Systems & Control Letters 179 (2023) 105588

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

A sufficient condition for the super-linearization of polynomial systems

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ARTICLE INFO

Article history:
Received 9 January 2023
Received in revised form 7 May 2023
Accepted 28 June 2023
Available online 31 July 2023

Keywords:
Carleman linearization
Koopman linearization
Super-linearization
Nonlinear systems

ABSTRACT

We provide in this paper a sufficient condition for a polynomial dynamical system $\dot{x}(t) = f(x(t))$ to be super-linearizable, i.e., to be such that all its trajectories are linear projections of the trajectories of a linear dynamical system. The condition is expressed in terms of the hereby introduced weighted dependency graph G, whose nodes v_i correspond to variables x_i and edges v_iv_j have weights $\frac{\partial f_j}{\partial x_i}$. We show that if the product of the edge weights along any cycle in G is a constant, then the system is super-linearizable. The proof is constructive, and we provide an algorithm to obtain super-linearizations and illustrate it on an example. Our result also provides a partial answer to an open question about polyflows.

1. Introduction

The idea of linearizing system dynamics via embeddings dates back at least to the works of Carleman [1] and Koopman [2,3]. These embeddings are still actively studied a century later, and have found applications in nonlinear control [4,5], and data-driven methods in control [6,7].

We derive in this paper a sufficient condition under which a polynomial system can be globally linearized by embedding it into a higher, yet finite-dimensional vector space. In particular, the contribution of this paper is to provide a generalized converse of the result established in [8]. We elaborate on this below.

To proceed, we consider the following dynamical system:

$$\dot{x} = f(x) \tag{1.1}$$

where $x \in \mathbb{R}^n$. This system is said to admit a *super-linearization* (see Definition 2.1 below) if there exist $m \geq 0$ functions, called observables, which when adjoined to the original system would permit its linearization. A typical example [9] is the following two-dimensional system

$$\begin{cases} \dot{x} = -x + y^2 \\ \dot{y} = -y \end{cases} \tag{1.2}$$

Adding the observable $w := y^2$, whose total time derivative is given by $\dot{w} = 2y\dot{y} = -2y^2 = -2w$, we obtain the three-dimensional *linear* system:

$$\begin{cases} \dot{x} = -x + w \\ \dot{y} = -y \\ \dot{w} = -2w. \end{cases} \tag{1.3}$$

Observe that the variables on which the nonlinear part of the dynamics (1.2) depend (here, the variable y) evolve in a linear, autonomous (i.e., independent from x) manner. In a recent paper [8], we showed that *if* a polynomial system admits a super-linearization with only *one* so-called visible observable [10], then there exists a linear change of variables under which the nonlinear part of the dynamics depends solely on variables evolving linearly and autonomously. The dynamics resulting from the change of variable are termed the *canonical form* [8] for the polynomial system (explicitly, the canonical form is given in (3.1) below), and its existence provides a *necessary* condition for the super-linearization of that special class of polynomial systems.

Conversely, we exhibit in this paper a *sufficient* condition for the super-linearization of general polynomial systems, *without* any restriction on the number of visible observables. In particular, the result of this paper, combined with the ones of [8], provide a necessary and sufficient condition for the class of polynomial systems with a single visible observable to be super-linearizable.

The remainder of the paper is organized as follows: We describe the relevant terminology and notation at the end of this

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https://doi.org/10.1016/j.sysconle.2023.105588
0167-6911/

section. We present the main result in Section 2 and its proof in Section 3. The paper ends with a summary and outlook.

Terminology and notation used. We let $G = (V, E)$ be a directed graph (possibly with self-loops), with V the node set and E the edge set. We use $e = v_iv_j$ to denote a directed edge of G from node v_i to node v_j (if $v_i = v_j$, then e is a self-loop). A *walk* is a sequence of nodes $w = v_{i_1}v_{i_2}\dots v_{i_k}$ such that $v_{i_\ell}v_{i_{\ell+1}}$ is an edge of G for each $\ell = 1, \dots, k-1$. The *length* of a walk is the number of edges it traverses. A path is a walk which does not visit a node more than once. We call the *depth* of G the length of the longest path in G.

For a dynamical system $\dot{x}(t) = f(x(t))$, we denote by $e^{tf}x_0$ the solution of the system at time t with initial state x_0. For a vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ and a differentiable vector-valued function $p : \mathbb{R}^n \to \mathbb{R}^k$, we denote the *Lie derivative* of p along g by $\mathcal{L}_g p := \frac{\partial p}{\partial x} g$.

2. Statement of the result

We start by defining what it means for system (1.1) to be super-linearizable. Let $m \geq 0$ be an integer, and $\Pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the canonical projection onto the first n variables, namely, we have for $z \in \mathbb{R}^{n+m}$ that $\Pi(z) = (z_1, \dots, z_n)$. We reproduce the following definition from [10]:

Definition 2.1 (*Super-linearization*)**.** The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is *super-linearizable* to the system $\dot{z} = Az + D$ with $A \in \mathbb{R}^{(n+m)\times(n+m)}$ and $D \in \mathbb{R}^{n+m}$ if there exists an injective map $p : \mathbb{R}^n \to \mathbb{R}^m$ so that for all $x_0 \in \mathbb{R}^n$, the following holds:

$$\Pi\left(e^{t(Az+D)}z_0\right) = e^{tf}x_0 \text{ with } z_0 = (x_0, p(x_0)). \tag{2.1}$$

We call the functions $p : \mathbb{R}^n \to \mathbb{R}^m$ the *observables*.

The data of A, D and p is referred to as a super-linearization of f. We can express the relation (2.1) as the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{e^{tf}} & \mathbb{R}^n \\ \Big\downarrow{\scriptstyle (\mathrm{id},\ p)} & & \Big\uparrow{\scriptstyle \Pi} \\ \mathbb{R}^{n+m} & \xrightarrow[e^{t(Az+D)}]{} & \mathbb{R}^{n+m} \end{array}$$

The presentation of our main result is natural in terms of graph-theoretic notions. To this end, we introduce the following notion:

Definition 2.2 (*Weighted Dependency Graph*)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. The **weighted dependency graph (WDG)** $G = (V, E, \gamma)$ of f is a weighted directed graph (with self-loop) on n nodes $v_1, \dots, v_n$. For every ordered pair (v_i, v_j), we define the scalar function:

$$\gamma_{ij}(x) := \frac{\partial f_j(x)}{\partial x_i} \quad \text{for} \quad 1 \leq i, j \leq n.$$

There is an edge v_iv_j in G if $\gamma_{ij} \neq 0$, and its weight is γ_{ij}.

We illustrate the definition on the following example:

Example 2.3. Consider the following polynomial system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \\ \dot{x}_3 = x_2^2 \\ \dot{x}_4 = x_3 + x_1x_2^2 \\ \dot{x}_5 = -x_5 + x_3^2 + x_1^2x_2. \end{cases} \tag{2.2}$$

Its weighted dependency graph is depicted in Fig. 1. □

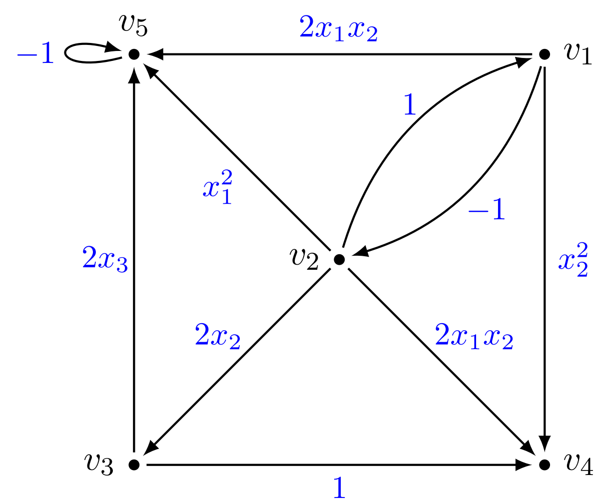

Fig. 1. The weighted dependency graph of system (2.2).

Next, for each directed walk $w = v_{i_1}\dots v_{i_k}$ in G, we let

$$\gamma_w := \prod_{j=1}^{k-1} \gamma_{i_j i_{j+1}}.$$

In the sequel, we will assume that f is a polynomial vector field. It should be clear that $\gamma_w(x)$, for any walk w, is then a polynomial function in x. Also, we assume, without loss of generality, that G is weakly connected (otherwise, the original system can be decoupled into sub-systems of lower dimensions and our result, stated below, can be applied to each sub-system independently).

Super-linearizable polynomial vector fields are easily seen to admit polynomial observables. Because their flow maps can be expressed as $\Pi e^{At}z_0$, with $z_0 = (x_0, p(x_0))$, we observe that for each fixed t, the flow map of any such vector field is a *polynomial* in the initial state x_0. These flows have been studied in the literature [11,12] under the name of *polyflow*. In [12, Question 1, p. 672], the author puts forth the open question of characterizing polynomial vector fields that generate polyflows. The main result of the paper, stated below, partially answers this open question in the form of a sufficient condition:

Theorem 2.4. *For a polynomial system $\dot{x}(t) = f(x(t))$, let G be the associated weighted dependency graph. If γ_c is a constant for every cycle c of G, then f is super-linearizable.*

The sufficient condition stated above implies that $\det(\partial f(x)/\partial x)$ is constant. Note that the weighted dependency graph of system (2.2), depicted in Fig. 1, satisfies the sufficient condition of Theorem 2.4, and thus system (2.2) is super-linearizable. As an illustration of the proof technique used, we will provide toward the end a super-linearization of this system.

3. Proof of Theorem 2.4 and an algorithm

3.1. Proof of the theorem

We start with a simple proposition, dealing with systems where the variables on which the nonlinear part of the dynamics depend evolve linearly and autonomously, and show that such systems are super-linearizable. This result provides a converse of the result of [8], and will be used as a building block to establish the general case.

Proposition 3.1. *Suppose that the system $\dot{x}(t) = f(x(t))$ takes the following form:*

$$\begin{cases} \dot{x}'(t) = A'x'(t) + D \\ \dot{x}''(t) = A''x''(t) + g(x'(t)), \end{cases} \tag{3.1}$$

where $x = (x'; x'')$, D is a constant vector, and g is a polynomial; then, system (3.1) is super-linearizable.

Proof. Let n' be the dimension of x' and d be the degree of g. Let P_d be the vector space of all polynomials in x' with real coefficients, whose dimension is given by

$$r := \dim P_d = \binom{n'+d}{d}.$$

Next, for convenience, we let $f'(x') := A'x' + D$. Since f' is affine, $\mathcal{L}_{f'}\phi \subseteq P_d$ for any $\phi \in P_d$ and, hence, $\mathcal{L}_{f'} : P_d \to P_d$ is a linear automorphism. Let the minimal polynomial associated with $\mathcal{L}_{f'}$ be given by

$$s^N + \alpha_{N-1}s^{N-1} + \cdots + \alpha_0$$

for some $N \le r$. In particular, for any $\phi \in P_d$, we have that

$$(\mathcal{L}^N_{f'}\phi) + \alpha_{N-1}(\mathcal{L}^{N-1}_{f'}\phi) + \cdots + \alpha_0\phi = 0.$$

Now, define

$$p(x) = \begin{bmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_N(x) \end{bmatrix} := \begin{bmatrix} g(x') \\ \mathcal{L}_{f'}g(x') \\ \vdots \\ \mathcal{L}^N_{f'}g(x') \end{bmatrix}. \tag{3.2}$$

It then follows that the time derivative of $p(x(t))$ is

$$\frac{d}{dt}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N-1} \\ p_N \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ -\alpha_0 I & -\alpha_1 I & \cdots & -\alpha_{N-2}I & -\alpha_{N-1}I \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N-1} \\ p_N \end{bmatrix}. \tag{3.3}$$

This completes the proof. □

We note here that relations similar to (3.3) have appeared several times in the literature. This is no coincidence as it can be shown that a system $\dot{x} = f(x)$ is super-linearizable if and only if there exists an $N < \infty$ such that the dynamics of $f(x), \mathcal{L}_f f(x), \ldots, \mathcal{L}^N_f f(x)$ obey an equation such as (3.3). This fact has been derived several times in the literature under different guises. We mention here that it appears in [13, Lemma 1] for observer design, and in [12, Propositions 2.2 and 2.3] for the study of polyflows. From that point of view, our contribution, Theorem 2.4, can also be restated as exhibiting a condition for which such a relation holds.

We next introduce two notions that are necessary for enabling the recursive use of Proposition 3.1 in the proof of the main theorem. The first is the notion of strong component decomposition.

Definition 3.2 (*Strong Component Decomposition*)**.** Let G be a weakly connected digraph. The subgraphs $G_i = (V_i, E_i)$, for $1 \le i \le q$, form *a strong component decomposition* of G if the following items hold:

1. The V_i's partition the vertex set as $V = \sqcup_{i=1}^q V_i$;
2. Each G_i is a subgraph induced by V_i and is strongly connected;

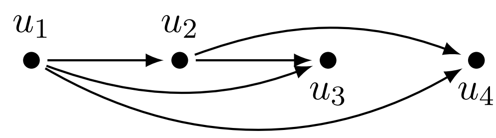

Fig. 2. The skeleton graph S of the WDG G of system (2.2), depicted in Fig. 1. Note that $\pi^{-1}(u_1) = \{v_1, v_2\}$, $\pi^{-1}(u_2) = \{v_3\}$, $\pi^{-1}(u_3) = \{v_4\}$, and $\pi^{-1}(u_4) = \{v_5\}$.

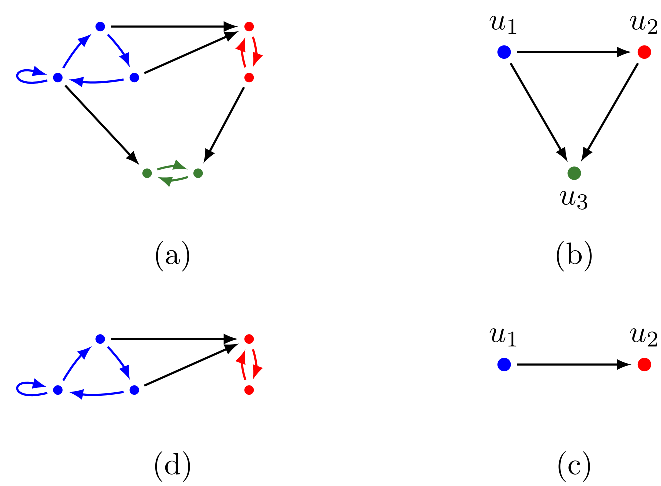

Fig. 3. Illustration of $G_{S'}$: (a) A weakly connected digraph $G = (V, E)$, with three strongly connected components highlighted in blue, red, and green, respectively; (b) The skeleton graph $S = (U, F)$ of G, with $U = \{u_1, u_2, u_3\}$ and $F = \{u_1u_2, u_1u_3, u_2u_3\}$; (c) A subgraph S' of S; and (d) The corresponding subgraph $G_{S'}$ of G. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Any strongly connected subgraph G' of G is a subgraph of some G_i, for $i \in \{1, \ldots, q\}$.

By treating the strongly connected components G_i as single nodes, we obtain the second notion, namely the one of skeleton graph S of G:

Definition 3.3. Let $G = (V, E)$ be a weakly connected digraph, and let $G_1, \ldots, G_q$ be the strong component decomposition of G. The *skeleton graph* $S = (U, F)$ is a digraph on q nodes $u_1, \ldots, u_q$, corresponding to $G_1, \ldots, G_q$. There is no self-loop in S. There is an edge u_iu_j, for $u_i \ne u_j$, only if there exist a node $v_{i'}$ in G_i and a node $v_{j'}$ in G_j such that $v_{i'}v_{j'}$ is an edge in G. Further, we denote by $\pi : V \to U$ the map that sends nodes $v_{i'}$ in V_i to u_i.

We illustrate the definition in Fig. 2.

A subgraph $S' = (U', F')$ of S induces a subgraph of G, obtained by only keeping the nodes of G contained in the strong components represented by nodes of S'; precisely, to S', we attach the subgraph $G_{S'}$ of G induced by $\pi^{-1}(U')$. See Fig. 3 for an illustration. Note that the skeleton graph S is acyclic because otherwise, it will contradict the third item of Definition 3.2.

Let ℓ be the depth of the graph S; we now introduce a node set decomposition, termed the *depth decomposition*, of S:

$$U = \sqcup_{m=0}^{\ell} U_m. \tag{3.4}$$

Starting with U_0, we simply let it be the subset of nodes of U *without* incoming edges. Since S is acyclic, U_0 is non-empty. Now to each node u_j in $U - U_0$, we assign the set P_j of paths from nodes in U_0 to u_j. It should be clear that P_j is non-empty. We define the *depth of the node* u_j, denoted by $\text{depth}(u_j)$, to be the maximal length of all paths in P_j, i.e.,

$$\text{depth}(u_j) := \max\{\text{length}(w) \mid w \in P_j\}.$$

The subset U_m is then the collection of all nodes in S of depth m. The subsets U_m, for $0 \leq m \leq \ell$, are all nonempty, pairwise disjoint, and their union is U.

With the preliminaries above, we establish Theorem 2.4.

Proof of Theorem 2.4. Let $G = (V, E, \gamma)$ be the weighted dependency graph of the polynomial vector field f. Let $S = (U, F)$ be the associated skeleton graph (obtained using Definition 3.3 and ignoring the weights γ of G), and ℓ be the depth of S. Because G is weakly connected by assumption, so is S. The proof will be carried out by induction on ℓ.

Base case $\ell = 0$: In this case, since S is weakly connected, it is a single node. It follows that G is strongly connected. Next, we claim that *all* the weights γ_{ij} for the edges v_iv_j of G are constant. To see this, for each edge v_iv_j in G, we let $c = v_{i_1}v_{i_2}\cdots v_{i_k}v_{i_1}$ be a cycle in G that contains this edge, with $v_{i_1}v_{i_2} = v_iv_j$. By the hypothesis of Theorem 2.4, it holds that γ_c is constant. We have that

$$\gamma_{v_iv_j}\gamma_{v_{i_2}\cdots v_{i_k}v_{i_1}} = \gamma_c.$$

Since γ_c is a constant and since both $\gamma_{v_iv_j}$ and $\gamma_{v_{i_2}\cdots v_{i_k}v_{i_1}}$ are polynomials (over $\mathbb{R}$), it must hold that they are also constants. This establishes the claim. As a consequence, the vector field f is an affine function. This completes the proof for the base case.

Inductive step: We assume that the statement holds for $\ell \geq 0$ and prove it for $(\ell+1)$. Let $\sqcup_{m=0}^{\ell+1}U_m$ be the node set decomposition of U introduced in (3.4). Consider the subgraph S' of S induced by the nodes in $\sqcup_{m=0}^{\ell}U_m$, and S'' the subgraph of S induced by nodes in $U_{\ell+1}$.

It should be clear that S' is itself an acyclic digraph whose depth is ℓ, and that S'' is a union of isolated nodes. To see that the latter statement holds, it suffices to observe that if S'' has an edge, then it necessarily has nodes with different depths. We let $x'(t)$ be the vector with entries taken from $x(t)$ corresponding to nodes in $G_{S'}$ and $x''(t)$ be the vector corresponding to $G_{S''}$. By construction of S', the dynamics of $x'(t)$ do not depend on $x''(t)$ and, hence, we can write the said dynamics as $\dot{x}'(t) = f'(x'(t))$. On the one hand, by applying the induction hypothesis to each connected component of S', we have that f' is super-linearizable. We set p' to be the associated observables, on which the super-linearization relies.

On the other hand, the dynamics of $x''(t)$ may depend on both $x'(t)$ and $x''(t)$, i.e., $\dot{x}''(t) = f''(x'(t), x''(t))$ for f'' a polynomial vector field. Since each connected component of $G_{S''}$ is strongly connected, every edge in $G_{S''}$ belongs to a cycle in $G_{S''}$. By the hypothesis of Theorem 2.4 and by the same arguments given in the base case, we then have that all the edge weights in $G_{S''}$ are constants. This implies that $f''(x', x'')$ is *affine* in x'' (note that edge weights in $G_{S''}$ only take into account differentiation of f'' with respect to x'', i.e., the variables corresponding to nodes $G_{S''}$). Combining the above, the dynamics can be expressed as

$$\begin{cases}\dot{z}'(t) = A'z'(t) + D\\ \dot{x}''(t) = A''x''(t) + g(z'(t)),\end{cases} \tag{3.5}$$

where, owing to Proposition 3.1, $z' := (x'; p')$, A' and A'' are constant matrices, D is a constant vector, and g is a polynomial vector field. By Proposition 3.1, system (3.5) is super-linearizable. This completes the proof. □

3.2. Algorithm for super-linearization

The steps outlined in the proof of Theorem 2.4 can be formalized as an algorithm, which we will present below. For ease of presentation, we introduce some notations.

Let G be the WDG of a given system $\dot{x} = f(x)$ and S be the corresponding skeleton graph. Let $U = \sqcup_{m=0}^{\ell}U_m$ be the depth decomposition, S_m be the subgraph of S induced by U_m. With a slight abuse of notation, we will use x_m to denote the "sub-vector" of x with entries corresponding to the nodes in G_{S_m}, and let $f_m(x)$ be defined such that $\dot{x}_m(t) = f_m(x(t))$.

The algorithm for super-linearization is as follows:

Input: A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ for the system $\dot{x}(t) = f(x(t))$.

Step 1: Compute the WDG G of the system and terminate if G does not satisfy the conditions of Theorem 2.4.

Step 2: Compute the skeleton graph $S = (U, F)$, its depth ℓ, and the depth decomposition $U = \sqcup_{m=0}^{\ell}U_m$.

Step 3: Set $\ell' := 0$ and $z_0 := x_0$. While $\ell' < \ell$, repeat:

3.1: Perform the super-linearization of the following system:

$$\begin{cases}\dot{z}_{\ell'}(t) = A_{\ell'}z_{\ell'}(t) + D_{\ell'},\\ \dot{x}_{\ell'+1}(t) = f_{\ell'+1}(x(t)).\end{cases} \tag{3.6}$$

and obtain the super-linearized dynamics of (3.6)

$$\dot{z}_{\ell'+1}(t) = A_{\ell'+1}z_{\ell'+1}(t) + D_{\ell'+1} \tag{3.7}$$

with observables $p_{\ell'+1}$.

3.2: Increase ℓ' by 1.

Output: The data (A_ℓ, D_ℓ, p_ℓ) as a super-linearization of the original system.

Remark 3.4. We elaborate below on a few points of Step 3.1 in the Algorithm:

1. When $\ell' = 0$, (3.6) implies that the dynamics of x_0 are necessarily affine. It is indeed the case, and was argued in the proof of Theorem 2.4 (the base case).
2. In (3.6), the dynamics of $x_{\ell'+1}$ depend only on $x_0, \ldots, x_{\ell'+1}$ and, moreover, linearly in $x_{\ell'+1}$ as was argued in the proof of Theorem 2.4 (the inductive step). Note that $z_{\ell'+1}$ contains the variables $x_0, \ldots, x_{\ell'+1}$ and the observables $p_{\ell'+1}$.
3. In order to obtain the super-linearized dynamics (3.7), one can follow, e.g., the steps of the proof of Proposition 3.1. The fact that (3.6) is in the same form as (3.1) is argued in the second item of this remark. More specifically, the first step is then to determine the degree d of the polynomial vector field $f_{\ell'+1}$. Next, upon choosing a basis for P_d, determine the matrix of the linear operator $\mathcal{L}_{\bar{f}_{\ell'}} : P_d \to P_d$ where $\bar{f}_{\ell'}(z) := A_{\ell'}z + D_{\ell'}$ and compute the minimal polynomial of this matrix. Finally, introduce the observables p as given in (3.2); they obey the linear dynamics (3.3).
There exist other ways to obtain a super-linearization of the system; we will in fact follow a slightly different approach in the example next.

We illustrate the algorithm on the polynomial system given in Example 2.3. Recall that the WDG G of the system is given in Fig. 1, and the corresponding skeleton graph $S = (U, F)$ is in Fig. 2.

We next compute the depth decomposition of U. The only node that has no incoming edges is u_1, and thus $U_0 = \{u_1\}$. The longest path joining u_1 to u_2 is of length 1, and the longest paths from u_1 to either u_3 or u_4 are of lengths 2; hence $U_1 = \{u_2\}$ and $U_2 = \{u_3, u_4\}$.

Now, for Step 3, there will be two iterations:

1. The first iteration considers the dynamics of the variables associated to U_0 (namely x_1, x_2) and U_1 (namely, x_3). We

have

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \\ \dot{x}_3 = x_2^2. \end{cases} \tag{3.8}$$

We observe that the dynamics associated to the nodes in U_0 are indeed linear. Following (3.2), we set $x = (x', x'')$ with $x' := (x_1, x_2)$ and $x'' := x_3$, $p_1(x) := x_2^2$, and $f'(x') := (x_2, -x_1)$. We obtain that

$$\mathcal{L}_{f'}p_1 = -2x_1x_2 =: p_2$$
$$\mathcal{L}_{f'}p_2 = 2(x_1^2 - x_2^2) =: p_3$$
$$\mathcal{L}_{f'}p_3 = 8x_1x_2 = -4p_2.$$

The super-linearized system is thus

$$\dot{z}_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \\ p_1 \\ p_2 \\ p_3 \\ -4p_2 \end{bmatrix} =: A_1 z_1. \tag{3.9}$$

2. The second iteration starts with the super-linearized system (3.9) with the dynamics of the variables in U_2 adjoined. Namely, with

$$\begin{cases} \dot{z}_1 = A_1 z_1 \\ \dot{x}_4 = x_3 + x_1x_2^2 \\ \dot{x}_5 = -x_5 + x_3^2 + x_1^2x_2 \end{cases}$$

To proceed, we could attempt to super-linearize the vector $(x_1x_2^2; x_3^2+x_1^2x_2)$ at once, or handle each entry consecutively. We choose the latter option, which deviates slightly from the procedure described in Proposition 3.1 but requires fewer computations. Also, note that there is some freedom in how one expresses the nonlinear terms. For example, $x_1x_2^2$ can also be written as x_1p_1 or $-\frac{1}{2}x_2p_2$, given the observables introduced in the first iteration.
We start by setting $p_4 := x_1x_2^2$ and $f'(z_1) := A_1z_1$. By computation, we obtain that

$$\mathcal{L}_{f'}p_4 = x_2^3 - 2x_1^2x_2 =: p_5$$
$$\mathcal{L}_{f'}p_5 = -7x_1x_2^2 + 2x_1^3 = -7p_4 + 2x_1^3 =: p_6$$
$$\mathcal{L}_{f'}p_6 = -7p_5 + 6x_1^2x_2 =: p_7$$
$$\mathcal{L}_{f'}p_7 = -7p_6 + 12x_1x_2^2 - 6x_1^3 = -7p_6 + 12p_4 - 3(p_6 + 7p_4) = -10p_6 - 9p_4.$$

Next, we set $p_8 := x_3^2 + x_1^2x_2$ and

$$\mathcal{L}_{f'}p_8 = 2x_3p_1 + 2x_1x_2^2 - x_1^3 = 2x_3p_1 - \frac{1}{2}(p_6 + 3p_4) =: p_9$$
$$\mathcal{L}_{f'}p_9 = 2p_1^2 + 2x_3p_2 - \frac{1}{2}(p_7 + 3p_5) =: p_{10}$$
$$\mathcal{L}_{f'}p_{10} = 6p_1p_2 + 2x_3p_3 + \frac{1}{2}(9p_4 + 7p_6) =: p_{11}$$
$$\mathcal{L}_{f'}p_{11} = 6p_2^2 + 8p_1p_3 - 8x_3p_2 + \frac{1}{2}(9p_5 + 7p_7) =: p_{12}$$
$$\mathcal{L}_{f'}p_{12} = 20p_2p_3 - 40p_1p_2 - 8x_3p_3 - \frac{1}{2}(63p_4 + 61p_6) =: p_{13}$$
$$\mathcal{L}_{f'}p_{13} = 20p_3^2 - 120p_2^2 - 48p_1p_3 + 32x_3p_2 - \frac{1}{2}(63p_5 + 61p_7) =: p_{14}$$
$$\mathcal{L}_{f'}p_{14} = -448p_2p_3 + 224p_1p_2 + 32x_3p_3 + \frac{1}{2}(549p_4 + 547p_6) =: p_{15}$$
$$\mathcal{L}_{f'}p_{15} = 2016p_2^2 - 448p_3^2 + 256p_1p_3 - 128x_3p_2 + \frac{1}{2}(549p_5 + 547p_7) =: p_{16}$$
$$\mathcal{L}_{f'}p_{16} = 7872p_2p_3 - 1152p_1p_2 - 128x_3p_3 - \frac{1}{2}(4923p_4 + 4921p_6) = \frac{1}{2}(1485p_4 + 1215p_6) - 256p_{11} - 144p_{13} - 24p_{15}.$$

We thus obtain the following super-linearization of the original system (2.2):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \\ \dot{x}_3 = p_1 \\ \dot{x}_4 = x_3 + p_4 \\ \dot{x}_5 = -x_5 + p_7 \\ \dot{p}_i = p_{i+1}, \text{ for } i = 1, 2, 4, 5, 6, 8, \dots, 15 \\ \dot{p}_3 = -4p_2 \\ \dot{p}_7 = -10p_6 - 9p_4 \\ \dot{p}_{16} = \frac{1485}{2}p_4 + \frac{1215}{2}p_6 - 256p_{11} - 144p_{13} - 24p_{15}. \end{cases}$$

4. Summary and outlook

We provided in this paper a sufficient condition for a system $\dot{x}(t) = f(x(t))$, with f a polynomial vector field, to be super-linearizable. The condition is simply expressed in terms of cycles in what we called the weighted dependency graph of the system. The proof of the main result is constructive, and we have sketched an algorithm based on it that produces a super-linearization of vector fields meeting the sufficient condition. The algorithm was also illustrated on an example.

The main result of this paper provides a generalized converse of the results in [8]. Indeed, while the canonical form exhibited there entails that in the original dynamics, the variables on which the nonlinear terms depend have to evolve linearly, it is easy to see that this fact does not hold for the system (2.2). The gap of course lies in the fact that [8] restricts its scope to systems with only one visible observable, which precludes the nested super-linearizations that arise in the inductive step of the proof. In terms of the vocabulary introduced in this paper, the results of [8] only deal with skeleton graphs of *depth* 1. We will address the converse of the results presented in this paper, similarly generalize the results of [8], in future work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

M.-A. Belabbas was supported partially by grants AFOSR, United States of America FA9550-20-1-0333 and National Science Foundation, United States of America CCF-2106358. Xudong Chen was supported partially by grants National Science Foundation, United States of America ECCS-2042360 and AFOSR, United States of America FA9550-20-1-0076.

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