REPRESENTING TOPOLOGICAL FULL GROUPS IN STEINBERG ALGEBRAS AND C*-ALGEBRAS

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Abstract. We study the natural representation of the topological full group of an ample Hausdorff groupoid in the groupoid's complex Steinberg algebra and in its full and reduced C*-algebras. We characterise precisely when this representation is injective and show that it is rarely surjective. We then restrict our attention to discrete groupoids, which provide unexpected insight into the behaviour of the representation of the topological full group in the full and reduced groupoid C*-algebras. We show that the image of the representation is not dense in the full groupoid C*-algebra unless the groupoid is a group, and we provide an example showing that the image of the representation may still be dense in the reduced groupoid C*-algebra even when the groupoid is not a group.

1. Introduction

Topological full groups of ample Hausdorff groupoids were introduced by Matui [16] as a generalisation of the topological full groups studied by Giordano, Putnam, and Skau in the context of Cantor minimal systems [8]. Matui showed in [17, Theorem 3.10] that for any two minimal effective Hausdorff étale groupoids whose unit spaces are Cantor sets, the groupoids are isomorphic if and only if their topological full groups are isomorphic. This is equivalent to there being a diagonal-preserving isomorphism of the Steinberg algebras of the groupoids; see [1, Theorem 3.1]. It is therefore clear that there are strong connections between the topological full groups and Steinberg algebras of ample Hausdorff groupoids.

In addition to being a groupoid invariant, topological full groups have enticing connections to some infamous open questions. For example, they give presentations of Thompson's groups [13, 15, 17, 28], and have already been used to solve several important problems in group theory; see [2, 11, 12, 20, 26]. Recent results also reveal interesting connections between topological full groups and the elusive simplicity problem for group C*-algebras; see [3, 14, 24]. It is this latter problem that motivates our study.

For every ample Hausdorff groupoid $\mathcal{G}$ with compact unit space, there are natural representations of the topological full group of $\mathcal{G}$ in the complex Steinberg algebra of $\mathcal{G}$ and in the full and reduced C*-algebras of $\mathcal{G}$. It is known that these representations often fail to be injective. We make this statement precise by showing that the representation of the

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topological full group taking values in the Steinberg algebra of the groupoid is almost never injective. In particular, we show in Theorem 3.2 that injectivity fails when

(1) the groupoid is all isotropy and has at least 2 nontrivial isotropy groups; or
(2) the groupoid is not all isotropy and has at least 3 non-unit elements.

We then show that this representation is almost never surjective as a map into the complex Steinberg algebra. In fact, we show in Corollary 4.4 that the representation is surjective onto the Steinberg algebra if and only if $\mathcal{G}$ is a group. However, strangely, the image of the representation of the topological full group may still be dense in the full or reduced groupoid C*-algebras. For example, density of the image holds for the representation of the topological full group associated to the Cuntz groupoid (that is, the boundary-path groupoid of the directed graph with a single vertex and two edges) into the Cuntz algebra $\mathcal{O}_2$; see [3, Remark 4.7] and [10, Proposition 5.3]. Example 5.5 provides another such example.

Our proof techniques for the results in Sections 3 and 4 were developed by first considering these questions for discrete groupoids. The arguments in our proof of Theorem 3.2 in particular are quite combinatorial in nature.

In Section 5 we demonstrate that surprising things can happen in the setting of discrete groupoids. In Theorem 5.3, we show that the image of the representation of the topological full group of a discrete groupoid with finite unit space is dense in the full groupoid C*-algebra if and only if the groupoid is a group. (Note that the Cuntz groupoid mentioned above is not discrete, and thus this result does not hold for ample Hausdorff groupoids in general; see Remark 5.4.) In Example 5.5 we demonstrate that it is possible for the image of the representation of the topological full group of a discrete groupoid to be dense in the reduced groupoid C*-algebra even when the groupoid is not a group. Finally, in Corollary 5.6 we combine our results from Sections 3, 4, and 5 to show that the representation of the topological full group of an ample Hausdorff groupoid $\mathcal{G}$ with compact unit space is an isomorphism into the Steinberg algebra of $\mathcal{G}$ if and only if $\mathcal{G}$ is a group, and that when $\mathcal{G}$ is discrete with finite unit space, the same result holds for the extension of this representation to the full C*-algebra.

2. Preliminaries

2.1. **Groupoids.** A *groupoid* $\mathcal{G}$ is a small category in which every morphism $\gamma \in \mathcal{G}$ has a unique inverse $\gamma^{-1} \in \mathcal{G}$. Throughout, we assume that all groupoids are nonempty. We define the *range* and *source* of each $\gamma \in \mathcal{G}$ by $r(\gamma) \coloneqq \gamma\gamma^{-1}$ and $s(\gamma) \coloneqq \gamma^{-1}\gamma$, respectively, where composition is read from right to left. We write

$$\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$$

for the set of *composable pairs* in $\mathcal{G}$, and we write $\mathcal{G}^{(0)} = r(\mathcal{G}) = s(\mathcal{G})$ for the *unit space* of $\mathcal{G}$. Note that a groupoid $\mathcal{G}$ is a group if and only if $\mathcal{G}^{(0)}$ is a singleton. A *topological groupoid* is a groupoid endowed with a topology under which composition and inversion are continuous. A *Hausdorff groupoid* is a topological groupoid with a locally compact Hausdorff topology. If $\mathcal{G}$ is a Hausdorff groupoid, then $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$. A topological groupoid $\mathcal{G}$ is *étale* if the range and source maps $r, s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms. A subset $B \subseteq \mathcal{G}$ is called a *bisection* of $\mathcal{G}$ if $r|_B$ and $s|_B$ are injective. If B is an open bisection of an étale groupoid $\mathcal{G}$, then $r|_B$ and $s|_B$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Every étale groupoid has a basis consisting of open bisections; see [6, Proposition 3.5]. We say that an étale groupoid is *ample* if it has a basis of *compact* open bisections. By [7, Proposition 4.1], a Hausdorff étale groupoid is ample if and only if its unit space is totally disconnected. If $\mathcal{G}$ is an étale groupoid, then the unit space $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$, and for all $u, v \in \mathcal{G}^{(0)}$, each of the sets

$$\mathcal{G}^u \coloneqq r^{-1}(u), \ \mathcal{G}_v \coloneqq s^{-1}(v), \ \text{and} \ \ \mathcal{G}_v^u \coloneqq \mathcal{G}^u \cap \mathcal{G}_v$$

is discrete with respect to the relative topology induced by $\mathcal{G}$. The *isotropy group* of a unit $u \in \mathcal{G}^{(0)}$ is the group

$$\mathcal{G}_u^u = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma) = u\},$$

and the *isotropy subgroupoid* of $\mathcal{G}$ is the collection

$$\mathrm{Iso}(\mathcal{G}) \coloneqq \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}.$$

Let $\mathcal{G}$ be a Hausdorff étale groupoid. For each continuous function $f\colon \mathcal{G} \to \mathbb{C}$, we define $\mathrm{supp}(f) \coloneqq \overline{\{\gamma \in \mathcal{G} : f(\gamma) \neq 0\}}$. We write $C_c(\mathcal{G})$ for the collection of continuous compactly supported complex-valued functions on $\mathcal{G}$. This is a $*$-algebra with respect to the convolution product

$$(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$$

and $*$-involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$ for $f, g \in C_c(\mathcal{G})$ and $\gamma \in \mathcal{G}$. Given a Hilbert space $\mathcal{H}$, we write $B(\mathcal{H})$ for the C*-algebra of bounded linear operators on $\mathcal{H}$. The *full groupoid C*-algebra* $C^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the *full C*-norm*

$$\|f\|_{\max} \coloneqq \sup\{\|\pi(f)\| \mid \pi\colon C_c(\mathcal{G}) \to B(\mathcal{H}) \text{ is a } *\text{-representation for some } \mathcal{H}\}.$$

For each $u \in \mathcal{G}^{(0)}$, there is a $*$-representation $\pi_u\colon C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}_u))$, called the *regular representation* of $C_c(\mathcal{G})$ associated to u, such that

$$\pi_u(f)\delta_\gamma = \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma} \quad \text{for } f \in C_c(\mathcal{G}) \text{ and } \gamma \in \mathcal{G}_u.$$

The *reduced groupoid C*-algebra* $C_r^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the *reduced C*-norm*

$$\|f\|_r \coloneqq \sup\{\|\pi_u(f)\| \mid u \in \mathcal{G}^{(0)}\}.$$

See [23, Chapter II] or [25, Chapter 9] for details.

The *(complex) Steinberg algebra* of an ample Hausdorff groupoid $\mathcal{G}$ is the collection

$$\begin{aligned} A(\mathcal{G}) &\coloneqq \mathrm{span}\{1_U\colon \mathcal{G} \to \mathbb{C} \mid U \text{ is a compact open bisection of } \mathcal{G}\} \\ &= \{f \in C_c(\mathcal{G}) \mid f \text{ is locally constant}\} \end{aligned}$$

equipped with the convolution product and $*$-involution defined above. If $\mathcal{G}$ is discrete, then $A(\mathcal{G}) = C_c(\mathcal{G})$. In general, $A(\mathcal{G})$ is dense in $C_c(\mathcal{G})$ with respect to both the full and reduced C*-norms (see [4, Proposition 4.2]), and for all $f \in A(\mathcal{G})$, we have

$$\|f\|_{\max} \coloneqq \sup\{\|\pi(f)\| \mid \pi\colon A(\mathcal{G}) \to B(\mathcal{H}) \text{ is a } *\text{-representation for some } \mathcal{H}\} \tag{2.1}$$

(see [5, Theorem 7.1]). Note that a discrete group G may be viewed as an ample Hausdorff groupoid, and in this case the singletons in G are all compact open bisections, and so the Steinberg algebra $A(G)$ is just the complex group ring $\mathbb{C}G$, which is generated by the point-mass functions $\delta_g \coloneqq 1_{\{g\}}$ for $g \in G$. See [4, 27] for further details on Steinberg algebras.

2.2. **Topological full groups.** Let $\mathcal{G}$ be an ample groupoid with compact unit space $\mathcal{G}^{(0)}$. We write $B^{\mathrm{co}}(\mathcal{G})$ for the inverse semigroup of compact open bisections of $\mathcal{G}$, and we say that a bisection B of $\mathcal{G}$ is *full* if $r(B) = s(B) = \mathcal{G}^{(0)}$. We define the *topological full group* of $\mathcal{G}$ to be the (discrete) group

$$F(\mathcal{G}) \coloneqq \{B \in B^{\mathrm{co}}(\mathcal{G}) \mid B \text{ is full}\}$$

equipped with the operations

$$AB \coloneqq \{\alpha\beta \mid (\alpha, \beta) \in (A \times B) \cap \mathcal{G}^{(2)}\} \quad \text{and} \quad B^{-1} \coloneqq \{\gamma^{-1} \mid \gamma \in B\}$$

for all $A, B \in F(\mathcal{G})$. Note that if $\mathcal{G}$ is a discrete group, then

$$F(\mathcal{G}) = B^{\mathrm{co}}(\mathcal{G}) = \{\{g\} \mid g \in \mathcal{G}\} \cong \mathcal{G}.$$

See [19, 21] for further details on topological full groups.

Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space. Given compact open bisections U and V of $\mathcal{G}$, we have

$$1_U * 1_V = 1_{UV} \quad \text{and} \quad (1_U)^* = 1_{U^{-1}}.$$

It follows that there is a $*$-homomorphism $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ satisfying $\pi(\delta_U) = 1_U$, which we call the *representation* of $F(\mathcal{G})$ in $A(\mathcal{G})$. This representation is studied extensively in [3], as are the C*-completions $\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}}$ and $\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_r}$ of its image. In this paper we investigate the necessary and sufficient conditions under which π is injective and surjective.

Remark 2.1. In [22, Definition 3.2] Nyland and Ortega define the topological full group of an (effective) ample Hausdorff groupoid with a unit space that is not necessarily compact. Since the Steinberg algebra of an ample Hausdorff groupoid $\mathcal{G}$ is unital (with unit $1_{\mathcal{G}^{(0)}}$) if and only if the unit space $\mathcal{G}^{(0)}$ is compact, it is impossible to represent the topological full group of $\mathcal{G}$ in $A(\mathcal{G})$ (or in $C^*(\mathcal{G})$ or $C^*_r(\mathcal{G})$) unless $\mathcal{G}^{(0)}$ is compact. It is for this reason that we restrict our attention in this paper to ample Hausdorff groupoids with compact unit space.

3. Lack of injectivity for ample Hausdorff groupoids

In this section we characterise precisely when the representation $\pi\colon \delta_U \mapsto 1_U$ of $\mathbb{C}F(\mathcal{G})$ in $A(\mathcal{G})$ is injective. In particular, we show in Theorem 3.2 that π is injective if and only if either $\mathcal{G}$ consists entirely of isotropy and has at most one nontrivial isotropy group, or $\mathcal{G}$ contains exactly 2 non-unit elements outside its isotropy.

Proposition 3.1. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. Suppose that either*

(1) *$\mathcal{G} = \operatorname{Iso}(\mathcal{G})$ and $\mathcal{G}$ has at least two nontrivial isotropy groups; that is, there exist $u, v \in \mathcal{G}^{(0)}$ such that $u \neq v$ and $|\mathcal{G}_u^u|, |\mathcal{G}_v^v| > 1$; or*
(2) *$\mathcal{G} \neq \operatorname{Iso}(\mathcal{G})$ and $\left|\mathcal{G} \setminus \mathcal{G}^{(0)}\right| \geq 3$.*

Then the representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is not injective.

Proof. We first assume that condition (1) holds. Fix $\gamma_1, \gamma_2 \in \mathcal{G} \setminus \mathcal{G}^{(0)}$ such that $r(\gamma_1) \neq r(\gamma_2)$. Since $\mathcal{G}^{(0)}$ is Hausdorff and $\mathcal{G}$ is all isotropy, we can find disjoint compact open bisections B_1 and B_2 containing γ_1 and γ_2, respectively, such that

$$r(B_1) = s(B_1), \quad r(B_2) = s(B_2), \quad \text{and} \quad r(B_1) \cap r(B_2) = \varnothing.$$

Set $R := \mathcal{G}^{(0)} \setminus \big(r(B_1) \cup r(B_2)\big)$, and note that R is a compact open bisection of $\mathcal{G}$. Consider the following disjoint unions:

$$\begin{aligned} U_1 &:= B_1 \cup r(B_2) \cup R, \\ U_2 &:= B_2 \cup r(B_1) \cup R, \text{ and} \\ U_3 &:= B_1 \cup B_2 \cup R. \end{aligned}$$

It is straightforward to verify that U_1, U_2, and U_3 are distinct elements of $F(\mathcal{G})$. Define $a := \delta_{U_1} + \delta_{U_2} - \delta_{U_3} - \delta_{\mathcal{G}^{(0)}}$. Then

$$\pi(a) = 1_{U_1} + 1_{U_2} - 1_{U_3} - 1_{\mathcal{G}^{(0)}} = 1_{r(B_2)} + 1_{r(B_1)} + 1_R - 1_{\mathcal{G}^{(0)}} = 0,$$

and so $0 \neq a \in \ker \pi$. Hence π is not injective.

We now assume that condition (2) holds instead. Then there exist

$$\gamma_1 \in \mathcal{G} \setminus \mathcal{G}^{(0)} \text{ and } \gamma_2 \in \mathcal{G} \setminus \operatorname{Iso}(\mathcal{G}) \text{ such that } \gamma_1 \neq \gamma_2 \text{ and } \gamma_1 \neq \gamma_2^{-1}. \tag{3.1}$$

For any γ_1, γ_2 satisfying condition (3.1), we have $r(\gamma_2) \neq s(\gamma_2)$, and either $\gamma_1 \notin \operatorname{Iso}(\mathcal{G})$ or $\gamma_1 \in \operatorname{Iso}(\mathcal{G})$. By replacing γ_1 with γ_1^{-1} or γ_2 with γ_2^{-1} if necessary, we can summarise all possible cases as follows:

(i) $\gamma_1 \notin \mathrm{Iso}(\mathcal{G})$ and $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) \neq r(\gamma_1)$;
(ii) $\gamma_1 \notin \mathrm{Iso}(\mathcal{G})$ and $r(\gamma_1)$, $s(\gamma_1)$, $r(\gamma_2)$, and $s(\gamma_2)$ are all distinct;
(iii) $\gamma_1 \in \mathrm{Iso}(\mathcal{G})$ and $s(\gamma_1)$, $r(\gamma_2)$, and $s(\gamma_2)$ are all distinct;
(iv) $\gamma_1 \in \mathrm{Iso}(\mathcal{G})$ and $s(\gamma_1) = r(\gamma_2)$; and
(v) $\gamma_1 \notin \mathrm{Iso}(\mathcal{G})$ and $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_1)$.

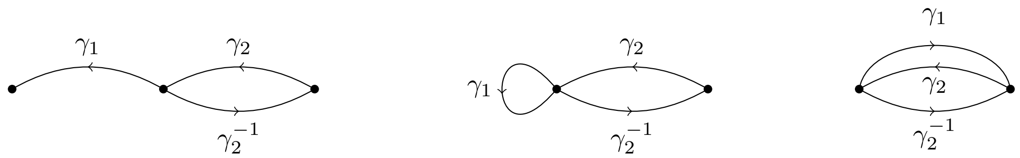

FIGURE 1. From left to right: cases (i), (iv), and (v).

Moreover, we can reduce case (v) to case (iv) by replacing γ_1 with $\gamma_2\gamma_1$. Therefore, it suffices to show that $\ker \pi$ is nontrivial in each of the cases (i) to (iv).

Case (i): Suppose that the hypotheses of case (i) hold, and let B_1 and B_2 be compact open bisections containing γ_1 and γ_2, respectively. Since $\mathcal{G}^{(0)}$ is Hausdorff and since $r(\gamma_1)$, $s(\gamma_1)$, and $s(\gamma_2)$ are all distinct, we may assume that $r(B_1)$, $s(B_1)$, and $s(B_2)$ are mutually disjoint by shrinking B_1 and B_2 if necessary. Moreover, since $s(\gamma_1) = r(\gamma_2)$, we can replace B_1 with $B_1\big(s(B_1) \cap r(B_2)\big)$ and B_2 with $\big(s(B_1) \cap r(B_2)\big)B_2$, and thus without loss of generality we may assume that $s(B_1) = r(B_2)$. Set $R \coloneqq \mathcal{G}^{(0)} \setminus \big(r(B_1) \cup s(B_1) \cup s(B_2)\big)$, and note that R is a compact open bisection of $\mathcal{G}$. Consider the following disjoint unions that are distinct elements of $F(\mathcal{G})$:

$$\begin{aligned} U &\coloneqq B_1 \cup B_2 \cup (B_1B_2)^{-1} \cup R, \\ U^{-1} &= B_1^{-1} \cup B_2^{-1} \cup (B_1B_2) \cup R, \\ U_1 &\coloneqq B_1 \cup B_1^{-1} \cup s(B_2) \cup R, \\ U_2 &\coloneqq B_2 \cup B_2^{-1} \cup r(B_1) \cup R, \text{ and} \\ U_3 &\coloneqq B_1B_2 \cup (B_1B_2)^{-1} \cup s(B_1) \cup R. \end{aligned}$$

Define $a \coloneqq \delta_U + \delta_{U^{-1}} - \delta_{U_1} - \delta_{U_2} - \delta_{U_3} + \delta_{\mathcal{G}^{(0)}}$. Then

$$\pi(a) = 1_U + 1_{U^{-1}} - 1_{U_1} - 1_{U_2} - 1_{U_3} + 1_{\mathcal{G}^{(0)}} = -1_{s(B_2)} - 1_{r(B_1)} - 1_{s(B_1)} - 1_R + 1_{\mathcal{G}^{(0)}} = 0,$$

and so $a \in \ker \pi \setminus \{0\}$. Hence π is not injective.

Case (ii): Now suppose that the hypotheses of case (ii) hold, and let B_1 and B_2 be compact open bisections containing γ_1 and γ_2, respectively. Since $\mathcal{G}^{(0)}$ is Hausdorff and since $r(\gamma_1)$, $s(\gamma_1)$, $r(\gamma_2)$, and $s(\gamma_2)$ are all distinct, we may assume that $r(B_1)$, $s(B_1)$, $r(B_2)$, and $s(B_2)$ are mutually disjoint by shrinking B_1 and B_2 if necessary. Set

$$R \coloneqq \mathcal{G}^{(0)} \setminus \big(r(B_1) \cup s(B_1) \cup r(B_2) \cup s(B_2)\big),$$

and note that R is a compact open bisection of $\mathcal{G}$. Consider the following disjoint unions that are distinct elements of $F(\mathcal{G})$:

$$\begin{aligned} U_1 &\coloneqq B_1 \cup B_1^{-1} \cup r(B_2) \cup s(B_2) \cup R, \\ U_2 &\coloneqq B_2 \cup B_2^{-1} \cup r(B_1) \cup s(B_1) \cup R, \text{ and} \\ U_3 &\coloneqq B_1 \cup B_1^{-1} \cup B_2 \cup B_2^{-1} \cup R. \end{aligned}$$

It is straightforward to verify that $a \coloneqq \delta_{U_1} + \delta_{U_2} - \delta_{U_3} - \delta_{\mathcal{G}^{(0)}} \in \ker \pi \setminus \{0\}$, and hence π is not injective.

Case (iii): Next, suppose that the hypotheses of case (iii) hold, and let B_1' and B_2 be compact open bisections containing γ_1 and γ_2, respectively. Since $\mathcal{G}^{(0)}$ is Hausdorff and since $s(\gamma_1)$, $r(\gamma_2)$, and $s(\gamma_2)$ are all distinct, we may assume that $s(B_1')$, $r(B_2)$, and $s(B_2)$ are mutually disjoint by shrinking B_1' and B_2 if necessary. Let $V := r(B_1') \cap s(B_1')$, and define $B_1 := VB_1'V$. Then B_1 is a compact open bisection containing γ_1, because $\mathcal{G}$ is an ample Hausdorff groupoid and $r(\gamma_1) = s(\gamma_1) \in V$. Suppose that $r(B_1) \neq s(B_1)$. Then there exists $\alpha \in B_1$ such that $r(\alpha) \notin s(B_1)$ or $s(\alpha) \notin r(B_1)$. In either case, $\alpha \notin \operatorname{Iso}(\mathcal{G})$ and $r(\alpha), s(\alpha) \in V \subseteq s(B_1')$. Thus, since $\gamma_2 \in \mathcal{G} \setminus \operatorname{Iso}(\mathcal{G})$ and $r(\gamma_2), s(\gamma_2) \in \mathcal{G}^{(0)} \setminus s(B_1')$, we deduce that $\alpha \neq \gamma_2$ and $\alpha \neq \gamma_2^{-1}$, and that $r(\alpha)$, $s(\alpha)$, $r(\gamma_2)$, and $s(\gamma_2)$ are all distinct. So if $r(B_1) \neq s(B_1)$, then case (iii) can be reduced to case (ii) by replacing γ_1 with α. Now suppose that $r(B_1) = s(B_1)$. Since $r(B_1) \subseteq V \subseteq s(B_1')$, we know that $r(B_1)$, $r(B_2)$, and $s(B_2)$ are mutually disjoint. Set $R := \mathcal{G}^{(0)} \setminus \big(r(B_1) \cup r(B_2) \cup s(B_2)\big)$, and note that R is a compact open bisection of $\mathcal{G}$. Consider the following disjoint unions that are distinct elements of $F(\mathcal{G})$:

$$\begin{aligned} U_1 &:= r(B_1) \cup B_2 \cup B_2^{-1} \cup R, \\ U_2 &:= B_1 \cup r(B_2) \cup s(B_2) \cup R, \text{ and} \\ U_3 &:= B_1 \cup B_2 \cup B_2^{-1} \cup R. \end{aligned}$$

It is straightforward to verify that $a := \delta_{U_1} + \delta_{U_2} - \delta_{U_3} - \delta_{\mathcal{G}^{(0)}} \in \ker \pi \setminus \{0\}$, and hence π is not injective in this case either.

Case (iv): Finally, suppose that the hypotheses of case (iv) hold, and let B_1' and B_2' be compact open bisections containing γ_1 and γ_2, respectively. Since $\mathcal{G}^{(0)}$ is Hausdorff and since $r(\gamma_2) \neq s(\gamma_2)$, we may assume that $r(B_2') \cap s(B_2') = \varnothing$. Let $W := r(B_1') \cap s(B_1') \cap r(B_2')$, and define $B_1 := WB_1'W$ and $B_2 := s(B_1)B_2'$. Then B_1 and B_2 are compact open bisections containing γ_1 and γ_2, respectively, because $\mathcal{G}$ is an ample Hausdorff groupoid and $r(\gamma_1) = s(\gamma_1) = r(\gamma_2) \in W$. Since $s(B_1) \subseteq W \subseteq r(B_2')$, we have $r(B_2) = s(B_1) \cap r(B_2') = s(B_1)$. Suppose that $r(B_1) \neq s(B_1)$. Then there exists $\alpha \in B_1$ such that $r(\alpha) \notin s(B_1)$ or $s(\alpha) \notin r(B_1)$. In either case, $\alpha \notin \operatorname{Iso}(\mathcal{G})$ and $r(\alpha), s(\alpha) \in W \subseteq r(B_2')$, so there exists $\beta \in B_2'$ such that $r(\beta) = s(\alpha) \in W$. Since $s(\beta) \in s(B_2') \subseteq \mathcal{G}^{(0)} \setminus r(B_2') \subseteq \mathcal{G}^{(0)} \setminus W$, we know that $s(\beta) \neq r(\alpha)$, $s(\beta) \neq s(\alpha)$, and $s(\beta) \neq r(\beta)$. Thus $\beta \in \mathcal{G} \setminus \operatorname{Iso}(\mathcal{G})$, $\alpha \neq \beta$, and $\alpha \neq \beta^{-1}$. So if $r(B_1) \neq s(B_1)$, then case (iv) can be reduced to case (i) by replacing γ_1 and γ_2 with α and β, respectively. Now suppose that $r(B_1) = s(B_1) = r(B_2)$. Since $r(B_2) \subseteq r(B_2')$ and $s(B_2) \subseteq s(B_2')$, we know that $r(B_2) \cap s(B_2) = \varnothing$. Set $R := \mathcal{G}^{(0)} \setminus \big(r(B_2) \cup s(B_2)\big)$, and note that R is a compact open bisection of $\mathcal{G}$. Consider the following disjoint unions that are elements of $F(\mathcal{G})$:

$$\begin{aligned} U_1 &:= B_2 \cup (B_1B_2)^{-1} \cup R, \\ U_2 &:= B_1B_2 \cup B_2^{-1} \cup R, \\ U_3 &:= B_2 \cup B_2^{-1} \cup R, \text{ and} \\ U_4 &:= B_1B_2 \cup (B_1B_2)^{-1} \cup R. \end{aligned}$$

To see that U_1, U_2, U_3, and U_4 are distinct elements of $F(\mathcal{G})$, note that since $\gamma_1 \notin \mathcal{G}^{(0)}$, we have $\gamma_2 \neq \gamma_1\gamma_2$, and hence $\gamma_2 \notin B_1B_2$, because $\gamma_1\gamma_2$ is the unique element of the bisection B_1B_2 with source $s(\gamma_2)$. It is straightforward to verify that $a := \delta_{U_1} + \delta_{U_2} - \delta_{U_3} - \delta_{U_4} \in \ker \pi \setminus \{0\}$, and hence π is not injective. $\square$

We conclude this section by proving that the converse of Proposition 3.1 also holds.

Theorem 3.2. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space. The representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is injective if and only if*

(1) *$\mathcal{G} = \operatorname{Iso}(\mathcal{G})$ and $\mathcal{G}$ has at most one nontrivial isotropy group; or*
(2) *$\mathcal{G} \neq \operatorname{Iso}(\mathcal{G})$ and $\big|\mathcal{G} \setminus \mathcal{G}^{(0)}\big| < 3$.*

Proof. If π is injective, then the result follows by the contrapositive of Proposition 3.1. For the converse, first suppose that condition (1) holds. If $\mathcal{G} = \mathcal{G}^{(0)}$, then $F(\mathcal{G}) = \{\mathcal{G}^{(0)}\}$, and so $\mathbb{C}F(\mathcal{G}) \cong \mathbb{C}$, and hence π is injective. Suppose that $\mathcal{G} \neq \mathcal{G}^{(0)}$. Then there exists a nontrivial discrete group Γ with identity e_Γ such that $\mathcal{G} = \Gamma \sqcup X$, where $X = \mathcal{G}^{(0)} \backslash \{e_\Gamma\}$. Since $\mathcal{G}$ is Hausdorff and étale, $X = (\mathcal{G} \backslash \{e_\Gamma\}) \cap \mathcal{G}^{(0)}$ is open in $\mathcal{G}$. We claim that X is compact. To see this, first observe that since $\mathcal{G}$ is Hausdorff, $\mathcal{G}^{(0)}$ is closed, and so $\Gamma \backslash \{e_\Gamma\} = \mathcal{G} \backslash \mathcal{G}^{(0)}$ is open in $\mathcal{G}$. Thus, since $\mathcal{G}$ is étale, $\{e_\Gamma\} = r(\Gamma \backslash \{e_\Gamma\})$ is open in $\mathcal{G}$, and so $X = (\mathcal{G} \backslash \{e_\Gamma\}) \cap \mathcal{G}^{(0)}$ is closed. Now, since $X \subseteq \mathcal{G}^{(0)}$ and $\mathcal{G}^{(0)}$ is compact by hypothesis, X must also be compact, as claimed. For each $\gamma \in \Gamma$, choose a compact open bisection U_γ of $\mathcal{G}$ containing γ. Then $U_\gamma \cap \Gamma = \{\gamma\}$. Since $X = \mathcal{G}^{(0)} \backslash \{e_\Gamma\}$ is compact and open in $\mathcal{G}$, we have $V_\gamma \coloneqq U_\gamma \cup X = \{\gamma\} \sqcup X \in F(\mathcal{G})$, and it follows that $F(\mathcal{G}) = \big\{\{\gamma\} \sqcup X \mid \gamma \in \Gamma\big\}$. Now, let $f \in \ker \pi \subseteq \mathbb{C}F(\mathcal{G})$. Then for some $m \in \mathbb{N}$, there exist $c_1, \dots, c_m \in \mathbb{C}$ and $\gamma_1, \dots, \gamma_m \in \Gamma$ such that $\gamma_i \neq \gamma_j$ whenever $i \neq j$, and $f = \sum_{i=1}^m c_i \, \delta_{\{\gamma_i\} \sqcup X}$. Since $\pi(f) = 0$, we have

$$c_k = \Big(\sum_{i=1}^m c_i 1_{\{\gamma_i\}} + \big(\sum_{i=1}^m c_i \big) 1_X \Big)(\gamma_k) = \pi(f)(\gamma_k) = 0$$

for each $k \in \{1, \dots, m\}$, and so $f = 0$. Thus π is injective.

Now suppose that condition (2) holds. Since $\mathcal{G} \neq \operatorname{Iso}(\mathcal{G})$, there exists $\gamma \in \mathcal{G} \setminus \operatorname{Iso}(\mathcal{G})$, and it follows that γ and γ^{-1} are distinct elements of $\mathcal{G} \setminus \mathcal{G}^{(0)}$. Thus $\big|\mathcal{G} \setminus \mathcal{G}^{(0)}\big| = 2$, and so $\mathcal{G} = \mathcal{G}^{(0)} \sqcup \{\gamma, \gamma^{-1}\}$. In particular, $\mathcal{G}$ is compact. Since $\mathcal{G}$ is Hausdorff, $\mathcal{G}^{(0)}$ is closed, and so $\{\gamma, \gamma^{-1}\} = \mathcal{G} \setminus \mathcal{G}^{(0)}$ is open. Thus $\{r(\gamma), s(\gamma)\} = r(\{\gamma, \gamma^{-1}\})$ is open since $\mathcal{G}$ is étale. Therefore, $U \coloneqq \mathcal{G} \backslash \{r(\gamma), s(\gamma)\}$ is a closed subset of $\mathcal{G}$, and by the compactness of $\mathcal{G}$ it follows that U is a full compact open bisection containing γ and γ^{-1}. In fact, given $V \in F(\mathcal{G})$ with $\gamma \in V$, we must have $r(\gamma), s(\gamma) \notin V$, and so $V = U$. It follows that $F(\mathcal{G}) = \{U, \mathcal{G}^{(0)}\}$. Suppose that $f = a\delta_U + b\delta_{\mathcal{G}^{(0)}} \in \ker(\pi)$ for some $a, b \in \mathbb{C}$. Then $a = \pi(f)(\gamma) = 0$ and $b = \pi(f)(r(\gamma)) = 0$, and so $f = 0$. Thus π is injective. $\square$

4. Lack of surjectivity for ample Hausdorff groupoids

In this section we study the image of the representation $\pi\colon \delta_U \mapsto 1_U$ of $\mathbb{C}F(\mathcal{G})$ in $A(\mathcal{G})$. In particular, we show in Corollary 4.4 that π is surjective if and only if $\mathcal{G}$ is a group.

We begin by proving certain properties for elements of the image of π. Recall (for instance, from [18, Section 2.2]) that there are linear maps $r_*, s_*\colon A(\mathcal{G}) \to A(\mathcal{G}^{(0)})$ given by

$$r_* f(u) \coloneqq \sum_{\gamma \in \mathcal{G}^u} f(\gamma) \quad \text{and} \quad s_* f(u) \coloneqq \sum_{\gamma \in \mathcal{G}_u} f(\gamma), \quad \text{for all } f \in A(\mathcal{G}) \text{ and } u \in \mathcal{G}^{(0)};$$

and there is a linear map $\delta_1\colon A(\mathcal{G}) \to A(\mathcal{G}^{(0)})$ given by $\delta_1 \coloneqq s_* - r_*$.

Proposition 4.1. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. Then*
(a) $\pi(\mathbb{C}F(\mathcal{G})) \subseteq \{f \in A(\mathcal{G}) \mid r_* f(u) = s_* f(v) \text{ for all } u, v \in \mathcal{G}^{(0)}\} \subseteq \ker \delta_1$; *and*
(b) $r_*\big(\pi(\mathbb{C}F(\mathcal{G}))\big) = \mathbb{C}1_{\mathcal{G}^{(0)}} = s_*\big(\pi(\mathbb{C}F(\mathcal{G}))\big)$.

Proof. For part (a), fix $f \in \pi(\mathbb{C}F(\mathcal{G}))$. Then there exist $U_1, \dots, U_m \in F(\mathcal{G})$ and $c_1, \dots, c_m \in \mathbb{C}$ such that

$$f = \pi\Big(\sum_{i=1}^m c_i \, \delta_{U_i} \Big) = \sum_{i=1}^m c_i 1_{U_i}.$$

Fix $u, v \in \mathcal{G}^{(0)}$. For each $i \in \{1, \dots, m\}$, the sets $U_i \cap \mathcal{G}^u$ and $U_i \cap \mathcal{G}_v$ are singletons because U_i is a full bisection of $\mathcal{G}$. Thus

$$r_* f(u) = \sum_{\gamma \in \mathcal{G}^u} f(\gamma) = \sum_{\gamma \in \mathcal{G}^u} \sum_{i : \gamma \in U_i} c_i = \sum_{i=1}^m c_i = \sum_{\gamma \in \mathcal{G}_v} \sum_{i : \gamma \in U_i} c_i = \sum_{\gamma \in \mathcal{G}_v} f(\gamma) = s_* f(v).$$

It follows that $r_*f(x) = s_*f(x)$ for all $x \in \mathcal{G}^{(0)}$, and so $\delta_1(f) = 0$.

We now prove part (b). Routine calculations show that for all $B \in B^{\text{co}}(\mathcal{G})$, we have $r_*(1_B) = 1_{r(B)}$ and $s_*(1_B) = 1_{s(B)}$. Thus, for all $B \in F(\mathcal{G})$, we have $r_*(1_B) = 1_{\mathcal{G}^{(0)}} = s_*(1_B)$. Since r_*, s_*, and π are all linear maps, it follows that

$$r_*\big(\pi(\mathbb{C}F(\mathcal{G}))\big) = \mathbb{C}1_{\mathcal{G}^{(0)}} = s_*\big(\pi(\mathbb{C}F(\mathcal{G}))\big). \qquad \square$$

In order to prove Corollary 4.4, we first utilise Proposition 4.1(b) to prove the following result. We thank the anonymous referee for suggesting this simple proof.

Proposition 4.2. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. If B is a nonempty compact open bisection of $\mathcal{G}$ such that $1_B \in \pi(\mathbb{C}F(\mathcal{G}))$, then $B \in F(\mathcal{G})$.*

Proof. Let B is a nonempty compact open bisection of $\mathcal{G}$ such that $1_B \in \pi(\mathbb{C}F(\mathcal{G}))$. By Proposition 4.1(b), we know that $1_{r(B)} = r_*(1_B)$ and $1_{s(B)} = s_*(1_B)$ are both nonzero elements of $\mathbb{C}1_{\mathcal{G}^{(0)}}$. It follows that $r(B) = \mathcal{G}^{(0)} = s(B)$, and hence $B \in F(\mathcal{G})$. $\square$

The following result is an immediate corollary of Proposition 4.2, because $A(\mathcal{G})$ is the span of characteristic functions on compact open bisections of $\mathcal{G}$.

Corollary 4.3. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. If there exists a nonempty compact open bisection B of $\mathcal{G}$ such that $B \notin F(\mathcal{G})$, then the representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is not surjective.*

As the following corollary shows, it turns out that the hypothesis of Corollary 4.3 is very easily satisfied, as it holds whenever $\mathcal{G}$ is not a group.

Corollary 4.4. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. The representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is surjective if and only if $\mathcal{G}$ is a group.*

Proof. If $\mathcal{G}$ is a group, then $F(\mathcal{G}) \cong \mathcal{G}$, so $\mathbb{C}F(\mathcal{G}) = A(\mathcal{G})$, and $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is the identity map and hence is surjective. For the converse, suppose that $\mathcal{G}$ is not a group, and fix distinct units $u, v \in \mathcal{G}^{(0)}$. Since $\mathcal{G}$ is an ample Hausdorff groupoid, there exist disjoint compact open sets $U, V \subseteq \mathcal{G}^{(0)}$ containing u and v, respectively. But then $v \notin U$, so $U \in B^{\text{co}}(\mathcal{G}) \backslash F(\mathcal{G})$, and hence Corollary 4.3 implies that $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is not surjective. $\square$

5. Representations of topological full groups of discrete groupoids

In this section we restrict our attention to discrete groupoids, and to the images of the representations of their topological full groups in the full and reduced groupoid C*-algebras. In particular, we prove an analogue of Corollary 4.4 for the extension of the representation π with respect to the full C*-norm (see Theorem 5.3), and we show in Example 5.5 that Theorem 5.3 does not hold in the reduced setting. We conclude the section by connecting our results from Sections 3, 4, and 5 in Corollary 5.6.

Let $\mathcal{G}$ be a discrete groupoid with finite unit space $\mathcal{G}^{(0)} = \{a_1, \dots, a_n\}$. Recall that, for a groupoid $\mathcal{G}$ and $a, b \in \mathcal{G}^{(0)}$, we define $\mathcal{G}_b^a \coloneqq \{\gamma \in \mathcal{G} \mid r(\gamma) = a \text{ and } s(\gamma) = b\}$. Thus $\mathcal{P}_{\mathcal{G}} \coloneqq \big\{\mathcal{G}_{a_j}^{a_i} : i, j \in \{1, \dots, n\}\big\}$ is a partition of $\mathcal{G}$ into disjoint sets. For $\gamma \in \mathcal{G}$, write $1_\gamma \coloneqq 1_{\{\gamma\}} \in A(\mathcal{G})$. Given $f \in A(\mathcal{G})$ and $i, j \in \{1, \dots, n\}$, we define a map $f_{i,j}\colon \mathcal{G} \to \mathbb{C}$ by

$$f_{i,j}(\gamma) \coloneqq \begin{cases} f(\gamma) & \text{if } \gamma \in \mathcal{G}_{a_j}^{a_i} \\ 0 & \text{otherwise.} \end{cases}$$

Then each $f_{i,j} \in A(\mathcal{G})$, and since $\mathcal{P}_{\mathcal{G}}$ is a partition of $\mathcal{G}$, it follows that $f = \displaystyle\sum_{i,j=1}^{n} f_{i,j}$.

Define $T\colon A(\mathcal{G}) \to M_n(\mathbb{C})$ by

$$T(f)_{ij} := \sum_{\gamma \in \mathcal{G}^{a_i}_{a_j}} f(\gamma), \quad \text{for each } i, j \in \{1, \dots, n\}.$$

We will use this map T to study the image of the representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$. We first show that T is a $*$-representation of $A(\mathcal{G})$.

Lemma 5.1. *Let $\mathcal{G}$ be a discrete groupoid with finite unit space $\mathcal{G}^{(0)} := \{a_1, \dots, a_n\}$. The map $T\colon A(\mathcal{G}) \to M_n(\mathbb{C})$ defined above is a $*$-representation of $A(\mathcal{G})$.*

Proof. It is straightforward to verify that T is linear. Fix $f, g \in A(\mathcal{G})$. For all $i, j \in \{1, \dots, n\}$, we have

$$\begin{aligned} T(f * g)_{ij} &= \sum_{\gamma \in \mathcal{G}^{a_i}_{a_j}} (f * g)(\gamma) = \sum_{\gamma \in \mathcal{G}^{a_i}_{a_j}} \sum_{\alpha\beta = \gamma} f(\alpha) g(\beta) \\ &= \sum_{k=1}^{n} \sum_{\alpha \in \mathcal{G}^{a_i}_{a_k}} f(\alpha) \sum_{\beta \in \mathcal{G}^{a_k}_{a_j}} g(\beta) = \sum_{k=1}^{n} T(f)_{ik} T(g)_{kj} = \big(T(f)T(g)\big)_{ij}, \end{aligned}$$

and

$$T(f^*)_{ij} = \sum_{\gamma \in \mathcal{G}^{a_i}_{a_j}} f^*(\gamma) = \sum_{\gamma \in \mathcal{G}^{a_i}_{a_j}} \overline{f(\gamma^{-1})} = \overline{\sum_{\eta \in \mathcal{G}^{a_j}_{a_i}} f(\eta)} = \overline{T(f)_{ji}} = \big(T(f)^*\big)_{ij}.$$

Thus $T(f * g) = T(f)T(g)$ and $T(f^*) = T(f)^*$, and so T is a $*$-homomorphism. □

The following result is a corollary of Proposition 4.1(a).

Corollary 5.2. *Let $\mathcal{G}$ be a discrete groupoid with finite unit space $\mathcal{G}^{(0)} := \{a_1, \dots, a_n\}$. Then*

$$\pi(\mathbb{C}F(\mathcal{G})) \subseteq \big\{ f \in A(\mathcal{G}) \mid \exists\, c_f \in \mathbb{C} \text{ such that all row and column sums of } T(f) \text{ are } c_f \big\}.$$

Proof. Fix $f = \sum_{i,j=1}^{n} f_{i,j} \in \pi(\mathbb{C}F(\mathcal{G}))$. Then, for each $i, j \in \{1, \dots, n\}$,

$$\text{the } i^{\text{th}} \text{ row sum of } T(f) = \sum_{k=1}^{n} T(f)_{ik} = \sum_{k=1}^{n} \sum_{\gamma \in \mathcal{G}^{a_i}_{a_k}} f(\gamma) = \sum_{\gamma \in \mathcal{G}^{a_i}} f(\gamma) = r_* f(a_i),$$

and

$$\text{the } j^{\text{th}} \text{ column sum of } T(f) = \sum_{k=1}^{n} T(f)_{kj} = \sum_{k=1}^{n} \sum_{\gamma \in \mathcal{G}^{a_k}_{a_j}} f(\gamma) = \sum_{\gamma \in \mathcal{G}_{a_j}} f(\gamma) = s_* f(a_j).$$

By Proposition 4.1(a), it follows that for all $i, j \in \{1, \dots, n\}$,

$$\text{the } i^{\text{th}} \text{ row sum of } T(f) \;=\; \text{the } j^{\text{th}} \text{ column sum of } T(f).$$ □

We now use Corollary 5.2 to study the completions of $\pi(\mathbb{C}F(\mathcal{G}))$ in the full and reduced groupoid C*-algebras. In Theorem 5.3 we prove that for a discrete groupoid $\mathcal{G}$, an analogue of Corollary 4.4 holds for the full groupoid C*-algebra $C^*(\mathcal{G})$.

Theorem 5.3. *Let $\mathcal{G}$ be a discrete groupoid with finite unit space $\mathcal{G}^{(0)}$. Then*

$$\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}} = C^*(\mathcal{G})$$

if and only if $\mathcal{G}$ is a group.

Proof. If $\mathcal{G}$ is a group, then $F(\mathcal{G}) \cong \mathcal{G}$, so $\mathbb{C}F(\mathcal{G}) = A(\mathcal{G})$, and hence

$$\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}} = \overline{A(\mathcal{G})}^{\|\cdot\|_{\max}} = C^*(\mathcal{G}).$$

Suppose that $\mathcal{G}$ is not a group. We show that $\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}} \neq C^*(\mathcal{G})$ by proving an even stronger result: that $1_\gamma \notin \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}}$ for each $\gamma \in \mathcal{G}$. Write $\mathcal{G}^{(0)} = \{a_1, \dots, a_n\}$, and note that $n \geq 2$ since $\mathcal{G}$ is not a group. Fix $\gamma \in \mathcal{G}$, and suppose for contradiction that $1_\gamma \in \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}}$. Then there exists a sequence $(\varphi_m)_{m=0}^\infty$ of functions in $\pi(\mathbb{C}F(\mathcal{G}))$ such that $\|\varphi_m - 1_\gamma\|_{\max} \to 0$ as $m \to \infty$. By Lemma 5.1, $T\colon A(\mathcal{G}) \to M_n(\mathbb{C})$ is a $*$-representation of $A(\mathcal{G})$, and hence equation (2.1) on page 3 implies that T is bounded. Thus

$$\|T(\varphi_m) - T(1_\gamma)\|_{M_n(\mathbb{C})} = \|T(\varphi_m - 1_\gamma)\|_{M_n(\mathbb{C})} \to 0 \quad \text{as } m \to \infty. \tag{5.1}$$

Let ℓ and k be the unique elements of $\{1, \dots, n\}$ such that $\gamma \in \mathcal{G}_{a_k}^{a_\ell}$. Note that each $T(\varphi_m)$ has $n \geq 2$ rows, and it follows from equation (5.1) that for each $i \in \{1, \dots, n\}$, we have

$$i^{\text{th}} \text{ row sum of } T(\varphi_m) \to i^{\text{th}} \text{ row sum of } T(1_\gamma) = \sum_{j=1}^n T(1_\gamma)_{ij} = T(1_\gamma)_{ik} = \begin{cases} 1 & \text{if } i = \ell \\ 0 & \text{otherwise} \end{cases}$$

as $m \to \infty$. But this contradicts Corollary 5.2, which says that for each $m \in \mathbb{N}$, all of the row (and column) sums of $T(\varphi_m)$ are equal. So we must have $1_\gamma \notin \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}}$. □

Remark 5.4. It is known that Theorem 5.3 does not hold for ample Hausdorff groupoids in general. For example, if $\mathcal{G}$ is the Cuntz groupoid (that is, the boundary-path groupoid of the directed graph with a single vertex and two edges), then $F(\mathcal{G})$ is Thompson's group V_2, and the representation $\pi\colon \mathbb{C}(F(\mathcal{G})) \to A(\mathcal{G})$ extends to a surjective representation of $F(\mathcal{G})$ in the Cuntz algebra $\mathcal{O}_2$; see [3, Remark 4.7] and [10, Proposition 5.3].

It turns out that Theorem 5.3 does not hold in the reduced setting. We provide an example demonstrating this fact below.

Example 5.5. Let $\mathcal{G} = \mathbb{F}_2 \sqcup \mathbb{F}_2$. Then each element of $\mathcal{G}$ is of the form (g, k), where $g \in \mathbb{F}_2$, and $k \in \{1, 2\}$ identifies whether g belongs to the first or the second copy of $\mathbb{F}_2$. Since $\mathcal{G}$ is not a group, we know by Theorem 5.3 that $\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}} \neq C^*(\mathcal{G})$. We show that despite this, we still have $\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_r} = C_r^*(\mathcal{G})$. To do so, it suffices to show that for each $g \in \mathbb{F}_2$, we have $1_{(g,1)} \in \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_r}$, because a symmetric argument then shows that $1_{(g,2)} \in \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_r}$. Fix $t \in \mathbb{F}_2$, and for each $m \in \mathbb{N}$, let E_m denote the set of (reduced) elements of $\mathbb{F}_2$ with length m. List elements of $\mathbb{F}_2$ in increasing order of their lengths; that is, write $\mathbb{F}_2 = \{g_1, g_2, g_3, \dots\}$, with $|g_i| \leq |g_{i+1}|$ for all $i \geq 1$. Now define a sequence of functions $(\phi_n)_{n=1}^\infty \subseteq \pi(\mathbb{C}F(\mathcal{G})) \subseteq A(\mathcal{G})$ by

$$\phi_n := \pi\Big(\delta_{(t,1)} + \sum_{i=1}^n \tfrac{1}{n}\, \delta_{(g_i,2)}\Big) = 1_{(t,1)} + \frac{1}{n}\Big(\sum_{i=1}^n 1_{(g_i,2)}\Big).$$

We claim that $\phi_n \to 1_{(t,1)}$ in $C_r^*(\mathcal{G})$. Since the map $1_{g_i} \mapsto 1_{(g_i,2)}$ extends to an embedding of $C_r^*(\mathbb{F}_2)$ in $C_r^*(\mathcal{G})$, it suffices to show that $\psi_n := \dfrac{1}{n}\Big(\displaystyle\sum_{i=1}^n 1_{g_i}\Big) \to 0$ in $C_r^*(\mathbb{F}_2)$.

By [9, Lemma 1.5], we know that for all $f \in C_c(\mathbb{F}_2)$,

$$\|f\|_r \leq 2\Big(\sum_{s\in\mathbb{F}_2} |f(s)|^2 \left(1 + |s|^4\right)\Big)^{\frac{1}{2}}. \tag{5.2}$$

For each $m \geq 1$, we have $|E_m| = 4 \times 3^{m-1}$. Thus, for each $n \geq 1$, we have

$$\sum_{m=0}^{\lceil \log_3 n \rceil} |E_m| = |E_0| + 4 \sum_{m=1}^{\lceil \log_3 n \rceil} 3^{m-1} = 1 + \frac{4\big(3^{\lceil \log_3 n \rceil} - 1\big)}{3-1} \geq 1 + \frac{4(n-1)}{2} \geq n,$$

and it follows that $\operatorname{supp}(\psi_n) = \{g_1, \dots, g_n\} \subseteq \bigcup_{m=0}^{\lceil \log_3 n \rceil} E_m$.

Now, for each $n \geq 1$, inequality (5.2) implies that

$$\begin{aligned}
\|\psi_n\|_r &\leq 2\Big(\sum_{s \in \mathbb{F}_2} |\psi_n(s)|^2 \left(1 + |s|^4\right)\Big)^{\frac{1}{2}} = 2\Big(\sum_{m=0}^{\lceil \log_3 n \rceil} \sum_{s \in E_m} |\psi_n(s)|^2 \left(1 + |s|^4\right)\Big)^{\frac{1}{2}} \\
&\leq 2\Big(\sum_{m=0}^{\lceil \log_3 n \rceil} \frac{|E_m|}{n^2} \left(1 + m^4\right)\Big)^{\frac{1}{2}} \leq 2\Big(\sum_{m=0}^{\lceil \log_3 n \rceil} \frac{4 \times 3^{m-1} \times 2m^4}{n^2}\Big)^{\frac{1}{2}} \\
&\leq 2\Big(\frac{8\lceil \log_3 n \rceil^4}{n^2} \sum_{m=0}^{\lceil \log_3 n \rceil} 3^{m-1}\Big)^{\frac{1}{2}} = 2\Big(\frac{8\lceil \log_3 n \rceil^4 \left(3^{1+\lceil \log_3 n \rceil} - \frac{1}{3}\right)}{n^2\,(3-1)}\Big)^{\frac{1}{2}} \\
&\leq \frac{4\lceil \log_3 n \rceil^2}{n} \left(3^{2+\log_3 n}\right)^{\frac{1}{2}} = \frac{12\lceil \log_3 n \rceil^2}{\sqrt{n}}.
\end{aligned}$$

Since $\dfrac{12\lceil \log_3 n \rceil^2}{\sqrt{n}} \to 0$ as $n \to \infty$, we deduce that $\psi_n \to 0$ in $C_r^*(\mathbb{F}_2)$, as required.

We conclude the paper with a corollary of Theorem 3.2, Corollary 4.4, and Theorem 5.3.

Corollary 5.6. *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. The representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is an isomorphism if and only if $\mathcal{G}$ is a group. Similarly, if $\mathcal{G}$ is discrete, then the extension $\overline{\pi}_{\max}\colon C^*(F(\mathcal{G})) \to C^*(\mathcal{G})$ of π is an isomorphism if and only if $\mathcal{G}$ is a group.*

Proof. If $\mathcal{G}$ is a group, then $\mathcal{G}$ satisfies condition (1) of Theorem 3.2, so Theorem 3.2 and Corollary 4.4 together imply that $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is an isomorphism. If $\mathcal{G}$ is not a group, then Corollary 4.4 implies that π is not an isomorphism. Now suppose that $\mathcal{G}$ is discrete. Since $\pi\colon \mathbb{C}F(\mathcal{G}) \to C^*(\mathcal{G})$ is a $*$-homomorphism, it extends uniquely to a $*$-homomorphism $\overline{\pi}_{\max}\colon C^*(F(\mathcal{G})) \to C^*(\mathcal{G})$. If $\mathcal{G}$ is a group, then $F(\mathcal{G}) \cong \mathcal{G}$, so the representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is the identity map, and thus the extension $\overline{\pi}_{\max}$ is an isomorphism. If $\mathcal{G}$ is not a group, then Theorem 5.3 implies that $\overline{\pi}_{\max}$ is not an isomorphism. □

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