

Quantum Computing with Dartboards

Published as part of *The Journal of Physical Chemistry* virtual special issue "Physical Chemistry of Quantum Information Science".

Ishaan Ganti and Srinivasan S. Iyengar*



Cite This: *J. Phys. Chem. A* 2023, 127, 7853–7857



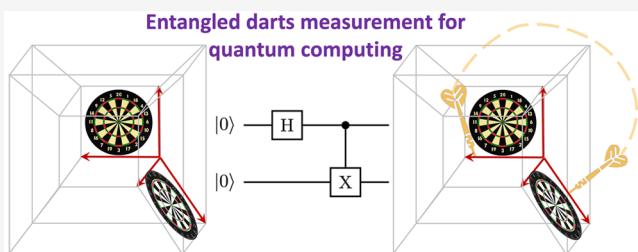
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ABSTRACT: We present a physically appealing and elegant picture for quantum computing, using rules constructed for a game of darts. A dartboard is used to represent the state space in quantum mechanics, and the act of throwing the dart is shown to have close similarities to the concept of measurement or collapse of the wave function in quantum mechanics. The analogy is constructed in arbitrary dimensional spaces, that is, using arbitrary dimensional dartboards, and for such arbitrary spaces this also provides us a "visual" description of uncertainty. Finally, connections between qubits and quantum computing algorithms are also made, opening the possibility to construct analogies between quantum algorithms and coupled dart throws.



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I. INTRODUCTION

The promise of solving exponentially complex problems efficiently using quantum technologies^{1–3} and the development of the associated software is a rapidly evolving research frontier. While we are in the early stages of this emerging quantum revolution, there is already a diverse set of problems that can benefit from such developments. However, true progress can only be achieved by a rigorous study facilitated by the development of a competitive quantum workforce. As a result, research and workforce development in the strongly interdisciplinary area of quantum information sciences has been noted by the U.S. National Science Foundation (NSF) as one of the "Big Ideas" and recognized through the introduction of the National Quantum Initiative (NQI) from the White House. Furthermore, the 2019 National Academies Report entitled *Quantum Computing: Progress and Prospects*⁴ observes that "[a]dvances in QC theory and devices will require contributions from many fields beyond physics, including mathematics, computer science, materials science, chemistry, and multiple areas of engineering." By contrast, regarding the present state of the quantum information science (QIS) workforce, the September 2018 National Science and Technology Council report⁵ notes that "America's current educational system typically focuses on discrete disciplinary tracks, rarely emphasizing cross-disciplinary study that equips graduates for complex modern questions and challenges, prominently including QIS." Emphasizing this point, Jeremy Hilton, Quantum Engineering Lead at Google AI, wrote in *Forbes*⁶ that "one issue has everyone united: There's a shortage of quantum computing talent. This shortage has a significant impact on the future of the industry. A

trained, well-rounded quantum workforce is the key to realizing the full practical value of quantum computing. And yet, pundits describe time and again the difficulty in recruiting talent. There isn't a direct pipeline from universities, and there's fierce competition for the limited workforce that is available...To continue expanding the quantum ecosystem, we need to grow the number of quantum-literate experts—now."

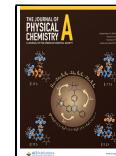
Given these critical challenges, it is vital that we develop new paradigms for quantum education that are accessible at multiple levels of pedagogy. Strongly influenced by other similar initiatives,^{7–11} here we present an approach where we utilize a game of darts to discover the complexities of quantum mechanics and eventually quantum computing. The first author of this article is a high school senior who has deeply benefited from this interdisciplinary initiative. In fact, the goal of this effort can be labeled as "quantum computing without physics." We make the case that it may be possible to introduce quantum theory without any background in physics at all.

This work is organized as follows. We will begin by discussing Schrödinger's cat problem along with a two-dimensional game of darts with rules constructed to reproduce the cat-state problem. Then, we will generalize this game of darts to multiple dimensions and eventually to a Fourier space, which yields one

Received: June 25, 2023

Revised: August 14, 2023

Published: September 7, 2023



visualization of uncertainty. Finally, we connect the game of darts to measurements performed on a set of qubit states and thus quantum computing and analyze a simple quantum circuit by constructing an analogy to a game of darts.

II. FROM CATS TO DARTBOARDS: SETTING THE STAGE

Schrödinger's cat is a central thought experiment detailing the following scenario: a cat is trapped in a box with a lethal device for a certain duration of time. During this time, the device has some probability of activation. Then, by the end of a fixed duration of time, the box is opened, and the state of the cat (alive or dead) is observed.

Schrödinger's cat is as famous as it is because according to the principles of quantum mechanics, at the moment just before the box is opened, the cat is in a "super-position" state, that is, both dead and alive. And somehow, when the box is opened, the cat "collapses" to one of the two dead or alive states. Of course, this sounds extremely odd. Even Schrödinger thought so. Yet, if we consider this scenario as an analogy to the behavior of subatomic particles, this thought experiment accurately describes and presents the basis for much of contemporary nonrelativistic physics and also provides us with an algorithm to computationally inspect almost all phenomena in chemistry, biochemistry, and material science.

To rationalize this idea and to set the stage for more general prescriptions, we may begin with a two-dimensional space where one axis is labeled as "alive" and the other is labeled as "dead". Any point in this 2D space represents the unobserved state of the cat prior to the box being opened. Thus, we may say

$$cat = \alpha \cdot (alive) + \beta \cdot (dead) \quad (1)$$

or formally

$$|cat\rangle = \alpha|alive\rangle + \beta|dead\rangle \quad (2)$$

This can be equivalently represented in a two-dimensional vector space as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (3)$$

assuming the first axis is the "alive" axis and the second is the "dead" axis. The only additional condition on choosing the specific point on the two-dimensional space is that the sum of the squares of the "alive" and "dead" components, that is, the length of the vector in eq 2, must equal 1. In the Copenhagen interpretation of quantum mechanics, this implies that the probability that the cat is alive or dead is equal to 1. In other words, the probabilities of the cat being found "alive" or "dead" when the box is opened correspond to α^2 and β^2 , respectively, and we have

$$\alpha^2 + \beta^2 = 1 \quad (4)$$

II.A. A Dartboard for Our Cat. Imagine a game of darts for the above scenario. We will define the rules of the game as follows:

Rule 1 The action of throwing the dart is designed to correspond to the action of opening the box in Schrödinger's scenario. We thus require that the dart can only land directly on one of the axes that defines our space. While this requirement appears odd at this stage, we will see later that this corresponds to a normal game of darts when the number of dimensions grow.

Rule 2 When the dart lands on a certain axis, the distance from the origin is noted. This distance, given eq 2, represents one measurement of the quantity α or β depending on the axis the dart lands on. Indeed, as per the discussion that follows eq 3, the probability that the dart lands on any given axis is proportional to the length α (or β).

Rule 3 Given eq 4, the net probability of the dart landing on either axis is 1.

Rule 4 Whether the dart lands on the horizontal or vertical axes is determined by the magnitude of α and β . The greater a particular value is, the more likely that the dart lands on that specific axis. More specifically, these likelihoods are α^2 and β^2 as noted above.

Without loss of generality, our axes were chosen as the horizontal and vertical axes, and we labeled these as "dead" and "alive". This is consistent with the original thought experiment, where, when we observe the cat, it must be either dead or alive. Thus, our dart throw must also provide us with one of the two results.

Furthermore, the result of a single throw of the dart is completely random and can result in a dead state or an alive state with probabilities α^2 and β^2 . Thus, with a single dart throw, we do not know much about the state of the cat prior to the throw. However, with multiple such dart throws, the ratio of the number of darts landing on the "alive" axis to the number of darts landing on the "dead" axis would approach the ratio of the probabilities, namely, α^2/β^2 .

II.B. Game of Darts in *N*-Dimensions May Be Interpreted as a Monte Carlo Problem. We may express a function by enumerating its values at every single point on the number-line as

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad (5)$$

where the value y_i represents the function's value at x_i . This infinitely long vector is not dissimilar from that in eq 3.

The rules for the dartboard game in Section II.A when applied to this vector space lead to an appealing interpretation. In our game of darts discussed above, each value along the vector in eq 3 corresponds to one specific component that dictates the probability that the dart lands on a given axis. The same set of rules when applied to eq 5 yields a game of darts with n orthogonal dimensions where the probability of the dart landing on an axis labeled as x_i is given by y_i . However, since eq 5 also represents some function, we can re-express this multidimensional game of darts as being played on a two-dimensional coordinate system (see Figure 1), where every throw of the dart can land anywhere along the horizontal axis, with the probability of a given throw landing on a specific point along the horizontal axis being given by the square of the vertical axis value at that point as shown in Figure 1! This leads us to a natural interpretation that appears similar to a Monte Carlo problem.

III. FUNCTIONS IN FOURIER SPACE LEAD TO COMPLEMENTARY FOURIER DARTBOARDS

Now, consider two real functions: $\cos(kx)$ and $\sin(mx)$, such that k and m are integers. Since we've established above that we can treat these functions as vectors, as in eq 5, that is

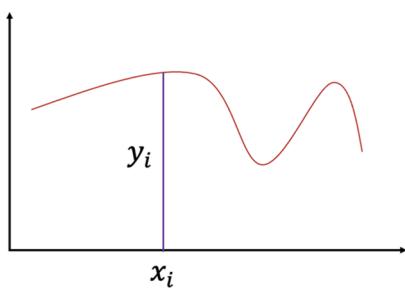


Figure 1. An alternative image of the dartboard problem for [eq 5](#). The probability of a dart landing at x_i is proportional to y_i .

$$\cos(kx) \equiv \begin{pmatrix} \cos(kx_1) \\ \cos(kx_2) \\ \vdots \\ \cos(kx_n) \end{pmatrix} \quad (6)$$

and similarly

$$\sin(mx) \equiv \begin{pmatrix} \sin(mx_1) \\ \sin(mx_2) \\ \vdots \\ \sin(mx_n) \end{pmatrix} \quad (7)$$

we can then express the dot product of these vectors as a sum of products; that is

$$\int_{-\pi}^{\pi} dx \cos(kx) \sin(mx) \quad (8)$$

which is identically zero. Thus, we may say that the sine and cosine functions are essentially vectors that are orthogonal to each other. A similar statement can be made for any two Cosine or any two Sine functions as well, that is

$$\int_{-\pi}^{\pi} dx \cos(kx) \cos(mx) = \delta_{k,m} \quad (9)$$

and

$$\int_{-\pi}^{\pi} dx \sin(kx) \sin(mx) = \delta_{k,m} \quad (10)$$

where $\delta_{k,m}$ is the Kronecker delta function. As a result, we may use a family of Sine and Cosine functions, with different values for k and m in [eqs 6](#) and [7](#), to create an n -dimensional vector space where each axis is labeled using [eq 6](#) or [7](#). For example, $\cos(x)$ and $\cos(2x)$ define a 2D vector space. If we add a third function, such as $\cos(3x)$, we have a three-dimensional space since $\cos(3x)$ is orthogonal to both $\cos(x)$ and $\cos(2x)$ and so on. With n such functions, we have now created an n -dimensional space, and as we let $n \rightarrow \infty$, we have an infinite-dimensional vector space. Thus, in a manner that is complementary to [eq 5](#), we may express any function as a linear combination of the vectors that make up our new infinite dimensional basis, which are the sinusoidal functions in the form $\cos(kx)$ and $\sin(kx)$, where $k \in \mathbb{Z}$:

$$f(x) = \sum_{i=0}^{\infty} [A_i \cos(k_i x) + B_i \sin(k_i x)] \quad (11)$$

or to present this in a fashion complementary to [eq 5](#):

$$\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \\ B_0 \\ B_1 \\ \vdots \\ B_n \end{pmatrix} \quad (12)$$

which exemplifies the many findings we just made. The coefficients $\{A_i, B_i\}$ are evaluated by using the “dot” product:

$$A_m = \int_{-\pi}^{\pi} dx f(x) \cos(mx) \quad (13)$$

As $f(x)$ is made up of a certain *amount* of each wave, and we have shown that these waves are independent of one another (their dot product is 0), taking the dot product of $f(x)$ and one wave component, $\cos(nx)$ or $\sin(nx)$, will yield a numerical value corresponding to the amount of that wave in $f(x)$.

III.A. A Complementary N -Dimensional Dartboard in Fourier Space. The rules for our game of darts discussed in [Section II.A](#) when applied to [eq 5](#) led to a Monte Carlo interpretation in [Section II.B](#). But we may apply the same rules to [eq 12](#) as represented by [Figure 2](#). Based on “Rule 4” above,

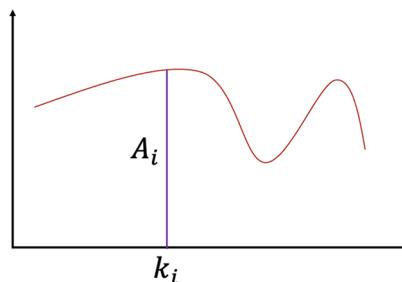


Figure 2. Dartboard problem for [eq 12](#). The probability of a dart landing at k_i is proportional to A_i . A similar figure may be constructed for the Sine functions.

this leads to the interpretation that the probability of the dart landing on any given axis of our new basis space (or any given point in the horizontal axis of [Figure 2](#)) is dictated by the extent to which a specific frequency captured by the sine or cosine wave is present in the function. Thus, rather than having just two axes (the “dead” and “alive” axes from the first example), in [Section II.B](#) and here, we now have an infinite number of orthogonal axes. Each axis is labeled here by a wave—specifically, its frequency. A point in this space, like in the first example, is simply a linear combination of these axes. From the discussion of Fourier components, we know that every function can be uniquely represented in this space.

For example, given the function

$$f(x) = a \cos(x) + b \cos(2x) + c \sin(5x) \quad (14)$$

the probability of our dart throw landing along the $\cos(x)$ axis, or at a point with $k_i = 1$ in [Figure 2](#), would be $\frac{a^2}{a^2 + b^2 + c^2}$. Similarly, the probability of our dart throw landing along the $\cos(2x)$ axis, or at a point with $k_i = 2$ in [Figure 2](#), would be $\frac{b^2}{a^2 + b^2 + c^2}$, and so on.

Notice how in the cat scenario the calculation of the probabilities for landing on a specific axis is the same as the calculation above. The “alive” probability would be $\frac{\alpha^2}{\alpha^2 + \beta^2}$, and the “dead” probability would be $\frac{\beta^2}{\alpha^2 + \beta^2}$. However, due to the normalization condition, where $\alpha^2 + \beta^2 = 1$, this simplifies to α^2 and β^2 , respectively.

IV. THE UNCERTAINTY DILEMMA BETWEEN THESE COMPLEMENTARY DARTBOARDS

Let us go back to Figure 1 and imagine that our function $f(x)$, represented by eq 5, is an exact Cosine wave, say $\cos(x)$. In such a situation, eq 12 would have exactly one value, $A_1 = 1$, and all other values in eq 12 would be identically zero. Consequently, while Figure 1 would display a wave with a single frequency, where the dart may land anywhere along the horizontal axis, the complementary form of Figure 2 would yield a single peak at $k_1 = 1$, zero everywhere else, thus yielding a probability of “1” that the dart lands at one precise point. Thus, while the outcome of every throw of a dart is precisely dictated, with certainty in Figure 2, the corresponding outcome of a single throw of darts may land anywhere along the horizontal axis in Figure 1 and hence is uncertain!

Of course the description above is general and does not really only apply to quantum mechanics. But if we were equipped with de Broglie’s wave–particle duality, where the momentum of a particle is dictated by the frequency associated with its wave-nature, it is clear from the above discussion that the function $\cos(x)$ has a precisely defined momentum. However, given that the game of darts in Figure 1 may land anywhere along the horizontal axis, the corresponding position is infinitely uncertain! Thus, one may say that using the game of darts, we have picturized in Figure 2 a momentum space, and in Figure 1, a complementary position space. But our development above did not need the principles of quantum mechanics and is hence expected to be more general. Indeed, it is well-known that a similar time–frequency uncertainty exists in signal processing.¹²

V. SCHRÖDINGER’S CAT AS A QUBIT, DARTS AS MEASUREMENTS

Now, let us move on to the connection to the quantum bits. Quantum bits, also known as qubits, are similar to the Schrödinger’s cat dartboard example that we began with earlier. This time, instead of our two states being “dead” and “alive”, we label the states as $|0\rangle$ and $|1\rangle$, which are again defined to be orthogonal to each other as before. If we define $|0\rangle$ along the horizontal axis and $|1\rangle$ along the vertical axis, then our qubit is similar to our cat-state from Section II. We can express it in an analogous fashion as in eq 2, and

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad (15)$$

where $|\psi\rangle$ represents a qubit state with components α and β which dictate the probabilities of “collapsing” the state to the $|0\rangle$ and $|1\rangle$ directions, respectively, exactly as in the game of darts.

Note how qubits differ from standard bits. Standard bits can only be 0 or 1. Qubits, on the other hand, can exist in a superposition, as in eq 15. However, when measured, equivalent to our earlier dart throws, these qubits collapse to a $|0\rangle$ or $|1\rangle$ state.

Now consider two qubits. Since these can both independently reside on their respective $|0\rangle$ and $|1\rangle$ states, the two-qubit system is a four-dimensional space with axes labeled as $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. How would we be able to express this pair of qubits?

The key is in recognizing that these four states are similar to (a) the $|0\rangle$ and $|1\rangle$ states for a single qubit, (b) the basis states for the function in eq 5, and (c) the states in eqs 11 and 12, that is, the states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ are orthogonal. Four orthogonal states implies a four-dimensional space, and a state

$$|q\rangle = q_{00}|00\rangle + q_{01}|01\rangle + q_{10}|10\rangle + q_{11}|11\rangle \quad (16)$$

is a point in this four dimensional space. Again, all of our discussion on the dartboards applies to this state. The square of the coefficients, as usual, corresponds to the probability of the pair of qubits collapsing to one specific dimension in this four-dimensional space. But, upon adding an extra qubit, we have an expanded game of darts. If we were to add a third then we have eight different possible basis states and in general we have 2^N orthogonal dimensions for N -qubits. In this fashion, the complexity and the ability to store (and process) information grows exponentially from a family of qubits.

We may also construct an analogy between eq 16 and the state corresponding to two cats that may be independently “dead” or “alive”. Thus

$$\begin{aligned} |2\text{cats}\rangle = & q_{\text{alive}_1\text{alive}_2}|\text{alive}_1\text{alive}_2\rangle + q_{\text{alive}_1\text{dead}_2}|\text{alive}_1\text{dead}_2\rangle \\ & + q_{\text{dead}_1\text{alive}_2}|\text{dead}_1\text{alive}_2\rangle + q_{\text{dead}_1\text{dead}_2}|\text{dead}_1\text{dead}_2\rangle \end{aligned} \quad (17)$$

where for example the probability of recovering both cats as living is proportional to $|q_{\text{alive}_1\text{alive}_2}|^2$.

VI. BELL STATES USING “ENTANGLED” DARTS

We now introduce a sophisticated concept in quantum information known as entanglement, which is used in the context of quantum teleportation. We developed a dartboard analogue to this concept. For this we first introduce a Bell state² which is created from a subset of the four vectors in eq 16, namely

$$|q_B\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle] \quad (18)$$

Thus, by comparison with eq 17, when cat-1 is recovered to be alive, this is also the case for cat-2 and vice versa. Thus, in some sense, the outcomes of the throw of darts corresponding to the two cats have not become “entangled”. That is, the outcome from the first dart throw also dictates and influences the second dart. A quantum circuit that creates such a state is shown in Figure 3. How does one create a picture of this using our dartboard analogy?

Let us imagine two darts that, as we will discover, behave in a very peculiar fashion. The two wires in Figure 3 represent the

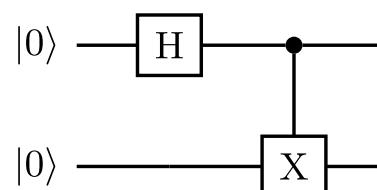
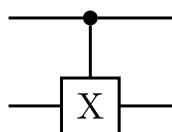


Figure 3. A quantum circuit to create the Bell state in eq 18. An initial state, $|0\rangle \otimes |0\rangle$, is propagated through a Hadamard transform of qubit one to yield $\left\{ \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] \otimes |0\rangle \right\}$. Following this, application of a CNOT gate yields the state shown in eq 18.

two different dartboards where these darts are supposed to land. We will also assume, as done in [eq 17](#), that there exist two cats whose lives are influenced by the outcome from these two darts and their respective boards. Dartboard-1, the top wire in [Figure 3](#), is preprocessed through a transformation given by the Hadamard transform H in [Figure 3](#). This Hadamard transform simply rotates the state that dictates the dart landing probabilities for Dartboard-1 by 45° . So our Dartboard-1 will now present dart-1 with equal probabilities of landing along the horizontal and vertical axis. Thus, in a sense, going back to our cat states in [eq 2](#), the Hadamard transform has created a state that looks like $\frac{1}{\sqrt{2}}[|alive\rangle \pm |dead\rangle]$ and notice then by comparison with [eq 2](#), that both α^2 and β^2 are equal. Thus, after the Hadamard transform, Dart-1 is free to land on either axis, with no bias.

Let us now presume that Dart-2 begins with a dartboard where the dart throws are dictated by the probability, using the cat-state analogy, $\alpha = 0$ and $\beta = 1$. Thus, we know with certainty that the state that dictates Dart-2 is, for example, the $|0\rangle$ state, or the cat represented here is, for example, $|alive\rangle$ with probability 1. Thus, without influence from Dart-1, Dart-2 will always provide an outcome where the cat is “alive”.

Now something very strange happens as a result of the operation



in [Figure 3](#). The two darts are not independent anymore. When Dart-1 lands on the state $|0\rangle$, that is, finds that the corresponding cat is “alive”, Dart-2, dictated by the probabilities that control Dartboard-2 discussed in the previous paragraph, recovers an $|alive\rangle$ or $|0\rangle$ too. However, when Dart-1 lands on the state $|1\rangle$, that is, finds that the corresponding cat is “dead”, Dart-2 promptly *flips* the probabilities in the state that dictates Dartboard-2 and also recovers a $|dead\rangle$ or $|1\rangle$ state. Thus, the behavior of Dart-2 is not independent from that of Dart-1, and this property is called entanglement. This bizarre result is one of the major hallmarks of quantum mechanics and is thought to be central to a presumed advantage that a computer created from the principles of quantum mechanics may have over those that we currently use. We have attempted to reproduce this rather bizarre result in our table-of-contents image on the first page of this paper. At the center of this TOC image is the image from [Figure 3](#). On the left side is an illustration of a four-dimensional object known as a tesseract. The two dartboards represent the two qubits and are shown along orthogonal dimensions of the tesseract. On the right side of the quantum circuit, we show two “entangled” darts that control each other and perform the role of the measurement discussed above. This is a visual representation of the entanglement process within our dartboard game formulated here.

VII. CONCLUSION

We develop an approach to discuss the fundamental underpinnings of quantum mechanics and quantum computing using the concept of dart throws onto the two axes of two-dimensional space. The concept of qubits and measurement is related to these dart throws, and the process, when generalized, yields visual descriptions of uncertainty and Fourier transforms. We follow the prescriptions of this game and develop a way to relate

these concepts to fundamental notions in quantum information storage and processing, including analogues to quantum circuits and Bell states.

■ AUTHOR INFORMATION

Corresponding Author

Srinivasan S. Iyengar — Department of Chemistry, and the Indiana University Quantum Science and Engineering Center (IU-QSEC), Indiana University, Bloomington, Indiana 47405, United States;  orcid.org/0000-0001-6526-2907; Email: iyengar@indiana.edu

Author

Ishaan Ganti — Mission San Jose High School, Fremont, California 94539, United States

Complete contact information is available at:

<https://pubs.acs.org/10.1021/acs.jpca.3c04262>

Notes

The authors declare no competing financial interest.

■ ACKNOWLEDGMENTS

This research was supported by the National Science Foundation grants OMA-1936353 and CHE-2102610 to SSI.

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