# Emergent modified gravity: Covariance regained

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In its canonical formulation, general relativity is subject to gauge transformations that are equivalent to space-time coordinate changes of general covariance only when the gauge generators, given by the Hamiltonian and diffeomorphism constraints, vanish. Since the specific form taken by Poisson brackets of the constraints and of the gauge transformations and equations of motion they generate is important for general covariance to be realized, modifications of the canonical theory, suggested for instance by approaches to quantum gravity, are not guaranteed to be compatible with the existence of a covariant space-time line element. This caveat applies even if the modification preserves the number of independent gauge transformations and the modified constraints remain first class. Here, a complete derivation of covariance conditions, regained from the canonical constraints without assuming that space-time has its classical structure, is presented and applied in detail to spherically symmetric vacuum models. As a broad application, the presence of structure functions in the constraint brackets plays a crucial role, which in an independent analysis has recently been shown to lead to higher algebraic structures in hypersurface deformations given by an $L_\infty$ bracket. The physical analysis of a related feature presented here demonstrates that, at least within the spherically symmetric setting, new theories of modified gravity are possible that are not of higher-curvature form.

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## I. INTRODUCTION

General relativity cannot be a complete fundamental theory valid on all scales because a large number of relevant solutions are limited by space-time singularities. Quantum effects might change this outcome, but they are also expected to modify general relativity away from singularities. Since general relativity has a large and nontrivial gauge content, which classically ensures general covariance, possible modifications that could describe quantum effects at least in an effective formulation are highly constrained. In a metric formulation based on space-time tensors, for instance, the class of admissible effective theories is given by higher-curvature actions. The observation that the speed of gravitational waves is very close to the speed of light puts strong constraints on phenomenologically viable higher-curvature actions [1–4]. It is therefore of interest to look for new alternatives.

Some aspects of classical gravity and, in particular, of possible equantizations are more conveniently expressed in a canonical formulation, in which space-time tensors are replaced by a combination of spatial tensors on spacelike hypersurfaces in a space-time foliation, with flow equations that determine how these fields change from hypersurface to nearby hypersurface. Depending on how they are applied, the flow equations may present a picture of evolution for the spatial tensors in a given foliation, or they may be used to determine how the spatial tensors and other quantities change if one transforms to a different foliation. For these hypersurface deformations to be equivalent to general covariance, the spatial tensors on any hypersurface must obey the Hamiltonian and diffeomorphism constraints of canonical general relativity [5]. These constraints, at the same time, generate the flow equations via their Hamiltonian vector fields. This equivalence is often used in practice when one interchangeably refers to coordinate invariance and slicing independence in an analysis of space-time solutions in general relativity. Both concepts are usually included in the condition of general covariance, but it turns out that there are subtle differences between them, owing to the requirement that constraints are to be imposed.

An immediate implication is that canonical gravity is a Hamiltonian gauge theory with first-class constraints. However, unlike in gauge theories encountered for instance in the standard model of particle physics, Poisson brackets of the constraints do not define a Lie algebra because they do not have structure constants: In ADM notation [6,7], the diffeomorphism constraint $\vec{H}[\vec{M}]$, depending on a spatial shift vector field $\vec{M}$ of an infinitesimal tangential deformation of a spatial hypersurface, and the Hamiltonian

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constraint $H[N]$, depending on a spatial lapse function $N$ that determines how much a hypersurface is deformed in its normal direction within space-time, have Poisson brackets

$$\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\}=\vec{H}[\mathcal{L}_{\vec{N}}\vec{M}], \quad (1)$$

$$\{H[N],\vec{H}[\vec{N}]\}=-H[N^b\partial_b N], \quad (2)$$

$$\{H[N],H[M]\}=\vec{H}[q^{ab}(N\partial_b M-M\partial_b N)] \quad (3)$$

that depend not only on $\vec{M}$ and $N$, but also on the inverse of the spatial metric $q_{ab}$ on a spatial hypersurface.

A closed set of brackets is obtained only if the deformation functions $\vec{M}$ and $N$ inserted in the constraints, including the diffeomorphism constraint on the right-hand side of (3), depend not only on space-time coordinates, but independently also on the spatial metric. Starting with phase-space independent $\vec{M}$ and $N$ and iterating Poisson brackets, such as

$$\begin{aligned}&\{H[N],\{H[N],H[M]\}\}\\&\quad=\{H[N],\vec{H}[q^{ab}(N\partial_b M-M\partial_b N)]\}\\&\quad=-H[q^{ab}(N\partial_b M-M\partial_b N)\partial_a N]\\&\qquad+\vec{H}[\{H[N],q^{ab}\}(N\partial_b M-M\partial_b N)],\end{aligned}$$

shows that not only the diffeomorphism constraint in (3) appears with a metric-dependent shift, but also the Hamiltonian constraint shows up with a metric-dependent lapse function. Iterating further, different dependencies on the metric are generated in each step that adds a new factor of the inverse metric. However, if we allow phase-space dependent lapse and shift in the Poisson brackets on the left-hand sides of (1)–(3), there are additional terms on the right-hand sides that, via the chain rule, depend on derivatives of the deformation functions by the spatial metric. These terms disappear only when the constraints are imposed and the theory is taken on shell, giving rise to the on-shell condition for an equivalence between the gauge symmetries of hypersurface deformations and coordinate changes. Off shell, however, there is a difference between hypersurface deformations and space-time coordinate transformations.

Mathematically, the dependence on the spatial metric is conveniently expressed in an algebroid picture, in which Eqs. (1)–(3) are related to a suitable bracket structure for sections of a fiber bundle over the base manifold of spatial metrics (or a suitable substitute or extension of this space), rather than a bracket for elements in a Lie algebra as it appears for constraint brackets without structure functions. Moreover, as has been shown in an explicit form only recently [9], the same metric dependence also implies that a consistent algebraic bracket corresponding to (1)–(3) is not Lie but rather an $L_\infty$ bracket in which the Jacobi identity is violated in a specific way. The gauge content of canonical gravity is therefore described by a higher algebraic structure.

The purpose of the present article, in brief, is to perform a complete physical analysis of geometrical consequences of structure functions in hypersurface deformation brackets. As suitable for physical evaluations of canonical gravity through Hamiltonian vector fields generated by the constraints, we will employ Poisson brackets, which by definition obey the Jacobi identity and, when directly applied to the constraint functions on phase space, do not show the algebraic features of an $L_\infty$ bracket. We will discuss how the structure function of hypersurface deformation brackets for a given set of modified constraints, such as a general expansion up to a certain order in derivatives of an effective field theory, can be used to derive a space-time geometry in which the corresponding constraints generate hypersurface deformations, and which is subject to a complete set of covariance conditions. In an analysis of new theories of modified gravity, the resulting spatial part of the space-time metric may be distinct from the original phase-space function $q_{ab}$ in which the constraints have been formulated. Since the precise form of the space-time metric then does not have a close relationship with the fundamental fields and must be derived using the form of gauge transformations, it is emergent within this broad set of emergent modified gravity.

This new possibility of modified gravity relies on the presence of structure functions, just like the higher algebraic structures found earlier. At this point, however, we are not aware of a more detailed relationship between these two properties. From a mathematical point of view, we are looking for different realizations of the classical algebraic structure underlying hypersurface deformations, which guarantees that new models will be amenable to standard space-time analysis using for instance line elements. We are not interested in modifications of hypersurface deformations or of the underlying $L_\infty$ structure. Our strategy is comparable to the well-known derivations of modified gravity in space-time form, which lead to different realizations of higher-curvature actions that all share the same space-time structure with standard covariance symmetries, expressed canonically through hypersurface deformations. The main difference with our approach is that we aim to derive modified theories fully on the canonical level, arriving at a space-time picture only at the very end through covariance conditions on the canonical constraints and their Poisson brackets. Rather surprisingly, we will show that new modified theories can be obtained in this form that are not of higher-curvature form. The crucial feature that makes such new theories possible is that we allow for the resulting (emergent) space-time metric to be different from the fundamental fields that enter the defining equations, given here by the constraints. In our case, the correct space-time metric cannot be identified before a detailed covariance analysis has been performed, in contrast to other theories of modified gravity in which a space-time metric must be known before the theory is defined through its curvature tensors. From a mathematical point of view, the identification of an emergent space-time makes use of a redefinition of the spatial metric, which could be formulated as a diffeomorphism on a suitable extension of the base manifold of the $L_\infty$ algebroid, and an application of nonconstant sections of the fiber bundle. These steps in our construction do not change the algebraic structure of an $L_\infty$ algebroid, but they can change the geometry and physics of space-time solutions of the constraints, equipped with the new emergent metric.

As a specific example of new theories, brackets of the form (1)–(3) may be obtained not only for metric-independent $\vec{M}$ and $N$, for which they have been derived from canonical general relativity via Poisson brackets, but also for $\vec{M}$ and $N$ with a specific dependence on the spatial metric and perhaps also extrinsic curvature of a hypersurface. For this to happen, any contributions from partial derivatives of $\vec{M}$ and $N$ that initially appear in a calculation of the Poisson brackets would have to cancel out. There is then a new on-shell interpretation as a gravity theory associated to these metric and extrinsic-curvature dependent $\vec{M}$ and $N$, but it need not be the same as the original theory of canonical general relativity because the structure function, used as the inverse spatial metric of an emergent space-time line element, need not be of the classical form where it is identical with one of the basic phase-space degrees of freedom. This identification of the emergent space-time metric through structure functions depends on the off-shell behavior of the theory, just as the higher structures in hypersurface deformations.

In a modified theory of gravity that has a chance of being generally covariant, the brackets (1)–(3) are of the classical form, but possibly with a modification of the structure function that classically equals inverse spatial metric $q^{ab}$. Uniqueness results [10,11] that show how classical general relativity follows from the brackets (1)–(3) on-shell can be circumvented by such a modification. Canonical gravity then has a potential to allow consistent modifications that are not of higher-curvature form. [All higher-curvature effective actions have the brackets (1)–(3) without a modification of $q^{ab}$ [12].] If $q^{ab}$ in (3) is replaced with a different phase-space function, however, it is not guaranteed that its inverse can still play the role of a spatial metric in some space-time line element, together with a lapse function and shift vector for the time components. Our analysis of modified gravity in canonical form therefore requires an extension of the classic results of [10,11] in which not only the dynamical equations, but also space-time structure (that is, the existence of a consistent space-time line element) must be derived, or regained from the constraints, their brackets, and from the gauge transformation they generate. As a general contribution of this paper, we present a complete set of covariance conditions in canonical form, building on previous constructions in [13].

As an example, the Poisson brackets of constraints with phase-space dependent lapse and shift are guaranteed to equal a linear combination of the constraints. They are first class and present a consistent gauge theory in canonical form. For the underlying gauge transformations to correspond to space-time symmetries via hypersurface deformations, we require in addition that new contributions to the brackets depending on partial derivatives of lapse and shift by phase-space degrees of freedom cancel out. A new set of brackets of the form (1)–(3) is then obtained from which a candidate for an emergent inverse spatial metric can be read off via the structure function. (We refer to this metric as "emergent" in this case because it is not one of the fundamental fields and must be derived from covariance conditions, unlike in standard general relativity.) As we will show, the appearance of a candidate spatial metric in the brackets does not guarantee that it can be part of a consistent and coordinate independent space-time line element. We will derive an additional, previously unrecognized condition on the gauge flow generated by the Hamiltonian constraint that guarantees matching symmetries and therefore an invariant emergent space-time line element. Together with the cancellation property, this covariance condition imposes strong restrictions on possible dependences of $\vec{M}$ and $N$ on the spatial metric or on extrinsic curvature. We will specify all these conditions and evaluate them in spherically symmetric models, demonstrating that new theories of modified gravity are indeed possible in this setting. Some of the new models we derive are closely related to recent constructions of consistent modified theories in canonical spherically symmetric models [14–17], and they explain the origin of these modifications.

As a part of our new discussion of general covariance from a canonical theory, we construct a complete procedure to derive an emergent space-time line element from canonical hypersurface-deformation brackets, extending previous results from [13]. In addition to the construction of emergent modified spherically symmeytric models based on phase-space dependent lapse and shift, we also construct more general consistent theories that include potential modifications possibly implied by quantum gravity, such as nonpolynomial terms in extrinsic curvature instead of the classical quadratic form. In this case as well, we will see that general covariance imposes previously unrecognized conditions on possible modifications of canonical gravity theories, in addition to the usual condition that the constraints remain first-class and resemble hypersurface-deformation brackets. General covariance in an emergent line element is therefore recognized as a restrictive condition on possible quantum space-time effects, required to be consistent with a geometrical continuum theory of space-time at low curvature. These general properties will be derived and discussed in Sec. II.

We will work out specific versions of generally covariant emergent modified gravity theories in Sec. III, using spherically symmetric reductions. We will first redefine the classical constraints by replacing them with phase-space dependent linear combinations. The resulting modified theories demonstrate that off-shell properties of hypersurface deformations are indeed relevant for physical implications because they make it possible to evaluate the cancellation condition and the structure function. We will also revisit the partial Abelianizations of brackets proposed in [18,19] in Sec. III D and, following [20,21], reanalyze their off-shell structure from our new perspective. Examples of modifications that obey the new conditions are those of [15,22]. Section IV will present the general case of modified Hamiltonian constraints up to second order in spatial derivatives that are consistent with the covariance condition, including a discussions of the freedom implied by applying canonical transformations. As an application, in Sec. IV F, we will use our constraints and related methods to derive new, nontrivial partial Abelianizations compatible with general covariance.

## II. MODIFIED GRAVITY IN A CANONICAL FORMULATION

As usual in canonical theories, we assume that space-time, or at least a region of interest, is globally hyperbolic: $M = \Sigma \times \mathbb{R}$ with a three-dimensional "spatial" manifold $\Sigma$. In a generally covariant theory, there is no unique embedding of $\Sigma$ in $M$, but we can parametrize different choices by working with foliations of $M$ into smooth families of spacelike hypersurfaces $\Sigma_t$, $t \in \mathbb{R}$, each of which is homeomorphic to $\Sigma$. For a given foliation, $\Sigma$ can be embedded in $M$ as a constant-time hypersurface: $\Sigma \cong \Sigma_{t_0} \cong (\Sigma_{t_0}, t_0) \hookrightarrow M$ for any fixed $t_0$.

### A. Canonical decomposition

Given a foliation into spacelike hypersurfaces $\Sigma_t$, a metric $g_{\mu\nu}$ on $M$ induces a unique spacelike metric $q_{ab}(t_0)$ on any $\Sigma_{t_0}$. (As in this example, we use greek letters for indices of space-time tensors, and latin letters for indices of spatial tensors.) Using the unit normal vector field $n^\mu$ on $\Sigma_{t_0}$, the induced spatial metric is obtained by restricting the space-time tensor $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ to vector fields tangential to the hypersurface, while $q_{\mu\nu} n^\nu = 0$. The space-time metric is therefore expressed as a time-dependent family of spatial metrics. Since spatial hypersurfaces within a foliation of a covariant theory are invariant under spatial diffeomorphisms, interpreting the time dependence of $q_{ab}(t)$ as unambiguous evolution requires an additional structure that relates points on infinitesimally related hypersurfaces defined by different values of $t$. This additional structure can be expressed as a time-evolution vector field

$$t^\mu = N n^\mu + N^a s_a^\mu \tag{4}$$

in space-time, with the lapse function $N$ and shift vector field $N^a$ [7]. The three vector fields $s_a^\mu(t_0)$ inject $T\Sigma_{t_0}$ into $TM$ such that $g_{\mu\nu} n^\mu s_a^\mu = 0$.

The new ingredients $N$ and $N^a$ describe the frame of an observer in curved space-time who measures the evolving $q_{ab}(t)$. In the four-dimensional picture, the frame corresponds to a choice of space-time coordinates which completes spatial coordinates on $\Sigma$ by a time coordinate $t$ in $M$ such that $\Sigma_{t_0} = M_{t=t_0}$. The space-time metric or line element is then in one-to-one correspondence with the family $(q_{ab}(t), N(t), N^a(t))$ of spatial tensors on the foliation $(\Sigma_t, t) \hookrightarrow M$. We have

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a \mathrm{d}t)(\mathrm{d}x^b + N^b \mathrm{d}t). \tag{5}$$

A hypersurface in the foliation has extrinsic curvature

$$K_{ab} = \frac{1}{2} \mathcal{L}_n q_{ab} \tag{6}$$

related to the Lie derivative of the spatial metric in the normal direction. Expressed through a "velocity" of $q_{ab}$ with respect to the time-evolution vector field $t^\mu$, it takes the form

$$K_{ab} = \frac{1}{2N} q_a{}^c q_b{}^d (\mathcal{L}_t q_{cd} - \mathcal{L}_N q_{cd}). \tag{7}$$

Evolution on a given foliation, defined by a choice of $t$ and $t^\mu$ (or $N$ and $N^a$), is Hamiltonian: Infinitesimal changes of $q_{ab}$ and $K_{ab}$ are obtained via Poisson brackets of these tensors with a Hamiltonian $H[N, N^a]$, where the Poisson bracket is defined by considering

$$p^{ab} = \frac{\sqrt{\det q}}{16\pi G} (K^{ab} - K^c_c q^{ab}) \tag{8}$$

as canonically conjugate momenta of $q_{ab}$. Given the original manifold $M$ as well as general covariance of the relativistic dynamics, evolution within a foliation is closely related to transformations of the foliation to a new one. We merely have to reinterpret $N$ and $N^a$ as gauge parameters $\epsilon^0$ and $\epsilon^a$ that parametrize an infinitesimal change of the foliation.

Evolution and gauge transformations are therefore described by the same flow, which implies that the dependence on $q_{ab}$ and $p^{ab}$ of the Hamiltonian $H[\epsilon^0, \epsilon^a]$ of the gauge flow is the same as the dependence of the Hamiltonian $H[N, N^a]$ for evolution. In its role as generator of a gauge flow, however, $H[\epsilon^0, \epsilon^a]$ must be a constraint, $H[\epsilon^0, \epsilon^a] = 0$ for all $\epsilon^0$ and $\epsilon^a$, in order to have a well-defined symplectic structure on gauge-invariant observables. The dynamics, therefore, is also fully constrained: $H[N, N^a] = 0$ for all $N$ and $N^a$. (We assume that our manifolds do not require nontrivial boundary conditions, in which case some choices of $N$ and $N^a$ may not be considered gauge.) Since spatial and normal deformations of hypersurfaces are independent, there are two different constraint functionals, the Hamiltonian constraint $H[N]$ and the diffeomorphism constraint $H_a[N^a]$ such that $H[N, N^a] = H[N] + H_a[N^a]$. Physical solutions of the theory are "on shell," that is, they have $q_{ab}$ and $p^{ab}$ on each hypersurface of a foliation such that $H[N] = 0$ and $H_a[N^a] = 0$.

The constraints generate gauge transformations for any given phase-space function $\mathcal{O}$, depending on $q_{ab}$ and $p^{ab}$, via the Poisson bracket $\delta_\epsilon \mathcal{O} = \{\mathcal{O}, H[\epsilon^0, \epsilon^a]\}$. These are indeed gauge transformations because the constraints obey the hypersurface-deformation brackets (1)–(3) and are therefore first class. Within each foliation related by a gauge transformation, the canonical fields $q_{ab}$ and $p^{ab}$, or any function $\mathcal{O}$ of them, evolve according to $\dot{\mathcal{O}} \equiv \delta_t \mathcal{O} = \{\mathcal{O}, H[N, N^a]\}$. Poisson brackets of this form do not immediately provide gauge transformations of $N$ and $N^a$ because they do not have momenta, and they do not physically evolve because they specify a frame with respect to which evolution is defined. However, $N$ and $N^a$ must be subject to gauge changes because the corresponding coefficients in the line element (5) depend on the foliation. These gauge transformations can be derived from the condition that the gauge transformation of an evolution equation should be consistently related to evolution of gauge-transformed phase-space variables. This condition refers to commutators of gauge transformations and evolution, and is therefore sensitive to the structure functions in (1)–(3). Gauge transformations obeying this condition are given by [13,23,24]

$$\delta_\epsilon N = \dot{\epsilon}^0 + \epsilon^a \partial_a N - N^a \partial_a \epsilon^0, \tag{9}$$

$$\delta_\epsilon N^a = \dot{\epsilon}^a + \epsilon^b \partial_b N^a - N^b \partial_b \epsilon^a + q^{ab}(\epsilon^0 \partial_b N - N \partial_b \epsilon^0), \tag{10}$$

where the structure function appears in the last term.

The final ingredient required for a discussion of general covariance in canonical form is a relationship between gauge transformations generated by a Hamiltonian and Lie derivatives along a space-time vector field $\xi^\mu$. Components of the latter refer to coordinate directions, while hypersurface deformations refer to the normal direction. These basis choices are linearly related by

$$\xi^\mu = \epsilon^0 n^\mu + \epsilon^a s_a^\mu = \xi^t t^\mu + \xi^a s_a^\mu, \tag{11}$$

or

$$\xi^t = \frac{\epsilon^0}{N}, \qquad \xi^a = \epsilon^a - \frac{\epsilon^0}{N} N^a, \tag{12}$$

if we assume that the same spatial coordinate systems are used, as in (5). If the constraints and equations of motion are satisfied (on shell or "O.S."), the gauge transformations

$$\{q_{ab}, \vec{H}[\vec{\epsilon}]\}|_{\text{O.S.}} = \mathcal{L}_{\vec{\epsilon}} q_{ab}, \quad \{q_{ab}, H[\epsilon^0]\}|_{\text{O.S.}} = \mathcal{L}_{\epsilon^0 n} q_{ab} \tag{13}$$

together with the gauge transformations of lapse and shift, (9), are equivalent to infinitesimal space-time diffeomorphisms of the metric in (5),

$$\delta_\epsilon g_{\mu\nu}|_{\text{O.S.}} = \mathcal{L}_\xi g_{\mu\nu}, \tag{14}$$

identifying time derivatives by using evolution equations generated by the same constraints.

Off-shell, however, hypersurface deformations are rather different from coordinate changes. The presence of structure functions in the description of hypersurface deformations implies that a closed set of brackets can be obtained from them only if lapse and shift are allowed to depend on the spatial metric, in addition to their dependence on space-time coordinates. However, if one computes Poisson brackets of the phase-space functions that provide the Hamiltonian and diffeomorphism constraints smeared with phase-space dependent lapse and shift, additional terms appear compared with (1)–(3), given by constraints evaluated with partial derivatives of lapse and shift by components of the spatial metric. These terms do not change the first-class nature of the constraints or their on-shell properties, but in general they are not compatible with the equivalence (14) if one attempts to extend it to off-shell metrics. Only specific phase-space functions for lapse and shift may be compatible with general covariance, provided they obey conditions that we will derive in what follows. Changing lapse and shift in this way is equivalent to redefining the Hamiltonian constraint or the normal direction in a corresponding space-time geometry. The normal direction together with the spatial metric determines the emergent space-time geometry. In order to evaluate whether this basic property could lead to new theories of modified gravity, we have to look look more closely at possible modifications of the constraints and their resulting brackets.

### B. Hypersurface-deformation brackets and covariance conditions

There are different sources for possible modifications of the constraints in models of canonical gravity. As just described, there may be new terms in their brackets if lapse and shift are allowed to be phase-space dependent. In addition, one may be interested in studying possible modifications of the dependence of the Hamiltonian and diffeomorphism constraints on the canonical fields. For instance, higher-order terms beyond the classically at most quadratic dependence of the constraints on momenta could be motivated by quantum effects, as a canonical version of higher-curvature effective actions but without higher time derivatives that usually accompany the latter. We will derive general conditions for a covariant modification, making two common assumptions: that the spatial structure of hypersurfaces is unmodified (governed by the classical diffeomorphism constraint) and that the theory remains spatially local (with a Hamiltonian constraint that depends on spatial derivatives up to some finite order).

Any modification of a canonical gauge theory is subject to consistency conditions. First, the constraints must remain first class, or vanish on shell, which is guaranteed if we try to modify the classical theory by using phase-space dependent lapse and shift, but not necessarily by modifications of the phase-space dependence of the constraints, as implied by higher-order terms in momenta. Second, for the modified theory to be considered a space-time theory, any modified brackets of the constraints must in some way exhibit an equivalence with space-time coordinate changes, at least (and usually only) on shell, as in (14). Since the structure functions of classical hypersurface deformations imply the correct transformations of lapse and shift via (9), the modified brackets must be of the classical form (1)–(3).

While this condition leads to brackets identical in form to the classical ones, an opening for new theories of modified gravity can be found in the possibility that lapse, shift, and spatial metric as they appear initially in $H[N]$ and $H[N^a]$ are not required to be identical to the same objects seen as components of the space-time metric (5), in which form they have been derived classically. There may be an emergent lapse $\tilde{N}$, shift $\tilde{N}^a$ and spatial metric $\tilde{q}_{ab}$ that depend on $N$, $N^a$, and $q_{ab}$ as they appear in the constraints and define the phase-space structure together with $p^{ab}$, but are not identical to them. An emergent extrinsic curvature $\tilde{K}_{ab}$ would then be derived as well for hypersurfaces in the emergent space-time line element defined by $\tilde{N}$, $\tilde{N}^a$ and $\tilde{q}_{ab}$. This possibility had been exploited in [13] to show, via (9), that a sign change of the classical structure function amounts, under certain conditions, to signature change in an emergent space-time consistent with general covariance in the modified theory. The additional conditions, however, were quite restrictive as they only allowed emergent spatial metrics obtained from $q_{ab}$ by multiplication with a spatially constant function [which was allowed to depend on time when used in a completion to a space-time metric as in (5)].

#### 1. Emergent space-time metric and general covariance

More generally, whenever modifications lead to brackets of the form

$$\{\vec{H}[\vec{N}],\vec{H}[\vec{M}]\}=\vec{H}[\mathcal{L}_{\vec{N}}\vec{M}], \tag{15}$$

$$\{\tilde{H}[N],\vec{H}[\vec{N}]\}=-\tilde{H}[N^b\partial_b N], \tag{16}$$

$$\{\tilde{H}[N],\tilde{H}[M]\}=-\vec{H}[\tilde{q}^{ab}(N\partial_b M-M\partial_b N)] \tag{17}$$

without additional terms off-shell for phase-space independent $N$ and $\vec{M}$, but with a modified structure function $\tilde{q}^{ab}\neq q^{ab}$ as some phase-space function, then $\tilde{q}_{ab}$ rather than $q_{ab}$ should be used as the spatial metric of an emergent space-time line element:

$$\mathrm{d}s^2=-N^2\mathrm{d}t^2+\tilde{q}_{ab}(\mathrm{d}x^a+N^a\mathrm{d}t)(\mathrm{d}x^b+N^b\mathrm{d}t). \tag{18}$$

(For now we assume that $\tilde{q}^{ab}$ is invertible; see Sec. II C for the more general case of $\tilde{q}^{ab}$ that may be noninvertible on submanifolds of codimension at least one in space-time.) Under gauge transformations, lapse and shift then transform as

$$\delta_\epsilon N=\dot{\epsilon}^0+\epsilon^a\partial_a N-N^a\partial_a\epsilon^0, \tag{19}$$

$$\delta_\epsilon N^a=\dot{\epsilon}^a+\epsilon^b\partial_b N^a-N^b\partial_b\epsilon^a+\tilde{q}^{ab}(\epsilon^0\partial_b N-N\partial_b\epsilon^0), \tag{20}$$

consistent with corresponding coordinate changes in the new emergent line element.

Here, our construction differs from that in [13], where factors that multiply the classical $q^{ab}$ in a modified structure functions were attempted to be absorbed in a redefined lapse function. Such a choice is more natural in a discussion of signature change, which is expected to affect the time components of the space-time metric where the lapse function appears, but it leads to strong conditions on allowed modifications. Redefining the spatial metric rather than the lapse function agrees with the constructions of [16,17] and earlier in [25], where signature change did not occur. Our general treatment here allows for signature change as well as redefined spatial metrics, as we will see.

A third, and final, condition appears because a modified structure function $\tilde{q}^{ab}$ is not guaranteed to gauge transform in a way compatible with an interpretation as the inverse of a spatial metric in a space-time line element. This condition has not been analyzed completely in previous studies. We say that the theory is generally covariant if there are sufficiently many independent fields $f$ (fundamental or composite) such that (i)

$$\delta_\epsilon f|_{\mathrm{O.S.}}=\mathcal{L}_\xi f|_{\mathrm{O.S.}}, \tag{21}$$

and (ii) they can be arranged as components of a space-time line element (18). The space-time geometry regained via (18) is then generally covariant:

$$\delta_\epsilon\tilde{g}_{\mu\nu}|_{\mathrm{O.S.}}=\mathcal{L}_\xi\tilde{g}_{\mu\nu}|_{\mathrm{O.S.}}. \tag{22}$$

This covariance condition is not automatically satisfied just by virtue of the hypersurface deformation brackets, (15)–(17), even after a redefinition of the spatial metric or lapse and shift. In order to see this, we look at each component of the covariance condition, using (15)–(17) and performing the ADM decomposition with (12). In what follows, it is understood that each covariance condition is required to hold only on-shell, but we drop the symbol "O.S." for the sake of simplicity.

Beginning with the $ta$ components, the left-hand side of the covariance condition (22) is

$$\delta_\epsilon \tilde{g}_{ta} = N^b \delta_\epsilon \tilde{q}_{ab} + \tilde{q}_{ab} \delta_\epsilon N^b. \tag{23}$$

If we assume that $\delta_\epsilon \tilde{q}_{ab} = \mathcal{L}_\xi \tilde{g}_{ab}$, then this covariance condition can be written as

$$\begin{aligned} \tilde{q}_{ab}\delta_\epsilon N^b &= \mathcal{L}_\xi \tilde{g}_{ta} - N^b \mathcal{L}_\xi \tilde{g}_{ab}, \\ &= \tilde{q}_{ab}(N^b\partial_t \xi^t + \partial_t \xi^b - N^b N^c \partial_c \xi^t \\ &\quad - N^c \partial_c \xi^b + \xi^\mu \partial_\mu N^b) - N^2 \partial_a \xi^t, \\ &= \tilde{q}_{ab}(\dot{\epsilon}^b + \epsilon^c \partial_c N^b - N^c \partial_c \epsilon^b) \\ &\quad + \epsilon^0 \partial_a N - N \partial_a \epsilon^0, \end{aligned} \tag{24}$$

where we have used (12) in the last step. This result is consistent with the canonical gauge transformation of the shift, (20). The derivation also shows that the classical relation (12) between a coordinate basis and one adjusted to hypersurfaces should not be modified: Because there is a term in the final result for $\tilde{q}_{ab}\delta_\epsilon N^b$ that depends on $\tilde{q}_{ab}$ and one that does not, all components of this relation have been used independently in its derivation. [This conclusion might be circumvented by using metric-dependent coefficients in a modified version of (12), but such a choice would complicate other equations.]

Similarly, the left-hand side of the $tt$ component is

$$\delta_\epsilon \tilde{g}_{tt} = -2N\delta_\epsilon N + N^a N^b \delta_\epsilon \tilde{q}_{ab} + 2\tilde{q}_{ab} N^a \delta_\epsilon N^b. \tag{25}$$

If we assume again that $\delta_\epsilon \tilde{q}_{ab} = \mathcal{L}_\xi \tilde{g}_{ab}$ and that the shift transforms as (20), then the $tt$ component of the covariance condition can be written as

$$2N\delta_\epsilon N = 2N(\dot{\epsilon}^0 + \epsilon^a \partial_a N - N^a \partial_a \epsilon^0), \tag{26}$$

which is consistent with the canonical gauge transformation (19) of the lapse function.

Lastly, the spatial components of the covariance condition are

$$\begin{aligned} \delta_\epsilon \tilde{q}_{ab} &= \mathcal{L}_\xi \tilde{g}_{ab}, \\ &= \frac{\epsilon^0}{N}\dot{\tilde{q}}_{ab} + \epsilon^c \partial_c \tilde{q}_{ab} + \tilde{q}_{ca}\partial_b \epsilon^c + \tilde{q}_{cb}\partial_a \epsilon^c \\ &\quad - \frac{\epsilon^0}{N}(N^c \partial_c \tilde{q}_{ab} + \tilde{q}_{ca}\partial_b N^c + \tilde{q}_{cb}\partial_a N^c) \end{aligned} \tag{27}$$

from (12). To proceed with our evaluation of this equation, we make the common assumption that the diffeomorphism constraint remains unmodified, which implies that $\tilde{q}_{ab}$ is a spatial tensor and that its Poisson bracket with the diffeomorphism constraint equals a spatial Lie derivative along the shift vector. The time derivative $\dot{\tilde{q}}_{ab} = \{\tilde{q}_{ab}, \tilde{H}[N, N^a]\}$ inserted on the right-hand side of (27) and the gauge transformation $\delta_\epsilon \tilde{q}_{ab} = \{\tilde{q}_{ab}, \tilde{H}[\epsilon^0, \epsilon^a]\}$ on the left-hand side of this equation then have matching terms for all spatial derivatives in (27) to cancel out. We are left with the equation

$$\{\tilde{q}_{ab}, \tilde{H}[\epsilon^0]\}|_{\text{O.S.}} = \frac{\epsilon^0}{N}\{\tilde{q}_{ab}, \tilde{H}[N]\}|_{\text{O.S.}}. \tag{28}$$

We now assume that the modified theory remains local, such that $\tilde{H}[\epsilon^0]$ depends on spatial derivatives of the phase-space degrees of freedom up to some finite order. As a local functional of $\epsilon_0$, the normal gauge transformation of the spatial metric takes the generic form

$$\{\tilde{q}_{ab}, \tilde{H}[\epsilon^0]\} = Q_{ab}\epsilon^0 + Q^c_{ab}\partial_c \epsilon^0 + Q^{cd}_{ab}\partial_c \partial_d \epsilon^0 + \cdots, \tag{29}$$

where the $Q$ tensors are phase-space dependent and the series truncates at some finite order. Substituting this expansion into (28), we obtain

$$\begin{aligned} & Q^c_{ab}\frac{\partial_c \epsilon^0}{\epsilon^0} + Q^{cd}_{ab}\frac{\partial_c \partial_d \epsilon^0}{\epsilon^0} + \cdots \bigg|_{\text{O.S.}} \\ &\quad = Q^c_{ab}\frac{\partial_c N}{N} + Q^{cd}_{ab}\frac{\partial_c \partial_d N}{N} + \cdots \bigg|_{\text{O.S.}} \end{aligned} \tag{30}$$

(neglecting boundary terms that may result after integrating by parts). For a generally covariant theory, the gauge generator functions $(\epsilon^0, \epsilon^a)$ must be independent of each other, and of $N$, $N^a$ as well as phase-space functions that they are supposed to transform. Thus, each $Q$ tensor in (29) must vanish independently, and we obtain a series of conditions on the gauge transformation of the emergent spatial metric or its inverse:

$$\frac{\partial(\delta_{\epsilon^0}\tilde{q}^{ab})}{\partial(\partial_c \epsilon^0)}\bigg|_{\text{O.S.}} = \frac{\partial(\delta_{\epsilon^0}\tilde{q}^{ab})}{\partial(\partial_c \partial_d \epsilon^0)}\bigg|_{\text{O.S.}} = \cdots = 0. \tag{31}$$

Since this tensor is determined by the structure function in the hypersurface-deformation brackets of a modified canonical theory, generally covariant modifications are subject to additional conditions that go beyond the basic requirement that the brackets remain first class. They must be first class with structure functions obeying (31). Because the structure function and its gauge transformation are both determined by the constraints, these are nontrivial conditions on the modified Hamiltonian constraint. This condition has been overlooked in several previous treatments of canonical gravity and possible modifications, such as models of loop quantum gravity, but it is essential for complete covariance.

### 2. Extrinsic curvature and the geometry of embedded hypersurfaces

In our discussion so far, we have used the constraint equations and the gauge flow (28) or the equation of motion of the emergent metric. The condition that this transformation is equivalent to the Lie derivative of a spatial metric led to a nontrivial consistency condition (31) for general covariance. Owing to modifications of the structure function, the emergent spatial metric that obeys this condition need not agree with the basic phase-space function $q_{ab}$ of the canonical theory, unlike in classical theories of gravity. In the same way, the momentum $p^{ab}$ canonically conjugate to $q_{ab}$ need not be linearly related to extrinsic curvature if the structure function is modified. Instead, the emergent space-time line element (18) may be used to derive a suitable extrinsic curvature tensor $\tilde{K}_{ab}$ in the same slicing in which the emergent metric $\tilde{q}_{ab}$ is induced by (18) as the spatial metric. Since extrinsic curvature by definition depends on normal derivatives of the spatial metric, equations of motion of the canonical theory could be used to relate $\tilde{K}_{ab}$ to the original canonical fields $q_{ab}$ and $p^{ab}$, just as $\tilde{q}_{ab}$ is not a basic canonical field but depends, in general, on both $q_{ab}$ and $p^{ab}$ in a nonlinear way.

Such an extrinsic curvature tensor $\tilde{K}_{ab}$ would be derived from a consistent space-time geometry and would therefore be a proper covariant two tensor. This property implies that there is no additional momentum-version of the covariance condition derived from $\tilde{q}_{ab}$. If one is interested in comparing canonical gauge transformations of $\tilde{K}_{ab}$, defined through the relationship between this tensor and the phase-space functions $q_{ab}$ and $p^{ab}$, with space-time coordinate transformations of this tensor, one would make use of the momentum version of the gauge transformations (or of the corresponding equations of motion) (28). The full set of gauge transformations is therefore used if one compares gauge transformations with space-time Lie derivatives for both $\tilde{q}_{ab}$ and $\tilde{K}_{ab}$. However, since the structure function of hypersurface-deformation brackets uniquely determines the complete space-time line element, only the gauge transformations of $\tilde{q}_{ab}$ yield nontrivial covariance conditions, while covariance of $\tilde{K}_{ab}$ is then implied. Heuristically, the correct transformation of $\tilde{K}_{ab}$ is implied because $\tilde{K}_{ab}$ is defined as a certain space-time coordinate change of $\tilde{q}_{ab}$. (This transformation is not a Lie derivative because extrinsic curvature depends on the slicing. It is a spatial tensor on a fixed hypersurface but not a space-time tensor.) A detailed derivation together with explicit equations for the correct coordinate transformations can be found in Appendix A.

Covariance of the spatial metric tensor therefore implies covariance of the extrinsic-curvature tensor. While all equations of motion are used if one derives explicit transformations for both $\tilde{q}_{ab}$ and $\tilde{K}_{ab}$, only the former lead to nontrivial covariance conditions. This result reinforces the heuristic understanding that the equations of motion for $\tilde{q}_{ab}$ determine geometrical properties of an embedded hypersurface (the relationship between extrinsic curvature and normal derivatives of the spatial metric), equations of motion for $\tilde{q}_{ab}$ determine the dynamics of the theory and therefore physical properties.

### 3. The necessity of emergence for modified gravity

As an example, consider a theory in metric variables, where the phase space is composed of the "bare" spatial metric $q_{ab}$ (used to define the phase-space structure) and its conjugate momenta $p^{ab}$, and the emergent spatial metric equals the bare spatial metric (that is, the structure function remains classical). The covariance condition (31) then implies, from $\{q_{ab}, \tilde{H}[\epsilon^0]\} = \delta\tilde{H}[\epsilon^0]/\delta p^{ab}$, that the Hamiltonian constraint must not contain spatial derivatives of $p^{ab}$. If we use only up to second-order spatial derivatives of $q_{ab}$, the Hamiltonian constraint is uniquely determined by the hypersurface deformation brackets, (15)–(17), up to the choice of Newton's and the cosmological constant, and assuming parity symmetry [10,11]. It must therefore be classical, and generally covariant modifications are ruled out under the stated conditions.

If the spatial metric is considered a composite function of the phase space, as it happens when the space-time line element has an emergent spatial metric $\tilde{q}_{ab}$ distinct from the phase-space function $q_{ab}$ (and not just obtained by directly applying a canonical transformation), the regaining procedure of [10,11] is modified and may result in new gravitational theories even at second derivative order. The covariance condition (31) is nontrivial in this situation. For instance, if the emergent spatial metric depends on the momenta, as it happens in the examples discussed in the next section, spatial derivatives of the bare metric, which always appear in the Hamiltonian constraint, also contribute to the covariance condition and could cancel out with unwanted terms from spatial derivatives of the momenta. The covariance condition and the concept of an emergent metric then present important ingredients in constructions of modified canonical gravity, in addition to the requirement that hypersurface-deformation brackets of the form (15) be realized. Although there are then different versions of $q_{ab}$ in such a theory, given by the bare metric and the emergent metric, it is not an example of bimetric gravity: A unique metric, the emergent one, is singled out by the covariance condition, if the latter can be solved at all.

Results similar to those of the present section can be formulated for triad variables, in which case the spatial metric has the status of a composite field even in the classical theory.

## C. Noninvertible structure functions and signature change

The definition of an emergent line element from a modified structure function $\tilde{q}^{ab}$ requires that this spatial tensor is invertible in a space-time region in which it is applied. If this condition is not strictly fulfilled but still holds on a dense submanifold in space-time, then there are hypersurfaces (possibly timelike or lightlike and not just spacelike) that separate regions in which $\tilde{q}^{ab}$ is invertible. Emergent line elements then exist only in these regions but not on the separating hypersurfaces. Moreover, they may differ from one another by certain sign factors of $\mathrm{sgn}\,\det\tilde{q}^{ab}$. Space-time then has distinct regions in which emergent line elements exist but no global line element. (A single space-time object is defined by solutions of the constraints before they are equipped with emergent line elements.) An example for such emergent space-times is given by models with dynamical signature change [26,27].

If we are in a region where $\mathrm{sgn}\,\det\tilde{q}^{ab} = -1$, then the emergent line element (18) is of (negative) Euclidean signature $(-1,-1,-1,-1)$ and no longer Lorentzian as in the classical limit. (We assume that the signature remains spatially isotropic in order to prevent the existence of a distinguished spatial direction in the resulting gravity theory.) Combining our constructions with those in [13], it follows that this emergent line element is equivalent to one with positive Euclidean signature $(+1,+1,+1,+1)$ because we may define the emergent spatial metric as

$$\tilde{\tilde{q}}_{ab} = \mathrm{sgn}(\det\tilde{q}^{ab})\tilde{q}_{ab} \tag{32}$$

and introduce an emergent line element

$$\mathrm{d}s^2 = -\mathrm{sgn}(\det\tilde{q}^{ab})N^2\mathrm{d}t^2 + \tilde{\tilde{q}}_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t). \tag{33}$$

Because $\mathrm{sgn}(\det\tilde{q}^{ab})$ is spatially constant in any region in which $\tilde{q}^{ab}$ is invertible, the conclusions of [13] apply and show that the new definitions guarantee general covariance.

## III. SPHERICALLY SYMMETRIC THEORY OF GRAVITY IN VACUUM

Following the general results of the previous section, we will now focus on spherically symmetric models. We choose the basic phase-space variables to be certain extrinsic-curvature components as the configuration variables and densitized-triad components as their conjugate momenta, as used frequently in models of loop quantum gravity [28,29].

### A. Classical spherically symmetric theory

In the spherically symmetric classical theory, the space-time metric is

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{xx}(\mathrm{d}x + N^r\mathrm{d}t)^2 + q_{\vartheta\vartheta}\mathrm{d}\Omega^2, \tag{34}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$. The spatial metric components are related to the radial and angular components of a densitized triad, $E^x$ and $E^\varphi$, respectively, via

$$q_{xx} = \frac{(E^\varphi)^2}{E^x}, \qquad q_{\vartheta\vartheta} = E^x, \tag{35}$$

which, for the purpose of the present paper, may be considered a part of a canonical transformation of the phase space in metric variables. Extrinsic-curvature components are then transformed to radial fields $K_x$ and $K_\varphi$ such that

$$\{K_x(x), E^x(y)\} = \{K_\varphi(x), E^\varphi(y)\} = \delta(x-y). \tag{36}$$

(We choose units such that $2G = 1$.) The geometrical interpretation of $K_x$ and $K_\varphi$ follows from their equations of motion, generated by the Hamiltonian and diffeomorphism constraints,

$$H[N] = \int \mathrm{d}x N\left[\frac{((E^x)')^2}{8\sqrt{|E^x|}E^\varphi} - \frac{E^\varphi}{2\sqrt{|E^x|}} - \frac{E^\varphi K_\varphi^2}{2\sqrt{|E^x|}} - 2K_\varphi\sqrt{|E^x|}K_x - \frac{\sqrt{|E^x|}(E^x)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{|E^x|}(E^x)''}{2E^\varphi}\right], \tag{37}$$

and

$$H_r[N^r] = \int \mathrm{d}x N^r(K_\varphi' E^\varphi - K_x(E^x)'), \tag{38}$$

where the primes are radial derivatives.

The hypersurface-deformation brackets in this case are

$$\{H_r[N^r], H_r[M^r]\} = H_r[N^r(M^r)' - M^r(N^r)'], \tag{39}$$

$$\{H[N], H_r[M^r]\} = -H[M^r N'], \tag{40}$$

$$\{H[N], H[M]\} = H_r[q^{xx}(NM' - N'M)] \tag{41}$$

with the structure function $q^{xx} = E^x/(E^\varphi)^2$, which indeed follows from Poisson brackets of the constraints for phase-space independent $N$ and $M^r$. The structure function determines off-shell gauge transformations for lapse and shift as

$$\delta_\epsilon N = \dot{\epsilon}^0 + \epsilon^r N' - N^r(\epsilon^0)', \tag{42}$$

$$\delta_\epsilon N^r = \dot{\epsilon}^r + \epsilon^r(N^r)' - N^r(\epsilon^r)' + q^{xx}(\epsilon^0 N' - N(\epsilon^0)'). \tag{43}$$

Condition (14) for the covariance of the metric is satisfied, and the gauge generator functions are related to the two-component vector generator of infinitesimal diffeomorphisms by

$$\xi^\mu = \epsilon^0 n^\mu + \epsilon^r s^\mu = \xi^t t^\mu + \xi^r s^\mu \tag{44}$$

with components

$$\xi^t = \frac{\epsilon^0}{N}, \qquad \xi^r = \epsilon^r - \frac{\epsilon^0}{N} N^r. \tag{45}$$

### B. Modified spherically symmetric theory

We consider modifications to the spherically symmetric theory with canonical variables $(K_\varphi, E^\varphi)$ and $(K_x, E^x)$. If we modify the Hamiltonian constraint, then the brackets (39)–(41) determine the emergent radial spatial metric via $\tilde{q}_{xx} = (\tilde{q}^{xx})^{-1}$ if the modified structure function $\tilde{q}^{xx}$ is invertible everywhere, and therefore positive definite. The angular component of the metric $\tilde{q}_{\vartheta\vartheta}$ cannot be determined in this way because it does not appear in the classical brackets. We will therefore keep it unmodified in the present section. The emergent space-time line element is then given by

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \tilde{q}_{xx}(\mathrm{d}x + N^r \mathrm{d}t)^2 + E^x \mathrm{d}\Omega^2 \tag{46}$$

if $\tilde{q}^{xx} > 0$ is strictly positive. If $\tilde{q}^{xx}$ is not positive definite, then we define the emergent line element as

$$\mathrm{d}s^2 = -\mathrm{sgn}(\tilde{q}^{xx}) N^2 \mathrm{d}t^2 + \tilde{\tilde{q}}_{xx}(\mathrm{d}x + N^r \mathrm{d}t)^2 + E^x \mathrm{d}\Omega^2 \tag{47}$$

with $\tilde{\tilde{q}}_{xx} = |\tilde{q}^{xx}|^{-1}$, choosing the second option of Sec. II C in order to avoid a distinguished role played by the radial direction in space-time signature. This choice is determined by our decision to keep $q_{\vartheta\vartheta}$ unmodified.

Another immediate implication of this decision is that $q_{\vartheta\vartheta} = E^x$ is not a composite field in the emergent line element. As in our general discussion, we therefore conclude that modified constraints cannot depend on spatial derivatives of the variable $K_x$ canonically conjugate to $E^x$: The covariance condition (31), evaluated for the angular component of the metric, implies

$$\left.\frac{\partial \tilde{H}}{\partial K_x'}\right|_{\mathrm{O.S.}} = \left.\frac{\partial \tilde{H}}{\partial K_x''}\right|_{\mathrm{O.S.}} = \cdots = 0, \tag{48}$$

using $\delta_{\epsilon^0} E^x = -\delta \tilde{H}[\epsilon^0]/\delta K_x$.

Radial derivatives of $K_x$ in $\tilde{H}$ can be consistent with the covariance condition only if one considers a more general emergent angular metric component $\tilde{q}_{\vartheta\vartheta}$ that depends not only on $E^x$ but also on other phase-space variables such as extrinsic curvature. Within spherically symmetric models, a choice of $\tilde{q}_{\vartheta\vartheta} \neq E^x$ could therefore be justified if one would like to include a specific term, for instance with spatial derivatives of $K_x$, in a modified Hamiltonian constraint. Such terms have been considered in [30] but without finding a closed version of the modified constraints. Alternatively, a modified angular metric could potentially be determined by constraint brackets if they are derived from the spherical reduction of a consistently modified full theory, or from a model system with less symmetry than spherical models. We will leave these possibilities for future research.

The radial component of the covariance condition takes the form

$$\left.\frac{\partial(\delta_{\epsilon^0}\tilde{q}^{xx})}{\partial(\epsilon^0)'}\right|_{\mathrm{O.S.}} = \left.\frac{\partial(\delta_{\epsilon^0}\tilde{q}^{xx})}{\partial(\epsilon^0)''}\right|_{\mathrm{O.S.}} = \cdots = 0. \tag{49}$$

Since $\tilde{q}^{xx}$, like $q^{xx}$ itself in a triad formulation, is a composite field, this condition is more complicated than the angular version (48). We will consider specific modified constraints and their structure functions in our evaluations of this condition.

A direct application of the covariance condition (49) to the emergent space-times considered in [15,22] confirms that these two models are both covariant. The space-times proposed in several other works, among them [13,19], can be shown not to satisfy covariance, even though the underlying modified constraints are first class and have constraint brackets of hypersurface-deformation form. In the next subsection we construct a new example by using phase-space dependent gauge generator functions, as performed in a different way in [18,19].

### C. Linear combination of constraints with phase-space dependence

We have now specified conditions for general covariance of an emergent line element determined by the structure functions of hypersurface-deformation brackets. As a first application, we can now test whether it is possible, at least in spherically symmetric models, to construct modified gravity theories by using different versions of phase-space dependent lapse and shift in such a way that the structure function no longer agrees with a basic phase-space variable.

From the point of view of a canonical gravity theory, phase-space dependent lapse and shift matter because they imply additional terms in off-shell gauge transformations. Consider two phase space functions $Q$ and $B$, and a lapse function $N$ that is phase-space independent. The gauge transformation of $Q$ generated by the Hamiltonian constraint $H$ with gauge function $BN$ instead of $N$ is given by

$$\begin{aligned}\{Q, H[BN]\} &= \int \mathrm{d}y \{Q, H(y)B(y)\} N(y) \\ &= \int \mathrm{d}y (\{Q, H(y)\} B(y) \\ &\quad + \{Q, B(y)\} H(y)) N(y)\end{aligned} \tag{50}$$

with a new term $\{Q, B\} \neq 0$. While this new term is multiplied by $H(y)$ and therefore disappears on shell, it changes the form of off-shell gauge transformations. Off-shell gauge transformations, applied to the constraints themselves, are relevant for properties of hypersurface-deformation brackets and may contribute to their structure functions and thereby to emergent line elements. Our new methods from the previous section bring us in a position to evaluate these implications.

For the sake of simplicity, we will implement phase-space dependent lapse and shift in a way that does not change spatial diffeomorphisms. Only the Hamiltonian constraint will then have a phase-space dependent multiplier. As a generalization of (50), in addition to replacing $N$ with $BN$ we may also add a contribution from the diffeomorphism constraint to the new normal deformation. Formally, we may arrive at such linear combinations by the substitution

$$N \to BN, \qquad N^r \to AN + N^r \tag{51}$$

in the original constraints with phase-space independent $N$ and $N^r$. The original constraints obey the brackets (39), while phase-space dependent $A$ and $B$ imply additional terms in the brackets of $H_r[N^r]$ with a new Hamiltonian constraint

$$H^{(\mathrm{new})}[N] = H[BN] + H_r[AN] \tag{52}$$

derived from the complete Hamiltonian

$$H[BN, AN + N^r] = H[BN] + H_r[AN] + H_r[N^r] \tag{53}$$

after the substitution (51), collecting all $N$-dependent terms in the definition of $H^{(\mathrm{new})}[N]$. This procedure of implementing a phase-space dependent linear combination of the constraints follows a construction proposed in [18,19] that allows one to eliminate structure functions. Off-shell consistency conditions and covariance, however, had not been considered in these papers, which is made more difficult, if not impossible, by the very act of eliminating the structure function that determines consistent space-time structures. In the present paper we are not interested specifically in eliminating structure functions, but our methods are general enough to analyze covariance also in this context. We will briefly return to this question after our derivation of general consequences of phase-space dependent linear combinations.

In general, the off-shell Poisson brackets of $H^{(\mathrm{new})}[N]$ and $H_r[N^r]$ are not of the form (39). The existence of a covariant emergent line element is therefore not guaranteed. We will now use our methods from the previous section to derive new results that tell us under which conditions on $A$ and $B$ a covariant emergent line element exists, based on (48) and (49).

#### 1. Anomaly-freedom and the covariance condition

We now consider the same canonical variables $(K_\varphi, E^\varphi)$ and $(K_x, E^x)$ as well as the diffeomorphism constraint $H_r$ from (38) as used in spherically symmetric gravity. Our derivations in this subsection are general enough to allow for a generic initial Hamiltonian constraint $H^{(\mathrm{old})}$ that could, for instance, correspond to a dilaton gravity model. We will then implement a phase-space dependent linear transformation of the form just described, replacing $H^{(\mathrm{old})}$ with $H^{(\mathrm{new})}$ defined as in (52). Because $H^{(\mathrm{old})}$ and $H^{(\mathrm{new})}$ (before smearing) are densities of weight 1 and $H_r$ is a density of weight 2, it follows that $B$ has density weight 0 and that $A$ has density weight $-1$.

By construction, the gauge transformations $\delta_\epsilon^{(\mathrm{new})}$ generated by the new constraint (52) are equivalent to a combination of gauge transformations generated by the old constraint and the diffeomorphism constraint, with partially phase-space dependent generators:

$$\delta_\epsilon^{(\mathrm{new})} \equiv \delta_{\epsilon^0,\epsilon^r}^{(\mathrm{new})} = \delta_{B\epsilon^0, A\epsilon^0+\epsilon^r}^{(\mathrm{old})}. \tag{54}$$

In order to highlight new terms implied by phase-space dependent multipliers, we define for label="new" and label="old" the contribution

$$\begin{aligned}\not{\delta}_{F_0\epsilon^0, F_r\epsilon^r}^{(\mathrm{label})} Q \equiv& \int \mathrm{d}y \{Q, H^{(\mathrm{label})}(y)\} F_0(y)\epsilon^0(y)\\ &+ \int \mathrm{d}y \{Q, H_r(y)\} F_r(y) \epsilon^0(y)\end{aligned} \tag{55}$$

to normal gauge transformations, where $\epsilon^0$ and $\epsilon^r$ are phase-space independent, while $Q$, $F_0$, and $F_r$ are phase-space functions. Assuming that we already know the gauge transformations generated by the old constraints, we can then write the new transformations as

$$\begin{aligned}\delta_\epsilon^{(\mathrm{new})} Q(x) =& \not{\delta}_{B\epsilon^0, A\epsilon^0+\epsilon^r}^{(\mathrm{old})} Q(x)\\ &+ H^{(\mathrm{old})}[\{Q(x), B\}\epsilon^0 + \{Q(x), A\}\epsilon^r]\\ &+ H_r^{(\mathrm{old})}[\{Q(x), A\}\epsilon^0].\end{aligned} \tag{56}$$

We begin with the condition that the bracket (40) should be reobtained for $H^{(\mathrm{new})}$ and $H_r$ if the new constraints correspond to a realization of hypersurface deformations. A direct calculation shows

$$\begin{aligned}&\{H^{(\mathrm{new})}[N], H_r[M^r]\}\\ &= -H^{(\mathrm{new})}[M^r N'] + H_r[(\delta_{0,M^r}^{(\mathrm{old})} A)N - (A' M^r - A(M^r)')N]\\ &\quad + H^{(\mathrm{old})}[(\delta_{0,M^r}^{(\mathrm{old})} B)N - B' M^r N].\end{aligned} \tag{57}$$

For this to be of the hypersurface deformation form, we have the conditions

$$\delta_{0,M^r}^{(\mathrm{old})} A = M^r A' - A(M^r)', \tag{58}$$

$$\delta_{0,M^r}^{(\mathrm{old})} B = B' M^r. \tag{59}$$

which are satisfied provided $B$ is of density weight zero and $A$ of density weight $-1$.

The bracket of the new Hamiltonian constraint with itself contains several terms:

$$
\begin{aligned}
\{H^{(\mathrm{new})}[N], H^{(\mathrm{new})}[M]\} = \int \mathrm{d}x\mathrm{d}y N(x)M(y)(&\{H^{(\mathrm{old})}(x), H^{(\mathrm{old})}(y)\}B(x)B(y) + \{B(x), B(y)\}H^{(\mathrm{old})}(x)H^{(\mathrm{old})}(y) \\
&+ (\{B(x), H^{(\mathrm{old})}(y)\}H^{(\mathrm{old})}(x)B(y) - (N \leftrightarrow M)) \\
&+ \{H_r(x), H_r(y)\}A(x)A(y) + \{A(x), A(y)\}H_r(x)H_r(y) \\
&+ (\{A(x), H_r(y)\}H_r(x)A(y) - (N \leftrightarrow M)) \\
&+ (\{H_r(x), H^{(\mathrm{old})}(y)\}A(x)B(y) + \{A(x), H^{(\mathrm{old})}(y)\}H_r(x)B(y) - (N \leftrightarrow M)) \\
&+ (\{B(x), H_r(y)\}H(x)A(y) + \{A(x), B(y)\}H_r(x)H^{(\mathrm{old})}(y) - (N \leftrightarrow M))).
\end{aligned} \tag{60}
$$

The first term contains the old bracket $\{H^{(\mathrm{old})}(x), H^{(\mathrm{old})}(y)\}$ for which we can use the hypersurface-deformation result, but there are several additional terms which can be written as

$$
\begin{aligned}
\{H^{(\mathrm{new})}[N], H^{(\mathrm{new})}[M]\} = {}& H_r[B^2 q^{xx}_{(\mathrm{old})}(NM' - MN') + (\not\delta^{(\mathrm{old})}_{BM,0}A)N - (\not\delta^{(\mathrm{old})}_{BN,0}A)M] \\
&+ \int \mathrm{d}x\mathrm{d}y(\{B(x), A(y)\}N(x)H^{(\mathrm{old})}(x)M(y)H_r(y) - (N \leftrightarrow M)) \\
&+ \int \mathrm{d}x\mathrm{d}y\{A(x), A(y)\}N(x)H_r(x)M(y)H_r(y) \\
&+ \int \mathrm{d}x\mathrm{d}y\{B(x), B(y)\}N(x)H^{(\mathrm{old})}(x)M(y)H^{(\mathrm{old})}(y) \\
&+ H^{(\mathrm{old})}[(\not\delta^{(\mathrm{old})}_{BM,0}B)N - (\not\delta^{(\mathrm{old})}_{BN,0}B)M + AB(NM' - MN')].
\end{aligned} \tag{61}
$$

For this combination of terms to be of the required hypersurface-deformation form, it must include $H_r$ smeared by $q^{xx}_{(\mathrm{new})}(NM' - MN')$ where $q^{xx}_{(\mathrm{new})} = \tilde{q}^{xx}$ is the new structure function on which we will impose our condition for general covariance. These terms are contained in the first three lines of (61).

The last two lines in (61) do not contain $H_r$ as an overall factor, and thus they must vanish. To simplify the analysis we restrict ourselves to functions $B$ of the form

$$
B = B(E^x, K_\varphi, (E^x)'/E^\varphi). \tag{62}
$$

All arguments of such a function are of density weight zero and therefore fulfill the earlier condition (59) on $B$. With this choice, the fourth line in (61) vanishes because of antisymmetry of the Poisson bracket, and the last line can be written as $H^{(\mathrm{old})}[F(NM' - MN')]$, where

$$
F = \frac{\partial}{\partial M'}(\not\delta^{(\mathrm{old})}_{BM,0}B) + AB = B\left(\frac{\partial}{\partial M'}(\not\delta^{(\mathrm{old})}_{M,0}B) + A\right) \tag{63}
$$

is independent of $M$ and $N$. The condition $F = 0$ directly relates $A$ to $B$ via

$$
A = -\frac{\partial}{\partial M'}(\delta^{(\mathrm{old})}_{M,0}B), \tag{64}
$$

which is of density weight $-1$, and therefore fulfills (58).

The first three lines in (61) then determine the new structure function via $\{H^{(\mathrm{new})}[N], H^{(\mathrm{new})}[M]\} = H_r[q^{xx}_{(\mathrm{new})}(NM' - N'M)]$:

$$
\begin{aligned}
q^{xx}_{(\mathrm{new})} = {}& B^2 q^{xx}_{(\mathrm{old})} + B\frac{\partial}{\partial M_1'}(\delta^{(\mathrm{old})}_{M_1,0}A) + \left(\frac{\partial B}{\partial E^x}\frac{\partial A}{\partial K_x'} + \frac{\partial B}{\partial E^\varphi}\frac{\partial A}{\partial K_\varphi'} - \frac{\partial B}{\partial K_\varphi}\frac{\partial A}{\partial (E^\varphi)'} - \frac{\partial B}{\partial (E^x)'}\frac{\partial A}{\partial K_x}\right)H^{(\mathrm{old})} \\
&+ \frac{1}{2}\left(\frac{\partial A}{\partial E^x}\frac{\partial A}{\partial K_x'} + \frac{\partial A}{\partial E^\varphi}\frac{\partial A}{\partial K_\varphi'} - \frac{\partial A}{\partial K_x}\frac{\partial A}{\partial (E^x)'} - \frac{\partial A}{\partial K_\varphi}\frac{\partial A}{\partial (E^\varphi)'}\right)H_r,
\end{aligned} \tag{65}
$$

where we have neglected possible second-order derivative terms in $A$, for which a straightforward extension of the present analysis would be needed. The covariance condition (49) applied to the structure function (65) takes the form

$$\frac{\partial}{\partial(\epsilon^0)'}\delta^{(\text{new})}_{\epsilon^0}q^{xx}_{(\text{new})}\Big|_{\text{O.S.}}=\frac{\partial}{\partial(\epsilon^0)''}\delta^{(\text{new})}_{\epsilon^0}q^{xx}_{(\text{new})}\Big|_{\text{O.S.}}=\cdots=0. \tag{66}$$

### 2. Constraints of the spherically symmetric theory

Using the Hamiltonian constraints, (37), and the classical structure function, $q^{xx}_{(\text{old})}=E^x/(E^\varphi)^2$, the anomaly-free linear combination of the constraints of the form (52) is obtained from (62) and (64):

$$A=-\frac{\sqrt{E^x}(E^x)'}{2(E^\varphi)^2}\frac{\partial B}{\partial K_\varphi}-2K_\varphi\sqrt{E^x}\frac{\partial B}{\partial(E^x)'}. \tag{67}$$

The structure function (65) of the resulting anomaly-free brackets of hypersurface-deformation form then equals

$$\begin{aligned}q^{xx}_{(\text{new})}=&\frac{E^x}{(E^\varphi)^2}B^2-\frac{K_\varphi E^x}{(E^\varphi)^2}B\frac{\partial B}{\partial K_\varphi}-\left(\frac{\sqrt{E^x}(E^x)'}{2(E^\varphi)^2}\right)^2B\frac{\partial^2 B}{(\partial K_\varphi)^2}\\&+\frac{E^x(E^x)'}{(E^\varphi)^2}B\frac{\partial B}{\partial(E^x)'}+(2K_\varphi\sqrt{E^x})^2B\frac{\partial^2 B}{(\partial(E^x)')^2}\\&+\frac{2K_\varphi E^x(E^x)'}{(E^\varphi)^2}B\frac{\partial^2 B}{\partial K_\varphi\partial(E^x)'}.\end{aligned} \tag{68}$$

[The second and third line of (65) vanish identically in the present case.]

The covariance condition (66) requires that

$$\begin{aligned}&16K_\varphi(3K_\varphi\partial_zB(4K_\varphi\partial_z^2B+\partial_{K_\varphi}B)+B(K_\varphi(4K_\varphi\partial_z^3B+3\partial_{K_\varphi}\partial_zB)-3\partial_zB))\\&+12z[K_\varphi\partial_{K_\varphi}B(4K_\varphi\partial_z^2B+\partial_{K_\varphi}B)+4K_\varphi\partial_zB(\partial_zB+2K_\varphi\partial_{K_\varphi}\partial_zB)+B(K_\varphi(4\partial_z^2B+4K_\varphi\partial_{K_\varphi}\partial_z^2B+\partial^2_{K_\varphi}B)-\partial_{K_\varphi}B)]\\&+12z^2[\partial_{K_\varphi}B(\partial_zB+2K_\varphi\partial_{K_\varphi}\partial_zB)+K_\varphi\partial_zB\partial^2_{K_\varphi}B+B(\partial_{K_\varphi}\partial_zB+K_\varphi\partial^2_{K_\varphi}\partial_zB)]+z^3[3\partial_{K_\varphi}B\partial^2_{K_\varphi}B+B\partial^3_{K_\varphi}B]=0,\end{aligned} \tag{69}$$

where $z=(E^x)'/E^\varphi$. If we further simplify the form of $B$ to $B=B(K_\varphi,E^x)$, the long condition (69) reduces to two shorter equations for $B$:

$$K_\varphi\left(\frac{\partial B}{\partial K_\varphi}\right)^2+B\left(K_\varphi\frac{\partial^2B}{(\partial K_\varphi)^2}-\frac{\partial B}{\partial K_\varphi}\right)=0, \tag{70}$$

$$B\frac{\partial^3 B}{(\partial K_\varphi)^3}+3\frac{\partial B}{\partial K_\varphi}\frac{\partial^2 B}{(\partial K_\varphi)^2}=0. \tag{71}$$

These equations have the general solutions $B=c_1\sqrt{c_2\pm K_\varphi^2}$ and $B=\tilde c_1\sqrt{\tilde c_2\pm K_\varphi^2+\tilde c_3K_\varphi}$, respectively, where $c_i$ and $\tilde c_i$ are free functions of $E^x$. Consistency between the two solutions yields

$$B_s(K_\varphi,E^x)=\mu\sqrt{1-s\lambda^2K_\varphi^2}, \tag{72}$$

where $s=\pm1$, $\mu=\mu(E^x)$, $\lambda=\lambda(E^x)$. This result implies

$$A_s=\mu\frac{\sqrt{E^x}(E^x)'}{2(E^\varphi)^2}\frac{s\lambda^2K_\varphi}{\sqrt{1-s\lambda^2K_\varphi^2}} \tag{73}$$

via (67), and

$$q^{xx}_{(\text{new})}=\mu^2\left(1+\frac{s\lambda^2}{1-s\lambda^2K_\varphi^2}\left(\frac{(E^x)'}{2E^\varphi}\right)^2\right)\frac{E^x}{(E^\varphi)^2} \tag{74}$$

follows from (68). The classical constraint and structure function are recovered in the limit $\lambda\to0$, $\mu\to1$. The new structure function is always positive for $s=+1$ and therefore directly determines the radial metric of an emergent line element. (In this case, the modified theory is equivalent to what has been analyzed in [16,17] for constant $\lambda$ and a specific $\mu$ depending on $\lambda$. In these papers, the covariance conditions had been checked specifically for the modified constraints without a general underlying theory.) For $s=-1$ there may be regions of signature change.

The case of $s=1$ is interesting because the square root in (72) then implies a bounded curvature component $|K_\varphi|\le1/\lambda$. The modified Hamiltonian constraint in this case equals

$$\begin{aligned}H^{(\text{new})}=&\mu\sqrt{1-s\lambda^2K_\varphi^2}\left(\left(\frac{1}{8\sqrt{|E^x|}E^\varphi}-s\lambda^2\frac{\sqrt{E^x}}{2(E^\varphi)^2}\frac{K_\varphi K_x}{1-s\lambda^2K_\varphi^2}\right)((E^x)')^2-\frac{\sqrt{|E^x|}}{2(E^\varphi)^2}(E^x)'(E^\varphi)'+\frac{\sqrt{|E^x|}}{2E^\varphi}(E^x)''\right.\\&\left.+s\lambda^2\frac{\sqrt{E^x}}{2(E^\varphi)^2}\frac{E^\varphi K_\varphi}{1-s\lambda^2K_\varphi^2}(E^x)'K_\varphi'-\frac{E^\varphi}{2\sqrt{|E^x|}}-\frac{E^\varphi K_\varphi^2}{2\sqrt{|E^x|}}-2K_\varphi\sqrt{|E^x|}K_x\right).\end{aligned} \tag{75}$$

This constraint was first found in [14,15], up to a canonical transformation

$$K_\varphi \to \frac{\sin(\lambda K_\varphi)}{\lambda}, \qquad E^\varphi \to \frac{E^\varphi}{\cos(\lambda K_\varphi)} \tag{76}$$

(for constant $\lambda$) that preserves the diffeomorphism constraint.

The canonically transformed constraint equals

$$\begin{aligned} H_{\rm cc} = -\mu\frac{\sqrt{E^x}}{2}\Bigg[&\frac{E^\varphi}{E^x}\left(1+\frac{\sin^2(\lambda K_\varphi)}{\lambda^2}\right)+4K_x\frac{\sin(2\lambda K_\varphi)}{2\lambda}\\ &-\frac{((E^x)')^2}{4E^\varphi}\left(\frac{1}{E^x}\cos^2(\lambda K_\varphi)-\frac{K_x}{E^\varphi}2\lambda\sin(2\lambda K_\varphi)\right)\\ &+\cos^2(\lambda K_\varphi)\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2}-\cos^2(\lambda K_\varphi)\frac{(E^x)''}{E^\varphi}\Bigg]. \end{aligned} \tag{77}$$

The structure function (72) must then also be transformed, yielding

$$q^{xx}_{\rm cc} = \mu^2\cos^2(\lambda K_\varphi)\left(1+\left(\frac{\lambda(E^x)'}{2E^\varphi}\right)^2\right)\frac{E^x}{(E^\varphi)^2}. \tag{78}$$

By construction, this function implies a covariant emergent line element. Static space-time solutions of the dynamical equations generated by the constraint (77) and the classical diffeomorphism constraint have been studied in [16,17] for $\mu = 1/\sqrt{1+\lambda^2}$, where covariance was also demonstrated explicitly. In this space-time, the surface of maximum curvature $K_\varphi$ is a surface of reflection symmetry, as is readily seen from (77) and (78).

Physical properties of this modified space-time are independent of the canonical transformation because the evaluations of dynamics and covariance are based completely on Poisson brackets. This observation makes it clear that the curvature bound and the avoidance of the classical singularity in [16] must be a consequence of the phase-space dependent linear combination of hypersurface-deformation generators, which can lead to modified space-time solutions because it changes the emergent normal direction.

These physical implications are independent of the use of the canonical transformation or periodic functions originally intended to model holonomies of loop quantum gravity; they are more general properties of emergent modified gravity. (The canonical transformation is non-bijective, which has been argued in a different model to allow new physical effects [31], but this possibility has been ruled out by [32].) Nevertheless, the application of the canonical transformation (77) has a technical advantage because, by its nonbijective nature, the holonomylike variables can be extended to both sides of the reflection-symmetry surface. The constraint (75) diverges at the maximum-curvature surface, $\lambda K_\varphi \to 1$, while the constraint (77) remains finite, $H^{\rm (new)}_{\rm cc} \to -\mu E^\varphi/\sqrt{E^x}$ as $\lambda K_\varphi \to \pi/2$. This behavior is possible because the nonbijective canonical transformation maps all finite values of $E^\varphi$ to infinite values at $\lambda K_\varphi = \pi/2$, suppressing terms that originally diverge as the maximum-curvature surface is approached. The nonbijective nature of the canonical transformation therefore does have an implication, but only on the convenient parametrization of the surface and not on the surrounding space-time regions where $E^\varphi$ is finite in both descriptions.

Our solution for $s = -1$ has not been found before. It may be transformed canonically as in the $s = +1$ case, using hyperbolic instead of trigonometric functions. There is no curvature bound in this case, but the possibility of signature change might turn it into an interesting model system. We leave a detailed analysis to future work.

### D. Off-shell partial Abelianization

As another application of our general equations, we can systematically rederive the partial Abelianization of spherically symmetric constraints from [18,19]. To this end, we need to find a function $B$ in (68) that eliminates the structure function in an anomaly-free way: $q^{xx}_{\rm (new)} \equiv q^{xx}_{\rm (A)} = 0$. According to (68), this is possible if

$$B = K_\varphi B_x(E^x) \tag{79}$$

with some function $B_x$ that depends only on $E^x$, such that

$$A = -\frac{\sqrt{E^x}(E^x)'}{(E^\varphi)^2}B_x. \tag{80}$$

This solution of the Abelianization condition $q^{xx}_{\rm (new)} = 0$ is unique up to a choice of $B_x = B_x(E^x)$. The resulting Abelianized Hamiltonian constraint equals

$$\begin{aligned} \frac{H^{\rm (A)}}{B_x} = \Bigg(&\frac{((E^x)')^2}{8\sqrt{|E^x|}E^\varphi}(1+8E^xK_x)\\ &-\frac{E^\varphi}{2\sqrt{|E^x|}}(1+K_\varphi^2)-2K_\varphi\sqrt{|E^x|}K_x\\ &-\frac{\sqrt{|E^x|}(E^x)'(E^\varphi)'}{2(E^\varphi)^2}+\frac{\sqrt{|E^x|}(E^x)''}{2E^\varphi}\Bigg)K_\varphi\\ &-\frac{\sqrt{E^x}(E^x)'}{E^\varphi}K_\varphi'. \end{aligned} \tag{81}$$

Compared with the constructions in [18,19], our results provide a local and off-shell pathway to partial Abelianizations without the need of an additional integration by parts in the Hamiltonian constraint, integrating the lapse function. Our method therefore supports one of the motivations of [18,19], which is to simplify common quantization procedures that are often untractable in the presence of structure functions. The fully local construction given here provides further simplifications in a quantization procedure that also aims to supply solutions of the Abelianized theory with space-time interpretations. However, space-time considerations require a transformation back to brackets of hypersurface-deformation form: A vanishing structure function, as found in a partially Abelianized theory, makes it impossible to interpret solutions of the theory as emergent space-times because the theory does not provide an unambiguous choice of $q_{\rm eff}^{xx}$. Formally, our covariance condition is trivially satisfied in this case, but only because there is no emergent line element to begin with.

Instead of using a full quantization right away, one may begin with an analysis of modifications that are sometimes necessary in certain quantization approaches, such as "polymerization" or the substitution of periodic functions for extrinsic-curvature components in models of loop quantum gravity. To have a chance of being covariant, such modifications must be compatible with suitable corresponding modifications to the hypersurface-deformation constraint (37), in such a way that the latter still satisfies the covariance condition (49). If one is interested in a space-time picture with an emergent line element, this condition remains in place also for the Abelian constraint (81).

As an example, we may use the constraint (77) and its structure function (78), already modeling holonomy modifications, and then perform the partial Abelianization as done above using (62), (64), and (65). Considering $B = B(E^x, K_\varphi)$, the resulting Abelianized constraint is

$$\frac{H_{\rm cc}^{(\rm A)}}{B_x} = -\frac{\sqrt{E^x}}{2}\left[\frac{E^\varphi}{E^x}\left(1+\frac{\sin^2(\lambda K_\varphi)}{\lambda^2}\right)\frac{\tan(\lambda K_\varphi)}{\lambda} + 4K_x\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} - \frac{((E^x)')^2}{E^\varphi}\left(\frac{1}{4E^x}\frac{\sin(2\lambda K_\varphi)}{2\lambda} - \frac{K_x}{E^\varphi}\lambda^2\frac{\sin^2(\lambda K_\varphi)}{\lambda^2} + \frac{K_x}{E^\varphi}\right)\right.$$
$$\left. + \sec(\lambda K_\varphi)\frac{(E^x)'}{E^\varphi}\left(\frac{\sin(\lambda K_\varphi)}{\lambda}\right)' + \frac{\sin(2\lambda K_\varphi)}{2\lambda}\left(\frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi}\right)\right], \tag{82}$$

which is again unique up to a choice of $B_x = B_x(E^x)$. This constraint reintroduces the divergence at $\lambda K_\varphi = \pi/2$ owing to the function $\tan(\lambda K_\varphi)$. The appearance of $\tan(\lambda K_\varphi)$ in the first line and $\sec(\lambda K_\varphi)$ in the $K_\varphi'$ term, complicates a promotion of $H_{\rm cc}^{(\rm A)}$ to an operator and a corresponding discussion of covariance at the full quantum level.

Another key difference between (82) and the Abelianized constraint of [18,19] is the lack of $K_x$ in the latter, which facilitates loop quantization as no radial holonomies are needed. However, imposing the covariance condition, radial holonomy modifications are not allowed in the present spherically reduced model, presenting an ongoing challenge to a complete loop quantization of black holes. To see this, we will analyze the general case of modified Hamiltonian constraints in spherically symmetric models, which may describe combinations of possible covariant versions of holonomy modifications with phase-space dependent linear combinations of hypersurface-deformation generators.

## IV. GENERAL MODIFIED HAMILTONIAN CONSTRAINTS

In the preceding section we have demonstrated that phase-space dependent linear combinations of the constraints can give rise to modified gravity theories. We used several simplifying assumptions, such as in the limited dependence of one of the linear coefficients, $B$, on the phase-space fields, for this demonstration. In the present section, we continue to work with the same models, given by the phase space and diffeomorphism constraint of spherical symmetry, but aim to derive a more general form of modified Hamiltonian constraints consistent with general covariance according to our new condition.

The results can be understood as modified theories of gravity in which the Hamiltonian constraint may be subject to a number of different modifications, motivated for instance by canonical approaches to quantum gravity. For full generality, one should then also allow for possible phase-space dependent linear combinations with the diffeomorphism constraint since it is not certain that a theory of quantum space-time would follow the classical separation into tangential and normal deformations of spacelike hypersurfaces. Such linear combinations also provide additional free functions compared with modifications of the Hamiltonian constraint by itself. As we will demonstrate, these free functions help to regain general covariance in an emergent space-time description of modified constraints.

In this way, we consider general modifications to the spherically symmetric theory with canonical variables $(K_\varphi, E^\varphi)$ and $(K_x, E^x)$, without introducing any additional degrees of freedom as they would be implied by higher time derivatives in the action. We therefore explore possibilities of modified gravity that do not require new degrees of freedom, reducing the danger of instabilities that might otherwise arise as in higher-curvature effective actions; see for instance [33].

Once we modify the Hamiltonian constraint in a specific way, the constraint brackets (39)–(41) determine the radial metric component via $\tilde{q}_{xx} = 1/|\tilde{q}^{xx}|$. As before, the angular component of the metric cannot be determined by the constraint brackets. For now, we include a generic expression $\tilde{q}_{\vartheta\vartheta} = \tilde{q}_{\vartheta\vartheta}(E^x)$ for the modified angular component. As before, the covariance condition for the angular component of the emergent metric implies (48) and (49), using $\delta_{\epsilon^0} E^x = -\delta\tilde{H}[\epsilon^0]/\delta K_x$.

The emergent space-time metric is then given by

$$\mathrm{d}s^2 = -\mathrm{sgn}(\tilde{q}^{xx})N^2\mathrm{d}t^2 + |\tilde{q}_{xx}|(\mathrm{d}x + N^r\mathrm{d}t)^2 + \tilde{q}_{\vartheta\vartheta}\mathrm{d}\Omega^2. \tag{83}$$

Since the sign of $\tilde{q}_{\vartheta\vartheta}$ is not strictly determined within a spherically symmetric theory, we assume that this metric component remains positive. The emergent space-time line element then follows the second option presented in Sec. II C, applied to the $(1+1)$-dimensional radial space-time. In this way, the four-dimensional space-time line element that includes the angular term does not have a distinguished spatial direction based on signature.

### A. Modified constraint brackets

We follow the procedure employed in [14,15], but consider an expanded version of the Hamiltonian constraint to include solutions not contained in these papers. We use the general ansatz

$$H = a_0 + ((E^x)')^2 a_{xx} + ((E^\varphi)')^2 a_{\varphi\varphi} + (E^x)'(E^\varphi)' a_{x\varphi} + (E^x)'' a_2 + (K_\varphi')^2 b_{\varphi\varphi} + (K_\varphi)'' b_2 + (E^x)' K_\varphi' c_{x\varphi} + (E^\varphi)' K_\varphi' c_{\varphi\varphi} + (E^\varphi)'' c_2 \tag{84}$$

for our Hamiltonian constraint, where $a_0$, $a_{ij}$, $a_2$, $b_{\varphi\varphi}$, $b_2$, $c_2$, $c_{ij}$ are all functions of the phase space variables, but not of their derivatives. (Here and from now on, we drop the tilde on $H$ with the understanding that we are dealing with modified constraints.) We have included terms quadratic in first-order radial derivatives and linear in second-order radial derivatives of all the phase space variables, except of $K_x$ because this would break covariance as demanded by (48). Terms linear in $K_\varphi'$, with coefficient $c_{x\varphi}$ or $c_{\varphi\varphi}$, may be viewed as derivative corrections, or as contributions from the diffeomorphism constraint in a phase-space dependent linear combination with the Hamiltonian constraint.

Starting from this ansatz we will obtain the conditions for it to satisfy the hypersurface-deformation brackets (39)–(41), possibly with a modified structure function $\tilde{q}^{xx}$.

#### 1. $\{H,H_r\}$ bracket

The bracket $\{H[N], H_r[N^r]\}$ can be written as

$$\{H[N], H_r[N^r]\} = \int \mathrm{d}x N^r[N\mathcal{F}_0 + N'\mathcal{F}_1 + N''\mathcal{F}_2] \tag{85}$$

using integration by parts to avoid derivatives of $N^r$. For this result to match (40), we set $\mathcal{F}_1 = \mathcal{F}_2 = 0$ and $\mathcal{F}_1 + H = 0$. Since all the free functions in the Hamiltonian constraint (84) are, by definition, independent of derivatives of the phase space variables, any terms in the equations implied by (85) that multiply different kinds of derivatives must vanish independently.

For a generic Hamiltonian constraint as used here, there is a rather large number of such equations; see Appendix B for more details. They imply that the constraint must have the form

$$H = -\sqrt{E^x}\frac{g}{2}\left[E^\varphi A_0 + \frac{((E^x)')^2}{E^\varphi}A_{xx} + \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} + \frac{(K_\varphi')^2}{E^\varphi}B_{\varphi\varphi} + \frac{(E^x)'K_\varphi'}{E^\varphi}C_{x\varphi} + \left(\frac{(E^\varphi)'K_\varphi'}{(E^\varphi)^2} - \frac{(K_\varphi)''}{E^\varphi}\right)C_{\varphi\varphi}\right], \tag{86}$$

where $A_0$, $A_{ij}$, $B_\varphi$, $C_{ij}$, and $g$ are free functions of $E^x$, $K_\varphi$, and $K_x/E^\varphi$. (The function $g$ has been factored out for convenience.)

#### 2. $\{H,H\}$ bracket

The bracket $\{H[N], H[M]\}$ can be written as

$$\{H[N], H[M]\} = \int \mathrm{d}x[(NM' - MN')(\mathcal{G}_0 - \mathcal{G}_1' + \mathcal{G}_2'') - (NM''' - MN''')\mathcal{G}_2], \tag{87}$$

where we used several integration by parts. Specific expressions for $\mathcal{G}_0$, $\mathcal{G}_1$, and $\mathcal{G}_2$ can be obtained from an explicit calculation of the Poisson bracket of two Hamiltonian constraints. At this stage, they are quite long, but some of the terms have direct implications that simplify the allowed dependence of coefficients on phase-space degrees of freedom. We will indicate the simplifying implications first and then proceed to more complicated terms.

For the bracket to match (41), we must set $\mathcal{G}_2 = 0$ and $\mathcal{G} \equiv \mathcal{G}_0 - \mathcal{G}_1' = H_r\tilde{q}^{xx}$ for some function $\tilde{q}^{xx}$ of density weight $-2$. The first one of these equations, collecting the highest derivative terms, implies

$$C_{\varphi\varphi}g^2\frac{E^x}{4(E^\varphi)^3}((E^x)' + K_\varphi' C_{\varphi\varphi}) = 0, \tag{88}$$

which, for a nontrivial Hamiltonian constraint, is solved only by $C_{\varphi\varphi} = 0$.

The second equation, $\mathcal{G} = \tilde{q}^{xx}H_r$ for some $\tilde{q}^{xx}$, can again be separated into terms multiplying different derivatives of the phase space variables. The terms

$$\mathcal{G} \supset -\frac{1}{4}E^x g(\partial g/\partial K_x)(E^x)''' + G^x K_x' + G_\varphi (E^\varphi)', \tag{89}$$

with

$$G^x = -\frac{E^\varphi}{K_x} G_\varphi = \frac{g^2 E^x}{4(E^\varphi)^2} \frac{\partial^2 A_0}{\partial (K_x/E^\varphi)^2}, \tag{90}$$

are the only ones that cannot contribute to the diffeomorphism constraint and must therefore vanish separately. The first term in (89) immediately implies that $g$ does not depend on $K_x$, and therefore it does not depend on $E^\varphi$ either because such a dependence could occur only in the combination $K_x/E^\varphi$.

In the remaining terms of (89), $G^x$ and $G_\varphi$ are phase-space functions that do not depend on spatial derivatives. Since they appear in the bracket of two Hamiltonian constraints via terms with only two spatial derivatives, one in $NM' - MN'$ and one in $K_x'$ or $(E^\varphi)'$, they can result only from a Poisson bracket of the first term in (86), proportional to $E^\varphi A_0$, with some of the other terms. The presence of $(E^x)''$ in any Hamiltonian constraint that has the correct classical limit implies that $A_0$ can be at most linear in $K_x$, as also seen directly from (90), such that any spatial derivative of $\{A_0, E^x\}$ taken after integrating by parts no longer produces terms with $K_x'$. To summarize this step, $g$ does not depend on $K_x/E^\varphi$ and $A_0$ is linear in $K_x/E^\varphi$:

$$g = g(E^x, K_\varphi) \quad \text{and} \quad A_0 = f_0 + \frac{K_x}{E^\varphi} f_1 \tag{91}$$

where $f_0$ and $f_1$ are free functions of $E^x$ and $K_\varphi$.

The remaining nonzero terms in $\mathcal{G}$. They are of the form

$$\begin{aligned}\mathcal{G} = {} & G^\varphi K_\varphi' + G_x (E^x)' + (Fx\varphi (K_\varphi)' + F_{xx}(E^x)')(E^x)'' + (F^{\varphi\varphi} K_\varphi' + Fx2\varphi (E^x)')K_\varphi'' \\ & + (G^{\varphi\varphi x}(K_\varphi')^2 + G_x^{x\varphi} K_\varphi' (E^x)' + G_{xx}^x ((E^x)')^2) K_x' + (G_\varphi^{\varphi\varphi}(K_\varphi')^2 + G_{x\varphi}^\varphi (E^x)' K_\varphi' + G_{xx\varphi}((E^x)')^2)(E^\varphi)' \\ & + G^{\varphi\varphi\varphi}(K_\varphi')^3 + G_x^{\varphi\varphi}(K_\varphi')^2 (E^x)' + G_{xx}^\varphi ((E^x)')^2 K_\varphi' + G_{xxx}((E^x)')^3.\end{aligned} \tag{92}$$

Explicit expressions for all coefficients are given in Appendix C. All terms can in principle contribute to the diffeomorphism constraint. It is therefore convenient to rearrange the terms according to

$$\begin{aligned}\mathcal{G} = {} & (\tilde{q}_0 + \tilde{q}_{2x}(E^x)'' + \tilde{q}_{2\varphi} K_\varphi'' + (\tilde{q}_1{}^{\varphi x} K_\varphi' + \tilde{q}_1{}^x_x (E^x)') K_x' \\ & + (\tilde{q}_1{}^\varphi_\varphi K_\varphi' + \tilde{q}_{1x\varphi}(E^x)')(E^\varphi)' + \tilde{q}_{2\varphi\varphi}(K_\varphi')^2 \\ & + \tilde{q}_{2x\varphi}(E^x)' K_\varphi' + \tilde{q}_{2xx}((E^x)')^2) H_r,\end{aligned} \tag{93}$$

where all the $q$ coefficients contribute to the structure function of the resulting hypersurface-deformation bracket.

In order to obtain the required factorization as a multiple of the diffeomorphism constraint, the terms in $\mathcal{G}$ must satisfy the relations

$$\frac{G^\varphi}{K_\varphi} = -\frac{G_x}{K_x}, \tag{94}$$

$$\frac{F_x^\varphi}{E^\varphi} = -\frac{F_{xx}}{K_x}, \tag{95}$$

$$\frac{F^{\varphi\varphi}}{E^\varphi} = -\frac{F_x^{2\varphi}}{K_x}, \tag{96}$$

$$\frac{G^{\varphi\varphi x}}{E^\varphi} = -a \frac{G_x^{x\varphi}}{K_x}, \tag{97}$$

$$(1-a)\frac{G_x^{x\varphi}}{E^\varphi} = -\frac{G_{xx}^x}{K_x}, \tag{98}$$

$$\frac{G_\varphi^{\varphi\varphi}}{E^\varphi} = -b \frac{G_{x\varphi}^\varphi}{K_x}, \tag{99}$$

$$(1-b)\frac{G_{x\varphi}^\varphi}{E^\varphi} = -\frac{G_{xx\varphi}}{K_x}, \tag{100}$$

$$\frac{G^{\varphi\varphi\varphi}}{E^\varphi} = -a_1 \frac{G_x^{\varphi\varphi}}{K_x}, \tag{101}$$

$$\tilde{b}_1 \frac{G_{xx}^\varphi}{E^\varphi} = -\frac{G_{xxx}}{K_x}, \tag{102}$$

$$(1-a_1)\frac{G_x^{\varphi\varphi}}{E^\varphi} = -(1-\tilde{b}_1)\frac{G_{xx}^\varphi}{K_x}, \tag{103}$$

where $a$, $a_1$, and $\tilde{b}_1$ are arbitrary functions of $E^x$, $K_\varphi$, and $K_x/E^\varphi$. The first three equations can be solved for some of the coefficients in (86):

$$\begin{aligned}B_{\varphi\varphi} = {} & \frac{1}{z}\frac{1}{2f_0}\left(\frac{\partial f_0}{\partial K_\varphi} + 2f_0 \frac{\partial \ln g}{\partial K_\varphi} - \frac{\partial f_1}{\partial E^x} + 2f_1 f_2\right) \\ & + \frac{1}{z^2}(A_{xx} - f_2),\end{aligned} \tag{104}$$

$$C_{x\varphi} = -\frac{\partial \ln g}{\partial K_\varphi} - \frac{2}{z}(A_{xx} - f_2), \tag{105}$$

where $z = K_x/E^\varphi$ and $f_2 = f_2(E^x, K_\varphi)$. The remaining relations remain quite long and complicated.

The structure function is now given by

$$\tilde{q}^{xx} = q_0 + q_{2x} + q_{2\varphi} + q_1{}^x + q_{1\varphi} + q_3 \tag{106}$$

with

$$q_0 = \frac{G^\varphi}{E^\varphi} = \frac{E^x g^2}{4(E^\varphi)^2}\left(2f_0 B_{\varphi\varphi} - f_1 C_{x\varphi} + \frac{\partial f_1}{\partial K_\varphi}\right), \quad (107)$$

$$q_{2x} = \frac{F^\varphi_x}{E^\varphi}(E^x)'' = \frac{E^x g^2 (E^x)''}{4(E^\varphi)^4 z^2}\frac{\partial C_{x\varphi}}{\partial z}, \quad (108)$$

$$q_{2\varphi} = \frac{F^{\varphi\varphi}}{E^\varphi}K''_\varphi = \frac{E^x g^2 K''_\varphi}{2(E^\varphi)^4}\frac{\partial B_{\varphi\varphi}}{\partial z}, \quad (109)$$

$$\begin{aligned} q_1{}^x &= \left(\frac{G^{\varphi\varphi x}}{E^\varphi}K'_\varphi - \frac{G^x_{xx}}{K_x}(E^x)'\right)K'_x, \\ &= \frac{E^x g^2 K'_x}{2(E^\varphi)^6 z^4}\left(A_{xx} - f_2 - z\frac{\partial A_{xx}}{\partial z} + \frac{z^2}{2}\frac{\partial^2 A_{xx}}{\partial z^2}\right) \\ &\quad\times\left(\frac{a}{1-a}E^\varphi K'_\varphi - K_x(E^x)'\right), \end{aligned} \quad (110)$$

$$\begin{aligned} q_{1\varphi} &= \left(\frac{G^{\varphi\varphi}_\varphi}{E^\varphi}K'_\varphi - \frac{G_{xx\varphi}}{K_x}(E^x)'\right)(E^\varphi)' \\ &= \frac{E^x g^2 (E^\varphi)'}{4(E^\varphi)^6 z}\frac{\partial^2 A_{xx}}{\partial z^2}\left(\frac{b}{1-b}E^\varphi K'_\varphi - K_x(E^x)'\right), \end{aligned} \quad (111)$$

and

$$\begin{aligned} q_3 &= \frac{G^{\varphi\varphi\varphi}}{E^\varphi}(K'_\varphi)^2 - b_1\frac{G^\varphi_{xx}}{K_x}(E^x)'K'_\varphi - \frac{G_{xxx}}{K_x}((E^x)')^2, \\ &= -\frac{E^x g^2}{2(E^\varphi)^6 z^4}\left(A_{xx}(A_{xx} - f_2) - \frac{\partial}{\partial E^x}(A_{xx} - f_2) + \frac{z^2}{2}\frac{\partial \ln g}{\partial K_\varphi}\frac{\partial A_{xx}}{\partial z}\right. \\ &\quad\left. + \frac{z}{2}\left(\frac{\partial \ln g}{\partial K_\varphi}\frac{\partial \ln g}{\partial E^x} - \frac{1}{g}\frac{\partial^2 g}{\partial K_\varphi \partial E^x} + \frac{\partial^2 A_{xx}}{\partial E^x \partial z} - 2f_2\frac{\partial A_{xx}}{\partial z} - \frac{\partial A_{xx}}{\partial K_\varphi}\right)\right) \\ &\quad\times\left(\frac{a_1}{1-a_1}\frac{1-\tilde{b}_1}{\tilde{b}_1}(E^\varphi)^2(K'_\varphi)^2 - \frac{1-\tilde{b}_1}{\tilde{b}_1}E^\varphi K_x K'_\varphi (E^x)' + K_x^2((E^x)')^2\right). \end{aligned} \quad (112)$$

### B. Covariance condition

Now that we have the structure function (106), we can apply the covariance condition (49) which in our case is nontrivial up to the third-order derivative of the gauge function.

The covariance condition must be evaluated on shell. In particular, one has to pay attention to the on-shell property $H_r = 0$, which implies $E^\varphi K'_\varphi = K_x(E^x)'$ and can mix some derivative terms that were independent in the off-shell treatment so far. In order to solve all the covariance conditions, it is best to focus first on the highest derivative terms of each one.

The highest-order derivative term of the first covariance condition is

$$\begin{aligned} \left.\frac{\partial(\delta_\epsilon q^{xx})}{\partial(\epsilon^0)'}\right|_{\text{O.S.}} &\supset \frac{g^3(E^x)^{3/2}}{8(E^\varphi)^3 K_x^3 f_0}(E^x)''' \\ &\quad\times\left(2gf_2(2f_0 - zf_1) - 4f_0 g A_{xx} - 2f_0 z\frac{\partial g}{\partial K_\varphi}\right. \\ &\quad\left. + gz\left(2f_0\frac{\partial A_{xx}}{\partial z} + \frac{\partial f_1}{\partial E^x} - \frac{\partial f_0}{\partial K_\varphi}\right)\right). \end{aligned} \quad (113)$$

The on-shell condition does not affect this equation, which has the solution

$$\begin{aligned} A_{xx} &= f_2 - \frac{z}{2f_0}\left(2f_0\frac{\partial \ln g}{\partial K_\varphi} + \frac{\partial f_0}{\partial K_\varphi} - \frac{\partial f_1}{\partial E^x} + 2f_1 f_2\right) \\ &\quad + z^2 f_3 \end{aligned} \quad (114)$$

where $f_3 = f_3(E^x, K_\varphi)$. With this result for $A_{xx}$, the third covariance condition,

$$\begin{aligned} \frac{\partial(\delta_\epsilon q^{xx})}{\partial(\epsilon^0)'''} &= \frac{g^3(E^x)^{3/2}}{4(E^\varphi)^2 K_x^4}\left(-(E^x)'\left(2zA_{xx} - 2zf_2 + z^2\frac{\partial \ln g}{\partial K_\varphi}\right.\right. \\ &\quad\left. + \frac{z^2}{2f_0}\left(2f_1 f_2 + \frac{\partial f_0}{\partial K_\varphi} - \frac{\partial f_1}{\partial E^x}\right) - z^2\frac{\partial A_{xx}}{\partial z}\right) \\ &\quad + \left(\frac{a}{1-a}K'_\varphi - z(E^x)'\right) \\ &\quad\left.\times\left(A_{xx} - f_2 - z\frac{\partial A_{xx}}{\partial z} + \frac{z^2}{2}\frac{\partial^2 A_{xx}}{(\partial z)^2}\right)\right), \end{aligned} \quad (115)$$

vanishes. The highest-order derivative term of the second covariance condition is

$$\frac{\partial(\delta_\epsilon q^{xx})}{\partial(\epsilon^0)''} \supset \frac{g(E^x)^{3/2}}{E^\varphi}(E^\varphi K''_\varphi - K_x(E^x)'')f_3, \quad (116)$$

and we conclude that $f_3 = 0$.

We now go back to some of the anomaly-freedom equations. We note that we can write Eq. (102) in powers of $z$, which terminates at second order because of the quadratic dependence on $K_x$. The nonvanishing powers are

$$zG^{\varphi}_{xx} = zG^{\varphi(1)}_{xx} \tag{117}$$

and

$$G_{xxx} = G^{(0)}_{xxx} + zG^{(1)}_{xxx}, \tag{118}$$

where all $G^{\varphi}_{xx}{}^{(i)}$ and $G^{(i)}_{xxx}$ are independent of $z$. Thus, the anomaly-freedom equation (102) requires $\widetilde{b}_1 = 1$ and $G^{(0)}_{xxx} = 0$, which in turn implies

$$0 = f_0^2\left[\frac{\partial}{\partial E^x}\left(\frac{\partial \ln g}{\partial K_\varphi} - C_{x\varphi}\right) + 2\frac{\partial f_2}{\partial K_\varphi}\right]. \tag{119}$$

We note that with $\tilde{b}_1 = 1$, the function $a_1$ drops out of all equations. Using these results we obtain $B_{\varphi\varphi} = 0$, and the anomaly-freedom equations (97)–(101) and (103) are automatically satisfied.

The remaining equations are given by lower-order derivatives of $E^x$ in

$$\frac{\partial}{\partial(\epsilon^0)'}\delta_\epsilon q^{xx}|_{\text{O.S.}} = 0,$$

which has only two nonzero terms, with $(E^x)'$ and with $((E^x)')^3$, that must vanish separately. The vanishing of the $(E^x)'$ term implies

$$0 = \frac{\partial^2 \ln g}{(\partial K_\varphi)^2} f_1 - 2\frac{\partial \ln g}{\partial K_\varphi}\frac{\partial f_1}{\partial K_\varphi} - \frac{\partial^2 f_1}{(\partial K_\varphi)^2} - \frac{\partial(f_1 C_{x\varphi})}{\partial K_\varphi} + 3f_1\left(C_{x\varphi}^2 + \frac{\partial C_{x\varphi}}{\partial K_\varphi} + \frac{\partial \ln g}{\partial K_\varphi}C_{x\varphi}\right), \tag{120}$$

and the vanishing of the $((E^x)')^3$ term implies

$$0 = \left(\frac{\partial \ln g}{\partial K_\varphi} + 3C_{x\varphi}\right)\left(\frac{\partial^2 \ln g}{(\partial K_\varphi)^2} + C_{x\varphi}^2 + \frac{\partial C_{x\varphi}}{\partial K_\varphi} + \frac{\partial \ln g}{\partial K_\varphi}C_{x\varphi}\right) + \frac{\partial}{\partial K_\varphi}\left(\frac{\partial^2 \ln g}{\partial K_\varphi^2} + C_{x\varphi}^2 + \frac{\partial C_{x\varphi}}{\partial K_\varphi} + \frac{\partial \ln g}{\partial K_\varphi}C_{x\varphi}\right). \tag{121}$$

This exhausts all the anomaly-freedom equations (94)–(103). Using these values and condition (119), one can check that the covariance condition $(\partial\delta_\epsilon q^{xx}/\partial(\epsilon^0)'')|_{\text{O.S.}} = 0$ is automatically satisfied.

To summarize, the general form of an anomaly-free and covariant Hamiltonian constraint is

$$H = -\sqrt{E^x}\frac{g}{2}\left[E^\varphi\left(f_0 + \frac{K_x}{E^\varphi}f_1\right) + \frac{((E^x)')^2}{E^\varphi}\left(f_2 - \frac{1}{2}\frac{K_x}{E^\varphi}\left(\frac{\partial \ln g}{\partial K_\varphi} + C_{x\varphi}\right)\right) + \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} + \frac{(E^x)'K_\varphi'}{E^\varphi}C_{x\varphi}\right], \tag{122}$$

where $C_{x\varphi}$ is given by

$$C_{x\varphi} = \frac{1}{f_0}\left(f_0\frac{\partial \ln g}{\partial K_\varphi} + \frac{\partial f_0}{\partial K_\varphi} - \frac{\partial f_1}{\partial E^x} + 2f_1 f_2\right), \tag{123}$$

and $g$, $f_0$, $f_1$, and $f_2$ are functions of $E^x$ and $K_\varphi$ that must satisfy Eqs. (119)–(121). The structure function is

$$\tilde{q}^{xx} = \left(\frac{\partial f_1}{\partial K_\varphi} - f_1 C_{x\varphi} - \frac{1}{2}\left(\frac{\partial^2 \ln g}{(\partial K_\varphi)^2} + C_{x\varphi}^2 + \frac{\partial C_{x\varphi}}{\partial K_\varphi} + \frac{\partial \ln g}{\partial K_\varphi}C_{x\varphi}\right)\left(\frac{(E^x)'}{E^\varphi}\right)^2\right)\frac{g^2}{4}\frac{E^x}{(E^\varphi)^2}. \tag{124}$$

Unlike the structure function found in the preceding section by using only a phase-space dependent linear combination of the constraints but no further modification, this structure function is not guaranteed to be positive. Suitable sign choices are therefore necessary when using the inverse of this structure function in an emergent spacetime line element, as discussed in Sec. II C.

### C. Applying canonical transformations

By directly solving the required conditions, it can be shown that the set of canonical transformations preserving the diffeomorphism constraint (38) and leaving the variable $E^x$ invariant must have the form

$$K_\varphi = f_c(\tilde{E}^x, \tilde{K}_\varphi), \qquad E^\varphi = \tilde{E}^\varphi\left(\frac{\partial f_c}{\partial \tilde{K}_\varphi}\right)^{-1},$$
$$K_x = \tilde{K}_x + \tilde{E}^\varphi\frac{\partial f_c}{\partial \tilde{E}^x}\left(\frac{\partial f_c}{\partial \tilde{K}_\varphi}\right)^{-1}, \qquad E^x = \tilde{E}^x, \tag{125}$$

where the new variables are written with a tilde. This canonical transformation can be generalized by noting that the transformation

$$K_x = \frac{\partial(\alpha^2 E^x)}{\partial E^x}\tilde{K}_x, \qquad \tilde{E}^x = \alpha^2 E^x \tag{126}$$

is canonical and preserves the diffeomorphism constraint too, where $\alpha = \alpha(E^x)$. We will be using a combination of these as a subset of all diffeomorphism-preserving canonical transformations:

$$K_\varphi = f_c(E^x, \tilde{K}_\varphi),$$
$$E^\varphi = \tilde{E}^\varphi \left(\frac{\partial f_c}{\partial \tilde{K}_\varphi}\right)^{-1},$$
$$K_x = \frac{\partial(\alpha_c^2 E^x)}{\partial E^x}\tilde{K}_x + \tilde{E}^\varphi \frac{\partial f_c}{\partial E^x}\left(\frac{\partial f_c}{\partial \tilde{K}_\varphi}\right)^{-1},$$
$$\tilde{E}^x = \alpha_c^2(E^x)E^x, \tag{127}$$

with type-3 generating function

$$F_3(E^x, E^\varphi, \tilde{K}_\varphi, \tilde{K}_x) = -f_c(E^x, \tilde{K}_\varphi)E^\varphi - \alpha_c^2(E^x)E^x\tilde{K}_x. \tag{128}$$

Let us now consider the Hamiltonian constraint (122) and perform a canonical transformation of the form (127). By focusing on the $(E^x)'(E^\varphi)'$ and $(E^x)''$ terms, we see that the global factor transforms from $g(E^x, K_\varphi)$ to $g_{\rm cc}(E^x, K_\varphi)$, where

$$g_{\rm cc}(E^x, K_\varphi) = g_{\rm (old)}(E^x/\alpha_c^2, f_c)\left(1 - \frac{\partial \ln \alpha_c^2}{\partial E^x}\right)\frac{1}{\alpha_c^3}\frac{\partial f_c}{\partial K_\varphi}. \tag{129}$$

We then see that the $C_{x\varphi}(E^x, K_\varphi)$ coefficient transforms to

$$C_{x\varphi}^{\rm cc}(E^x, K_\varphi) = -\left(\frac{\partial f_c}{\partial K_\varphi}\right)^{-1}\frac{\partial^2 f_c}{\partial K_\varphi^2} + \frac{\partial f_c}{\partial K_\varphi}C_{x\varphi}^{\rm (old)}(E^x/\alpha_c^2, f_c(E^x, K_\varphi)). \tag{130}$$

By setting $f_c = f_c(K_\varphi)$ and $\alpha_c = 1$, it is therefore possible to find, at least locally in phase space, a diffeomorphism-preserving canonical transformation by solving an ordinary differential equation, such that $C_{x\varphi}^{\rm cc} = 0$. After such a canonical transformation, the Hamiltonian constraint and the structure function simplify by setting $C_{x\varphi} = 0$.

Moreover, any modified angular component of the form $\tilde{q}_{\vartheta\vartheta} = \alpha_c^{-2}(E^x)E^x$ can be mapped to its classical form, $\tilde{q}_{\vartheta\vartheta} \to E^x$, by using the canonical transformation (127) with $f_c = K_\varphi$ and the necessary $\alpha_c$, which preserves $C_{x\varphi} = 0$. With these choices, the residual canonical transformation has the form

$$K_\varphi \to f_x K_\varphi - \tilde{\mu}_\varphi, \qquad E^\varphi \to \frac{E^\varphi}{f_x},$$
$$K_x \to K_x + E^\varphi K_\varphi\left(\frac{\partial \ln f_x}{\partial E^x} - \frac{1}{f_x}\frac{\partial \tilde{\mu}_\varphi}{\partial E^x}\right), \qquad E^x \to E^x, \tag{131}$$

where $f_x = f_x(E^x)$ and $\tilde{\mu}_\varphi = \tilde{\mu}_\varphi(E^x)$.

Because the anomaly-freedom equations and the covariance condition are all based on Poisson brackets, canonical transformations leave them form invariant. Therefore, $C_{x\varphi} = 0$ becomes a new condition on the free functions through (123), greatly simplifying the remaining equations. This observation allows us to obtain exact solutions to all anomaly-freedom and covariance conditions as follows: We first solve Eq. (121) for $g$, then solve Eq. (120) for $f_1$, Eq. (119) for $f_2$, and Eq. (123), set equal to zero, for $f_0$.

For future convenience we write the residual canonical transformation of all terms in the Hamiltonian constraint according to (131) as

$$g_{\rm cc} = g(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)f_x, \tag{132}$$

$$g_{\rm cc}f_0^{\rm cc} = g(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)\left(\frac{f_0(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)}{f_x} + \left(\frac{\partial \ln f_x}{\partial E^x} - \frac{1}{f_x}\frac{\partial \tilde{\mu}_\varphi}{\partial E^x}\right)K_\varphi f_1(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)\right), \tag{133}$$

$$g_{\rm cc}f_1^{\rm cc} = g(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)f_1(E^x, f_x K_\varphi - \tilde{\mu}_\varphi), \tag{134}$$

$$g_{\rm cc}f_2^{\rm cc} = g(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)f_x\Bigg(f_2(E^x, f_x K_\varphi - \tilde{\mu}_\varphi) - K_\varphi\left(\frac{\partial \ln f_x}{\partial E^x} - \frac{1}{f_x}\frac{\partial \tilde{\mu}_\varphi}{\partial E^x}\right)\frac{1}{2}\frac{\partial \ln g(E^x, f_x K_\varphi - \tilde{\mu}_\varphi)}{\partial K_\varphi} - \left(\frac{\partial \ln f_x}{\partial E^x} - \frac{1}{f_x}\frac{\partial \tilde{\mu}_\varphi}{\partial E^x}\right)\Bigg). \tag{135}$$

### D. Classical limit

When solving the anomaly-freedom and covariance equations as outlined above with $C_{x\varphi} = 0$ we should keep in mind that the classical constraint, (37), must be recovered in an appropriate limit. The general solution to (121) is given by

$$g = \lambda_0 \cos^2(\lambda(K_\varphi + \mu_\varphi)), \tag{136}$$

where $\lambda_0$, $\lambda$, and $\mu_\varphi$ are free functions of $E^x$, and its classical limit is $g \to 1$ as $\lambda_0 \to 1$, $\lambda \to 0$. Using this, the general solution to (120), compatible with the classical limit, is

$$gf_1 = 4\lambda_0\left(c_f\frac{\sin(2\lambda(K_\varphi + \mu_\varphi))}{2\lambda} + q\cos(2\lambda(K_\varphi + \mu_\varphi))\right), \tag{137}$$

where $c_f$ and $q$ are free functions of $E^x$, and its classical limit is $f_1 \to 4K_\varphi$ as $\lambda_0, c_f \to 1$ and $\lambda, \mu_\varphi, q \to 0$. The general solution to (119), compatible with the classical limit, is then

$$f_2 = -\frac{\alpha_2}{4E^x} + \frac{\sin(2\lambda(K_\varphi + \mu_\varphi))}{2\lambda\cos^2(\lambda(K_\varphi + \mu_\varphi))}\left(\lambda\frac{\partial(\lambda\mu_\varphi)}{\partial E^x} + \lambda K_\varphi\frac{\partial\lambda}{\partial E^x}\right), \tag{138}$$

where $\alpha_2 = \alpha_2(E^x)$, and its classical limit is $f_2 \to -1/(4E^x)$ as $\alpha_2 \to 1$. The general solution to (123) equals

$$\begin{aligned} gf_0 = \lambda_0 \Bigg( & c_{f0} + \frac{\alpha_0}{E^x} + 2\frac{\sin^2(\lambda(K_\varphi+\mu_\varphi))}{\lambda^2}\frac{\partial c_f}{\partial E^x} + 4\frac{\sin(2\lambda(K_\varphi+\mu_\varphi))}{2\lambda}\frac{\partial q}{\partial E^x} \\ & + 4c_f\left(\frac{1}{\lambda}\frac{\partial(\lambda\mu_\varphi)}{\partial E^x}\frac{\sin(2\lambda(K_\varphi+\mu_\varphi))}{2\lambda} + \left(\frac{\alpha_2}{4E^x} - \frac{\partial\ln\lambda}{\partial E^x}\right)\frac{\sin^2(\lambda(K_\varphi+\mu_\varphi))}{\lambda^2}\right) \\ & + 8q\left(-\lambda\frac{\partial(\lambda\mu_\varphi)}{\partial E^x}\frac{\sin^2(\lambda(K_\varphi+\mu_\varphi))}{\lambda^2} + \left(\frac{\alpha_2}{4E^x} - \frac{1}{2}\frac{\partial\ln\lambda}{\partial E^x}\right)\frac{\sin(2\lambda(K_\varphi+\mu_\varphi))}{2\lambda}\right) \\ & + 4K_\varphi\frac{\partial\ln\lambda}{\partial E^x}\left(c_f\frac{\sin(2\lambda(K_\varphi+\mu_\varphi))}{2\lambda} + q\cos(2\lambda(K_\varphi+\mu_\varphi))\right)\Bigg), \end{aligned} \tag{139}$$

where $c_{f0}$ and $\alpha_0$ are undetermined functions of $E^x$. (They can be combined to a single free function, but it is convenient to separate them for the purpose of taking the classical limit.) Its classical limit is $gf_0 \to 1/E^x$ as $\alpha_0, \alpha_2, \lambda_0, c_f \to 1$, $\lambda, q, \mu_\varphi$.

These results completely determine the anomaly-free, covariant Hamiltonian constraint and the structure function for vacuum up to the undetermined functions of $E^x$. The classical constraint (37) can be recovered in different limits. The most straightforward way is to set all the parameters constant and then take the limits $\lambda_0, c_f, \alpha_i \to 1$ and $\lambda, c_{f0}, c_{f2}, \mu_\varphi \to 0$. The cosmological constant can be recovered by instead setting $c_{f0} \to -\Lambda$.

### E. Periodicity and bounded-curvature effects

The additional restrictions on consistent modified Hamiltonian constraints, implied by the covariance condition, allow us to clarify the question of possible functional dependences of the constraint on $K_\varphi$, with various properties of relevance for models of loop quantum gravity.

The parameter $\lambda$ can heuristically be interpreted as the holonomy angular length in models of loop quantum gravity. We may therefore restrict its form by referring to specific triangulations of space. For example, we can choose a fine lattice such that the spheres are triangulated by small squares of side length $\lambda$. Each plaquette at radius $\sqrt{E^x}$ then covers an area $E^x\lambda^2$. Requiring that the plaquettes at different radii have equal sizes, we obtain

$$\lambda = \frac{\bar{r}}{\sqrt{E^x}}\bar{\lambda}, \tag{140}$$

where $\bar{r}$ is a constant reference radius at which $\lambda = \bar{\lambda}$. This result satisfies $\lambda \to 0$ as $E^x \to \infty$, as desired to recover the classical limit at large distances.

Furthermore, the Hamiltonian constraint obtained from the previous results would be nonperiodic in $K_\varphi$ for nonconstant $\lambda$, owing to the last term in (138) and (139). In models of loop quantum gravity, a Hamiltonian periodic in $K_\varphi$ (if not $K_x$, which is harder to achieve) is often desired in order to motivate a well-defined quantization in a representation of the holonomy-flux algebra. In addition to being quite restricted by the covariance condition, such periodicity properties are not invariant under canonical transformations. The canonical transformations (132) can be used to reestablish periodicity if we start with a nonconstant $\lambda$ for which the last term in (138) and (139) is not periodic.

Upon such a transformation with $\tilde{\mu}_\varphi = \mu_\varphi$, the Hamiltonian constraint terms become

$$g_{\rm cc} = \lambda_0 f_x \cos^2(\lambda f_x K_\varphi), \tag{141}$$

$$\begin{aligned} g_{\rm cc}f_0^{\rm cc} = \frac{\lambda_0}{f_x}\Bigg( & c_{f0} + \frac{\alpha_0}{E^x} + 2\frac{\sin^2(\lambda f_x K_\varphi)}{\lambda^2}\frac{\partial c_f}{\partial E^x} + 4\frac{\sin(2\lambda f_x K_\varphi)}{2\lambda}\frac{\partial q}{\partial E^x} + 4c_f\left(\frac{\alpha_2}{4E^x} - \frac{\partial\ln\lambda}{\partial E^x}\right)\frac{\sin^2(\lambda f_x K_\varphi)}{\lambda^2} \\ & + 8q\left(\frac{\alpha_2}{4E^x} - \frac{1}{2}\frac{\partial\ln\lambda}{\partial E^x}\right)\frac{\sin(2\lambda f_x K_\varphi)}{2\lambda}\Bigg) + \frac{\partial\ln(\lambda f_x)}{\partial E^x}4\lambda_0 K_\varphi\left(c_f\frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} + q\cos(2\lambda f_x K_\varphi)\right), \end{aligned} \tag{142}$$

$$g_{\rm cc}f_1^{\rm cc} = 4\lambda_0\left(c_f\frac{\sin(2\lambda f_x K_\varphi)}{2\lambda} + q\cos(2\lambda f_x K_\varphi)\right), \tag{143}$$

$$g_{\rm cc}f_2^{\rm cc} = \lambda_0 f_x\cos^2(\lambda f_x K_\varphi)\left(-\frac{\alpha_2}{4E^x} - \frac{\partial\ln f_x}{\partial E^x} + \frac{\sin(2\lambda f_x K_\varphi)}{2\lambda\cos^2(\lambda f_x K_\varphi)}\lambda^2 f_x K_\varphi\frac{\partial\ln(\lambda f_x)}{\partial E^x}\right). \tag{144}$$

The phase $\mu_\varphi$ then drops out of the Hamiltonian constraint. Furthermore, the constraint is rendered periodic and bounded in $K_\varphi$ by choosing

$$f_x = \frac{\bar{\lambda}}{\lambda}, \tag{145}$$

where $\bar{\lambda}$ is a constant. However, this transformation also removes any $E^x$ dependence in coefficients of $K_\varphi$ in periodic functions.

Building upon the constraint obtained by the preceding canonical transformation, a further simplification consists in redefining the remaining parameters according to

$$\lambda_0 \to \lambda_0 \frac{\lambda}{\bar{\lambda}}, \qquad q \to q \frac{\bar{\lambda}}{\lambda}, \qquad c_{f0} \to \frac{\bar{\lambda}^2}{\lambda^2} c_{f0},$$
$$\alpha_0 \to \frac{\bar{\lambda}^2}{\lambda^2} \alpha_0, \qquad \alpha_2 \to \alpha_2 + 4E^x \frac{\partial \ln \lambda}{\partial E^x}. \tag{146}$$

Under these redefinitions, the most general Hamiltonian constraint, up to canonical transformations, is of the form

$$\begin{aligned} H = & -\lambda_0 \frac{\sqrt{E^x}}{2} \left[ E^\varphi \left( c_{f0} + \frac{\alpha_0}{E^x} + 2 \frac{\sin^2(\bar{\lambda} K_\varphi)}{\bar{\lambda}^2} \frac{\partial c_f}{\partial E^x} + 4 \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} \frac{\partial q}{\partial E^x} + 4 c_f \frac{\alpha_2}{4E^x} \frac{\sin^2(\bar{\lambda} K_\varphi)}{\bar{\lambda}^2} + 8 q \frac{\alpha_2}{4E^x} \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} \right) \right. \\ & + 4 K_x \left( c_f \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} + q \cos(2\bar{\lambda} K_\varphi) \right) + \frac{((E^x)')^2}{E^\varphi} \left( -\frac{\alpha_2}{4E^x} \cos^2(\bar{\lambda} K_\varphi) + \frac{K_x}{E^\varphi} \bar{\lambda}^2 \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} \right) \\ & \left. + \left( \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \cos^2(\bar{\lambda} K_\varphi) \right], \\ = & -\lambda_0 \frac{\sqrt{E^x}}{2} E^\varphi \left[ c_{f0} + \frac{\alpha_0}{E^x} + 2 \frac{\sin^2(\bar{\lambda} K_\varphi)}{\bar{\lambda}^2} \frac{\partial c_f}{\partial E^x} + 4 \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} \frac{\partial q}{\partial E^x} + \frac{\alpha_2}{\bar{\lambda}^2 E^x} c_f - \frac{(E^\varphi)^2}{\lambda_0^2 \bar{\lambda}^2 E^x} \left( \frac{\alpha_2}{E^x} \tilde{q}^{xx} + 2 \frac{K_x}{E^\varphi} \frac{\partial \tilde{q}^{xx}}{\partial K_\varphi} \right) \right. \\ & \left. - \left( \frac{(E^x)'((E^\varphi)^{-2})'}{2} + \frac{(E^x)''}{(E^\varphi)^2} \right) \cos^2(\bar{\lambda} K_\varphi) \right], \end{aligned} \tag{147}$$

with structure function

$$\tilde{q}^{xx} = \left( \left( c_f + \left( \frac{\bar{\lambda}(E^x)'}{2E^\varphi} \right)^2 \right) \cos^2(\bar{\lambda} K_\varphi) - 2q\bar{\lambda}^2 \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} \right) \times \lambda_0^2 \frac{E^x}{(E^\varphi)^2}. \tag{148}$$

The function $\lambda$ has been completely absorbed by the other parameters. Thus, we conclude that nonconstant $\lambda$ can always be traded in for nonclassical functions for the other parameters but not in an obvious way. Within the setting of modified gravity, there is no invariant meaning to specific $E^x$ dependencies or periodicity conditions in holonomies. The classical constraint is recovered in the limit $\lambda_0, c_f, \alpha_i \to 1$ and $\bar{\lambda}, c_{f0}, c_{f2} \to 0$. The cosmological constant can be recovered by instead setting $c_{f0} \to -\Lambda$. In the following sections we drop the bar in $\bar{\lambda}$ and write this constant parameter as $\lambda$ for simplicity.

The case $\lambda_0, c_f, \alpha_i \to 1$ and $c_{f0}, q \to 0$ for (147) was first found in [14] by demanding anomaly freedom and, since they had no knowledge of the covariance condition derived here, some functions were only proposed and the rest obtained by solving the anomaly-freedom equations for a less general constraint than (84). They also chose constant $\lambda$ in order to have a constraint periodic in $K_\varphi$.

In [14], it was shown that the Hamiltonian constraint in this case is the result of a specific linear combination of the classical constraints with phase-space dependent coefficients, after performing the diffeomorphism-constraint preserving canonical transformation

$$K_\varphi \to \frac{\sin(\lambda K_\varphi)}{\lambda}, \qquad E^\varphi \to \frac{E^\varphi}{\cos(\lambda K_\varphi)}. \tag{149}$$

The emergent space-time it implies was studied in [16], where it was shown that the classical singularity does not appear in a black-hole-like solution. Our analysis demonstrates that this outcome is an implication of the phase-space dependent linear combination, rather than of periodic and bounded functions in the Hamiltonian constraint. Our derivations strengthen this result by showing in Sec. III C that this constraint is the unique covariant linear combination of the constraints up to an overall function of $E^x$.

In the constraint (147), the terms containing $q$ are the only modifications with nontrivial holonomy effects allowed by anomaly freedom and covariance that have not previously been considered. However, they do not seem directly related to holonomy corrections in any obvious way as some of them survive the limit $\lambda \to 0$. Among all the modification functions found here, $c_f$, $\lambda$, $q$, and $\lambda_0$ are the most characteristic of emergent modified gravity: If we require the classical form of the structure function, these functions must all take their classical expressions. The freedom of choosing the remaining functions is related to the emergent metric only through their dependence on a nonconstant $\lambda$, but the freedom expressed by these functions appears even classically in the terms of a $(1+1)$-dimensional dilaton action. The same argument shows that the constraints depend at most quadratically on $K_\varphi$ unless a nonclassical emergent metric is considered, which requires $\lambda \neq 0$.

### F. General partial Abelianization

In a combination of our preceding results, we can now use linear combinations of the form $H^{(\mathrm{new})} = BH + AH_r$, where $A$ and $B$ are phase-space dependent functions as in Sec. III C, but insert the general modified Hamiltonian constraint $H$ from the present section, given in Eq. (147). Since the Hamiltonian constraint is general, this procedure should not result in new covariant theories, but we can use the construction as in Sec. III D and impose conditions other than general covariance on the resulting structure function $\tilde{q}^{xx}_{(\mathrm{new})}$. As an example, we derive new partial Abelianizations by requiring that the new structure function vanish. The resulting theory would be compatible with general covariance because this condition has been implemented on the original Hamiltonian constraint, and it might be more amenable to quantizations using, for instance, the loop representation because structure functions have been eliminated in the partial Abelianization.

Assuming some function $B = B(E^x, K_\varphi)$, the second function $A$ in the linear combination and the structure function are uniquely determined, now given by

$$A = -\lambda_0 \cos^2(\bar{\lambda} K_\varphi) \frac{\sqrt{E^x}(E^x)'}{2(E^\varphi)^2} \frac{\partial B}{\partial K_\varphi}, \tag{150}$$

and

$$\begin{aligned}\tilde{q}^{xx}_{(\mathrm{new})} = B \frac{E^x}{(E^\varphi)^2} \frac{\lambda_0 \cos(\lambda K_\varphi)}{2} \Bigg( & 2\lambda_0 B \left( c_f \cos(\bar{\lambda} K_\varphi) - 2\lambda^2 q \frac{\sin(\bar{\lambda} K_\varphi)}{\bar{\lambda}} \right) \\ & - 2\lambda_0 \frac{\partial B}{\partial K_\varphi} \cos(\bar{\lambda} K_\varphi) \left( c_f \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} + \frac{q}{2} (\cos(\lambda K_\varphi) + \cos(3\lambda K_\varphi)) \right) \\ & + \frac{((E^x)')^2}{(E^\varphi)^2} \lambda_0 \cos(\bar{\lambda} K_\varphi) \left( B\bar{\lambda}^2 - \frac{\partial B}{\partial K_\varphi} 3\bar{\lambda}^2 \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} + \frac{\partial^2 B}{(\partial K_\varphi)^2} \cos^2(\bar{\lambda} K_\varphi) \right) \Bigg)\end{aligned} \tag{151}$$

if we apply the procedure to the general expression of a modified Hamiltonian constraint.

Partial Abelianization requires $\tilde{q}^{xx}_{(\mathrm{new})} = 0$. Because $B$ is assumed to be independent of $(E^x)'$, the first two lines of (65) must vanish separately, such that

$$B = \frac{B_x}{\cos^2(\bar{\lambda} K_\varphi)} \left( c_f \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} + q \cos(2\bar{\lambda} K_\varphi) \right), \tag{152}$$

where $B_x = B_x(E^x)$. Inserting this result in the last two lines, the condition that they vanish too implies

$$B_x \bar{\lambda} q = 0. \tag{153}$$

For a nontrivial Abelianization with $\bar{\lambda} \neq 0$, this is realized only if $q = 0$. We arrive at the Abelianized constraint

$$\begin{aligned}\frac{H^{(\mathrm{A})}}{B_x} = -\frac{\sqrt{E^x}}{2} \frac{\tan(\bar{\lambda} K_\varphi)}{\bar{\lambda}} \Bigg[ & E^\varphi \left( c_{f0} + \frac{\alpha_0}{E^x} + 2 \frac{\sin^2(\bar{\lambda} K_\varphi)}{\bar{\lambda}^2} \frac{\partial c_f}{\partial E^x} + c_f \frac{\alpha_2}{E^x} \frac{\sin^2(\bar{\lambda} K_\varphi)}{\bar{\lambda}^2} \right) \\ & + 4 K_x c_f \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} + \frac{((E^x)')^2}{E^\varphi} \left( -\frac{\alpha_2}{4E^x} \cos^2(\bar{\lambda} K_\varphi) + \frac{K_x}{E^\varphi} 2\bar{\lambda}^2 \frac{\sin(2\bar{\lambda} K_\varphi)}{2\bar{\lambda}} \right) \\ & + \left( \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi} \right) \cos^2(\bar{\lambda} K_\varphi) \Bigg] - \frac{\sqrt{E^x}(E^x)'}{2(E^\varphi)^2} (E^\varphi K_\varphi' - K_x (E^x)'),\end{aligned} \tag{154}$$

where we redefined $B_x$ in order to absorb an overall term $c_f \lambda_0$. This result agrees with the partial Abelianization obtained in Sec. III D if we choose all parameters except for $\bar{\lambda}$ to take classical values.

The Abelianized constraint $H^{(\mathrm{A})}$ has a divergence at $\bar{\lambda} K_\varphi = \pi/2$ implied by the overall $\tan(\bar{\lambda} K_\varphi)$ multiplying the terms of the first line. Looking at the first three terms, the divergence can be resolved by setting

$$\frac{\partial c_f}{\partial E^x} = -\frac{\bar{\lambda}^2}{2}\left(c_{f0} + \frac{\alpha_0}{E^x}\right), \tag{155}$$

which is easily solved for $c_f$ if we use the classical values $\alpha_0 = 1$ and $c_{f0} = -\Lambda$:

$$c_f = 1 + \frac{\bar{\lambda}^2}{2}(\Lambda E^x - \ln(E^x/c_0)), \tag{156}$$

where $c_0$ is the integration constant. This nonclassical form of the function $c_f$, depending on $\bar{\lambda}$, may be considered an indirect effect of nonconstant holonomy length or a modified angular metric. [Trying to include the divergence of the last term of the first line in (154) in this derivation does not result in a $c_f$ compatible with the classical limit for $\bar{\lambda} \to 0$.]

The first three terms of the first line are then free of divergences in $K_\varphi$ while respecting the classical limit of the Hamiltonian constraint (147), a feat that could not be accomplished in Sec. III D because nonclassical forms of $c_f$ were not considered there. However, divergence of the last term in the first line remains unresolved.

The last line does not show an immediate divergence in $K_\varphi$, but there is one if we put $K_\varphi'$ into a manifestly periodic form:

$$K_\varphi' = \sec(\beta\bar{\lambda}K_\varphi)\frac{\sin(\beta\bar{\lambda}K_\varphi)'}{\beta\bar{\lambda}}, \tag{157}$$

with some integer $\beta$. The appearance of $\sec(\beta\bar{\lambda}K_\varphi)$ makes it difficult to promote $H_{\rm cc}^{({\rm A})}$ to an operator in a loop quantization because of its divergence at $\beta\bar{\lambda}K_\varphi = \pi/2$. One possible resolution might be to consider a modified diffeomorphism constraint as in [34,35], since $K_\varphi'$ is introduced by taking a linear combination with this constraint. This divergence problem is therefore related to the fact [36] that the diffeomorphism constraint, as the generator of infinitesimal diffeomorphisms, cannot be directly quantized in the usual loop representation but is replaced by the action of finite diffeomorphisms. We leave this problem for future work.

A notable difference between our result (154) and the Abelianization of [18,19] is that the latter does not depend on $K_x$, while the former does. This term is harder to express in a loop representation because so far no consistent modification periodic in $K_x$ has been found, but it demonstrates that our result is much more general than the previous partial Abelianizations. If the $K_x$ dependence is completely eliminated, while spatial derivatives of $E^\varphi$ are also removed by the construction of [18], the constraints trivially Poisson commute. However, removing derivatives of $E^\varphi$ requires integrating by parts, which introduces a certain degree of nonlocality. In our constraint, neither $K_x$ not $(E^\varphi)'$ have been eliminated, and we only used linear combinations without integrating by parts. Our construction is completely local and relies on highly nontrivial cancellations of several terms for the new constraints to Poisson commute.

## V. CONCLUSIONS

Our analysis of canonical gauge transformations acting on a space-time metric has revealed gaps in the widely held assumption that the constraint brackets in canonical models of modified gravity have full control over general covariance. These brackets ensure the correct transformation of the "time" components of a compatible space-time metric, as previously recognized, but by themselves they do not guarantee the correct transformation of the spatial metric, determined by the structure function in hypersurface-deformation brackets, to reproduce full space-time diffeomorphisms on shell. The new covariance condition formulated here is automatically satisfied if the structure function depends directly on a single phase-space variable, as in the classical case where the structure function is the inverse spatial metric. But this is no longer the case if the structure function is a composite field and depends on multiple phase-space functions, for instance in modified theories in which it may also depend on momentum components. The full covariance condition is essential in canonical theories of modified gravity, where it presents strong restrictions on the allowed modifications.

We have specifically applied the covariance condition to the spherically symmetric model in which the classical constraints are replaced by phase-space dependent linear combinations of the gauge generators. As we discussed in the introduction, modifications are possible because such linear combinations in the context of hypersurface deformations imply a redefinition of the normal direction. The normal, together with the spatial metric derived from the structure function of the constraint brackets, then determines an emergent space-time metric which need not be equivalent to the original classical geometry. Our explicit derivations in the spherically symmetric model, where the relevant equations that control general covariance can be solved exactly, confirm this expectation. We derived a new covariant model in which signature change may be possible, and confirmed the covariance of a recent model derived initially by different means [16]. Our results also demonstrate the covariance of older models, such as [22]. Other examples, such as [13,19], turned out not to be covariant.

In this process, we have developed a general method to obtain anomaly-free brackets from the linear combination of some of the original constraints that resulted in the computation of a new emergent space-time and a well-defined, off-shell partial Abelianization along the lines of [18,19]. This result opens the way to analyzing more complicated modified constraints and their emergent space-times, and it restricts the modifications to those compatible with general covariance, which had proved challenging until now.

Finally, we have derived a general expression for modified Hamiltonian constraints compatible with general covariance, extending the vacuum results of [14,15] by implementing the latter condition. A discussion of canonical transformations implied several simplifications and revealed redundancies in common choices of modifications in models of loop quantum gravity, in particular in the choice of periodic functions with phase-space dependent periods. As a byproduct, we derived new nontrivial partial Abelianizations of constraints compatible with general covariance.

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## APPENDIX A: COVARIANCE OF THE EMERGENT EXTRINSIC-CURVATURE TENSOR

Extrinsic curvature is defined as a Lie derivative of the spatial metric along the unit normal to a hypersurface. An analysis of how the tensor transforms therefore requires an equation for the transformation of the unit normal by a gauge transformation that changes the space-time slicing. In this appendix, we assume that there is a covariant space-time line element with space-time metric $g_{\mu\nu}$. In our canonical theories, $g_{\mu\nu}$ would be the emergent metric tensor, but here we drop tildes for the sake of convenience.

Starting with

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{A1}$$

both the unit normal vector and the spatial metric change to $n^\mu + \delta_\epsilon n^\mu$ and $q_{ab} + \delta q_{ab}$ under a gauge transformation. Since $n^\mu$ is assumed to be normalized, we have $n^\mu q_{\mu\nu} = 0$. It will remain normalized after the gauge transformation if and only if $\delta_\epsilon(N n^\mu q_{\mu\nu}) = 0$.

It is easier to evaluate this condition if we use the space-time metric instead of the spatial metric, in which case we can express normalization of $n^\mu$ as $N n^\mu g_{\mu\nu} \mathrm{d}x^\nu = \sigma N^2 \mathrm{d}t$ where $\sigma = 1$ for Lorentzian signature and $\sigma = -1$ for Euclidean signature. Normalization is then preserved by a gauge transformation $\delta_\epsilon$ if and only if

$$\delta_\epsilon(N n^\mu g_{\mu\nu}) dx^\nu = 2\sigma N \delta_\epsilon N dt, \tag{A2}$$

where we shall use the transformation of the lapse (9). We can now expand the left-hand side using the Leibniz rule to obtain $n^\mu \delta_\epsilon(N g_{\mu\nu}) + N g_{\mu\nu} \delta_\epsilon n^\mu$. The change of basis $n^\mu = N^{-1}(t^\mu - N^a s^\mu_a)$ and the component expression $g^{t\mu}$ of in terms of the lapse function and shift vector then imply

$$\delta_\epsilon n^\mu = -\frac{1}{N}\delta_\epsilon N n^\mu - \frac{1}{N}\delta_\epsilon N^a s^\mu_a. \tag{A3}$$

The normal vector is associated to the particular coordinates and foliation we choose, therefore, its transformation is not directly equivalent to a Lie derivative. (This is similar to how connections do not transform by a simple Lie derivative.) In fact, the Lie derivative of $n^\mu$ and its infinitesimal coordinate transformation are related by

$$\begin{aligned}
\mathcal{L}_\xi n^\mu &= \xi^\nu \partial_\nu n^\mu - n^\nu \partial_\nu \xi^\mu, \\
&= \frac{\epsilon^0}{N}\left(\partial_t\left(\frac{1}{N}\right)t^\mu - \partial_t\left(\frac{N^a}{N}\right)s^\mu_a\right) + \left(\epsilon^b - \frac{\epsilon^0}{N}N^b\right)\left(\partial_b\left(\frac{1}{N}\right)t^\mu - \partial_b\left(\frac{N^a}{N}\right)s^\mu_a\right) \\
&\quad - \frac{1}{N}\left(\partial_t\left(\frac{\epsilon^0}{N}\right)t^\mu + \left(\partial_t\epsilon^a - \partial_t\left(\frac{\epsilon^0}{N}N^a\right)\right)s^\mu_a\right) + \frac{N^b}{N}\left(\partial_b\left(\frac{\epsilon^0}{N}\right)t^\mu + \left(\partial_b\epsilon^a - \partial_b\left(\frac{\epsilon^0}{N}N^a\right)\right)s^\mu_a\right), \\
&= -\frac{1}{N}(\dot\epsilon^0 + \partial_b N - N^b\partial_b\epsilon^0)n^\mu - \frac{1}{N}(\dot\epsilon^a + \epsilon^b\partial_b N^a - N^b\partial_b\epsilon^a)s^\mu_a, \\
&= -\frac{\delta_\epsilon N}{N}n^\mu - \frac{\delta_\epsilon N^a - q^{ab}(\epsilon^0\partial_b N - N\partial_b\epsilon^0)}{N}s^\mu_a, \\
&= \delta_\epsilon n^\mu + \frac{q^{ab}}{N}(\epsilon^0\partial_b N - N\partial_b\epsilon^0)s^\mu_a.
\end{aligned} \tag{A4}$$

In addition to the normal vector, the spatial basis vectors $s^\mu_a$ appear in some of the expressions. Before we apply our results to extrinsic curvature, we make sure that $s^\mu_a$ does not change by a gauge transformation. For these vectors to remain spatial, we have $\delta_\epsilon(g_{\mu\nu} n^\mu s^\nu_b) = 0$. We evaluate this condition by decomposing

$$\delta_\epsilon s_b^\mu \equiv A_b n^\mu + B s_b^\mu \tag{A5}$$

into normal and spatial components. Using (A3) and the gauge transformations of lapse and shift, the condition that $s_a^\mu$ remain spatial directly implies

$$A_b = g_{\mu b}\delta_\epsilon n^\mu + n^\mu \delta_\epsilon g_{\mu b} = 0. \tag{A6}$$

Furthermore, $\delta_\epsilon q_{ab} = \delta_\epsilon(g_{\mu\nu}s_a^\mu s_b^\nu)$ implies $B = 0$.

Now, the extrinsic-curvature tensor is given by

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n q_{\mu\nu} \tag{A7}$$

and therefore depends on the slicing through the unit normal. Its transformation should reflect the slicing dependence because $n^\mu$ changes by a gauge transformation, as we saw. If we consider the infinitesimal changes

$$q_{\mu\nu} \to q_{\mu\nu} + \mathcal{L}_\xi q_{\mu\nu}, \tag{A8}$$

$$n^\mu \to n^\mu + \delta_\epsilon n^\mu, \tag{A9}$$

under a coordinate transformation, where $\delta_\epsilon n^\mu$ is given by (A3), the extrinsic-curvature tensor transforms as

$$\begin{aligned} K_{\mu\nu} &\to \frac{1}{2}\mathcal{L}_{n+\delta_\epsilon n}(q_{\mu\nu} + \mathcal{L}_\xi q_{\mu\nu}) \\ &= K_{\mu\nu} + \frac{1}{2}\mathcal{L}_{\delta_\epsilon n} q_{\mu\nu} + \frac{1}{2}\mathcal{L}_n \mathcal{L}_\xi q_{\mu\nu}, \end{aligned} \tag{A10}$$

where we have kept only the first-order term in $\xi^\mu$ for an infinitesimal transformation. For a gauge transformation of the same tensor, we have

$$\begin{aligned} \delta_\epsilon K_{\mu\nu} &= \delta_\epsilon\left(\frac{1}{2}\mathcal{L}_n q_{\mu\nu}\right), \\ &= \frac{1}{2}\mathcal{L}_{\delta_\epsilon n} q_{\mu\nu} + \frac{1}{2}\mathcal{L}_n(\delta_\epsilon q_{\mu\nu}). \end{aligned} \tag{A11}$$

If the covariance condition, $\delta_\epsilon q_{\mu\nu} = \mathcal{L}_\xi q_{\mu\nu}$, is satisfied, this is precisely the expression derived above. Therefore, the gauge transformation of extrinsic curvature, derived from the emergent space-time metric, gives the desired covariant transformation, keeping in mind that the coordinate transformation is not a Lie derivative which is similar to the transformation of the normal vector or connections, as all these cases depend on the foliation. (Extrinsic curvature is a spatial tensor on a given slice but not a space-time tensor.)

Since the spatial metric and extrinsic curvature form a complete set of spatial tensors that define the geometry of an embedded hypersurface, we conclude that the gauge transformation of all tensors derived from the spacetime metric will be equivalent to their infinitesimal coordinate transformations provided the covariance condition $\delta_\epsilon g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$ is satisfied.

## APPENDIX B: RESTRICTIONS ON THE GENERAL HAMILTONIAN CONSTRAINT FROM $\{H, H_r\}$

The third term in (85) must vanish, $\mathcal{F}_2 = 0$, which implies

$$0 = a_2 + \left(a_{x\varphi} - 3\frac{\partial d_2}{\partial E^x}\right)E^\varphi, \tag{B1}$$

$$0 = 2a_{\varphi\varphi} - 3\frac{\partial d_2}{\partial E^\varphi}, \tag{B2}$$

$$0 = b_2 + \left(c_{\varphi\varphi} - 3\frac{\partial d_2}{\partial K_\varphi}\right)E^\varphi, \tag{B3}$$

$$0 = \frac{\partial d_2}{\partial K_x}. \tag{B4}$$

Using this and $\mathcal{F}_0 = 0$, we obtain

$$\frac{\partial}{\partial E^\varphi}\left(E^\varphi \frac{\partial}{\partial E^\varphi}\left(a_{\varphi\varphi} - \frac{\partial d_2}{\partial E^\varphi}\right) - K_x \frac{\partial a_{\varphi\varphi}}{\partial K_x}\right) = 0 \tag{B5}$$

from the coefficient of $(E^\varphi)'''$,

$$\begin{aligned} &\frac{\partial}{\partial K_\varphi}\left(E^\varphi\left(\frac{\partial b_{\varphi\varphi}}{\partial E^\varphi} - \frac{\partial c_{\varphi\varphi}}{\partial K_\varphi} + \frac{\partial^2 d_2}{(\partial K_\varphi)^2}\right)\right. \\ &\left. + K_x \frac{\partial b_{\varphi\varphi}}{\partial K_x} + b_{\varphi\varphi} - \frac{\partial b_2}{\partial K_\varphi}\right) = 0 \end{aligned} \tag{B6}$$

from the coefficient of $K_\varphi'''$, and

$$\begin{aligned} &\frac{\partial}{\partial E^x}\left(E^\varphi\left(-\frac{\partial a_{x\varphi}}{\partial E^x} + \frac{\partial a_{xx}}{\partial E^\varphi} + \frac{\partial^2 d_2}{(\partial E^x)^2}\right)\right. \\ &\left. + K_x \frac{\partial a_{xx}}{\partial K_x} + a_{xx} - \frac{\partial a_2}{\partial E^x}\right) = 0 \end{aligned} \tag{B7}$$

from the coefficient of $(E^x)'''$.

Using these results and $\mathcal{F}_1 = -H$, we obtain

$$d_2 + 2\frac{\partial d_2}{\partial E^\varphi}E^\varphi = 0 \tag{B8}$$

from the coefficient of $(E^\varphi)''$,

$$\frac{\partial b_2}{\partial E^\varphi}E^\varphi + \frac{\partial b_2}{\partial K_x}K_x + b_2 = 3\frac{\partial d_2}{\partial K_\varphi}E^\varphi \tag{B9}$$

from the coefficient of $K_\varphi''$,

$$\frac{\partial a_2}{\partial E^\varphi}E^\varphi + \frac{\partial a_2}{\partial K_x}K_x + a_2 = 3\frac{\partial d_2}{\partial E^x}E^\varphi \qquad \text{(B10)}$$

from the coefficient of $(E^x)''$,

$$E^\varphi\left(\frac{\partial b_{\varphi\varphi}}{\partial E^\varphi} - 2\frac{\partial c_{\varphi\varphi}}{\partial K_\varphi} + 3\frac{\partial^2 b_2}{(\partial K_\varphi)^2}\right) + \frac{\partial b_{\varphi\varphi}}{\partial K_x}K_x + b_{\varphi\varphi} - 2\frac{\partial b_2}{\partial K_\varphi} = 0 \qquad \text{(B11)}$$

from the coefficient of $(K_\varphi')^2$,

$$3E^\varphi\left(\frac{\partial a_{\varphi\varphi}}{\partial E^\varphi} - \frac{\partial^2 d_2}{(\partial E^\varphi)^2}\right) + a_{\varphi\varphi} - \frac{\partial a_{\varphi\varphi}}{\partial K_x}K_x = 0 \qquad \text{(B12)}$$

from the coefficient of $((E^\varphi)')^2$, and

$$E^\varphi\left(-2\frac{\partial a_{x\varphi}}{\partial E^x} + \frac{\partial a_{xx}}{\partial E^\varphi} + 3\frac{\partial^2 d_2}{(\partial E^x)^2}\right) + \frac{\partial a_{xx}}{\partial K_x}K_x + a_{\mathrm{lxx}} - 2\frac{\partial a_2}{\partial E^x} = 0 \qquad \text{(B13)}$$

from the coefficient of $((E^x)')^2$.

## APPENDIX C: RESTRICTIONS FROM $\{H,H\}$

Using $z = K_x/E^\varphi$, the coefficients in (92) are

$$G^\varphi = \frac{g^2E^x}{4E^\varphi}\left(2A_0B_{\varphi\varphi} - (2zB_{\varphi\varphi} + C_{x\varphi})\frac{\partial A_0}{\partial z} + \frac{\partial^2 A_0}{\partial K_\varphi \partial z}\right), \qquad \text{(C1)}$$

$$G_x = \frac{g^2E^x}{4E^\varphi}\left(A_0\left(C_{x\varphi} - \frac{\partial \ln g}{\partial K_\varphi}\right) - (2A_{xx} + zC_{x\varphi})\frac{\partial A_0}{\partial z} - \frac{\partial A_0}{\partial K_\varphi} + \frac{\partial^2 A_0}{\partial E^x \partial z}\right), \qquad \text{(C2)}$$

$$F_{xx} = \frac{g^2E^x}{4(E^\varphi)^3}\left(C_{x\varphi} + \frac{\partial \ln g}{\partial K_\varphi} + 2\frac{\partial A_{xx}}{\partial z}\right), \qquad \text{(C3)}$$

$$F_x^\varphi = \frac{g^2E^x}{4(E^\varphi)^3}\frac{\partial C_{x\varphi}}{\partial z}, \qquad \text{(C4)}$$

$$F^{\varphi\varphi} = \frac{g^2E^x}{4(E^\varphi)^3}\frac{\partial B_{\varphi\varphi}}{\partial z}, \qquad \text{(C5)}$$

$$F_x^{2\varphi} = \frac{g^2E^x}{4(E^\varphi)^3}\left(2B_{\varphi\varphi} + \frac{\partial C_{x\varphi}}{\partial z}\right), \qquad \text{(C6)}$$

$$G^{\varphi\varphi x} = \frac{g^2E^x}{4(E^\varphi)^4}\frac{\partial^2 B_{\varphi\varphi}}{(\partial z)^2}, \qquad \text{(C7)}$$

$$G_x^{x\varphi} = \frac{g^2E^x}{4(E^\varphi)^4}\left(2\frac{\partial B_{\varphi\varphi}}{\partial z} + \frac{\partial^2 C_{x\varphi}}{(\partial z)^2}\right), \qquad \text{(C8)}$$

$$G_{xx}^x = \frac{g^2E^x}{4(E^\varphi)^4}\left(\frac{\partial C_{x\varphi}}{\partial z} + \frac{\partial^2 A_{xx}}{(\partial z)^2}\right), \qquad \text{(C9)}$$

$$G_\varphi^{\varphi\varphi} = -\frac{g^2E^x}{4(E^\varphi)^4}\left(2\frac{\partial B_{\varphi\varphi}}{\partial z} + z\frac{\partial^2 B_{\varphi\varphi}}{(\partial z)^2}\right), \qquad \text{(C10)}$$

$$G_{x\varphi}^\varphi = -\frac{g^2E^x}{4(E^\varphi)^4}\left(2\left(B_{\varphi\varphi} + z\frac{\partial B_{\varphi\varphi}}{\partial z} + \frac{\partial C_{x\varphi}}{\partial z}\right) + z\frac{\partial^2 C_{x\varphi}}{(\partial z)^2}\right), \qquad \text{(C11)}$$

$$G_{xx\varphi} = -\frac{g^2E^x}{4(E^\varphi)^4}\left(\frac{\partial \ln g}{\partial K_\varphi} + C_{x\varphi} + 2\frac{\partial A_{xx}}{\partial z} + z\frac{\partial C_{x\varphi}}{\partial z} + z\frac{\partial^2 A_{xx}}{(\partial z)^2}\right), \qquad \text{(C12)}$$

$$G^{\varphi\varphi\varphi} = \frac{g^2E^x}{4(E^\varphi)^3}\left(-2B_{\varphi\varphi}^2 - (2zB_{\varphi\varphi} + C_{x\varphi})\frac{\partial B_{\varphi\varphi}}{\partial z} + \frac{\partial^2 B_{\varphi\varphi}}{\partial K_\varphi \partial z}\right), \qquad \text{(C13)}$$

$$G_x^{\varphi\varphi} = -\frac{g^2E^x}{4(E^\varphi)^3}\left(B_{\varphi\varphi}\left(3C_{x\varphi} + \frac{\partial \ln g}{\partial K_\varphi} + 2z\frac{\partial C_{x\varphi}}{\partial z}\right) + (2A_{xx} + zC_{x\varphi})\frac{\partial B_{\varphi\varphi}}{\partial z} + C_{x\varphi}\frac{\partial C_{x\varphi}}{\partial z} - \frac{\partial B_{\varphi\varphi}}{\partial K_\varphi} - \frac{\partial^2 C_{x\varphi}}{\partial K_\varphi \partial z} - \frac{\partial^2 B_{\varphi\varphi}}{\partial E^x \partial z}\right), \qquad \text{(C14)}$$

$$G_{xx}^\varphi = -\frac{g^2E^x}{4(E^\varphi)^3}\left(C_{x\varphi}^2 + 2A_{xx}\left(B_{\varphi\varphi} + \frac{\partial C_{x\varphi}}{\partial z}\right) + C_{x\varphi}\left(\frac{\partial \ln g}{\partial K_\varphi} + \frac{\partial A_{xx}}{\partial z} + z\frac{\partial C_{x\varphi}}{\partial z}\right) + 2zB_{\varphi\varphi}\frac{\partial A_{xx}}{\partial z} - \frac{\partial^2 A_{xx}}{\partial K_\varphi \partial z} - 2\frac{\partial B_{\varphi\varphi}}{\partial E^x} - \frac{\partial^2 C_{x\varphi}}{\partial E^x \partial z}\right), \qquad \text{(C15)}$$

$$G_{xxx} = -\frac{g^2E^x}{4(E^\varphi)^3}\left(A_{xx}\left(\frac{\partial \ln g}{\partial K_\varphi} + C_{x\varphi} + 2\frac{\partial A_{xx}}{\partial z}\right) + zC_{x\varphi}\frac{\partial A_{xx}}{\partial z} + \frac{\partial A_{xx}}{\partial K_\varphi} - \frac{\partial C_{x\varphi}}{\partial E^x}\frac{\partial A_{xx}}{\partial E^x \partial z}\right), \qquad \text{(C16)}$$

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