

ON BOUNDEDNESS OF SINGULARITIES AND MINIMAL LOG DISCREPANCIES OF KOLLÁR COMPONENTS

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Abstract

Recent study in K-stability suggests that Kawamata log terminal (klt) singularities whose local volumes are bounded away from zero should be bounded up to special degeneration. We show that this is true in dimension three, or when the minimal log discrepancies of Kollár components are bounded from above. We conjecture that the minimal log discrepancies of Kollár components are always bounded from above, and verify it in dimension three when the local volumes are bounded away from zero. We also answer a question from Han, Liu, and Qi on the relation between log canonical thresholds and local volumes.

1. Introduction

In recent years, tremendous progress has been made towards the boundedness of Fano varieties in many different contexts, see for example [HMX14, Bir19, Bir21, Jia20]. In contrast, much less is known about the boundedness of Kawamata log terminal (klt) singularities, often viewed as the local analog of Fano varieties. Certainly, one needs to be more careful about what boundedness means in the local situation, as in general a (nonisolated) singularity can have an infinite dimensional versal deformation space. As a remedy, one considers boundedness up to special degeneration, see Definition 2.18; roughly speaking, a class of klt singularities is bounded up to special degeneration (or specially bounded) if they degenerate to a bounded family of klt singularities.

Typically, we expect a class of singularities to be specially bounded after fixing some interesting invariants. One example, studied in [Mor21, HLS], is the class of (ε, δ) -lc singularities. These are ε -lc singularities that admit δ -plt blowups (see Definition 2.3), and they are known to be bounded up to special

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degeneration [HLM20]. Another invariant that has attracted a lot of attention is the local volume of a klt singularity, originally introduced in [Li18] in the context of K-stability. It is expected that these two types of invariants are closely related: a positive lower bound on the local volumes would force the singularities to be (ε, δ) -lc for some fixed $\varepsilon, \delta > 0$ and in particular they should be bounded up to special degeneration. Since the minimal log discrepancy is always bounded from below by the local volume up to a positive dimensional constant [LLX20, Theorem 6.13], the main question is the existence of δ -plt blowup. The following is a precise formulation of the conjecture.

Conjecture 1.1 ([HLQ23, Conjectures 1.6 and 8.9]). *Let $n \in \mathbb{N}^*$ and let $\varepsilon, \eta > 0$. Then there exists some $\delta > 0$ depending only on n, ε, η such that: if $x \in (X, \Delta = \sum_{i=1}^m a_i \Delta_i)$ is an n -dimensional klt singularity such that*

- (1) $a_i \geq \eta$ for all i ,
- (2) each Δ_i is an effective Weil divisor, and
- (3) $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$,

then $x \in (X, \Delta)$ admits a δ -plt blowup. Moreover, the set of such singularities is log bounded up to special degeneration.

The special boundedness part of Conjecture 1.1 is also supported by the Stable Degeneration Conjecture [Li18, LX18] from the local K-stability theory of klt singularities. It predicts that every klt singularity admits a volume preserving special degeneration to a K-semistable log Fano cone singularity, and differential geometric considerations (see [SS17]) suggest that the set of K-semistable log Fano cone singularities may be bounded when the local volume is bounded from below. The recent proof [LXZ22] of the Higher Rank Finite Generation Conjecture further suggests that Conjecture 1.1 will be a crucial ingredient towards the proof of the Stable Degeneration Conjecture.¹ Despite these heuristics, Conjecture 1.1 is only known in dimension two, in some special cases in dimension three, and when $x \in X$ is already analytically bounded [HLQ23, Theorems 1.7 and 1.8].

The first result of this paper is the solution of Conjecture 1.1 in dimension three.

Theorem 1.2 (=Corollary 6.12). *Conjecture 1.1 holds in dimension three.*

In the course of the proof, we also discover the following statement.

Theorem 1.3 (=Theorem 3.1). *For any $n \in \mathbb{N}^*$, there exists some constant $c = c(n) > 0$ depending only on n such that for any n -dimensional*

¹Postscript note: the local version of the Higher Rank Finite Generation Conjecture is later settled in [XZ]. Combined with earlier works [Blu18, LX18, LWX21, Xu20, XZ21], the proof of the Stable Degeneration Conjecture is now complete.

\mathbb{Q} -Gorenstein klt singularity $x \in (X, \Delta)$ we have

$$\mathrm{lct}_x(X, \Delta; \Delta) \geq c(n) \cdot \widehat{\mathrm{vol}}(x, X, \Delta).$$

This gives a positive answer to [HLQ23, Question 8.1] as well as further evidence for Conjecture 1.1. Indeed, if we allow the above constant $c > 0$ to also rely on the local volume, then the statement is essentially implied by Conjecture 1.1 and the lower semi-continuity of log canonical thresholds.

Our approach to Conjecture 1.1 is through another invariant of the singularity: the minimal log discrepancy of Kollár components.

Definition 1.4. Let (X, Δ) be a klt pair and let $\eta \in X$ (not necessarily a closed point). The minimal log discrepancy of Kollár components, denoted by $\mathrm{mld}^K(\eta, X, \Delta)$, is the smallest log discrepancy $A_{X, \Delta}(E)$ among all Kollár components E over $\eta \in (X, \Delta)$.

It is not hard to see (Lemma 4.2) that in order for a class of singularities to admit δ -plt blowups, their mld^K are necessarily bounded from above. Our next result is the following converse.

Theorem 1.5 (=Theorem 4.1). *Let n be a positive integer, let $\varepsilon, A > 0$, and let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists some constant $\delta = \delta(n, \varepsilon, A, I) > 0$ such that any n -dimensional klt singularity $x \in (X, \Delta)$ with $\widehat{\mathrm{vol}}(x, X, \Delta) \geq \varepsilon$, $\mathrm{mld}^K(x, X, \Delta) \leq A$ and $\mathrm{Coef}(\Delta) \subseteq I$ admits a δ -plt blowup.*

As a direct application, we also obtain the toric case of Conjecture 1.1, proven independently by [MS] using convex geometry.

Corollary 1.6 (=Proposition 4.6). *Let $n \in \mathbb{N}^*$ and let $\varepsilon > 0$. Then there are only finitely many toric singularities $x \in X$ (up to isomorphism) that support a klt singularity $x \in (X, \Delta)$ with $\widehat{\mathrm{vol}}(x, X, \Delta) \geq \varepsilon$ (for some effective \mathbb{Q} -divisor Δ).*

If we compare Theorem 1.5 with Conjecture 1.1, it is natural to expect that the assumption on $\mathrm{mld}^K(x, X, \Delta)$ in Theorem 1.5 is redundant. This leads us to make Conjecture 1.7.

Conjecture 1.7 (BDD for mld^K when $\widehat{\mathrm{vol}} \geq \varepsilon$). *Let $n \in \mathbb{N}^*$, $\varepsilon > 0$ and let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists some $A = A(n, \varepsilon, I)$ such that*

$$\mathrm{mld}^K(\eta, X, \Delta) \leq A$$

for any n -dimensional klt pair (X, Δ) with $\mathrm{Coef}(\Delta) \subseteq I$ and any (not necessarily closed point) $\eta \in X$ with $\widehat{\mathrm{vol}}(\eta, X, \Delta) \geq \varepsilon$.

We will show that Conjecture 1.7 holds in dimensions up to three (Corollary 6.11), and in any dimension it implies Conjecture 1.1 (Theorem 4.8). Together with Theorem 1.5 they constitute the three major steps in the proof of Theorem 1.2.

Shokurov has conjectured that the set of minimal log discrepancies (mld) satisfies the ascending chain condition (ACC) [Sho04]. In particular, there should be an upper bound on the mlds that only depends on the dimension. This is known as the boundedness (BDD) conjecture for mld. On the other hand, in the local study of klt singularities it is often more natural to consider Kollár components rather than general exceptional divisors. Therefore, we are also tempted to propose the following stronger conjecture, although we are only able to partially verify it at codimension two points (Proposition 6.4).

Conjecture 1.8 (BDD for mld^K). *Let $n \in \mathbb{N}^*$ and let $I \subseteq [0, 1]$ be a DCC set. Then there exists some constant $A = A(n, I)$ depending only on n and I such that*

$$\text{mld}^K(\eta, X, \Delta) \leq A$$

for any klt pair (X, Δ) with $\text{Coef}(\Delta) \subseteq I$ and any point $\eta \in X$ of codimension n .

1.1. Outline of proofs. One of the technical steps in proving Theorem 1.2 is to verify Conjecture 1.7 in dimension three. While it is well known that $\text{mld}(x, X, \Delta) \leq 3$ for any 3-dimensional klt singularity $x \in (X, \Delta)$, the divisors that compute the mld are usually not Kollár components, thus bounding mld^K from above presents a very different problem. In fact, as Example 7.1 shows, already on smooth surfaces mld^K can be arbitrarily large if we allow the boundary Δ to vary.

Certainly, the advantage of working in dimension three is that we have a classification result for terminal singularities [Mor85]. Thus we need to find ways to reduce to the terminal case and to take care of the additional boundary. The first observation is that every Kollár component is an lc place of a bounded complement [Bir19]. If we take a bounded complement D of a klt singularity $x \in X$ and a terminal modification $\pi: Y \rightarrow X$, the lc places of (X, D) are the same as lc places of (Y, π^*D) , so we may hope to find Kollár components over X with bounded log discrepancy by taking suitable Kollár components over Y that are lc places of (Y, π^*D) . In general, Kollár components over a birational model do not descend to Kollár components over the singularity, but this is the case if they are lc places of *special* complements, a notion that first appears in the recent proof of the higher rank finite generation conjecture [LXZ22]. Therefore, a key step in our proof of Conjecture 1.7 is the construction of *bounded special complements*. This step works in any dimension and allows us to reduce Conjecture 1.7 to the terminal singularity case (with boundary). Another technical observation (see Section 5.3) involving special complements further reduces the question to the case where the terminal pair (X, Δ) has a reduced complement. From there we are able to use classification results to conclude Conjecture 1.7 in dimension three.

To achieve the bounded part of the special complements, we need uniform control of various invariants of the singularity, and this is where Theorem 1.3 plays a crucial role. The key to the proof of Theorem 1.3 is a uniform Izumi type estimate. Recall that the usual Izumi type inequality states that for any klt singularity $x \in (X, \Delta)$, there exists some constant $c > 0$ such that

$$\mathrm{lct}_x(X, \Delta; D) \geq \frac{c}{\mathrm{mult}_x D}$$

for any effective \mathbb{Q} -Cartier divisor D on X (see, e.g., [Li18, Theorem 3.1]). However, the constant c in general depends on the singularity $x \in (X, \Delta)$. Our discovery is that a uniform constant can be achieved if we replace mult_x by the minimizing valuation of the normalized volume function (Corollary 3.5). In particular, we can compare both sides of the inequality in Theorem 1.3 through the minimizing valuation.

Finally we make a few remarks on the proof of Theorem 1.5. The idea is to show that the Kollár components that have bounded log discrepancy belong to a bounded family of log Fano pairs, which is the case if the Cartier indices are bounded on the corresponding plt blowup. Using the finite degree formula proved in [XZ21], this boils down to a few estimates of local volumes on the plt blowup, see Section 4.

2. Preliminaries

2.1. Notation and conventions. Throughout this paper, we work over an algebraically closed field of characteristic 0. We follow the standard terminology from [KM98, Kol13].

A singularity $x \in (X, \Delta)$ consists of a pair (X, Δ) (i.e., a normal variety X together with an effective \mathbb{Q} -divisor Δ) and a closed point $x \in X$. We will always assume that X is affine and $x \in \mathrm{Supp}(\Delta)$ (whenever $\Delta \neq 0$). In general, when we discuss local properties of a pair (X, Δ) at a (not necessarily closed) point η , we will freely shrink X around η .

Suppose that X is a normal variety. A prime divisor F on some birational model $\pi: Y \rightarrow X$ (where Y is normal and π is proper) of X is called a *divisor over X* . Its center, denoted $C_X(F)$, is the generic point of its image in X .

A *valuation* v on X is an \mathbb{R} -valued valuation $v: K(X)^\times \rightarrow \mathbb{R}$ (where $K(X)$ denotes the function field of X) such that v has a center on X and $v|_{k^\times} = 0$. The set of valuations on X is denoted as Val_X .

For a valuation v on X and $m \in \mathbb{N}$, its *valuation ideal sheaf* $\mathfrak{a}_m(v)$ is

$$\mathfrak{a}_m(v) := \{f \in \mathcal{O}_X \mid v(f) \geq m\}.$$

When we refer to a constant A as $A = A(n, \varepsilon, \dots)$ we mean it only depends on n, ε, \dots .

2.2. Kollár components.

Definition 2.1. Let (X, Δ) be a subpair (i.e., Δ need not be effective) and let F be a divisor over X . We will say F is a divisor over $\eta \in (X, \Delta)$ if η is the center of F . When F is a divisor on X we write $\Delta = \Delta_1 + aF$ where $F \not\subseteq \text{Supp}(\Delta_1)$; otherwise let $\Delta_1 = \Delta$.

- (1) F is said to be primitive over X if there exists a projective birational morphism $\pi : Y \rightarrow X$ such that Y is normal, F is a prime divisor on Y and $-F$ is a π -ample \mathbb{Q} -Cartier divisor. We call $\pi : Y \rightarrow X$ the associated prime blowup (it is uniquely determined by F).
- (2) F is said to be of plt (resp. lc) type over (X, Δ) if it is primitive over X and the pair $(Y, \Delta_Y + F)$ is plt (resp. lc) in a neighbourhood of F , where $\pi : Y \rightarrow X$ is the associated prime blowup and Δ_Y is the strict transform of Δ_1 on Y . When (X, Δ) is klt (resp. lc) and F is exceptional over X , π is called a plt (resp. lc) blowup over X . If in addition $(Y, \Delta_Y + F)$ is δ -plt in a neighbourhood of F for some $\delta > 0$, we say that π is a δ -plt blowup.

The following result from [BCHM10] will be frequently used.

Lemma 2.2. *Let (X, Δ) be a klt pair and let E be a divisor over X . Assume that there exists an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor D such that $(X, \Delta + D)$ is lc and $A_{X, \Delta + D}(E) = 0$. Then E is of lc type.*

Proof. Since $(X, \Delta + (1 - \varepsilon)D)$ is klt and $0 < A_{X, \Delta + (1 - \varepsilon)D}(E) \ll 1$ when $0 < \varepsilon \ll 1$, we know that E is primitive by [BCHM10, Corollary 1.4.3] and the basepoint free theorem. Let $\pi : Y \rightarrow X$ be the associated prime blowup. By assumption we also have $\pi^*(K_X + \Delta + D) \geq K_Y + \Delta_Y + E$. Since $(X, \Delta + D)$ is lc, it follows that $(Y, \Delta_Y + E)$ is lc, i.e., E is of lc type. \square

Definition 2.3 (Kollár components). Let (X, Δ) be a klt pair and let $\eta \in X$ (not necessarily a closed point). A divisor over $\eta \in (X, \Delta)$ is called a *Kollár component* over $\eta \in (X, \Delta)$ if it's of plt type over $(U, \Delta|_U)$ for some neighbourhood $U \subseteq X$ of η . We say that $\eta \in (X, \Delta)$ admits a δ -plt blowup (for some $\delta > 0$) if it has a Kollár component whose associated prime blowup is a δ -plt blowup.

By [Xu14], every klt singularity has a Kollár component. More precisely we have:

Lemma 2.4. *Let (X, Δ) be a klt pair and let D be an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor such that $(X, \Delta + D)$ is strictly lc. Let $\pi : (Y, \Gamma) \rightarrow (X, \Delta)$ be a dlt modification, where $K_Y + \Gamma = \pi^*(K_X + \Delta + D)$. Then at least one component of $[\Gamma]$ is of plt type over (X, Δ) .*

Proof. This follows from the proof of [Xu14, Lemma 1] (cf. [LX20, Proposition 2.10]). The point is that if $W \rightarrow X$ is a log resolution of $(X, \Delta + D)$ then some prime divisor on W that is an lc place of $(X, \Delta + D)$ will be of plt type over (X, Δ) . The resolution can be chosen as a blowup of Y along non-SNC locus of (Y, Γ) , and the dlt assumption ensures that none of the exceptional divisors are lc places of (Y, Γ) . Thus the plt type divisor on W has to be a component of $[\Gamma]$. \square

We next describe a criterion which will be used to verify certain weighted blowups on hypersurfaces singularities provide Kollár components.

Lemma 2.5. *Let $X = (f(x_1, \dots, x_n) = 0) \subseteq \mathbb{A}^n$ be a hypersurface singularity. Let E be the exceptional divisor of the weighted blowup with $\text{wt}(x_i) = a_i > 0$. Assume that the hypersurface $X_0 = (\text{in}(f) = 0) \subseteq \mathbb{A}^n$ has only canonical singularity at the origin. Then E is a Kollár component over $0 \in X$.*

Here $\text{in}(f)$ denotes the sum of the monomials in f with lowest weights. Note that for nondegenerate hypersurfaces this is proved in [IP01, Proposition 3.3]. We essentially follow the same argument.

Proof. Let $f_0 = \text{in}(f)$ and let $\pi: Y \rightarrow X$ be the weighted blowup. Then $E \cong (f_0 = 0) \subseteq \mathbb{P}(a_1, \dots, a_n)$. Since f_0 is irreducible by assumption, we see that E is a primitive divisor on Y . It remains to show that (Y, E) is plt. By inversion of adjunction, this is equivalent to showing that $(E, \text{Diff}_E(0))$ is klt. Note that X_0 admits a good \mathbb{G}_m -action with $t \cdot x_i = t^{a_i} x_i$, hence $X_0 \setminus \{0\}$ is a Seifert \mathbb{G}_m -bundle in the sense of [Kol]. A direct calculation shows that its orbifold base is exactly $(E, \text{Diff}_E(0))$. Since X_0 has canonical singularities (hence is klt) in a neighbourhood of the origin, using the \mathbb{G}_m -action we see that X_0 is klt everywhere. Hence by [Kol, Proposition 19], the local orbifold covers of $(E, \text{Diff}_E(0))$ are all klt, hence $(E, \text{Diff}_E(0))$ is klt as well. This finishes the proof. \square

2.3. Local volumes. Given a pair (X, Δ) , the log discrepancy function

$$A_{X, \Delta}: \text{Val}_X \rightarrow \mathbb{R} \cup \{+\infty\}$$

is defined as in [JM12] and [BdFFU15, Theorem 3.1]. It is possible that $A_{X, \Delta}(v) = +\infty$ for some $v \in \text{Val}_X$, see, e.g., [JM12, Remark 5.12]. For a closed point $x \in X$, we denote by $\text{Val}_{X, x}^*$ the set of valuations $v \in \text{Val}_X$ with center x and $A_{X, \Delta}(v) < +\infty$.

Definition 2.6. Let X be an n -dimensional normal variety and let $x \in X$ be a closed point. The *volume* of a valuation $v \in \text{Val}_{X, x}^*$ is defined as

$$\text{vol}(v) = \text{vol}_{X, x}(v) = \limsup_{m \rightarrow \infty} \frac{\ell(\mathcal{O}_{X, x} / \mathfrak{a}_m(v))}{m^n / n!}.$$

Thanks to [ELS03, LM09, Cut13], the above limsup is actually a limit.

We now briefly recall the definition of the volumes of klt singularities [Li18].

Definition 2.7. Let $x \in (X, \Delta)$ be an n -dimensional klt singularity. For any $v \in \text{Val}_{X,x}^*$, we define the *normalized volume* of v as

$$\widehat{\text{vol}}_{X,\Delta}(v) := A_{X,\Delta}(v)^n \cdot \text{vol}_{X,x}(v).$$

The *local volume* of $x \in (X, \Delta)$ is defined as

$$\widehat{\text{vol}}(x, X, \Delta) := \inf_{v \in \text{Val}_{X,x}^*} \widehat{\text{vol}}_{X,\Delta}(v).$$

By [Blu18], the above infimum is a minimum. In fact, the minimizer is unique up to rescaling [XZ21], and is a quasi-monomial valuation [Xu20], but we do not need these facts in the sequel.

Since we also study the singularities at nonclosed points, we extend Definition 2.7 to those cases as follows.

Definition 2.8. Let (X, Δ) be a klt pair and let η be the generic point of a subvariety $W \subseteq X$. The local volume of (X, Δ) at η is defined to be

$$\widehat{\text{vol}}(\eta, X, \Delta) := \widehat{\text{vol}}(w, X, \Delta),$$

where $w \in W$ is a general closed point. This is well-defined since the local volume function $x \mapsto \widehat{\text{vol}}(x, X, \Delta)$ is constructible [Xu20].

We will frequently use the fact that the volume function $x \mapsto \widehat{\text{vol}}(x, X, \Delta)$ is lower semi-continuous [BL21]. We recall a few more useful properties of local volumes.

Lemma 2.9. *Let $x \in (X, \Delta)$ be a klt singularity of dimension n and let D be a \mathbb{Q} -Cartier Weil divisor on X . Then the Cartier index of D is at most $\frac{n^n}{\widehat{\text{vol}}(x, X, \Delta)}$.*

Proof. This follows directly from [XZ21, Corollary 1.4]. \square

Lemma 2.10. *Let $\pi: (Y, \Delta_Y) \rightarrow (X, \Delta)$ be a proper birational morphism between klt pairs. Assume that $K_Y + \Delta_Y \leq \pi^*(K_X + \Delta)$. Then $\widehat{\text{vol}}(y, Y, \Delta_Y) \geq \widehat{\text{vol}}(x, X, \Delta)$ for any $x \in X$ and any $y \in \pi^{-1}(x)$.*

Proof. This follows from the same proof of [LX19, Lemma 2.9(2)], which tackles the boundary-free case. \square

Lemma 2.11 ([HLQ23, Lemma 2.16]). *Let $x \in (X, \Delta)$ be a klt singularity of dimension n , let $\lambda > 0$, and let D be an effective \mathbb{Q} -Cartier divisor on X such that $(X, \Delta + (1 + \lambda)D)$ is lc. Then*

$$\widehat{\text{vol}}(x, X, \Delta + D) \geq \left(\frac{\lambda}{1 + \lambda} \right)^n \widehat{\text{vol}}(x, X, \Delta).$$

Proof. By assumption, for any $v \in \text{Val}_{X,x}^*$, we have $A_{X,\Delta}(v) \geq (1 + \lambda)v(D)$, thus

$$A_{X,\Delta+D}(v) = A_{X,\Delta}(v) - v(D) \geq \left(1 - \frac{1}{1 + \lambda} \right) A_{X,\Delta}(v)$$

and hence

$$A_{X,\Delta+D}(v)^n \cdot \text{vol}(v) \geq \left(\frac{\lambda}{1+\lambda} \right)^n A_{X,\Delta}(v)^n \cdot \text{vol}(v).$$

Taking the infimum over all $v \in \text{Val}_{X,x}^*$, the lemma follows. \square

2.4. Complements. In this subsection we recall some results about complements of singularities. A DCC set is a subset of \mathbb{R} that satisfies the descending chain condition.

Definition 2.12. A \mathbb{Q} -complement of an lc pair (X, Δ) is an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -(K_X + \Delta)$ such that $(X, \Delta + D)$ is lc. A \mathbb{Q} -complement of $\eta \in (X, \Delta)$ is an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor D such that $(X, \Delta + D)$ is lc at η and has η as the generic point of an lc center.² In either case, a \mathbb{Q} -complement D is called an N -complement for $N \in \mathbb{N}^*$ if $N(K_X + \Delta + D) \sim 0$.

Lemma 2.13. *Let $n \in \mathbb{N}^*$ and let $I = \bar{I} \subseteq [0, 1] \cap \mathbb{Q}$ be a DCC set. Then there exists some integer $N > 0$ depending only on n and I such that if (X, Δ) is a klt pair of dimension n with $\text{Coef}(\Delta) \subseteq I$ and $\eta \in X$, then any Kollár component E over $\eta \in (X, \Delta)$ is an lc place of some N -complement. In particular, every such $\eta \in (X, \Delta)$ has an N -complement.*

Proof. This should be well known to experts but we provide a proof for the readers' convenience. We may assume that $1 \in I$. Let $\pi: Y \rightarrow X$ be the plt blowup that extracts E . Since E is a Kollár component, $(Y, \Delta_Y + E)$ is plt, $-(K_Y + \Delta_Y + E)$ is π -ample and hence Y is of Fano type over X . By [HLS, Theorem 1.10] (which builds on [Bir19, Theorem 1.8]), after possibly replacing X by a neighbourhood of η , there exists an integer $N > 0$ that only depends on n and I , and an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor D_Y on Y such that $N(\Delta_Y + D_Y)$ has integer coefficients, $(Y, \Delta_Y + E + D_Y)$ is lc, and

$$N(K_Y + \Delta_Y + E + D_Y) \sim 0.$$

It follows that if we let $D = \pi_* D_Y$, then $K_Y + \Delta_Y + E + D_Y = \pi^*(K_X + \Delta + D)$ and hence $(X, \Delta + D)$ is lc with E as an lc place. Since η is the center of E , this implies that D is a \mathbb{Q} -complement of $\eta \in (X, \Delta)$. Moreover, the line bundle $N(K_Y + \Delta_Y + E + D_Y)$ descends to the line bundle $N(K_X + \Delta + D)$ by Shokurov's basepoint-free theorem (see, e.g., [KM98, Theorem 3.3]). Thus D is also an N -complement. Since every klt singularity has a Kollár component, this finishes the proof. \square

Lemma 2.14. *Let $f: (y \in (Y, \Delta_Y)) \rightarrow (x \in (X, \Delta))$ be a finite morphism between lc singularities such that $f^*(K_X + \Delta) = K_Y + \Delta_Y$. Let D be a*

²Morally speaking, this should be called a strictly lc \mathbb{Q} -complement since we also require that the complement has an lc center at η . For simplicity, we drop the phrase "strictly lc" when we talk about complement if the center η is specified.

divisor on X . Then D is a \mathbb{Q} -complement of $x \in (X, \Delta)$ if and only if f^*D is a \mathbb{Q} -complement of $y \in (Y, \Delta_Y)$.

Proof. This is a direct consequence of the proof of [KM98, Proposition 5.20(2)]. \square

2.5. Index one covers. The index one cover of a \mathbb{Q} -Cartier Weil divisor $D \subseteq X$ [KM98, Definition 5.19] is a cyclic cover

$$\tilde{X} = \mathbf{Spec} \bigoplus_{0 \leq m \leq r-1} \mathcal{O}_X(mD),$$

where r is the Cartier index of D . It has the property that the preimage of D becomes Cartier. We will need a similar construction for multiple divisors.

Lemma-Definition 2.15. Let $x \in X$ be a normal singularity and let D_1, \dots, D_m be \mathbb{Q} -Cartier Weil divisors on X . Let H be the subgroup of the local class group of $x \in X$ generated by all the D_i 's and consider

$$\tilde{X} := \mathbf{Spec} \bigoplus_{D \in H} \mathcal{O}_X(D).$$

It comes with a natural quasi-étale Galois morphism $\pi: \tilde{X} \rightarrow X$ with abelian covering group $\hat{H} := \mathrm{Hom}(H, \mathbb{C}^*)$. The preimage $\pi^{-1}(x)$ consists of a single point \tilde{x} and π^*D_i is Cartier for all $1 \leq i \leq m$. We call $\tilde{x} \in \tilde{X}$ the (simultaneous) index one cover of D_1, \dots, D_m .

Proof. The only nontrivial statement is that $\pi^{-1}(x)$ consists of a single point \tilde{x} and π^*D_i is Cartier. The first claim can be proved as in [Kol13, 2.48(1)]: the evaluation map $\mathcal{O}_X(D) \otimes \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X/\mathfrak{m}$ is zero for all $D \in H \setminus \{0\}$, thus every $f \in \bigoplus_{D \in H \setminus \{0\}} \mathcal{O}_X(D)$ is nilpotent in $\mathcal{O}_{\tilde{X}}/\mathfrak{m}\mathcal{O}_{\tilde{X}}$ and this implies that the preimage of x is a single point in \tilde{X} . To see the other claim, note that the map $\pi: \tilde{X} \rightarrow X$ factors through the index one cover of D_i , thus π^*D_i is Cartier. \square

2.6. Family of singularities and special boundedness. In this subsection, we recall the definition for special boundedness of singularities, following [HLQ23].

Definition 2.16. We call $B \subseteq (\mathcal{X}, \mathcal{D}) \rightarrow B$ a \mathbb{Q} -Gorenstein family of klt singularities (over a normal but possibly disconnected base B) if

- (1) \mathcal{X} is flat over B , and $B \subseteq \mathcal{X}$ is a section of the projection,
- (2) For any closed point $b \in B$, \mathcal{X}_b is connected, normal and is not contained in $\mathrm{Supp}(\mathcal{D})$,
- (3) $K_{\mathcal{X}/B} + \mathcal{D}$ is \mathbb{Q} -Cartier and $b \in (\mathcal{X}_b, \mathcal{D}_b)$ is a klt singularity for any $b \in B$.

Given a \mathbb{Q} -Gorenstein family $B \subseteq \mathcal{X} \rightarrow B$ of klt singularities and a klt singularity $x \in X$, we denote by $(x \in X^{\text{an}}) \in (B \subseteq \mathcal{X}^{\text{an}} \rightarrow B)$, if $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{\mathcal{X}_b,b}}$ for some closed point $b \in B$.

Definition 2.17. A special test configuration of a klt singularity $x \in (X, \Delta)$ is a \mathbb{Q} -Gorenstein family $\mathbb{A}^1 \subseteq (\mathcal{X}, \mathcal{D}) \rightarrow \mathbb{A}^1$ of klt singularities over \mathbb{A}^1 , together with a \mathbb{G}_m -action on $(\mathcal{X}, \mathcal{D})$ that commutes with the standard \mathbb{G}_m -action on \mathbb{A}^1 , such that $(t \in (\mathcal{X}_t, \mathcal{D}_t)) \cong (x \in (X, \Delta))$ for all $t \in \mathbb{A}^1 \setminus \{0\}$. Its central fiber $0 \in (\mathcal{X}_0, \mathcal{D}_0)$ is called a *special degeneration* of $x \in (X, \Delta)$.

Definition 2.18. A set \mathcal{P} of klt singularities is said to be *log bounded up to special degeneration* if there is a log bounded set \mathcal{C} of pairs, such that the following holds.

For any klt singularity $x \in (X, \Delta)$ in \mathcal{P} , there exist a special degeneration $x_0 \in (X_0, \Delta_0)$ of $x \in (X, \Delta)$, a pair $(Y, D) \in \mathcal{C}$ and a closed point $y \in Y$, such that $(y \in (Y, \text{Supp}(D))) \cong (x_0 \in (X_0, \text{Supp}(\Delta_0)))$ in some neighbourhoods of $y \in Y$ and $x_0 \in X_0$ respectively.

When \mathcal{P} is log bounded and the coefficients set $I \subseteq \mathbb{Q}$ is finite, we will simply say that \mathcal{P} is bounded up to special degeneration, since in this case there is a \mathbb{Q} -Gorenstein family of klt singularities such that every $x \in (X, \Delta)$ in \mathcal{P} specially degenerates to at least one of them.

The following result is useful when showing that a class of klt singularities is log bounded up to special degeneration.

Lemma 2.19. *Let $n \in \mathbb{N}^*$ and let $\varepsilon, \delta, c > 0$. Then the set of n -dimensional ε -lc singularities $x \in (X, \Delta)$ with $\text{Coef}(\Delta) \geq c$ that admits a δ -plt blowup is log bounded up to special degeneration.*

Proof. This is essentially [HLM20, Theorem 4.1], at least when X is \mathbb{Q} -Gorenstein. In general, let $\pi: Y \rightarrow X$ be a δ -plt that extracts a Kollár component E , and let $L = -E|_E$ be the \mathbb{Q} -divisor defined by [HLS, Definition A.4]. Then there is a special degeneration of $x \in (X, \Delta)$ to the orbifold cone over $(E, \Delta_E := \text{Diff}_E(\Delta_Y))$ with polarization L (see, e.g., [LX20, Section 2.4] or [LZ22, Proposition 2.10]). Thus it suffices to show that the triple $(E, \text{Supp}(\Delta_E), L)$ is bounded.

Since (Y, E) is δ -plt, there exists some integer m depending only on δ such that mE is Cartier away from a codimension two set in E (it suffices to inspect the codimension one points of E as in the proof of [HLM20, Theorem 4.1]). Thus mL has integer coefficients. By adjunction, we also see that (E, Δ_E) is δ -klt and log Fano, and $\text{Coef}(\Delta_E) \geq \frac{c}{m}$. By [Bir21, Theorem 1.1], we first deduce that E belongs to a bounded family. But as $\text{Coef}(\Delta_E) \geq \frac{c}{m}$ and $-K_E - \Delta_E$ is ample, we further see that the degree of $\text{Supp}(\Delta_E)$ is bounded from above, thus $(E, \text{Supp}(\Delta_E))$ is log bounded. Finally $-(K_E + \Delta_E) \sim_{\mathbb{Q}}$

$A_{X,\Delta}(E) \cdot L \geq \varepsilon L$, thus the Weil divisor mL also has bounded degree, hence the triple $(E, \text{Supp}(\Delta_E), L)$ is bounded as desired. \square

3. Comparison between lct and volume

In this section we prove the following statement which gives a positive answer to [HLQ23, Question 8.1]. It will play an important role in many of the reduction steps that we will carry out in the next few sections.

Theorem 3.1. *For any $n \in \mathbb{N}^*$, there exists some constant $c(n) > 0$ depending only on n such that for any n -dimensional \mathbb{Q} -Gorenstein klt singularity $x \in (X, \Delta)$ we have*

$$\text{lct}_x(X, \Delta; \Delta) \geq c(n) \cdot \widehat{\text{vol}}(x, X, \Delta).$$

The proof will be divided into several steps, but first we shall consider an interpolation (based on a construction from [XZ21]) of the \mathbb{Q} -divisor Δ and the valuation ideals of the minimizing valuations of the normalized volume function. For this purpose we first revisit some results from [XZ21].

3.1. A multiplicity formula. Let $x \in (X, \Delta)$ be a singularity of dimension n and let $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet$ be two graded sequence of ideals. Following [XZ21, Section 3.3], we define $\mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet$ to be the graded sequence of ideals with

$$(\mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet)_m = \sum_{i=0}^m \mathfrak{a}_i \cap \mathfrak{b}_{m-i}.$$

In this section, we give slight generalizations of some results from [XZ21, Section 3.3].

Lemma 3.2. *Assume that $x \in (X, \Delta)$ is klt. Then $\text{lct}_x(\mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet) \leq \text{lct}_x(\mathfrak{a}_\bullet) + \text{lct}_x(\mathfrak{b}_\bullet)$.*

Proof. This follows from the same proof of [XZ21, Theorem 3.11]: the \mathfrak{m}_x -primary assumption there is only used to ensure that all the lct are taken at x . \square

Assume next that both sequences $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet$ are decreasing and that \mathfrak{a}_\bullet is \mathfrak{m}_x -primary. Note that the latter condition implies that $\mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet$ is also \mathfrak{m}_x -primary. We will give a formula for the multiplicity of $\mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet$. Let $R_m := \mathcal{O}_{X,x}/\mathfrak{a}_m$. The decreasing sequence \mathfrak{b}_\bullet induces an \mathbb{N} -filtration \mathcal{F} on all R_m :

$$\mathcal{F}^j R_m := (\mathfrak{b}_j + \mathfrak{a}_m)/\mathfrak{a}_m,$$

which also induces a filtration (still denoted as \mathcal{F}) on the subspaces $\mathfrak{a}_{m-1}/\mathfrak{a}_m \subseteq R_m$. It is not hard to check that $\mathcal{F}^j(\mathfrak{a}_{m-1}/\mathfrak{a}_m) \cong \mathfrak{a}_{m-1} \cap \mathfrak{b}_j/\mathfrak{a}_m \cap \mathfrak{b}_j$. We further assume that:

(†) the limit

$$\mathrm{vol}(\mathfrak{a}_\bullet; \mathfrak{b}_\bullet^t) := \lim_{m \rightarrow \infty} \frac{\ell(\mathcal{F}^{mt} R_m)}{m^n/n!}$$

exists for all $t > 0$.

Note that under this assumption, the function $t \mapsto \mathrm{vol}(\mathfrak{a}_\bullet; \mathfrak{b}_\bullet^t)$ is decreasing and hence continuous at almost all t .

Lemma 3.3. *Under the above assumptions, we have*

$$\mathrm{mult}(\mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet) = \mathrm{mult}(\mathfrak{a}_\bullet) - (n+1) \int_0^\infty \frac{\mathrm{vol}(\mathfrak{a}_\bullet; \mathfrak{b}_\bullet^t)}{(1+t)^{n+2}} dt.$$

Proof. This follows from the same proof of [XZ21, Lemma 3.13]. We sketch the main steps for the reader's convenience. Let $R = \mathcal{O}_{X,x}$ and let $\mathfrak{c}_\bullet = \mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet$. We have

$$\mathrm{mult}(\mathfrak{c}_\bullet) = \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{c}_m)}{m^n/n!} = \lim_{m \rightarrow \infty} \frac{\sum_{j=1}^m \ell(R/\mathfrak{c}_j)}{m^{n+1}/(n+1)!}.$$

From the short exact sequence

$$0 \rightarrow \frac{\mathfrak{a}_{j-\ell-1} \cap \mathfrak{b}_{\ell+1}}{\mathfrak{a}_{j-\ell} \cap \mathfrak{b}_{\ell+1}} \rightarrow \frac{R}{\sum_{i=0}^\ell \mathfrak{a}_{j-i} \cap \mathfrak{b}_i} \rightarrow \frac{R}{\sum_{i=0}^{\ell+1} \mathfrak{a}_{j-i} \cap \mathfrak{b}_i} \rightarrow 0$$

for all ℓ , we get

$$\ell(R/\mathfrak{c}_j) = \ell(R/\mathfrak{a}_j) - \sum_{i=1}^j \ell(\mathcal{F}^i(\mathfrak{a}_{j-i}/\mathfrak{a}_{j-i+1})).$$

Summing over $j = 0, 1, \dots, m$ we obtain

$$\sum_{j=1}^m \ell(R/\mathfrak{c}_j) = \sum_{j=1}^m \ell(R/\mathfrak{a}_j) - \sum_{i=1}^m \ell(\mathcal{F}^i(R/\mathfrak{a}_{m-i+1})).$$

Note that $\lim_{m \rightarrow \infty} \frac{\ell(\mathcal{F}^{\lceil my \rceil}(R/\mathfrak{a}_{m-\lceil my \rceil+1}))}{m^n/n!} = (1-y)^n \mathrm{vol}(\mathfrak{a}_\bullet; \mathfrak{b}_\bullet^{\frac{y}{1-y}})$ for almost all $0 < y < 1$ (i.e., wherever the right hand side is continuous). Thus after dividing the above equality by $\frac{m^{n+1}}{(n+1)!}$ and using the dominated convergence theorem, we get

$$\begin{aligned} \mathrm{mult}(\mathfrak{c}_\bullet) &= \mathrm{mult}(\mathfrak{a}_\bullet) - (n+1) \int_0^1 (1-y)^n \mathrm{vol}(\mathfrak{a}_\bullet; \mathfrak{b}_\bullet^{\frac{y}{1-y}}) dy \\ &= \mathrm{mult}(\mathfrak{a}_\bullet) - (n+1) \int_0^\infty \frac{\mathrm{vol}(\mathfrak{a}_\bullet; \mathfrak{b}_\bullet^t)}{(1+t)^{n+2}} dt \end{aligned}$$

as desired. □

3.2. Izumi inequality. The next ingredient in the proof of Theorem 3.1 is an Izumi type inequality as follows.

Lemma 3.4. *Let $x \in (X, \Delta)$ be an n -dimensional klt singularity, and let $v_0 \in \text{Val}_{X,x}^*$ be a valuation. Then there exists some constant $c_0 = c_0(n) > 0$ depending only on n such that*

$$\text{lct}_x(X, \Delta; D) \geq c_0 \cdot \frac{\widehat{\text{vol}}(x, X, \Delta)}{\widehat{\text{vol}}_{X, \Delta}(v_0)} \cdot \frac{A_{X, \Delta}(v_0)}{v_0(D)}$$

for any effective \mathbb{Q} -Cartier divisor D on X .

As an immediate corollary, we have

Corollary 3.5. *Let $n \in \mathbb{N}^*$. Then there exists some constant $c_0 > 0$ depending only on n such that for any n -dimensional klt singularity $x \in (X, \Delta)$ and any effective \mathbb{Q} -Cartier divisor D on X , we have*

$$\text{lct}_x(X, \Delta; D) \geq c_0 \cdot \frac{A_{X, \Delta}(v_0)}{v_0(D)},$$

where v_0 is the minimizing valuation of the normalized volume function, i.e., $\widehat{\text{vol}}_{X, \Delta}(v_0) = \widehat{\text{vol}}(x, X, \Delta)$.

This can be seen as a uniform Izumi type estimate. Recall that the classical Izumi inequality (see, e.g., [Laz04, Proposition 9.5.13]) says that if $x \in X$ is a smooth point and D is an effective divisor on X , then $\text{lct}_x(X; D) \geq \frac{1}{\text{mult}_x D}$. More generally, for any klt singularity $x \in (X, \Delta)$, there exists some constant $c > 0$ such that

$$\text{lct}_x(X, \Delta; D) \geq \frac{c}{\text{mult}_x D}$$

for any effective \mathbb{Q} -Cartier divisor D on X (see, e.g., [Li18, Theorem 3.1]). However, the constant c in general depends on the singularity $x \in (X, \Delta)$. Therefore, Corollary 3.5 asserts that a uniform constant can be achieved if we replace mult_x by the minimizing valuation of the normalized volume function. Similarly, Lemma 3.4 suggests that the constant in the Izumi inequality tends to get worse if we choose a valuation that's further away from the normalized volume minimizer.

Proof of Lemma 3.4. After rescaling the coefficient, we may assume that $D = (f = 0)$ is Cartier. For ease of notation, we will abbreviate $\text{lct}_x(X, \Delta; \cdot)$ as $\text{lct}(\cdot)$. Let $\mathbf{a}_\bullet = \mathbf{a}_\bullet(v_0)$ be the graded sequence of valuation ideals of v_0 , i.e., $\mathbf{a}_m = \{s \in \mathcal{O}_{X,x} \mid v_0(s) \geq m\}$. For each $t > 0$, we also set $\mathbf{b}_{m,t} = (f)^{\lceil \frac{m}{t} \rceil}$ and $\mathbf{c}_{\bullet,t} = \mathbf{a}_\bullet(v_0) \boxplus \mathbf{b}_{\bullet,t}$. Roughly speaking, $\mathbf{c}_{m,t}$ is generated by $s \in \mathcal{O}_{X,x}$ such that $v_0(s) + t \cdot \text{ord}_D(s) \geq m$, so as t varies they interpolate between $\mathbf{a}_\bullet(v_0)$ and the ideals $\mathcal{O}_X(-mD)$ ($m \in \mathbb{N}$). The idea of the proof is to analyze the inequality (see [Liu18, Theorem 27])

$$\text{lct}(\mathbf{c}_{\bullet,t})^n \cdot \text{mult}(\mathbf{c}_{\bullet,t}) \geq \widehat{\text{vol}}(x, X, \Delta)$$

for some suitably chosen value of t . Note that $\text{lct}(\mathbf{b}_{\bullet,t}) = t \cdot \text{lct}(D)$ and $\text{lct}(\mathbf{a}_{\bullet}) \leq A_{X,\Delta}(v_0)$ (the latter follows from the fact that $v_0(\mathbf{a}_{\bullet}(v_0)) = 1$). We take

$$t = \frac{A_{X,\Delta}(v_0)}{\text{lct}(D)} > 0$$

and write $\mathbf{b}_{\bullet,t}$, $\mathbf{c}_{\bullet,t}$ simply as \mathbf{b}_{\bullet} , \mathbf{c}_{\bullet} from now on. We then have

$$(3.1) \quad \text{lct}(\mathbf{c}_{\bullet}) \leq \text{lct}(\mathbf{a}_{\bullet}) + \text{lct}(\mathbf{b}_{\bullet}) \leq 2A_{X,\Delta}(v_0)$$

by Lemma 3.2 and the above discussions. We claim that

$$(3.2) \quad \frac{\text{mult}(\mathbf{c}_{\bullet})}{\text{vol}(v_0)} = 1 - (n+1) \int_0^{1/c} \frac{(1-cu)^n}{(1+u)^{n+2}} du,$$

where $c = \frac{v_0(D)}{t} = \text{lct}(D) \cdot \left(\frac{A_{X,\Delta}(v_0)}{v_0(D)} \right)^{-1}$. Granting this for the moment, let us finish the proof of the lemma. To this end, denote the right hand side of (3.2) by $S(c)$ and treat it as a function of $c > 0$. Note that $S(c) \geq 1 - \int_0^{1/c} \frac{n+1}{(1+u)^{n+2}} du = \int_{1/c}^{+\infty} \frac{n+1}{(1+u)^{n+2}} du$, hence $S(c) > 0$ and $\lim_{c \rightarrow \infty} S(c) = 1$. We also have

$$(3.3) \quad \lim_{c \rightarrow 0} \frac{S(c)}{c} = 1.$$

Indeed, we have

$$\begin{aligned} S(c) &= 1 - (n+1) \int_0^{1/c} \frac{(1+c-c(1+u))^n}{(1+u)^{n+2}} du \\ &= 1 - (n+1) \sum_{i=0}^n \int_0^{1/c} \frac{\binom{n}{i} (1+c)^{n-i} (-c)^i}{(1+u)^{n+2-i}} du \\ &= 1 - (n+1) \int_0^{1/c} \left(\frac{1+cn}{(1+u)^{n+2}} - \frac{cn}{(1+u)^{n+1}} \right) du + O(c^2), \end{aligned}$$

from which (3.3) follows easily (after a direct calculation). It then follows that there exists some constant $M > 0$ depending only on n such that $S(c) \leq M \cdot c$ for all $c > 0$.

On the other hand, by (3.1) and (3.2) we have

$$\widehat{\text{vol}}(x, X, \Delta) \leq \text{lct}(\mathbf{c}_{\bullet})^n \cdot \text{mult}(\mathbf{c}_{\bullet}) \leq 2^n S(c) \cdot \widehat{\text{vol}}_{X,\Delta}(v_0) \leq 2^n M c \cdot \widehat{\text{vol}}_{X,\Delta}(v_0).$$

Therefore,

$$c \geq \frac{1}{2^n M} \cdot \frac{\widehat{\text{vol}}(x, X, \Delta)}{\widehat{\text{vol}}_{X,\Delta}(v_0)}.$$

Recall that $c = \text{lct}(D) \cdot \left(\frac{A_{X,\Delta}(v_0)}{v_0(D)} \right)^{-1}$, this gives the statement of the lemma with $c_0 = \frac{1}{2^n M}$ (which only depends on the dimension n).

It remains to prove the multiplicity formula (3.2) using Lemma 3.3. First let us verify that the assumption (\dagger) in Lemma 3.3 is satisfied. Recall that (using the notation in Lemma 3.3)

$$\mathcal{F}^j R_m = (\mathfrak{b}_j + \mathfrak{a}_m) / \mathfrak{a}_m \cong \mathfrak{b}_j / (\mathfrak{b}_j \cap \mathfrak{a}_m).$$

Since $\mathfrak{b}_j = (f)^{\lceil \frac{j}{t} \rceil}$ and $\mathfrak{a}_m = \{s \in R \mid v_0(s) \geq m\}$, we see that $s \in \mathfrak{b}_j \cap \mathfrak{a}_m$ if and only if $s = f^{\lceil \frac{j}{t} \rceil} \cdot s_1$ for some $s_1 \in R$ with $v_0(s_1) \geq m - \lceil \frac{j}{t} \rceil v_0(f)$. Thus

$$\mathcal{F}^j R_m \cong (f)^{\lceil \frac{j}{t} \rceil} / (f)^{\lceil \frac{j}{t} \rceil} \cdot \mathfrak{a}_{m - \lceil \frac{j}{t} \rceil v_0(f)} \cong R / \mathfrak{a}_{m - \lceil \frac{j}{t} \rceil v_0(f)}.$$

It is then clear that

$$\lim_{m \rightarrow \infty} \frac{\ell(\mathcal{F}^{mu} R_m)}{m^n / n!} = \max\{0, (1 - cu)^n\} \cdot \text{vol}(v_0),$$

hence Lemma 3.3 applies and gives

$$\text{mult}(\mathfrak{c}_\bullet) = \text{mult}(\mathfrak{a}_\bullet) - (n+1) \int_0^{1/c} \frac{(1 - cu)^n \cdot \text{vol}(v_0)}{(1 + u)^{n+2}} du.$$

As $\text{mult}(\mathfrak{a}_\bullet) = \text{vol}(v_0)$, this is exactly (3.2). The proof is now complete. \square

3.3. Completion of the proof. We are now ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. By [LX19, Lemma A.1], there exists some $v_0 \in \text{Val}_{X,x}^*$ such that $\widehat{\text{vol}}_X(v_0) \leq n^n$. Since (X, Δ) is klt, we have $v_0(\Delta) < A_X(v_0)$. Clearly we also have $A_{X,\Delta}(v_0) \leq A_X(v_0)$. Hence by Lemma 3.4 we have

$$\begin{aligned} \text{lct}_x(X, \Delta; \Delta) &\geq c_0(n) \cdot \frac{\widehat{\text{vol}}(x, X, \Delta)}{\widehat{\text{vol}}_{X,\Delta}(v_0)} \cdot \frac{A_{X,\Delta}(v_0)}{v_0(\Delta)} \\ &= \frac{c_0(n)}{A_{X,\Delta}(v_0)^{n-1} \cdot v_0(\Delta) \cdot \text{vol}(v_0)} \cdot \widehat{\text{vol}}(x, X, \Delta) \\ &\geq \frac{c_0(n)}{A_X(v_0)^{n-1} \cdot A_X(v_0) \cdot \text{vol}(v_0)} \cdot \widehat{\text{vol}}(x, X, \Delta) \\ &= \frac{c_0(n)}{\widehat{\text{vol}}_X(v_0)} \cdot \widehat{\text{vol}}(x, X, \Delta) \\ &\geq \frac{c_0(n)}{n^n} \cdot \widehat{\text{vol}}(x, X, \Delta) \end{aligned}$$

for some $c_0(n) > 0$ that only depends on n . Thus the theorem holds with $c(n) = \frac{c_0(n)}{n^n}$. \square

4. Boundedness of singularities

In this section, we study the boundedness of klt singularities using the minimal log discrepancies of their Kollár components. The first result is Theorem 4.1, which gives a criterion for special boundedness of singularities.

Theorem 4.1. *Let $\varepsilon, A > 0$, let n be a positive integer, and let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists some constant $\delta = \delta(n, \varepsilon, A, I) > 0$ such that any n -dimensional klt singularity $x \in (X, \Delta)$ with $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, $\text{mld}^K(x, X, \Delta) \leq A$ and $\text{Coef}(\Delta) \subseteq I$ admits a δ -plt blowup.*

Let us first note that having bounded mld^K is a necessary condition for the above conclusion.

Lemma 4.2. *Let $n \in \mathbb{N}^*$ and let $\delta > 0$. Then there exists some $A = A(n, \delta) > 0$ such that $\text{mld}^K(x, X, \Delta) \leq A$ for any n -dimensional klt singularity $x \in (X, \Delta)$ that admits a δ -plt blowup.*

Proof. By [HLS, Proposition 4.3(1)], there exists some constant $A = A(n, \delta) > 0$ such that for any δ -plt blowup $\pi: Y \rightarrow X$ of an n -dimensional klt singularity $x \in (X, \Delta)$ with exceptional divisor E , we have $A_{X, \Delta}(E) \leq A$. In particular, $\text{mld}^K(x, X, \Delta) \leq A$. \square

The key to the proof of Theorem 4.1 is to bound the Cartier index of divisors on the plt blowups that extract the Kollár components. Using [XZ21], this essentially boils down to the following volume estimate.

Lemma 4.3. *Let (X, Δ) be a klt pair of dimension n and let $\pi: Y \rightarrow X$ be an lc blowup with exceptional divisor E . Then*

$$\widehat{\text{vol}}(y, Y, \Delta_Y) \geq \frac{\widehat{\text{vol}}(\pi(y), X, \Delta)}{\max\{1, A_{X, \Delta}(E)^n\}}$$

for all $y \in Y$, where $\Delta_Y = \pi_*^{-1}\Delta$.

Proof. Let $a = A_{X, \Delta}(E) - 1$ and let $x = \pi(y)$. If $a \leq 0$, then $K_Y + \Delta_Y \leq \pi^*(K_X + \Delta)$, hence by Lemma 2.10, we get $\widehat{\text{vol}}(y, Y, \Delta_Y) \geq \widehat{\text{vol}}(x, X, \Delta)$ for all $y \in E$. Thus we may assume that $a > 0$. Let $m > 0$ be a sufficiently divisible integer, and let $\mathfrak{a} = \pi_* \mathcal{O}_Y(-mE)$. We further let $D = \frac{1}{mp}(\{f_1 = 0\} + \cdots + \{f_p = 0\})$ for some sufficiently large integer p and some general $f_1, \dots, f_p \in \mathfrak{a}$. Since $(Y, \Delta_Y + E)$ is lc by assumption, and

$$K_Y + \Delta_Y + E = \pi^*(K_X + \Delta + \mathfrak{a}^{\frac{a+1}{m}}),$$

we see that $\text{lct}(X, \Delta; D) = \text{lct}(X, \Delta; \mathfrak{a}^{1/m}) \geq a + 1$ and $\text{lct}(X, \Delta; aD) \geq 1 + \frac{1}{a}$. By Lemma 2.11 we obtain

$$\widehat{\text{vol}}(x, X, \Delta + aD) \geq \frac{\widehat{\text{vol}}(x, X, \Delta)}{A_{X, \Delta}(E)^n}.$$

Notice that $\pi^*(K_X + \Delta + aD) = K_Y + \Delta_Y + aD_Y$ where $D_Y = \pi_*^{-1}D$, hence another application of Lemma 2.10 implies

$$\begin{aligned}\widehat{\text{vol}}(y, Y, \Delta_Y) &\geq \widehat{\text{vol}}(y, Y, \Delta_Y + aD_Y) \\ &\geq \widehat{\text{vol}}(x, X, \Delta + aD) \geq A_{X,\Delta}(E)^{-n} \cdot \widehat{\text{vol}}(x, X, \Delta)\end{aligned}$$

for all $y \in E$. This proves the lemma. \square

Proof of Theorem 4.1. Without loss of generality we may assume that $A \geq 1$. By assumption, there exists a Kollár component E over $x \in (X, \Delta)$ such that $A_{X,\Delta}(E) \leq A$. Let $\pi: Y \rightarrow X$ be the corresponding plt blowup and let Δ_Y be the strict transform of Δ . By Lemma 4.3, we have

$$(4.1) \quad \widehat{\text{vol}}(y, Y, \Delta_Y) \geq \frac{\varepsilon}{A^n}$$

for all $y \in \pi^{-1}(x) = E$. Thus by Lemma 2.9, we deduce that there exists a positive integer N depending only on ε, n, A and the coefficient set I such that $N\Delta$ has integer coefficients and ND is Cartier for any \mathbb{Q} -Cartier Weil divisor D on Y . In particular, $N^2(K_Y + \Delta_Y + E)$ is Cartier. Since $(Y, \Delta_Y + E)$ is plt, this implies that it is $\frac{1}{N^2}$ -plt and we are done. \square

Using Lemma 2.19, this immediately implies:

Corollary 4.4. *Let $\varepsilon, A > 0$, $n \in \mathbb{N}^*$ and let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Then the set of n -dimensional klt singularities $x \in (X, \Delta)$ with $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, $\text{mld}^K(x, X, \Delta) \leq A$ and $\text{Coef}(\Delta) \subseteq I$ is bounded up to special degeneration.*

For toric singularities we have the following uniform bound on mld^K . As an application, we get the boundedness of toric singularities whose volumes are bounded from below.

Lemma 4.5. *Let $x \in (X, \Delta)$ be a klt toric singularity of dimension n . Then there exists a torus invariant Kollár component E such that $A_{X,\Delta}(E) \leq n$. In particular,*

$$\text{mld}^K(x, X, \Delta) \leq n.$$

Proof. This follows from two well-known results:

- (1) $\text{mld}(x, X, \Delta) \leq n$ for toric singularities (see [Bor97, Amb06]) and the mld can be computed by some torus invariant divisor over $x \in (X, \Delta)$;
- (2) every torus invariant divisor over $x \in (X, \Delta)$ is a Kollár component.

For the reader's convenience we sketch the proof. We refer to [Ful93] for basics on toric varieties. We have $X = X(\sigma)$ for some rational convex polyhedral cone $\sigma \subseteq N_{\mathbb{R}} = \mathbb{R}^n$. Let $\rho_i \in N = \mathbb{Z}^n$ be the primitive generator of the 1-dimensional faces of σ . We may write $\Delta = \sum a_i \Delta_i$ where the Δ_i 's are torus invariant prime divisors corresponding to ρ_i . The pair (X, Δ) is klt if and only if $0 \leq a_i < 1$ for all i and there exists a linear function $f: N_{\mathbb{R}} \rightarrow \mathbb{R}$ such that $f(\rho_i) = 1 - a_i$. Every torus invariant divisor E_w over $x \in (X, \Delta)$ comes

from a primitive vector $w \in N \cap \text{int}(\sigma)$ and the corresponding log discrepancy is $A_{X,\Delta}(E_w) = f(w)$. In particular, if w is the primitive generator in the direction $\rho_1 + \cdots + \rho_n$ then clearly $A_{X,\Delta}(E_w) = f(w) \leq f(\rho_1) + \cdots + f(\rho_n) \leq n$.

Next, let E be a torus invariant divisor over $x \in (X, \Delta)$ that corresponds to $w \in N \cap \text{int}(\sigma)$. After adding the vertex w , any simplicial subdivision of σ gives rise to a birational morphism $Y_1 \rightarrow X$ such that Y_1 is \mathbb{Q} -factorial and E is the only exceptional divisor (since $\mathbb{C}_{\geq 0} \cdot w$ is the only 1-dimensional face that's new). Let Y be the ample model of $-E$ over X , i.e., $Y_1 \dashrightarrow Y$ is a birational contraction over X and $-E$ is π -ample (where $\pi: Y \rightarrow X$); in other words, Y is the outcome of the $(-E)$ -MMP over X followed by the corresponding relative ample model (we can run any MMP on toric varieties). Note that the toric pair $(Y, \Delta_Y + E)$ is plt (where $\Delta_Y = \pi_*^{-1}\Delta$) since $\text{Coef}(\Delta_Y + E) \leq 1$ and E is the only prime divisor with coefficient 1. It follows that E is a Kollár component. This completes the proof. \square

Proposition 4.6. *Let $n \in \mathbb{N}^*$ and let $\varepsilon > 0$. Then there are only finitely many toric singularities $x \in X$ (up to isomorphism) that supports a klt singularity $x \in (X, \Delta)$ with $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$ (for some effective \mathbb{Q} -divisor Δ).*

Note that we do not require the boundary Δ to be torus invariant.

Proof. First we treat the $\Delta = 0$ case, i.e., when $x \in X$ itself is \mathbb{Q} -Gorenstein and $\widehat{\text{vol}}(x, X) \geq \varepsilon$. By the proof of Theorem 4.1 and Lemma 4.5, we see that there exists some $\delta > 0$ depending only on n, ε such that $x \in X$ admits a δ -plt toric blowup. If E is the corresponding torus invariant Kollár component, then by adjunction $(E, \text{Diff}_E(0))$ is δ -klt. By [BB92], there are only finitely many such variety E . As in the proof of Lemma 2.19, the divisor E induces a degeneration of $x \in X$ to the orbifold cone over E . But E is torus invariant, so the degeneration is trivial (the corresponding test configuration is $X \times \mathbb{A}^1$ with a diagonal \mathbb{G}_m -action), thus X itself is an orbifold cone over E . The finiteness of E then implies the finiteness of $x \in X$ as in proof of Lemma 2.19.

For the general case, let $\pi: Y \rightarrow X$ be a small birational modification such that $-K_Y$ is \mathbb{Q} -Cartier and π -ample. Such modification exists and is torus equivariant by Lemma 4.7. Let $y \in \pi^{-1}(x)$ be a torus invariant closed point. By Lemma 2.10, we have $\widehat{\text{vol}}(y, Y) \geq \widehat{\text{vol}}(y, Y, \Delta_Y) \geq \widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, thus from the boundary-free case treated above we know that there are only finitely many such toric singularities $y \in Y$. Hence there exists some constant $M \gg 0$ depending only on n, ε and some torus invariant divisor E over $y \in Y$ such that $A_Y(E) \leq M$ and $\text{vol}_{Y,y}(\text{ord}_E) \leq M$.

We now view E as a divisor over $x \in X$. Let Γ be the sum of all torus invariant prime divisors on X , and let $\varphi: Z \rightarrow X$ be the associated blowup that extracts the divisor E . Note that $K_X + \Gamma \sim 0$ and E is an lc place of (X, Γ) . Since E is torus invariant, we know from the same proof above that

$x \in X$ is an orbifold cone over E . We aim to show that there exists some integer $N > 0$ depending only on n, ε such that $N \cdot E$ is Cartier on Z . Since $-E|_E$ is ample and $(-E|_E^{n-1}) = \text{vol}_{X,x}(\text{ord}_E) \leq \text{vol}_{Y,y}(\text{ord}_E) \leq M$ (the first inequality comes from [LX19, Lemma 2.9(1)]), it would follow that there are only finitely many such toric varieties E , and we get the finiteness of $x \in X$.

To this end, we choose some sufficiently divisible integer m , let $D_Y = \frac{1}{m}D_0 \sim_{\mathbb{Q}} -K_Y$ for some general $D_0 \in |-mK_Y|$, and let $G = \frac{M-1}{M}\Gamma + \frac{1}{M}\pi_*D_Y$. Then $K_X + G \sim_{\mathbb{Q}} 0$ and by Bertini theorem we know that $(Y, \Gamma_Y + D_Y)$ is lc (as usual Γ_Y etc. denotes the strict transform of Γ etc. on Y), hence for any valuation $v \in \text{Val}_{X,x}^*$, we get

$$A_{Y,G_Y}(v) \geq A_Y(v) - \frac{M-1}{M}v(\Gamma_Y + D_Y) \geq \frac{1}{M}A_Y(v)$$

(the first inequality holds as long as $M \geq 2$). It follows that $A_{X,G}(v) = A_{Y,G_Y}(v) \geq \frac{1}{M}A_Y(v) \geq \frac{1}{M}A_{X,\Delta}(v)$, thus $\widehat{\text{vol}}(x, X, G) \geq M^{-n}\widehat{\text{vol}}(x, X, \Delta) \geq M^{-n}\varepsilon$ as in Lemma 2.11. On the other hand, recall that E is an lc place of (Y, Γ_Y) , hence $A_{X,G}(E) = A_{Y,G_Y}(E) \leq A_{Y, \frac{M-1}{M}\Gamma_Y}(E) = \frac{1}{M}A_Y(E) \leq 1$ by our choice of M . This means that $K_Z + G_Z \leq \varphi^*(K_X + G)$, and an application of Lemma 2.10 gives $\widehat{\text{vol}}(z, Z, G_Z) \geq M^{-n}\varepsilon$ for all $z \in E$. By Lemma 2.9 we deduce that the Cartier index of E on Z is bounded from above by some constant that only depends on n, ε . As explained earlier, this concludes the proof. \square

The following result is used in the above proof, and will be needed again in the proof of Theorem 4.8.

Lemma 4.7. *Let (X, Δ) be a klt pair and let D be a Weil \mathbb{Q} -divisor on X . Then there exists a unique small birational modification $\pi: Y \rightarrow X$ such that the strict transform $D_Y = \pi_*^{-1}D$ is \mathbb{Q} -Cartier and π -ample.*

Proof. The uniqueness part follows from the existence since we necessarily have

$$Y = \mathbf{Proj}_X \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_Y(mrD_Y) = \mathbf{Proj}_X \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(mrD)$$

for some sufficiently divisible $r \in \mathbb{N}$. Every Weil divisor D on X can be written as $D = A - B$ where A is effective and B is Cartier. Since $\pi_*^{-1}(A - B) = \pi_*^{-1}A - \pi_*^{-1}B$ is π -ample if and only if $\pi_*^{-1}A$ is π -ample, we may assume that D is effective. Let $\pi_1: Y_1 \rightarrow X$ be a small \mathbb{Q} -factorial modification of X , which exists by [BCHM10, Corollary 1.4.3] (and the remarks thereafter). Let Δ_1 (resp. D_1) be the strict transform of Δ (resp. D). Note that (Y_1, Δ_1) is klt (it is crepant to (X, Δ)), thus the pair $(Y_1, \Delta_1 + \varepsilon D_1)$ is also klt for $0 < \varepsilon \ll 1$. By [BCHM10, Theorem 1.2(2)], it has a log canonical model over X , i.e., there exists a birational contraction $Y_1 \dashrightarrow Y$ over X such that $K_Y + \Delta_Y + \varepsilon D_Y$ is ample over X (where Δ_Y, D_Y are the strict transforms of Δ, D). Denote the

map $Y \rightarrow X$ by π . Clearly it is a small birational contraction (as π_1 is), and hence $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ and

$$K_Y + \Delta_Y + \varepsilon D_Y \sim_{\pi, \mathbb{Q}} \varepsilon D_Y.$$

It follows that D_Y is also π -ample and we finish the proof. \square

We next show that Conjecture 1.7 and Theorem 4.1 together imply Conjecture 1.1. The main ingredients come from Theorem 3.1 and Kollár's effective basepoint-free theorem [Kol93].

Theorem 4.8. *Assume that Conjecture 1.7 holds in dimension n . Then for any $\varepsilon > 0$ and any n -dimensional klt singularity $x \in (X, \Delta = \sum_{i=1}^m a_i \Delta_i)$ such that*

- (1) $a_i \geq \varepsilon$ for all i ,
- (2) each Δ_i is an effective Weil divisor, and
- (3) $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$,

there exists some $\delta = \delta(n, \varepsilon) > 0$ such that $x \in (X, \Delta)$ admits a δ -plt blowup. In particular, the set of such singularities is log bounded up to special degeneration.

Proof. We focus on the existence of δ -plt blowup, since the special boundedness would then follow from Lemma 2.19. If $\Delta^+ \geq \Delta$ and (X, Δ^+) is klt, then any δ -plt blowup of (X, Δ^+) is also a δ -plt blowup of (X, Δ) . By Conjecture 1.7 and Theorem 4.1, if $r\Delta^+$ has integral coefficients for some integer $r > 0$ and $\widehat{\text{vol}}(x, X, \Delta^+) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$, then we may choose δ to only depend on n, r and ε_0 . So our goal is to prove that we can find Δ^+ , r and ε_0 as above such that r, ε_0 only depends on n, ε .

To this end, let $\gamma = c(n) \cdot \varepsilon$ where $c(n) > 0$ is the constant from Theorem 3.1. Let $\ell = \lceil \frac{2+\gamma}{\varepsilon\gamma} \rceil$, and let

$$\Delta' = \sum_{i=1}^m \frac{1}{\ell} \left\lfloor \frac{(1+\gamma)\ell a_i}{1 + \frac{\gamma}{2}} \right\rfloor \Delta_i.$$

Using the assumption that $a_i \geq \varepsilon$, it is not hard to check that $\Delta \leq \Delta'$ and

$$(4.2) \quad \left(1 + \frac{\gamma}{2}\right) \Delta' \leq (1 + \gamma) \Delta.$$

If both K_X and Δ' were \mathbb{Q} -Cartier, we could simply take $r = \ell$, $\Delta^+ = \Delta'$: by Theorem 3.1, the pair $(X, (1 + \gamma)\Delta)$ is klt, hence so is $(X, (1 + \frac{\gamma}{2})\Delta')$ by (4.2). By Lemma 2.11 (with $\lambda = \frac{\gamma}{2}$), we get $\widehat{\text{vol}}(x, X, \Delta') \geq \left(\frac{\gamma}{\gamma+2}\right)^n \widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon_0$ for some constant ε_0 that depends only on n, ε . In other words, the desired conditions on $r, \Delta^+ := \Delta'$ and ε are satisfied.

To deal with the general case, the idea is to find another divisor $D \geq 0$ such that $(X, \Delta' + D)$ is klt (in particular, $K_X + \Delta' + D$ is \mathbb{Q} -Cartier), while

keeping the coefficients in a fixed finite set. To this end, we take a \mathbb{Q} -factorial modification $\pi: Y \rightarrow X$ such that $-(K_Y + \Delta'_Y)$ is π -nef where $\Delta'_Y = \pi_*^{-1}\Delta'$. Such map exists by Lemma 4.7: first take a small birational modification such that the strict transform of $-(K_X + \Delta')$ is ample over X , then take a further small \mathbb{Q} -factorial modification using [BCHM10, Corollary 1.4.3]. We also set $\Delta_Y = \pi_*^{-1}\Delta$. The strict transform of the sought divisor D should be in the \mathbb{Q} -linear system $|-(K_Y + \Delta'_Y)|_{\mathbb{Q}}$. To control the coefficient of D we first proceed to verify the effective basepoint-freeness of this linear system. Since (Y, Δ_Y) is crepant to (X, Δ) , we have

$$\widehat{\text{vol}}(y, Y, \Delta_Y) \geq \widehat{\text{vol}}(\pi(y), X, \Delta) \geq \varepsilon$$

for all $y \in Y$ by Lemma 2.10. Since $\ell\Delta'_Y$ has integer coefficients, we deduce from Lemma 2.9 that there exists a positive integer N_0 depending only on n and ε such that $L := -N_0(K_Y + \Delta'_Y)$ is Cartier. Note that L and $L - (K_Y + \Delta'_Y)$ are both π -nef and π -big by our construction of Y , therefore by Kollár's effective base-point-free theorem [Kol93, Theorem 1.1] (see also [Fuj09, Theorem 1.3] for the relative version), there exists another positive integer m_0 depending only on the dimension n such that m_0L is π -generated. In particular, we get an integer $r_0 = m_0N_0$ which only depends on n and ε such that $-r_0(K_Y + \Delta'_Y)$ is Cartier and π -generated. By the same argument as in the special case above (where K_X and Δ' are assumed to be \mathbb{Q} -Cartier), we also know that $(Y, (1 + \frac{\gamma}{2})\Delta'_Y)$ is klt. Thus by Bertini's theorem, we can choose some effective \mathbb{Q} -divisor $D_Y \sim_{\pi, \mathbb{Q}} -(K_Y + \Delta'_Y)$ such that $2r_0D_Y$ has integral coefficients and the pair $(Y, (1 + \frac{\gamma_0}{2})(\Delta'_Y + D_Y))$ is klt for $\gamma_0 = \min\{1, \gamma\}$.

As $K_Y + \Delta'_Y + D_Y \sim_{\pi, \mathbb{Q}} 0$, we have

$$K_Y + \Delta'_Y + D_Y = \pi^*(K_X + \Delta' + D),$$

where $D = \pi_*D_Y$. Note that $(X, \Delta' + D)$ is klt since the same holds for $(Y, \Delta'_Y + D_Y)$. We also know that $2r_0\ell(\Delta' + D)$ has integral coefficients by our construction. According to the discussion at the beginning of the proof, it remains to check that $\widehat{\text{vol}}(x, X, \Delta' + D) \geq \varepsilon_0$ for some constant $\varepsilon_0 > 0$ that only depends on n and ε . But by construction, it is not hard to see that $\Delta' + D - \Delta$ is \mathbb{Q} -Cartier and that $(X, \Delta + (1 + \frac{\gamma_0}{2})(\Delta' + D - \Delta))$ is klt: indeed, the strict transform of the boundary on Y is at most $(1 + \frac{\gamma_0}{2})(\Delta'_Y + D_Y)$. Hence by Lemma 2.11 (with $\lambda = \frac{\gamma_0}{2}$) we obtain

$$\widehat{\text{vol}}(x, X, \Delta' + D) \geq \left(\frac{\gamma_0}{2 + \gamma_0}\right)^n \widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon_0$$

for some constant that only depends on n and ε (as the same holds for γ_0). This finishes the proof. \square

Remark 4.9. While the results in this paper are stated for pairs with rational coefficients, the general real coefficient case can often be reduced to the rational case by perturbing the coefficients. In Conjecture 1.7 we can even relax the assumption that $\text{Coef}(\Delta)$ belongs to a fixed finite set I to $\text{Coef}(\Delta) \geq \varepsilon$ and get an equivalent conjecture. The reason is as follows: by perturbation, we may further assume $\text{Coef}(\Delta) \subseteq \mathbb{Q}$; if the original version of Conjecture 1.7 holds, then by Theorem 4.8 and Lemma 4.2, for any klt pair (X, Δ) with $\text{Coef}(\Delta) \geq \varepsilon$ and any $\eta \in X$ with $\widehat{\text{vol}}(\eta, X, \Delta) \geq \varepsilon$, we have $\text{mld}^K(\eta, X, \Delta) \leq A$ for some constant $A = A(n, \varepsilon) > 0$.

Using [HLS], we can also extend Theorem 4.1 to the real coefficient case. Indeed, given any finite set $I \subseteq [0, 1]$ (not necessarily $\subseteq \mathbb{Q}$), by [HLS, Theorem 5.6] we can find a finite set $I' \subseteq [0, 1] \cap \mathbb{Q}$ such that: for any n -dimensional klt singularity $x \in (X, \Delta)$ with $\text{Coef}(\Delta) \subseteq I$, there exists some effective \mathbb{Q} -divisor $\Delta' \geq \Delta$ on X with $\text{Coef}(\Delta') \subseteq I'$, such that $x \in (X, \Delta')$ is klt, $\widehat{\text{vol}}(x, X, \Delta') \geq 2^{-n} \widehat{\text{vol}}(x, X, \Delta)$ and $\text{mld}^K(x, X, \Delta') \leq 2 \cdot \text{mld}^K(x, X, \Delta)$ (cf. [Zhu, Lemma 2.17]). Since $\Delta' \geq \Delta$, any δ -plt blowup of (X, Δ') is also a δ -plt blowup of (X, Δ) . The real coefficient case of Theorem 4.1 then follows from the rational case applied to the coefficient set I' .

5. Reduction steps

In this section, we work out some reduction steps for Conjectures 1.7 and 1.8 which apply in any dimension. They will be combined with classification results in the next section to prove both conjectures in low dimensions.

5.1. Special complements. One of our main tools is the notion of special complements. It helps us descend Kollár components over birational models of the singularity to Kollár components of the singularity itself. This notion was introduced in [LXZ22] to prove the Higher Rank Finite Generation Conjecture, here we need a slight variant.

Definition 5.1. Let (X, Δ) be a klt pair and let $\pi: Y \rightarrow X$ be a proper birational morphism. A \mathbb{Q} -complement Γ of (X, Δ) is said to be *special* (with respect to π) if for any $y \in \text{Ex}(\pi)$, there exists some effective \mathbb{Q} -Cartier π -ample \mathbb{Q} -divisor $G \leq \pi^* \Gamma$ such that $y \notin \text{Supp}(G)$.

Lemma 5.2. *Let (X, Δ) be a klt pair, let $\pi: Y \rightarrow X$ be a proper birational morphism and let (Y, Δ_Y) be the crepant pullback of (X, Δ) . Let Γ be a special \mathbb{Q} -complement with respect to π , and let E be an lc place of $(X, \Delta + \Gamma)$. Assume that E is of plt type over (Y, Δ_Y) . Then E is also of plt type over (X, Δ) .*

Proof. This basically follows from the same proof of [LXZ22, Lemma 3.5], which we provide here for reader's convenience. By Lemma 2.4 it suffices to

find an effective divisor D such that E is the unique lc place of $(X, \Delta + \lambda D)$, where $\lambda = \text{lct}(X, \Delta; D)$.

Let $\rho: Z \rightarrow Y$ be the plt blowup of E . By assumption, there exists an effective π -ample \mathbb{Q} -divisor G on Y such that $\pi^*\Gamma \geq G$ and $\rho(E) \not\subseteq \text{Supp}(G)$. Let H be a sufficiently ample divisor on X such that $G + \pi^*H$ is ample. Since $-E$ is ρ -ample, there exists some rational number $\varepsilon > 0$ such that $\rho^*(G + \pi^*H) - \varepsilon E$ is ample on Z . Let G_1 be a general divisor in the \mathbb{Q} -linear system $|\rho^*(G + \pi^*H) - \varepsilon E|_{\mathbb{Q}}$ and consider the effective divisor D on X satisfying $\rho^*\pi^*D = \rho^*(\pi^*\Gamma - G) + G_1 + \varepsilon E$ (this is possible since the right hand side is $\sim_{\mathbb{Q}} 0$ over X). We claim that this divisor D satisfies the desired condition.

Let $K_Z + \Delta_Z = \rho^*(K_Y + \Delta_Y)$ be the crepant pullback. We first note that the above claim is a consequence of the following two properties:

- (1) $(Y, \Delta_Y + \pi^*\Gamma - G)$ is sub-lc and E is an lc place of this subpair;
- (2) E is the only divisor that computes $\text{lct}(Y, \Delta_Y; \rho_*(G_1 + \varepsilon E))$.

This is because, (1) implies that

$$A_{X, \Delta}(F) = A_{Y, \Delta_Y}(F) \geq \text{ord}_F(\pi^*\Gamma - G)$$

for divisor F over X , and the equality holds when $F = E$; on the other hand, if we let $\mu = \text{lct}(Y, \Delta_Y; \rho_*(G_1 + \varepsilon E)) > 0$, then (2) implies that

$$A_{X, \Delta}(F) = A_{Y, \Delta_Y}(F) \geq \mu \cdot \text{ord}_F(\rho_*(G_1 + \varepsilon E)),$$

and equality holds if and only if $F = E$. Combining the two inequalities we have

$$\text{ord}_F(\pi^*D) = \text{ord}_F(\pi^*\Gamma - G + \rho_*(G_1 + \varepsilon E)) \leq (1 + \mu^{-1})A_{X, \Delta}(F),$$

and equality holds if and only if $F = E$. In particular, $\text{lct}(X, \Delta; D) = \frac{1}{1 + \mu^{-1}}$ and E is the unique lc place that computes this lct, which is exactly what we want.

It remains to prove the two properties above. Point (1) is quite straightforward since by assumption E is an lc place of the sub-lc subpair $(Y, \Delta_Y + \pi^*\Gamma)$ and G does not contain the center of E . To see point (2), we note that by assumption the subpair $(Z, \Delta_Z \vee E)$ is plt. Here we denote by $D_1 \vee D_2$ the smallest \mathbb{Q} -divisor D such that $D \geq D_i$ for $i = 1, 2$. Let $t = \frac{A_{Y, \Delta_Y}(E)}{\varepsilon}$. Then

$$\rho^*(K_Y + \Delta_Y + t\rho_*(G_1 + \varepsilon E)) = K_Z + \Delta_Z \vee E + tG_1$$

by construction. Since G_1 is general, the pair $(Z, \Delta_Z \vee E + tG_1)$ is also plt. This implies that $\text{lct}(Y, \Delta_Y; \rho_*(G_1 + \varepsilon E)) = t$ and E is the only divisor that computes the lct. In particular (2) holds. The proof is now complete. \square

As an application, we have:

Lemma 5.3. *Let (X, Δ) be a klt pair, let $\pi: Y \rightarrow X$ be an lc blowup with exceptional divisor E , and let $\Delta_Y = \pi_*^{-1}\Delta$. Then any lc place of $(Y, \Delta_Y + E)$ that is of plt type over (Y, Δ_Y) is also of plt type over (X, Δ) .*

Proof. By assumption $(Y, \Delta_Y + E)$ is lc and its lc centers are contained in E , thus (Y, Δ_Y) is klt. Let $c = A_{X, \Delta}(E) > 0$ and let $\Gamma = \Delta_Y + (1-c)E < \Delta_Y + E$. Let F be an lc place of $(Y, \Delta_Y + E)$ that is of plt type over (Y, Δ_Y) . Since Γ is a convex combination of Δ_Y and $\Delta_Y + E$, by interpolation we know that F is also of plt type over (Y, Γ) . Note that $K_Y + \Gamma = \pi^*(K_X + \Delta)$. By Lemma 5.2, it suffices to show that F is an lc place of some special complement with respect to π . Since $-E$ is π -ample, by Bertini theorem we can choose some general effective \mathbb{Q} -divisor $D_i \sim_{\pi, \mathbb{Q}} -E$ ($i = 1, \dots, n = \dim X$) on Y such that $(Y, \Delta_Y + E + cD_i)$ remains lc for all i and that for any $y \in \text{Ex}(\pi)$ we have $y \notin \text{Supp}(D_i)$ for some $i \in \{1, \dots, n\}$. Let $D_Y = \frac{1}{n}(D_1 + \dots + D_n) \sim_{\pi, \mathbb{Q}} -E$ and let $D = \pi_* D_Y$. Then $\pi^* D = D_Y + E$ and hence $\pi^*(K_X + \Delta + cD) = K_Y + \Delta_Y + cD_Y + E$. By construction, locally on X the divisor cD is a special \mathbb{Q} -complement that has F as an lc place, hence we are done. \square

In the rest of this subsection, we include two auxiliary results that will be useful later.

Lemma 5.4. *Let (X, Δ) be a klt pair, let D be a \mathbb{Q} -complement of (X, Δ) and let $W \subseteq X$ be a minimal lc center of $(X, \Delta + D)$. Let E be an lc place of $(X, \Delta + D)$ with center W . Assume that E is of plt type over (X, Δ) in a neighbourhood of the generic point of W . Then E is of plt type over (X, Δ) .*

Proof. Let $\pi: Y \rightarrow X$ be the prime blowup of E (which exists by Lemma 2.2). Since E is an lc place of the lc pair $(X, \Delta + D)$, we have

$$K_Y + \Delta_Y + D_Y + E = \pi^*(K_X + \Delta + D)$$

and $(Y, \Delta_Y + D_Y + E)$ is lc. In particular, the pair $(Y, \Delta_Y + E)$ is also lc. By assumption, it is also plt over the generic point of W . If it is not plt everywhere, then it has some lc center that does not dominate W . However, any such lc center of $(Y, \Delta_Y + E)$ is also an lc center of $(Y, \Delta_Y + D_Y + E)$ and therefore maps to an lc center of $(X, \Delta + D)$ that is strictly contained in W . This contradicts the assumption that W is a minimal lc center. Hence $(Y, \Delta_Y + E)$ is plt. \square

Lemma 5.5. *Let $n \in \mathbb{N}^*$ and let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Let (X, Δ) be an lc pair of dimension n , let D be a \mathbb{Q} -complement of (X, Δ) and let $W \subseteq X$ be an lc center of $(X, \Delta + D)$. Assume that $\text{Coef}(\Delta), \text{Coef}(D) \subseteq I$. Then there exists some rational number $\varepsilon_0 \in (0, 1)$ depending only on n and I such that every lc type divisor over $(X, \Delta + (1 - \varepsilon_0)D)$ with center W is also an lc place of $(X, \Delta + D)$.*

Proof. ³ We may assume that $1 \in I$. By the ACC of log canonical thresholds [HMX14], there exists some $\varepsilon_0 \in (0, 1)$ such that if (X_0, Δ_0) is an n -dimensional lc pair and D_0 is an effective \mathbb{Q} -Cartier divisor such that

$$\text{Coef}(\Delta_0) \cup \text{Coef}(D_0) \subseteq I$$

and $(X_0, \Delta_0 + (1 - \varepsilon_0)D_0)$ is lc, then $(X_0, \Delta_0 + D_0)$ is also lc. Let us show that this ε_0 satisfies the statement of the lemma. Let E be an lc type divisor over $(X, \Delta + (1 - \varepsilon_0)D)$ with center W and let $\pi: Y \rightarrow X$ be the associated prime blowup. Then $(Y, \Delta_Y + (1 - \varepsilon_0)D_Y + E)$ is lc. By our choice of ε_0 , this implies that $(Y, \Delta_Y + D_Y + E)$ is lc. Suppose that E is not an lc place of $(X, \Delta + D)$. Then we have

$$K_Y + \Delta_Y + D_Y + \lambda E = \pi^*(K_X + \Delta + D)$$

for some $\lambda < 1$. In particular, $(Y, \Delta_Y + D_Y + \lambda E)$ is lc. Moreover, E contains some lc center of $(Y, \Delta_Y + D_Y + \lambda E)$ since $\pi(E) = W$ is an lc center of $(X, \Delta + D)$. As $\lambda < 1$, it follows that $(Y, \Delta_Y + D_Y + E)$ cannot be lc, a contradiction. Thus E is an lc place of $(X, \Delta + D)$ as desired. \square

5.2. Plt type lc place of bounded complements. The goal of this subsection is to reduce Conjecture 1.7 to the following special case, whose statement is motivated by Lemma 2.13.

Conjecture 5.6. *Let $n, N \in \mathbb{N}^*$ and let $\varepsilon > 0$. Then there exists some $A = A(n, N, \varepsilon)$ such that for any N -complement D of an n -dimensional klt variety X and any lc center W of (X, D) such that $\widehat{\text{vol}}(\eta, X) \geq \varepsilon$ (where η is the generic point of W), there exists a Kollár component E over $\eta \in X$ such that*

$$A_{X,D}(E) = 0, \quad \text{and} \quad A_X(E) \leq A.$$

Roughly speaking, we expect that all bounded complements have plt type lc places of bounded log discrepancy. Its connection with Conjecture 1.7 is given by the following result.

Proposition 5.7. *For any fixed dimension n , Conjecture 1.7 and Conjecture 5.6 are equivalent.*

As the first step towards the proof, we show that Conjecture 1.7 can be reduced to the \mathbb{Q} -Gorenstein case.

Proposition 5.8. *Fix the dimension n . Assume that Conjecture 1.7 holds when X is \mathbb{Q} -Gorenstein. Then it holds in general.*

Proof. Starting with any klt pair (X, Δ) in dimension n whose coefficients lie in a finite set I and any $\eta \in X$ with $\widehat{\text{vol}}(\eta, X, \Delta) \geq \varepsilon$, our plan is to find a small birational modification $\pi: Y \rightarrow X$ such that Y is \mathbb{Q} -Gorenstein, together with an N -complement D of $\eta \in (X, \Delta)$ that is special with respect

³The author learned this argument from Yuchen Liu.

to π , where the integer N only depends on n, I and ε . Let us first explain why this is enough.

After shrinking X , we may assume that $\overline{\eta}$ (the closure of η) is the minimal lc center of $(X, \Delta + D)$. Let Δ_Y (resp. D_Y) be the strict transform of Δ (resp. D), let $W \subseteq \pi^{-1}(\overline{\eta})$ be a minimal lc center of $(Y, \Delta_Y + D_Y)$ and let ξ be the generic point of W . Note that $\text{Coef}(\Delta_Y) = \text{Coef}(\Delta) \subseteq I$ and $\text{Coef}(D_Y) \subseteq \frac{1}{N}\mathbb{Z} \cap [0, 1]$. By Lemma 5.5, there exists some rational number $\varepsilon_0 \in (0, 1)$ depending only on n, I and N such that if E is a Kollár component over $\xi \in (Y, \Delta_Y + (1 - \varepsilon_0)D_Y)$, then it is also an lc place of $(Y, \Delta_Y + D_Y)$. By Lemma 5.4, we further deduce that E is of plt type over $(Y, \Delta_Y + (1 - \varepsilon_0)D_Y)$ and hence also over (Y, Δ_Y) . By Lemma 5.2, E is a Kollár component over $\eta \in (X, \Delta)$. Note that as N only depends on n, I, ε so does ε_0 . On the other hand, by Lemma 2.10 we have $\widehat{\text{vol}}(\xi, Y, \Delta_Y) \geq \widehat{\text{vol}}(\eta, X, \Delta) \geq \varepsilon$. By Lemma 2.11 we further obtain

$$\widehat{\text{vol}}(\xi, Y, \Delta_Y + (1 - \varepsilon_0)D_Y) \geq \left(\frac{\varepsilon_0}{1 + \varepsilon_0} \right)^n \widehat{\text{vol}}(\xi, Y, \Delta_Y) \geq C_1$$

for some constant $C_1 > 0$ that only depends on n, I, ε . Since we assume Conjecture 1.7 to hold for \mathbb{Q} -Gorenstein singularities, the Kollár component E above can be chosen so that

$$A_{Y, \Delta_Y + (1 - \varepsilon_0)D_Y}(E) \leq A_0$$

for some constant A_0 that only depends on n, I, ε (a priori it also depends on ε_0 and C_1 but these two constants only depend on n, I and ε). As E is automatically a Kollár component over $\eta \in (X, \Delta)$ and an lc place of $(Y, \Delta_Y + D_Y)$ from the above discussion, we get

$$A_{Y, \Delta_Y + (1 - \varepsilon_0)D_Y}(E) = \varepsilon_0 \cdot A_{Y, \Delta_Y}(E) + (1 - \varepsilon_0)A_{Y, \Delta_Y + D_Y}(E) = \varepsilon_0 \cdot A_{Y, \Delta_Y}(E),$$

hence $\text{mld}^K(\eta, X, \Delta) \leq A_{X, \Delta}(E) = A_{Y, \Delta_Y}(E) \leq \frac{A_0}{\varepsilon_0}$. Since the right hand side only depends on n, I, ε , this proves Conjecture 1.7 in the general (non- \mathbb{Q} -Gorenstein) case.

We now return to construct the map $\pi: Y \rightarrow X$ and the N -complement D . The argument is very similar to that for Theorem 4.8. Let $\pi: Y \rightarrow X$ be a small birational modification such that K_Y is \mathbb{Q} -Cartier and π -ample (existence is guaranteed by Lemma 4.7). In particular, Y is \mathbb{Q} -Gorenstein. By Lemma 2.10 we have $\widehat{\text{vol}}(y, Y, \Delta_Y) \geq \varepsilon$ for all $y \in Y$. Thus by Theorem 3.1, there exists some rational number $\gamma > 0$ depending only on n, ε such that $(Y, (1 + \gamma)\Delta_Y)$ is klt. In addition:

- By Lemma 2.9, there exists a positive integer N_0 depending only on n and ε such that $L := N_0 \cdot K_Y$ is Cartier.

- By Kollár's effective base-point-free theorem, there exists another positive integer m_0 depending only on the dimension n such that m_0L is π -generated.

Putting these two facts together with Bertini's theorem, we deduce that there exists an integer $r_0 = nm_0N_0$ that only depends on n, ε , and an effective divisor $D_Y \sim_{\pi, \mathbb{Q}} K_Y$ such that

- r_0D_Y has integer coefficients,
- $(Y, (1 + \gamma)\Delta_Y + \gamma D_Y)$ is klt, and
- for any $y \in \text{Ex}(\pi)$ there exists some irreducible component of D_Y that's \mathbb{Q} -Cartier and ample over X such that y is not contained in its support.

For example, one can take $D_Y = \frac{1}{r_0}(D_1 + \cdots + D_n)$ where D_1, \dots, D_n are general members of $|m_0N_0K_Y|$. Note that by construction,

$$K_Y + (1 + \gamma)\Delta_Y + \gamma D_Y \sim_{\pi, \mathbb{Q}} (1 + \gamma)(K_Y + \Delta_Y) \sim_{\pi, \mathbb{Q}} 0,$$

thus $K_Y + (1 + \gamma)\Delta_Y + \gamma D_Y = \pi^*(K_X + (1 + \gamma)\Delta + \gamma D)$ where $D = \pi_*D_Y$, and the pair $(X, (1 + \gamma)\Delta + \gamma D)$ is also klt. Since the coefficients of $(1 + \gamma)\Delta + \gamma D$ are contained in a finite set that only depends on n, I and ε , by Lemma 2.13 we see that $\eta \in (X, (1 + \gamma)\Delta + \gamma D)$ admits an N -complement G for some integer N that only depends on n, I, ε . The divisor $\Gamma = \gamma(\Delta + D) + G$ is then an N -complement of $\eta \in (X, \Delta)$. It remains to show that Γ is special with respect to π . But as $\Gamma \geq \gamma D$, this is clear from our choice of D_Y . \square

We now return to the proof of Proposition 5.7.

Proof of Proposition 5.7. By Proposition 5.8, it suffices to show that Conjecture 5.6 is equivalent to the \mathbb{Q} -Gorenstein case of Conjecture 1.7. First assume that Conjecture 5.6 holds. Let (X, Δ) be an n -dimensional klt pair where X is \mathbb{Q} -Gorenstein and let $\eta \in X$ be such that $\widehat{\text{vol}}(\eta, X, \Delta) \geq \varepsilon$. By Theorem 3.1, there exists some constant $c = c(n, \varepsilon) > 0$ such that $(X, (1 + c)\Delta)$ is still klt. We may assume that $c \in \mathbb{Q}$. By Lemma 2.13, there exists an integer N depending only on n and $\text{Coef}(\Delta)$ such that $\eta \in (X, (1 + c)\Delta)$ has an N -complement D (a priori N also depends on c but we already know that c only depends on n and ε). Since Conjecture 5.6 holds, we see that there exist some $A = A(n, N, \varepsilon)$ and some Kollár component E over $\eta \in X$ such that $A_{X, (1+c)\Delta+D}(E) = 0$ and $A_X(E) \leq A$. In particular, the constant A only depends on n, ε and $\text{Coef}(\Delta)$, and clearly $A_{X, \Delta}(E) \leq A_X(E) \leq A$. Thus to prove Conjecture 1.7 for \mathbb{Q} -Gorenstein singularities it suffices to show that E is also a Kollár component over $x \in (X, \Delta)$. But this is straightforward: if $\pi: Y \rightarrow X$ is the plt blowup that extracts E , then (Y, E) is plt and $(Y, E + (1 + c)\Delta_Y + D_Y)$ is lc (because E is an lc place of $(X, (1 + c)\Delta + D)$). By

interpolation, it follows that $(Y, E + \Delta_Y)$ is also plt and thus E is also a Kollár component over $\eta \in (X, \Delta)$. This proves one direction of the equivalence.

Next assume that Conjecture 1.7 holds. Let X, D, W, η be as in Conjecture 5.6. By Lemma 5.5, there exists some rational $\varepsilon_0 \in (0, 1)$ depending only on N such that every Kollár component over $\eta \in (X, (1 - \varepsilon_0)D)$ is also an lc place of (X, D) . Since $\widehat{\text{vol}}(\eta, X, (1 - \varepsilon_0)D) \geq \left(\frac{\varepsilon_0}{1 + \varepsilon_0}\right)^n \widehat{\text{vol}}(\eta, X) \geq \frac{\varepsilon_0^n \varepsilon}{(1 + \varepsilon_0)^n}$ by Lemma 2.11, and Conjecture 1.7 holds by assumption, we see that there exists some constant $A > 0$ depending only on n, N and ε and some Kollár component E over $\eta \in (X, (1 - \varepsilon_0)D)$ such that $A_{X, (1 - \varepsilon_0)D}(E) \leq A$ (a priori A also depends on ε_0 and $\text{Coef}(D)$, but these only depend on n, N, ε by the above construction). From the above discussion, we have $A_{X, D}(E) = 0$. Thus $A_X(E) = \frac{A_{X, (1 - \varepsilon_0)D}(E)}{\varepsilon_0} \leq \frac{A}{\varepsilon_0}$ is bounded from above by some constant that only depends on n, N and ε . This proves the other direction of the desired equivalence. \square

5.3. Further reduction. We further break Conjecture 5.6 into two smaller parts. The first part is the following weaker version of Conjecture 5.6 (the difference is that we don't require the lc place of the complement to be of plt type over the singularity). Note that it is a special case of the uniform boundedness conjecture of mlds [HLL, Conjecture 7.2], which has been verified in dimension two [HL23].

Conjecture 5.9. *Let $n, N \in \mathbb{N}^*$ and let $\varepsilon > 0$. Then there exists some $A_0 = A_0(n, N, \varepsilon)$ such that for any N -complement D of an n -dimensional klt variety X and any lc center W of (X, D) such that $\widehat{\text{vol}}(\eta, X) \geq \varepsilon$ (where η is the generic point of W), there exists a divisor E over $\eta \in X$ such that*

$$A_{X, D}(E) = 0, \quad \text{and} \quad A_X(E) \leq A_0.$$

The other part is Conjecture 5.6 for reduced complement. More precisely:

Lemma 5.10. *Fix the dimension n . Assume that Conjecture 5.9 holds and that Conjecture 5.6 holds when the complement D is reduced (i.e., all its coefficients are 1). Then Conjecture 5.6 (equivalently: Conjecture 1.7) holds.*

Proof. We use the notation and assumptions in Conjecture 5.9. Let E be a divisor over X with center W such that E is an lc place of (X, D) and $A_X(E) \leq A_0 := A_0(n, N, \varepsilon)$. We may assume that $A_0 \geq 1$. By Lemma 2.2, there exists a birational morphism $\pi: Y \rightarrow X$ such that E is the unique exceptional divisor and $-E$ is ample. Since E is an lc place of (X, D) we have $\pi^*(K_X + D) = K_Y + D_Y + E \geq K_Y + E$, hence (Y, E) is lc and any lc place of (Y, E) is also an lc place of (X, D) . By Lemma 4.3, we have $\widehat{\text{vol}}(y, Y) \geq A_0^{-n} \varepsilon$ for all $y \in Y$. If (Y, E) is already plt then there is nothing to prove. Otherwise let $Z \subseteq Y$ be a minimal lc center of (Y, E) and let ξ be the generic point of Z . Note that $\pi(\xi) = \eta$. Since Conjecture 5.6 holds for $\xi \in Y$ with the

reduced complement E by assumption, and $\widehat{\text{vol}}(\xi, Y) \geq A_0^{-n}\varepsilon$, we see that there exists some Kollár component F over $\xi \in Y$ such that $A_{Y,E}(F) = 0$ and $A_Y(F) \leq A_1$ for some constant $A_1 = A_1(n, A_0, \varepsilon) = A_1(n, N, \varepsilon) > 0$. In particular, F is an lc place of (Y, E) and hence is also an lc place of (X, D) . Since $\pi^*K_X = K_Y + (1 - A_X(E))E$, it follows that $A_X(F) = A_Y(F) + (A_X(E) - 1) \cdot \text{ord}_F(E) = A_X(E) \cdot A_Y(F) \leq A_0 A_1$. Since Z is a minimal lc center of (Y, E) , Lemma 5.4 implies that F is of plt type over (Y, E) . By Lemma 5.3, F is a Kollár component over X . Thus Conjecture 5.6 holds with $A(n, N, \varepsilon) = A_0 A_1$. \square

6. Boundedness of mld^K

In this section, we study Conjecture 1.8 in codimension two, as well as Conjecture 1.7 (equivalently: Conjecture 5.6) in dimension 3. In particular, we confirm the special boundedness of threefold klt singularities whose volumes are bounded from below. As a preliminary step, we first show that Conjecture 5.6 holds for singularities $x \in X$ that belong to an analytically bounded family.

Lemma 6.1. *Let N be a positive integer and let $B \subseteq \mathcal{X} \rightarrow B$ be a \mathbb{Q} -Gorenstein family of klt singularities. Then there exists some $A > 0$ depending only on N and the family $B \subseteq \mathcal{X} \rightarrow B$ such that for any klt singularity $x \in X$ with $(x \in X^{\text{an}}) \in (B \subseteq \mathcal{X}^{\text{an}} \rightarrow B)$ and any N -complement D of $x \in X$, there exists some Kollár component E over $x \in X$ such that*

$$A_{X,D}(E) = 0, \quad \text{and} \quad A_X(E) \leq A.$$

Proof. By assumption, $\text{Coef}(D) \subseteq \frac{1}{N}\mathbb{N} \cap [0, 1]$. Thus by Lemma 5.5 there also exists some $\varepsilon_0 > 0$ depending only on N and $n = \dim X$ such that every Kollár component E over $x \in (X, (1 - \varepsilon_0)D)$ is automatically an lc place of (X, D) and in particular we have

$$\begin{aligned} A_{X,(1-\varepsilon_0)D}(E) &= A_X(E) - (1 - \varepsilon_0)\text{ord}_E(D) \\ &= A_X(E) - (1 - \varepsilon_0)A_X(E) = \varepsilon_0 A_X(E). \end{aligned}$$

Therefore, to prove the lemma, it suffices to show that $\text{mld}^K(x, X, (1 - \varepsilon_0)D)$ is bounded from above by some constant that only depends on N and $B \subseteq \mathcal{X} \rightarrow B$.

Since the volume function is constructible [Xu20, Theorem 1.3], there exists some $\varepsilon > 0$ depending only on the family $B \subseteq \mathcal{X} \rightarrow B$ such that $\widehat{\text{vol}}(x, X) \geq \varepsilon$. By Lemma 2.11, we then deduce that $\widehat{\text{vol}}(x, X, (1 - \varepsilon_0)D) \geq \varepsilon_0^n \widehat{\text{vol}}(x, X) \geq \varepsilon_0^n \varepsilon$. By [HLQ23, Theorem 1.7] and Lemma 4.2, this further implies that there exists some $A = A(n, \varepsilon_0, \varepsilon) > 0$ such that $\text{mld}^K(x, X, (1 - \varepsilon_0)D) \leq A$. Tracing

back the proof, we see that the constant A only depends on N and the family $B \subseteq \mathcal{X} \rightarrow B$. \square

6.1. Codimension two case. We next prove Conjecture 1.8 in codimension 2 when $\bar{I} \subseteq \mathbb{Q}$. This result is also needed when we prove Conjecture 1.7 for threefold singularities.

After localizing at the codimension two point, we immediately reduce to the surface case, albeit over a nonalgebraically closed field. Thus, throughout this subsection, all the objects (singularities, divisors, etc.) we consider are defined over a field \mathbb{k} of characteristic 0 that is not necessarily algebraically closed. The main technical result is the following:

Proposition 6.2. *Let $N \in \mathbb{N}^*$ and let $x \in X$ be a smooth surface germ. Then there exists some constant $A_1 > 0$ depending only on N such that for any $G \subseteq \text{Aut}(x \in X)$ and any G -invariant N -complement D of $x \in X$, there exists a G -invariant divisor E (defined over \mathbb{k}) over $x \in X$ with $A_{X,D}(E) = 0$ and $A_X(E) \leq A_1$.*

Note that when $\mathbb{k} = \bar{\mathbb{k}}$ and G is trivial, this is already given by Lemma 6.1, so it remains to check that the corresponding Kollár component can be chosen so that it is G -invariant and descends to the base field \mathbb{k} . The key is the following uniqueness result.

Lemma 6.3. *Let $x \in X$ be a smooth surface germ over $\bar{\mathbb{k}}$ and let D be a \mathbb{Q} -complement of $x \in X$. Then the log discrepancy $A_X(E)$ of divisorial lc places of $x \in (X, D)$ is minimized by a unique divisor E . In particular, if $x \in (X, D)$ is defined over \mathbb{k} then E is also defined over \mathbb{k} and it is invariant under $\text{Aut}(x \in (X, D))$.*

Proof. Every divisor E over $x \in X$ can be extracted via successive blowups of its centers (in particular, the sequence of blowups is canonical):

$$(6.1) \quad E \subseteq X_m \rightarrow \cdots \rightarrow X_1 := \text{Bl}_x X \rightarrow X_0 := X.$$

For any such sequence, if E_i is the exceptional divisor of $X_i \rightarrow X_{i-1}$, then $C_{X_i}(E_{i+1}) \in E_i$ and hence $A_X(E_{i+1}) > A_X(E_i)$ for all i . We denote by $\ell(E) := m$ the number of blowups in this sequence and let

$$\ell(x, X, D) := \min\{\ell(E) \mid C_X(E) = \{x\}, E \text{ is an lc place of } (X, D)\}.$$

We say that a divisor E over $x \in X$ dominates another E' , if E' appears as an exceptional divisor in the canonical sequence (6.1) of blowups associated to E . By the above discussion, $A_X(E) > A_X(E')$ if E dominates E' and $E \neq E'$. Thus it suffices to show that there exists a divisor over $x \in (X, D)$ that is an lc place of (X, D) and that at the same time is dominated by all other divisorial lc place of (X, D) .

We prove this stronger statement by induction on $\ell(x, X, D)$. Let E be a divisorial lc place of (X, D) that computes $\ell(x, X, D)$, i.e., $\ell(E) = \ell(x, X, D)$. Let E_1 be the exceptional divisor of the ordinary blowup $\pi_1: X_1 = \text{Bl}_x X \rightarrow X$. When $\ell(E) = 1$, we have $E = E_1$ and it is dominated by all divisorial lc place of (X, D) , since all blowup sequence has to begin with the ordinary blowup. Thus the statement holds when $\ell(x, X, D) = 1$. If $\ell(E) > 1$, then E_1 is not an lc place of (X, D) . Since x is an lc center of (X, D) , we have $\text{mult}_x D > 1$ (otherwise (X, D) is canonical by [KM98, Theorem 4.5] and the lc centers would have dimension one) and $\pi_1^*(K_X + D) = K_{X_1} + D_1$, where $D_1 = \pi_{1*}^{-1} D + (\text{mult}_x D - 1)E_1 \geq 0$. By assumption E_1 is not an lc place of (X_1, D_1) , thus the lc center W of (X_1, D_1) intersects E_1 at a finite number of points. By Kollár-Shokurov connectedness [KM98, Theorem 5.48], we deduce that $W \cap E = \{x_1\}$ consists of a single point (as a set). It follows that $\ell(x, X, D) = \ell(x_1, X_1, D_1) + 1$ and all lc places of $x \in (X, D)$ dominate the ordinary blowup of x_1 on X_1 . The result now follows by induction. \square

Proof of Proposition 6.2. Let $(X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}}) := (X, D) \times_{\mathbb{k}} \bar{\mathbb{k}}$. By Lemma 6.1, there exists some constant $A_1 > 0$ depending only on N and some divisorial lc place F of $x \in (X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}})$ such that $A_{X_{\bar{\mathbb{k}}}}(F) \leq A_1$. By Lemma 6.3, the log discrepancy of divisorial lc places of $x \in (X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}})$ is minimized by some G -invariant divisor E that's defined over \mathbb{k} . In particular, $A_X(E) \leq A_1$. This proves the statement of the proposition. \square

We are now ready to prove a version of Conjecture 1.8 at codimension two points.

Proposition 6.4. *Let $I = \bar{I} \subseteq [0, 1] \cap \mathbb{Q}$ be a DCC set. Then there exists some constant A depending only on I such that*

$$\text{mld}^K(\eta, X, \Delta) \leq A$$

for any klt pair (X, Δ) with $\text{Coef}(\Delta) \subseteq I$ and any codimension two point $\eta \in X$.

Proof. After localizing at $\eta \in X$, we may assume that η is a closed point x and $x \in (X, \Delta)$ is a surface klt singularity (over a field \mathbb{k} that's not necessarily algebraically closed). By Lemma 2.13, there exists an N -complement D of $x \in (X, \Delta)$ for some integer N that only depends on the coefficient set I . Let $\pi: (\tilde{x} \in \tilde{X}) \rightarrow (x \in X)$ be the universal cover. Let $\tilde{D} = \pi^* D$ and $\tilde{\Delta} = \pi^* \Delta$. Then $\tilde{x} \in \tilde{X}$ is a smooth surface germ and $\tilde{D} + \tilde{\Delta}$ is an N -complement of $\tilde{x} \in \tilde{X}$ by Lemma 2.14.

First consider a special case, i.e., when D is reduced. If (X, D) is not plt, then x is an lc center of (X, D) and $\Delta = 0$. In this case \tilde{D} is a 1-complement of $\tilde{x} \in \tilde{X}$, thus it is nodal and the exceptional divisor \tilde{E} of the ordinary blowup of \tilde{x} is an lc place of (\tilde{X}, \tilde{D}) . Clearly \tilde{E} descends to a Kollár component E

over $x \in X$ and $A_X(E) \leq A_{\tilde{X}}(\tilde{E}) \leq 2$ by [KM98, Proof of Proposition 5.20]. If (X, D) is plt, then \tilde{D} is smooth, $X = \tilde{X}/\mu_m$ for some $m \in \mathbb{N}$, and we can choose coordinates u, v at $\tilde{x} \in \tilde{X}$ that diagonalize the μ_m -action and with $\tilde{D} = (u = 0)$. Since \tilde{x} is an lc center of $(\tilde{X}, \tilde{\Delta} + \tilde{D})$, by adjunction we see that $(\tilde{D}, \tilde{\Delta}|_{\tilde{D}})$ is strictly lc, hence $\tilde{\Delta}|_{\tilde{D}} = \tilde{x}$. Since $\tilde{\Delta} + \tilde{D}$ is an N -complement of $\tilde{x} \in \tilde{X}$ and in particular $N\tilde{\Delta}$ has integer coefficients, we deduce from $\tilde{\Delta}|_{\tilde{D}} = \tilde{x}$ that the local defining equation of $N\tilde{\Delta}$ has the form $v^N + u\phi(u, v) = 0$ for some $\phi \in \mathbb{k}[[u, v]]$. Note that $\phi \neq 0$ since $(\tilde{X}, \tilde{\Delta})$ is klt.

We seek a weighted blowup up with $\text{wt}(u) = a$, $\text{wt}(v) = b$ that provides a Kollár component over $\tilde{x} \in (\tilde{X}, \tilde{\Delta})$ which is also an lc place of $(\tilde{X}, \tilde{\Delta} + \tilde{D})$. For this we look at the Newton polygon (denoted as Q) of $v^N + u\phi(u, v)$, i.e., the convex hull of the union of $(p, q) + \mathbb{R}_{\geq 0}^2$ where $(p, q) \in \mathbb{N}^2$ varies among pairs for which $u^p v^q$ has nonzero coefficients in $v^N + u\phi(u, v)$. Certainly $(0, N) \in Q$. Let $(r, s) \in Q$ be the vertex that lies on the same edge as $(0, N)$. We choose the weights a, b so that $\gcd(a, b) = 1$, $ar + bs = bN$. In particular, $a \leq N - s \leq N$. It is not hard to check that the corresponding exceptional divisor \tilde{E} is a Kollár component over $\tilde{x} \in (\tilde{X}, \tilde{\Delta})$ (essentially because it induces a degeneration of $\tilde{x} \in (\tilde{X}, \tilde{\Delta})$ to the klt singularity $\tilde{x} \in (\tilde{X}, \frac{1}{N}(v^N + u^r v^s = 0))$, cf. Lemma 2.5). Since the coordinates u, v diagonalize the μ_m -action, the divisor \tilde{E} is μ_m -invariant, hence it descends to a Kollár component over $x \in (X, \Delta)$ by [LX20, Lemma 2.13]. By [KM98, Proposition 5.20] and a direct calculation, we then have $A_{X, \Delta}(E) \leq A_{\tilde{X}, \tilde{\Delta}}(\tilde{E}) = a \leq N$. As N only depends on I , we conclude the proof in the case when the \mathbb{Q} -complement D is reduced.

In the general case, we apply Proposition 6.2 to get a constant A_1 that only depends on N (thus it depends on I only), and an $\text{Aut}(\tilde{x} \in \tilde{X})$ -invariant divisor \tilde{E} over $\tilde{x} \in \tilde{X}$ such that $A_{\tilde{X}, \tilde{\Delta} + \tilde{D}}(\tilde{E}) = 0$ and $A_{\tilde{X}}(\tilde{E}) \leq A_1$. By [KM98, Proposition 5.20], the divisor \tilde{E} induces a divisor E over $x \in X$ that is an lc place of $(X, \Delta + D)$ and still satisfies $A_X(E) \leq A_1$. In particular, $A_{X, \Delta}(E) \leq A_X(E) \leq A_1$ and E is of lc type over (X, Δ) . If E is already a Kollár component then we are done. Otherwise, if $Y \rightarrow X$ is the prime blowup of E then the minimal lc centers of $(Y, \Delta_Y + E)$ are 0-dimensional and supported on $[\text{Diff}_E(\Delta_Y)]$. But as $-(K_Y + \Delta_Y + E)|_E = -(K_E + \text{Diff}_E(\Delta_Y))$ is ample, we see that $E \cong \mathbb{P}^1$ and $\deg[\text{Diff}_E(\Delta_Y)] = 1$, hence the minimal lc center of $(Y, \Delta_Y + E)$ is a \mathbb{k} -rational closed point $y \in Y$. Note that E is a reduced N -complement of (Y, Δ_Y) , thus by the special case treated above we may choose a Kollár component E over $y \in (Y, \Delta_Y)$ such that $A_{Y, \Delta_Y}(E) \leq A_0$ for some constant A_0 that only depends on I . By Lemma 5.3, we know that E is also a Kollár component over $x \in (X, \Delta)$. As in the proof of Lemma 5.10,

we also have $A_{X,\Delta}(E) \leq A_0 A_1$. Since both constants A_0, A_1 only depend on I , we are done. \square

Corollary 6.5. *Conjecture 5.6 holds at codimension 2 points.*

Proof. This is an immediate by Proposition 6.4 and the proof of Proposition 5.7. \square

6.2. Threefold singularities. In this subsection, we will prove Conjecture 1.7 in dimension 3. Note that it suffices to consider the case when $\eta \in X$ is a closed point, since we already proved Conjecture 1.8 at codimension two points. According to Proposition 5.7 and Lemma 5.10, the first step is to verify Conjecture 5.9 for 3-dimensional klt singularities.

Proposition 6.6. *Let $N \in \mathbb{N}^*$ and let $\varepsilon > 0$. Then there exists some $A_0 = A_0(3, N, \varepsilon)$ such that for any 3-dimensional klt singularity $x \in X$ with $\widehat{\text{vol}}(x, X) \geq \varepsilon$ and any N -complement D of $x \in X$, there exists some divisor E over $x \in X$ such that*

$$A_{X,D}(E) = 0, \quad \text{and} \quad A_X(E) \leq A_0.$$

In other words, Conjecture 5.9 holds in dimension 3.

We first consider the terminal singularity case. Eventually we will reduce the proof to this special case.

Lemma 6.7. *Let $N \in \mathbb{N}^*$ and let $\varepsilon > 0$. Then there exists some $A_1 = A_1(3, N, \varepsilon)$ such that for any 3-dimensional terminal singularity $x \in X$ with $\widehat{\text{vol}}(x, X) \geq \varepsilon$ and any N -complement D of $x \in X$, there exists some divisor E over $x \in X$ such that*

$$A_{X,D}(E) = 0, \quad \text{and} \quad A_X(E) \leq A_1.$$

Proof. Certainly the plan is to use the classification of terminal singularities [Mor85] (see also [Rei87, Section 6]). Assume that $x \in X$ has index r and let $p: (\tilde{x} \in \tilde{X}) \rightarrow (x \in X)$ be the index 1 cover. By [XZ21, Theorem 1.3] and Lemma 2.9, we have $r \leq \frac{27}{\varepsilon}$ and $\widehat{\text{vol}}(\tilde{x}, \tilde{X}) = r \cdot \widehat{\text{vol}}(x, X) \geq \varepsilon$. Let $\tilde{D} = p^*D$. By assumption, $N\tilde{D}$ has integer coefficient; another application of Lemma 2.9 then gives an integer N_1 depending only on N, ε such that $N_1\tilde{D}$ is Cartier. By [Rei87, Theorem in (6.1)], there exists a linear μ_r -action on \mathbb{A}^4 and an eigenfunction $f \in \mathcal{O}_{\mathbb{A}^4,0}$ such that analytically locally $x \in X$ is isomorphic to $0 \in (f=0)/\mu_r$. Since the statement of the lemma is analytically local, we may assume that $(x \in X) = (0 \in (f=0)/\mu_r)$. In particular $\tilde{X} \cong (f=0) \subseteq \mathbb{A}^4$. Since $N_1\tilde{D}$ is Cartier and μ_r -invariant, we may also write $N_1\tilde{D} = (g=0)|_{\tilde{X}}$ for some eigenfunction $g \in \mathcal{O}_{\mathbb{A}^4,0}$.

Let $\tilde{\Delta} = (g=0) \subseteq \mathbb{A}^4$ and consider the pair $y \in (Y, \Delta) := 0 \in (\mathbb{A}^4, \tilde{X} + \frac{1}{N_1}\tilde{\Delta})/\mu_r$, namely $Y = \mathbb{A}^4/\mu_r$ and if $q: \mathbb{A}^4 \rightarrow Y$ is the quotient map then the divisor Δ is chosen such that $K_{\mathbb{A}^4} + \tilde{X} + \frac{1}{N_1}\tilde{\Delta} = q^*(K_Y + \Delta)$. By Lemma 2.14,

\tilde{D} is a \mathbb{Q} -complement of $\tilde{x} \in \tilde{X}$. By inversion of adjunction (see, e.g., [Kol13, Theorem 4.9]), this implies that $(\mathbb{A}^4, \tilde{X} + \frac{1}{N_1}\tilde{\Delta})$ is lc in a neighbourhood of \tilde{X} . Since \tilde{X} is clearly an lc center of this pair, and the origin is an lc center of (\tilde{X}, \tilde{D}) , we see that the minimal lc center of $(\mathbb{A}^4, \tilde{X} + \frac{1}{N_1}\tilde{\Delta})$ is the origin. Hence $\tilde{X} + \frac{1}{N_1}\tilde{\Delta}$ is a \mathbb{Q} -complement of $0 \in \mathbb{A}^4$, and Δ is a \mathbb{Q} -complement of $y \in Y$ (again by Lemma 2.14).

Note that $\text{Coef}(\Delta) \subseteq \frac{1}{rN_1}\mathbb{Z} \cap [0, 1]$ and $\Delta = X + \Gamma$ for some effective divisor Γ such that $\Gamma|_X = D$. Since $r \leq \frac{27}{\varepsilon}$ and for each $r \in \mathbb{N}$ there are only finitely many nonisomorphic linear actions of μ_r on \mathbb{A}^4 , the singularities $y \in Y$ that arise from the above construction belong to a bounded family, hence by Lemma 6.1 we deduce that there exist some constant $A_1 = A_1(r, N_1) = A_1(\varepsilon, N) > 0$ and some Kollár component F over $y \in Y$ that is also an lc place of (Y, Δ) such that $A_Y(F) \leq A_1$. Let $\pi: Y' \rightarrow Y$ be the associated plt blowup and let X', Γ' be the strict transform of X, Γ on Y' . Let E be an irreducible component of $F|_{X'}$. By adjunction, E is an lc place of $(X, \Gamma|_X) = (X, D)$. We claim that E is the sought divisor. Clearly E is centered at $x \in X$. Since F is an lc place of (Y, Δ) we have $K_{Y'} + F + X' + \Gamma' = \pi^*(K_Y + \Delta)$ and the pair $(Y', F + X' + \Gamma')$ is lc. By the classification of lc surface pairs (see, e.g., [Kol13, Example 3.28]), it follows that X' is normal at the generic point of E and $\text{mult}_E(F|_{X'}) \leq 1$. We also have $K_{Y'} + F + X' = \pi^*(K_Y + X) + A_{Y,X}(F) \cdot F$ which by adjunction gives $A_X(E) = A_{Y,X}(F) \cdot \text{mult}_E(F|_{X'}) \leq A_Y(F) \leq A_1$. This completes the proof. \square

We now prove the general case of Proposition 6.6 by reducing it to the terminal case.

Proof of Proposition 6.6. Let $\pi: Y \rightarrow X$ be a terminal modification of $x \in X$ (whose existence is given by e.g., [Kol13, Theorem 1.33]). It has the property that $\pi^*K_X = K_Y + \Gamma$ for some effective exceptional divisor Γ , Y is \mathbb{Q} -Gorenstein and has only terminal singularities. Moreover, by Lemma 2.10 we have $\widehat{\text{vol}}(y, Y) \geq \widehat{\text{vol}}(x, X) \geq \varepsilon$ for all $y \in Y$. Let $D_Y = \Gamma + \pi^*D$. Since $K_Y + D_Y = \pi^*(K_X + D)$ and D is an N -complement of $x \in X$, we see that D_Y is an N -complement of Y , the pair (Y, D_Y) is strictly log canonical, and it has a minimal lc center W that's contained in $\pi^{-1}(x)$. If $\dim W = 2$, then we may take $E = W$ as $A_X(W) = A_{Y,\Gamma}(W) \leq 1$. If $\dim W = 1$ then it has codimension 2 in Y . By Corollary 6.5, there exist some constant $A_1 > 0$ depending only on N and some divisor E over $x \in X$ (in fact it has center W on Y) such that $A_{X,D}(E) = A_{Y,D_Y}(E) = 0$ and $A_X(E) = A_{Y,\Gamma}(E) \leq A_Y(E) \leq A_1$. Finally if $\dim W = 0$ then $W = \{y\}$ for some $y \in Y$. Since $y \in Y$ has terminal singularity, we know by Lemma 6.7 that there exist another constant $A_2 > 0$ (depending only on N and ε) and a divisor E over $y \in Y$

such that $A_{X,D}(E) = A_{Y,D_Y}(E) = 0$ and $A_X(E) = A_{Y,\Gamma}(E) \leq A_Y(E) \leq A_2$. Therefore, the proposition holds with $A_0 = \max\{1, A_1, A_2\}$. \square

Going back to the proof of Conjecture 1.7, the second step is to verify Conjecture 5.6 for reduced \mathbb{Q} -complements.

Proposition 6.8. *Let $\varepsilon > 0$. Then there exists some constant $A > 0$ depending only on ε such that for any 3-dimensional klt singularity $x \in X$ with $\widehat{\text{vol}}(x, X) \geq \varepsilon$ and any reduced \mathbb{Q} -complement D , there exists some Kollár component E over $x \in X$ such that*

$$A_{X,D}(E) = 0, \quad \text{and} \quad A_X(E) \leq A.$$

In other words, Conjecture 5.6 holds for reduced \mathbb{Q} -complements in dimension 3.

Again we first treat a special case. Recall that a singularity $x \in X$ is called a cDV singularity if by taking general complete intersections of hypersurfaces that contains x one gets a Du Val singularity.

Lemma 6.9. *Proposition 6.8 holds when the simultaneous index one cover of K_X and D (see Section 2.5) is a cDV singularity.*

Proof. Let $\tilde{x} \in \tilde{X}$ be the simultaneous index one cover of K_X and D so that $(x \in X) \cong (\tilde{x} \in \tilde{X})/H$ for some finite abelian group H . By Lemma 2.9, we have $|H| \leq \frac{27}{\varepsilon}$. Let $\tilde{D} \subseteq \tilde{X}$ be the preimage of D . Then \tilde{D} is a Cartier divisor and we may write $\tilde{D} = (g = 0)$ for some $g \in \mathfrak{m}$ (where $\mathfrak{m} \subseteq \mathcal{O}_{\tilde{X},\tilde{x}}$ denotes the maximal ideal). We divide the proof into the following three cases:

- (1) $\tilde{x} \in \tilde{X}$ is a smooth point.
- (2) $\tilde{x} \in \tilde{X}$ is singular and $g \in \mathfrak{m}^2$.
- (3) $\tilde{x} \in \tilde{X}$ is singular and $g \notin \mathfrak{m}^2$.

Case 1. $\tilde{x} \in \tilde{X}$ is a smooth point. Then analytically $(x \in X) \cong (0 \in \mathbb{A}^3)/H$ belongs to a bounded family of singularities (since the size of H is bounded from above and for each choice of H there are only finitely many nonisomorphic actions of H on $0 \in \mathbb{A}^3$). By Lemma 6.1, we see that there exists some constant $A_1 > 0$ that only depends on ε (since the upper bound of the index r only relies on ε) and some Kollár component E over $x \in X$ such that $A_{X,D}(E) = 0$ and $A_X(E) \leq A_1$. Thus the lemma holds in this case.

Case 2. $\tilde{x} \in \tilde{X}$ is singular and $g \in \mathfrak{m}^2$. Since $\tilde{x} \in \tilde{X}$ is a cDV singularity, analytically we may identify it with $0 \in (f = 0) \subseteq \mathbb{A}^4$ for some $f \in \hat{\mathcal{O}}_{\mathbb{A}^4,0}$ with $\text{mult}(f) = 2$. Moreover, since H is abelian, we can assume that it acts diagonally on the coordinates y, z, w, t of \mathbb{A}^4 , all monomials in f have the same eigenvalue, and the isomorphism $\tilde{X} \cong (f = 0)$ is H -equivariant. Since (\tilde{X}, \tilde{D}) is lc, we also know that $(\tilde{X}, \mathfrak{m}^2)$ is lc and any lc place of $(\tilde{X}, \mathfrak{m}^2)$ is also an lc place of (\tilde{X}, \tilde{D}) . Therefore, it suffices to find an H -invariant lc place of $(\tilde{X}, \mathfrak{m}^2)$ with bounded log discrepancy that is a Kollár component.

By adjunction [Kol13, Theorem 4.9], we see that a general plane section of X has semi-log canonical singularities, hence is a nodal curve. It follows that the quadratic part in the Taylor expansion of f has rank at least 2.

First assume that the rank is at least 3. Let \tilde{E} be the exceptional divisor of the ordinary blowup of $\tilde{x} \in \tilde{X}$. It is straightforward to check that \tilde{E} is an lc place of $(\tilde{X}, \mathfrak{m}^2)$, $A_{\tilde{X}}(\tilde{E}) = 2$, and \tilde{E} is a Kollár component (\tilde{E} is a quadric surface and in particular has klt singularities). Clearly \tilde{E} is H -invariant (in fact it is invariant under $\text{Aut}(\tilde{x} \in \tilde{X})$), thus by [LX20, Lemma 2.13] it descends to a Kollár component E over $x \in X$ with $A_{X,D}(E) = 0$ and $A_X(E) \leq 2$. Hence the lemma holds in this case.

Assume next that the rank of the quadratic part of f is two. We claim that there exists a finite group $G \subseteq \text{Aut}(\tilde{x} \in \tilde{X})$ of order at most $2|H|^4$ containing H such that $W := \tilde{X}/G$ belongs to an analytically bounded family of singularities (that depends only on ε). If f contains the monomial t^2 , then after an H -equivariant change of variables (cf. the proof of [Rei87, Page 395, Proposition], especially [Rei87, Page 394, Rule III]), we may assume that $f = t^2 + h(y, z, w)$ for some $\text{mult}(h) \geq 2$. Let τ be the involution $(y, z, w, t) \mapsto (y, z, w, -t)$ and let $G \subseteq \text{Aut}(\tilde{x} \in \tilde{X})$ be the (abelian) subgroup generated by H and τ . Clearly $|G| \leq 2|H|$ and \tilde{X}/τ is a smooth point, thus $\tilde{X}/G = (\tilde{X}/\tau)/(G/\tau)$ is a quotient singularity. Since $|G/\tau| \leq |H| \leq \frac{27}{\varepsilon}$ we see that there are only finitely many nonisomorphic quotients as before and hence $W = \tilde{X}/G$ belongs to an analytically bounded family of singularities.

If f does not contain any of y^2, z^2, w^2, t^2 , then as its quadratic part has rank 2 we have $f = yz + h(w, t)$ after a change of variable (again by [Rei87, Page 395, Proposition]). This time let τ be the involution $(y, z, w, t) \mapsto (z, y, w, t)$ and let $G \subseteq \text{Aut}(\tilde{x} \in \tilde{X})$ be the subgroup generated by H and τ . It is not hard to see that for every $\varphi \in G$ either φ or $\tau\varphi$ acts diagonally on (y, z, w, t) with order at most $|H|$, hence $|G| \leq 2|H|^4$. Let $G_0 < G$ be the subgroup generated by all the conjugates of τ ; it is also the smallest normal subgroup that contains τ . By a theorem of Chevalley, $\mathbb{A}^4/G_0 \cong \mathbb{A}^4$ and in fact it is not hard to check that the quotient map is given by $(y, z, w, t) \mapsto (y^{r_0} + z^{r_0}, yz, w, t)$ for some $r_0 \in \mathbb{N}^*$. It then follows from the equation of f that $(\tilde{x} \in \tilde{X})/G_0$ is a smooth point and $\tilde{X}/G = (\tilde{X}/G_0)/(G/G_0)$ is a quotient singularity. As $|G/G_0|$ is at most $|H|^4 \leq \frac{27^4}{\varepsilon^4}$, we conclude as before that there are only finitely many nonisomorphic quotients and $W = \tilde{X}/G$ belongs to an analytically bounded family of singularities which only depends on ε . This proves the claim.

Let $(W, \Delta_W + \mathfrak{a})$ be the crepant G -quotient of $(\tilde{X}, \mathfrak{m}^2)$, where \mathfrak{a} is a \mathbb{Q} -ideal. Note that $(\tilde{X}, \mathfrak{m}^2)$ is strictly lc, thus the same holds for $(W, \Delta_W + \mathfrak{a})$. By construction the coefficients of $\Delta_W + \mathfrak{a}$ lie in $\frac{1}{|G|}\mathbb{N}$. By Lemma 6.1, there exist

some constant $A'_2 > 0$ depending only on ε and a Kollár component \tilde{E} over W that is an lc place of $(W, \Delta_W + \mathfrak{a})$ such that $A_W(\tilde{E}) \leq A'_2$. By [LX20, Lemma 2.13], it pulls back to a G -invariant Kollár component over \tilde{X} and hence also induces a Kollár component E over X . By [KM98, Proposition 5.20] we know that E is an lc place of (X, D) and $A_X(E) \leq |G| \cdot A_{W, \Delta_W}(\tilde{E}) \leq A_2$ for some constant $A_2 > 0$ that only depends on ε . Therefore we are done in this case.

Case 3. $\tilde{x} \in \tilde{X}$ is singular and $g \notin \mathfrak{m}^2$. Then up to a change of coordinate we may assume that $g = t$ and $\tilde{X} \cong (f = 0) \subseteq \mathbb{A}^4$ with $\text{mult}(f) = 2$. We may write $f = tf_1(y, z, w, t) + h(y, z, w)$ and in particular $\tilde{D} \cong (h = 0) \subseteq \mathbb{A}^3$. By adjunction, the surface \tilde{D} has slc singularity and the origin is an lc center. Thus $2 \leq \text{mult}(h) \leq 3$.

Suppose first that $\text{mult}(h) = 3$. Let h_1 be the homogeneous term of degree 3 in h . Let $q: \tilde{D}' \rightarrow \tilde{D}$ be the ordinary blowup of the origin. Then $q^*K_{\tilde{D}} = K_{\tilde{D}'} + F$ where $F \cong (h_1(y, z, w) = 0) \subseteq \mathbb{P}^2$ is the q -exceptional divisor. Since \tilde{D} is slc, so is the pair (\tilde{D}', F) , thus by adjunction we know that the cubic curve F is at most nodal (for us it is enough to know that h_1 is irreducible). Note that $\text{mult}(f_1) = 1$ since $\text{mult}(f) = 2$. If the linear term in f_1 is not proportional to t , then we may apply an H -equivariant change of coordinates between y, z, w so that $f_1 = ay + bt + (\text{mult} \geq 2)$ ($a \neq 0$). Consider the weighted blowup with $\text{wt}(y, z, w, t) = (1, 1, 1, 2)$. By Lemma 2.5, the exceptional divisor E is a Kollár component, since the corresponding initial term $ayt + h_1(y, z, w)$ gives a cA -type singularity. Note that E is an H -invariant lc place of (X, D) and $A_X(E) = 2$.

The other possibility is that the linear term in f_1 is proportional to t , i.e., $f_1 = at + (\text{mult} \geq 2)$ ($a \neq 0$). Consider the weighted blowup with $\text{wt}(y, z, w, t) = (2, 2, 2, 3)$. The corresponding initial term gives the hypersurface $(at^2 + h_1(y, z, w) = 0) \subseteq \mathbb{A}^4$ which is of cD -type by Lemma 6.10. Thus the exceptional divisor E of the weighted blowup is a Kollár component by Lemma 2.5. It is also an H -invariant lc place of (X, D) and $A_X(E) = 3$.

Next suppose that $\text{mult}(h) = 2$. Then the quadratic part of h has rank 1, otherwise \tilde{D} is either a union of two planes or has A_m singularity for some $m \in \mathbb{N}^*$, contradicting the assumption that the origin is an lc center of \tilde{D} . Thus we may choose coordinates so that $h = y^2 + h_0(z, w)$ and $\text{mult}(h_0) \geq 3$. By Weierstrass preparation theorem (applied to f), we then have $f = (\text{unit}) \cdot (y^2 + ay + b)$ for some $a, b \in (z, w, t) \subseteq \mathbb{C}[[z, w, t]]$, which can be turned into $f = (\text{unit}) \cdot (y^2 + c(z, w, t))$ with another change of variable $y \mapsto y - \frac{a}{2}$ which leaves t fixed. Here we need to further justify that our application of the Weierstrass preparation theorem is H -equivariant. Indeed, one way to prove the Weierstrass preparation theorem is through the Weierstrass division theorem: there exist some unique $s_1 \in \mathbb{C}[[y, z, w, t]]$ and $s_2 \in \mathbb{C}[[z, w, t]][y]$

such that $y^2 = f \cdot s_1 + s_2$ and the degree of y in s_2 is ≤ 1 ; one then checks that s_1 is a unit and if $s_2 = -ay - b$ then we get the desired form $f = (\text{unit}) \cdot (y^2 + ay + b)$. Since all monomials in f have the same eigenvalues under the H -action, the same holds for both s_1 and s_2 by their uniqueness. Thus with an H -equivariant change of coordinates we may assume that f has the form $f = y^2 + c(z, w, t)$ and at the same time $g = t$. We can now argue as in Case 2: if τ is the involution $(y, z, w, t) \mapsto (-y, z, w, t)$ then it leaves \tilde{D} invariant and \tilde{X}/τ is a smooth point. Hence if $G \subseteq \text{Aut}(\tilde{x} \in \tilde{X})$ is the (abelian) subgroup generated by H and τ , then $|G/\tau| \leq |H| \leq \frac{27}{\varepsilon}$ and $W := \tilde{X}/G = (\tilde{X}/\tau)/(G/\tau)$ is a quotient singularity and belongs to an analytically bounded family of singularities. The crepant G -quotient (W, Δ_W) of (\tilde{X}, \tilde{D}) has coefficients in $\frac{1}{|G|}\mathbb{N}$ and therefore as in Case 2 we get some constant A_3 that depends only on ε and some Kollár component E over $x \in X$ that is an lc place of (X, D) and satisfies $A_X(E) \leq A_3$.

Putting Cases 1–3 together, we see that the proof of the lemma is finished by setting $A = \max\{3, A_1, A_2, A_3\}$. \square

The following result is used in the above proof.

Lemma 6.10. *Let $n \geq 3$ and let $f \in \mathbb{C}[[x_2, \dots, x_n]]$. Assume that $\text{mult}(f) = 3$ and let f_3 be the leading term of f . Then the hypersurface singularity $(x_1^2 + f(x_2, \dots, x_n) = 0) \subseteq \mathbb{A}^n$ is of cD-type if and only if its singular locus has codimension at least 2 and f_3 is not a cube.*

Proof. The conditions are preserved by taking general hyperplane sections of the form $a_2x_2 + \dots + a_nx_n = 0$. Thus we may assume that $n = 3$, where the statement follows from [KM98, Step 4 on Page 126]. \square

Proof of Proposition 6.8. Let $N = \lfloor \frac{27}{\varepsilon} \rfloor!$. By Lemma 2.9, we know that D is an N -complement. Let A_1 (resp. A_2) be the constant from Corollary 6.5 (resp. Lemma 6.9). We will show that the proposition holds with $A = \max\{A_1, A_2\}$. We do this by induction on $d(x, X)$, the number of divisors E over $x \in X$ such that $A_X(E) \leq 1$. By [KM98, Proposition 2.36(2)], $d(x, X) < +\infty$. Let $\tilde{x} \in \tilde{X}$ be the simultaneous index one cover of K_X and D so that $(x \in X) \cong (\tilde{x} \in \tilde{X})/H$ for some finite abelian group H . Let $\tilde{D} \subseteq \tilde{X}$ be the preimage of D . Then $\tilde{x} \in \tilde{X}$ is a Gorenstein canonical singularity and \tilde{D} is Cartier. If \tilde{X} is a cDV singularity, then the statement already follows from Lemma 6.9. In particular, the proposition holds when $d(x, X) = 0$ (i.e., when $x \in X$ is terminal), since the index one cover of a threefold terminal singularity is cDV. If $\tilde{x} \in \tilde{X}$ is not a cDV singularity, then by [KM98, Lemma 5.30 and Theorem 5.35] there exists an H -invariant divisor \tilde{F} over $\tilde{x} \in \tilde{X}$ with $A_{\tilde{X}}(\tilde{F}) = 1$. Since \tilde{D} is Cartier, we obtain $\text{ord}_{\tilde{F}}(\tilde{D}) \geq 1$ and hence $A_{\tilde{X}, \tilde{D}}(\tilde{F}) \leq 0$. As (\tilde{X}, \tilde{D}) is lc, it follows that \tilde{F} is an lc place of (\tilde{X}, \tilde{D}) . Let F be the induced divisor over X . Then by [KM98, Proposition 5.20] we

have $A_X(F) \leq 1$ and $A_{X,D}(F) = 0$. By Lemma 2.2, there exists a birational morphism $\pi: Y \rightarrow X$ with a unique exceptional divisor F such that $-F$ is π -ample. Let $D_Y = \pi_*^{-1}D$. Since F is an lc place of (X, D) , any lc place E of (Y, F) is also an lc place of (X, D) . Moreover, if E is of plt type over Y , then it is a Kollár component over $x \in X$ by Lemma 5.3. By construction, $K_Y \leq \pi^*K_X$, hence $A_X(E) \leq A_Y(E)$ for any divisor E over Y . Thus it suffices to find an lc place E of (Y, F) that is of plt type over Y and has log discrepancy $A_Y(E) \leq A$. If the minimal lc center of (Y, F) has codimension ≤ 2 this follows from Corollary 6.5 and our choice of the constant A . Otherwise F is a reduced \mathbb{Q} -complement of $y \in Y$ for some $y \in \pi^{-1}(x)$. By Lemma 2.10, we have $\widehat{\text{vol}}(y, Y) \geq \varepsilon$. From the definition we also have $d(y, Y) \leq d(x, X) - 1$, hence in this case the result follows from the induction hypothesis and we are done. \square

Corollary 6.11. *Conjecture 1.7 holds in dimension three.*

Proof. This is a combination of Proposition 6.4, Lemma 5.10, Proposition 6.6 and Proposition 6.8. \square

Corollary 6.12. *Conjecture 1.1 holds in dimension three.*

Proof. This is a combination of Theorem 4.8 and Corollary 6.11. \square

7. Examples and discussions

In general one should be more careful when making analogy between mld^K and the usual mld . In this section, we give some examples that illustrate the difference between them, and discuss some related questions. Our first example shows that mld^K can be arbitrarily large in a given dimension, and may fail to be lower semi-continuous.

Example 7.1. Let $1 < a < b$ be coprime positive integers. Consider the klt surface singularity $0 \in (X = \mathbb{A}_{xy}^2, \Delta = \frac{1}{a}(x^a + y^b = 0))$ and the weighted blowup $\pi: Y \rightarrow X$ with $\text{wt}(x) = b$ and $\text{wt}(y) = a$. The exceptional divisor E is a Kollár component and by adjunction one can check that $\text{Diff}_E(\Delta_Y) = (1 - \frac{1}{a})P + (1 - \frac{1}{b})Q + \frac{1}{a}R$ where P and Q are the two singular points of Y and $R \neq P, Q$ is a smooth point. It is then not hard to see that the alpha invariant $\alpha(E, \text{Diff}_E(\Delta_Y)) = 1$ and hence E is the only Kollár component over $0 \in (X, \Delta)$ by [Pro00, Theorem 4.3]. It follows that

$$\text{mld}^K(0, X, \Delta) = A_{X,\Delta}(E) = a.$$

By [LX20, Theorem 1.3], we also have $\widehat{\text{vol}}(x, X, \Delta) = A_{X,\Delta}(E)^2 \cdot (- (E^2)) = \frac{a}{b}$. Thus unlike the boundedness conjecture for mld , the upper bound on mld^K in Conjecture 1.7, if exists, necessarily depends on the coefficient set I . Note

also that $\mathrm{mld}^K(x, X, \Delta) \leq 2$ when $x \neq 0$. Hence the example also shows that mld^K is not lower semi-continuous.

When X is a surface and $\Delta = 0$, one can check that $\mathrm{mld}^K(x, X)$ coincides with the usual minimal log discrepancy $\mathrm{mld}(x, X)$ (see [LX]). Example 7.2 shows that starting in dimension three $\mathrm{mld}^K(x, X, \Delta)$ differs from $\mathrm{mld}(x, X, \Delta)$, even when $\Delta = 0$.

Example 7.2. Let $n \geq 3$ and consider the canonical hypersurface singularity

$$0 \in (x_0^{n+1} + x_1^n + x_2^n \cdots + x_n^n = 0) \subseteq \mathbb{C}^{n+1}.$$

It can be resolved by the ordinary blow up, and the corresponding exceptional divisor E is the only divisor that computes the mld. But E is not a Kollár component: it has a singular point of multiplicity n , in particular it's not klt.

Although Example 7.1 shows that mld^K does not satisfy lower semi-continuity, we may still ask if it satisfies the other conjectural property of the mld, i.e., the ascending chain condition (ACC).

Question 7.3. Let $n \in \mathbb{N}^*$ and let $I \subseteq [0, 1]$ be a DCC set. Does the set

$$\{\mathrm{mld}^K(x, X, \Delta) \mid \dim X = n, \mathrm{Coef}(\Delta) \in I, \text{ and } x \in (X, \Delta) \text{ is klt} \}$$

satisfy the ACC?

Recall that the Weil index of a Fano variety X is the largest integer q such that $-K_X \sim_{\mathbb{Q}} qA$ for some Weil divisor A on X . A Fano variety X is said to be weakly special if (X, D) is log canonical for every effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_X$. If Conjecture 1.8 were true, it would imply Conjecture 7.4; otherwise the orbifold cone construction will produce counterexamples to Conjecture 1.8.

Conjecture 7.4. *Let $n \in \mathbb{N}^*$. Then there exists some constant $N > 0$ depending on n such that the Weil index of any n -dimensional weakly special Fano variety is at most N .*

We suspect that Conjecture 7.4 may hold even for K-semistable Fano varieties. Our main intuition comes from toric examples: while a weighted projective space can have arbitrarily large Weil index, it is K-semistable only when it's \mathbb{P}^n . If this is true, it may provide strong evidence towards Shokurov's boundedness conjecture for mld, since by [XZ21] every klt singularity admits a unique K-semistable valuation.

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