HIGHER ORDER KIRILLOV-RESHETIKHIN MODULES, IMAGINARY MODULES AND MONOIDAL CATEGORIFICATION FOR $\mathbf{U}_q(A_n^{(1)})$

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Abstract.

1. The main results

In this section we introduce the basic notation and state our main results. We assume throughout that q is a non–zero complex number and not a root of unity. As usual $\mathbb{C}$ (resp. $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$) will denote the set of complex numbers (integers, non-negative integers, positive integers). Let $\mathbb{C}^\times$ be the set of non–zero complex numbers.

1.1. The essential notation.

1.1.1. *The algebra* $\hat{\mathbf{U}}_n$. Given $n \in \mathbb{N}$ let $[1,n]$ be the set of integers $\{1,\cdots,n\}$. Let $\hat{\mathbf{U}}_n$ be the quantum loop algebra over $\mathbb{C}$ of type $A_n^{(1)}$; we refer the reader to [5] for precise definitions. For our purposes, it is enough to recall that $\hat{\mathbf{U}}_n$ is a Hopf algebra and is generated as an algebra by elements $x_{i,s}^\pm$, $k_i^{\pm 1}$, $\phi_{i,r}^\pm$ $i \in [1,n]$ and $s \in \mathbb{Z}$, $r \in \mathbb{Z}\setminus\{0\}$. The algebra generated by the elements $k_i^{\pm 1}$, $\phi_{i,r}^\pm$, $i \in [1,n]$, $r \in \mathbb{Z}\setminus\{0\}$ is denoted $\hat{\mathbf{U}}_n^0$ and is a commutative subalgebra of $\hat{\mathbf{U}}_n$.

Given $a \in \mathbb{C}^\times$ let $\tau_a : \hat{\mathbf{U}}_n \to \hat{\mathbf{U}}_n$ be the Hopf algebra homomorphism given by

$$x_{i,s}^\pm \to a^s x_{i,s}^\pm, \quad \tau_a(k_i^{\pm 1}) = k_i^{\pm 1}, \quad \tau_a(\phi_{i,r}^\pm) = a^r \phi_{i,r}^\pm,$$

where $i \in [1,n]$, $s \in \mathbb{Z}$ and $0 \neq r \in \mathbb{Z}$.

1.1.2. *The group* $\mathcal{P}_n$. Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^+$) be the (multiplicative) free abelian group (resp. monoid) generated by elements of the set $\{\boldsymbol{\omega}_{i,a} : i \in [1,n], \quad a \in \mathbb{Z}\}$. The elements of $\mathcal{P}_n$ are called ℓ–weights and those of $\mathcal{P}_n^+$ the dominant ℓ–weights. Let P be the free (additive) abelian group on generators $\{\omega_i : i \in [1,n]\}$ and P^+ the corresponding monoid. Define a morphism of groups by extending the assignment

$$\mathrm{wt} : \mathcal{P} \to P, \quad \mathrm{wt}\,\boldsymbol{\omega}_{i,a} = \omega_i, \quad i \in [1,n],\ a \in \mathbb{Z}.$$

It is also convenient to identify $\mathcal{P}_n^+$ with the monoid consisting of n-tuple of polynomials by extending the assignment

$$\boldsymbol{\omega}_{i,a} \mapsto (1 - \delta_{i,j} q^a u)_{j \in [1,n]}$$

to a multiplicative homomorphism.

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1.1.3. *The category $\mathscr{F}_n$.* type 1? also mention that q is not a root of unity? Let $\mathscr{F}_n$ be the category of finite–dimensional representations of $\hat{\mathbf{U}}_n$. The Hopf algebra structure on $\hat{\mathbf{U}}_n$ makes $\mathscr{F}_n$ a monoidal rigid tensor category. For $a \in \mathbb{C}^\times$ and $V \in \mathscr{F}_n$, let $\tau_a V$ be the corresponding object of $\mathscr{F}_n$. Clearly

$$\tau_a(V \otimes W) \cong \tau_a V \otimes \tau_a W.$$

Let $\mathcal{K}_0(\mathscr{F}_n)$ be the corresponding Grothendieck ring of $\mathscr{F}_n$ and let $[V]$ denote the class of an object V of $\mathscr{F}_n$.

1.1.4. *The modules $W(\boldsymbol{\omega})$ and $V(\boldsymbol{\omega})$ and the category $\mathscr{F}_{n,\mathbb{Z}}$.* For $\boldsymbol{\omega} \in \mathcal{P}_n^+$ let $W(\boldsymbol{\omega})$ be the $\hat{\mathbf{U}}_n$–module generated by an element $v_{\boldsymbol{\omega}}$ satisfying the relations

$$x_{i,s}^+ v_{\boldsymbol{\omega}} = 0 = (x_{i,0}^-)^{\deg \pi_i(u)+1} v_{\boldsymbol{\omega}}, \quad k_i v_{\boldsymbol{\omega}} = q^{\deg \pi_(u)} v_{\boldsymbol{\omega}}, \quad \phi_{i,r}^\pm v_{\boldsymbol{\omega}} = \gamma_{i,r}^\pm v_{\boldsymbol{\omega}}, \quad s, r \in \mathbb{Z}, \quad r \neq 0,$$

where $\gamma_{i,r}^\pm \in \mathbb{C}(q)$ are defined by

$$\sum_{r=0}^{\infty} \gamma_{i,\pm r}^\pm u^{\pm r} = q^{\deg \pi_i} \frac{\pi_i(q^{-1}u)}{\pi_i(qu)}, \qquad \boldsymbol{\omega} = (\pi_i(u))_{i \in I}.$$

Any quotient of $W(\boldsymbol{\omega})$ is called an ℓ–highest weight module with highest ℓ–weight $\boldsymbol{\omega}$ and we will continue to denote by $v_{\boldsymbol{\omega}}$ the image of the generator of $W(\boldsymbol{\omega})$ in any quotient. The module $W(\boldsymbol{\omega})$ is finite–dimensional and has an unique irreducible quotient which we denote as $V(\boldsymbol{\omega})$. Finally, any irreducible module in $\mathscr{F}_n$ is isomorphic to a tensor product of objects of the form $\tau_b V(\boldsymbol{\omega})$ for some $b \in \mathbb{C}^\times$ and $\boldsymbol{\omega} \in \mathcal{P}_n^+$.

Let $\mathscr{F}_{n,\mathbb{Z}}$ be the full subcategory of $\mathscr{F}_n$ whose Jordan–Holder constituents are of the form $V(\boldsymbol{\omega})$, $\boldsymbol{\omega} \in \mathcal{P}_n^+$. It is well–known that $\mathscr{F}_{n,\mathbb{Z}}$ is a rigid tensor subcategory of $\mathscr{F}_n$ and we let $\mathcal{K}_0(\mathscr{F}_{n,\mathbb{Z}})$ be the corresponding Grothendieck ring.

1.2. Higher order KR–modules and the first main theorem.

1.2.1. *The set $S_{i,n}$ and (i,n)–segments.* For $i \in [1,n]$, let

$$S_{i,n} = \{2j : 1 \le j \le \min\{i, n+1-i\}\} = S_{n+1-i,n}. \tag{1.1}$$

Definition. We shall say that an element $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$ is an (i,n)-segment of length r if $a_p - a_{p-1} \in S_{i,n}$ for all $2 \le p \le r$. □

Since $0 \notin S_{i,n}$ the entries of $\mathbf{a}$ are all distinct and so in what follows we will also think of segments as sets.

Example. If $n = 3$ we have

$$S_{1,3} = \{2\} = S_{3,3}, \quad S_{2,3} = \{2,4\}.$$

Moreover, the element $\mathbf{a} = (0,4,6,10)$ is the union of three $(1,3)$-segments (and also $(3,3)$–segments): (0), $(4,6)$ and (10). However $\mathbf{a}$ is a $(2,3)$–segment of length 4.

1.2.2. *General and special position of segments.*

Definition. Say that two (i,n)-segments $\mathbf{a} = (a_1, \cdots, a_r)$ and $\mathbf{b} = (b_1, \cdots, b_s)$ are in general position if their union does not contain an (i,n)–segment of length greater than $\max\{r, s\}$. Otherwise we say that they are in special position. □

Examples.

(i) An (i,n)-segment is in general position with itself.

(ii) Suppose that $n = 3$ and consider the $(2,3)$–segments

$$\mathbf{a} = (0, 2, 6, 10), \quad \mathbf{b} = (4, 10), \quad \mathbf{c} = (16, 18).$$

Then $\mathbf{a}$ and $\mathbf{b}$ are in special position since their union contains the $(2,3)$-segment $(0, 2, 4, 6, 10)$ while $\mathbf{a}, \mathbf{c}$ (and also $\mathbf{b}$, $\mathbf{c}$) are in general position.

1.2.3. *The KR–modules of type* (i,n). Given $i \in [1, n]$ let $\mathcal{P}^+_{i,n}$ be the submonoid of $\mathcal{P}^+_n$ generated by the elements $\boldsymbol{\omega}_{i,a}$, $a \in \mathbb{Z}$. For $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$ set

$$\boldsymbol{\omega}_{i,\mathbf{a}} = \boldsymbol{\omega}_{i,a_1} \cdots \boldsymbol{\omega}_{i,a_r} \in \mathcal{P}^+_{i,n}.$$

Definition. Given $\boldsymbol{\omega} \in \mathcal{P}^+_{i,n}$ we say that $V(\boldsymbol{\omega})$ is a KR–module of type (i,n) if there exists an (i,n)–segment $\mathbf{a}$ such that $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{a}}$. □

We note that the usual KR–module for $\hat{\mathbf{U}}_n$ is of the form $\boldsymbol{\omega}_{i,\mathbf{a}}$ where $\mathbf{a} = (a, a+2, \cdots a+2r-2)$ for some $a \in \mathbb{Z}$ and $r \geq 1$. We refer to the KR–modules associated with more general segments as the higher order KR–modules since they encode the reducibility data of $V(\boldsymbol{\omega}_{i,a}) \otimes V(\boldsymbol{\omega}_{i,b})$ in higher rank.

Remark. The modules $V(\boldsymbol{\omega}_{i,\mathbf{a}})$ where $\mathbf{a}$ is an (i,n)–segment are a special family of snake modules studied in [20, 21].

1.2.4. *A prime factorization result.* We can now state our first main theorem, which generalizes the result of [4] in the rank one case.

Theorem 1. Given $\boldsymbol{\omega} \in \mathcal{P}^+_{i,n}$ there exists a unique integer $k \geq 1$ and unique (upto permutation) (i,n)-segments $\mathbf{a}_1, \cdots, \mathbf{a}_k$ which are in pairwise general position such that

$$V(\boldsymbol{\omega}) \cong V(\boldsymbol{\omega}_{i,\mathbf{a}_1}) \otimes \cdots \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}_k}).$$

In particular $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega})$ is irreducible for all $\boldsymbol{\omega} \in \mathcal{P}^+_{i,n}$. Moreover $V(\boldsymbol{\omega})$ is prime iff there exists an (i,n) segment $\mathbf{a}$ with $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{a}}$.

1.3. The second main theorem: an inflation of Grothendieck rings.

1.3.1. *The ℓ–weight space decomposition and q–characters.* Given any object V of $\mathscr{F}_n$ we can regard it as a module for the commutative subalgebra $\hat{\mathbf{U}}^0_n$. It follows that we can write V as a direct sum of generalized eigenspaces for the action of this subalgebra. The generalized eigenspaces are called ℓ–weight spaces and it was proved in [10] that if V is an object of $\mathscr{F}_{n,\mathbb{Z}}$ then the ℓ–weight spaces are indexed by elements of $\mathcal{P}_n$ in a natural way and so,

$$V = \bigoplus_{\boldsymbol{\omega} \in \mathcal{P}_n} V_{\boldsymbol{\omega}}, \quad \mathrm{wt}_\ell\, V = \{\boldsymbol{\omega} \in \mathcal{P}_n : V_{\boldsymbol{\omega}} \neq 0\}.$$

The q–character of V is the element $\chi_{q,n}(V)$ of the group ring of $\mathbb{Z}[\mathcal{P}_n]$ given by

$$\chi_{q,n}(V) = \sum_{\boldsymbol{\omega}\in\mathcal{P}_n} \dim V_{\boldsymbol{\omega}} e(\boldsymbol{\omega}).$$

One also has the usual weight space decomposition of V in terms of the generators $K_i^{\pm 1}$; namely

$$V = \bigoplus_{\lambda\in P} V_\lambda, \qquad V_\lambda = \bigoplus_{\substack{\boldsymbol{\omega}\in\mathcal{P}\\ \operatorname{wt}\boldsymbol{\omega}=\lambda}} V_{\boldsymbol{\omega}}.$$

Given any subgroup $\mathcal{G}$ of $\mathcal{P}_n$ the corresponding truncated q–character is given as follows:

$$\chi_{q,n}^{\mathcal{G}}(V) = \sum_{\boldsymbol{\omega}\in\mathcal{G}} \dim V_{\boldsymbol{\omega}} e(\boldsymbol{\omega}) \in \mathbb{Z}[\mathcal{G}].$$

1.3.2. *The subgroups $\mathcal{G}_n$ and the category $\mathscr{C}_n^-$.* Given $n\in\mathbb{N}$ let $\mathcal{G}_n$ be the subgroup of $\mathcal{P}_n$ generated by the elements

$$\{\boldsymbol{\omega}_{p,r} : r-p\in 2\mathbb{Z},\ p\in[1,n], \quad r\in(-\infty,0]\},$$

and let $\mathcal{G}_n^+$ be the corresponding monoid.

The category $\mathscr{C}_n^-$ is defined to be the full subcategory of $\mathscr{F}_n$ whose simple factors are of the form $V(\boldsymbol{\omega})$, $\boldsymbol{\omega}\in\mathcal{G}_n^+$. For $V\in\mathscr{C}_n^-$, set

$$\chi_{q,n}^-(V) = \chi_{q,n}^{\mathcal{G}_n}(V).$$

The following was proved in [15].

Proposition. *The category $\mathscr{C}_n^-$ is a monoidal tensor category and the assignment*

$$[V(\boldsymbol{\omega})] \mapsto \chi_{q,n}^-(V(\boldsymbol{\omega})), \quad \boldsymbol{\omega}\in\mathcal{G}_n^+,$$

defines an injective homomorphism of rings $\chi_{q,n}^- : \mathcal{K}_0(\mathscr{C}_n^-)\to\mathbb{Z}[\mathcal{G}_n]$. In particular the image of $\chi_{q,n}^-$ is a polynomial subalgebra of $\mathbb{Z}[\mathcal{G}_n]$. □

1.3.3. *The homomorphism $\Phi_{\bar{i},n}$, the category $\mathcal{C}_{i,n}^-$ and the second main result. Assume that $n\in\mathbb{N}$ and $i\in[1,n]$ are such that $n+1 = i(\bar{i}+1)$ for some $\bar{i}$.*

Define a homomorphism $\Phi_{\bar{i},n} : \mathcal{P}_{\bar{i}}^+ \to \mathcal{P}_n^+$ by

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{j_1,a_1}\cdots\boldsymbol{\omega}_{j_k,a_k} \in \mathcal{P}_{\bar{i}}^+ \mapsto \Phi_{\bar{i},n}(\boldsymbol{\omega}) = \boldsymbol{\omega}_{ij_1,ia_1}\cdots\boldsymbol{\omega}_{ij_k,ia_k} \in \mathcal{P}_n^+.$$

Let $\mathscr{C}_{i,n}^-$ be the full subcategory of $\mathscr{F}_n$ consisting of objects whose Jordan–Holder components are of the form $V(\phi(\boldsymbol{\omega}))$, $\boldsymbol{\omega}\in\mathcal{G}_{\bar{i}}^+$.

Our second main result is the following.

Theorem 2. The category $\mathscr{C}_{i,n}^-$ is a monoidal tensor category and we have an isomorphism of Grothendieck rings

$$\Phi_{\bar{i},n} : \mathcal{K}_0(\mathscr{C}_{\bar{i}}^-) \to \mathcal{K}_0(\mathscr{C}_{i,n}^-) \ \text{ such that } \ \Phi_{\bar{i},n}[V(\boldsymbol{\omega})] = [V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))], \quad \boldsymbol{\omega}\in\mathcal{G}_{\bar{i}}^+.$$

In particular, we have a geometric q–character formula for $V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))$.

1.3.4. *Monoidal Categorification.* Recall from [14] the category $\mathscr{C}_r$ which consists of objects V in $\mathscr{F}_n$ whose Jordan–Holder constituents lie in the submonoid of $\mathcal{G}_n$ generated by the elements $\{\boldsymbol{\omega}_{j,a} : j \in [1,n],\ -2r-1 \leq a \leq 0\}$. It was proved in [14] for $r = 1$ and in [23] in general that the Grothendieck ring $\mathcal{K}_0(\mathscr{C}_r)$ is the monoidal categorification of a cluster algebra. In other words, there exists a cluster algebra $\mathcal{A}$ and an injective homomorphism $\mathcal{A} \to \mathcal{K}_0(\mathscr{C}_r)$ which maps a cluster variable to the class of a prime representation and a cluster monomial to the class of irreducible tensor product of representations. Define $\mathscr{C}_{i,r}$ in the obvious way. The following is now an immediate consequence of Theorem 2.

Proposition. *The category $\mathscr{C}_{i,r}$ is a monoidal tensor category and hence the ring $\mathcal{K}_0(\mathscr{C}_{i,r})$ is a monoidal categorification of a cluster algebra.*

1.4. Imaginary Modules. Recall that a module $V(\boldsymbol{\omega})$ for $\hat{\mathbf{U}}_n$ is said to be real if its tensor square is irreducible. Otherwise, we call the module imaginary.

The first example of imaginary modules was given by Leclerc in [19] where he showed that if

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{2,6}\boldsymbol{\omega}_{1,3}\boldsymbol{\omega}_{3,3}\boldsymbol{\omega}_{2,0}$$

then the module $V(\boldsymbol{\omega})$ for $\hat{\mathbf{U}}_4$ is imaginary. In [18] further examples of real and imaginary modules can be found. In both cases the examples come from representations of affine Hecke algebras by using Schur–Weyl duality.

As an illustration of the possible applications of our main results we construct new examples (which do not fit into the framework of [18]) of imaginary irreducible modules and we work entirely inside $\mathscr{F}_n$.

Proposition. *Suppose that $i \in [1,n]$ and $\mathbf{b} = (b_1, \cdots, b_r)$ with $r \geq i$ is an $(i, 2i-3)$–segment and let $\mathbf{a} = (b_1 - 2i, \cdots, b_r - 2i)$. Then one of the factors of Jordan–Holder series of the module $V(\boldsymbol{\omega}_{i,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}})$ is an imaginary irreducible module.*

Remark. We shall see that Leclerc's example corresponds to the case $i = 2$ and $n = 3$ and $\mathbf{b} = (6, 4)$.

We have the following general conjecture.

Conjecture. Retain the assumptions of the proposition set

$$s_j = \frac{1}{2}(b_{j-1} - b_j + 2i), \quad 2 \leq j \leq r,$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i,b_r}\left(\boldsymbol{\omega}_{i-s_r,b_{k-1}-s_r}\boldsymbol{\omega}_{i+s_r,b_{k-1}-s_r}\right)\cdots\left(\boldsymbol{\omega}_{i-s_2,b_1-s_2}\boldsymbol{\omega}_{i+s_2,b_1-s_2}\right)\boldsymbol{\omega}_{i,b_1-n-1}.$$

Then $V(\boldsymbol{\omega})$ is an imaginary module occurring in the Jordan–Holder series of $V(\boldsymbol{\omega}_{i,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}})$.

We shall prove the conjecture in certain special cases which will show that this family of examples are very rarely of the type given in [18].

Note that Proposition **??**(i) proves the conjecture for $i \geq 2$ and $\mathbf{b} = (2k, \cdots, 2i)$ while part (ii) proves it in the case when $i = 3$ and $\mathbf{b} = (12, 10, 6)$. Comments to be made for arbitrary n and the inflation maps

We prove these results in the subsequent sections. Sections 5 and Section 6 are devoted to the proof of Theorem 1. Theorem 2 and Proposition 1.3.4 are proved in Section 2. The proof of Proposition **??** can be found in Section 3.

1.5. Further known facts on the structure of $\mathscr{F}_n$. We conclude this section by stating known results on the category $\mathscr{F}_n$, the ℓ–highest weight modules and their q-characters which will be needed for our study.

1.5.1. *Duals and the Cartan involution.* Let V be an object of $\mathscr{F}_n$. Then V has a left and right dual denoted by V^* and ${}^*V$ respectively, and we have $\hat{\mathbf{U}}_n$–maps

$$\mathbb{C} \hookrightarrow V^* \otimes V, \quad V \otimes {}^*V \to \mathbb{C} \to 0.$$

Moreover if we set

$$\boldsymbol{\omega}_{i,a}^* = \boldsymbol{\omega}_{n+1-i,a+n+1}, \quad {}^*\boldsymbol{\omega}_{i,a} = \boldsymbol{\omega}_{n+1-i,a-n-1},$$

we get corresponding endomorphisms $\boldsymbol{\omega} \mapsto \boldsymbol{\omega}^*$ and $\boldsymbol{\omega} \mapsto {}^*\boldsymbol{\omega}$ of $\mathcal{P}$ and

$$V(\boldsymbol{\omega})^* \cong V(\boldsymbol{\omega}^*), \quad {}^*V(\boldsymbol{\omega}) \cong V({}^*\boldsymbol{\omega}).$$

We shall freely use properties of duals, in particular, the isomorphisms

$$(U \otimes V)^* \cong V^* \otimes U^*, \quad {}^*(U \otimes V) \cong {}^*V \otimes {}^*U,$$

$$\operatorname{Hom}_{\hat{\mathbf{U}}_n}(V \otimes U, W) \cong \operatorname{Hom}_{\hat{\mathbf{U}}_n}(U, V^* \otimes W), \quad \operatorname{Hom}_{\hat{\mathbf{U}}_n}(U \otimes V, W) \cong \operatorname{Hom}_{\hat{\mathbf{U}}_n}(U, W \otimes {}^*V).$$

Motivated by this, we have the following definition.

Definition. Given an (i, n)–segment $\mathbf{a} = (a_1, \cdots, a_k)$ we define the $(n + 1 - i, n)$–segments $\mathbf{a}^* = (a_1 + n + 1, \cdots, a_k + n + 1)$ and ${}^*\mathbf{a} = (a_1 - n - 1, \cdots, a_k - n - 1)$. □

The quantum affine analog of the Cartan involution of A_n is the algebra homomorphism and coalgebra anti automorphism $\Omega : \hat{\mathbf{U}}_n \to \hat{\mathbf{U}}_n$ given by

$$\Omega(x_{i,s}^{\pm}) = -x_{i,-s}^{\mp}, \quad \Omega(\phi_{i,r}^{\pm}) = \phi_{i,-r}^{\mp}, \quad \Omega(k_i^{\pm 1}) = k_i^{\mp 1},$$

for $i \in [1, n]$, $s \in \mathbb{Z}$ and $0 \neq r \in \mathbb{Z}$. If U, V are objects of $\mathscr{F}_n$ and $\boldsymbol{\omega} \in \mathcal{P}_n^+$, we have isomorphisms

$$\Omega(U \otimes V) \cong \Omega(V) \otimes \Omega(U), \quad \Omega(V(\boldsymbol{\omega})) \cong V(\Omega(\boldsymbol{\omega})), \quad \Omega(\boldsymbol{\omega}_{i,a}) = \boldsymbol{\omega}_{n+1-i,-a}.$$

1.5.2. *Tensor products.* Part (i) of the next proposition was proved in [5, 6]. Part (ii) was proved independently in [1] and [2] and part (iii) in [2].

Proposition. *Suppose that $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \in \mathcal{P}_n^+$ with $a_1 \leq \cdots \leq a_k$.*

(i) Let $\boldsymbol{\omega}' \in \mathcal{P}_n^+$. The module $V(\boldsymbol{\omega}\boldsymbol{\omega}')$ occurs with multiplicity one in the Jordan–Holder series of $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega}')$. Moreover, $V(\boldsymbol{\omega}\boldsymbol{\omega}')$ is isomorphic to $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega}')$ iff $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega}')$ and its left (or right) dual are ℓ–highest weight modules.

(ii) We have,

$$W(\boldsymbol{\omega}) \cong V(\boldsymbol{\omega}_{i_k,a_k}) \otimes \cdots \otimes V(\boldsymbol{\omega}_{i_1,a_1})$$

and hence for all $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}_n^+$ the following holds in $\mathcal{K}_0(\mathscr{F}_n)$:

$$[W(\boldsymbol{\omega}\boldsymbol{\omega}')] = [W(\boldsymbol{\omega})][W(\boldsymbol{\omega}')].$$

(iii) The module $V(\boldsymbol{\omega}_{i_1,a_1})\otimes V(\boldsymbol{\omega}_{i_2,a_2})$ is ℓ–highest weight iff

$$a_2-a_1\notin\{2p+2-i_1-i_2:\max\{i_1,i_2\}\le p<\min\{i_1+i_2-1,n\}\}.$$

□

1.5.3. *q–characters and a result of Frenkel–Reshetikhin.* Recall from Section 1.3.1 the definition of the q–character of an object of $\mathscr{F}_{n,\mathbb{Z}}$. The following was proved in [10].

Theorem 3. The assignment $[V]\mapsto\chi_q(V)$ defines an injective homomorphism of rings

$$\chi_q:\mathcal{K}_0(\mathscr{F}_{n,\mathbb{Z}})\to\mathbb{Z}[\mathcal{P}_n].$$

Moreover, $\mathcal{K}_0(\mathscr{F}_{n,\mathbb{Z}})$ is a polynomial ring in the generators $[V(\boldsymbol{\omega}_{i,a})]$ with $i\in[1,n]$ and $a\in\mathbb{Z}$. In particular if V and V' are objects of $\mathscr{F}_{n,\mathbb{Z}}$ we have,

$$\mathrm{wt}_\ell(V\otimes V')=\mathrm{wt}_\ell V\,\mathrm{wt}_\ell V'.$$

□

1.5.4. *ℓ–lowest weight modules.* An ℓ–lowest weight module is defined in the obvious way; it is generated by an element v which is an eigenvector for the elements $\phi^{\pm}_{i,r}$ and $x^-_{i,r}v=0$.

Proposition. *(i) Any ℓ–highest weight module with ℓ–highest weight $\boldsymbol{\omega}$ in $\mathscr{F}_n$ is also a lowest ℓ–weight module with lowest weight $(\boldsymbol{\omega}^*)^{-1}$.*

(ii) Let V,V' be ℓ–highest weight modules with ℓ–highest weight $\boldsymbol{\omega},\boldsymbol{\omega}'\in\mathcal{P}_n^+$ respectively. Let v^- and v^+ be non–zero lowest and highest ℓ–weights of V and V'. Then $v^-\otimes v^+$ is an ℓ–weight vector with ℓ–weight $(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega}'$ and

$$V\otimes V'=\hat{\mathbf{U}}_n(v^-\otimes v^+).$$

In particular if U is a proper quotient of $V\otimes V'$ then $\dim U_{(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega}'}\neq 0$.

Proof. We sketch a proof. Let $\lambda=\mathrm{wt}\,\boldsymbol{\omega}$. Since V is an ℓ–highest weight module we have $\mathrm{wt}\,V\subset\lambda-Q^+$ and $\dim V_\lambda=1$. Since V is a finite–dimensional module for $\hat{\mathbf{U}}_n$ and hence also for $\mathbf{U}_n$ $\mathbf{U}_n$ hasnt been defined it follows that $\dim V_{w_\circ\lambda}=1$ where $w_\circ$ is the longest element of the Weyl group S_{n+1} of A_n; in particular any non–zero element of $V_{w_\circ\lambda}$ is an ℓ–weight vector. It was shown in [2] that if $V=V(\boldsymbol{\omega})$ then $V_{w_\circ\lambda}$ was an ℓ–weight space with ℓ–weight $(\boldsymbol{\omega}^*)^{-1}$. Since $V(\boldsymbol{\omega})$ is a quotient of any ℓ–highest weight module with ℓ–weight $\boldsymbol{\omega}$, part (i) follows. Part (ii) is immediate from the formulae for the comultiplication [7] (see also [2]). □

1.5.5. *The ℓ–root lattice and diagram subalgebras.* For $i\in[1,n]$ and $r\in\mathbb{Z}$ let $\boldsymbol{\alpha}_{i,r}\in\mathcal{P}_n$ be defined by

$$\boldsymbol{\alpha}_{i,a}=\boldsymbol{\omega}_{i-1,a}^{-1}\boldsymbol{\omega}_{i,r-1}\boldsymbol{\omega}_{i,a+1}\boldsymbol{\omega}_{i+1,a}^{-1},$$

and let $\mathcal{Q}_n$ be the subgroup of $\mathcal{P}_n$ generated by these elements and let $\mathcal{Q}_n^+$ be defined in the obvious way. define Q and Q^+.Then it is known (see for instance, [3]) that

$$\mathrm{wt}_\ell\,W(\boldsymbol{\omega})\subset\boldsymbol{\omega}(\mathcal{Q}_n^+)^{-1}.$$

Given $J\subset[1,n]$ let $\hat{\mathbf{U}}_{n,J}$ be the subalgebra of $\hat{\mathbf{U}}_n$ generated by the elements $x^{\pm}_{j,r},k_j^{\pm1},\phi^{\pm}_{j,s}$ with $j\in J$. Let $\mathcal{P}_{n,J}$ be the subgroup of $\mathcal{P}_n$ generated by the elements $\boldsymbol{\omega}_{j,c}$ with $j\in J$ and $c\in\mathbb{Z}$ and define $\mathcal{P}^+_{n,J}$, $\mathcal{Q}_{n,J}$ and $\mathcal{Q}^+_{n,J}$ in the obvious way. Define a homomorphism $\mathcal{P}_n\to\mathcal{P}_{n,J}$ sending $\boldsymbol{\omega}\mapsto\boldsymbol{\omega}_J$ by extending the assignment

$$\boldsymbol{\omega}_{i,c}\mapsto\boldsymbol{\omega}_{i,c},\quad i\in J,\quad \boldsymbol{\omega}_{i,c}\mapsto\mathbf{1},\quad i\notin J.$$

Let $V(\boldsymbol{\omega}_J)$ be an irreducible ℓ–highest weight module for $\hat{\mathbf{U}}_{n,J}$ with ℓ–highest weight $\boldsymbol{\omega}_J$.

The following well–known lemma is immediate from the results of [9].

Lemma. *Suppose that V is an object of $\mathcal{F}_{n,\mathbb{Z}}$ and let $J \subset [1,n]$. Suppose $\boldsymbol{\omega} \in \mathcal{P}_n$ and $0 \neq v \in V_{\boldsymbol{\omega}}$ are such that $\hat{\mathbf{U}}_{n,J}v$ is an ℓ-highest weight module for $\hat{\mathbf{U}}_{n,J}$. Then*

$$\boldsymbol{\alpha} \in \mathcal{Q}^+_{n,J}, \quad \boldsymbol{\omega}_J\boldsymbol{\alpha}_J^{-1} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_J) \implies \boldsymbol{\omega}\boldsymbol{\alpha}^{-1} \in \mathrm{wt}_\ell\, \hat{\mathbf{U}}_n v.$$

□

1.5.6. *Tensor products and diagram subalgebras.* Given $J \subset [1,n]$ and $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}_n^+$ we have an isomorphism of $\hat{\mathbf{U}}_{n,J}$–modules $V(\boldsymbol{\omega}_J) \cong \hat{\mathbf{U}}_{n,J}v_{\boldsymbol{\omega}} \subset V(\boldsymbol{\omega})$.

$$V(\boldsymbol{\omega}_J) \otimes V(\boldsymbol{\omega}'_J) \cong \hat{\mathbf{U}}_{n,J}v_{\boldsymbol{\omega}} \otimes \hat{\mathbf{U}}_{n,J}v_{\boldsymbol{\omega}'}.$$

In particular if $v \in V(\boldsymbol{\omega}_J) \otimes V(\boldsymbol{\omega}'_J)$ is an ℓ–highest weight vector with ℓ–highest $\boldsymbol{\omega}_J\boldsymbol{\omega}'_J\boldsymbol{\alpha}_J^{-1}$ with $\boldsymbol{\alpha} \in \mathcal{Q}^+_{n,J}$ then $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega}')$ has an ℓ–highest weight vector with ℓ–highest $\boldsymbol{\omega}\boldsymbol{\omega}'\boldsymbol{\alpha}^{-1}$,

1.6. Some results of Mukhin and Young. We recall special cases of the results of Mukhin and Young established in [20, 21] which will play an important role in the paper.

1.6.1. *The sets $\mathbb{P}_{i,a}$ and $\mathbb{P}_{i,\mathbf{a}}$.* For $i \in [1,n]$ and $a \in \mathbb{Z}$, let $\mathbb{P}_{i,a}$ be the set of all functions $p : [0, n+1] \to \mathbb{Z}$ satisfying the following:

$$p(0) = i + a, \quad p(r+1) - p(r) \in \{-1, 1\}, \quad 0 \le r \le n, \quad p(n+1) = n + 1 - i + a.$$

For $p \in \mathbb{P}_{i,a}$ set

$$\mathbf{c}_p^{\pm} = \{r \in [1,n] : p(r-1) = p(r) \pm 1 = p(r+1)\},$$

$$\boldsymbol{\omega}(p) = \prod_{r \in \mathbf{c}_p^+} \boldsymbol{\omega}_{r,p(r)} \prod_{r \in \mathbf{c}_p^-} \boldsymbol{\omega}^{-1}_{r,p(r)} \in \mathcal{P}_n.$$

In particular $\boldsymbol{\omega}(p)$ is in the subgroup of $\mathcal{P}_n$ generated by the elements $\{\boldsymbol{\omega}_{j,p(j)} : j \in \mathbf{c}_p^+ \cup \mathbf{c}_p^-\}$.

Let $p_{i,a}$ and $p^*_{i,a}$ be the elements of $\mathbb{P}_{i,a}$ given as follows:

$$p_{i,a}(j) = \begin{cases} i - j + a, & 0 \le j \le i, \\ j - i + a, & i < j \le n+1, \end{cases} \qquad p^*_{i,a}(j) = \begin{cases} a + i + j, & 0 \le j \le n + 1 - i, \\ a + 2n + 2 - i - j, & n + 2 - i \le j \le n + 1. \end{cases}$$

Then

$$\boldsymbol{\omega}(p_{i,a}) = \boldsymbol{\omega}_{i,a}, \quad \boldsymbol{\omega}(p^*_{i,a}) = \boldsymbol{\omega}^{-1}_{n+1-i,a+n+1}.$$

The following is a simple calculation.

Lemma. *Let a, b, c be integers $b - a = 2m_1$ and $c - b = 2m_2$ for some $m_1, m_2 \in \mathbb{N}$. Then, for all $j \in [0, n+1]$ we have*

$$p_{i,a}(j) < p(j) < p^*_{i,c}(j), \quad p \in \mathbb{P}_{i,b}.$$

□

Given $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$ of length $r \geq 1$, set

$$\mathbb{P}_{i,\mathbf{a}} = \{(p_1, \cdots, p_r) : p_j \in \mathbb{P}_{i,a_j}, \ \ p_j(k) < p_s(k) \text{ for all } k \in [0, n+1], \ \ 1 \leq j < s \leq r\}, \quad (1.2)$$

$$\boldsymbol{\omega}(\underline{p}) = \boldsymbol{\omega}(p_1) \cdots \boldsymbol{\omega}(p_r), \ \ \underline{p} = (p_1, \cdots, p_r) \in \mathbb{P}_{i,\mathbf{a}}. \quad (1.3)$$

Remark. The condition that $\underline{p} \in \mathbb{P}_{i,\mathbf{a}}$ guarantees that the expression on the right hand side of (1.3) is a reduced word in $\mathcal{P}_n^+$.

1.6.2. Given $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$ and $\mathbf{b} = (b_1, \cdots, b_s) \in \mathbb{Z}^s$ set

$$\mathbf{a} \vee \mathbf{b} = (a_1, \cdots, a_r, b_1, \cdots, b_s).$$

The next proposition is a special case of the main result of [20, 21]. We remark that those papers do not use the language of (i,n)–segments. However it is not hard to see that the module $V(\boldsymbol{\omega}_{i,\mathbf{a}})$ associated to a (i,n)–segment $\mathbf{a}$ satisfies the restrictions of that paper.

Proposition. *Let $\mathbf{a} = (a_1, \cdots, a_r)$ be an (i,n)-segment.*

(i) Suppose that $\mathbf{c}$ is an (i,n)–segment such that either $\mathbf{a} \vee \mathbf{c}$ or $\mathbf{c} \vee \mathbf{a}$ is also an (i,n)–segment. Then $V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{c}})$ is reducible.

(ii) We have

$$\mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}}) = \{\boldsymbol{\omega}(\underline{p}) : \underline{p} \in \mathbb{P}_{i,\mathbf{a}}\}, \quad \dim V(\boldsymbol{\omega}_{i,\mathbf{a}}) = \# \, \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}}),$$

and

$$\mathcal{P}^+ \cap \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}}) = \{\boldsymbol{\omega}_{i,a_1} \cdots \boldsymbol{\omega}_{i,a_r}\}.$$

(iii) Let $1 \leq m_r \leq \min\{i, n+1-i\}$ and take $\mathbf{b} = (a_2, \cdots, a_r, a_r + 2m_r)$. Then $\mathbf{b}$ is an (i,n)–segment and the following equality holds in $\mathcal{K}_0(\mathscr{F}_n)$:

$$[V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}})] = [V(\boldsymbol{\omega}_{i,\mathbf{a}} \boldsymbol{\omega}_{i,a_r+2m_r})][V(\boldsymbol{\omega}_{i,\mathbf{b}} \boldsymbol{\omega}_{i,a_r+2m_r}^{-1})] + [V(\boldsymbol{\omega}^+)][V(\boldsymbol{\omega}^-)]$$

where

$$\boldsymbol{\omega}^{\pm} = \boldsymbol{\omega}_{i \pm m_1, a_1+m_1} \cdots \boldsymbol{\omega}_{i \pm m_r, a_r + m_r}, \quad 2m_j = a_{j+1} - a_j, \quad 1 \leq j \leq r-1.$$

Moreover,

$$\boldsymbol{\omega}^+ \boldsymbol{\omega}^- \notin \mathrm{wt}_\ell (V(\boldsymbol{\omega}_{i,\mathbf{a}} \boldsymbol{\omega}_{i,a_r+2m_r}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}} \boldsymbol{\omega}_{i,a_r+2m_r}^{-1})).$$

□

1.6.3. We note some consequences of Lemma 1.6.1 and Proposition 1.6.2 for later use.

Proposition. *Suppose that $\mathbf{a} = (a_1, \cdots, a_r)$ is an (i,n)-segment and for $1 \leq j \leq s \leq r$ let $\mathbf{a}_{j,s} = (a_j, \cdots, a_s)$.*

(i) We have

$$(p_1, \cdots, p_r) \in \mathbb{P}_{i,\mathbf{a}} \implies (p_j, \cdots, p_s) \in \mathbb{P}_{i,\mathbf{a}_{j,s}}.$$

(ii) Suppose that

$$\boldsymbol{\omega}' = \boldsymbol{\omega}_{i,a_1} \cdots \boldsymbol{\omega}_{i,a_{j-1}} \boldsymbol{\omega} \boldsymbol{\omega}_{n+1-i, a_{s+1}+n+1}^{-1} \cdots \boldsymbol{\omega}_{n+1-i, a_r+n+1}^{-1},$$

where $\boldsymbol{\omega}$ is in the subgroup of $\mathcal{P}_n$ generated by $\boldsymbol{\omega}_{m,b}$, $m \in [1,n]$ and $a_j \leq b \leq a_s + n + 1$. Then

$$\boldsymbol{\omega} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}_{j,s}}) \iff \boldsymbol{\omega}' \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}}).$$

Proof. Part (i) is immediate from Proposition 1.6.2(ii). If $\boldsymbol{\omega} \in \operatorname{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}_{j,s}})$ let $\underline{p}_{j,s} = (p_j, \cdots, p_s) \in \mathbb{P}_{i,\mathbf{a}_{j,s}}$ be such that $\boldsymbol{\omega}(\underline{p}_{j,s}) = \boldsymbol{\omega}$. It is immediate from Lemma 1.6.1(i) that

$$\underline{p} = (p_{i,0}^{a_1}, \cdots p_{i,0}^{a_{j-1}}, p_j, \cdots, p_s, p_{i,*}^{a_{s+1}}, \cdots p_{i,*}^{a_r}) \in \mathbb{P}_{i,\mathbf{a}},$$

and hence by Proposition 1.6.2(ii) again, we have $\boldsymbol{\omega}' = \boldsymbol{\omega}(\underline{p}) \in \operatorname{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}})$. To prove the converse let $\underline{p} = (p_1, \cdots, p_r)$ be such that $\boldsymbol{\omega}(\underline{p}) = \boldsymbol{\omega}'$. Note that by our assumption on $\boldsymbol{\omega}$ we have that $\boldsymbol{\omega}_{i,a_1} \cdots \boldsymbol{\omega}_{i,a_{j-1}}$ must occur in $\boldsymbol{\omega}(\underline{p})$. This can only happen if $(p_1, \cdots, p_j) = (p_{i,0}^{a_1}, \cdots, p_{i,0}^{a_{j-1}})$. Similarly, $\boldsymbol{\omega}_{n+1-i,a_{s+1}+n+1-i}^{-1} \cdots \boldsymbol{\omega}_{n+1-i,a_r+n+1-i}^{-1}$ must occur in $\boldsymbol{\omega}(\underline{p})$ and this can only happen if $(p_{s+1}, \cdots, p_r) = (p_{i,*}^{a_{s+1}}, \cdots, p_{i,*}^{a_r})$. Therefore, by item (i) of this proposition we have $\underline{p}' = (p_j, \cdots, p_s) \in \mathbb{P}_{i,\mathbf{a}_{j,s}}$ and hence $\boldsymbol{\omega} = \boldsymbol{\omega}(\underline{p}')$. Proposition 1.6.2(ii) implies that $\boldsymbol{\omega} \in V(\boldsymbol{\omega}_{i,\mathbf{a}_{j,s}})$ which completes the proof. □

2. Proof of Theorem 2

Throughout this section we fix $i \in [1, n]$ with $n + 1 = 0 \mod i$ and write $n + 1 = i(\bar{i} + 1)$. Let $\mathcal{H}_{\bar{i}}$ (resp. $\mathcal{H}_{\bar{i}}^+$) be the subgroup (resp. submonoid) of $\mathcal{P}_{\bar{i}}$ generated by the elements of the set

$$\{\boldsymbol{\omega}_{j,a} : j \in [1, \bar{i}], \ j - a \in 2\mathbb{Z}, \quad a \in (-\infty, 0]\}.$$

2.1. The map $\Phi_{\bar{i},n}$. Let $\Phi_{\bar{i},n} : \mathcal{P}_{\bar{i}} \to \mathcal{P}_n$ (resp. $\phi_{\bar{i},n} : P_{\bar{i}} \to P_n$) be the group homomorphism defined by extending the assignment

$$\Phi_{\bar{i},n}(\boldsymbol{\omega}_{j,a}) = \boldsymbol{\omega}_{ij,ia}, \qquad (\text{resp. } \phi_{\bar{i},n}(\omega_j) = \omega_{ij}), \quad j \in [1, \bar{i}], \ a \in \mathbb{Z}$$

Clearly $\operatorname{wt} \circ \Phi_{\bar{i},n} = \phi_{\bar{i},n} \circ \operatorname{wt}$.

Lemma. *We have $\Phi_{\bar{i},n}(\mathcal{Q}_{\bar{i}}^+) \subset \mathcal{Q}_n^+$ and $\phi_{\bar{i},n}(Q_{\bar{i}}^+) \subset Q_n^+$. Moreover for $\boldsymbol{\pi}, \boldsymbol{\omega} \in \mathcal{P}_{\bar{i}}$,*

$$\operatorname{wt} \boldsymbol{\omega} - \operatorname{wt} \boldsymbol{\pi} \in Q_{\bar{i}}^+ \iff \phi_{\bar{i},n}(\operatorname{wt} \boldsymbol{\omega}) - \phi_{\bar{i},n}(\operatorname{wt} \boldsymbol{\pi}) \in Q_n^+.$$

Proof. It is easily checked that

$$\phi_{\bar{i},n}(\alpha_j) = \sum_{p=1}^{i} \sum_{s=p-i}^{i-p} \alpha_{ij+s}.$$

formulate the ℓ–root version similarly.
For the second assertion of the lemma, the forward direction is now immediate. For the converse assume that $\boldsymbol{\omega} - \boldsymbol{\pi} = \sum_{j=1}^n s_j \alpha_j$ with $s_j < 0$ for some j. The assertion follows once we notice that the coefficient of α_{ij} in $(\phi_{\bar{i},n}(\operatorname{wt} \boldsymbol{\omega}) - \phi_{\bar{i},n}(\operatorname{wt} \boldsymbol{\pi}))$ is is_j.

□

More generally, one computes that for $j \in [1, \bar{i}]$ and $a \in \mathbb{Z}$ we that $\Phi_{\bar{i},n}(\boldsymbol{\alpha}_{j,a})$ is equal to

$$\left(\prod_{k=1}^{i-1} \prod_{p=1}^{k} \boldsymbol{\alpha}_{i(j-1)+k,\, i(a+1)-k+2p-2} \boldsymbol{\alpha}_{i(j+1)-k,\, i(a+1)-k+2p-2} \right) \prod_{p=1}^{i} \boldsymbol{\alpha}_{ij,\, i(a+1)-i+2p-2}.$$

In the rest of the section we shall, for ease of notation, set $\Phi = \Phi_{\bar{i},n}$ and $\phi = \phi_{\bar{i},n}$.

2.2. The set $\Phi(\mathrm{wt}_\ell V(\boldsymbol{\omega}_{k,\mathbf{a}}))$. Given $p \in \mathbb{P}_{k,a}$ with $1 \le k \le \bar{i}$, define $\Phi(p) : [0, n+1] \to \mathbb{Z}$ as follows: for $0 \le j < \bar{i}+1$ and $0 \le j' < i$ we have

$$\Phi(p)(n+1) = n+1-ik+ia, \qquad \Phi(p)(ij+j') = \begin{cases} ip(j)+j', & p(j+1)-p(j)=1, \\ ip(j)-j', & p(j+1)-p(j)=-1. \end{cases}$$

A straightforward checking hows that

$$\Phi(p) \in \mathbb{P}_{ik,ia}, \quad \text{and} \quad \boldsymbol{\omega}(\Phi(p)) = \Phi(\boldsymbol{\omega}(p)).$$

Conversely, suppose that $g \in \mathbb{P}_{ik,ia}$ is such that $\boldsymbol{\omega}(g) \in \Phi(\mathcal{P}_{\bar{i}})$. We claim that $g(ir)$ is a multiple of i for all $r \in [0, \bar{i}]$. This follows from the assumption on $\boldsymbol{\omega}(g)$ if $g(ir-1) = g(ir+1)$; otherwise there exists $m_1, m_2 \in [1, n]$ with $m_1 < ir < m_2$ and $g(m_s - 1) = g(m_s + 1)$ for $s = 1, 2$. Since $g(ir) = g(m_1) \pm (ir - m_1)$ the result again follows from our assumptions.

Hence we can define

$$\Phi^{-1}(g) : [0, \bar{i}+1] \to \mathbb{Z}, \quad \Phi^{-1}(r) = g(ir)/i, \quad r \in [0, \bar{i}+1].$$

It is straightforward to see that $\Phi^{-1}(g) \in \mathbb{P}_{k,a}$ and that

$$\boldsymbol{\omega}(g) = \Phi(\boldsymbol{\omega}(\Phi^{-1}(g))).$$

The following is now a trivial checking using Proposition 1.6.2.

Proposition. *Let $k \in [1, \bar{i}]$ and assume that $\mathbf{a} = (a_1, \cdots, a_r)$ is a $(k, \bar{i})$–segment and assume that $\boldsymbol{\pi} \in \mathcal{P}_{\bar{i}}$. Then*

$$\boldsymbol{\pi} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{k,\mathbf{a}}) \iff \Phi(\boldsymbol{\pi}) \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{ik,i\mathbf{a}}).$$

□

The following corollary is immediate by using Proposition 1.5.2.

Corollary. *For all $\boldsymbol{\omega} \in \mathcal{P}_{\bar{i}}^+$, we have*

$$\boldsymbol{\pi} \in \mathrm{wt}_\ell W(\boldsymbol{\omega}) \implies \Phi(\boldsymbol{\pi}) \in \mathrm{wt}_\ell W(\Phi(\boldsymbol{\omega})).$$

Remark. In fact Proposition 2.2 holds for all the modules studied in [21] and the proof is identical to the one given above for segments.

2.3. We prove a partial converse to Corollary 2.2.

2.3.1. *The set* $\mathrm{wt}_\ell V(\Phi(\boldsymbol{\omega}_{k,a})) \setminus \Phi(\mathrm{wt}_\ell V(\boldsymbol{\omega}_{k,a})$.

Lemma. *Let $\boldsymbol{\omega}_{k,a} \in \mathcal{H}_{\bar{i}}$ and suppose that $\boldsymbol{\omega} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{ik,ia})$ and $\boldsymbol{\omega} \notin \Phi(\mathcal{H}_{\bar{i}})$ with reduced expression*

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{j_1,c_1}^{\epsilon_1} \cdots \boldsymbol{\omega}_{j_r,c_r}^{\epsilon_r}, \quad c_s \in \mathbb{Z}, \quad 1 \le j_1 < \cdots < j_r \le n, \quad \epsilon_s \in \{-1, 1\}, \quad 1 \le s \le r.$$

Let $s \in [1, r]$ be such that c_s is maximal with the property that $\boldsymbol{\omega}_{j_s,c_s} \notin \Phi(\mathcal{H}_{\bar{i}})$. Then $\epsilon_s = -1$.

Proof. Let $p \in \mathbb{P}_{ik,ia}$ be such that $\boldsymbol{\omega}(p) = \boldsymbol{\omega}$, in particular this means that if $p(j-1) = p(j+1)$ then $j = j_k$ for some $k \in [1, r]$ and $p(j_k) = c_k$ and also that $c_k - c_{k-1} = \epsilon_k(j_{k-1} - j_k)$ for $2 \le k \le r$.

Assume that there exists m such that $\epsilon_m = 1$ and $\boldsymbol{\omega}_{j_m,c_m} \notin \Phi(\mathcal{H}_{\bar{i}})$ (otherwise there is nothing to prove). This means immediately that there exists $m' \in [1,n]$ with $\epsilon_{m'} = -1$ and so $r \geq 2$.

Setting $(j_0, c_0, \epsilon_0) = (0, ik+ia, 0)$ and $(j_{r+1}, c_{r+1}, \epsilon_{r+1}) = (n+1, n+1-ik+ia, 0)$ we see that

$$\epsilon_{m-1} = -1 + \delta_{m,1} + \delta_{m,r} = \epsilon_{m+1}, \quad c_{m-1} - c_m = j_m - j_{m-1}, \quad c_{m+1} - c_m = j_{m+1} - j_m$$

and so

$$c_{m-1} + j_{m-1} = c_m + j_m = c_{m+1} - j_{m+1} + 2j_m.$$

It is now simple to see that at least one of $\boldsymbol{\omega}_{m\pm1,c_{m\pm1}} \notin \Phi(\mathcal{H}_{\bar{i}})$ and the Lemma follows since $c_{m\pm1} > c_m$. □

2.3.2. The next proposition gives the partial converse to Proposition 2.2.

Proposition. *Suppose that*

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{j_1,a_1} \cdots \boldsymbol{\omega}_{j_k,a_k} \in \mathcal{H}_{\bar{i}}^+, \quad \tilde{\boldsymbol{\pi}} = \boldsymbol{\omega}(p_1) \cdots \boldsymbol{\omega}(p_k) \in \operatorname{wt}_\ell W(\Phi(\boldsymbol{\omega})), \quad p_s \in \mathbb{P}_{ij_s, ia_s}, \quad 1 \leq s \leq k.$$

Then

$$\tilde{\boldsymbol{\pi}} \in \Phi(\mathcal{H}_{\bar{i}}) \implies \boldsymbol{\omega}(p_s) = \Phi(\boldsymbol{\omega}(g_s)), \quad g_s \in \mathbb{P}_{j_s,a_s}, \ 1 \leq s \leq k. \tag{2.1}$$

$$\tilde{\boldsymbol{\pi}} \in \mathcal{P}_n^+ \implies \tilde{\boldsymbol{\pi}} \in \Phi(\mathcal{H}_{\bar{i}}). \tag{2.2}$$

Proof. We proceed by induction on k with Proposition 2.2 showing that induction begins at $k = 1$.

For the inductive step, suppose for a contradiction that $\boldsymbol{\omega}(p_1) \notin \Phi(\mathcal{H}_{\bar{i}})$. Choose s_1 as in Lemma 2.3.1, i.e., $\boldsymbol{\omega}_{s_1,p_1(s_1)}^{-1}$ occurs in $\boldsymbol{\omega}(p_1)$ and if $p_1(j+1) = p_1(j-1)$ and $\boldsymbol{\omega}_{j,p(j)} \notin \Phi(\mathcal{H}_{\bar{i}})$ then $p_1(s_1) > p_1(j)$. Then we must have that $\boldsymbol{\omega}_{s_1,p_1(s_1)}$ occurs in $\boldsymbol{\omega}(p_m)$ for some $2 \leq m \leq k$ and assume without loss of generality that $m = 2$. In particular this means that $\boldsymbol{\omega}(p_2) \notin \Phi(\mathcal{H}_{\bar{i}})$. Repeating we find that this process can never stop which is clearly absurd.

Hence $\boldsymbol{\omega}(p_1) \in \Phi(\mathcal{H}_{\bar{i}})$ and so by Proposition 2.2 $\boldsymbol{\omega}(p_1) = \Phi(\boldsymbol{\omega}(g_1))$ for some $g_1 \in \mathbb{P}_{j_1,a_1}$. It follows that

$$\boldsymbol{\omega}(p_2) \cdots \boldsymbol{\omega}(p_k) = \Phi(\boldsymbol{\pi}\boldsymbol{\omega}(g_1)^{-1}) \in \operatorname{wt}_\ell W(\Phi(\boldsymbol{\omega}\boldsymbol{\omega}_{j_1,a_1}^{-1})).$$

The inductive hypothesis applies and the proposition follows. □

Corollary. *Suppose that*

$$\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathcal{H}_{\bar{i}}, \quad \tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{\pi}}_1 \tilde{\boldsymbol{\pi}}_2, \quad \tilde{\boldsymbol{\pi}}_s \in \operatorname{wt}_\ell(V(\Phi(\boldsymbol{\omega}_s)), \quad s = 1, 2.$$

Then,

$$\tilde{\boldsymbol{\pi}} \in \mathcal{P}_n^+ \implies \tilde{\boldsymbol{\pi}} \in \Phi(\mathcal{H}_{\bar{i}}),$$

$$\tilde{\boldsymbol{\pi}} \in \Phi(\mathcal{H}_{\bar{i}}) \iff \tilde{\boldsymbol{\pi}}_s \in \Phi(\mathcal{H}_{\bar{i}}), \quad s = 1, 2.$$

Proof. Recall from Section 1.5.2 that $V(\boldsymbol{\omega})$ is the unique irreducible quotient of $W(\boldsymbol{\omega})$ and that

$$\operatorname{wt}_\ell V(\boldsymbol{\omega}) \subset \operatorname{wt}_\ell W(\boldsymbol{\omega}), \quad \operatorname{wt}_\ell W(\boldsymbol{\omega}_1) \operatorname{wt}_\ell W(\boldsymbol{\omega}_2) = \operatorname{wt}_\ell W(\boldsymbol{\omega}_1 \boldsymbol{\omega}_2).$$

The corollary is now an immediate consequence of the proposition. □

2.4. Let $\mathcal{H}_{i,n} = \Phi_{\bar{i},n}(\mathcal{H}_{\bar{i}})$. In what follows we shall continue to denote by $\Phi_{\bar{i},n}$ the induced map from $\mathbb{Z}[\mathcal{H}_{\bar{i}}] \to \mathbb{Z}[\mathcal{H}_{i,n}]$. Let $\mathscr{C}_{\bar{i}}$ (resp. $\mathscr{C}_{i,n}$) be the full subcategory of $\mathscr{F}_{\bar{i},\mathbb{Z}}$ (resp. $\mathscr{F}_{n,\mathbb{Z}}$) consisting of objects whose Jordan–Holder constituents are of the form $V(\boldsymbol{\omega})$ (resp. $V(\Phi_{\bar{i},n}(\boldsymbol{\omega})))$ with $\boldsymbol{\omega} \in \mathcal{H}_{\bar{i}}^+$.

2.5. For $m \geq 1$ and a subgroup $\mathcal{G}$ of $\mathcal{P}_m$ and an object V of $\mathscr{F}_{m,\mathbb{Z}}$ set

$$\chi^{\mathcal{G}}(V) = \sum_{\boldsymbol{\omega} \in \mathcal{G}} \dim V_{\boldsymbol{\omega}} e(\boldsymbol{\omega}) \in \mathbb{Z}[\mathcal{G}].$$

It was proved in [10] that if $\mathcal{G} = \mathcal{P}_m$ then one has an injective homomorphism

$$\chi^{\mathcal{P}_m} : \mathcal{K}_0(\mathscr{F}_{m,\mathbb{Z}}) \to \mathbb{Z}[\mathcal{P}_m].$$

Moreover the image of $\chi^{\mathcal{P}_m}$ is a polynomial algebra generated by the elements $\chi^{\mathcal{P}_m}(V(\boldsymbol{\omega}_{j,b}))$ with $j \in [1, m]$ and $b \in \mathbb{Z}$.

An analogous result was proved in [16] for $\mathscr{C}_{\bar{i}}$ with $\mathcal{G} = \mathcal{H}_{\bar{i}}$. In fact one has the following more general statement.

2.6. We now prove the following proposition.

Proposition. *Retain the notation established so far.*

(i) The category $\mathscr{C}_{i,n}$ is a monoidal tensor category.

(ii) The assignment

$$\chi^{\mathcal{H}_{i,n}} : K_0(\mathscr{C}_{i,n}) \to \mathbb{Z}[\mathcal{H}_{i,n}],$$

is an injective homomorphism and the image is the polynomial algebra generated by the elements $\chi^{\mathcal{H}_{i,n}}(V(\Phi_{\bar{i},n}(\boldsymbol{\omega}_{j,a})))$, $\boldsymbol{\omega}_{j,a} \in \mathcal{H}_{\bar{i}}^+$. Moreover, we have

$$\Phi_{\bar{i},n} \circ \chi^{\mathcal{H}_{\bar{i}}}(V(\boldsymbol{\omega}_{j,a})) = \chi^{\mathcal{H}_{i,n}}(V(\Phi_{\bar{i},n}(\boldsymbol{\omega}_{j,a})), \quad \boldsymbol{\omega}_{j,a} \in \mathcal{H}_{\bar{i}}^+.$$

Corollary. *We have an isomorphism $\mathcal{K}_0(\mathscr{C}_i) \to \mathcal{K}_0(\mathscr{C}_{i,n})$ defined by $[V(\boldsymbol{\omega}_{j,a})] \mapsto [V(\Phi_{\bar{i},n}(\boldsymbol{\omega}_{j,a}))]$. Moreover this map takes snake modules to snake modules.*

Proof. The isomorphism is clear from the proposition. Suppose that $V(\boldsymbol{\omega})$ is a snake module in $\mathscr{C}_i$. Then it is clear that $\Phi_{\bar{i},n}(\boldsymbol{\omega})$ also defines a snake. Using Remark 2.2 we have

$$\Phi_{\bar{i},n}(\mathrm{wt}_\ell\, V(\boldsymbol{\omega})) = \mathrm{wt}_\ell(V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))) \cap \Phi_{\bar{i},n}(\mathcal{H}_{\bar{i}}).$$

Since ℓ–weight spaces are one–dimensional in snake modules we have that $\dim V(\boldsymbol{\omega})_{\boldsymbol{\pi}} = \dim V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))_{\Phi_{\bar{i},n}(\boldsymbol{\pi})}$ and so we get

$$\Phi_{\bar{i},n} \circ \chi^{\mathcal{H}_{\bar{i}}} V(\boldsymbol{\omega}) = \chi^{\mathcal{H}_{i,n}} V(\Phi(\boldsymbol{\omega})).$$

Using injectivity we get the corollary.

□

The proposition is established in several steps. Since i, n are fixed from now on for ease of notation we set

$$\Phi = \Phi_{\bar{i},n}, \quad \mathcal{H} = \mathcal{H}_{i,n}, \quad \mathcal{H}^+ = \mathcal{H}_{i,n}^+, \quad \bar{\mathcal{H}} = \mathcal{H}_{\bar{i}}, \quad \bar{\mathcal{H}}^+ = \mathcal{H}_{\bar{i}}^+.$$

2.6.1. *Proof of Proposition 2.6.* To prove that $\mathscr{C}_{i,n}$ is a monodial tensor category it suffices to prove that the Jordan–Holder constituents of $V(\Phi(\boldsymbol{\omega}_1)) \otimes V(\Phi(\boldsymbol{\omega}_2))$ are of the form $V(\Phi(\boldsymbol{\pi}))$ with $\boldsymbol{\pi} \in \mathcal{H}^+_{\bar{i}}$. But this is immediate from the second assertion of Corollary 2.3.2.

To prove that $\chi^{\mathcal{H}}$ is a homomorphism of rings we must show that

$$\left(\sum_{\boldsymbol{\omega}\in\mathcal{H}} \dim V(\boldsymbol{\omega}_1)_{\boldsymbol{\omega}} e(\boldsymbol{\omega})\right)\left(\sum_{\boldsymbol{\omega}\in\mathcal{H}} \dim V(\boldsymbol{\omega}_2)_{\boldsymbol{\omega}} e(\boldsymbol{\omega})\right) = \sum_{\boldsymbol{\omega}\in\mathcal{H}} \dim\left(V(\boldsymbol{\omega}_1)\otimes V(\boldsymbol{\omega}_2)\right)_{\boldsymbol{\omega}} e(\boldsymbol{\omega}).$$

But this is immediate from the first assertion of Corollary 2.3.2 and the fact that $\chi^{\mathcal{P}_n}$ is a homomorphism.

To prove that $\chi^{\mathcal{H}}$ is injective we use the following elementary fact. Suppose that V_1 and V_2 are two objects of $\mathcal{F}_n$ such that $(\dim V_1)_{\boldsymbol{\omega}} = (\dim V_2)_{\boldsymbol{\omega}}$ for all $\boldsymbol{\omega} \in \mathcal{P}_n^+$. Then $[V_1] = [V_2]$. Since any ℓ–dominant weight of an object of $\mathscr{C}_{i,n}$ is in $\mathcal{H}^+$ by Corollary 2.3.2 we see that $\chi^{\mathcal{H}}$ is injective.

Part (i) of this proposition shows that $\mathcal{K}_0(\mathscr{C}_{i,n})$ is generated by the classes of $V(\Phi(\boldsymbol{\omega}_{j,a}))$ with $\boldsymbol{\omega}_{j,a}$. Since $\Phi(\boldsymbol{\omega}_{j,a})$ are part of the generators of $\mathcal{K}_0(\mathscr{F}_{n,\mathbb{Z}})$ it follows that $\mathcal{K}_0(\mathscr{C}_{i,n})$ and hence also the image of $\chi^{\mathcal{H}}$ are polynomial algebras. The final statement of (ii) is obvious from (2.1).

2.7.

Proposition. *We have an isomorphism of algebras $\mathcal{K}_0(\mathscr{C}_i) \to \mathcal{K}_0(\mathscr{C}_{i,n})$ which maps $V(\boldsymbol{\omega})$ to $V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))$.*

Proof. It is clear from the proposition and the results of [16] that the assignment

$$[V(\boldsymbol{\omega}_{j,a})] \to [V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))]$$

defines an isomorphism of rings $\Psi : \mathcal{K}_0(\mathscr{C}_i) \to \mathcal{K}_0(\mathscr{C}_{i,n})$. Writing $\boldsymbol{\omega} = \boldsymbol{\omega}_{j_1,a_1}\cdots\boldsymbol{\omega}_{j_k,a_k}$ we prove by induction on $\operatorname{wt}\boldsymbol{\omega}$ that this isomorphism maps $V(\boldsymbol{\omega})$ to $V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))$. The definition of the isomorphism shows that induction begins and we prove the inductive step. Let $\boldsymbol{\omega}' = \boldsymbol{\omega}_{i_2,a_2}\cdots\boldsymbol{\omega}_{i_k,a_k}$ and write

$$[V(\boldsymbol{\omega}_{i_k,a_k})\otimes V(\boldsymbol{\omega}')] = [V(\boldsymbol{\omega})] + \sum_{\boldsymbol{\pi}\in\mathcal{H}^+_{\bar{i}}} a_{\boldsymbol{\pi}}[V(\boldsymbol{\pi})], \quad a_{\boldsymbol{\pi}} \in \mathbb{Z}_+, \quad a_{\boldsymbol{\pi}} = 0 \;\text{ if }\; \operatorname{wt}\boldsymbol{\pi} \nleq \operatorname{wt}\boldsymbol{\omega}.$$

Using Corollary 2.3.2 we can also write,

$$[V(\Phi_{\bar{i},n}(\boldsymbol{\omega}_{i_k,a_k}))\otimes V(\Phi_{\bar{i},n}(\boldsymbol{\omega}'))] = [V(\Phi_{\bar{i},n}(\boldsymbol{\omega}))] + \sum_{\boldsymbol{\pi}\in\mathcal{H}^+_{\bar{i}}} b_{\boldsymbol{\pi}}[V(\Phi_{\bar{i},n}(\boldsymbol{\pi}))], \;\; b_{\boldsymbol{\pi}} \in \mathbb{Z}_+, \;\; b_{\boldsymbol{\pi}} = 0 \;\text{ if }\; \operatorname{wt}\boldsymbol{\pi} \nleq \operatorname{wt}\boldsymbol{\omega}.$$

Using Proposition 2.2 and using the induction hypothesis we get

$$\Phi_{\bar{i},n}\circ\chi^{\mathcal{H}_{\bar{i}}}[V(\boldsymbol{\omega}_{i_k,a_k})\otimes V(\boldsymbol{\omega}')] = \chi^{\mathcal{H}_{i,n}}[V(\Phi_{\bar{i},n}(\boldsymbol{\omega}_{i_k,a_k}))\otimes V(\Phi_{\bar{i},n}(\boldsymbol{\omega}'))]$$

∎

3. Imaginary modules

In this section we study tensor products of pairs of modules defined by dual (i,n)–segments. We first state a general conjecture that such tensor products always have an imaginary module in its Jordan–Holder series. We then give an approach to proving this conjecture. Finally, we show that the conjecture is always true if we work with Kirillov–Reshetikhin modules and we give examples involving higher order Kirillov–Reshetikhin modules as well. Finally, we show that the methods of this section can be used to construct a rather simple example of an imaginary module in D_4.

3.1. A conjecture. Let $i \geq 2$ and assume that $\mathbf{b} = (b_1, \cdots, b_r)$ with $r \geq 2$ is a (i,p)–segment for some $p \leq 2i-3$. Define integers $2s_j = b_{j-1} - b_j + 2i$ and set

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i,b_r}(\boldsymbol{\omega}_{i-s_r,b_{r-1}-s_r}\boldsymbol{\omega}_{i+s_r,b_{r-1}-s_r})\cdots(\boldsymbol{\omega}_{i-s_2,b_1-s_2}\boldsymbol{\omega}_{i+s_2,b_1-s_2})\boldsymbol{\omega}_{i,b_1-2i}. \tag{3.1}$$

Conjecture. The $\hat{\mathbf{U}}_n$–module $V(\boldsymbol{\omega})$ is imaginary for all $n \geq 2i-1$.

Remark.

(i) It suffices to prove the conjecture for $n = 2i-1$ by Proposition 1.5.6.

(ii) Taking $\mathbf{b} = (0,2,4,6)$ as a $(2,1)$–segment we see that $\boldsymbol{\omega} = \boldsymbol{\omega}_{2,6}\boldsymbol{\omega}_{1,3}\boldsymbol{\omega}_{3,3}\boldsymbol{\omega}_{2,0}$ and this is the original example of Leclerc.

(iii) If $\mathbf{b} = (0,4,6,10)$ and $i = 3$ then $\boldsymbol{\omega} = \boldsymbol{\omega}_{3,0}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{3,5}\boldsymbol{\omega}_{3,10}$. This example shows up in the work of [18] but in general the examples considered in this section are not part of their theory.

3.2. The following is the main result of this section.

Theorem 4. Let $i, r \geq 2$ and assume that p divides i for some $1 \leq p < i$. Let $\mathbf{b} = (b_1, \cdots, b_r) \in \mathbb{Z}^r$. Conjecture 3.1 is true if $b_s - b_{s-1} = 2p$ for all $2 \leq s \leq r$.

Remark. In Section 3.9 we show that the conjecture holds if $i = 3$ and the $(3,2)$–segment $\mathbf{b} = (12,10,6)$. Notice that this case is not covered by the theorem.

3.3. Let $i \in [1,n]$, $\mathbf{b} = (b_1, \cdots, b_r)$ be an $(i, 2i-1)$ segment and set

$$\mathbf{a} = (b_1 - 2i, \cdots, b_r - 2i), \quad V = V(\boldsymbol{\omega}_{i,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}}).$$

Lemma. *We have.*

$$\dim(V \otimes V)_{\mathbf{1}} = 1 = \dim V_{\mathbf{1}}.$$

Proof. It suffices to prove that for $\epsilon = 0, 1$,

$$\boldsymbol{\omega} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{b}})^{\otimes(1+\epsilon)} \text{ and } \boldsymbol{\omega}^{-1} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}})^{\otimes(1+\epsilon)} \implies \boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{b}}^{1+\epsilon}.$$

If $\epsilon = 0$ write

$$\boldsymbol{\omega} = \boldsymbol{\omega}(p_1)\cdots\boldsymbol{\omega}(p_r), \quad \boldsymbol{\omega}^{-1} = \boldsymbol{\omega}(p_1')\cdots\boldsymbol{\omega}(p_r'),$$

and if $\epsilon = 1$

$$\boldsymbol{\omega} = \boldsymbol{\omega}(p_1)\cdots\boldsymbol{\omega}(p_r)\boldsymbol{\omega}(g_1)\cdots\boldsymbol{\omega}(g_r), \quad \boldsymbol{\omega}^{-1} = \boldsymbol{\omega}(p_1')\cdots\boldsymbol{\omega}(p_r')\boldsymbol{\omega}(g_1')\cdots\boldsymbol{\omega}(g_r'),$$

with $(p_1, \cdots, p_r), (g_1, \cdots, g_r) \in \mathbb{P}_{i,\mathbf{b}}$, $(p_1', \cdots, p_r'), (g_1', \cdots, g_r') \in \mathbb{P}_{i,\mathbf{a}}$.

Since $\boldsymbol{\omega}(p_j')$ and $\boldsymbol{\omega}(g_j')$ are in the subgroup of $\mathcal{P}_n^+$ generated generated by the elements $\boldsymbol{\omega}_{k,c}$ with $c < b_r$ if $j < r$ and $c \le b_r$ if $j = r$ it follows that

$$\epsilon = 0 \implies \boldsymbol{\omega}(p_r) = \boldsymbol{\omega}_{i,b_r} = \boldsymbol{\omega}(p_r')^{-1}.$$

In the case $\epsilon = 1$ we claim that

$$\boldsymbol{\omega}(p_r)\boldsymbol{\omega}(g_r) = \boldsymbol{\omega}_{i,b_r}^2 = \boldsymbol{\omega}(p_r')\boldsymbol{\omega}(g_r').$$

If not, assume that $\boldsymbol{\omega}_{j_1,c_1}^{-1}$ appears in $\boldsymbol{\omega}(p_r)$ with $c_1 > b_r$ maximal. Then $\boldsymbol{\omega}_{j_1,c_1}$ must occur in $\boldsymbol{\omega}(g_s)$ for some $s \le r$ and hence $\boldsymbol{\omega}(g_s)$ has $\boldsymbol{\omega}_{k_1,d_1}^{-1}$ with $d_1 > c_1$. Again this means that $\boldsymbol{\omega}_{k_1,d_1}$ must occur in $\boldsymbol{\omega}(p_k)$ for some $k < r$. Repeating we see that this process never stops which is absurd and the claim follows. In particular the result holds for $r = 1$ and if $r > 1$ we have proved that

$$\boldsymbol{\omega}_1 = \boldsymbol{\omega}\boldsymbol{\omega}_{i,b_r}^{-1-\epsilon} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{b}_1})^{\otimes(1+\epsilon)}, \quad \boldsymbol{\omega}_1^{-1-\epsilon} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}_1})^{\otimes(1+\epsilon)},$$

where $\mathbf{b}_1 = (b_1, \cdots, b_{r-1})$, $\mathbf{a}_1 = (a_1, \cdots, a_{r-1})$. An obvious induction gives the result.

□

3.4. We shall need the following result.

Lemma. *Suppose that* $\boldsymbol{\pi} \in \mathcal{P}$ *and* $\boldsymbol{\pi}^2 \in \mathrm{wt}_\ell\, V$. *Then* $\boldsymbol{\pi} = 1$.

Proof. Suppose that $\boldsymbol{\pi}^2 \in \mathrm{wt}_\ell\, V$. Write

$$\boldsymbol{\omega}_{j_1,d_1}^{2\epsilon_1} \cdots \boldsymbol{\omega}_{j_m,d_m}^{2\epsilon_m} = \boldsymbol{\pi}^2 = \boldsymbol{\omega}(\underline{p})\boldsymbol{\omega}(\underline{p}'), \quad \underline{p} \in \mathbb{P}_{i,\mathbf{b}},\ \underline{p}' \in \mathbb{P}_{i,\mathbf{a}}, \quad \epsilon_1, \cdots, \epsilon_m \in \{-1, 1\}. \tag{3.2}$$

By Proposition 1.6.2 we know that $\boldsymbol{\omega}(\underline{p})$ is a weight of the form $\boldsymbol{\omega}_{i_1,c_1}^{\epsilon_1} \cdots \boldsymbol{\omega}_{i_r,c_r}^{\epsilon_k}$ with $\epsilon \in \{-1, 0, 1\}$ and a similar assertion for $\boldsymbol{\omega}(\underline{p}')$. Suppose that $\boldsymbol{\omega}(p_r) \ne \boldsymbol{\omega}_{i,b_r}$. Since $\mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}})$ is in the subgroup of $\mathcal{P}$ generated by elements $\boldsymbol{\omega}_{j,c}$ with $c \le b_r$ we would get a contradiction to (3.2). Hence $\boldsymbol{\omega}(p_r) = \boldsymbol{\omega}_{i,b_r}$. Again to avoid a contradiction we must have $\boldsymbol{\omega}(p_r') = \boldsymbol{\omega}_{i,b_r}^{-1}$. But this means that $\boldsymbol{\pi}^2 \in V(\boldsymbol{\omega}_{i,\mathbf{b}_1}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}_1})$ with $\mathbf{a}_1 = (a_1, \cdots, a_{r-1})$ and $\mathbf{b}_1 = (b_1, \cdots, b_{r-1})$ and an obvious induction proves the Lemma.

□

3.5.

Proposition. *Suppose that* $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,c_1} \cdots \boldsymbol{\omega}_{i_k,c_k} \in \mathcal{P}_n^+$ *with* $c_1 \le c_2 \le \cdots \le c_k$ *and assume that* $(i_j, c_j) \ne (i_1 + n + 1, c_1 + n + 1)$ *for all* $1 \le j \le k$ *Then*

$$(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega} \notin \mathrm{wt}_\ell\, W(\boldsymbol{\omega})$$

In particular if M *is any* ℓ*–highest weight module with* ℓ*–highest weight* $\boldsymbol{\omega}$*, we have*

$$\mathrm{Hom}_{\hat{\mathbf{U}}_{2i-1}}(M \otimes M, V(\boldsymbol{\omega})) = 0.$$

Proof. Let $m \le k$ be maximal such that $c_m < c_1 + n + 1$. The assumptions of the Lemma mean that $\boldsymbol{\omega}_{i_1,c_1} \cdots \boldsymbol{\omega}_{i_m,c_m}\boldsymbol{\omega}_{n+1-i_1,c_1+n+1}^{-1}$ occur in any reduced expression for $(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega}$. This means that if we choose $p_j \in \mathbb{P}_{i_j,c_j}$, $1 \le j \le k$ such that

$$\boldsymbol{\omega}(p_1) \cdots \boldsymbol{\omega}(p_k) = \boldsymbol{\omega}_{n+1-i_1,c_1+n+1}^{-1} \cdots \boldsymbol{\omega}_{n+1-i_k,c_k+n+1}^{-1}\boldsymbol{\omega}_{i_1,c_1} \cdots \boldsymbol{\omega}_{i_k,c_k},$$

we can assume without loss of generality that $\boldsymbol{\omega}(p_j) = \boldsymbol{\omega}_{i_j,c_j}$ for all $1 \le j \le m$. On the other hand there must exist $1 \le s \le k$ so that $n+1-i \in \mathbf{c}_{p_s}^-$ and $p_s(n+1-i) = c_1+n+1$. This means that $c_s < c_1+n+1$ and hence $c_s \le c_m$. But this is a contradiction since we saw that we must also have $\boldsymbol{\omega}(p_s) = \boldsymbol{\omega}_{i_s,c_s}$ if $s \le m$. □

The following is immediate.

Corollary. *Let $n = 2i-1$ and $\mathbf{b} = (b_1, \cdots, b_r)$ be an $(i, 2i-3)$ segment with $r \ge 2i-1$ and let*

$$2s_j = b_{j-1} + 2i - b_j, 2 \le j \le r.$$

Setting

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i,b_r}(\boldsymbol{\omega}_{i-s_r,b_{r-1}-s_r}\boldsymbol{\omega}_{i+s_r,\ \ b_{r-1}-s_r})\cdots(\boldsymbol{\omega}_{i-s_2,b_1-s_2}\boldsymbol{\omega}_{i+s_2,b_1-s_2})\boldsymbol{\omega}_{i,b_1-2i}, \tag{3.3}$$

we have

$$(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega} \notin \mathrm{wt}_\ell\, W(\boldsymbol{\omega}).$$

□

3.6.

Proposition. *Let $n = 2i-1$. Retain the notation established so far and assume that $\mathbf{b}$ is an $(i, 2i-3)$–segment and let $\boldsymbol{\omega}$ be as in (3.3). Suppose that $V(\boldsymbol{\omega})$ occurs in the Jordan–Holder series of V. Suppose also that if $V(\boldsymbol{\pi})$ with $\boldsymbol{\pi} \in \mathcal{P}^+$ occurs in the Jordan-Holder series of V then $\mathrm{wt}\,\boldsymbol{\pi} < \mathrm{wt}\,\boldsymbol{\omega}$ if and only if $\boldsymbol{\pi} = \mathbf{1}$. Then $V(\boldsymbol{\omega})$ is imaginary.*

Proof. Set $V' = V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}})$. By Section 1.5.1 we have maps of $\hat{\mathbf{U}}_n$–modules

$$\mathbb{C} \hookrightarrow V, \quad <\ ,\ >: V' \to \mathbb{C} \to 0.$$

Since $V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}})$ is irreducible (in other words real) by Theorem 1 (see [8, Theorem 3.4]) we can use [17, Corollary 3.16] to conclude that $\mathbb{C}$ is the socle (resp. head) of V (resp. V').

Since $V(\boldsymbol{\omega})$ occurs in the Jordan–Holder series for V, we can choose $M \subset V$ si that there exists a surjective map $M \to V(\boldsymbol{\omega}) \to 0$. Since V has simple socle the assumptions on $\boldsymbol{\omega}$ guarantee that we have a non–split short exact sequence

$$0 \to \mathbb{C} \to M \to V(\boldsymbol{\omega}) \to 0$$

and that M is an ℓ–highest weight module with $\boldsymbol{\omega}$ as its ℓ-highest weight.

Let $\Phi : V \otimes V \to V$ be the map $\mathrm{id}_{\mathbf{b}} \otimes <\ ,> \otimes\, \mathrm{id}_{\mathbf{a}}$. Clearly Φ is surjective and Lemma 3.3 gives

$$\Phi(V \otimes V)_{\mathbf{1}} = V_{\mathbf{1}} \ne 0 \quad \text{and so} \quad \Phi(M \otimes M) \ne 0.$$

Using Lemma 3.4 and Lemma 3.5 and Proposition 1.5.4 we see $\Phi(M \otimes M)$ must have an irreducible quotient $V(\boldsymbol{\pi})$ with $\boldsymbol{\pi} \notin \{1, \boldsymbol{\omega}, \boldsymbol{\omega}^2\}$. It follows that

$$\Phi\left(M \otimes \mathbb{C} + \mathbb{C} \otimes M\right) = 0,$$

and hence we have an induced map $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega}) \to V(\boldsymbol{\pi}) \to 0$ and the proof is complete .

□

3.7. We prove Theorem 4 by showing that $\boldsymbol{\omega}$ satisfies the conditions of Proposition 3.6.

For $1 \le m \le i$ and $a \in \mathbb{Z}$ let $g_{i,m}^a \in \mathbb{P}_{i,a}$ be defined by requiring

$$\boldsymbol{\omega}(g_{i,m}^a) = \boldsymbol{\omega}_{i-m,a+m}\boldsymbol{\omega}_{i,a+2m}^{-1}\boldsymbol{\omega}_{i+m,a+m}.$$

Equivalently

$$\{g_{i,m}^a : 1 \le m \le i\} = \{g \in \mathbb{P}_{i,a} : \mathbf{c}_g^- = \{i\}\}. \tag{3.4}$$

It is easy to check that $\underline{g} = (g_{i,0}^{a_1}, g_{i,i-1}^{a_2}, \cdots, g_{i,i-1}^{a_r}) \in \mathbb{P}_{i,\mathbf{a}}$ and hence $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}(\underline{g}) \in \mathrm{wt}_\ell V$.

We first prove that $V(\boldsymbol{\omega})$ occurs in the Jordan–Holder series of V. Proposition 1.5.2(iii) gives the following sequence of inclusions of $\hat{\mathbf{U}}_{2i-2p-1}$–modules,

$$\mathbb{C} \hookrightarrow V(\boldsymbol{\omega}_{i-p,\{b_1-2p\}\vee\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i-p,\mathbf{a}\vee\{b_r-2i+2p\}})$$
$$\hookrightarrow V(\boldsymbol{\omega}_{i-p,b_1-2p}) \otimes V(\boldsymbol{\omega}_{i-p,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i-p,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i-p,b_r-2i+2p}),$$

and so

$$\mathrm{Hom}_{\hat{\mathbf{U}}_{2i-2p-1}}(V(\boldsymbol{\omega}_{i-p,b_1-2i}) \otimes V(\boldsymbol{\omega}_{i-p,b_r}),\ V(\boldsymbol{\omega}_{i-p,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i-p,\mathbf{a}})) \ne 0.$$

By Proposition 1.5.2 the $\hat{\mathbf{U}}_{2i-2p-1}$–module $V(\boldsymbol{\omega}_{i-p,b_1-2i}) \otimes V(\boldsymbol{\omega}_{i-p,b_r})$ is irreducible and so $V(\boldsymbol{\omega}_{i-p,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{i-p,\mathbf{a}})$ contains an ℓ–highest weight vector of ℓ-weight $\boldsymbol{\omega}_{i-p,b_r}\boldsymbol{\omega}_{i-p,b_1-2i}$ and

$$\boldsymbol{\omega}_{i-p,b_1-2i}\boldsymbol{\omega}_{i-p,b_r} = \boldsymbol{\omega}_{i-p,\mathbf{b}}\boldsymbol{\omega}(p_{i-p,i-p}^{a_r}) \cdots \boldsymbol{\omega}(p_{i-p,i-p}^{a_2})\boldsymbol{\omega}_{i-p,b_1-2i}.$$

It follows from Section 1.5.6 that V contains an ℓ–highest vector of weight

$$\boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}(p_{i,i-p}^{a_r}) \cdots \boldsymbol{\omega}(p_{i,i-p}^{a_2})\boldsymbol{\omega}_{i,b_1-2i}$$

and it is trivial to check that this is precisely $\boldsymbol{\omega}$.

We now show that if $1 \ne \boldsymbol{\pi} \in \mathrm{wt}_\ell V \cap \mathcal{P}_{2i-1}^+$ is such that $\mathrm{wt}\,\boldsymbol{\omega} - \mathrm{wt}\,\boldsymbol{\pi} \in Q_{2i-1}^+$ then $\boldsymbol{\pi} \in \mathrm{wt}_\ell V(\boldsymbol{\omega})$. For this we set

$$\mathbf{c}_k = (b_1 - i + p, \cdots b_{k-1} - i + p), \quad 1 \le k \le r$$

and first show that $\boldsymbol{\pi}$ is in the following set:

$$\{\boldsymbol{\omega}_{i,b_1}\boldsymbol{\omega}_{i,b_1-2i}\} \cup \ \{\boldsymbol{\omega}_{i,b_k}\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}_{2i-p,\mathbf{c}_k}\boldsymbol{\omega}_{i,b_1-2i} : 1 < k \le r\} \tag{3.5}$$

We first reduce the problem to the case when $p = 1$. Assume that $i - b_r \in 2\mathbb{Z}$, $b_r \le 0$ and that p divides b_r. Set

$$i = i^\diamond p, \quad i^\diamond \ge 2, \qquad b_s^\diamond = b_s/p, \quad \mathbf{b}^\diamond = (b_1^\diamond, \cdots, b_s^\diamond), \quad \Phi = \Phi_{2i^\diamond-1,2i-1},$$
$$\mathbf{a}^\diamond = (b_1^\diamond - 2i^\diamond, \cdots, b_r^\diamond - 2i^\diamond),\ V^\diamond = V(\boldsymbol{\omega}_{i,\mathbf{b}^\diamond}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}^\diamond}).$$

Let $\boldsymbol{\omega}^\diamond$ be the element defined in (3.1) associated with $i^\diamond$ and $\mathbf{b}^\diamond$. It is obvious that

$$\Phi(\boldsymbol{\omega}_{i^\diamond,\mathbf{b}^\diamond}) = \boldsymbol{\omega}_{i,\mathbf{b}}, \quad \Phi(\boldsymbol{\omega}^\diamond) = \boldsymbol{\omega}.$$

Since $\boldsymbol{\pi} \in \mathcal{P}_{2i-1}^+$ we can use Corollary 2.3.2 and write

$$\boldsymbol{\pi} = \Phi(\boldsymbol{\pi}_1)\Phi(\boldsymbol{\pi}_2), \quad \boldsymbol{\pi}_s \in \mathcal{H}_{2i^\diamond-1}, \quad \Phi(\boldsymbol{\pi}_1) \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{b}}), \quad \Phi(\boldsymbol{\pi}_2) \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}}).$$

Proposition 2.2 now shows that

$$\boldsymbol{\pi}_1 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i^\diamond,\mathbf{b}^\diamond}), \quad \boldsymbol{\pi}_2 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i^\diamond,\mathbf{a}^\diamond}) \implies \boldsymbol{\pi}^\diamond = \boldsymbol{\pi}_1\boldsymbol{\pi}_2 \in \mathrm{wt}_\ell V^\diamond.$$

By Lemma 2.1 we have

$$\operatorname{wt}\boldsymbol{\omega}-\operatorname{wt}\boldsymbol{\pi}\in Q^+_{2i-1}\iff \operatorname{wt}\boldsymbol{\omega}^\diamond-\operatorname{wt}\boldsymbol{\pi}^\diamond\in Q^+_{i^\diamond}$$

and hence we have shown that

$$\{\boldsymbol{\pi}\in\mathcal{P}^+_{2i-1}:\operatorname{wt}\boldsymbol{\pi}<\operatorname{wt}\boldsymbol{\omega}\}\cap\operatorname{wt}_\ell V=\Phi\{\boldsymbol{\pi}\in\mathcal{P}^+_{2i^\diamond-1}:\operatorname{wt}\boldsymbol{\pi}^\diamond<\operatorname{wt}\boldsymbol{\omega}^\diamond\}\cap\operatorname{wt}_\ell V^\diamond. \tag{3.6}$$

It remains to prove (3.5) for $p=1$. This is done as follows. Write

$$\boldsymbol{\pi}=\boldsymbol{\omega}(\underline{p}')\boldsymbol{\omega}(\underline{p}),\quad \underline{p}'=(p'_1,\cdots,p'_r)\in\mathbb{P}_{i,\mathbf{b}},\quad \underline{p}=(p_1,\cdots,p_r)\in\mathbb{P}_{i,\mathbf{a}}.$$

By Proposition 1.6.2 we see that $\operatorname{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}})$ is contained in the subgroup generated by the elements $\boldsymbol{\omega}_{j,c}$ with $c\le b_r$. Since $\boldsymbol{\pi}\in\mathcal{P}^+_{2i-1}$ we have $\boldsymbol{\omega}(p'_r)=\boldsymbol{\omega}_{i,b_r}$. Proposition 1.6.2 implies that $\boldsymbol{\omega}(\underline{p}')=\boldsymbol{\omega}_{i,\mathbf{b}}$ and hence we get

$$\mathbf{c}^-_{p_s}=\{i\},\quad \text{i.e. } p_s=p^{a_s}_{i,m_s},\quad \text{for some } 0\le m_1\le\cdots\le m_s\le i,\quad 1\le s\le r.$$

Since $\operatorname{wt}\boldsymbol{\pi}<\operatorname{wt}\boldsymbol{\omega}$ we see that if $m_2<i-1$ then $a_2+2m_2=b_2-2i+2m_2=b_1+2-2i+2m_2<b_1$ and hence $\boldsymbol{\omega}^{-1}_{i,a_2+2m_2}$ would occur in a reduced expression for $\boldsymbol{\pi}$ contradicting $\boldsymbol{\pi}\in\mathcal{P}^+$. It follows from Proposition 1.6.2 that $m_j\ge i-1$ for all $2\le j\le r$. Equation 3.5 is now a simple calculation.

We prove that any $\boldsymbol{\pi}$ in the set in (3.5) is in $\operatorname{wt}_\ell V(\boldsymbol{\omega})$. Let $J=\{1,\cdots,2p-1\}$ and let $\mathbf{U}_{q,J}$–submodule $V_J(\boldsymbol{\omega})$ of $V(\boldsymbol{\omega})$ generated by $v_{\boldsymbol{\omega}}$. Since $\boldsymbol{\omega}_J=\boldsymbol{\omega}_{p,\mathbf{c}_r}\in\mathcal{P}^+_{i-1}$ we have

$$V_J(\boldsymbol{\omega})\cong_{\mathbf{U}_{q,J}}V(\boldsymbol{\omega}_{p,\mathbf{c}_r}).$$

The module on the right has the following ℓ–weight:

$$\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}^{-1}_{p,\tilde{\mathbf{c}}_k},\quad \tilde{\mathbf{c}}_k=(b_k+i+p,\cdots,b_{r-1}+i+p)$$

and hence the module $V(\boldsymbol{\omega})$ has an ℓ–weight vector v of ℓ–weight

$$\boldsymbol{\omega}_{i,b_1-2i}\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}^{-1}_{p,\tilde{\mathbf{c}}_k}\boldsymbol{\omega}_{2p,?}\boldsymbol{\omega}_{2i-p,\mathbf{c}_r}\boldsymbol{\omega}_{i,b_r}.$$

The element v is an ℓ–highest weight vector for $J'=\{2i-2p+1,\cdots,2i-1\}$ with corresponding ℓ–highest weight $\boldsymbol{\omega}_{2i-p,\mathbf{c}_r}$. A similar argument now proves that $V(\boldsymbol{\omega})$ contains an ℓ–highest weight vector of ℓ-weight given buy

$$\boldsymbol{\omega}_{i,b_1-2i}\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}^{-1}_{p,\tilde{\mathbf{c}}_k}\boldsymbol{\omega}_{2p,?}\boldsymbol{\omega}_{2i-2p,?}\boldsymbol{\omega}^{-1}_{2i-p,\tilde{c}_k}\boldsymbol{\omega}_{2i-p,\mathbf{c}_k}\boldsymbol{\omega}_{i,b_r}.$$

Now one has to drop i in the subalgebra $(2p,\cdots,2i-p)$ to get the result.

3.8. The case $i>p>1$.

Let $v_{\boldsymbol{\omega}}$ be the ℓ-highest weight of $V(\boldsymbol{\omega})$ and $J=[1,i-1]$. Using the discussion in Section 1.5.6 we have

$$\hat{\mathbf{U}}_J v_{\boldsymbol{\omega}}\cong V_J(\boldsymbol{\omega}_J)=V_J(\boldsymbol{\omega}_{p,\mathbf{c}_r}).$$

Using Proposition 1.6.2 it is easy to check that

$$\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}^{-1}_{i-p,\tilde{\mathbf{c}}_k} \in \mathrm{wt}_\ell\, V_J(\boldsymbol{\omega}_{p,\mathbf{c}_r}), \quad \tilde{\mathbf{c}}_k = (b_k+p,\cdots,b_{r-1}+p).$$

In particular, there exists $\boldsymbol{\beta}_k \in \mathcal{Q}^+_J$ such that $\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}^{-1}_{i-p,\tilde{\mathbf{c}}_k} = (\boldsymbol{\omega}\boldsymbol{\beta}_k^{-1})_J$ and hence Lemma 1.5.5 implies that

$$\boldsymbol{\omega}' = \boldsymbol{\omega}\boldsymbol{\beta}_k^{-1} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}), \quad \dim V(\boldsymbol{\omega})_{\boldsymbol{\omega}'} = 1.$$

Working similarly as before, with $v_{\boldsymbol{\omega}'} \in V(\boldsymbol{\omega})_{\boldsymbol{\omega}'}$ non–zero and $J' = [i+1, 2i-1]$, we have that $v_{\boldsymbol{\omega}'}$ is ℓ-highest weight vector for the subalgebra $\hat{\mathbf{U}}_{J'}$ and $(\boldsymbol{\omega}')_{J'} = \boldsymbol{\omega}_{2i-p,\mathbf{c}_r}$. Then there exists $\boldsymbol{\beta}'_k \in \mathcal{Q}^+_{J'}$ such that

$$\boldsymbol{\omega}_{2i-p,\mathbf{c}_k}\boldsymbol{\omega}^{-1}_{i+p,\tilde{\mathbf{c}}_k} = (\boldsymbol{\omega}'\boldsymbol{\beta}'^{-1}_k)_{J'} \in \mathrm{wt}_\ell\, \hat{\mathbf{U}}_{J'}v_{\boldsymbol{\omega}'} \implies \boldsymbol{\omega}(\boldsymbol{\beta}_k\boldsymbol{\beta}'_k)^{-1} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}).$$

One checks that put in more details

$$\boldsymbol{\omega}(\boldsymbol{\beta}_k\boldsymbol{\beta}'_k)^{-1} = \boldsymbol{\omega}_{i,b_r}\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}^2_{i,\tilde{\mathbf{c}}_k-p}\boldsymbol{\omega}_{2i-p,\mathbf{c}_k}\boldsymbol{\omega}_{i,b_1-2i}(\boldsymbol{\omega}_{i+p,\tilde{\mathbf{c}}_k}\boldsymbol{\omega}_{i-p,\tilde{\mathbf{c}}_k})^{-1}.$$

3.8.1. For each $1 \le k \le r$ let

$$V_k = V(\boldsymbol{\omega}_{i,b_r}\boldsymbol{\omega}_{p,\mathbf{d}_k}\boldsymbol{\omega}_{2i-p,\mathbf{d}_k}) \otimes V(\boldsymbol{\omega}_{i,b_1-2i}\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}_{2i-p,\mathbf{c}_k}),$$

where $\mathbf{d}_k$ is such that $\mathbf{c}_r = \mathbf{c}_k \vee \mathbf{d}_k$. It follows from Proposition 1.5.2 that V_k is ℓ-highest weight and hence $V(\boldsymbol{\omega})$ is its irreducible quotient. From now on we fix such k and for easy of notation we write

$$\boldsymbol{\pi}_1 = \boldsymbol{\omega}_{i,b_r}\boldsymbol{\omega}_{p,\mathbf{d}_k}\boldsymbol{\omega}_{2i-p,\mathbf{d}_k}, \quad \boldsymbol{\pi}_2 = \boldsymbol{\omega}_{i,b_1-2i}\boldsymbol{\omega}_{p,\mathbf{c}_k}\boldsymbol{\omega}_{2i-p,\mathbf{c}_k}.$$

For $K \subset [1,n]$ let $w_{\circ,K}$ be the longest element of the subgroup of $\mathcal{W}$ generated by the simple reflections $\{s_j : j \in K\}$. Let $w = w_{\circ,J}w_{\circ,J'} \in \mathcal{W}$, where J and J' are as before. The discussion in the previous section gives

$$T_w(\boldsymbol{\pi}_1) = \boldsymbol{\pi}_1(\boldsymbol{\beta}_k\boldsymbol{\beta}'_k)^{-1} \in \mathrm{wt}_\ell\, V(\boldsymbol{\pi}_1),$$

$$x^+_{j,s}V(\boldsymbol{\pi}_1)_{T_w(\boldsymbol{\pi}_1)} = 0, \quad j \ne i-p, i+p, \quad s \in \mathbb{Z}. \tag{3.7}$$

In particular, since $\dim V(\boldsymbol{\pi}_1)_{T_w\boldsymbol{\pi}_1} = 1$ it follows that

$$\dim(V_k)_{\boldsymbol{\omega}(\boldsymbol{\beta}_k\boldsymbol{\beta}'_k)^{-1}} = 1 = \dim V(\boldsymbol{\omega})_{\boldsymbol{\omega}(\boldsymbol{\beta}_k\boldsymbol{\beta}'_k)^{-1}}.$$

Therefore, if $v \in V(\boldsymbol{\omega})_{\boldsymbol{\omega}(\boldsymbol{\beta}_k\boldsymbol{\beta}'_k)^{-1}}$ is non zero, by (3.7) v is an ℓ-highest weight vector for the action of $\hat{\mathbf{U}}_K$, $K = [i-p+1, i+p-1]$ and hence, using Section 1.5.6 and Theorem 1 we have

$$\hat{\mathbf{U}}_k v \cong V_K(\boldsymbol{\omega}_{i,b_r}\boldsymbol{\omega}^2_{i,\tilde{\mathbf{c}}_k-p}\boldsymbol{\omega}_{i,b_1-2i}) \cong V_K(\boldsymbol{\omega}_{i,b_r}\boldsymbol{\omega}_{i,\tilde{\mathbf{c}}_k-p}\boldsymbol{\omega}_{i,b_1-2i}) \otimes V_K(\boldsymbol{\omega}_{i,\tilde{\mathbf{c}}_k-p}).$$

Since $\boldsymbol{\omega}^{-1}_{i,\tilde{\mathbf{c}}_k+p} \in \mathrm{wt}_\ell\, V_K(\boldsymbol{\omega}_{i,\tilde{\mathbf{c}}_k-p})$, using Lemma 1.5.6 it follows that $\boldsymbol{\omega}_k \in V(\boldsymbol{\omega})$ as desired.

3.9. We give an example where the conjecture holds for the higher rank KR–modules. The simplest example is when $i = 3$ and we work with the $(3, 5)$–segments $\mathbf{b} = (10, 6)$ and$\mathbf{a} = (4, 0)$ when once can prove that the module $\boldsymbol{\omega}_{3,10}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{4,5}\boldsymbol{\omega}_{3,0}$. But this is an example which appears in the work of Lapid. So instead, we give an example which does not appear in their work and comes from the $(3, 5)$–segments

$$\mathbf{b} = (12, 10, 6), \quad \mathbf{a} = (6, 4, 0), \quad \boldsymbol{\omega} = \boldsymbol{\omega}_{3,12}\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{4,5}\boldsymbol{\omega}_{5,8}\boldsymbol{\omega}_{3,0}.$$

We consider $\hat{\mathbf{U}}_5$–module $V = V(\boldsymbol{\omega}_{3,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{3,\mathbf{a}})$ and prove that $V(\boldsymbol{\omega})$ satisfies the conditions of Proposition 3.6.

Suppose that $\boldsymbol{\pi} = \boldsymbol{\omega}(\underline{p}')\boldsymbol{\omega}(\underline{p})$ is an ℓ–dominant weight of V, with $\underline{p}' = (p_1', p_2', p_3') \in \mathbb{P}_{3,\mathbf{b}}$ and $\underline{p} = (p_1, p_2, p_3) \in \mathbb{P}_{3,\mathbf{a}}$. Since $\boldsymbol{\omega}(\underline{p})$ is in the subgroup of $\mathcal{P}_5^+$ generated by $\boldsymbol{\omega}_{j,c}$ with $c \le 12$ it follows that $\boldsymbol{\omega}(p_3') = \boldsymbol{\omega}_{3,12}$ and hence also, by using Proposition 1.6.2 that $\boldsymbol{\omega}(p_2') = \boldsymbol{\omega}_{3,10}$. Suppose that $\boldsymbol{\omega}(p_1') \neq \boldsymbol{\omega}_{3,6}$; then there exists $j \in \mathbf{c}^-_{p_3'}$ with $p_3'(j) \ge 8$. It follows that $j \in \mathbf{c}^+_{p_s}$ for some $s = 2, 3$ and $p_s(j) = p_3'(j)$. If $s = 2$ then the only possibility is $\boldsymbol{\omega}(p_2) = \boldsymbol{\omega}_{2,9}^{-1}\boldsymbol{\omega}_{3,8}\boldsymbol{\omega}_{4,9}^{-1}$ and so $j = 3$ and $p_3'(j) = 8$. This means $\boldsymbol{\omega}(p_3') = \boldsymbol{\omega}_{2,7}\boldsymbol{\omega}_{3,8}^{-1}\boldsymbol{\omega}_{4,7}$ and now we see that $\boldsymbol{\omega}_{2,9}^{-1}\boldsymbol{\omega}_{4,9}^{-1}$ occurs in the reduced word for $\boldsymbol{\pi}$ which is a contradiction. Hence we have

$$\boldsymbol{\pi} = \boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}(\underline{p}), \quad \mathbf{c}^-_{p_s} \subset \{3\}, \quad s = 1, 2, 3.$$

In particular we have proved that all ℓ–dominant weights in V occur with multiplicity 1.

Assume that $\boldsymbol{\pi} \neq \boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}_{i,\mathbf{a}}$. Then

$$\mathbf{c}^-_{p_1} = \{3\} \implies p_1(3) = 6 \implies p_2(3) = 8, 10.$$

If $p_2(3) = 8$ then $\mathbf{c}^-_{p_2} = \emptyset$ which is impossible. Hence we must have $p_2(3) = 10$ and $\mathbf{c}^-_{p_2} = \{3\}$. It follows that $\boldsymbol{\omega}(p_3) = \boldsymbol{\omega}_{3,12}^{-1}$ and so $\boldsymbol{\pi} = 1$. If $\mathbf{c}^-_{p_1} = \emptyset$ then

$$\mathbf{c}^-_{p_2} = \{3\} \implies p_2(3) = 6, 10.$$

If $p_2(3) = 10$ then we must have $p_3(3) = 12$ and so we get the weight $\boldsymbol{\omega}_{3,6}\boldsymbol{\omega}_{3,0}$. If $p_2(3) = 6$ then $p_3(3) \in \{10, 12\}$ and

$$\boldsymbol{\pi} \in \{\boldsymbol{\omega} = \boldsymbol{\omega}_{3,12}\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{5,8}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{4,5}\boldsymbol{\omega}_{3,0}, \quad \boldsymbol{\omega}_{3,10}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{4,5}\boldsymbol{\omega}_{3,0}\}.$$

Finally if $\mathbf{c}^-_{p_1} = \mathbf{c}^-_{p_2} = \emptyset$ and $\mathbf{c}^-_{p_3} = \{3\}$ then we must have $p_3(3) = 10, 12$ and we get

$$\boldsymbol{\pi} \in \{\boldsymbol{\omega}_{3,12}\boldsymbol{\omega}_{3,6}\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{5,8}\boldsymbol{\omega}_{3,4}\boldsymbol{\omega}_{3,0}, \quad \boldsymbol{\omega}_{3,10}\boldsymbol{\omega}_{3,6}\boldsymbol{\omega}_{3,4}\boldsymbol{\omega}_{3,0}\}.$$

The usual application of Proposition 1.5.2 and Theorem 5 shows that V is ℓ–highest weight. Theorem 1 shows that the irreducible quotient of V is $\bar{V} := V(\boldsymbol{\omega}_{3,(0,4)\vee\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{3,6})$. It follows from Proposition 1.6.2 that if $\boldsymbol{\pi}$ is such that $\mathbf{c}^-_{p_1} = \mathbf{c}^-_{p_2} = \emptyset$ then $\boldsymbol{\pi} \in \mathrm{wt}_\ell \bar{V}$. Moreover it also follows that $\boldsymbol{\omega} \notin \mathrm{wt}_\ell \bar{V}$. Therefore if v is an ℓ–weight vector of weight $\boldsymbol{\omega}$ then v generates a proper submodule, say M of V. We claim that v is an ℓ–highest weight vector. Otherwise M contains an ℓ–highest weight vector $\boldsymbol{\omega}'$ with $\mathrm{wt}\,\boldsymbol{\omega}' > \mathrm{wt}\,\boldsymbol{\omega}$. But the preceding analysis shows that the only possibility is $\boldsymbol{\omega}_{i,\mathbf{a}}\boldsymbol{\omega}_{i,\mathbf{b}}$ which would mean $M = V$ and gives a contradiction. Hence $V(\boldsymbol{\omega})$ is the unique irreducible quotient of M. It remains to prove that M has Jordan–Holder series of length two. If not it has a Jordan-Holder consitiuent $V(\boldsymbol{\pi})$

with $\boldsymbol{\pi} \in \{\boldsymbol{\omega}_{3,10}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{4,5}\boldsymbol{\omega}_{3,0},\ \boldsymbol{\omega}_{3,6}\boldsymbol{\omega}_{3,0}\}$ and in that case $\boldsymbol{\pi} \notin \mathrm{wt}_\ell V(\boldsymbol{\omega})$. By Lemma 1.5.5 we get

$$\boldsymbol{\omega}_{3,10}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{4,5}\boldsymbol{\omega}_{3,0} = \boldsymbol{\omega}\boldsymbol{\alpha}_{1,9}^{-1}\boldsymbol{\alpha}_{5,9}^{-1}\boldsymbol{\alpha}_{2,10}^{-1}\boldsymbol{\alpha}_{4,10}^{-1}\boldsymbol{\alpha}_{3,11}^{-1} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}).$$

Hence the only possibility is that $\boldsymbol{\pi} = \boldsymbol{\omega}_{3,6}\boldsymbol{\omega}_{3,0}$. If $\boldsymbol{\pi} \notin \mathrm{wt}_\ell V(\boldsymbol{\omega})$ let $v' \in M$ be an ℓ–weight vector of weight $\boldsymbol{\pi}$. Then v' generates a proper submodule of M. This means that v' must be an ℓ–highest weight vector since otherwise the submodule that it generates would be M. Hence we have a non–zero map

$$V(\boldsymbol{\omega}_{3,6}) \otimes V(\boldsymbol{\omega}_{3,0}) \to M \hookrightarrow V.$$

This means that there exists a non–zero map

$$V(\boldsymbol{\omega}_{3,0}) \otimes V(\boldsymbol{\omega}_{3,\mathbf{b}}) \to V(\boldsymbol{\omega}_{3,12}) \otimes V(\boldsymbol{\omega}_{3,\mathbf{b}}).$$

The module on the right is irreducible by Theorem 1 and hence the map is surjective and an isomorhpism. But this is absurd since they have different ℓ–weights.

3.10. An example in D_4. Using the methods of this section we give the first example outside quantum affine $\mathfrak{sl}_n$ of an imaginary module. In D_4 it is known that the left dual of $V(\boldsymbol{\omega}_{j,a})$ is $V(\boldsymbol{\omega}_{j,a+6})$ and we assume that 2 is the trivalent node. We take

$$\mathbf{b} = (10,8,6), \quad \mathbf{a} = (4,2,0), \quad \boldsymbol{\omega} = \boldsymbol{\omega}_{1,10}\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{1,2}\boldsymbol{\omega}_{1,0},$$
$$V = V(\boldsymbol{\omega}_{1,\mathbf{b}}) \otimes V(\boldsymbol{\omega}_{1,\mathbf{a}}), \quad \bar{V} = V(\boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}_{i,\mathbf{a}}),$$

and prove that the module $V(\boldsymbol{\omega})$ is imaginary.

We first discuss the analogs of the results of Sections 3.3–3.5. The arguments are all much easier in this case but we include the details since this is the first example of an imaginary module in type different from A. It is convenient to recall that $\mathrm{wt}_\ell V(\boldsymbol{\omega}_{1,0})$ is the set with elements

$$\boldsymbol{\omega}_{1,0},\ \boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{1,2}^{-1},\ \boldsymbol{\omega}_{2,3}^{-1}\boldsymbol{\omega}_{3,2}\boldsymbol{\omega}_{4,2},\ \boldsymbol{\omega}_{3,4}^{-1}\boldsymbol{\omega}_{4,2},\ \boldsymbol{\omega}_{4,4}^{-1}\boldsymbol{\omega}_{3,2},$$
$$\boldsymbol{\omega}_{2,3}\boldsymbol{\omega}_{3,4}^{-1}\boldsymbol{\omega}_{4,2}^{-1},\quad \boldsymbol{\omega}_{1,4}\boldsymbol{\omega}_{2,5}^{-1},\quad \boldsymbol{\omega}_{1,6}^{-1}.$$

In particular we see that $\mathrm{wt}_\ell V(\boldsymbol{\omega}_{1,0})$ are in the subgroup generated by $\boldsymbol{\omega}_{j,c}$ with $c \le 5$ and $\boldsymbol{\omega}_{1,6}$. It follows from this and [11, Lemma 4.4] that if $\boldsymbol{\pi} \in \mathrm{wt}_\ell V$ and $\boldsymbol{\pi} \in \mathcal{P}^+$ then $\boldsymbol{\pi} = \boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}_1$ with $\boldsymbol{\omega}_1 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}})$. We shall use this freely in what follows.

Suppose that $\boldsymbol{\pi} = \boldsymbol{\omega}_1\boldsymbol{\omega}_2 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{b}})^{\otimes 2}$ and assume that $\boldsymbol{\pi} \ne \boldsymbol{\omega}_{i,\mathbf{b}}^2$. Then without loss of generality we have $\boldsymbol{\omega}_1 \ne \boldsymbol{\omega}_{i,\mathbf{b}}$ and hence it follows from [11] that a reduced expression for $\boldsymbol{\omega}$ must contain $\boldsymbol{\omega}_{j,c}^{-1}$ for some j, c with $c \ge 12$ and c maximal with this property. It is then immediate that $\boldsymbol{\omega}_2 \ne \boldsymbol{\omega}_{i,\mathbf{b}}$ either and a reduced expression for it must have $\boldsymbol{\omega}_{j,c}$ and so also $\boldsymbol{\omega}_{k,d}^{-1}$ with $d \ge 13$. Since the $\mathrm{wt}_\ell V(\boldsymbol{\omega}_{1,a})$ with $a = 6, 8$ does not contain a term of the form $\boldsymbol{\omega}_{k,d}$ with $d \ge 13$ we see that $\boldsymbol{\pi} \notin \mathcal{P}^+$. In other words $\boldsymbol{\omega}_{i,\mathbf{b}}^2$ is the unique ℓ dominant weight of $V(\boldsymbol{\omega}_{i,\mathbf{b}})^{\otimes 2}$.

Suppose that $\boldsymbol{\omega}_1\boldsymbol{\omega}_2 = \mathbf{1}$ with $\boldsymbol{\omega}_1 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{b}})$ and $\boldsymbol{\omega}_2 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}})$. Since $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_{i,\mathbf{b}}$ it follows that $\dim V_{\mathbf{1}} = 1$. Similarly if $\boldsymbol{\omega}_1\boldsymbol{\omega}_2 = \mathbf{1}$ with $\boldsymbol{\omega}_1 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{b}})^{\otimes 2}$ and $\boldsymbol{\omega}_2 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}})^{\otimes 2}$ one proves that $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_{i,\mathbf{b}}^2$ and hence $\dim(V \otimes V)_{\mathbf{1}} = 1$.

Next we prove that $\boldsymbol{\pi} \in \mathcal{P}^+$ is such that $\boldsymbol{\pi}^2 \in \mathrm{wt}_\ell V$ then $\boldsymbol{\pi} = \mathbf{1}$. But this is clear by inspection since $\boldsymbol{\pi}^2 = \boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}_1$ with $\boldsymbol{\omega}_1 \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{i,\mathbf{a}})$.

We prove that $(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega} \notin \mathrm{wt}_\ell W(\boldsymbol{\omega})$. But this follows since if we write $(\boldsymbol{\omega}^*)^{-1}\boldsymbol{\omega}$ as a product of ℓ–weights from each fundamental module, then we must take $\boldsymbol{\omega}_{1,a} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{1,a})$ for $a = 0, 2$. But then $\boldsymbol{\omega}_{1,6}^{-1}$ does not occur as part of an ℓ–weight of the other modules.

The final thing to prove is that $\boldsymbol{\omega}$ satisfies the conditions of Proposition 3.6. Once that is done the proof that $V(\boldsymbol{\omega})$ is imaginary is identical to the one given in the A_n case. For this we proceed as follows.

Note that we have a map $\mathbb{C} \hookrightarrow V$. Moreover, it was proved in [2] that V is ℓ–highest weight. Using [22] we know that the irreducible quotient $\bar{V}$ of V has a unique ℓ–dominant weight. Since $\boldsymbol{\omega} = \boldsymbol{\omega}_{1,10}\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{1,2}\boldsymbol{\omega}_{1,0}$ is an ℓ–dominant weight in V and if $\boldsymbol{\pi}$ is any other ℓ–weight in V then $\mathrm{wt}\,\boldsymbol{\pi} < \mathrm{wt}\,\boldsymbol{\omega} = 6\omega_1 - \alpha_1$ it follows that $V(\boldsymbol{\omega})$ must occur in the Jordan–Holder series of V.

For this, it suffices to prove that $\boldsymbol{\pi} \notin \{\boldsymbol{\omega}_{1,\mathbf{b}}\boldsymbol{\omega}_{1,\mathbf{a}}, \boldsymbol{\omega}, \mathbf{1}\}$ is an ℓ–dominant weight of V then $\boldsymbol{\pi} \in \mathrm{wt}_\ell V(\boldsymbol{\omega})$. Writing $\boldsymbol{\pi} = \boldsymbol{\omega}_{i,\mathbf{b}}\boldsymbol{\omega}_1$ a simple computation shows that the following are the only possibilities for $\boldsymbol{\pi}$:

$$\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{1,6}\boldsymbol{\omega}_{1,2}\boldsymbol{\omega}_{1,0}, \quad \boldsymbol{\omega}_{1,10}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{1,0} \quad \boldsymbol{\omega}_{1,6}\boldsymbol{\omega}_{1,0}.$$

By the appropriate analog of Lemma 1.5.5 we see that

$$\boldsymbol{\omega}_{1,8}\boldsymbol{\omega}_{1,6}\boldsymbol{\omega}_{1,2}\boldsymbol{\omega}_{1,0} = \boldsymbol{\omega}\boldsymbol{\alpha}_{2,6}^{-1}\boldsymbol{\alpha}_{3,7}^{-1}\boldsymbol{\alpha}_{4,7}^{-1}\boldsymbol{\omega}_{2,8}^{-1}\boldsymbol{\omega}_{1,9}^{-1}\boldsymbol{\omega}_{1,7}^{-1} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}).$$

Next we prove that $\boldsymbol{\omega}_{1,10}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{1,0} \notin \mathrm{wt}_\ell V$; equivalently that $\boldsymbol{\omega}_{1,8}^{-1}\boldsymbol{\omega}_{1,6}^{-1}\boldsymbol{\omega}_{2,5}\boldsymbol{\omega}_{1,0}$ is not an ℓ–weight of $V(\boldsymbol{\omega}_{1,\mathbf{a}})$. In turn this is equivalent to proving that $\boldsymbol{\omega}_{1,8}^{-1}\boldsymbol{\omega}_{1,6}^{-1}\boldsymbol{\omega}_{2,5}$ is not an ℓ–weight of $V(\boldsymbol{\omega}_{1,4}\boldsymbol{\omega}_{1,2})$. Since

$$[V(\boldsymbol{\omega}_{1,4}) \otimes V(\boldsymbol{\omega}_{1,2})] = [V(\boldsymbol{\omega}_{1,4}\boldsymbol{\omega}_{1,2})] + [V(\boldsymbol{\omega}_{2,3})],$$

and $\boldsymbol{\omega}_{1,8}^{-1}\boldsymbol{\omega}_{1,6}^{-1}\boldsymbol{\omega}_{2,5} \in \mathrm{wt}_\ell V(\boldsymbol{\omega}_{2,3})$ and it occurs in $V(\boldsymbol{\omega}_{1,4}) \otimes V(\boldsymbol{\omega}_{1,2})$ with multiplicity one it follows that $\boldsymbol{\omega}_{1,8}^{-1}\boldsymbol{\omega}_{1,6}^{-1}\boldsymbol{\omega}_{2,5}$ is not an ℓ–weight of $V(\boldsymbol{\omega}_{1,4}\boldsymbol{\omega}_{1,2})$.

Finally we prove that $\boldsymbol{\omega}_{1,6}\boldsymbol{\omega}_{1,0} \in \mathrm{wt}_\ell V(\boldsymbol{\omega})$. Using Lemma 1.5.5 and the corresponding result for $\hat{\mathbf{U}}_J$ with $J = \{1\}$ and then with $J = \{2, 3, 4\}$ we see that

$$\boldsymbol{\pi} = \boldsymbol{\omega}_{1,10}\boldsymbol{\omega}_{1,8}^2\boldsymbol{\omega}_{1,6}^2\boldsymbol{\omega}_{2,7}^{-1}\boldsymbol{\omega}_{2,9}^{-1}\boldsymbol{\omega}_{1,0} = \boldsymbol{\omega}(\boldsymbol{\alpha}_{1,3}\boldsymbol{\alpha}_{2,6}\boldsymbol{\alpha}_{2,4}\boldsymbol{\alpha}_{3,7}\boldsymbol{\alpha}_{3,5}\boldsymbol{\alpha}_{4,7}\boldsymbol{\alpha}_{4,5}\boldsymbol{\alpha}_{2,8}\boldsymbol{\alpha}_{2,6})^{-1}.$$

Notice that $\mathrm{wt}\,\boldsymbol{\pi} + \alpha_1 = 8\omega_1 - 3\omega_2$. This is W–conjugate to $2\omega_1 + 3\omega_2 \nleq 4\omega_1 + \omega_2 = \mathrm{wt}\,\boldsymbol{\omega}$. It follows that any element in $V(\boldsymbol{\omega})_{\boldsymbol{\pi}}$ is an ℓ–highest weight vector for $\hat{\mathbf{U}}_J$ with $J = \{1\}$ and hence by Lemma 1.5.5 again we get

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{1,6}\boldsymbol{\omega}_{1,0} = \boldsymbol{\pi}\boldsymbol{\alpha}_{1,9}^{-1}\boldsymbol{\alpha}_{1,7}^{-1} \in \mathrm{wt}_\ell V(\boldsymbol{\omega})$$

and the proof is complete.

4. Combinatorics of (i,n)–segments

In this section we shall give an alternate formulation for a pair of segments to be in special or general position. We then use this to prove that any element of $\mathcal{P}_{i,n}^+$ can be written uniquely, upto a renumbering as a product of elements associated to segments in general position.

4.1. Given $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$ and $\mathbf{b} = (b_1, \cdots, b_s) \in \mathbb{Z}^s$ recall the notation $\mathbf{a} \vee \mathbf{b}$ established in Section 1.6.2. In the rest of this section we shall prove the following proposition.

Proposition. *Suppose that $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$. There exists a unique integer $k \geq 1$ and (i,n)–segments $\mathbf{a}_1, \cdots, \mathbf{a}_k$ which are pairwise in general position and a permutation $\sigma \in S_r$ such that*

$$(a_{\sigma(1)}, \cdots, a_{\sigma(r)}) = \mathbf{a}_1 \vee \mathbf{a}_2 \cdots \vee \mathbf{a}_k.$$

Moreover k is unique and the (i,n)–segments are also unique up to a permutation by an element of S_k.

Example. Let $\mathbf{a} = (0, 6, 4, 2, 10, 16, 10) \in \mathbb{Z}^7$, then the associated $(2,3)$-segments are

$$\mathbf{a}_1 = (0, 2, 4, 6, 10), \quad \mathbf{a}_2 = \{10\}, \quad \mathbf{a}_3 = \{16\},$$

or any permutation of these by an element of S_3. The following corollary is immediate.

Corollary. *Suppose that $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{a}} \in \mathcal{P}_{i,n}^+$ for some $\mathbf{a} \in \mathbb{Z}^r$. Then $\boldsymbol{\omega}$ can be written uniquely (upto a permutation) as a product $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{a}_1} \cdots \boldsymbol{\omega}_{i,\mathbf{a}_k}$ where $\mathbf{a}_1, \cdots, \mathbf{a}_k$ are (i,n)–segments in general position.*

4.2. To prove Proposition 4.1 it is useful to have a more explicit formulation of segments in general or special position.

Proposition. *Let $\mathbf{a} = (a_1, \cdots, a_r)$ and $\mathbf{b} = (b_1, \cdots, b_m)$ with $r \geq m$ be (i,n)–segments.*

(i) The segments $\mathbf{a}$ and $\mathbf{b}$ are in general position if and only if one of the following holds:
  - *(a) $b_1 - a_r > 2i$,*
  - *(b) $a_1 - b_m > 2i$*
  - *(c) $b_1 - a_1 \notin 2\mathbb{Z}$,*
  - *(d) $\{b_1, \cdots, b_m\} \subset \{a_1, \cdots, a_r\}$.*

(ii) The segments $\mathbf{a}$ and $\mathbf{b}$ are in special position if and only there exists $1 \leq j \leq m$ such that one of the following hold:
  - *(a) $b_j - a_r \in S_{i,n}$ or $a_1 - b_j \in S_{i,n}$,*
  - *(b) there exists $1 \leq k < r$ such that $a_k < b_j < a_{k+1}$.*

Proof. For part (i) note that it is clear that if one of conditions (a)-(d) hold then $\mathbf{a}$ and $\mathbf{b}$ are are in general position. For the converse we suppose that none of the conditions $(a)-(d)$ are satisfied and show that $\mathbf{a} \cup \mathbf{b}$ contains a segment of length $r+1$. If $b_1 > a_r$ then $b_1 - a_r \in S_{i,n}$ which means that $(a_1, \cdots, a_r, b_1)$ is an (i,n)-segment. The case $b_m < a_1$ is similar. Hence to complete the proof of (i) we must consider the case when all of the following hold:

$$b_1 \leq a_r, \quad b_m \geq a_1, \quad b_1 - a_1 \in 2\mathbb{Z}.$$

If $a_r < b_k$ or $b_k < a_1$ for some $1 \leq k \leq m$ then $(a_1, \cdots, a_r, b_k)$ or $(b_1, a_1, \cdots a_r)$ is an (i,n)-segment. Otherwise, we have $a_1 \leq b_1 < b_2 < \cdots < b_m \leq a_r$. Since condition (d) does not hold, it follows that $b_p \notin \{a_1, \cdots, a_r\}$ for some $1 \leq p \leq m$; in other words, there exists

$2 \leq k \leq r$ such that $a_{k-1} < b_p < a_k$. This means that $b_p - a_{k-1}$ and $a_k - b_p$ are elements of $S_{i,n}$ and so $(a_1, \cdots, a_{k-1}, b_k, a_k, \cdots a_r)$ is an (i,n)-segment. The proof of part (i) is complete.

If either of the conditions in part (ii) hold then $\{a_1, \cdots a_r, b_j\}$ is an (i,n)–segment after applying a suitable permutation and hence **a** and **b** are in special position. Suppose that **a** and **b** are in special position and assume that there does not exist $1 \leq j \leq m$ and $1 \leq k < r$ with $a_k < b_j < a_{k+1}$. By part (i) we see that we cannot have $\{b_1, \cdots, b_m\} \subset \{a_1, \cdots, a_r\}$. Hence we are forced to have either j maximal such that $b_j < a_1$ or k minimal with $b_k > a_r$. In the first case either $j = m$ or $j < m$ and $b_j < a_1 \leq b_{j+1}$. If $j = m$ then the segments can be in special position only if $a_1 - b_m \in S_{i,n}$. If $j < m$ then $b_{j+1} - b_j \in S_{i,n}$ and so $a_1 - b_j \in S_{i,n}$. The proof in the second case is identical. □

Corollary. *Suppose that* $\mathbf{a} = (a_1, \cdots, a_r)$ *and* $\mathbf{b} = (b_1, \cdots, b_m)$ *are* (i,n)*–segments in general position with* $r \geq m$. *Then,*

$$\{a_1, \cdots, a_r\} \cap \{b_1 \cdots, b_m\} \neq \emptyset \implies \{b_1 \cdots, b_m\} \subset \{a_1, \cdots, a_r\}.$$

□

4.3. Proof of Proposition 4.1. Let $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r$. We proceed by induction on r with induction beginning trivially at $r = 1$. After applying an element of S_r if needed, we may assume without loss of generality that $a_1 \leq a_s$ for all $1 \leq s \leq r$ and that $r_1 \leq r$ is maximal so that $\mathbf{a}_1 = (a_1, a_2 \cdots, a_{r_1})$ is an (i,n)-segment. If $r_1 = r$ we are done and otherwise we let $\mathbf{b} = (a_{r_1+1}, \cdots, a_r)$. The inductive hypothesis applies to **b** and we write $\mathbf{b} = \mathbf{a}_2 \vee \cdots \vee \mathbf{a}_k$ as in the proposition.

We prove that $\mathbf{a}_1$ and $\mathbf{a}_s$ are in general position for $2 \leq s \leq k$. Suppose that $\mathbf{a}_s = (b_1, \cdots, b_m)$ and recall that $a_1 \leq b_1$. Assume for a contradiction that $\mathbf{a}_s$ and $\mathbf{a}_1$ are in special position. By Proposition 4.2(ii) there exists $1 \leq p \leq m$ such that either $a_{k-1} < b_p < a_k$ for some $2 \leq k \leq r_1$ or $a_{r_1} < b_p$ with $b_p - a_{r_1} \in S_{i,n}$. In either case after applying a suitable permutation if needed we see that $\{a_1, \cdots, a_{r_1}, b_p\}$ is an (i,n)–segment contradicting our choice of r_1.

It remains to prove that k is unique and the segments are unique up to an element of S_k. For this, suppose that $\mathbf{c}_1, \cdots, \mathbf{c}_\ell$ is another set of (i,n)–segments in general position with $\mathbf{a} = \mathbf{c}_1 \vee \cdots \vee \mathbf{c}_\ell$. Since a_1 is minimal it must occur as the first term in some segment. Assume without loss of generality that a_1 is the first term in $\mathbf{c}_1$ and also that $\mathbf{c}_1$ has maximal length say s_1 amongst those $\mathbf{c}_s$ with first term a_1. Since r_1 is the maximum length of an (i,n)-segment starting at a_1 we have $s_1 \leq r_1$. We claim that $\mathbf{c}_1 = \mathbf{a}_1$. Otherwise, there exists $1 < p \leq r_1$ minimal such that a_p does not occur in $\mathbf{c}_1$. All the other segments $\mathbf{c}_s$ whose initial term is a_1 have length at most s_1 and hence by Corollary 4.2 must be contained in $\mathbf{c}_1$. Hence none of these segments contain a_p and so there must exist an (i,n)–segment $\mathbf{c}_j$ of length s_j whose minimal term is a_m for some $1 < m \leq p$ and contains a_p. Consider $\mathbf{c}_1 \cup \mathbf{c}_j$. If $s_1 \geq s_j$ then since $a_{p-1} \in \mathbf{c}_1$ it follows that $\mathbf{c}_1 \cup \{a_p\}$ is a longer (i,n)-segment in the union while if $s_j > s_1$ then $\{a_{m-1}\} \cup \mathbf{c}_j$ is a longer segment in the union. In both cases we have a contradiction to the fact that $\mathbf{c}_1$ and $\mathbf{c}_j$ are in general position. It follows that $\mathbf{a}_1 = \mathbf{c}_1$ and that $\mathbf{b} = \mathbf{c}_2 \vee \cdots \vee \mathbf{c}_\ell$. The uniqueness is now immediate by the inductive hypothesis.

5. Proof of Theorem 1: Reducibility

In this section we assume that $\mathbf{a} = (a_1, \cdots, a_r)$ and $\mathbf{b} = (b_1, \cdots, b_s)$ are (i, n)–segments in special position and assume that $r \geq s$.

5.1. The following elementary Lemma is helpful.

Lemma. *Suppose that $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}_n$ are in the subgroup of $\mathcal{P}_n$ generated by elements $\boldsymbol{\omega}_{j,c}$ with $c_1 \leq c \leq c_2$. Suppose that $\boldsymbol{\omega}_1, \boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2, \boldsymbol{\omega}_2'$ are elements of the subgroup generated by $\boldsymbol{\omega}_{j,c}$ with $c < c_1$ and $c > c_2$ respectively. Then*

$$\boldsymbol{\omega}_1 \boldsymbol{\omega} \boldsymbol{\omega}_2 = \boldsymbol{\omega}_1' \boldsymbol{\omega}' \boldsymbol{\omega}_2' \iff \boldsymbol{\omega}_1 = \boldsymbol{\omega}_1', \quad \boldsymbol{\omega} = \boldsymbol{\omega}', \quad \boldsymbol{\omega}_2 = \boldsymbol{\omega}_2'.$$

□

5.2. We prove that $V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}})$ is reducible. We shall do this by showing that

$$\mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}}) \neq \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}} \boldsymbol{\omega}_{i,\mathbf{b}}). \tag{5.1}$$

Recall from Proposition 1.6.2 that this statement is true in the special case when

$$r = s, \quad a_j = b_{j-1}, \quad 1 < j \leq r. \tag{5.2}$$

We shall prove the general case by showing that we can always find (i, n)–segments

$$\mathbf{a}_1 = (a_j, \cdots, a_{j+p}) \subset \mathbf{a}, \quad \mathbf{b}_1 = (b_m, \cdots, b_{m+p}) \subset \mathbf{b}$$

which satisfy the conditions in (5.2) and also

$$\max\{a_{j-1}, b_{m-1}\} < \min\{a_j, b_m\}, \quad \min\{a_{j+p+1}, b_{m+p+1}\} > \max\{a_{j+p}, b_{m+p}\}. \tag{5.3}$$

Once this is done the proof is completed as follows. Choose

$$\boldsymbol{\pi} \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}_1}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}_1}) \setminus \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}_1} \boldsymbol{\omega}_{i,\mathbf{b}_1})$$

and set

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i,\mathbf{a}_0} \boldsymbol{\omega}_{i,\mathbf{b}_0} \boldsymbol{\pi} \boldsymbol{\omega}^{-1}_{i^* \mathbf{a}_2^*} \boldsymbol{\omega}^{-1}_{i^*, \mathbf{b}_2^*}$$

where $\mathbf{a}_0 = (a_1, \cdots, a_{j-1})$, $\mathbf{a}_2 = (a_{j+p+1}, \cdots, a_r)$ and $\mathbf{b}_0$, $\mathbf{b}_2$ defined similarly, where understand that these segments can be empty. Writing $\boldsymbol{\pi} = \boldsymbol{\omega}_1 \boldsymbol{\omega}_2$ with $\boldsymbol{\omega}_1 \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}_1})$ and $\boldsymbol{\omega}_2 \in V(\boldsymbol{\omega}_{i,\mathbf{b}_1})$ we see by using Proposition 1.6.3 that

$$\boldsymbol{\omega}_{i,\mathbf{a}_0} \boldsymbol{\pi} \boldsymbol{\omega}^{-1}_{i^*, \mathbf{a}_2^*} \boldsymbol{\omega}_{i,\mathbf{b}_0} \boldsymbol{\omega}^{-1}_{i^*, \mathbf{b}_2^*} = \boldsymbol{\omega} = (\boldsymbol{\omega}_{i,\mathbf{a}_0} \boldsymbol{\omega}_1 \boldsymbol{\omega}^{-1}_{i^*, \mathbf{a}_2^*})(\boldsymbol{\omega}_{i,\mathbf{b}_0} \boldsymbol{\omega}_2 \boldsymbol{\omega}^{-1}_{i^*, \mathbf{b}_2^*}) \in \mathrm{wt}_\ell\, V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}}).$$

To prove (5.1) it is enough to show that

$$\boldsymbol{\omega} \notin \mathrm{wt}_\ell \left(V(\boldsymbol{\omega}_{i,\mathbf{a}_0} \boldsymbol{\omega}_{i,\mathbf{b}_0}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}_1} \boldsymbol{\omega}_{i,\mathbf{b}_1}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{a}_2} \boldsymbol{\omega}_{i,\mathbf{b}_2}) \right),$$

since the module $V(\boldsymbol{\omega}_{i,\mathbf{a}} \boldsymbol{\omega}_{i,\mathbf{b}})$ occurs in the Jordan–Holder series of the triple tensor product. Suppose for a contradiction that

$$\boldsymbol{\omega} = \boldsymbol{\pi}_1 \boldsymbol{\pi}_2 \boldsymbol{\pi}_3$$

where $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$, $\boldsymbol{\pi}_3$ are ℓ–weights of the corresponding modules in the tensor product. Since $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_2$ are in the subgroup of $\mathcal{P}_n$ generated by elements $\boldsymbol{\omega}_{p,c}$ with $p \in [1, n]$ and

$$\max\{b_{m+p} + n + 1, a_{j+p} + n + 1\} \geq c \geq \min\{b_m, a_{j+m}\} > a_{j-1}$$

we see that together with (5.3) and Lemma 5.1 that $\boldsymbol{\pi} = \boldsymbol{\pi}_2$ which contradicts our choice.

We prove the existence of $\mathbf{a}_1$ and $\mathbf{b}_1$ by induction on s. Suppose that $s = 1$. If $r = 1$ the result is immediate from (5.2). If $r > 1$ then by Proposition 4.2 we have to consider the case $a_j < b_1 < a_{j+1}$ for some $0 \le j \le r$, where we understand that the first or last inequality holds vacuously if $j = 0$ or $j = r$ and we take

$$\mathbf{a}_1 = (a_{j+\delta_{j,0}}), \quad \mathbf{b}_1 = (b_1).$$

For the inductive step we assume that the existence of the appropriate segments for all $s' < s$ and for all $r \ge s'$. Suppose that either $b_2 < a_1$ or $b_1 = a_1$ or $b_1 < a_1$ and there exists $j \ge 2$ and k with $a_k < b_j < a_{k+1}$. Set $\mathbf{b}' = (b_2, \cdots, b_s)$. The inductive hypothesis applies to $\mathbf{a}$ and $\mathbf{b}'$ in the first case and to $\mathbf{a}' = (a_2, \cdots, a_r)$ and $\mathbf{b}'$ in the other cases. It follows that we can take $\mathbf{a}_1 = \mathbf{a}'_1$ and $\mathbf{b}_1 = \mathbf{b}'_1$.

Next let $b_1 < a_1 \le b_2$ and assume that there does not exist $j \ge 2$ with $a_k < b_j < a_{k+1}$. Since $r \ge s$ there exists $p \in [2, s+1]$ maximal so that $a_{m-1} = b_m$ if $2 \le m \le p-1$ and $a_{p-1} \ne b_p$ where we understand $b_{s+1} = 0$. This time we take $\mathbf{b}_1 = (b_1, \cdots, b_{p-1})$ and $\mathbf{a}_1 = (a_1, \cdots, a_{p-1})$.

Finally, it remains to consider the case when $b_1 > a_1$. Working from the other side we see that we can further reduce to the case when $b_s < a_r$ as well. Choose p maximal and p' minimal so that $a_p < b_1 < b_s < a_{p'}$. Since $\mathbf{a}$ and $\mathbf{b}$ are in special position there exists $j \in [1, s]$ minimal such that $a_k < b_j < a_{k+1}$ for some $k \in [p, p']$. Define an integer m as follows: if either $j = 1$ or $j > 1$ and $b_{j-1} < a_k$ we take $m = 0$; otherwise we take m so that $a_{k-m'} = b_{j-m'-1}$ for all $0 \le m' < m$ and $a_{k-m} > b_{j-m-1}$. Notice that m must exist since $a_p < b_1$ and j was chosen minimal. This time we take

$$\mathbf{a}_1 = (a_{k-m}, \cdots, a_k), \quad \mathbf{b}_1 = (b_{j-m}, \cdots, b_j),$$

and the proof of the inductive step is complete.

6. Proof of Theorem 1: Irreducibility

We complete the proof of Theorem 1.

6.1. We recall the main results of [12] and [13].

Theorem 5. Let $\boldsymbol{\omega}_1, \cdots, \boldsymbol{\omega}_r \in \mathcal{P}_n^+$. The module $V(\boldsymbol{\omega}_1) \otimes \cdots \otimes V(\boldsymbol{\omega}_r)$ is an ℓ–highest weight module (resp. irreducible) if the modules $V(\boldsymbol{\omega}_j) \otimes V(\boldsymbol{\omega}_k)$ are ℓ–highest weight (resp. irreducible) for all $1 \le j < k \le r$. □

6.2. We shall use the following consequence of Proposition 1.5.2 and Theorem 5 repeatedly.

Lemma. *Suppose that* $\mathbf{c} = (c_1, \cdots, c_k)$ *and* $\mathbf{d} = (d_1, \cdots, d_p)$ *are* (i, n) *segments with* $c_j - d_m \notin S_{i,n}$ *(resp.* $\pm(c_j - d_m) \notin S_{i,n}$*) for all* $1 \le j \le k$ *and* $1 \le m \le p$. *Then* $V(\boldsymbol{\omega}_{i,\mathbf{d}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{c}})$ *is* ℓ*–highest weight (resp. irreducible).* □

Proof. Using Proposition 1.5.2(ii), (iii) and Theorem 5 we see that $W(\boldsymbol{\omega}_{i,\mathbf{d}}) \otimes W(\boldsymbol{\omega}_{i,\mathbf{c}})$ is ℓ–highest weight and hence so is the quotient $V(\boldsymbol{\omega}_{i,\mathbf{d}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{c}})$. If in addition we also have $d_m - c_j \notin S_{i,n}$ for $1 \le j \le k$ and $1 \le m \le p$, then it follows that $W(\boldsymbol{\omega}_{i^*,\mathbf{c}^*}) \otimes W(\boldsymbol{\omega}_{i^*,\mathbf{d}^*})$ and hence $V(\boldsymbol{\omega}_{i^*,\mathbf{c}^*})\otimes V(\boldsymbol{\omega}_{i^*,\mathbf{d}^*})$ are also ℓ–highest weight. The irreducibility of $V(\boldsymbol{\omega}_{i,\mathbf{d}})\otimes V(\boldsymbol{\omega}_{i,\mathbf{c}})$ then follows from Proposition 1.5.2(i). □

6.3. In view of Theorem 5 the irreducbility statement of Theorem 1 is immediate from the next proposition.

Proposition. *Suppose that* $\mathbf{a}$ *and* $\mathbf{b}$ *are* (i,n)*–segments in general position. Then* $V(\boldsymbol{\omega}_{i,\mathbf{a}})\otimes V(\boldsymbol{\omega}_{i,\mathbf{b}})$ *is irreducible.*

Proof. We shall use the equivalent formulation given in Proposition 4.2 for a pair of segments to be in general position.

Suppose that $a_r < b_1$ and $b_1 - a_r > 2i$. Then $b_m - a_p > 2i$ and $0 > a_p - b_m \notin S_{i,n}$ for all $1 \le m \le s$ and $1 \le p \le r$. It follows from Lemma 6.2 that the module $V(\boldsymbol{\omega}_{i,\mathbf{b}})\otimes V(\boldsymbol{\omega}_{i,\mathbf{a}})$ is irreducible. If $b_s < a_1$ and $a_1 - b_s \notin S_{i,n}$ or is $b_s - a_1 \notin 2\mathbb{Z}$ the proof is identical and we omit the details.

It remains to consider the case when $s \le r$ and $\{b_1,\cdots,b_s\}\subset\{a_1,\cdots,a_r\}$. We proceed by induction on s and for each s by a further induction on r. Suppose that $s=1$ and $b_1 = a_m$ for some $1\le m\le r$. If $m=r$ the result is immediate if we prove the following claim:

$$\mathrm{wt}_\ell(V(\boldsymbol{\omega}_{i,\mathbf{a}})\otimes V(\boldsymbol{\omega}_{i,a_r}))\cap\mathcal{P}_n^+ = \{\boldsymbol{\omega}_{i,\mathbf{a}}\boldsymbol{\omega}_{i,a_r}\}.$$

To prove the claim, let $\boldsymbol{\omega}$ be an element of the intersection and write $\boldsymbol{\omega} = \boldsymbol{\omega}(\underline{p})\boldsymbol{\omega}(p') \in \mathcal{P}_n^+$ with $\underline{p} = (p_1,\cdots,p_r)\in\mathbb{P}_{i,\mathbf{a}}$ and $p'\in\mathbb{P}_{i,a_r}$. If $\boldsymbol{\omega}(p')\neq\boldsymbol{\omega}_{i,a_r}$ then there exists $j\in\mathbf{c}_{p'}^-$ with $p'(j)\ge p'(s)$ for all $s\in\mathbf{c}_{p'}^+\cup\mathbf{c}_{p'}^-$ and $p'(j) > a_r$. It follows that $j\in\mathbf{c}_{p_k}^+$ for some $1\le k\le r$ with $p_k(j) = p'(j) > a_r > a_k$. This implies that $\boldsymbol{\omega}(p_k)\neq\boldsymbol{\omega}_{i,a_k}$ and so there exists $j'\in\mathbf{c}_{p_k}^-$ with $p_k(j') > p'(j)$. It follows from Proposition 1.6.2 that $\boldsymbol{\omega}_{j',p_k(j')}^{-1}$ occurs in any reduced expression for $\boldsymbol{\omega}$ which contradicts $\boldsymbol{\omega}\in\mathcal{P}_n^+$. Hence $\boldsymbol{\omega} = \boldsymbol{\omega}(\underline{p})\boldsymbol{\omega}_{i,a_r}$.

If $\boldsymbol{\omega}(\underline{p})\neq\boldsymbol{\omega}_{i,\mathbf{a}}$ then there must exist $1\le k\le r$ with $\mathbf{c}_{p_k}^- = \{i\}$ and $p_k(i) = a_r$ and $\mathbf{c}_j^- = \emptyset$ for all $1\le k\neq j\le r$. But this means that $k<r$ and then we have a contradiction to the definition of the set $\mathbb{P}_{i,\mathbf{a}}$. In particular, we have proved the case $s=r=1$.

Applying the Cartan involution Ω we see using the discussion in Section 1.5.1 that the module $V(\boldsymbol{\omega}_{i,a_1})\otimes V(\boldsymbol{\omega}_{i,\mathbf{a}})$ is also irreducible.

Hence to complete the proof that induction begins at $s=1$ case we must consider $a_1 < b_1 = a_m < a_r$. We assume moreover that the inductive hypothesis hold for $r' < r$. Let $\mathbf{a}' = (a_1,\cdots,a_m)$ and $\mathbf{a}'' = (a_{m+1},\cdots,a_r)$. These are both (i,n)–segments of length at most $r-1$ and consider

$$V(\boldsymbol{\omega}_{i,\mathbf{a}''})\otimes V(\boldsymbol{\omega}_{i,\mathbf{a}'})\otimes V(\boldsymbol{\omega}_{i,a_m}).$$

The tensor products $V(\boldsymbol{\omega}_{i,\mathbf{a}''})\otimes V(\boldsymbol{\omega}_{i,\mathbf{a}'})$ and $V(\boldsymbol{\omega}_{i,a_m})\otimes V(\boldsymbol{\omega}_{i,\mathbf{a}'})$ are ℓ–highest weight by Lemma 6.2. The inductive hypothesis shows that the tensor product of the second and third module is irreducible and hence ℓ–highest weight. Theorem 5 shows that the entire tensor product is ℓ–highest weight and hence so is its quotient $V(\boldsymbol{\omega}_{i,\mathbf{a}})\otimes V(\boldsymbol{\omega}_{i,a_m})$. Working with the

$(n+1-i,n)$–segments, $\mathbf{a}_1' = (a_1+n+1, \cdots, a_{m-1}+n+1)$, and $\mathbf{a}_1'' = (a_m+n+1, \cdots, a_r+n+1)$ and the tensor product

$$V(\boldsymbol{\omega}_{n+1-i,a_m+n+1}) \otimes V(\boldsymbol{\omega}_{n+1-i,\mathbf{a}_1''}) \otimes V(\boldsymbol{\omega}_{n+1-i,\mathbf{a}_1'}),$$

we see similarly that every pair of modules in the tensor product is ℓ–highest weight and hence so is $V(\boldsymbol{\omega}_{n+1-i,a_m+n+1}) \otimes V(\boldsymbol{\omega}_{n+1-i,\mathbf{a}^*})$. The irreducibilty of the tensor product follows from Proposition 1.5.2(i) and completes the proof that induction begins is when $s = 1$.

Assuming the result for $1 \leq s' < s$ and for all $r \geq s'$ we prove it for s and all $r \geq s$. Writing $\mathbf{b}' = (b_1, \cdots, b_{s-1})$ we consider

$$V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,b_s}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}'}).$$

The tensor product $V(\boldsymbol{\omega}_{i,b_s}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}'})$ is ℓ-highest weight by Lemma 6.2 and the tensor product of the remaining two pairs is irreducible by the inductive hypothesis on s. Hence the module $V(\boldsymbol{\omega}_{i,\mathbf{a}}) \otimes V(\boldsymbol{\omega}_{i,\mathbf{b}})$ is ℓ–highest weight. A similar argument with $(n+1-i,n)$–segments as in the $s = 1$ case also proves that the dual of this module is ℓ–highest weight and the irreducibility of the tensor product follows by Proposition 1.5.2(i). This completes the proof of the inductive step and the proof of the proposition is complete.

□

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