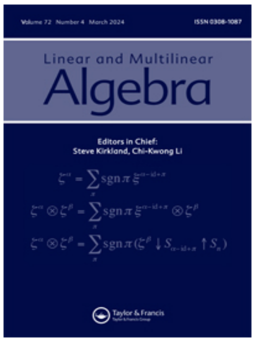

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Lie triple centralizers of the algebra of dominant block upper triangular matrices

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ABSTRACT

Let $\mathcal{N}$ be the algebra of all $n \times n$ dominant block upper triangular matrices over a field. In this paper, we explicitly describe all Lie triple centralizers of $\mathcal{N}$. We also describe Lie triple centralizers of the algebra $\mathcal{B}$ of block upper triangular matrices over a field.

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1. Introduction

Let $\mathfrak{g}$ be an algebra over a field $\mathbb{F}$. A linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ is called *Lie centralizer* of $\mathfrak{g}$ if

$$\phi([a, b]) = [\phi(a), b] \tag{1}$$

for all $a, b \in \mathfrak{g}$, where $[a, b] = ab - ba$ is the usual Lie product of a and b. It is easy to check that ϕ is a Lie centralizer on $\mathfrak{g}$ if and only if $\phi[a, b] = [a, \phi(b)]$ for any $a, b \in \mathfrak{g}$.

Fošner and Jing studied the Lie centralizers on triangular rings and nest algebras in [1] and presented characterizations of Lie centralizers on triangular rings and nest algebras. Ghomanjani and Bahmani dealt with the structure of Lie centralizers of trivial extension algebras in [2]. In [3] Ahmed characterized the non-additive Lie centralizers of strictly upper triangular matrices over a field of zero characteristic. Recently, Ghimire explicitly described linear Lie centralizers of the algebra of dominant block upper triangular matrices in [4] and of the strictly block upper triangular matrices in [5]. The Lie centralizer on the

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algebras is also called Lie centroid, which play an important role in studying the structure of algebras see [6,7].

Another important class of mappings on algebras is Lie triple centralizers. Let $\mathfrak{g}$ be an algebra over a field $\mathbb{F}$. A linear map $f : \mathfrak{g} \to \mathfrak{g}$ is called *Lie triple centralizer* of $\mathfrak{g}$ if

$$f([[a,b],c]) = [[f(a),b],c] \tag{2}$$

for all $a,b,c \in \mathfrak{g}$. It is easy to check that ϕ is a Lie triple centralizer on $\mathfrak{g}$ if and only if f([[a, b], c]) = [[a, f(b)], c] for any $a,b,c \in \mathfrak{g}$. Lie triple centralizers were first introduced by Behrooz et al. in [8] and characterized the Lie triple centralizers of the generalized matrix algebra. It is clear that each Lie centralizer is a Lie triple centralizer, but the converse may not be true in general see (Example 1.2, [8]). Therefore, the concept of the Lie triple centralizer generalizes the concept of the Lie centralizer.

The main goal of this paper is to explicitly describe the Lie triple centralizers of the algebra of dominant block upper triangular matrices and the algebra of block upper triangular matrices over a field $\mathbb{F}$. In recent years, significant progress has been made in studying the subalgebras of the algebra of upper triangular matrices over a field or a ring. Some results on the study of the subaglebras of the algebra of upper triangular matrices are given in [4,5,9,10–15].

Fix a field $\mathbb{F}$. Let $M_{m,n}$ be the set of all $m \times n$ matrices over $\mathbb{F}$, and put $M_n = M_{n,n}$. Let $\mathcal{N}$ (resp. $\mathcal{B}$) denote the set of all dominant block upper triangular matrices (resp. block upper triangular matrices) in M_n relative to a given partition. Then $\mathcal{N}$ and $\mathcal{B}$ are subalgebras of $\mathfrak{gl}(n,\mathbb{F})$, i.e. M_n with the usual matrix multiplication. In this paper, we explicitly describe the Lie triple centralizers of $\mathcal{B}$ and $\mathcal{N}$ over $\mathbb{F}$, which are as follows:

- Theorem 2.1 shows that every Lie triple centralizer of $\mathcal{B}$ is a sum of a linear mapping $\gamma : \mathcal{B} \to \mathcal{B}$ given by $\gamma(A) = \lambda A$ where $\lambda \in \mathbb{F}$ and a linear mapping that maps $\mathcal{B}$ to its center and vanishes at $[[\mathcal{B},\mathcal{B}],\mathcal{B}]$.
- Theorem 2.2 shows that every Lie triple centralizer of $\mathcal{N}$ is a sum of a linear mapping $\gamma : \mathcal{N} \to \mathcal{N}$ given by $\gamma(A) = \lambda A$ where $\lambda \in \mathbb{F}$, a linear mapping that maps $\mathcal{N}$ to its center and vanishes at $[[\mathcal{N},\mathcal{N}],\mathcal{N}]$, and two special linear mappings.

The main motivation of this work comes from Behrooz et al. result on the Lie triple centralizers on generalized matrix algebras in [8], Ghimire and Huang's work on the Lie triple derivations of the Lie algebra of dominant block upper triangular matrices in [11], and Ghimire's work on the Linear Lie centralizers of the algebra of dominant block triangular matrices in [4]. Our work on the Lie triple centralizers of $\mathcal{N}$ not only generalizes the result of Behrooz et al., but also uses a promising new approach for finding the Lie triple centralizers of other matrix algebras with appropriate block forms. The essential tools are Lemmas 3.1–3.3, where four types of product preserving linear maps between matrix spaces are determined.

Section 2 gives the basic notations and presents the main results, i.e. characterizations of the Lie triple centralizers of $\mathcal{B}$ and $\mathcal{N}$. Section 3 determines four types of product preserving linear maps between matrix spaces that will play essential roles in finding the Lie triple centralizers of $\mathcal{N}$. Section 4 presents some other lemmas and proves Theorem 2.2.

2. Main results

The Lie triple centralizers of the algebra $\mathcal{N}$ (resp. $\mathcal{B}$) of dominant block upper triangular matrices (resp. of block upper triangular matrices) will be determined in this section.

2.1. Notations

Let $[n] = \{1, 2, \dots, n\}$. Fix a field $\mathbb{F}$. Let $M_{m,n}$ (resp. M_n) be the set of $m \times n$ (resp. $n \times n$) matrices over $\mathbb{F}$. Let I_n denote the identity matrix in M_n. A $t \times t$ *block matrix form* in M_n is represented by a sequence $(n_1, n_2, \dots, n_t)$, where $n_i \in \mathbb{Z}^+$ for $i \in [t]$ and $n_1 + \cdots + n_t = n$. Fixing a $t \times t$ block matrix form in M_n represented by a sequence $(n_1, n_2, \dots, n_t)$, each $A \in M_n$ can be expressed as

$$A = \left[A_{i,j} \right]_{t\times t}$$

where the (i,j) block $A_{i,j} \in M_{n_i,n_j}$. The matrix A can also be expressed as

$$A = \sum_{(i,j)\in[t]\times[t]} A^{i,j}$$

such that each $A^{i,j} \in M_n$ has $A_{i,j}$ on the (i,j) block and 0's elsewhere. A is called

- *block upper triangular* if $A_{i,j} = 0$ for all $1 \le j < i \le t$,
- *dominant block upper triangular* if A is block upper triangular and $A_{i,i} = 0$ for $i \in S$, where S is a subset of $[t]$ that consists of nonconsecutive integers.

When A is not given in advance, $A^{i,j}$ and similar expressions may be used to express generic matrices in M_n with 0's outside of the (i,j) block.

Let $\mathcal{B}_n^{\bar{n}} = \mathcal{B}$ (resp. $\mathcal{N}_n^{\bar{n}} = \mathcal{N}$) denote the set of all block upper triangular matrices (resp. dominant block upper triangular matrices) in M_n with fixed $t \times t$ block matrix form $\bar{n} = (n_1, \dots, n_t)$. They are subalgebras of the algebra M_n with the usual matrix multiplication.

For $i, j \in [t]$, let $M_n^{i,j}$ denote the set of matrices in M_n with 0's outside of the (i,j) block. Define the *block index set* of $\mathcal{N}$ as

$$\Gamma_{\mathcal{N}} = \Omega \cup \Omega_1, \tag{3}$$

where

$$\Omega := \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i < j \le t\} \quad \text{and}$$
$$\Omega_1 := \{(i,i) \in \mathbb{Z}^2 \mid 1 \le i \le t\} \setminus \{(i,i) \in \mathbb{Z}^2 \mid \mathcal{N}^{ii} = 0\}.$$

2.2. Lie triple centralizers of $\mathcal{N}$

We describe the Lie triple centralizers of $\mathcal{N}$ when $t \ge 3$. The main results are given in Theorem 2.1 and in Theorem 2.2 below. Theorem 2.1 describes the Lie triple centralizers of the algebra $\mathcal{B}$ of block upper triangular matrices.

Theorem 2.1: *Let $\mathcal{B}$ be the algebra of block upper triangular matrices over a field $\mathbb{F}$. Let $f : \mathcal{B} \to \mathcal{B}$ be a Lie triple centralizer. Then*

$$f(A) = \lambda A + \tau(A) \tag{4}$$

for all $A \in \mathcal{B}$, where $\lambda \in \mathbb{F}$ and $\tau : \mathcal{B} \to \mathrm{Z}(\mathcal{B})$, $\mathrm{Z}(\mathcal{B})$ is the center of $\mathcal{B}$, is a linear mapping that vanishes at $[[\mathcal{B}, \mathcal{B}], \mathcal{B}]$.

Proof: Let $\mathcal{B}$ be the algebra of block upper triangular matrices. Then $\mathcal{B}$ is a triangular algebra (see [16], Block upper triangular matrix algebras' in Section 4). Since $\mathcal{B}$ is a triangular algebra, by ([8], Corollary 3.4 and Remark 4.3), we have

$$f(A) = \lambda A + \tau(A) \tag{5}$$

for all $A \in \mathcal{B}$, where $\lambda \in \mathbb{F}$ and $\tau : \mathcal{B} \to \mathrm{Z}(\mathcal{B})$ is a linear map satisfying $\tau[[\mathcal{B}, \mathcal{B}], \mathcal{B}] = 0$. ■

The next theorem explicitly describes the Lie triple centralizers of the algebra $\mathcal{N}$ of dominant block upper triangular matrices.

Theorem 2.2: *Let $t \geq 3$ and let $\mathcal{N}$ be the algebra of dominant block upper triangular matrices over a field $\mathbb{F}$. Let $f : \mathcal{N} \to \mathcal{N}$ be a Lie triple centralizer. Then*

$$f(A) = \lambda A + \delta(A) + \phi_1(A) + \phi_2(A) \tag{6}$$

for all $A \in \mathcal{N}$, where $\lambda \in \mathbb{F}$ and other summand components are given below:

(1) $\delta : \mathcal{N} \to \mathrm{Z}(\mathcal{N})$, $\mathrm{Z}(\mathcal{N})$ *is the center of* $\mathcal{N}$, *is a linear map that vanishes at* $[[\mathcal{N}, \mathcal{N}], \mathcal{N}]$.
(2) ϕ_1 *is an element of* $\mathrm{End}\,(\mathcal{N})$ *that satisfies:*
   (a) *when* $\mathcal{N}^{2,2} \neq 0$ *or* $\mathcal{N}^{t,t} \neq 0$, $\phi_1 = 0$;
   (b) *when* $\mathcal{N}^{2,2} = 0$, $\mathcal{N}^{t,t} = 0$, *and* $n_1 \geq 2$, $\phi_1 = 0$;
   (c) *when* $\mathcal{N}^{2,2} = 0$, $\mathcal{N}^{t,t} = 0$, *and* $n_1 = 1$, $\mathcal{N}^{i,j} \subseteq \mathrm{Ker}\,\phi_1$ *for all* $(i,j) \notin \{(1,1),(1,2)\}$, $\mathrm{Im}\,\phi_1 \subseteq \mathcal{N}^{1,t} + \mathcal{N}^{2,t}$, *and* $E^{12}_{1j} f(E^{11}_{11})^{2t} = -E^{11}_{11} f(E^{12}_{1j})^{1t}$, $j \in [n_2]$.
(3) ϕ_2 *is an element of* $\mathrm{End}\,(\mathcal{N})$ *that satisfies:*
   (a) *when* $\mathcal{N}^{1,1} \neq 0$ *or* $\mathcal{N}^{t-1,1-t} \neq 0$, $\phi_2 = 0$;
   (b) *when* $\mathcal{N}^{1,1} = 0$, $\mathcal{N}^{t-1,t-1} = 0$, *and* $n_t \geq 2$, $\phi_2 = 0$;
   (c) *when* $\mathcal{N}^{1,1} = 0$, $\mathcal{N}^{t-1,t-1} = 0$, *and* $n_t = 1$, $\mathcal{N}^{i,j} \subseteq \mathrm{Ker}\,\phi_2$ *for all* $(i,j) \notin \{(t-1,t),(t,t)\}$, $\mathrm{Im}\,\phi_2 \subseteq \mathcal{N}^{1,t-1} + \mathcal{N}^{1,t}$, *and* $f(E^{tt}_{11})^{1,t-1} E^{t-1,t}_{i,1} = -f(E^{t-1,t}_{i1})^{1,t} E^{t,t}_{1,1}$, $i \in [n_{t-1}]$.

We will give a proof of Theorem 2.2 in Section 4.

3. Linear maps preserving matrix products

The Lie triple centralizer property (2) over a matrix algebra is closely related to some matrix product preserving properties. These relationships are much more obvious when the algebra consists of block matrices. Here we will determine linear maps that preserve four different types of matrix products. These maps play essential roles in exploring the Lie triple centralizers of $\mathcal{N}$ as well as other algebras of block matrices.

In Lemmas 3.1–3.3, let $E^{p,q}_{m\times n}$ denote the $m \times n$ matrix that has the only nonzero entry 1 in the (p,q) position.

Lemma 3.1: *Suppose $\mathbb{F}$ is an arbitrary field. If $X \in M_m$ and $Y \in M_n$ satisfy that*

$$XA = AY \tag{7}$$

for all $A \in M_{mn}$, then $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$.

Proof: Suppose $X = (x_{ip}) \in M_m$ and $Y = (y_{qj}) \in M_n$, where $i, p \in [m]$ and $q, j \in [n]$. For any $(i,j) \in [m] \times [n]$, by (7),

$$XE^{i,j}_{m\times n} = E^{i,j}_{m\times n}Y. \tag{8}$$

Comparing the (i,j) entry of the matrices in (8), we get $x_{ii} = y_{j,j}$. Similarly, comparing the (p,j) entry for $p \neq i$, we get $x_{pi} = 0$; and comparing the (i,q) entry for $q \neq j$, we get $0 = y_{jq}$. Therefore, $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$. ∎

Lemma 3.2: *If linear maps $\phi : M_{m,p} \to M_{m,q}$ and $\varphi : M_{n,p} \to M_{n,q}$ satisfy that*

$$\phi(AB) = A\varphi(B) \quad \textit{for all } A \in M_{m,n},\ B \in M_{n,p},$$

then there is $X \in M_{p,q}$ such that $\phi(C) = CX$ for $C \in M_{m,p}$ and $\varphi(D) = DX$ for $D \in M_{n,p}$.

Proof: For any $j \in [n]$ and $B \in M_{n,p}$,

$$\phi(E^{1,j}_{m\times n}B) = E^{1,j}_{m\times n}\varphi(B).$$

Let $R^1_{m,p}$ denote the subspace of $M_{m,p}$ consisting of matrices with 0's outside of the first row. Similarly for $R^1_{m,q}$. Then for every $j \in [n]$, $R^1_{m,p} = E^{(1,j)}_{m\times n}M_{n,p}$, so that

$$\phi(R^1_{m,p}) = \phi(E^{(1,j)}_{m\times n}M_{n,p}) = E^{(1,j)}_{m\times n}\varphi(M_{n,p}) \subseteq R^1_{m,q}.$$

There exists an $X \in M_{p,q}$ such that the linear transformation $\phi|_{R^1_{m,p}} : R^1_{m,p} \to R^1_{m,q}$ can be expressed as

$$\phi|_{R^1_{m,p}}(T) = TX, \quad \text{for all } T \in R^1_{m,p}.$$

Then for every $B \in M_{n,p}$,

$$E^{(1,j)}_{m\times n}\varphi(B) = \phi(E^{(1,j)}_{m\times n}B) = E^{(1,j)}_{m\times n}BX.$$

Therefore, $\varphi(B) = BX$ for $B \in M_{n,p}$. Hence $\phi(AB) = A\varphi(B) = ABX$ for every $A \in M_{m,n}$ and $B \in M_{n,p}$. The linear combinations of all such AB form $M_{m,p}$. So $\phi(C) = CX$ for all $C \in M_{m,p}$. ∎

Lemma 3.3: *If linear maps $\phi : M_{m,p} \to M_{n,p}$ and $\varphi : M_{m,q} \to M_{n,q}$ satisfy that*

$$\phi(BA) = \varphi(B)A \quad \text{for all } A \in M_{q,p},\ B \in M_{m,q},$$

then there is $X \in M_{n,m}$ such that $\phi(C) = XC$ for $C \in M_{m,p}$ and $\varphi(D) = XD$ for $D \in M_{m,q}$.

Proof: The proof (omitted) is similar to that of Lemma 3.2. ■

4. Proofs of main results

The main goal of this section is to prove Theorem 2.2. We always assume that $t \geq 3$ and $\mathcal{N} \neq \mathcal{B}$ except where explicitly noted otherwise.

4.1. Lie triple centralizer image locations

First, we will give several auxiliary results on the image locations of $f(\mathcal{N}^{i,j})$ for a Lie triple centralizer f and $\mathcal{N}^{i,j} \subseteq \mathcal{N}$. Recall $\mathcal{N}^{i,j}$ refers to the collection of matrices in $\mathcal{N}$ whose entries outside the (i,j) block are 0. We will observe the following interesting fact: most nonzero blocks of $f(A^{i,j})$ for $A^{i,j} \in \mathcal{N}^{i,j}$ are located on the (i,j)th block and the center

$$\mathrm{Z}(\mathcal{N}) = \begin{cases} \mathbb{F}I & \text{if } \mathcal{N} = \mathcal{B}; \\ \mathcal{N}^{1,t} & \text{if } \mathcal{N}^{1,1} = 0 \text{ and } \mathcal{N}^{t,t} = 0; \\ 0 & \text{otherwise.} \end{cases}$$

of $\mathcal{N}$.

The first lemma discusses the Lie triple centralizer image on $\mathcal{N}^{1,1}$ and $\mathcal{N}^{t,t}$.

Lemma 4.1: *Let f be a Lie triple centralizer of $\mathcal{N}$. Then*

$$f(\mathcal{N}^{1,1}) \subseteq \mathcal{N}^{1,1} + \mathcal{N}^{2,t} + \mathrm{Z}(\mathcal{N}) \tag{9}$$

$$f(\mathcal{N}^{t,t}) \subseteq \mathcal{N}^{t,t} + \mathcal{N}^{1,t-1} + \mathrm{Z}(\mathcal{N}) \tag{10}$$

where $\mathrm{Z}(\mathcal{N})$ is the center of $\mathcal{N}$. Furthermore,

(1) *if $\mathcal{N}^{2,2} \neq 0$ or $\mathcal{N}^{t,t} \neq 0$, then $f(A^{1,1})^{2,t} = 0$.*
(2) *if $\mathcal{N}^{1,1} \neq 0$ or $\mathcal{N}^{t-1,t-1} \neq 0$, then $f(A^{t,t})^{1,t-1} = 0$.*

Proof: Suppose $A^{1,1} \in \mathcal{N}^{1,1}$ and $i,j \in [t]$. To prove (9), we consider the following cases:

(1) We first investigate $f(A^{1,1})^{j,j}$. We now consider the following two sub-cases:
- $t = 3$: By definition of $\mathcal{N}$, $(2,2) \in \Gamma_{\mathcal{N}}$ or $(3,3) \in \Gamma_{\mathcal{N}}$. Suppose $(3,3) \in \Gamma_{\mathcal{N}}$. Then for $A^{23} \in \mathcal{N}^{23}$ and $A^{33} \in \mathcal{N}^{33}$,

$$\begin{aligned} 0 = f([[A^{1,1}, A^{23}], A^{33}])^{23} &= ([[f(A^{1,1}), A^{23}], A^{33}])^{23} \\ &= f(A^{1,1})^{22}A^{23}A^{33} - A^{23}f(A^{1,1})^{33}A^{33} \end{aligned}$$

Therefore,

$$f(A^{1,1})^{22}A^{23} = A^{23}f(A^{1,1})^{33} \quad \text{for } A^{23} \in \mathcal{N}^{23}. \tag{11}$$

Lemma 3.1 implies $f(A^{1,1})^{22} = \lambda I_n^{22}$ and $f(A^{1,1})^{33} = \lambda I_n^{33}$ for some $\lambda \in \mathbb{F}$. By the assumption $\mathcal{N} \neq \mathcal{B}$, $\mathcal{N}^{22} = 0$, which forces $f(A^{1,1})^{22} = 0$. Thus, by (11), $f(A^{1,1})^{33} = 0$ as well. Similarly, we can conclude that $f(A^{1,1})^{22} = f(A^{1,1})^{33} = 0$ if $(2,2) \in \Gamma_{\mathcal{N}}$. Thus $f(A^{1,1})^{j,j} = 0$ for all $1 < j \leq 3$.

- $t \geq 4$: Suppose $1 < j < r \leq t$. Then for $A^{11} \in \mathcal{N}^{11}$, $A^{jr} \in \mathcal{N}^{j,r}$, and $A^{rt} \in \mathcal{N}^{rt}$, $[[A^{11}, A^{j,r}], A^{rt}] = 0$, so that

$$\begin{aligned} 0 = f([[A^{1,1}, A^{jr}], A^{rt}])^{jt} &= ([[f(A^{1,1}), A^{jr}], A^{rt}])^{jt} \\ &= -A^{jr}f(A^{1,1})^{rr}A^{rt} + f(A^{1,1})^{jj}A^{jr}A^{rt} \end{aligned}$$

Therefore,

$$f(A^{1,1})^{j,j}A^{j,r} = A^{j,r}f(A^{1,1})^{r,r} \quad \text{for } A^{j,r} \in \mathcal{N}^{j,r}. \tag{12}$$

Lemma 3.1 implies $f(A^{1,1})^{j,j} = \lambda I_n^{j,j}$ and $f(A^{1,1})^{r,r} = \lambda I_n^{r,r}$ for some $\lambda \in \mathbb{F}$. In the situation $\mathcal{N} \neq \mathcal{B}$, there exists $(p,p) \in [t] \times [t]$ such that $\mathcal{N}^{p,p} = 0$, which forces $f(A^{1,1})^{p,p} = 0$. Thus $f(A^{1,1})^{j,j} = 0$ for all $1 < j \leq t$.

(2) Next we show that $f(A^{1,1})^{i,j} = 0$ for $A^{1,1} \in \mathcal{N}^{1,1}$, $(i,j) \in \Gamma_{\mathcal{N}}$, $i<j$, and $(i,j) \notin \{(2,t)\}$. By assumption, either $i > 2$ or $j < t$.

- Suppose $i > 2$. For $A^{2,i} \in \mathcal{N}^{2,i}$ and $A^{j,t} \in \mathcal{N}^{j,t}$

$$0 = f([[A^{1,1}, A^{2,i}], A^{j,t}])^{2,t} = [[f(A^{1,1}), A^{2,i}], A^{j,t}]^{2,t} = -A^{2,i}f(A^{1,1})^{i,j}A^{jt},$$

which implies that $f(A^{1,1})^{i,j} = 0$.

- Suppose $j < t$. By assumption $j > 1$. For $A^{1,i} \in \mathcal{N}^{1,i}$ and $A^{j,t} \in \mathcal{N}^{j,t}$,

$$0 = f([[A^{1,1}, A^{1,i}], A^{jt}])^{1,t} = [[f(A^{1,1}), A^{1,i}], A^{jt}]^{1,t} = -A^{1i}f(A^{1,1})^{i,j}A^{j,t}$$

which implies that $f(A^{1,1})^{i,j} = 0$.

- It remains to show that $f(A^{1,1})^{1,t} = 0$. Suppose $(i,j) = (1,t)$. For $A^{1,1}, I_n^{1,1} \in \mathcal{N}^{1,1}$,

$$0 = f([[A^{1,1}, I_n^{1,1}], I_n^{1,1}])^{1,t} = [f(A^{1,1}), I_n^{1,1}], I_n^{1,1}]^{1,t} = f(A^{1,1})^{1,t}.$$

Thus $f(A^{1,1})^{1,t} = 0$ for $A^{1,1} \in \mathcal{N}^{1,1}$, proving (9).

Next, when $\mathcal{N}^{2,2} \neq 0$ or $\mathcal{N}^{t,t} \neq 0$, we show that $f(A^{1,1})^{2,t} = 0$. Suppose $\mathcal{N}^{2,2} \neq 0$. Then for $I_n^{2,2}, A^{2,2} \in \mathcal{N}^{2,2}$,

$$0 = f([[A^{1,1}, A^{2,2}], I_n^{2,2}])^{2,t} = [[f(A^{1,1}), A^{2,2}], I_n^{2,2}]^{2,t} = A^{2,2}f(A^{1,1})^{2,t}$$

which implies that $f(A^{1,1})^{2,t} = 0$. Simliarly, we can show that $f(A^{1,1})^{2,t} = 0$ when $\mathcal{N}^{t,t} \neq 0$.

The proofs of (10) and $f(A^{t,t})^{1,t-1} = 0$ when $\mathcal{N}^{1,1} \neq 0$ or $\mathcal{N}^{t-1,t-1} \neq 0$ are similar.

■

The next lemma discusses the Lie triple centralizer image on $\mathcal{N}^{k,k} \subseteq \mathcal{N}$ when $1 < k < t$.

Lemma 4.2: *Let f be a Lie triple centralizer of $\mathcal{N}$ and $1 < k < t$. Then*

$$f(\mathcal{N}^{k,k}) \subseteq \mathcal{N}^{k,k} + \mathrm{Z}(\mathcal{N}), \tag{13}$$

where $\mathrm{Z}(\mathcal{N})$ is the center of $\mathcal{N}$.

Proof: Suppose $A^{k,k} \in \mathcal{N}^{k,k}$ and $i,j \in [t]$. To prove (13), we consider the following cases:

(1) We first investigate $f(A^{k,k})^{j,j}$. Suppose $r < k < j \le t$. Then for $A^{r,j} \in \mathcal{N}^{r,j}$ and $A^{j,t} \in \mathcal{N}^{j,t}$, $[[A^{r,j}, A^{k,k}], A^{j,t}] = 0$, so that

$$\begin{aligned} 0 = f([[A^{r,j}, A^{k,k}], A^{j,t}])^{r,t} &= ([[A^{r,j}, f(A^{k,k}), A^{j,t}])^{r,t} \\ &= A^{r,j} f(A^{k,k})^{j,j} A^{j,t} - f(A^{k,k})^{r,r} A^{r,j} A^{j,t} \end{aligned}$$

Therefore,

$$f(A^{k,k})^{r,r} A^{r,j} = A^{r,j} f(A^{k,k})^{j,j} \quad \text{for } A^{r,j} \in \mathcal{N}^{r,j}. \tag{14}$$

Lemma 3.1 implies $f(A^{k,k})^{j,j} = \lambda I_n^{j,j}$ and $f(A^{k,k})^{r,r} = \lambda I_n^{r,r}$ for some $\lambda \in \mathbb{F}$. Equation (14) is also true for $j < k$. In the situation $\mathcal{N} \neq \mathcal{B}$, there exists $(p,p) \in [t] \times [t]$ such that $\mathcal{N}^{p,p} = 0$, which forces $f(A^{k,k})^{p,p} = 0$. Thus $f(A^{k,k})^{j,j} = 0$ for all $j \in [t]$ and $k \neq j$. When $\mathcal{N} = \mathcal{B}$, we have $f(A^{k,k})^{j,j} = \lambda I_n^{j,j}$ for all $j \in [t]$ and $k \neq j$.

(2) Next we show that $f(A^{k,k})^{i,j} = 0$ for $A^{k,k} \in \mathcal{N}^{k,k}$, $(i,j) \in \Gamma_{\mathcal{N}}$, and $(i,j) \notin \{(k,k), (1,t)\}$. We consider the following subcases:

- First we show that $f(A^{k,k})^{i,j} = 0$ for $A^{k,k} \in \mathcal{N}^{k,k}$, $(i,j) \in \Gamma_{\mathcal{N}}$, $i \neq k$, $j \neq k$, and $(i,j) \neq (1,t)$. Either $i > 1$ or $j < t$. Suppose $j < t$. Then for $A^{1,i} \in \mathcal{N}^{1,i}$ and $A^{j,t} \in \mathcal{N}^{j,t}$,

$$0 = f([[A^{1,i}, A^{k,k}], A^{j,t}])^{1,t} = ([[A^{1,i}, f(A^{k,k})], A^{j,t}])^{1,t} = A^{1,i} f(A^{k,k})^{i,j} A^{j,t}. \tag{15}$$

  So $f(A^{k,k})^{i,j} = 0$.Similarly, we can conclude that $f(A^{k,k})^{i,j} = 0$ if $i > 1$.

- Now we show that $f(A^{k,k})^{i,k} = 0$ for $1 \le i < k$ and $f(A^{k,k})^{k,j} = 0$ for $k < j \le t$. Suppose $1 \le i < k$. Then for $I_n^{k,k} \in \mathcal{N}^{k,k}$, $A^{k,r} \in \mathcal{N}^{k,r}$ and $k < r \le t$,

$$f(A^{k,r})^{i,t} = f([[A^{k,k}, A^{k,r}], I_n^{k,k}])^{i,t} = [[A^{k,k}, f(A^{k,r})], I_n^{k,k}]^{i,t} = 0. \tag{16}$$

  By (16),

$$\begin{aligned} 0 = f(A^{k,r})^{i,r} = f([[A^{k,k}, A^{k,r}], A^{r,t}])^{i,t} &= [[f(A^{k,k}), A^{k,r}], A^{r,t}]^{i,t} \\ &= f(A^{k,k})^{i,k} A^{k,r} A^{r,t} \quad \text{for } A^{r,t} \in \mathcal{N}^{r,t} \end{aligned}$$

  So, $f(A^{k,k})^{i,k} = 0$ for $1 \le i < k$.Similarly, $f(A^{k,k})^{kj} = 0$ for $k < j \le t$.

(3) Finally, if $(1,1) \in \Gamma_{\mathcal{N}}$ or $(t,t) \in \Gamma_{\mathcal{N}}$, say $(1,1) \in \Gamma_{\mathcal{N}}$, then for $k \notin \{1,t\}$,

$$0 = f([[A^{1,1}, A^{k,k}], I_n^{1,1}])^{1,t} = [[A^{1,1}, f(A^{k,k})], I_n^{1,1}]^{1,t} = A^{1,1} f(A^{k,k})^{1,t}.$$

So, $f(A^{k,k})^{1,t} = 0$.

Thus, (13) has been proved. ■

Next, we consider the Lie triple centralizer image on $\mathcal{N}^{1,2}$ and $\mathcal{N}^{t-1,t}$.

Lemma 4.3: *Let f be a Lie triple centralizer of $\mathcal{N}$. Then*

$$f(\mathcal{N}^{1,2}) \subseteq \mathcal{N}^{1,2} + \mathcal{N}^{1,t}, \tag{17}$$

$$f(\mathcal{N}^{t-1,t}) \subseteq \mathcal{N}^{t-1,t} + \mathcal{N}^{1,t}. \tag{18}$$

Furthermore,

(1) *if $\mathcal{N}^{2,2} \neq 0$ or $\mathcal{N}^{t,t} \neq 0$, then $f(A^{1,2})^{1,t} = 0$.*
(2) *if $\mathcal{N}^{1,1} \neq 0$ or $\mathcal{N}^{t-1,t-1} \neq 0$, then $f(A^{t-1,t})^{1,t} = 0$.*

Proof: We first prove (17). By definition of $\mathcal{N}$, $(1,1) \in \Gamma_{\mathcal{N}}$ or $(2,2) \in \Gamma_{\mathcal{N}}$. Suppose $(1,1) \in \Gamma_{\mathcal{N}}$. Then $\mathcal{N}^{1,2} = \mathcal{N}^{1,1}\mathcal{N}^{1,2} = [\mathcal{N}^{1,1}, \mathcal{N}^{1,2}]$. For $I_n^{1,1}, A^{1,1} \in \mathcal{N}^{1,1}$ and $A^{1,2} \in \mathcal{N}^{1,2}$, according to Lemma 4.1,

$$f[A^{1,1}, A^{1,2}] = f[[A^{1,1}, A^{1,2}], I_n^{1,1}] = [[f(A^{1,1}), A^{1,2}], I_n^{1,1}] \in \mathcal{N}^{1,2} + \mathcal{N}^{1,t}. \tag{19}$$

Thus, (17) is proved. Similarly, we can prove (17) if $(2,2) \in \Gamma_{\mathcal{N}}$.

Next we show that $f(A^{1,2})^{1,t} = 0$ when $\mathcal{N}^{2,2} \neq 0$ or $\mathcal{N}^{t,t} \neq 0$. Suppose $\mathcal{N}^{2,2} \neq 0$ or $\mathcal{N}^{t,t} \neq 0$. By Lemma 4.1, $f(A^{1,1})^{2,t} = 0$. So, (19) implies that $f(A^{1,2})^{1,t} = 0$.

The proofs of (18) and $f(A^{t-1,t})^{1,t} = 0$ when $\mathcal{N}^{1,1} \neq 0$ or $\mathcal{N}^{t-1,t-1} \neq 0$ are similar. ■

By Lemmas 4.1 and 4.3, we know that the $(2,t)$ block of $f(\mathcal{N}^{1,1})$ and $(1,t)$ block of $f(\mathcal{N}^{1,2})$ are zero when $\mathcal{N}^{2,2} \neq 0$ or $\mathcal{N}^{t,t} \neq 0$. The next lemma explicitly describes the $(2,t)$ block of $f(\mathcal{N}^{1,1})$ and $(1,t)$ block of $f(\mathcal{N}^{1,2})$when $\mathcal{N}^{2,2} = 0$ and $\mathcal{N}^{t,t} = 0$.

Lemma 4.4: *Let f be a Lie triple centralizer of $\mathcal{N}$, $\mathcal{N}^{2,2} = 0$, and $\mathcal{N}^{t,t} = 0$. Then the image $f(\mathcal{N}^{12})^{1t}$ and $f(\mathcal{N}^{11})^{2t}$ satisfy the following:*

(1) *If $n_1 \geq 2$, then $f(\mathcal{N}^{12})^{1t} = f(\mathcal{N}^{11})^{2t} = 0$.*
(2) *If $n_1 = 1$, then $E_{1j}^{12} f(E_{11}^{11})^{2t} = -E_{11}^{11} f(E_{1j}^{12})^{1t}$, $j \in [n_2]$.*

Furthermore, any $f \in \operatorname{End}\mathcal{N}$ that satisfies $\mathcal{N}^{i,j} \subseteq \operatorname{Ker} f$ for all $(i,j) \notin \{(1,1),(1,2)\}$, $\operatorname{Im} f \subseteq \mathcal{N}^{1,t} + \mathcal{N}^{2,t}$, and the hypothesis (1) and (2) is a Lie triple centralizer.

Proof: Let f be a Lie triple centralizer of $\mathcal{N}$. Since $\mathcal{N}^{2,2} = 0$, by definition of $\mathcal{N}$, $(1,1) \in \Gamma_{\mathcal{N}}$. Then for $I_n^{1,1}, A^{1,1} \in \mathcal{N}^{1,1}$ and $A^{1,2} \in \mathcal{N}^{1,2}$,

$$f[[A^{1,2}, A^{1,1}], I_n^{1,1}]^{1,t} = [[f(A^{1,2}), A^{1,1}], I_n^{1,1}]^{1,t} \tag{20}$$

and

$$f[[A^{1,2}, A^{1,1}], I_n^{1,1}]^{1,t} = [[A^{1,2}, f(A^{1,1})], I_n^{1,1}]^{1,t} \tag{21}$$

By (20) and (21), we get

$$[f(A^{1,2}), A^{1,1}], I_n^{1,1}]^{1,t} = [[A^{1,2}, f(A^{1,1})], I_n^{1,1}]^{1,t} \tag{22}$$

(22) implies that $A^{11}f(A^{12})^{1,t} = -A^{12}f(A^{11})^{2,t}$. In particular,

$$E_{rs}^{11}f(E_{i,j}^{12})^{1,t} = -E_{i,j}^{1,2}f(E_{r,s}^{1,1})^{2,t} \tag{23}$$

We now consider the following two cases:

1. when $n_1 \geq 2$, for any $E_{i,j}^{1,2} \in \mathcal{N}^{1,2}$ we can chose $r \in [n_1] \setminus \{i\}$. So, (23) implies that $f(E_{i,j}^{1,2})^{1,t} = 0$. Thus, $f(A_{1,2})^{1,t} = 0$ and $f(A_{1,1})^{2,t} = 0$.
2. when $n_1 = 1$, by (23) we get $E_{1j}^{12}f(E_{11}^{11})^{2t} = -E_{11}^{11}f(E_{1j}^{12})^{1t}$ for $j \in [n_2]$.

The last statement is easy to verify. ■

The next lemma describes the $(1,t)$ block of $f(\mathcal{N}^{t-1,t})$ and $(1,t-1)$ block of $f(\mathcal{N}^{t,t})$ when $\mathcal{N}^{1,1} = 0$ and $\mathcal{N}^{t-1,t-1} = 0$.

Lemma 4.5: *Let f be a Lie triple centralizer of $\mathcal{N}$, $\mathcal{N}^{1,1} = 0$, and $\mathcal{N}^{t-1,t-1} = 0$. Then the image $f(\mathcal{N}^{t-1,t})^{1,t}$ and $f(\mathcal{N}^{t,t})^{1,t-1}$ satisfy the following:*

1. *If $n_t \geq 2$, then $f(\mathcal{N}^{t-1,t})^{1t} = f(\mathcal{N}^{t,t})^{1,t-1} = 0$.*
2. *If $n_t = 1$, then $f(E_{11}^{t,t})^{1,t-1t}E_{i,1}^{t-1,t} = -f(E_{i,1}^{t-1,t})^{1t}E_{1,1}^{t,t}$, $i \in [n_{t-1}]$.*

Furthermore, any $f \in \operatorname{End}\mathcal{N}$ that satisfies $\mathcal{N}^{i,j} \subseteq \operatorname{Ker} f$ for all $(i,j) \notin \{(t-1,t),(t,t)\}$, $\operatorname{Im} f \subseteq \mathcal{N}^{1,t-1} + \mathcal{N}^{1,t}$, and the hypothesis (1) *and* (2) *is a Lie triple centralizer.*

Proof: The proof (omitted) is similar to that of Lemma 4.4. ■

We now discuss the Lie triple centralizer image on $\mathcal{N}^{2,3}, \mathcal{N}^{3,4}, \ldots, \mathcal{N}^{t-2,t-1}$ in next Lemma.

Lemma 4.6: *For a Lie triple centralizer f of $\mathcal{N}$ and $1 < i < t-1$,*

$$f(\mathcal{N}^{i,i+1}) \subseteq \mathcal{N}^{i,i+1} \tag{24}$$

Proof: Suppose $1 < i < t-1$. By definition of $\mathcal{N}$, $(i,i) \in \Gamma_{\mathcal{N}}$ or $(i+1,i+1) \in \Gamma_{\mathcal{N}}$. Without loss of generality, suppose $(i,i) \in \Gamma_{\mathcal{N}}$. Then for $I_n^{i,i} \in \mathcal{N}^{i,i}$ and $A^{i,i+1} \in \mathcal{N}^{i,i+1}$, according to Lemma 4.2,

$$f(A^{i,i+1}) = f[[A^{i,i+1}, I_n^{i,i}], I_n^{i,i}] = [[A^{i,i+1}, f(I_n^{i,i})], I_n^{i,i}] \in \mathcal{N}^{i,i+1}. \tag{25}$$

Thus,(24) is proved. ■

Now we consider the Lie triple centralizer image on the other $\mathcal{N}^{i,j}$.

Lemma 4.7: *For a Lie triple centralizer f of $\mathcal{N}$ and $i,j \in [t]$ and $j > i+1$, the image $f(\mathcal{N}^{i,j})$ satisfies*

$$f(\mathcal{N}^{i,j}) \subseteq \mathcal{N}^{i,j}. \tag{26}$$

Proof: Suppose $j > i+1$. By definition of $\mathcal{N}$, $(i,i) \in \Gamma_{\mathcal{N}}$ or $(i+1, i+1) \in \Gamma_{\mathcal{N}}$. Without loss of generality, suppose $(i,i) \in \Gamma_{\mathcal{N}}$. Then for $I_n^{i,i} \in \mathcal{N}^{i,i}$, $A^{i,i+1} \in \mathcal{N}^{i,i+1}$ and $A^{i+1,j} \in \mathcal{N}^{i+1,j}$, according to Lemma 4.6,

$$f(A^{i,j}) = f[[I_n^{i,i}, A^{i,i+1}], A^{i+1,j}] = [[I_n^{i,i}, f(A^{i,i+1})], A^{i+1,j}] \in \mathcal{N}^{i,j}. \tag{27}$$

Thus,(26) is proved. ■

The above lemmas determine all possibly nonzero blocks of $f(A^{i,j})$ for a Lie triple centralizer f and $A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}$. The next goal is to describe the f-images on these blocks.

Lemma 4.8: *Let f be a Lie triple centralizer of $\mathcal{N}$. Then there exist $\lambda \in \mathbb{F}$ such that*

$$f(A^{i,j})^{i,j} = \lambda A^{i,j} \quad \textit{for all } A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}. \tag{28}$$

Proof: Suppose $1 \le i < j \le t$. By definition of $\mathcal{N}$, $(i,i) \in \Gamma_{\mathcal{N}}$ or $(i+1, i+1) \in \Gamma_{\mathcal{N}}$. Without of loss generality, suppose $(i,i) \in \Gamma_{\mathcal{N}}$. For any $I_n^{ii}, A^{i,i} \in \mathcal{N}^{i,i}$ and $A^{i,j} \in \mathcal{N}^{i,j}$,

$$f(A^{i,i}A^{i,j})^{i,j} = f[[A^{i,j}, A^{i,i}], I_n^{i,i}]^{i,j} = [[A^{i,j}, f(A^{i,i})], I_n^{i,i}]^{i,j} = f(A^{i,i})^{i,i}A^{i,j}.$$

Applying Lemma 3.3, there exist $X^{i,i} \in M_n^{i,i}$ such that

$$\begin{aligned} f(A^{i,i})^{i,i} &= X^{i,i}A^{i,i} \quad \text{for all } A^{i,i} \in \mathcal{N}^{i,i}, \\ f(A^{i,j})^{i,j} &= X^{i,i}A^{i,j} \quad \text{for all } A^{i,j} \in \mathcal{N}^{i,j}. \end{aligned} \tag{29}$$

Since f is a Lie triple centralizer, for any $1 \le i < j \le t$, $I_n^{i,i}, A^{i,i} \in \mathcal{N}^{i,i}$ and $A^{i,j} \in \mathcal{N}^{i,j}$,

$$f(A^{i,i}A^{i,j})^{i,j} = f[[A^{i,j}, A^{i,i}], I_n^{i,i}]^{i,j} = [[f(A^{i,j}), A^{i,i}], I_n^{i,i}]^{i,j} = A^{i,i}f(A^{i,j})^{i,j}.$$

Applying Lemma 3.2, there exist $X^{j,j} \in M_n^{j,j}$ such that

$$f(A^{i,j})^{i,j} = A^{i,j}X^{j,j} \quad \text{for all } A^{i,j} \in \mathcal{N}^{i,j}, \tag{30}$$

By (29) and (30), we have

$$X^{i,i}A^{i,j} = A^{i,j}X^{j,j} \quad \text{for } A^{i,j} \in \mathcal{N}^{i,j}.$$

Applying Lemma 3.1, there exists $\lambda \in \mathbb{F}$ such that $X^{i,i} = \lambda I^{i,i}$ and $X^{j,j} = \lambda I^{j,j}$.

Therefore, $f(A^{i,i})^{i,i} = \lambda A^{i,i}$ for $A^{i,i} \in \mathcal{N}^{i,i}$ and $f(A^{i,j})^{i,j} = \lambda A^{i,j}$ for $A^{i,j} \in \mathcal{N}^{i,j}$, $i < j$. ■

4.2. Proof of Theorem 2.2

We are ready to prove our main result.

Proof of Theorem 2.2: By Lemma 4.8, there exists $\lambda \in \mathbb{F}$ such that

$$f(A^{i,j})^{i,j} = \lambda A^{i,j} \quad \text{for } A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}.$$

Define $f_0 := f - \lambda I_n$. Thus, f_0 is a Lie triple centralizer. Then (28) implies that $f_0(A^{i,j})^{i,j} = 0$ for all $A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}$. By Lemmas 4.1 and 4.2, for any $\mathcal{N}^{i,i} \subseteq \mathcal{N}$, the possibly non zero blocks of $f_0(\mathcal{N}^{i,i})$ are the following:

- $(1,t)$ block when $\mathcal{N}^{1,1} = \mathcal{N}^{t,t} = 0$,
- $(2,t)$ block when $(i,i) = (1,1)$, and
- $(1,t-1)$ block when $(i,i) = (t,t)$.

and by Lemmas 4.3, 4.6, and 4.7, for any $\mathcal{N}^{i,j} \subseteq \mathcal{N}$, $i<j$, the only possibly non zero block of $f_0(\mathcal{N}^{i,j})$ is the $(1,t)$ block when $(i,j) \in \{(1,2),(t-1,t)\}$.

Define $\phi_1, \phi_2 \in \operatorname{End}\mathcal{N}$ such that for $A \in \mathcal{N}$,

$$\begin{aligned}
\phi_1(A) &:= f_0(A^{1,1})^{2,t} + f_0(A^{1,2})^{1,t} = f(A^{1,1})^{2,t} + f(A^{1,2})^{1,t}\\
\phi_2(A) &:= f_0(A^{t,t})^{1,t-1} + f_0(A^{t-1,t})^{1,t} = f(A^{t,t})^{1,t-1} + f(A^{t-1,t})^{1,t}.
\end{aligned}$$

Then Lemmas 4.4 and 4.5 show that ϕ_1 and ϕ_2 are Lie triple centralizers. Define $f_1 := f_0 - \phi_1 - \phi_2 = (f - \lambda I_n) - \phi_1 - \phi_2$. Thus, f_1 is a Lie triple centralizer, and only the possible non-zero block of $f_1(\mathcal{N}^{i,i})$ is the $(1,t)$ block when $\mathcal{N}^{11} = \mathcal{N}^{tt} = 0$ and also $f_1(A^{i,j}) = 0$ for $i<j$.

Define a linear map $\delta : \mathcal{N} \to \mathcal{N}$ such that for $A \in \mathcal{N}$,

$$\delta(A) := \sum_{(i,i)\in\Gamma_{\mathcal{N}}} f_1(A^{i,i})^{1,t} = \sum_{(i,i)\in\Gamma_{\mathcal{N}}} f(A^{i,i})^{1,t}.$$

Then $\delta(A) \in Z(\mathcal{N})$ and $\delta[[\mathcal{N},\mathcal{N}],\mathcal{N}] = 0$. Thus, δ is a Lie triple centralizer. Now we get a new Lie triple centralizer

$$f_2 := f_1 - \delta = (f - \lambda I_n) - \phi_1 - \phi_2,$$

where $f_2(A^{i,j}) = 0$ for $A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}$. Therefore,

$$f(A) = \lambda A + \phi_1(A) + \phi_2(A) + \delta(A)$$

for $A \in \mathcal{N}$. Hence Theorem 2.2 is proved. ∎

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