dilations* defined for any $a > 0$ by

$$\delta_a(X) := a^i X, \qquad \text{for } X \in V_i.$$

For each $a > 0$, δ_a is a Lie algebra isomorphism, and the family of all dilations $\{\delta_a\}_{a>0}$ forms a one-parameter group of Lie algebra isomorphisms. We again can use the fact that exp is a global diffeomorphism to define the automorphisms D_a on G. The maps

$D_a := \exp \circ \delta_a \circ \log : \mathbb{G} \longrightarrow \mathbb{G}$ satisfy the following properties.

$$\begin{aligned} & D_a \circ \exp = \exp \circ \delta_a \quad \text{for any } a > 0, \\ & D_{a_1} \circ D_{a_2} = D_{a_1 a_2}, D_1 = I \quad \text{for any } a_1, a_2 > 0, \\ & D_a(g_1) D_a(g_2) = D_a(g_1 g_2) \quad \text{for any } a > 0 \text{ and } g_1, g_2 \in G, \\ & dD_a = \delta_a. \end{aligned} \tag{2.3}$$

That is, the group $\mathbb{G}$ has a family of dilations, which is adapted to its stratified structure. Actually D_a is the unique Lie group automorphism corresponding to δ_a. On a homogeneous Carnot group $\mathbb{R}^N$ the dilation D_a can be described explicitly by

$$D_a(x_1, \dots, x_N) := (a^{\sigma_1} x_1, \dots, a^{\sigma_N} x_N),$$

where $\sigma_j \in \mathbb{N}$ is called the *homogeneity* of x_j, with

$$\sigma_j = i \quad \text{for } \sum_{k=0}^{i-1} d_k + 1 \leqslant j \leqslant \sum_{k=1}^{i} d_k,$$

with $i = 1, \dots, r$ and recalling that $d_0 = 0$. That is, $\sigma_1 = \dots = \sigma_{d_1} = 1, \sigma_{d_1+1} = \dots = \sigma_{d_1+d_2} = 2$, and so on.

We assume that $\mathcal{H}$ is equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, in which case the Carnot group has a natural sub-Riemannian structure. Namely, one may use left translation to define a *horizontal distribution* $\mathcal{D}$ as a sub-bundle of the tangent bundle $T\mathbb{G}$, and a metric on $\mathcal{D}$. First, we identify the space $\mathcal{H} \subset \mathfrak{g}$ with $\mathcal{D}_e \subset T_e\mathbb{G}$. Then, for $g \in \mathbb{G}$, let L_g denote left translation $L_g h = gh$, and define $\mathcal{D}_g := (L_g)_* \mathcal{D}_e$ for any $g \in \mathbb{G}$. A metric on $\mathcal{D}$ may then be defined by

$$\begin{aligned} \langle u, v \rangle_{\mathcal{D}_g} &:= \langle (L_{g^{-1}})_* u, (L_{g^{-1}})_* v \rangle_{\mathcal{D}_e} \\ &= \langle (L_{g^{-1}})_* u, (L_{g^{-1}})_* v \rangle_{\mathcal{H}} \qquad \text{for all } u, v \in \mathcal{D}_g. \end{aligned}$$

We will sometimes identify the horizontal distribution $\mathcal{D}$ and $\mathcal{H}$. Vectors in $\mathcal{D}$ are called *horizontal*.

2.2 Sub-Laplacians and Dirichlet forms

For sub-Laplacians on sub-Riemannian complete manifolds [39, pp. 41–42] claimed that these operators are essentially self-adjoint, though without a complete proof or indication of how to choose the reference measure. If the manifold is a Lie group, a natural choice is a Haar measure, and we review relevant results below. For a more recent approach, we refer to [14]. To tackle more general sub-Riemannian manifolds in the future, one might also use the semigroup approach in [19, 20] combined with the Dirichlet form theory in [17, 31].

Suppose G is a real connected Lie group and $\{X_i\}_{i=1}^m$ is a family of left-invariant vector fields on G satisfying Hörmander's condition, then the sum of squares operator

$$\mathcal{L} := \sum_{i=1}^{m} X_i^2$$

is essentially self-adjoint on $C_c^\infty(G)$ in $L^2(G, dx)$ according to [10, p. 950]. Here, dx is a (right) Haar measure on G. In particular, the following integration by parts formula holds:

$$\int_G \sum_{i=1}^{m} X_i^2 f \mathrm{d}x = -\langle \mathcal{L}^* f, f \rangle_{L^2(G,dx)} < \infty$$

for any $f \in C^\infty(G) \cap \mathcal{D}_{\mathcal{L}^*}$. If $G = \mathbb{G}$ is a homogeneous Carnot group and $\{X_1, ..., X_{d_1}\}$ is an orthonormal basis of $\mathcal{H}$ of left-invariant vector fields and $m = d_1$, then by [10, Section 3], the operator $\mathcal{L}$ depends only on the inner product on $\mathcal{H}$ and not on the choice of the basis. This is what [3, Example 1.5.2] calls a canonical sub-Laplacian. More background on sub-Laplacians can be found in [19, Section 3.1]. This is not the subject of this paper, though we mention that by [19, Corollary 3.4] all sub-Laplacians differ only by 1st-order terms. We abuse notation and denote by X_j both the vector in $\mathcal{H}$ and the unique extension of this vector to the left-invariant vector field on $\mathbb{G}$.

Remark 2.5 (Choice of the measure). We chose a Haar measure as the reference measure for several reasons. First of all, the sub-Laplacians we are interested in are essential self-adjoint on $C_c^\infty(\mathbb{G})$ in $L^2(\mathbb{G}, dy)$ by [10, Section 3]. Secondly, a Haar measure on the metric space $(\mathbb{G}, |\cdot|)$ is doubling which is needed for using heat kernel estimates in [40]. The fact that a Haar measure is doubling in this setting follows from [10, Theorem 3.2] and [25, 32, 44]. For more comments on how the sub-Laplacians might depend on the choice of a reference measure including on when we can write it as $\operatorname{div} \nabla_{\mathcal{H}}$ we refer to [19, Section 4].

Before listing basic properties of this second-order differential operator, recall that $\mathbb{G}$ is unimodular, and so we can assume that it is equipped with a (bi-invariant) Haar measure dx. Moreover, if μ is the push-forward of the N-dimensional Lebesgue measure $\mathcal{L}^N$ via the exponential map, then it is a bi-invariant Haar measure on $\mathbb{G}$ and

$$d\mu\left(y \circ \delta_\lambda\right) = \lambda^Q d\mu\left(y\right). \tag{2.4}$$

If we identify $\mathbb{G}$ with the homogeneous Carnot group $\left(\mathbb{R}^N, \star, \delta_\lambda\right)$ via exponential coordinates in Definition 2.4, then for a measurable set $E \subset \mathbb{G}$, its Haar measure is explicitly given by $\mu\left(E\right) = \mathcal{L}^N\left(\exp^{-1}\left(E\right)\right)$. We will abuse notation and use the same notation for both measures.

As observed in [10, Section 3], the operator $\mathcal{L}$ is essentially self-adjoint on $C_c^\infty\left(\mathbb{G}\right)$ in $L^2\left(\mathbb{G}, dy\right)$. Thus, the corresponding semigroup $e^{t\mathcal{L}}$ can be defined by the spectral theorem. This semigroup commutes with left translations since its infinitesimal generator is left-invariant as well. Namely, for every $y \in \mathbb{G}$, we have

$$\mathcal{L}(f(y \circ x)) = (\mathcal{L}f)(y \circ x) \quad \text{for every } x \in \mathbb{G} \quad \text{and } f \in C_c^\infty(\mathbb{G}).$$

We now recall some basic properties of such sub-Laplacians, for more properties including regularity of the corresponding heat kernel, parabolic Harnack inequality, Gaussian upper, and lower bounds we refer to [10, Theorem 3.4] in a more general setting. Some of these properties rely on the fact that $\mathcal{L}$ is hypoelliptic by Hörmander's hypoellipticity theorem [24, Theorem 1.1] and doubling property for the metric we describe below. In addition, on Carnot groups the operator $\mathcal{L}$ is δ_λ-homogeneous of degree two, that is, for every fixed $\lambda > 0$

$$\mathcal{L}(f(\delta_\lambda(x))) = \lambda^2(\mathcal{L}f)(\delta_\lambda(x)) \quad \text{for every } \ x \in \mathbb{G} \quad \text{and } f \in C_c^\infty(\mathbb{G}),$$

since vector fields X_js are δ_λ-homogeneous of degree one. For more details, we refer to [3, p. 63] on homogeneous Carnot groups.

The sub-Laplacian is symmetric since the group is unimodular, and hence the integration by parts formula reads

$$\int_{\mathbb{G}} \sum_{j=1}^{d_1} X_j f \cdot X_j g \mathrm{d}y = -\int_{\mathbb{G}} \mathcal{L}f \cdot g \mathrm{d}y, f, g \in C_c^\infty\left(\mathbb{G}\right). \tag{2.5}$$

The symmetric form corresponding to the semigroup $e^{t\mathcal{L}}$ is

$$f \to \mathcal{E}\left(f\right) := \int_{\mathbb{G}} |\nabla_{\mathcal{L}} f(y)|^2_{\mathbb{R}^{d_1}} \mathrm{d}y, \tag{2.6}$$

where $\nabla_{\mathcal{L}} := \left(X_1, \ldots, X_{d_1}\right)$

is the horizontal gradient and

$$\begin{aligned}\mathcal{D}_{\mathcal{E}} &:= W_2^1(\mathbb{G}) \\ &:= \left\{ f \in L^2(\mathbb{G}, dx) : X_i f \in L^2(\mathbb{G}, dx), \text{ for all } i = 1, \ldots, d_1 \right\},\end{aligned}$$

where $X_i f$ is to understood in the distributional sense. The form $\mathcal{E}$ is a Dirichlet form by [10, p. 951]. Note that $\mathcal{E}$ is a closed form, that is, $\mathcal{D}_{\mathcal{E}}$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{W_2^1(\mathbb{G})} = \mathcal{E}(f, g) + \langle f, g \rangle_{L^2(\mathbb{G}, dx)},$$

where $\mathcal{E}(f, g)$ is obtained from (2.6) by polarization. Moreover, by (2.5), we have that for $f \in \mathcal{D}_{\mathcal{E}}$

$$\int_{\mathbb{G}} |\nabla_{\mathcal{L}} f(y)|_{\mathcal{H}}^2 \mathrm{d}y = -\int_{\mathbb{G}} f(y) \mathcal{L} f(y) \mathrm{d}y. \tag{2.7}$$

The next step is to check that (2.6) can be extended to a regular Dirichlet form. Recall that a Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ is called *regular* if it admits a core, that is, if there exists a subset $\mathcal{C}$ of $\mathcal{D}_{\mathcal{E}} \cap C_0(\mathbb{G})$ that is dense both in $\mathcal{D}_{\mathcal{E}}$ with respect to the Sobolev norm $\| \cdot \|_{W_2^1(\mathbb{G})}$, and in $C_0(\mathbb{G})$, with respect to the sup-norm. Note that $C_c^\infty(\mathbb{G})$ is dense in $L^2(G, dx)$, and it is a core for the bilinear form $\mathcal{E}$. Thus, $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ is a regular Dirichlet form. In addition, it is strongly local as defined in [22, Definition 1.2].

Definition 2.6. Suppose $\mathbb{G} = (\mathbb{R}^N, \star, \delta_\lambda)$ is a homogeneous Carnot group and $\rho : \mathbb{G} \to [0, \infty)$ is a continuous function with respect to the Euclidean topology. Then ρ is a *homogeneous norm* if it satisfies the following properties:

$$\begin{aligned}&\rho(\delta_\lambda(x)) = \lambda \rho(x) \text{ for every } \lambda > 0 \text{ and } x \in \mathbb{G}, \\ &\rho(x) > 0 \text{ if and only if } x \neq 0.\end{aligned}$$

The norm ρ is called *symmetric* if it satisfies $\rho(x^{-1}) = \rho(x)$ for every $x \in \mathbb{G}$.

If ρ_1 and ρ_2 are two homogeneous norms, then there exists a constant $c > 0$ such that

$$c^{-1} \rho_1(x) \leqslant \rho_2(x) \leqslant c \rho_1(x), \text{ for every } x \in \mathbb{G}, \tag{2.8}$$

see, for example, [3, Proposition 5.1.4, p. 230]. On every homogeneous Carnot group, there exist distinguished symmetric homogeneous norms related to a sub-Laplacian as follows.

Definition 2.7. A homogeneous symmetric norm ρ on $\mathbb{G}$ is called an *$\mathcal{L}$-gauge* if it is smooth everywhere except at the origin and

$$\mathcal{L}(\rho^{2-Q}(x)) = 0, x \in \mathbb{G} \setminus \{0\}, \tag{2.9}$$

where Q is the homogeneous dimension of $\mathbb{G}$.

By [3, Section 5.3], we know that there exists a unique fundamental solution Γ for the Poisson equation with a sub-Laplacian $\mathcal{L}$, that is, $\Gamma \in C^{\infty}(\mathbb{G} \setminus \{0\}) \bigcap L^1_{\mathrm{loc}}(\mathbb{G}, dx)$ and

$$\mathcal{L}\Gamma = -\operatorname{Dirac}_0,$$

where Dirac_0 is the Dirac measure supported at $\{0\}$.

$\mathcal{L}$-Gauges and the fundamental solution for $\mathcal{L}$ are related as follows. If $\mathbb{G}$ is a Carnot group of homogeneous dimension Q and Γ is a fundamental solution for $\mathcal{L}$, then

$$\rho(x) := \begin{cases} \Gamma(x)^{\frac{1}{2-Q}}, & x \in \mathbb{G}\backslash\{0\}; \\ 0, & x = 0 \end{cases} \tag{2.10}$$

is an $\mathcal{L}$-gauge on $\mathbb{G}$. By [3, Section 5.5], if ρ is an $\mathcal{L}$-gauge on $\mathbb{G}$, then there exists a constant α_d such that $\Gamma = \alpha_\rho \rho^{2-Q}$ is Green's function for $\mathcal{L}$. As a consequence, the $\mathcal{L}$-gauge is unique up to a multiplicative constant.

2.3 Regular boundary points: a probabilistic approach

We now recall the notion of regular points for a differential operator $\mathcal{L}$ in a bounded open connected set $\Omega \subset \mathbb{R}^N$ as found in a number of references, in particular for Brownian motion in [37, Chapter 8]. The connection with classical potential theory goes back to Doob *et al.* [9, 11, 27].

Suppose that $g_{\cdot} := \{g_t\}_t$ is a $\mathbb{G}$-valued diffusion process whose infinitesimal generator is $\mathcal{L}$. Using exponential coordinates of the 1st type, we can view $g_{\cdot}$ as an $\mathbb{R}^N$-valued process. We start with a probabilistic definition of regular points as found in

[28, Definition 8.2.1], and this is the definition used in [18] for $\mathcal{L} = -\frac{1}{2}\Delta_{\mathcal{H}}$ on $\mathbb{R}^3 \cong \mathbb{H}$, the three-dimensional Heisenberg group.

Definition 2.8. Let $g^x_\cdot$ be the diffusion process with generator $\mathcal{L}$ started at $x \in \partial\Omega$, where Ω is a bounded open connected set, and define

$$\tau^x_\Omega := \inf\left\{t > 0 : g^x_t \in \Omega^c\right\}.$$

We call x a *regular point* of $\partial\Omega$ if $\mathbb{P}^x\left(\tau^x_\Omega = 0\right) = 1$. We call the set Ω *regular* if every boundary point of Ω is regular. If $\mathbb{P}^x\left(\tau^x_\Omega = 0\right) < 1$, the point x is called a *singular point* of the boundary.

In Section 4, we compare Definition 2.8 with an analytic definition of regular points used for hypoelliptic operators such as sub-Laplacians on homogeneous Carnot groups. Moreover, we prove that the two notion of regular points are equivalent.

3 Spectral Properties of the Sub-Laplacian $\mathcal{L}$ Restricted to a Set Ω

We rely on the Dirichlet form theory to describe a restriction of $\mathcal{L}$ to a set Ω in $\mathbb{G}$. Our main reference here is [17], and in a more relevant setting of metric measure spaces [21, Section 6.1] and [22, p. 173]. In particular, we use this approach to show that $\mathcal{L}_\Omega$ has a discrete spectrum with minimal assumptions on the boundary of Ω.

Let Ω be a bounded open connected set in $\mathbb{G}$, and define

$$\mathcal{D}_{\mathcal{E}}(\Omega) := \overline{\left\{f \in \mathcal{D}_{\mathcal{E}} : \operatorname{supp} f \subset \Omega\right\}}^{W^1_2},$$

where $\|f\|^2_{W^1_2} = \mathcal{E}(f) + \|f\|^2_{L^2(\mathbb{G},dx)}$ and $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ is given in (2.6). Then, by [22, Section 3.2 Theorem 3.3], we know that $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega)\right)$ is a regular Dirichlet form on $L^2(\Omega, dx)$. Note that $\mathcal{D}_{\mathcal{E}}(\Omega)$ is dense in $L^2(\Omega, dx)$ since $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega)\right)$ is a regular Dirichlet form. We denote by $-\mathcal{L}_\Omega$ the nonnegative self-adjoint operator on $L^2(\Omega, dx)$ corresponding to the Dirichlet form $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega)\right)$.

Notation 3.1. We denote the *semigroups* corresponding to the Dirichlet forms $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}\right)$ and $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega)\right)$ by

$$P_t := e^{t\mathcal{L}},$$

$$P^\Omega_t := e^{t\mathcal{L}_\Omega},$$

respectively.

The domains of the corresponding infinitesimal generators are given by

$$\mathcal{D}(-\mathcal{L}) := \left\{ f \in L^2(\mathbb{G}, dx) : \lim_{t \downarrow 0} \frac{P_t f - f}{t} \text{ exists} \right\},$$
$$\mathcal{D}(-\mathcal{L}_\Omega) := \left\{ f \in L^2(\Omega, dx) : \lim_{t \downarrow 0} \frac{P_t^\Omega f - f}{t} \text{ exists} \right\}.$$

We now give a probabilistic description of the semigroups P_t and P_t^Ω. Homogeneous Carnot groups are complete metric spaces; therefore, by [40, Proposition 3.1], the semigroup P_t has a heat kernel $p_t(x, y)$, which is continuous in $x, y \in \mathbb{G}$. We also know that by Chow–Rashevskii's theorem that the intrinsic metric induced by the Dirichlet form $\mathcal{E}$ coincides with the original topology, so the Dirichlet form is *strongly regular*. The heat kernel $p_t(x, y)$ is the transition density of g_t, that is,

$$\mathbb{P}^x \left(g_t \in E \right) = \int_E p_t(x, y) \, \mathrm{d}y, \tag{3.1}$$

for any Borel set E in $\mathbb{G}$, where g_t is the Markov process whose generator is the sub-Laplacian $\mathcal{L}$. The strongly continuous semigroup associated with the Dirichlet form $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}\right)$ is given in terms of g_t by

$$P_t f(x) = \mathbb{E}^x[f(g_t)], \quad x \in \mathbb{G}, \; f \in L^2(\mathbb{G}, dx).$$

Definition 3.2. Let g_t be the $\mathbb{G}$-valued Markov process with the transition density given by the heat kernel $p_t(x, y)$. Then we refer to g_t as the *hypoelliptic Brownian motion*.

The heat kernel is the fundamental solution to the heat equation

$$\left(\partial_t - \frac{1}{2} \mathcal{L} \right) p_t(x, \cdot) = 0,$$
$$p_t(x, y) dy \to \mathrm{Dirac}_x(dy) \quad \text{weakly as } t \to 0, \tag{3.2}$$

where $\mathrm{Dirac}_x(dy)$ is the Dirac measure centered at $\{x\}$, see [10, Equation (3.6)].

Following [4, Section 2.11], we let Ω be a bounded open connected set and

$$\tau_\Omega := \inf \left\{ t > 0 \, : \, g_t \notin \Omega \right\}.$$

Then we can use Hunt's formula

$$p_t^{\Omega}(x,y) := p_t(x,y) - \mathbb{E}^x\left[\mathbb{1}_{\{\tau_\Omega<t\}}\, p_{t-\tau_\Omega}\left(g_{\tau_\Omega}, y\right)\right] \tag{3.3}$$

for the transition density $p_t^{\Omega}(x,y)$ of the killed Markov process g_t^{Ω} given by

$$g_t^{\Omega} := \begin{cases} g_t & t<\tau_\Omega, \\ \partial & t \geqslant \tau_\Omega, \end{cases}$$

where ∂ is the *cemetery* point. More precisely, we have that

$$\mathbb{P}^x\left(g_t^{\Omega} \in E\right) = \mathbb{P}^x\left(g_t \in E,\, t<\tau_\Omega\right) = \int_E p_t^{\Omega}(x,y)\,\mathrm{d}y, \tag{3.4}$$

for any $x \in \Omega$ and any Borel subset E of Ω. In particular,

$$\mathbb{P}^x\left(\tau_\Omega > t\right) = \int_\Omega p_t^{\Omega}(x,y)\,\mathrm{d}y. \tag{3.5}$$

We refer to p_t^{Ω} as the Dirichlet heat kernel. Note that regularity of the Dirichlet form implies that g_t^{Ω} is a Hunt process as well. Then the semigroup P_t^{Ω} can be viewed as

$$\begin{aligned} &P_t^{\Omega} : L^2(\Omega, dx) \longrightarrow L^2(\Omega, dx), \\ &P_t^{\Omega} f(x) = \mathbb{E}^x\left[f(g_t^{\Omega})\right] = \mathbb{E}^x\left[f(g_t),\, t<\tau_\Omega\right] = \int_\Omega p_t^{\Omega}(x,y) f(y)\,\mathrm{d}y, \end{aligned} \tag{3.6}$$

for any $f \in L^2(\Omega, dx)$. Note that by [40, Proposition 3.1] applied to the Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega))$ on $L^2(\Omega, dx)$ it follows that the function $p_t^{\Omega}(x,y)$ is Hölder continuous on $[T,\infty) \times \Omega \times \Omega$.

The following theorem is the main result of this section, where we collect spectral properties of the operator $\mathcal{L}_\Omega$.

Theorem 3.3. Let Ω be a bounded open connected subset of $\mathbb{G}$ and P_t^{Ω} the semigroup associated with the Dirichlet form $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega)\right)$ with generator $\mathcal{L}_\Omega$. Then,

(1) for every $t > 0$, P_t^{Ω} is a Hilbert–Schmidt operator on $L^2(\Omega, dx)$ and its spectrum is given by

$$\sigma(P_t^{\Omega}) \backslash \{0\} = \left\{e^{-\lambda_n t}\right\}_{n\in\mathbb{N}},$$

where $\{\lambda_n\}_{n\in\mathbb{N}} = \sigma_{pp}(-\mathcal{L}_\Omega)$ is the point spectrum of $-\mathcal{L}_\Omega$, with $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$.

(2) The operator $-\mathcal{L}_\Omega$ has a spectral gap, that is, $\lambda_1 > 0$.

(3) There exists an orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ of $L^2(\Omega, dx)$ such that, for every $n \in \mathbb{N}$, $t > 0$,

$$P_t^\Omega \phi_n = e^{-\lambda_n t}\phi_n.$$

Moreover, for every $n \in \mathbb{N}$, $\phi_n \in \mathcal{D}(-\mathcal{L}_\Omega)$ and

$$-\mathcal{L}_\Omega \phi_n = \lambda_n \phi_n.$$

Corollary 3.4. The operator $\mathcal{L}_\Omega$ has a pure point spectrum.

Proof of Corollary 3.4. By [34, Theorem XIII.64 p. 245] and Theorem 3.3 part (3), it follows that $(-\mu - \mathcal{L}_\Omega)^{-1}$ is a compact operator for every μ in the resolvent set of $\mathcal{L}_\Omega$, proving that $\mathcal{L}_\Omega$ has a pure point spectrum. ■

Proof of Theorem 3.3. (1) Let us first show that the semigroup P_t^Ω is a Hilbert–Schmidt operator for each t. We rely on [40] for a heat kernel estimate that only requires volume doubling and Poincaré's inequality without compactness assumption on the underlying metric space, unlike in [26]. Doubling and Poincaré's inequality are known to hold in our setting, for example, by [10, Theorem 3.2]. By [40, Equation (4.2)] with $\varepsilon = 1$, there exists a constant $A > 0$ such that

$$p_t(x, y) \leqslant A t^{-\frac{Q}{2}}, \tag{3.7}$$

for all $t > 0$ and $x, y \in \mathbb{G}$, where Q is the homogeneous dimension of $\mathbb{G}$. Thus, for any $t > 0$,

$$\begin{aligned}\|p_t^\Omega\|^2_{L^2(\Omega\times\Omega)} &= \int_\Omega\int_\Omega p_t^\Omega(x, y)^2 \mathrm{d}x\mathrm{d}y \leqslant \int_\Omega\int_\Omega p_t(x, y)^2 \mathrm{d}x\mathrm{d}y \\ &\leqslant A^2|\Omega|^2 t^{-Q} < \infty,\end{aligned}$$

where we used that $p_t^\Omega(x, y) \leqslant p_t(x, y)$ for almost all $x, y \in \Omega$. Indeed, for any Borel set $E \subset \Omega$, $t > 0$, and all $x \in \Omega$ we have that

$$\begin{aligned}\int_E p_t^\Omega(x, y)\mathrm{d}y &= \mathbb{P}^x\left(g_t^\Omega \in E\right) = \mathbb{P}^x\left(g_t \in E, \tau_\Omega > t\right) \\ &\leqslant \mathbb{P}^x\left(g_t \in E\right) = \int_E p_t(x, y)\mathrm{d}y,\end{aligned}$$

and hence $p_t^\Omega(x,y) \leqslant p_t(x,y)$ for any $x \in \Omega$ and for a.e. $y \in \Omega$. The estimate then follows for every $y \in \Omega$ since both p_t and p_t^Ω are continuous on $\Omega \times \Omega$.

The operator P_t^Ω is then Hilbert–Schmidt since $p_t^\Omega \in L^2(\Omega \times \Omega)$. In particular, for every $t > 0$, the operator P_t^Ω is compact, and hence by the spectral theorem for compact operators, there exists a sequence of decreasing eigenvalues $\{\lambda_n(t)\}_{n\in\mathbb{N}}$ and corresponding eigenfunctions $\{\phi_{n,k}^{(t)}\}_{n,k\in\mathbb{N}}$ such that for every $t > 0$,

$$\begin{aligned} &\sigma(P_t^\Omega)\backslash\{0\} = \sigma_{pp}(P_t^\Omega)\backslash\{0\} = \{\lambda_n(t)\}_{n\in\mathbb{N}}, \\ &\ker\left(\lambda_n(t) - P_t^\Omega\right) = \overline{\mathrm{Span}\{\phi_{n,k}^{(t)}, k \in \mathbb{N}\}} \text{ for every } n \in \mathbb{N}, \\ &L^2(\Omega, dx) = \bigoplus_{n=1}^{\infty} \overline{\mathrm{Span}\{\phi_{n,k}^{(t)}, k \in \mathbb{N}\}}. \end{aligned} \tag{3.8}$$

The semigroup P_t^Ω is strongly continuous because the Dirichlet form $\left(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(\Omega)\right)$ is regular, and hence by the spectral mapping theorem for semigroups [1, Theorem 6.3]

$$\sigma_{pp}(P_t^\Omega)\backslash\{0\} = \exp\left(t\,\sigma_{pp}(\mathcal{L}_\Omega)\right), \text{ for any } t > 0. \tag{3.9}$$

Thus, the eigenvalues of P_t^Ω are given by $e^{\mu_n t}$ for $\mu_n \in \sigma_{pp}(\mathcal{L}_\Omega)$. By the theory of Dirichlet forms [17], the operator $\mathcal{L}_\Omega$ is nonpositive definite, and hence we can write $\mu_n = -\lambda_n$, where $\lambda_n \to \infty$ as $n \to \infty$, which completes the proof of (1).

(2) Let us now prove that $\lambda_1 > 0$. Assume that $\lambda_1 = 0$, then by the spectral mapping theorem (3.9), we have that $1 \in \sigma_{pp}\left(P_t^\Omega\right)$ for all $t > 0$, and hence

$$1 \leqslant \|P_t^\Omega\|,$$

since $\sigma_{pp}(P_t^\Omega) \subset \{z \in \mathbb{C},\ |z| \leqslant \|P_t^\Omega\|\}$. By (3.7), we have that

$$1 \leqslant \|P_t^\Omega\|^2 \leqslant \|p_t^\Omega\|^2_{L^2(\Omega\times\Omega)} \leqslant A^2|\Omega|^2 t^{-Q},$$

for some finite constant $A > 0$. Thus, $t^Q \leqslant A^2|\Omega|^2$ for any $t > 0$, which is a contradiction.

(3) Let $-\lambda_n \in \sigma_{pp}(\mathcal{L}_\Omega)$, and let $\{\phi_{n,k}\}_{n,k\in\mathbb{N}}$ be an orthonormal basis of $\ker\left(-\lambda_n - \mathcal{L}_\Omega\right)$. By [1, Corollary 6.4], it follows that

$$\begin{aligned} \overline{\mathrm{Span}\{\phi_{n,k}^{(t)}, k \in \mathbb{N}\}} &= \ker\left(e^{-\lambda_n t} - P_t^\Omega\right) \\ &= \overline{\mathrm{Span}\left\{\ker\left(-\lambda_n + \frac{2\pi j}{t} i - \mathcal{L}_\Omega\right), j \in \mathbb{Z}\right\}} \end{aligned}$$

for any $t > 0$. The point spectrum of $\mathcal{L}_\Omega$ is real since $\mathcal{L}_\Omega$ is self-adjoint, and hence

$$\ker\left(-\lambda_n + \frac{2\pi j}{t} i - \mathcal{L}_\Omega\right) = \{0\}$$

for all $j \neq 0$. Thus, we have that

$$\begin{aligned}\overline{\operatorname{Span}\{\phi^{(t)}_{n,k}, k \in \mathbb{N}\}} &= \ker\left(e^{-\lambda_n t} - P^\Omega_t\right)\\ &= \ker\left(-\lambda_n - \mathcal{L}_\Omega\right) = \overline{\operatorname{Span}\{\phi_{n,k}, k \in \mathbb{N}\}}, \text{ for all } n \in \mathbb{N}.\end{aligned}$$

The operator P^Ω_t is compact, and thus for every $n \in \mathbb{N}$, the eigenspace $\ker\left(e^{-\lambda_n t} - P^\Omega_t\right)$ is finite-dimensional. Therefore, for every $n \in \mathbb{N}$, there exists an M_n such that for every $t > 0$

$$\ker\left(e^{-\lambda_n t} - P^\Omega_t\right) = \operatorname{Span}\{\phi_{n,k}, k = 1, \dots, M_n\}. \tag{3.10}$$

We proved that for every $n \in \mathbb{N}$, there exists an orthonormal basis $\{\phi_{n,k}\}_{k=1}^{M_n}$ of $\ker\left(e^{-\lambda_n t} - P^\Omega_t\right)$ for any $t > 0$ such that for any $k = 1, \dots, M_n$

$$\begin{aligned}P^\Omega_t \phi_{n,k} &= e^{-\lambda_n t}\phi_{n,k},\\ \mathcal{L}_\Omega \phi_{n,k} &= -\lambda_n \phi_{n,k}.\end{aligned}$$

By (3.8) and (3.10), it follows that

$$L^2(\Omega, dx) = \bigoplus_{n=1}^{\infty} \operatorname{Span}\{\phi_{n,k}, k = 1, \dots, M_n\}, \tag{3.11}$$

and hence $\{\phi_{n,k}\}_{k=1,n=1}^{M_n,\infty}$ is the desired orthonormal basis of $L^2(\Omega, dx)$. ■

Notation 3.5. Throughout the paper instead of using the orthonormal basis $\{\phi_{n,k}\}_{k=1,n=1}^{M_n,\infty}$ given in the proof of Theorem 3.3, we denote by $\{\phi_n\}_{n=1}^{\infty}$ the same orthonormal basis, where for each repeated eigenvalue λ_n, we index the corresponding eigenfunctions consequently according to its (finite) multiplicity M_n. In particular, we have that

$$P^\Omega_t f = \sum_{n=1}^{\infty} e^{-\lambda_n t} \sum_{k=1}^{M_n} \langle f, \phi_{n,k}\rangle \phi_{n,k} = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle f, \phi_n\rangle \phi_n \tag{3.12}$$

for every $f \in L^2(\Omega, dx)$.

We next prove regularity properties of the eigenfunctions of $-\mathcal{L}_\Omega$.

Proposition 3.6. Let Ω be a bounded open connected subset of $\mathbb{G}$ and $\{\phi_n\}_{n=1}^\infty$ be the eigensystem of $-\mathcal{L}_\Omega$ with eigenvalues $\{\lambda_n\}_{n=1}^\infty$. Then,

(1) there exists a constant $d(\Omega)$ such that for any $1 \leqslant p \leqslant \infty$,

$$\|\phi_n\|_{L^p(\Omega,dx)} \leqslant d(\Omega)\lambda_n^{\frac{Q}{2}},$$

where Q is the homogeneous dimension of $\mathbb{G}$;

(2) for every $n \in \mathbb{N}$, the function ϕ_n is continuous in Ω;

(3) the series

$$\sum_{n=1}^\infty e^{-\lambda_n t}\phi_n(x)\phi_n(y),$$

converges uniformly on $\Omega \times \Omega \times [\varepsilon, \infty)$, for any $\varepsilon > 0$;

(4) if Ω is regular in the sense of Definition 2.8, then for every $n \in \mathbb{N}$ and $z \in \partial\Omega$,

$$\lim_{x\to z} \phi_n(x) = 0. \tag{3.13}$$

Proof. (1) First, note that $P_t^\Omega : L^2(\Omega, dx) \to L^2(\Omega, dx)$ is a self-adjoint operator since its generator $\mathcal{L}_\Omega$ is self-adjoint. Thus,

$$(P_t^\Omega)^* f = P_t^\Omega f, \text{ for any } f \in L^2(\Omega, dx), \tag{3.14}$$

where $(P_t^\Omega)^*$ denotes the adjoint of P_t^Ω. By (3.7), there exists a constant $A > 0$ such that, for any $x, y \in \Omega$

$$p_t^\Omega(x,y) \leqslant p_t(x,y) \leqslant At^{-\frac{Q}{2}}.$$

Therefore, for any $1 \leqslant p \leqslant \infty$ and any $x \in \Omega$,

$$\|p_t^\Omega(x,\cdot)\|_{L^p(\Omega)}^p = \int_\Omega p_t^\Omega(x,y)^p \mathrm{d}y \leqslant |\Omega| A^p t^{-\frac{Q}{2}p},$$

that is,

$$\|p_t^\Omega(x,\cdot)\|_{L^p(\Omega)} \leqslant |\Omega|^{\frac{1}{p}} At^{-\frac{Q}{2}}.$$

Therefore, for any $1 \leqslant q \leqslant \infty$ and $f \in L^q(\Omega, dx)$,

$$\|P_t^\Omega f\|^2_{L^2(\Omega)} = \int_\Omega \left[\int_\Omega f(y) p_t^\Omega(x,y)\mathrm{d}y\right]^2 \mathrm{d}x$$
$$\leqslant \int_\Omega \|f\|^2_{L^q(\Omega)} \|p_t^\Omega(x,\cdot)\|^2_{L^{\frac{q}{q-1}}(\Omega)} \mathrm{d}x \leqslant \|f\|^2_{L^q(\Omega)} |\Omega|^{1+\frac{2q-2}{q}} A^2 t^{-Q}.$$

Thus, the operator $P_t^\Omega : L^q(\Omega, dx) \to L^2(\Omega, dx)$ is well defined for every $1 \leqslant q \leqslant \infty$, and it satisfies

$$\|P_t^\Omega\|_{L^q(\Omega)\to L^2(\Omega)} \leqslant c(q,\Omega) t^{-\frac{Q}{2}}, \tag{3.15}$$

where $c(q,\Omega) := |\Omega|^{\frac{1}{2}+\frac{q-1}{q}} A \leqslant A \max(|\Omega|^{\frac{1}{2}}, |\Omega|^{\frac{3}{2}}) =: c(\Omega)$. The adjoint $(P_t^\Omega)^* : L^2(\Omega, dx) \to L^p(\Omega, dx)$ then satisfies

$$\|(P_t^\Omega)^*\|_{L^2(\Omega)\to L^p(\Omega)} \leqslant c(\Omega) t^{-\frac{Q}{2}},$$

where p is the conjugate of q. Thus, by (3.14),

$$\|P_t^\Omega\|_{L^2(\Omega)\to L^p(\Omega)} \leqslant c(\Omega) t^{-\frac{Q}{2}},$$

for any $1 \leqslant p \leqslant \infty$.

Let ϕ_n be an eigenfunction for P_t^Ω with the eigenvalue $e^{-\lambda_n t}$, then it follows that

$$\|\phi_n\|_{L^p(\Omega)} \leqslant c(\Omega) t^{-\frac{Q}{2}} e^{\lambda_n t} \|\phi_n\|_{L^2(\Omega)},$$

and taking the infimum over $t > 0$, we see that

$$\|\phi_n\|_{L^p(\Omega)} \leqslant c(\Omega) \left(\frac{2e}{Q}\right)^{\frac{Q}{2}} \|\phi_n\|_{L^2(\Omega)} \lambda_n^{\frac{Q}{2}}.$$

(2) Note that, for any $f \in L^2(\Omega, dx)$, the function

$$x \longrightarrow \int_\Omega f(y) p_t^\Omega(x,y)\mathrm{d}y$$

is continuous in x since $p_t^\Omega(x,y)$ is continuous in x and y in Ω. Then,

$$\phi_n(x) = e^{\lambda_n t}(P_t^\Omega \phi_n)(x) = e^{\lambda_n t} \int_\Omega \phi_n(y) p_t^\Omega(x,y)\mathrm{d}y$$

is continuous for any $x \in \Omega$.

(3) Let $\varepsilon > 0$. Then, for every $x, y \in \Omega$, $t \geqslant \varepsilon$, we have that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} |\phi_n(x)\phi_n(y)| \leqslant \sum_{n=1}^{\infty} e^{-\lambda_n t} \|\phi_n\|^2_{L^\infty(\Omega,dx)}$$
$$\leqslant d(\Omega)^2 \sum_{n=1}^{\infty} e^{-\lambda_n t} \lambda_n^{\Omega} \leqslant d(\Omega)^2 \sum_{n=1}^{\infty} e^{-\lambda_n \varepsilon} \lambda_n^{\Omega},$$

which is convergent.

(4) For any $t > 0$, $n \in \mathbb{N}$, and $x \in \Omega$, we have that

$$|\phi_n(x)| = e^{\lambda_n t} \left| \int_\Omega p_t^{\Omega}(x,y)\phi_n(y) \mathrm{d}y \right| \leqslant e^{\lambda_n t} \|\phi_n\|_{L^\infty(\Omega,dx)} \int_\Omega p_t^{\Omega}(x,y) \mathrm{d}y$$
$$= e^{\lambda_n t} \|\phi_n\|_{L^\infty(\Omega,dx)} \mathbb{P}^x \left(\tau_\Omega > t\right).$$

By [6, Proposition 1, p. 163], we have that the function $x \to \mathbb{P}^x \left(\tau_\Omega > t\right)$ is upper semi-continuous for any $x \in \mathbb{G}$. Though their proof is for a standard Brownian motion, it only relies on the semigroup property, and thus the argument applies in our setting. If Ω is regular, then for any $z \in \partial\Omega$, we have that

$$\lim_{x \to z} |\phi_n(x)| \leqslant e^{\lambda_n t} \|\phi_n\|_{L^\infty(\Omega,dx)} \limsup_{x \to z} \mathbb{P}^x \left(\tau_\Omega > t\right)$$
$$\leqslant e^{\lambda_n t} \|\phi_n\|_{L^\infty(\Omega,dx)} \mathbb{P}^z \left(\tau_\Omega > t\right)$$
$$\leqslant e^{\lambda_n t} \|\phi_n\|_{L^\infty(\Omega,dx)} \mathbb{P}^z \left(\tau_\Omega > 0\right) = 0.$$

■

Regularity of Ω was only used in the proof of Part (4) of Proposition 3.6 to check that the eigenfunctions vanish on $\partial\Omega$. For the rest of this section, we do not assume regularity of the set Ω.

Corollary 3.7. Let $x, y \in \Omega$, and $t > 0$, then

$$p_t^{\Omega}(x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x)\phi_n(y),$$
$$\mathbb{P}^x \left(\tau_\Omega > t\right) = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \phi_n(x),$$

where $c_n := \int_\Omega \phi_n(y) \mathrm{d}y$.

Proof. By (3.12) and Proposition 3.6 part (3), we have that

$$\int_\Omega f(y)p_t^\Omega(x,y)\mathrm{d}y = P_t^\Omega f(x) = \sum_{n=1}^\infty e^{-\lambda_n t}\langle f,\phi_n\rangle_{L^2(\Omega,dx)}\phi_n(x)$$
$$= \sum_{n=1}^\infty e^{-\lambda_n t}\int_\Omega f(y)\phi_n(y)\mathrm{d}y\phi_n(x) = \int_\Omega f(y)\sum_{n=1}^\infty e^{-\lambda_n t}\phi_n(y)\phi_n(x)\mathrm{d}y,$$

for any $f \in L^2(\Omega,dx)$, and hence $p_t^\Omega(x,y) = \sum_{n=1}^\infty e^{-\lambda_n t}\phi_n(x)\phi_n(y)$. Then,

$$\mathbb{P}^x\left(\tau_\Omega > t\right) = \int_\Omega p_t^\Omega(x,y)\mathrm{d}y = \int_\Omega \sum_{n=1}^\infty e^{-\lambda_n t}\phi_n(x)\phi_n(y)\mathrm{d}y$$
$$= \sum_{n=1}^\infty e^{-\lambda_n t}\phi_n(x)\int_\Omega \phi_n(y)\mathrm{d}y = \sum_{n=1}^\infty e^{-\lambda_n t}c_n\phi_n(x),$$

where $c_n := \int_\Omega \phi_n(y)\mathrm{d}y$ is finite by Proposition 3.6 part (1). ■

Our next goal is to prove that the first eigenvalue λ_1 is a simple eigenvalue for $-\mathcal{L}_\Omega$ or equivalently, by Theorem 3.3, that $e^{-\lambda_1 t}$ is a simple eigenvalue for P_t^Ω. This will follow from the irreducibility of the semigroup P_t^Ω. The definition of irreducibility of Dirichlet forms and corresponding semigroups can be found in [17, p. 55]. For a definition of irreducible semigroups on Banach lattices, we refer to [2, Section 14.3]. We will use the following characterization of irreducible semigroups [2, Example 14.11]. Let T_t be a strongly continuous semigroup on $L^p(\Omega,dx)$, $1 \leqslant p < \infty$ with generator A. Let $s(A) := \sup\{\mathrm{Re}(\lambda),\lambda \in \sigma(A)\}$ and $R_\mu = (A-\mu)^{-1}$ for μ in the resolvent set of A.

Lemma 3.8 (Example 14.11 in [2]). The semigroup T_t is irreducible if and only if for any positive $f \in L^p(\Omega,dx)$, we have that

$$R_\mu f(x) > 0, \text{ for a.e.} x \in \Omega \text{ and some } \mu > s(A).$$

Theorem 3.9. The semigroup P_t^Ω is irreducible.

Proof. We first prove that

$$p_t^\Omega(x,y) > 0, \tag{3.16}$$

for every $t > 0$ and $x, y \in \Omega$. We claim that for every $y \in \Omega$ and r small enough, there exists a time t_0 such that for any $x \in B_r(y)$, $z \in \partial\Omega$ and $s < t < t_0$ one has that

$$p_t(x,y) - p_s(z,y) > 0. \tag{3.17}$$

Indeed, if we assume (3.17), then

$$\begin{aligned} p_t^{\Omega}(x,y) &:= p_t(x,y) - \mathbb{E}^x\left[\mathbb{1}_{\{\tau_\Omega < t\}} p_{t-\tau_\Omega}\left(g_{\tau_\Omega}, y\right)\right] \\ &\geqslant p_t(x,y) - \mathbb{E}^x\left[p_{t-\tau_\Omega}\left(g_{\tau_\Omega}, y\right)\right] = \mathbb{E}^x\left[p_t(x,y) - p_{t-\tau_\Omega}\left(g_{\tau_\Omega}, y\right)\right] > 0 \end{aligned}$$

for any $t < t_0$ and $x \in B_r(y)$. The result would then follow for any $x \in \Omega$ by a standard chaining argument. Let us now prove (3.17). By [10, Equation (3.7)], for any $k \in (0,1)$, there exists a $c_k \in (0, \infty)$ such that

$$p_s(z,y) \leqslant c_k \left(1 + \frac{1}{s}\right)^{\frac{\theta}{2}} e^{c_k s} e^{-k\frac{d(z,y)^2}{s}} \leqslant c_k \left(1 + \frac{1}{s}\right)^{\frac{\theta}{2}} e^{c_k s} e^{-k\frac{d(y,\partial\Omega)^2}{s}},$$

where the last inequality follows from the fact that $d(y, \partial\Omega) \leqslant d(z,y)$ since $z \in \partial\Omega$. By [10, Equation (3.8)], there exist constants $c_1, c_2 \in (0, \infty)$ such that

$$p_t(x,y) \geqslant c_1 \left(1 + \frac{1}{t}\right)^{\frac{\theta}{2}} e^{-c_2 t} e^{-c_2\frac{d(x,y)^2}{t}},$$

where θ is an integer defined in [10, eq. (3.4)]. For the sake of conciseness, set

$$\begin{aligned} &u := 1 + \frac{1}{t}, \qquad v := 1 + \frac{1}{s}, \\ &\alpha := \frac{\theta}{2}, \qquad \beta := c_2 d(x,y)^2, \qquad \gamma := k d(y, \partial\Omega)^2. \end{aligned}$$

Then,

$$\begin{aligned} p_t(x,y) - p_s(z,y) &\geqslant c_1 u^\alpha e^{-c_2 t} e^{-\frac{\beta}{t}} - c_k v^\alpha e^{c_k s} e^{-\frac{\gamma}{s}} \\ &= c_k v^\alpha e^{c_k s} e^{-\frac{\gamma}{s}} \left(\frac{c_1}{c_k} \frac{u^\alpha}{v^\alpha} \frac{e^{-c_2 t}}{e^{c_k s}} \frac{e^{-\frac{\beta}{t}}}{e^{-\frac{\gamma}{s}}} - 1\right), \end{aligned}$$

and hence, it is enough to prove that

$$\frac{u^\alpha}{v^\alpha}\frac{e^{-\frac{\beta}{t}}}{e^{-\frac{\gamma}{s}}} > \frac{c_k e^{c_k s}}{c_1 e^{-c_2 t}},$$

for all $0 < s < t$ small enough, that is,

$$\frac{u^\alpha}{v^\alpha} e^{-\beta u+\gamma v+\beta-\gamma} > \frac{c_k e^{c_k s}}{c_1 e^{-c_2 t}}, \tag{3.18}$$

for all $0 < s < t$ small enough. If we let $F(v) := \frac{u^\alpha}{v^\alpha} e^{-\beta u+\gamma v+\beta-\gamma}$ for $v > u > 0$ and u fixed, then

$$F'(v) = F(v)\left(\gamma - \frac{\alpha}{v}\right) > 0$$

for v large enough, and

$$F(u) = e^{\frac{\gamma-\beta}{t}},$$

Thus, if we choose r small enough so that $\gamma - \beta > 0$, then we can find a $t_0 = t_0(x, y, c_1, c_k, \Omega)$ such that (3.18) is satisfied and the proof of (3.16) is complete.

We can now prove the irreducibility of P_t^Ω. We can use [17, Exercise 1.3.1] to express the resolvent R_μ in terms of the semigroup P^Ω. Then, by Lemma 3.8 and (3.16) and for any $\mu \in \mathbb{R}$, for any $f > 0$, and for a.e. $x \in \Omega$, we have that

$$R_\mu f(x) = \int_0^\infty e^{-\mu t}(P_t^\Omega f)(x)\mathrm{d}x = 0$$

if and only if $(P_t^\Omega f)(x) = 0$ for a.e. $t > 0$, since P_t^Ω is a positive operator. Thus, $R_\mu f(x) = 0$ if and only if

$$\int_\Omega f(y)p_t^\Omega(x, y)\mathrm{d}y = 0,$$

that is, if and only if for a. e. $x \in \Omega$, $p_t^\Omega(x, y)$ is zero on a set of positive Haar measure, which is not possible by (3.16). ■

Theorem 3.10. Let λ_1 be the first nonzero eigenvalue of $-\mathcal{L}_\Omega$. Then λ_1 is a simple eigenvalue and there exists a corresponding eigenfunction ϕ such that $\phi(x) > 0$ for every $x \in \Omega$.

Proof. For every $t > 0$, the operator P_t^Ω is compact with spectral radius given by $e^{-\lambda_1 t}$, and $K := \{f \in L^2(\Omega, dx) f \geqslant 0 \text{ a.s.}\}$ is a cone in $L^2(\Omega, dx)$ such that $P_t^\Omega(K) \subset K$. Thus, by Krein–Rutman theorem [30], there exists an eigenfunction ϕ of P_t^Ω with eigenvalue $e^{-\lambda_1 t}$ such that $\phi \in K\backslash\{0\}$. By Theorem 3.3, we know that ϕ is an eigenfunction of $-\mathcal{L}_\Omega$ with eigenvalue λ_1. Let us assume that $\phi(x) = 0$ for some $x \in \Omega$. Then,

$$0 = \phi(x) = e^{\lambda_1 t} \int_\Omega \phi(y) p_t^\Omega(x, y) \mathrm{d}y \geqslant 0,$$

and hence $\phi(y) p_t^\Omega(x, y) = 0$ for a.e. $y \in \Omega$. The set

$$A := \{z \in \Omega : \phi(z) > 0\}$$

has positive Haar measure since $\phi \in K\backslash\{0\}$. Thus, $p_t^\Omega(x, y) = 0$ for almost every $y \in A$, which leads to a contradiction by (3.16).

The semigroup P_t^Ω is irreducible by Theorem 3.9, and its generator $\mathcal{L}_\Omega$ is self-adjoint, and we proved that there exists $\phi \in \ker(-\lambda_1 - \mathcal{L}_\Omega)$ such that $\phi > 0$. Thus, by [2, Proposition 14.42 (c)], it follows that $\dim \ker(-\lambda_1 - \mathcal{L}_\Omega) = 1$. ■

4 Regular Boundary Points: An Analytic Approach

In this section, we compare the probabilistic notion of regular points in Definition 2.8 with an analytic definition used for hypoelliptic operators. The main goal is to prove that these two notions are indeed equivalent.

Let $\mathcal{L}$ be a diffusion operator and Ω be a bounded open connected subset of a homogeneous Carnot group $\mathbb{G} \cong \mathbb{R}^N$. Consider the boundary value problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = \phi & \text{in } \partial\Omega, \end{cases}$$

where $\phi : \partial\Omega \longrightarrow \mathbb{R}$ is a continuous function. If Ω is an open set with compact closure and non-empty boundary, then there exists a generalized solution H_ϕ^Ω in the sense of Perron–Wiener–Brelot, which in this setting is described in [3, II.6.7, p. 359]. We now recall an analytic definition of regular points that can be found in [3, II.7.11].

Definition 4.1. A point $x \in \partial\Omega$ is called *regular* (or $\mathcal{L}$*-regular*) if

$$\lim_{\Omega \ni z \to x} H_\phi^\Omega(z) = \phi\,(x)$$

for every continuous function $\phi : \partial\Omega \longrightarrow \mathbb{R}$. We call the set Ω *regular* (or $\mathcal{L}$*-regular*) if every boundary point of Ω is regular.

The notion of regular points depends on the operator. The Euclidean space $\mathbb{R}^N$ is an example of a homogeneous Carnot group with respect to the Euclidean dilation, and the corresponding differential operator is the standard Laplacian $\Delta_{\mathbb{R}^N}$. If Ω is any bounded domain in $\mathbb{R}^N$ with a C^2-smooth boundary, then Ω is $\Delta_{\mathbb{R}^N}$-regular in the sense of Definition 4.1 since it satisfies the exterior ball condition [3, Proposition 7.1.5]. In [23], it was shown that this is not true for more general Carnot groups. In particular, there are sub-Laplacians $\mathcal{L}$ on Carnot groups and bounded convex domains with smooth boundary that are not $\mathcal{L}$-regular. Nonetheless, given a Carnot group, it is always possible to construct nice regular domains. More precisely, in [3, Proposition 7.2.8], it is shown that on a homogeneous Carnot group $\mathbb{G}$ the balls $B_r(x)$, $r > 0, x \in \mathbb{G}$, with respect to the $\mathcal{L}$-gauge are regular in the sense of Definition 4.1.

Example 4.2 (Heisenberg group). Suppose $\mathbb{H}$ is the Heisenberg group with the group operation given by

$$(x_1, x_2, x_3) \star (y_1, y_2, y_3) := \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1)\right),$$

then by [3, Example 5.4.7], the $\mathcal{L}$-gauge is given by

$$|x| := \sqrt[4]{(x_1^2 + x_2^2)^2 + 16x_3^2}.$$

We can endow $\mathbb{H}$ with a different homogeneous norm by

$$\rho(x) := \sqrt[4]{(x_1^2 + x_2^2)^2 + x_3^2}$$

and denote by B_r the corresponding ball of radius r centered at the identity. Then, in [18], it is shown that B_r is a regular set in the sense of Definition 2.8.

Let us recall some notions from potential theory [3, Chapter 7]. If V is $\mathcal{L}$-regular, then for every fixed $x \in V$, the map

$$C(\partial V, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$\phi \longmapsto H_\phi^V(x)$$

is a linear positive functional on $C(\partial V, \mathbb{R})$, and hence by the Riesz representation theorem, there exists a Radon measure μ_x^V supported on ∂V such that

$$H_\phi^V(x) = \int_{\partial V} \phi(y) \mathrm{d}\mu_x^V(y).$$

The measure μ_x^V is called the $\mathcal{L}$-*harmonic measure* related to V and x.

Definition 4.3. Let Ω be an bounded open connected set. A function $u : \Omega \to (-\infty, +\infty]$ is called $\mathcal{L}$-*superharmonic* in Ω if

(1) u is lower semi-continuous and $u < \infty$ in a dense subset of Ω;
(2) for every $\mathcal{L}$-regular open set V with $\overline{V} \subset \Omega$ and for ever $x \in V$

$$u(x) \geqslant \int_{\partial V} u(y) \mathrm{d}\mu_x^V(y).$$

The following result can be found in [6, Theorem 1, p. 177], for a standard Brownian motion on $\mathbb{R}^d$. The proof relies on the Markov property of the process, the semigroup property of the associated semigroup, and the definition of superharmonic functions. Thus, it carries over to the setting of the current paper, and therefore, we do not give a proof. We recall that $\{g_t\}_t$ refers to the hypoelliptic Brownian motion, that is, the diffusion associated with $\mathcal{L}$.

Theorem 4.4. Suppose D is a set such that $\overline{D} \subset \Omega$, and u is an $\mathcal{L}$-superharmonic function defined in Ω. Then,

$$\left\{u\left(g_{\tau_D \wedge t}\right)\right\}_{t \geqslant 0}$$

is a supermartingale under $\mathbb{P}^x$ for any $x \in D$ for which $u(x) < \infty$.

Theorem 4.5. Suppose Ω is an open bounded set and u is an $\mathcal{L}$-superharmonic function defined on Ω. Then,

$$\mathbb{E}^x\left[u\left(g_{\tau_D}\right)\right] \leqslant u(x) \tag{4.1}$$

for every $x \in \Omega$.

Proof. Let $\{\Omega_n\}_{n\geqslant 1}$ be a family of open bounded sets such that $\overline{\Omega}_n \subset \Omega$ and $\cup_{n=1}^{\infty}\Omega_n = \Omega$, and let $\tau_n := \tau_{\Omega_n}$. By Theorem 4.4 with $D = \Omega_n$, it follows that $\left\{u\left(g_{\tau_n\wedge t}\right)\right\}_{t\geqslant 0}$ is a supermartingale, and hence for any $t > 0$,

$$\mathbb{E}^x\left[u\left(g_{t\wedge\tau_n}\right)\right] \leqslant u(x).$$

Note that $\{t < \tau_n\} \nearrow \{t < \tau_\Omega\}$ as $n \to \infty$, and hence, if we let $n \to \infty$ by Fatou's lemma, the previous estimate becomes

$$\mathbb{E}^x\left[u\left(g_{t\wedge\tau_\Omega}\right)\right] \leqslant u(x). \tag{4.2}$$

Note that $\tau_\Omega < \infty$ $\mathbb{P}^x$-a.s. for any $x \in \Omega$. Indeed, $\{\tau_\Omega = \infty\} = \cap_{M=1}^{\infty}\{\tau_\Omega > M\}$, and hence by (3.7), for any $x \in \Omega$,

$$\begin{aligned}\mathbb{P}^x\left(\tau_\Omega = \infty\right) &\leqslant \mathbb{P}^x\left(\tau_\Omega > M\right)\\ &= \int_\Omega p_M^\Omega(x,y)\mathrm{d}y \leqslant A|\Omega|M^{-\frac{Q}{2}},\end{aligned}$$

and by letting $M \to \infty$, it follows that $\mathbb{P}^x\left(\tau_\Omega = \infty\right) = 0$. Thus, the proof is completed by letting $t \to \infty$ in (4.2). ■

We now need the following version of [29, Theorem 2.12, p. 245].

Proposition 4.6. Let $y \in \partial\Omega$, and assume that

$$\lim_{x\to y}\mathbb{E}^x\left[f\left(g_{\tau_\Omega}\right)\right] = f(y),$$

for every bounded measurable function $f : \partial\Omega \to \mathbb{R}$ that is continuous at y. Then y is a regular point in the sense of Definition 2.8.

Proof. The proof given in [29, Theorem 2.12, p. 245] holds for a standard Brownian motion in $\mathbb{R}^d$ with $d \geqslant 2$, but it only uses the Markov property and the fact that a standard Brownian motion never returns to its starting point when $d \geqslant 2$. The hypoelliptic Brownian motion $g_\cdot$ is a Markov process that never returns to its starting point. Indeed, one can write

$$g_t = \left(B_t, A_2(t), \ldots, A_r(t)\right),$$

where B_t is a d_1-dimensional standard Brownian motion and $A_j(t) \in \mathbb{R}^{d_j}$ is an iterated stochastic integral for $j = 2, \ldots, r$. Thus, if g_t were to return to its starting point so would B_t, and that is not possible since $d_1 \geqslant 2$. The proof of Proposition 4.6 then follows as in [29, Theorem 2.12, p. 245]. ■

Definition 4.7. Let $y \in \partial\Omega$. An $\mathcal{L}$-barrier at y in Ω is a superharmonic map $w : \Omega \to (-\infty, +\infty]$ such that

(1) $w(x) > 0$ for every $x \in \Omega$,
(2) $\lim_{x\to y} w(x) = 0$.

In [3, Theorem 6.10.4], it is shown that a point $y \in \partial\Omega$ is regular in the sense of Definition 4.1 if and only if there exists an $\mathcal{L}$-barrier at y in Ω. More precisely, for every regular point $y \in \partial\Omega$, one can construct an $\mathcal{L}$-barrier s_y^Ω such that

(1) s_y^Ω is $\mathcal{L}$-harmonic in Ω;
(2) $\inf_{\Omega\setminus U} s_y^\Omega > 0$ for every neighborhood U of y.

In particular, for every $z \in \partial\Omega$ with $z \neq y$, we have that

$$\liminf_{x\to z} s_y^\Omega(x) > 0.$$

We can now prove the main theorem of this section.

Theorem 4.8. Let Ω be an open bounded connected set, and let $y \in \partial\Omega$ be fixed. Then y is regular in the sense of Definition 2.8 if and only if is regular in the sense of Definition 4.1.

Proof. To simplify the notation, we say that a point $y \in \partial\Omega$ is P-regular (A-regular) if it is regular in the sense of Definition 2.8 (Definition 4.1).

Let $y \in \partial\Omega$ be an A-regular point and $w(x) := s_y^\Omega(x)$ be the $\mathcal{L}$-barrier defined above. By Proposition 4.6, it is enough to show that

$$\lim_{x\to y} \mathbb{E}^x\left[f\left(g_{\tau_\Omega}\right)\right] = f(y),$$

for every bounded measurable function $f : \partial\Omega \to \mathbb{R}$ that is continuous at y. The following argument is a modification of [29, Proposition 2.15, p. 248]. Set $M := \sup_{z\in\partial\Omega} |f(z)|$, and for any $\varepsilon > 0$, let $\delta > 0$ be such that $|f(z)-f(y)| \leqslant \varepsilon$ for any $z \in \partial\Omega$ with $d(z, y) < \delta$, where d is a homogeneous distance. We know that $w(x) > 0$ for every $x \in \Omega$ and $\liminf_{x\to z} w(x) > 0$

for any $z \in \partial\Omega$ with $z \neq y$. Thus, there exists a k such that $kw(x) \geqslant 2M$ for any $x \in \overline{\Omega}$ with $d(x, y) \geqslant \delta$. Thus, for any $z \in \partial\Omega$, we have that $|f(z) - f(y)| \leqslant \varepsilon$ if $d(z, y) < \delta$ and $|f(z) - f(y)| \leqslant 2M \leqslant kw(z)$ if $d(z, y) \geqslant \delta$. Hence, for any $z \in \partial\Omega$,

$$|f(z) - f(y)| \leqslant \max(\varepsilon, kw(z)).$$

$\mathcal{L}$-Barriers are superharmonic, and thus by Theorem 4.5, it follows that

$$\begin{aligned} |\mathbb{E}^x \left[f\left(g_{\tau_\Omega}\right) \right] - f(y)| &\leqslant \mathbb{E}^x \left[|f\left(g_{\tau_\Omega}\right) - f(y)| \right] \\ &\leqslant \mathbb{E}^x \left[\max\left(\varepsilon, kw\left(g_{\tau_\Omega}\right)\right) \right] = \max\left(\varepsilon, k\mathbb{E}^x \left[w\left(g_{\tau_\Omega}\right) \right]\right) \\ &\leqslant \max\left(\varepsilon, kw(x)\right), \end{aligned}$$

and then

$$\limsup_{x \to y} |\mathbb{E}^x \left[f\left(g_{\tau_\Omega}\right) \right] - f(y)| \leqslant \max\left(\varepsilon, k \limsup_{x \to y} w(x)\right) = \max(\varepsilon, 0) = \varepsilon,$$

for any $\varepsilon > 0$ and for any bounded measurable function $f : \partial\Omega \to \mathbb{R}$, which is continuous at y.

We now need to prove that P-regularity implies A-regularity. Let $y \in \partial\Omega$ be a P-regular point. By [3, Theorem 6.10.4], it is enough to construct an $\mathcal{L}$-barrier at y in Ω. Following [6, Exercise 10, p. 188], it is easy to prove that $w(x) := \mathbb{E}^x \left[\tau_\Omega\right]$ is the desired $\mathcal{L}$-barrier. ∎

5 Applications

5.1 Small deviations

Let $\Omega \subset \mathbb{G}$ be an open bounded connected set such that $e \in \Omega$, and for every $\varepsilon > 0$, let

$$\Omega_\varepsilon := \delta_\varepsilon\left(\Omega\right), \tag{5.1}$$

where $\delta_\varepsilon : \mathbb{G} \longrightarrow \mathbb{G}$ is the group dilation. In this section, we describe how the spectral results from Section 3 can be applied to find the asymptotic of the exit time $\tau_{\Omega_\varepsilon}$ of g_t from Ω_ε as $\varepsilon \to 0$.

First, let p_t be the heat kernel given by (3.1). Then, p_t satisfies the following scaling property [13, Theorem 3.1 (i)], for any $\varepsilon > 0$,

$$p_{\frac{t}{\varepsilon^2}}(x, y) = \varepsilon^Q p_t\left(\delta_\varepsilon(x), \delta_\varepsilon(y)\right), \tag{5.2}$$

for any $x, y \in \mathbb{G}$, where Q denotes the homogeneous dimension of $\mathbb{G}$.

Remark 5.1 (Space-time scaling in homogeneous Carnot groups). Let g_t be a hypoelliptic Brownian motion. Then, for any $x \in \mathbb{G}$ and for any $\varepsilon > 0$, we have that

$$g^x_{\frac{t}{\varepsilon^2}} \overset{(d)}{=} \delta_{\frac{1}{\varepsilon}}\left(g_t^{\delta_\varepsilon(x)}\right). \tag{5.3}$$

Indeed, by (5.2) for any Borel, set $A \subset \mathbb{G}$

$$\begin{aligned}
&\mathbb{P}\left(g^x_{\frac{t}{\varepsilon^2}} \in A\right) = \mathbb{P}^x\left(g_{\frac{t}{\varepsilon^2}} \in A\right) \\
&= \int_A p_{\frac{t}{\varepsilon^2}}(x,y)\mathrm{d}y = \varepsilon^Q \int_A p_t\left(\delta_\varepsilon(x), \delta_\varepsilon(y)\right)\mathrm{d}y \\
&= \int_{\delta_\varepsilon(A)} p_t\left(\delta_\varepsilon(x), z\right)\mathrm{d}z = \mathbb{P}^{\delta_\varepsilon(x)}\left(g_t \in \delta_\varepsilon(A)\right) = \mathbb{P}\left(\delta_{\frac{1}{\varepsilon}}\left(g_t^{\delta_\varepsilon(x)}\right) \in A\right).
\end{aligned}$$

Lemma 5.2. Let Ω be an open set and p_t^Ω be the Dirichlet heat kernel. Then, for any $\varepsilon > 0$, and any $x, y \in \Omega$

$$p^\Omega_{\frac{t}{\varepsilon^2}}(x,y) = \varepsilon^Q p_t^{\Omega_\varepsilon}\left(\delta_\varepsilon(x), \delta_\varepsilon(y)\right), \tag{5.4}$$

where Ω_ε is defined in (5.1).

Proof. First, note that

$$\mathbb{1}_{\left\{\tau^x_\Omega > \frac{t}{\varepsilon^2}\right\}} \overset{(d)}{=} \mathbb{1}_{\left\{\tau^{\delta_\varepsilon(x)}_{\Omega_\varepsilon} > t\right\}}. \tag{5.5}$$

Indeed, by Remark 5.1,

$$\begin{aligned}
&\mathbb{P}\left(\tau^x_\Omega > \frac{t}{\varepsilon^2}\right) = \mathbb{P}^x\left(g_s \in \Omega \text{ for all } 0 \leqslant s \leqslant \frac{t}{\varepsilon^2}\right) \\
&= \mathbb{P}\left(X^x_{\frac{s}{\varepsilon^2}} \in \Omega \text{ for all } 0 \leqslant s \leqslant t\right) \\
&= \mathbb{P}\left(\delta_{\frac{1}{\varepsilon}}\left(g_s^{\delta_\varepsilon(x)}\right) \in \Omega \text{ for all } 0 \leqslant s \leqslant t\right) \\
&= \mathbb{P}^{\delta_\varepsilon(x)}\left(g_s \in \Omega_\varepsilon \text{ for all } 0 \leqslant s \leqslant t\right) = \mathbb{P}\left(\tau^{\delta_\varepsilon(x)}_{\Omega_\varepsilon} > t\right).
\end{aligned}$$

Thus, for any $f \in L^2(\Omega, dx)$, we have that

$$
\begin{aligned}
&\int_\Omega f(y) p^\Omega_{\frac{t}{\varepsilon^2}}(x,y)\mathrm{d}y = \mathbb{E}^x\left[f\left(g_{\frac{t}{\varepsilon^2}}\right), \tau_\Omega > \frac{t}{\varepsilon^2}\right] = \mathbb{E}\left[f\left(X^x_{\frac{t}{\varepsilon^2}}\right), \tau^x_\Omega > \frac{t}{\varepsilon^2}\right] \\
&= \mathbb{E}\left[f\left(\delta_{\frac{1}{\varepsilon}}(g_t^{\delta_\varepsilon(x)})\right), \tau^{\delta_\varepsilon(x)}_{\Omega_\varepsilon} > t\right] = \mathbb{E}^{\delta_\varepsilon(x)}\left[f\left(\delta_{\frac{1}{\varepsilon}}(g_t)\right), \tau_{\Omega_\varepsilon} > t\right] \\
&= \int_{\Omega_\varepsilon} f\left(\delta_{\frac{1}{\varepsilon}}(z)\right) p^{\Omega_\varepsilon}_t(\delta_\varepsilon(x), z)\mathrm{d}z = \int_\Omega f(y)\varepsilon^Q p^{\Omega_\varepsilon}_t\left(\delta_\varepsilon(x), \delta_\varepsilon(y)\right)\mathrm{d}y,
\end{aligned}
$$

which completes the proof. ■

We conclude with an application to small deviations.

Theorem 5.3. Let $\mathbb{G}$ be a homogeneous Carnot group with the sub-Laplacian $\mathcal{L}$ and Ω be a bounded open connected set containing the identity e, and set $\Omega_\varepsilon := \delta_\varepsilon(\Omega)$. Let g_t be a hypoelliptic Brownian motion such that $g_0 = e$ a.s. Then,

$$
\lim_{\varepsilon \to 0} e^{\frac{\lambda_1}{\varepsilon^2}t}\mathbb{P}^e\left(\tau_{\Omega_\varepsilon} > t\right) = c\phi(e),
$$

where λ_1 is the spectral gap of $-\mathcal{L}_\Omega$ given by Theorem 3.3 and ϕ is the corresponding positive eigenfunction given by Theorem 3.10, and $c = \int_\Omega \phi(y)\mathrm{d}y$.

Corollary 5.4. Under the same assumption of Theorem 5.3, we have that

$$
\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}^e\left(\tau_{\Omega_\varepsilon} > t\right) = \lambda_1 t, \tag{5.6}
$$

for every $t > 0$.

Example 5.5. Let $|\cdot|$ be a homogeneous norm on $\mathbb{G}$. Then,

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} e^{\frac{\lambda_1}{\varepsilon^2}t}\mathbb{P}^e\left(\max_{0\leqslant s\leqslant t}|g_s| < \varepsilon\right) = c\phi(e), \\
&\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}^e\left(\max_{0\leqslant s\leqslant t}|g_s| < \varepsilon\right) = \lambda_1 t,
\end{aligned}
$$

where $\lambda_1 > 0$ is the spectral gap of $-\mathcal{L}_B$ and $B := \{x \in \mathbb{G},\ |x| < 1\}$.

Remark 5.6 (Spectral gap estimates). If $\mathbb{G} = \mathbb{H}$ is the Heisenberg group, it is shown in [5, Theorem 3.4] that

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}^e \left(\max_{0 \leqslant s \leqslant t} |g_s| < \varepsilon \right) = c^2 t,$$

for some finite constant $c > 0$. Moreover, Example 5.5 and [5, Theorem 5.6] provide an explicit estimate for the 1st Dirichlet eigenvalue $\lambda_1 = c^2$ for the sub-Laplacian on $\mathbb{H}$ in the Korányi ball. More precisely,

$$\lambda_1^{(2)} \leqslant \lambda_1 \leqslant c\left(\lambda_1^{(1)}, \lambda_1^{(2)}\right),$$

where

$$c\left(\lambda_1^{(1)}, \lambda_1^{(2)}\right) := f\left(x^*\right) = \inf_{x \in (0,1)} f(x),$$
$$f(x) = \frac{\lambda_1^{(2)}}{\sqrt{1-x}} + \frac{\lambda_1^{(1)}\sqrt{1-x}}{4x},$$
$$x^* = \frac{\sqrt{\left(\lambda_1^{(1)}\right)^2 + 32\lambda_1^{(1)}\lambda_1^{(2)}} - 3\lambda_1^{(1)}}{2\left(4\lambda_1^{(2)} - \lambda_1^{(1)}\right)},$$

and $\lambda_1^{(n)}$ are the lowest Dirichlet eigenvalues of $-\frac{1}{2}\Delta_{\mathbb{R}^n}$ in the unit ball in $\mathbb{R}^n$.

Proof of Theorem 5.3. By (5.5), we have that

$$\mathbb{P}^{\delta_\varepsilon(x)}\left(\tau_{\Omega_\varepsilon} > t\right) = \mathbb{P}^x\left(\tau_\Omega > \frac{t}{\varepsilon^2}\right)$$

for any $x \in \Omega$. Thus, by Corollary 3.7, we have that

$$\mathbb{P}^e\left(\tau_{\Omega_\varepsilon} > t\right) = \sum_{n=1}^{\infty} e^{-\lambda_n \frac{t}{\varepsilon^2}} c_n \phi_n(e),$$

where $\{\phi_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are defined as in Notation 3.5 with $c_n = \int_\Omega \phi_n(y)\mathrm{d}y$. By Theorems 3.10 and 3.3, there exists a $\phi > 0$ such that $\ker(e^{-\lambda_1 t} - P_t^\Omega) = \ker(-\lambda_1 - \mathcal{L}_\Omega) =$

Span$\{\phi\}$. Thus,

$$e^{\lambda_1 \frac{t}{\varepsilon^2}} \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right) = c\phi(e) + \sum_{n=2}^{\infty} e^{-(\lambda_n-\lambda_1)\frac{t}{\varepsilon^2}} c_n \phi_n(e),$$

where $\lambda_n \geqslant \lambda_2$ for all $n \geqslant 3$ and $\lambda_2 > \lambda_1$ since $\dim\ker(-\lambda_1 - \mathcal{L}_\Omega) = 1$. Thus, the result follows by letting ε go to zero. ■

Proof of Corollary 5.4. By Theorem 5.3, we know that

$$\lim_{\varepsilon\to 0} e^{\frac{\lambda_1}{\varepsilon^2}t} \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right) = c\phi(e),$$

with $c\phi(e) > 0$. Then,

$$\begin{aligned}
\log(c\phi(e)) &= \lim_{\varepsilon\to 0} \log\left(e^{\frac{\lambda_1}{\varepsilon^2}t} \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right)\right) \\
&= \lim_{\varepsilon\to 0}\left(\log \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right) + \frac{\lambda_1}{\varepsilon^2}t\right) = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon^2 \log \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right) + \lambda_1 t}{\varepsilon^2}\right),
\end{aligned}$$

which is finite if and only if

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2 \log \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right) + \lambda_1 t\right) = 0,$$

that is,

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}^e \left(\tau_{\Omega_\varepsilon} > t\right) = \lambda_1 t.$$

■

5.2 Large time behavior of the heat content

In this section, we use the spectral analysis from Section 3 to describe the large time behavior of the heat content. Let Ω be a bounded open connected regular set. We consider the Dirichlet problem for the heat equation on Ω

$$\begin{aligned}
&\left(\partial_t - \mathcal{L}\right) u(x,t) = 0, && (t,x) \in (0,\infty)\times\Omega, \\
&u(t,x) = 0, && (t,x) \in (0,\infty)\times\partial\Omega, \\
&u(0,x) = 1, && x \in \Omega.
\end{aligned} \tag{5.7}$$

Definition 5.7. Let u be the solution to the boundary value problem (5.7). The heat content associated with Ω is given by

$$Q_\Omega(t) := \int_\Omega u(t,x)\mathrm{d}x,$$

for $t > 0$.

If Ω is regular, it is easy to see that $\mathbb{P}^x\left(\tau_\Omega > t\right)$ is the solution to (5.7), and hence we can write

$$Q_\Omega(t) = \int_\Omega \int_\Omega p_t^\Omega(x,y)\mathrm{d}y\mathrm{d}x.$$

By Corollary 3.7, we have that

$$Q_\Omega(t) = \int_\Omega \sum_{n=1}^\infty e^{-\lambda_n t} c_n \phi_n(x)\mathrm{d}x, \tag{5.8}$$

where $c_n = \int_\Omega \phi_n(y)\mathrm{d}y$. Note that the series $\sum_{n=1}^\infty e^{-\lambda_n t} c_n \phi_n(x)$ converges uniformly on $[\varepsilon, \infty) \times \Omega$ for any $\varepsilon > 0$. Indeed, by Proposition 3.6, we have that, for any $x \in \Omega$ and $t \geqslant \varepsilon$

$$\begin{aligned} |e^{-\lambda_n t} c_n \phi_n(x)| &\leqslant e^{-\lambda_n t} c_n \|\phi_n\|_{L^\infty(\Omega,dx)} \\ &\leqslant |\Omega| e^{-\lambda_n t} \|\phi_n\|^2_{L^\infty(\Omega,dx)} \leqslant |\Omega| d(\Omega) e^{-\lambda_n \varepsilon} \lambda_n^Q \end{aligned}$$

for any $n \in \mathbb{N}$. Thus, $\sum_{n=1}^\infty e^{-\lambda_n t} c_n \phi_n(x)$ converges uniformly on the set $[\varepsilon, \infty) \times \Omega$ for any $\varepsilon > 0$ by Weierstraß' M-test since the series $\sum_{n=1}^\infty e^{-\lambda_n \varepsilon} \lambda_n^Q$ is convergent.

Thus, by (5.8), it follows that

$$\begin{aligned} Q_\Omega(t) &= \int_\Omega \sum_{n=1}^\infty e^{-\lambda_n t} c_n \phi_n(x)\mathrm{d}x \\ &= \sum_{n=1}^\infty e^{-\lambda_n t} c_n \int_\Omega \phi_n(x)\mathrm{d}x = \sum_{n=1}^\infty e^{-\lambda_n t} c_n^2, \end{aligned}$$

for any $t > 0$. We can then deduce the following large time asymptotics for the heat content:

$$\lim_{t\to\infty} e^{\lambda_1 t} Q_\Omega(t) = c_1^2.$$

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