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Model reference adaptive control for nonlinear time-varying hybrid dynamical systems

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Summary

This paper presents the first model reference adaptive control system for nonlinear, time-varying, hybrid dynamical plants affected by matched and parametric uncertainties, whose resetting events are unknown functions of time and the plant's state. In addition to a control law and an adaptive law, which resemble those of the classical model reference adaptive control framework for continuous-time dynamical systems, the proposed framework allows imposing instantaneous variations in the reference model's trajectory to rapidly steer the trajectory tracking error to zero, while retaining the closed-loop system's ability to follow a user-defined signal. These results are enabled by the first extension of the classical LaSalle–Yoshizawa theorem to time-varying hybrid dynamical systems, which is presented in this paper as well. A numerical simulation shows the key features of the proposed adaptive control system and highlights its ability to reduce both the control effort and the trajectory tracking error over a classical model reference adaptive control system applied to the same problem.

KEYWORDS

hybrid dynamical systems, LaSalle–Yoshizawa theorem, model reference adaptive control

1 | INTRODUCTION

This paper presents the first model reference adaptive control system for nonlinear, time-varying, hybrid plants affected by parametric and matched uncertainties. Similarly to the classical model reference adaptive control framework for continuous-time plants, the proposed control system applies to plants, whose nonlinearities and matched uncertainties are captured by the product of an unknown constant matrix by a user-defined regressor vector, which is an explicit function of the plant state and time, and both these terms may vary instantaneously whenever some resetting event occurs. The proposed control system also applies to plants, whose linearized uncontrolled dynamics are captured by an unknown constant matrix, and whose entries may vary instantaneously due to resetting events. Additionally, the plant state may experience instantaneous variations whenever some resetting event occurs. Finally, the resetting events affecting the plant state and the plant dynamics are not controllable and may occur whenever some unknown conditions on time and the plant's state are met. For these reasons, the proposed control system applies to broad classes of dynamical systems, including mechanical systems, which are affected by uncertainties, discontinuities in their dynamical models, as it occurs, for instance, in the transitions between static and and dynamic friction models, and discontinuities in their state, as it occurs, for instance, due to elastic collisions.

Similarly to classical model reference adaptive control systems, the proposed control system allows to steer the plant trajectory toward the trajectory of a user-defined, linear, hybrid reference model that captures the closed-loop system's ideal behavior. In particular, the proposed model reference adaptive control laws guarantee uniform boundedness of both the trajectory tracking error and the adaptive gains and asymptotic convergence of the trajectory tracking error to zero uniformly both in the initial time and any sequence of resetting times. A unique feature of the proposed adaptive control system is that the hybrid nature of the closed-loop system is leveraged to define resetting events for the reference model that instantaneously reduce the trajectory tracking error. The resetting events in the reference model, however, are designed not to disrupt the trajectory tracking error's asymptotic convergence to zero, and, hence, retain the ability of the closed-loop system to steer the plant trajectory toward a user-defined signal. Additionally, the proposed adaptive laws are continuous functions of time, which eases their implementations in problems of practical interest.

The proposed adaptive system has been deduced leveraging an extension of the LaSalle–Yoshizawa theorem to nonlinear, time-varying, hybrid dynamical systems. To the author's knowledge, this is the first extension of the classical LaSalle–Yoshizawa theorem[1(Th. 4.7)] to such a class of systems; existing works apply to nonlinear, time-invariant, hybrid dynamical systems[2] or time-varying, switched dynamical systems.[3] Similarly to the classical LaSalle–Yoshizawa theorem, the proposed extension provides sufficient conditions on the asymptotic convergence of the system's trajectory to the set of roots of a nonnegative-definite function, which serves as an upper bound on the total time derivative of a Lyapunov-like function along the system's trajectories. Despite the classical LaSalle–Yoshizawa theorem, which applies to continuous-time dynamical systems, the proposed result requires some boundedness conditions on the variations of the Lyapunov-like function across resetting events.

As recently discussed in Reference 4, the problem of controlling hybrid dynamical systems using an adaptive control framework has been addressed by a handful of authors. Among existing results on the adaptive control of hybrid dynamical systems, it is worthwhile recalling Reference 5, where the authors consider time-invariant dynamical systems and rely on the user's ability to find a Lyapunov-like function to construct their adaptive controller. Worthy of mention are also References 4 and 6, where the authors consider time-invariant dynamical systems and require stronger assumptions on the functional shape of the parametric uncertainties than those considered herein. In Reference 7, the authors presented a model reference adaptive control system for a subclass of hybrid plants, namely switched plants, that is, plants whose dynamics may change instantaneously and whose trajectory is a continuous function of time. The results presented in this paper differ from those presented in Reference 7 for multiple reasons. Firstly, in this paper we address a broader class of of plants, namely hybrid plants. Secondly, this work does not consider common Lyapunov functions, whereas in Reference 7 a set of linear matrix inequalities need to be solved to prove the existence of a common Lyapunov function that certifies the effectiveness of the adaptive control laws. Additionally, in this paper, we consider Krasovskii solutions, whereas in Reference 7, Carathéodory and Filippov solutions are considered. Thus, in the case of switched dynamical plants, the results presented in this paper provide an alternative to those presented in Reference 7. However, Carathéodory and Filippov solutions of differential inclusions are unable to capture instantaneous variations in the plant state due to resetting events. In the absence of instantaneous variations in both the plant's dynamical model and the plant state, the proposed control system reduces to the classical model reference adaptive control[8(Ch. 9)].

In this work, the use of a common Lyapunov function is avoided by leveraging user-defined convergence series to design the reference model's switching law and, hence, indirectly set the trajectory tracking error's dwell time. In this regard, the proposed work extends the results presented in Reference 9, where the authors presented the first model reference adaptive control system that does not leverage a common Lyapunov function, but a condition on the dwell time of the trajectory tracking error dynamics.

The result presented in this paper also provide an extension of the results shown in Reference 10, where the authors address the model reference adaptive control problem for linear, switched plants and switched reference models. Similarly to the present work, in Reference 10, the underlying Lyapunov function experiences instantaneous jumps at each point of discontinuity of the trajectory tracking error dynamics. However, the results in Reference 10 rely on the persistency of excitation of the control input to ensure boundedness of the variations in the Lyapunov function, whereas in this work, the reference model's switching law guarantees boundedness of the variations in the Lyapunov function across discontinuities; in general, persistently exciting control inputs tend to stress actuators and their use should be limited.

Finally, worthy of mention are the results discussed in References 11-13, where the model reference adaptive control strategy is discussed for switched, affine plants following switched linear reference models. The authors in References 11 and 12 employ a direct framework to regulate plants with a scalar control input. To this goal, they do not impose any constraints on the plant's or the reference model's switching law, and leverage a common Lyapunov function and some passivity assumptions on the reference model to prove effectiveness of their framework. The proposed work does not rely on passivity-based assumptions and on the use of common Lyapunov function but, as already discussed, leverages a user-defined resetting law based on a convergent series. The authors in Reference 13 propose both a direct and an indirect model reference adaptive control framework to control switched linear and affine plants controlled by multi-variable control inputs. Under the assumption of persistently exciting reference command inputs, the results in Reference 13 guarantee asymptotic convergence of the adaptive gains to the plant's counterpart; in this paper, no assumption is made on the reference command input since estimating the plant parameters is not a scope of this work. An extension of the proposed results to an indirect model reference adaptive control framework will be addressed in future works.

An adaptive system for the parameter estimation of linear, hybrid output maps affected by matched uncertainties is presented in Reference 14. Future work directions involve the extension of the proposed direct model reference adaptive control framework to an indirect framework. In that context, the results presented in Reference 14 will be useful to enable the parameter estimation mechanisms.

This paper is organized as follows. In Section 2, we present the mathematical notation used in this paper, and in Section 3 we present essential preliminary results on hybrid systems theory. Successively, in Section 4, we present an extension of the LaSalle–Yoshizawa theorem to nonlinear, time-varying, hybrid dynamical systems, and, in Section 5, we leverage this result to propose a model reference adaptive control system for nonlinear, time-varying hybrid plants affected by matched and parametric uncertainties. Section 6 presents a numerical example that illustrates the applicability and the advantages of the proposed control framework. This example shows how, in the presence of resetting events in the plant dynamics, the proposed model reference adaptive control system provides a smaller trajectory tacking error and requires a smaller control effort than a classical model reference adaptive control system. Finally, Section 7 draws conclusions and outlines future research directions.

2 | NOTATION

Let $\mathbb{N}$ denote the *set of positive integers*, $\overline{\mathbb{N}}$ denote the *set of nonnegative integers*, $\mathbb{R}$ the *set of real numbers*, $\mathbb{R}^n$ the *set of $n \times 1$ real column vectors*, and $\mathbb{R}^{n\times m}$ the *set of $n \times m$ real matrices*. The *boundary of $\mathcal{D} \subset \mathbb{R}^n$* is denoted by $\partial\mathcal{D}$, and the *closure of $\mathcal{D}$* is denoted by $\overline{\mathcal{D}}$.

The *open ball of radius $\rho > 0$ centered at $x \in \mathbb{R}^n$* is denoted by $\mathcal{B}_\rho(x)$. Given $\rho > 0$ and the bounded set $\mathcal{A} \subset \mathbb{R}^n$, let $\mathcal{B}_\rho(\mathcal{A}) \triangleq \cup_{x\in\mathcal{A}}\mathcal{B}_\rho(x)$ denote the union of all open balls of radius ρ centered at the points of $\mathcal{A}$.

The Lebesgue measure of the set $\mathcal{D}$ is denoted by $\mu(\mathcal{D})$. A property $\mathfrak{P}$ is verified *almost everywhere* with respect to the Lebesgue measure $\mu(\cdot)$ on a set $\mathcal{X} \subseteq \mathbb{R}^n$ if there exists $\mathcal{N} \subset \mathcal{X}$ such that $\mu(\mathcal{N}) = 0$ and $\mathfrak{P}$ is verified by all $x \in \mathcal{X}\setminus\mathcal{N}$. In this case, we write "$\mathfrak{P}$ is verified for $x \in \mathcal{X}$ a.e.." The *indicator function* of the set $\mathcal{A} \subset \mathbb{R}^n$ is denoted by $\chi_\mathcal{A} : \mathcal{A} \to \{0, 1\}$ and is defined so that if $x \in \mathcal{A}$, then $\chi_\mathcal{A}(x) = 1$, and if $x \notin \mathcal{A}$, then $\chi_\mathcal{A}(x) = 0$. Integrals are always meant in the sense of Lebesgue. The *transpose of $B \in \mathbb{R}^{n\times m}$* is denoted by B^T.

The *zero vector* in $\mathbb{R}^n$ is denoted by 0_n, the *zero $n \times m$ matrix* in $\mathbb{R}^{n\times m}$ is denoted by $0_{n\times m}$, the *identity matrix* in $\mathbb{R}^{n\times n}$ is denoted by $\mathbf{1}_n$, and the *ith element of the canonical basis of* $\mathbb{R}^n$ is denoted by $\mathbf{e}_{i,n} \triangleq [0, \ldots, 1, \ldots, 0]^\mathrm{T}$; for instance, $\mathbf{e}_{1,n} = [1, 0, \ldots, 0]^\mathrm{T}$, $\mathbf{e}_{2,n} = [0, 1, 0, \ldots, 0]^\mathrm{T}$, and $\mathbf{e}_{n,n} = [0, \ldots, 0, 1]^\mathrm{T}$.

We write $\|\cdot\|$ for the *Euclidean vector norm* and the corresponding *equi-induced matrix norm*[15(Def. 9.4.1)]. The distance between $x \in \mathbb{R}^n$ and the set $\mathcal{A} \subset \mathbb{R}^n$ is defined as $\|x\|_\mathcal{A} \triangleq \inf_{a\in\mathcal{A}} \|x - a\|$[16(Def. 3.5)].

3 | FUNDAMENTALS OF HYBRID SYSTEMS THEORY

In this paper, we consider nonlinear, time-varying, hybrid dynamical systems, that is, nonlinear time-varying dynamical system that experience instantaneous changes both in their trajectory and in their dynamics whenever resetting events occur. Such systems are captured by a set of differential and difference equations in the form

$$\dot{x}(t) = f_\mathrm{c}(t, x(t)), \qquad (t, x(t)) \notin \mathcal{D}, \tag{1}$$

$$x(t^+) = g_\mathrm{d}(t, x(t)), \qquad (t, x(t)) \in \mathcal{D}, \tag{2}$$

with $x(t_0) = x_0$.

The subscript c denotes quantities related to the system's continuous-time dynamics (1) between resetting events and the subscript d denotes quantities related to the system's discrete-time dynamics (2) at resetting events. The *initial time* is denoted by $t_0 \in [0, \infty)$. The open set, where the solutions of (1) and (2) are defined, is denoted by $\mathcal{S} \subseteq \mathbb{R}^n$, and we assume that $0 \in \mathcal{S}$. The *vector field*, also known as the *flow map*, $f_c : [t_0, \infty) \times \mathcal{S} \to \mathbb{R}^n$ is Lebesgue integrable, locally bounded, and such that $f_c(t, 0_n) = 0_n$ for all $t \in [t_0, \infty)$. The *switching law*, also known as the *jump map*, $g_d : [t_0, \infty) \times \mathcal{S} \to \mathbb{R}^n$ is continuous in its arguments and locally bounded. The *set of resetting events* $\mathcal{D} \subset [t_0, \infty) \times (\mathcal{S} \setminus \{0\})$ is also known as the *jump set*.

The *flow* of solutions of (1) and (2) is denoted by $s_c : [t_0, \infty) \times [0, \infty) \times \mathcal{S} \to \mathcal{S}$. Furthermore, the *resetting time before* $t \geq t_0$ is defined iteratively as $t_1 \triangleq \min\{t \geq t_0 : (t, s_c(t, t_0, x_0)) \notin \mathcal{D}\}$ and $t_k \triangleq \min\{t > t_{k-1} : (t, s_c(t, t_{k-1}, x_{k-1})) \notin \mathcal{D}\}$ for all $k \in \mathbb{N} \setminus \{1\}$. We assume that the system (1) and (2) is *left-continuous*, that is, the following three conditions are verified:

$$\lim_{\tau \to t^-} x(\tau) = x(t), \qquad t \in (t_0, \infty); \tag{3}$$

if $(t_0, x_0) \notin \mathcal{D}$, then, for all $t \in [t_0, \infty) \setminus \bigcup_{k \in \mathbb{N}} \{t_k\}$, it holds that

$$\lim_{\tau \to t^+} x(\tau) = x(t), \tag{4}$$

and, for all $k \in \mathbb{N}$, it holds that

$$x(t_k^+) = \lim_{\tau \to t_k^+} x(\tau) = g_d(t_k, x_k); \tag{5}$$

and if $(t_0, x_0) \in \mathcal{D}$, then (4) is verified for all $t \in (t_0, \infty) \setminus \bigcup_{k \in \mathbb{N}} \{t_k\}$ and (5) is verified for all $k \in \overline{\mathbb{N}}$. In the remainder of this paper, for brevity, we assume that $(t_0, x_0) \notin \mathcal{D}$; the case whereby $(t_0, x_0) \in \mathcal{D}$ is not addressed explicitly and can be deduced from the arguments provided. Several authors, such as, for instance, those in References 17-19 to name a few, assume that (1) and (2) is right-continuous, that is, (2) is rewritten as

$$x(t) = g_d(t, x(t)), \qquad (t^-, x(t^-)) \in \mathcal{D},$$

and $\lim_{\tau \to t^+} x(\tau) = x(t)$ for all $t \in [t_0, \infty)$; $\lim_{\tau \to t^-} x(\tau) = x(t)$ for all $t \in (t_0, \infty) \setminus \bigcup_{k \in \mathbb{N}} \{t_k\}$; and, for all $k \in \mathbb{N}$, $\lim_{\tau \to t_k^-} x(t_k) = g_d(t_k, x_k)$. The results provided in this paper can be seamlessly modified to match these assumptions.

In several problems of practical interest, such as those discussed in this paper, at each resetting event, the changes in the system's dynamics are tracked by a discrete variable $\sigma \in \Sigma \subseteq \overline{\mathbb{N}}$ known as *mode*. In this case, the hybrid systems system's dynamics is captured by

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} \hat{f}_{c,\sigma(t)}(t, y(t)) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} y(t_0) \\ \sigma(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ \sigma_0 \end{bmatrix}, \qquad (t, y(t)) \notin \mathcal{D}_{\sigma(t)}, \tag{6}$$

$$\begin{bmatrix} y(t^+) \\ \sigma(t^+) \end{bmatrix} = \hat{g}_{d,\sigma(t)}(t, y(t)), \qquad (t, y(t)) \in \mathcal{D}_{\sigma(t)}, \tag{7}$$

where $f_{c,\sigma} : [t_0, \infty) \times \widetilde{\mathcal{S}} \to \mathbb{R}^n$, $\sigma \in \Sigma$, is Lebesgue integrable, locally bounded, and such that $f_{c,\sigma}(t, 0_n) = 0_n$ for all $t \in [t_0, \infty)$, $\hat{g}_{d,\sigma} : [t_0, \infty) \times \widetilde{\mathcal{S}} \to \mathbb{R}^n$ is continuous in its arguments and locally bounded for all $\sigma \in \Sigma$, and $\widetilde{\mathcal{S}} \subseteq \mathbb{R}^{n-1}$ is an open set. As highlighted in Reference 17, (6) and (7) can be reduced to the same form as (1) and (2) with $\mathcal{S} = \widetilde{\mathcal{S}} \times \Sigma$, $x = \left[y^{\mathrm{T}}, \sigma\right]^{\mathrm{T}}$, $f_c(t, x) = \left[\hat{f}_{c,\sigma}^{\mathrm{T}}(t, y), 0\right]^{\mathrm{T}}$, $g_d(t, x) = \hat{g}_{d,\sigma}(t, y)$, and $\mathcal{D} = \bigcup_{\sigma \in \Sigma} (\mathcal{D}_\sigma \times \{\sigma\})$.

Next, we introduce the notion of *Krasovskii solution of* (1) and (2), which is derived from Reference 17. For the statement of this definition, let $\mathcal{I} \subseteq [t_0, \infty)$ be connected and such that $t_0 \in \mathcal{I}$ and $\mathcal{T} \triangleq (\mathcal{I} \times \mathcal{S}) \setminus \mathcal{D}$.

Definition 1. Assume that $x : \mathcal{I} \to \mathcal{S}$ is piecewise absolutely continuous, has a finite number of discontinuities on any compact subinterval of $\mathcal{I}$, and is such that if $(t, x(t)) \in \overline{\mathcal{T}}$, then

$$\dot{x}(t) \in K[f_c](t, x(t)), \tag{8}$$

and if $(t, x(t)) \in \overline{\mathcal{D}}$, then

$$x(t^+) \in K[g_d](t, x(t)), \tag{9}$$

where

$$K[f_{\rm c}](t,x) \triangleq \bigcap_{\delta>0} \overline{\rm co}\left(f_{\rm c}\left(\overline{\mathcal{B}}_\delta\left(\begin{bmatrix} t \\ x \end{bmatrix}\right) \cap \mathcal{T}\right)\right), \quad (t,x) \in \overline{\mathcal{T}}, \tag{10}$$

$$K[g_{\rm d}](t,x) \triangleq \bigcap_{\delta>0} g_{\rm d}\left(\overline{\mathcal{B}}_\delta\left(\begin{bmatrix} t \\ x \end{bmatrix}\right) \cap \mathcal{D}\right), \quad (t,x) \in \overline{\mathcal{D}}, \tag{11}$$

denote the *Krasovskii regularizations of* (1) and (2), respectively, and $\overline{\rm co}(\cdot)$ denotes the convex closure of its argument. Then, $x(\cdot)$ is a *Krasovskii solution of* (1) and (2). If there do not exist a connected set $\mathcal{J} \subseteq [t_0, \infty)$ and a Krasovskii solution $\overline{x} : \mathcal{J} \to \mathcal{S}$ of (1) and (2) such that $\mathcal{I} \subset \mathcal{J}$ and $\overline{x}(t) = x(t)$, $t \in \mathcal{I}$, then $x : \mathcal{I} \to \mathcal{S}$ is a *maximal Krasovskii solution* of (1) and (2). If $\mathcal{I} = [t_0, \infty)$, then a Krasovskii solution $x : \mathcal{I} \to \mathcal{S}$ of (1) and (2) is *complete*.

In this paper, it is assumed that the points of discontinuities of Krasovskii solutions of (1) and (2) occur at the resetting times only. The existence and uniqueness of Krasovskii solutions of hybrid dynamical systems is discussed in Reference 17. For dynamical systems in the same form as (1) and (2), it can be shown that, for all $x_0 \in \mathcal{S}$, there exists a non-trivial Krasovskii solution. Additionally, every maximal solution is either defined on $\mathcal{I} = [t_0, \infty)$ or $x(\cdot)$ leaves every compact subset of $\mathcal{S}$ in finite time. An alternative notion of solution of (1) and (2) is the Hermes solution[16(Ch. 4)]; for additional details about this notion and its relation with Krasovskii solutions, see References 17 and 16(p. 75). In the following, solutions of (1) and (2) are always meant in the sense of Krasovskii. In the reminder of this paper, the following two assumptions are made on solutions of (1) and (2).

Assumption 1. If $(t, x(t)) \in \overline{\mathcal{D}} \setminus \mathcal{D}$, then there exists $\varepsilon > 0$ such that, for all $\delta \in (0, \varepsilon)$, $s_{\rm c}(t + \delta, t, x(t)) \notin \mathcal{D}$.

Assumption 2. If $(t_k, x(t_k)) \in \partial\mathcal{D} \cap \mathcal{D}$, then there exists $\varepsilon > 0$ such that, for all $\delta \in (0, \varepsilon)$, $s_{\rm c}(t_k + \delta, t_k, x(t_k^+)) \notin \mathcal{D}$.

As discussed in Reference 20(pp. 419–420), Assumption 1 guarantees that if a trajectory of (1) and (2) reaches the closure of $\mathcal{D}$ at a point that does not belong to $\mathcal{D}$, then the trajectory must move away from $\mathcal{D}$. Assumption 2 guarantees that if a trajectory reaches the boundary of $\mathcal{D}$ at a point that belongs to $\mathcal{D}$, then the trajectory moves away from any resetting event, and, hence, the continuous dynamics takes over for a non-trivial time interval. Thus, since piecewise absolutely continuous solutions of (1) and (2) are considered and Krasovskii solutions of (1) and (2) are discontinuous at resetting times only, Assumptions 1 and 2 imply that the solutions of (1) and (2) can not enter the interior of $\mathcal{D}$. Assumptions 1 and 2 do not prevent solutions of (1) and (2) from being Zeno, that is, from going through infinitely many resetting events in finite time. Additionally, Assumptions 1 and 2 do not prevent solutions of (1) and (2) to suffer from confluence, that is, from coinciding with other solutions after some point in time. However, Assumption 2 prevents solutions from beating, that is, from encountering the boundary of the same resetting event a finite or infinite number of times in zero time.

Assumptions 1 and 2 are targeted at the scopes of this work, namely, the design model reference adaptive control laws for plants, whose trajectory and dynamics experience instantaneous variations, and whose continuous-time dynamics resumes immediately after a resetting event. Future work directions involve the design of model reference adaptive control systems, whose controlled trajectory tracking error may intersect the jump set multiple consecutive times before the continuous-time dynamics is resumed; see Reference 16(Ch. 2) for a more general framework on hybrid systems theory.

Next, we recall the notions of directional derivatives, generalized directional derivatives, and regular functions. For the statement of these definitions, let $[z, z + a) \triangleq \{z + \theta a, \theta \in [0, 1)\}$, $(z, a) \in \mathbb{R}^l \times \mathbb{R}^l$, denote a *line segment in* $\mathbb{R}^l$ and let

$$\text{vcone}(\mathcal{R}, z) \triangleq \left\{\xi \in \mathbb{R}^l : \exists \alpha > 0 \text{ such that } [z, z + \alpha\xi) \subset \mathcal{R}\right\} \tag{12}$$

denote the *variational cone* of $\mathcal{R} \subseteq \mathbb{R}^l$ at z.

Definition 2 (21(pp. 63–64) and 22(p. 39)). Let $W : \mathcal{R} \to \mathbb{R}$ be Lipschitz continuous, where $\mathcal{R} \subseteq \mathbb{R}^l$. The *right directional derivative of* $W(\cdot)$ *at* $z \in \mathcal{R}$ *along the direction of* $q \in \text{vcone}(\mathcal{R}, z)$ is defined as

$$W'(z, \sigma) \triangleq \lim_{\tau \to 0^+} \frac{W(z + \tau q) - W(z)}{\tau}, \qquad (z, \sigma) \in \mathcal{R} \times \text{vcone}(\mathcal{R}, z). \tag{13}$$

The *generalized directional derivatives of* $W(\cdot)$ *at* $z \in \mathcal{R}$ *along the direction of* $q \in \mathrm{vcone}(\mathcal{R}, z)$ is defined as

$$W^0(z,\sigma) \triangleq \limsup_{\substack{y\to z\\ \tau\to 0^+}} \frac{W(y+\tau q)-W(y)}{\tau}, \qquad (z,\sigma)\in\mathcal{R}\times\mathrm{vcone}(\mathcal{R},z). \tag{14}$$

If $W'(z,\sigma) = W^0(z,\sigma)$ for all $q \in \mathrm{vcone}(\mathcal{R}, z)$, then $W(\cdot)$ is *regular at* $z \in \mathcal{R}$.

Finally, we recall the following generalization of Barbalat's lemma[23(Lemma 8.2)].

Lemma 1 (24). Let $h : [t_0, \infty) \to \mathbb{R}$ be piecewise continuously differentiable and let $\{t_k\}_{k\in\mathbb{N}} \subset [t_0, \infty)$ denote the sequence of points of discontinuity of $h(\cdot)$. Suppose that $\inf_{k\in\mathbb{N}} |t_k - t_{k-1}| > 0$ and that both $h(\cdot)$ and $\dot{h}(\cdot)$ are bounded on $(t_{k-1}, t_k]$ uniformly in $\{t_k\}_{k\in\mathbb{N}}$. If $\lim_{t\to\infty} \int_0^t h(\tau)\mathrm{d}\tau$ exists and is finite, then $\lim_{t\to\infty} h(t) = 0$ uniformly in $\{t_k\}_{k\in\mathbb{N}}$.

4 | LASALLE–YOSHIZAWA CONDITIONS FOR HYBRID DYNAMICAL SYSTEMS

In this section, we present a result that, to the author's knowledge, is the first generalization of the LaSalle–Yoshizawa theorem to nonlinear, time-varying, hybdrid dynamical systems in the same form as (1) and (2). For the statement of this result, let $\mathcal{I} \subseteq [t_0, \infty)$ be connected and such that $t_0 \in \mathcal{I}$, and $x : \mathcal{I} \to \mathcal{S}$ be a solution of (1) and (2). Furthermore, let $V : \mathcal{I} \times \mathcal{S} \to \mathbb{R}$ be absolutely continuous in its first argument over compact intervals of $\mathcal{I}$ that do not contain resetting times in their interior for each $x \in \mathcal{S}$ and Lipschitz continuous and regular in the second argument for each $t \in \mathcal{I}$. Additionally, let $W : \mathcal{S} \to \mathbb{R}$ be absolutely continuous and nonnegative definite, and define $\bar{t}_k \in \mathcal{I}$, $k \in \overline{\mathbb{N}}$, such that $\bar{t}_0 = t_0$, $\bar{t}_1 = t_1$, if $\sum_{j=1}^{k-1} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right] > 0$, $k \in \mathbb{N} \setminus \{1\}$, along a solution of (1) and (2), then

$$\bar{t}_k = \inf\left\{ t \in \mathcal{I} : \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau \geq \sum_{j=1}^{k-1} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right] \right\}, \tag{15}$$

and if $\sum_{j=1}^{k-1} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right] \leq 0$, then $\bar{t}_k = t_1$. Finally, define

$$\hat{t}_k \triangleq \max\left\{ t_k, \bar{t}_k \right\}, \tag{16}$$

and consider the following preliminary result.

Lemma 2. Let $\mathcal{I} \subseteq [t_0, \infty)$ be connected and such that $t_0 \in \mathcal{I}$. Let $x : \mathcal{I} \to \mathcal{S}$ be a solution of (1) and (2), let $V : \mathcal{I} \times \mathcal{S} \to \mathbb{R}$ be absolutely continuous in its first argument over compact intervals of $\mathcal{I}$ that do not contain resetting times in their interior for each $x \in \mathcal{S}$ and Lipschitz continuous and regular in the second argument for each $t \in \mathcal{I}$, and let $W : \mathcal{S} \to \mathbb{R}$ be continuously differentiable and nonnegative definite. If $\hat{t}_k < t_{k+1}$ for each $k \in \mathbb{N}$ such that $\hat{t}_k \in \mathcal{I}$ and

$$\dot{V}(t, x(t)) \leq -W(x(t)), \qquad t \in \mathcal{I} \text{ a.e.}, \tag{17}$$

then, for each $k \in \overline{\mathbb{N}}$ such that $\hat{t}_k \in \mathcal{I}$, it holds that

$$V(t_k^+, x(t_k^+)) - V(t_k, x(t_k)) + V(t_0, x_0) \geq V(t, x(t)), \qquad t \in \left(\hat{t}_k, t_{k+1}\right] \cap \mathcal{I}. \tag{18}$$

Lemma 2, whose proof is provided in the Appendix, shows that if two consecutive resetting events are sufficiently apart from each other and the total time derivative of the function $V(\cdot, x(\cdot))$ along solutions of (1) and (2) is bounded from above by a piecewise absolutely continuous nonnegative-definite function, then $V(\cdot, x(\cdot))$ is bounded from above by a constant over a subset of the real line. In particular, if the first $k-1$ resetting events, $k \in \mathbb{N} \setminus \{1\}$, contribute to decreasing the value of $V(\cdot, x(\cdot))$ at resetting times between t_0 and t_{k-1}, that is, if $\sum_{j=1}^{k-1}\left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right] < 0$, then it follows from Lemma 2 that $V(t, x(t))$ is bounded for all $t \in (t_k, t_{k+1}] \cap \mathcal{I}$, that is, (18) is verified with $\hat{t}_k = t_k$. Alternatively, if the combined effect of the first $k-1$ resetting events, $k \in \mathbb{N} \setminus \{1\}$, on the value of $V(\cdot, x(\cdot))$ is positive but sufficiently small, that is, if $\bar{t}_k < t_{t+1}$ and $\int_{t_0}^{\bar{t}_k} W(x(\tau))\mathrm{d}\tau \geq \sum_{j=1}^{k-1}\left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right]$, then $V(t, x(t))$ is bounded for all $t \in \left((\bar{t}_k, t_{k+1}] \cap \mathcal{I}\right) \subset ((t_k, t_{k+1}] \cap \mathcal{I})$, that is, (18) is verified with $\hat{t}_k = \bar{t}_k$.

In the absence of discontinuities in $x(\cdot)$, Lemma 2 implies that if $\mathcal{I} = [t_0, \infty)$ and $\dot{V}(t, x(t)) < -W(x(t))$, $t \in [t_0, \infty)$ a.e., then, $V(t_0, x_0) \geq V(t, x(t))$; this result was proven as Lemma 2 of Reference 3. In light of this result, we also note that $V(t_k^+, x(t_k^+)) \geq V(t, x(t))$ for all $t \in (t_k, t_{k+1}] \cap \mathcal{I}$ and for each $k \in \mathbb{N}$ along the trajectories of (1) and (2). Next, we present the first main result of this section.

Theorem 1. Consider the nonlinear, time-varying, hybrid dynamical system given by (1) and (2), and assume that all maximal solutions of (1) and (2) are defined over $[t_0, T_{\max}) \subseteq [t_0, \infty)$. Let $\overline{\mathcal{A}} \subset \mathcal{S}$ be compact and such that $\left(\{t\} \times \overline{\mathcal{A}}\right) \not\subset \mathcal{D}$ for all $t \in [t_0, T_{\max})$. Let $V : [t_0, \infty) \times \mathcal{S} \to \mathbb{R}$ be absolutely continuous in its first argument over compact intervals of $[t_0, \infty)$ that do not contain resetting times in their interior for each $x \in \mathcal{S}$ and Lipschitz continuous and regular in the second argument for each $t \in [t_0, \infty)$. Assume that $\hat{t}_k \leq t_k$ for all $k \in \mathcal{N}$, where $0 \in \mathcal{N} \subseteq \overline{\mathbb{N}}$ and $\{t_k\}_{k \in \mathcal{N}} \subset [t_0, T_{\max})$, $\sum_{k \in \mathcal{N}}\left[V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))\right]$ exists and is finite. Assume that

$$W_1(x) \leq V(t, x) \leq W_2(x), \qquad (t, x) \in [t_0, \infty) \times (\mathcal{S} \setminus \mathcal{A}), \tag{19}$$

$$\dot{V}(t, x) \leq -W(x), \qquad (t, x) \notin \mathcal{D}, \tag{20}$$

where $W_1, W_2 : \mathcal{S} \setminus \mathcal{A} \to \mathbb{R}$ are such that $W_1(x) = W_2(x) = 0$ for all $x \in \partial\mathcal{A}$ and $W_1(x) > 0$ and $W_2(x) > 0$ for all $x \in \mathcal{S} \setminus \overline{\mathcal{A}}$, and $W : \mathcal{S} \to \mathbb{R}$ is continuously differentiable, nonnegative-definite, and such that $\overline{\mathcal{A}} \subset \{x \in \mathcal{S} : W(x) = 0\}$. Let $r > 0$ and $c > 0$ be such that $\mathcal{B}_r(\overline{\mathcal{A}}) \subset \mathcal{S}$ and $c < \min_{x \in \partial\mathcal{B}_r(\overline{\mathcal{A}})} W_1(x)$. Then, there exists $\mathcal{X}_0 \subset \mathbb{R}^n$ such that $\overline{\mathcal{A}} \subset \mathcal{X}_0 \subseteq \{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_2(x) \leq c\}$ such that if $x(t_0) \in \mathcal{X}_0$, then every maximal solution $x(t)$, $t \in [t_0, T_{\max})$, of (1) and (2) is bounded uniformly in both $t_0 \in [0, T_{\max})$ and $\{t_k\}_{k \in \mathcal{N}}$ and such that $\lim_{t \to T_{\max}} W(x(t)) = 0$ uniformly in both $t_0 \in [0, T_{\max})$ and $\{t_k\}_{k \in \mathcal{N}}$.

Theorem 1, whose proof is provided in the Appendix, provides sufficient conditions for the uniform pre-attractivity of $\{x \in \mathcal{S} : W(x) = 0\}$ for solutions of (1) and (2)[16(Def. 3.6)]. This result requires that, outside the compact set $\overline{\mathcal{A}}$, a Lyapunov-like function $V(\cdot, \cdot)$ is comprised between two positive-definite functions, namely, $W_1(\cdot)$ and $W_2(\cdot)$, and its total time derivative is bounded from above by a non-positive-definite function, namely $-W(x(\cdot))$, along the system's trajectories. To account for discontinuities in the solutions of (1) and (2), Theorem 2 requires two additional conditions. The first of these conditions is that $\hat{t}_k \leq t_k$, $k \in \mathbb{N}$, so that, by Lemma 2, $V(\cdot, x(\cdot))$ is bounded almost everywhere on $[t_0, \infty)$ along the solutions of (1) and (2). The second of these conditions is that the series of the variations of the Lyapunov-like function $V(\cdot, x(\cdot))$ along the solutions of (1) and (2) across resetting events, that is, $\sum_{j=1}^{\infty}[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))]$, exist and is finite. As shown by (A1) in the Appendix, together with (20), this condition is sufficient for $V(t, x(t))$ to be bounded for all $t \in [t_0, \infty)$.

In general, the set $\mathcal{X}_0$, wherein, according to Theorem 1, initial conditions of (1) and (2) must lie to assure pre-attractivity of $\{x \in \mathcal{S} : W(x) = 0\}$, can not be characterized without any additional assumption on $W(\cdot)$ such as, for instance, a characterization of $\mathcal{X}_0 \cap \{x \in \mathcal{S} : W(x) = 0\}$. The next theorem provides a stronger result than Theorem 1 under the assumption that the solutions of (1) and (2) are complete.

Theorem 2. Consider the nonlinear, time-varying, hybrid dynamical system given by (1) and (2), and assume that all solutions of (1) and (2) are complete. Let $\overline{\mathcal{A}} \subset \mathcal{S}$ be compact and such that $\left(\{t\} \times \overline{\mathcal{A}}\right) \not\subset \mathcal{D}$ for all $t \in [t_0, \infty)$. Let $V : [t_0, \infty) \times \mathcal{S} \to \mathbb{R}$ be absolutely continuous in its first argument over compact intervals of $[t_0, \infty)$ that do not contain resetting times in their interior for each $x \in \mathcal{S}$ and Lipschitz continuous and regular in the second argument for each $t \in [t_0, \infty)$. Assume that $\hat{t}_k \leq t_k$ for all $k \in \overline{\mathbb{N}}$, $\sum_{k=1}^{\infty}\left[V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))\right]$ exists and is finite, and (19) and (20) are verified. Let $r > 0$ and $c > 0$ be such that $\mathcal{B}_r(\overline{\mathcal{A}}) \subset \mathcal{S}$ and $c < \min_{x \in \partial\mathcal{B}_r(\overline{\mathcal{A}})} W_1(x)$. If $x(t_0) \in \{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_2(x) \leq c\}$, then every maximal solution $x(t)$, $t \geq t_0$, of (1) and (2) is bounded uniformly in both $t_0 \in [0, \infty)$ and $\{t_k\}_{k \in \mathbb{N}}$ and such that $\lim_{t \to \infty} W(x(t)) = 0$ uniformly in both $t_0 \in [0, \infty)$ and $\{t_k\}_{k \in \mathbb{N}}$. Furthermore, if $\mathcal{S} = \mathbb{R}^n$ and both $W_1(\cdot)$ and $W_2(\cdot)$ are radially unbounded, then every maximal solution $x(\cdot)$ of (1) and (2) is uniformly bounded in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$ and such that $\lim_{t\to\infty} W(x(t)) = 0$ for all $x_0 \in \mathbb{R}^n$ uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$.

Theorems 1 and 2 provide generalizations of the LaSalle–Yoshizawa theorem, since they allow to forecast the asymptotic behavior of the trajectory of nonlinear, time-varying, hybrid dynamical systems. In its original formulation, the LaSalle–Yoshizawa theorem provides sufficient conditions for the asymptotic convergence of a nonlinear dynamical system to the origin. To mirror this formulation, the next result specializes Theorem 2 to the case wherein $\overline{\mathcal{A}} = \{0\}$.

Theorem 3. Consider the nonlinear, time-varying, hybrid dynamical system given by (1) and (2), and assume that all maximal solutions of (1) and (2) are complete. Let $V : [t_0,\infty) \times \mathcal{S} \to \mathbb{R}$ be absolutely continuous in its first argument over compact intervals of $[t_0,\infty)$ that do not contain resetting times in their interior for each $x \in \mathcal{S}$ and Lipschitz continuous and regular in the second argument for each $t \in [t_0,\infty)$. Assume that $\hat{t}_k \le t_k$ for all $k \in \overline{\mathbb{N}}$, $\sum_{j=1}^{\infty}\left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right]$ exists and is finite, and

$$W_1(x) \le V(t,x) \le W_2(x), \qquad (t,x) \in [t_0,\infty) \times \mathcal{S}, \tag{21}$$

$$\dot{V}(t,x) \le -W(x), \qquad (t,x) \notin \mathcal{D}, \tag{22}$$

where $W_1, W_2 : \mathcal{S} \to \mathbb{R}$ are positive-definite and $W : \mathcal{S} \to \mathbb{R}$ is continuously differentiable and nonnegative-definite. Let $r > 0$ and $c > 0$ be such that $\mathcal{B}_r(0) \subset \mathcal{S}$ and $c < \min_{x\in\partial\mathcal{B}_r(0)} W_1(x)$. If $x(t_0) \in \{x \in \mathcal{B}_r(0) : W_2(x) \le c\}$, then every maximal solution $x(t)$, $t \ge t_0$, of (1) and (2) is bounded uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$ and such that $\lim_{t\to\infty} W(x(t)) = 0$ uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$. Furthermore, if $\mathcal{S} = \mathbb{R}^n$ and both $W_1(\cdot)$ and $W_2(\cdot)$ are radially unbounded, then every maximal solution $x(\cdot)$ of (1) and (2) is uniformly bounded in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$ and such that $\lim_{t\to\infty} W(x(t)) = 0$ for all $x_0 \in \mathbb{R}^n$ uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$.

In the case of switched dynamical systems, that is, dynamical systems in the same form as (6) and (7) with $[\mathbf{1}_n, 0_n]\, g_{\mathrm{d},\sigma}(t,y) = 0_n$, $(\sigma,t,y) \in \Sigma \times [t_0,\infty) \times \widetilde{\mathcal{S}}$, Filippov solutions of (1) and (2) can be employed. In this case, with relatively minor changes to the mechanism of their proofs, Lemma 2 reduces to Lemma 2 of Reference 3, and Theorem 3 reduces to Theorem 2 of Reference 7. Indeed, in the case of switched dynamical systems, the conditions whereby $\hat{t}_k \le t_k$, $k \in \mathbb{N}$, and $\sum_{j=1}^{\infty}\left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right]$ exists and is finite do not need to be verified. For additional details, see Remark 1 of Reference 3.

Future work directions involve multiple generalizations of Theorems 1 and 2. For instance, these theorems can be generalized by assuming that $\left([t_0,\infty) \times \overline{\mathcal{A}}\right) \cap \mathcal{D} \ne \{\emptyset\}$. Furthermore, along the lines of Theorem 3.17 of Reference 16 and Theorem 3.1 of Reference 25, Theorems 1 and 2 can be generalized by replacing the conditions whereby $\hat{t}_k \le t_k$, $k \in \mathbb{N}$, and $\sum_{j=1}^{\infty}[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))]$ exist and is finite, with some conditions on the upper bound of the variation of $V(\cdot,\cdot)$ across resetting events.

5 MODEL REFERENCE ADAPTIVE CONTROL FOR HYBRID SYSTEMS

5.1 Problem formulation

In this section, we present a model reference adaptive control law for nonlinear, time-varying, hybrid plants affected by matched and parametric uncertainties. In particular, we consider the problem of regulating uncertain, time-varying dynamical systems, whose trajectory tracking error dynamics are captured by

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{ref},\sigma(t)} e(t) + B_{\sigma(t)}\left[u(t) - \Theta_{\sigma(t)}^{\mathrm{T}} \Phi_{\sigma(t)}(t, x(t))\right] \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} e(t_0) \\ \sigma(t_0) \end{bmatrix} = \begin{bmatrix} e_0 \\ \sigma_0 \end{bmatrix}, \qquad \left((t, x(t)) \notin \mathcal{D}_{\sigma(t)}\right) \wedge \left((t, x_{\mathrm{ref}}(t)) \notin \mathcal{D}_{\mathrm{ref},\sigma(t)}\right), \tag{23}$$

$$\begin{bmatrix} e(t^+) \\ \sigma(t^+) \end{bmatrix} = g_{\mathrm{d},\sigma(t)}(t, e(t)), \qquad \left((t, x(t)) \in \mathcal{D}_{\sigma(t)}\right) \vee \left((t, x_{\mathrm{ref}}(t)) \in \mathcal{D}_{\mathrm{ref},\sigma(t)}\right), \tag{24}$$

where $e : [t_0, \infty) \to \mathbb{R}^n$ denotes the *trajectory tracking error*, $\sigma : [t_0, \infty) \to \Sigma$, $\Sigma \subset \mathbb{N}$, is bounded and, hence, without loss of generality, comprises the first $\sigma_{\max}$ positive integers, the piecewise continuous function $u : [t_0, \infty) \to \mathbb{R}^m$ denotes the *control input*, $A_{\mathrm{ref},\sigma} \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_\sigma \in \mathbb{R}^{n\times m}$ is such that $(A_{\mathrm{ref},\sigma}, B_\sigma)$ is controllable, $\Theta_\sigma \in \mathbb{R}^{N_\sigma\times m}$ is unknown and the mapping $\sigma \mapsto \Theta_\sigma$ is unknown, the *regressor vector* $\Phi_\sigma : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{N_\sigma}$ is Lipschitz continuous and captures matched uncertainties, $\wedge$ denotes the conjunction logic operator *and*, $\vee$ denotes the disjunction logic operator *or*, $x(t) \triangleq e(t) + x_{\mathrm{ref}}(t)$ denotes the *plant state*, $x_{\mathrm{ref}} : [t_0, \infty) \to \mathbb{R}^n$ denotes the *reference model's trajectory* and verifies the *reference model*

$$\begin{bmatrix} \dot{x}_{\mathrm{ref}}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{ref},\sigma(t)} x_{\mathrm{ref}}(t) + B_{\mathrm{ref},\sigma(t)} r(t) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_{\mathrm{ref}}(t_0) \\ \sigma(t_0) \end{bmatrix} = \begin{bmatrix} x_{\mathrm{ref},0} \\ \sigma_0 \end{bmatrix}, \qquad (t, x_{\mathrm{ref}}(t)) \notin \mathcal{D}_{\mathrm{ref},\sigma(t)}, \tag{25}$$

$$\begin{bmatrix} x_{\mathrm{ref}}(t^+) \\ \sigma(t^+) \end{bmatrix} = f_{\mathrm{d,ref},\sigma(t)}(t, e(t)), \qquad (t, e(t)) \in \mathcal{D}_{\mathrm{ref},\sigma(t)}, \tag{26}$$

in the sense of Krasovskii, $B_{\mathrm{ref},\sigma} \in \mathbb{R}^{n\times m}$ is such that

$$B_{\mathrm{ref},\sigma} = B_\sigma K_{r,\sigma}^{\mathrm{T}} \tag{27}$$

for some $K_{r,\sigma} \in \mathbb{R}^{m\times m}$, and the *reference command input* $r : [t_0, \infty) \to \mathbb{R}^m$ is bounded and piecewise continuous. The resetting events $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$ are unknown and such that if $x_{\mathrm{ref}}(t) \equiv 0$, $t \geq t_0$, then Assumptions 1 and 2 are verified by (23) and (24) with any piecewise continuous control input $u(\cdot)$. The definitions of the resetting events $\left\{\mathcal{D}_{\mathrm{ref},\sigma}\right\}_{\sigma\in\Sigma}$, which are considered design variables, and of the associated sequence of resetting times are provided in Section 5.2 below. Note that since $r(\cdot)$ is bounded and piecewise continuous and $A_{\mathrm{ref},\sigma(\cdot)}$ is Hurwitz, $x_{\mathrm{ref}}(\cdot)$ is bounded between resetting events.

In this paper, the switching law $g_{\mathrm{d},\sigma}(\cdot,\cdot)$, $\sigma \in \Sigma$, is assumed to be a known, uncontrollable property of the system's dynamics. The case whereby $g_{\mathrm{d},\sigma}(\cdot,\cdot)$, $\sigma \in \Sigma$, is a control input is omitted for brevity and can be deduced from the results provided in the following.

The problem of designing control laws for uncertain dynamical systems, whose trajectory tracking error dynamics are in the same form as (23) and (24), occurs, for instance, in the case the plant dynamics are captured by a hybrid dynamical model in the same form as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} A_{\sigma(t)} x(t) + B_{\sigma(t)} \left[u(t) + \tilde{\Theta}_{\sigma(t)}^{\mathrm{T}} \tilde{\Phi}_{\sigma(t)}(t, x(t))\right] \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} x(t_0) \\ \sigma(t_0) \end{bmatrix} = \begin{bmatrix} e_0 + x_{\mathrm{ref},0} \\ \sigma_0 \end{bmatrix}, \qquad (t, x(t)) \notin \mathcal{D}_{\sigma(t)}, \tag{28}$$

$$\begin{bmatrix} x(t^+) \\ \sigma(t^+) \end{bmatrix} = g_{\mathrm{d},\sigma(t)}(t, e(t)) + \begin{bmatrix} x_{\mathrm{ref}}(t^+) \\ 0 \end{bmatrix}, \qquad (t, x(t)) \in \mathcal{D}_{\sigma(t)}, \tag{29}$$

where $x : [t_0, \infty) \to \mathbb{R}^n$ denotes the *plant state*, $A_\sigma \in \mathbb{R}^{n\times n}$, $\sigma \in \Sigma$, is unknown and such that

$$A_{\mathrm{ref},\sigma} = A_\sigma + B_\sigma K_{x,\sigma}^{\mathrm{T}} \tag{30}$$

for some $K_{x,\sigma} \in \mathbb{R}^{n\times m}$, the mapping $\sigma \mapsto A_\sigma$ is unknown, $\tilde{\Phi}_\sigma(t, x)$ is such that $\Phi_\sigma(t, x) = \left[x^{\mathrm{T}}, r^{\mathrm{T}}(t), -\tilde{\Phi}_\sigma^{\mathrm{T}}(t, x)\right]^{\mathrm{T}}$, and $\Theta_\sigma = \left[K_{x,\sigma}^{\mathrm{T}}, K_{r,\sigma}^{\mathrm{T}}, \tilde{\Theta}_\sigma^{\mathrm{T}}\right]^{\mathrm{T}}$. In several problems of practical interest, the plant dynamics can be captured by (28) and (29). For example, the dynamics of mechanical systems affected by parametric uncertainties and subject to instantaneous changes in both its dynamical model and its trajectory because of elastic collisions and instantaneous transitions between static and dynamic friction models can be captured by (28) and (29). Having imposed that Assumptions 1 and 2 are verified by (23) and (24) with any piecewise continuous control input $u(\cdot)$ implies that (28) and (29) with any piecewise continuous control input $u(\cdot)$ verify Assumptions 1 and 2. In the case of mechanical systems subject to elastic collisions, for example, these assumptions reduce to requiring that the control input does not induce collisions in any arbitrarily small amount of time, which is a sufficiently realistic assumption in many problems of practical interest.

Designing adaptive control laws for trajectory tracking error dynamics in the same form as (23) and (24) provides a generalized solution of the model reference adaptive control design problem for switched dynamical systems discussed in Reference 7 and of the classical model reference adaptive control design problem for continuous-time dynamical systems[8(Ch. 9)]. Future work directions involve the analysis of trajectory tracking error dynamics in the same form as (23) and (24) and plants in the same form as (28) and (29) that are not complete.

5.2 | Definition of the control law, the adaptive law, and the reference model's resetting events

To present adaptive control laws that regulate (23) and (24), let $\Sigma_1, \ldots, \Sigma_p \subseteq \Sigma$, $p \in \{1, \ldots, \sigma_{\max}\}$, denote partitions of Σ, define $\overline{\Phi}_\sigma(t,x) \triangleq \left[\chi_{\Sigma_1}(\sigma)\Phi_1^{\mathrm{T}}(t,x), \ldots, \chi_{\Sigma_p}(\sigma)\Phi_p^{\mathrm{T}}(t,x)\right]^{\mathrm{T}}$, $(\sigma, t, x) \in \Sigma \times [t_0, \infty) \times \mathbb{R}^n$, and define $\Theta \triangleq \left[\Theta_1^{\mathrm{T}}, \ldots, \Theta_p^{\mathrm{T}}\right]^{\mathrm{T}}$ and $N \triangleq \sum_{\sigma=1}^{p} N_\sigma$. In this case, (23) is equivalent to

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\sigma}(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{ref},\sigma(t)}e(t) + B_{\sigma(t)}\left[u(t) - \Theta^{\mathrm{T}}\overline{\Phi}_{\sigma(t)}(t,x(t))\right] \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} e(t_0) \\ \sigma(t_0) \end{bmatrix} = \begin{bmatrix} e_0 \\ \sigma_0 \end{bmatrix}, \qquad \left((t,x(t)) \notin \mathcal{D}_{\sigma(t)}\right) \wedge \left((t,x_{\mathrm{ref}}(t)) \notin \mathcal{D}_{\mathrm{ref},\sigma(t)}\right). \tag{31}$$

Thus, we consider the *control law*

$$\eta(\hat{\Theta}, \overline{\Phi}_\sigma(t,x)) = \hat{\Theta}^{\mathrm{T}}\overline{\Phi}_\sigma(t,x), \qquad (\sigma, t, x, \hat{\Theta}) \in \Sigma \times [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{N\times m}, \tag{32}$$

and the *adaptive law*

$$\dot{\hat{\Theta}}(t) = \Gamma\overline{\Phi}_{\sigma(t)}(t,x(t))e^{\mathrm{T}}(t)P_{\sigma(t)}B_{\sigma(t)}, \qquad \hat{\Theta}(t_0) = \hat{\Theta}_0, \qquad t \geq t_0, \tag{33}$$

where $\Gamma \in \mathbb{R}^{N\times N}$ is user-defined, symmetric, and positive-definite, $P_\sigma \in \mathbb{R}^{n\times n}$, $\sigma \in \Sigma$, denotes the symmetric, positive-definite solution of the *algebraic Lyapunov equation*

$$0_{n\times n} = A_{\mathrm{ref},\sigma}^{\mathrm{T}}P_\sigma + P_\sigma A_{\mathrm{ref},\sigma} + Q_\sigma, \tag{34}$$

and $Q_\sigma \in \mathbb{R}^{n\times n}$ is user-defined, symmetric, and positive-definite.

Next, consider the *Lyapunov function candidate*

$$V(t, e, \Delta\Theta) \triangleq e^{\mathrm{T}}P_{\sigma(t)}e + \mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}\Gamma^{-1}\Delta\Theta\right), \qquad (t, e, \Delta\Theta) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^{N\times m}, \tag{35}$$

where $\Delta\Theta(t) \triangleq \hat{\Theta}(t) - \Theta$. Additionally, let

$$W(e) = \overline{\lambda}_{\min}\left(\{Q_\sigma\}_{\sigma\in\Sigma}\right)\|e\|^2, \qquad e \in \mathbb{R}^n, \tag{36}$$

where $\overline{\lambda}_{\min}\left(\{Q_\sigma\}_{\sigma\in\Sigma}\right) \triangleq \min\{\lambda_{\min}(Q_\sigma), \sigma \in \Sigma\}$ and $\lambda_{\min}(Q_\sigma)$, $\sigma \in \Sigma$, denotes the smallest eigenvalue of Q_σ. We note that the right-hand side of (33) is piecewise Lipschitz continuous with points of discontinuity at the resetting times t_k, $k \in \mathbb{N}$. Thus, (33) is a switched dynamical system, and we consider Carathéodory solutions of (33); recall that this class of solutions is absolutely continuous on $[t_0, \infty)$. Consequently, any instantaneous variation of $V(t, e(t), \hat{\Theta}(t))$, $t \geq t_0$, is due to variations of $e^{\mathrm{T}}(t)P_{\sigma(t)}e(t)$ across resetting events. As an alternative to Carathéodory solutions of (33), the proposed formulation can be modified to account for Filippov solutions.

We let $\{t_k\}_{k\in\mathbb{N}} = \{t_{s,i}\}_{i\in\mathbb{N}} \cup \left(\bigcup_{i\in\mathbb{N}}\{t_{\text{ref},i_w}\}_{w\in\mathbb{N}}\right)$, that is, we partition the set of resetting times of (23) and (24) into the union of the resetting times due to $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$ and the resetting times due to $\left\{\mathcal{D}_{\text{ref},\sigma}\right\}_{\sigma\in\Sigma}$. The i-th resetting time of the resetting event $\mathcal{D}_{\sigma_{i-1}}$, $(i,\sigma)\in\mathbb{N}\times\Sigma$, is given by $t_{s,i} = \min\{t \geq t_{i-1} : (t, s_{c,\sigma_{i-1}}(t, t_{i-1}, x_{i-1})) \notin \mathcal{D}_{\sigma_{i-1}}\}$. The resetting events of the reference model are defined so that $\mathcal{D}_{\text{ref},\sigma_{i_w}} \triangleq \{t_{\text{ref},i_w}\}\times\mathbb{R}^n$, $(i,w)\in\mathbb{N}\times\mathbb{N}$, where

$$t_{\text{ref},i_w} \triangleq \inf\left\{t > \max\{t_{s,i}, t_{\text{ref},i_w-1}\} : \int_{t_0}^{t} W(e(\tau))\mathrm{d}\tau \geq \sum_{j=1}^{k-1}\left[V(t_j^+, e(t_j^+), \hat{\Theta}(t_j)) - V(t_j, e(t_j), \hat{\Theta}(t_j))\right]\right\} \tag{37}$$

denotes the *w-th resetting time due to the reference model after the ith resetting event of the plant*, k denotes the index for the generic resetting times, i denotes the index for the resetting times due to $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$, and i_w denotes the index for the resetting times due to $\left\{\mathcal{D}_{\text{ref},\sigma}\right\}_{\sigma\in\Sigma}$ after the ith resetting time. The switching law $f_{\text{d,ref},\sigma}(t,e)$, $(\sigma,t,e)\in\Sigma\times[t_0,\infty)\times\mathbb{R}^n$, is defined so that

$$x_{\text{ref}}(t^+_{\text{ref},i_w}) = x(t_{\text{ref},i_w}) - \sqrt{\frac{e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w}) - s_{\text{ref},i_w}}{e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w})}}\, P^{-\frac{1}{2}}_{\sigma(t^+_{\text{ref},i_w})}P^{\frac{1}{2}}_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w}), \qquad (i,w)\in\mathbb{N}\times\mathbb{N}, \tag{38}$$

or, equivalently,

$$\begin{aligned}\Delta x_{\text{ref}}(t^+_{\text{ref},i_w}) &= x_{\text{ref}}(t^+_{\text{ref},i_w}) - x_{\text{ref}}(t_{\text{ref},i_w}) \\ &= \left(1 - \sqrt{\frac{e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w}) - s_{\text{ref},i_w}}{e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w})}}\right)P^{-\frac{1}{2}}_{\sigma(t^+_{\text{ref},i_w})}P^{\frac{1}{2}}_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w}),\end{aligned} \tag{39}$$

where $s_{\text{ref},i_w} \in \left(0, e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w})\right)$ is user-defined and $\sum_{i=1}^{\infty}\sum_{w=1}^{\infty}s_{\text{ref},i_w}$ is convergent. Figure 1 provides a representation of the sequence of resetting times $\{t_{s,i}\}_{i\in\mathbb{N}}$ and $\bigcup_{i\in\mathbb{N}}\{t_{\text{ref},i_w}\}_{w\in\mathbb{N}}$.

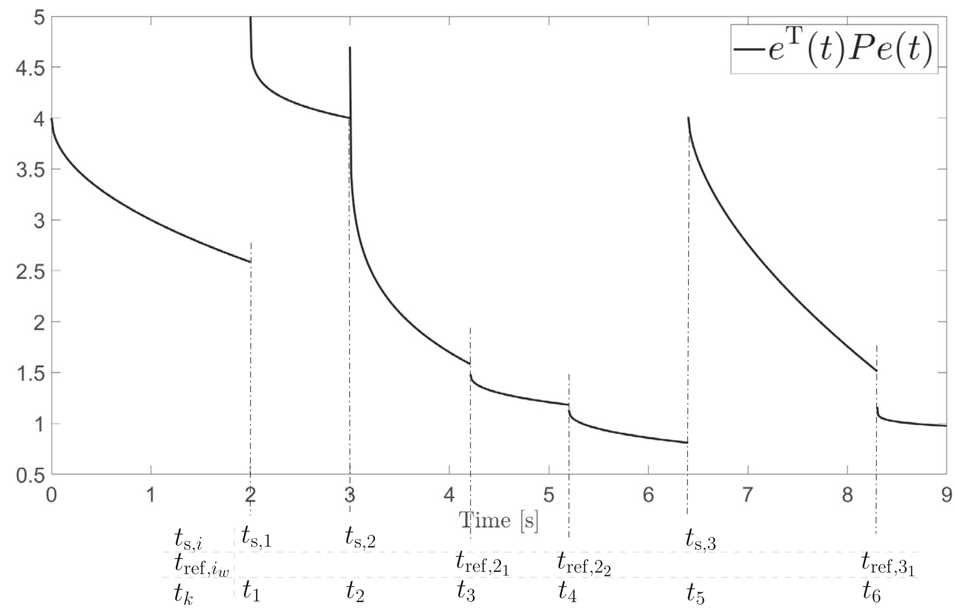

FIGURE 1 Graphical representation of the sequence of resetting times due to $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$ and $\left\{\mathcal{D}_{\text{ref},\sigma}\right\}_{\sigma\in\Sigma}$ and variations in $e^{\mathrm{T}}(t)Pe(t)$, $t\in[t_0,\infty)$, due to resetting events.

To construct $s_{\text{ref},i_w} \in \left(0, e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w})\right)$, $(i,w) \in \mathbb{N}\times\mathbb{N}$, such that the series $\sum_{i=1}^{\infty}\sum_{w=1}^{\infty} s_{\text{ref},i_w}$ is convergent, note that it is always possible to rearrange indexes and span $\bigcup_{i\in\mathbb{N}}\{i_w\}_{w\in\mathbb{N}}$ using a single index. Thus, assume, for instance, that $s_{\text{ref},j} = j^{-2}$, where $j = \max\{z \in \mathbb{N} : z > j-1, z^{-2} < e^{\mathrm{T}}(t_j)P_{\sigma(t_j)}e(t_j)\}$. Since $\sum_{j=1}^{\infty} j^{-2} = \pi^2/6$ and the sequence $\left\{\sum_{j=1}^{k} j^{-2}\right\}_{k\in\mathbb{N}}$ is sign-definite and monotonically decreasing, the limit of any series deduced from a subsequence of $\left\{\sum_{j=1}^{k} j^{-2}\right\}_{k\in\mathbb{N}}$ exists and is finite. Thus, it is always possible to extract a series from $\sum_{j=1}^{\infty} j^{-2}$ such that $\sum_{i=1}^{\infty}\sum_{w=1}^{\infty} s_{\text{ref},i_w}$, $s_{\text{ref},i_w} \in \left(0, e^{\mathrm{T}}(t_{\text{ref},i_w})P_{\sigma(t_{\text{ref},i_w})}e(t_{\text{ref},i_w})\right)$, and $\{j\}_{j\in\mathbb{N}} = \bigcup_{i\in\mathbb{N}}\{i_w\}_{w\in\mathbb{N}}$.

We remark that, for (39) to be defined, it must hold that $e(t_{i_w}) \neq 0_n$, $(i,w) \in \mathbb{N}\times\mathbb{N}$. Indeed, it follows from (37) that if $e(t_{i_w}) = 0_n$ for some $(i,w) \in \mathbb{N}\times\mathbb{N}$, then, t_{ref,i_w} must not be computed and the reference model must not introduce any additional resetting event, that is, (39) must not be computed as well.

The dwell time of the controlled trajectory tracking error dynamics, that is, the minimum difference between any two resetting times of (31) with $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_{\sigma(t)}(t, x(t)), t \geq t_0$, can be computed as $\min_{k\in\mathbb{N}}(t_k - t_{k-1})$, where $\{t_k\}_{k\in\mathbb{N}} = \{t_{s,i}\}_{i\in\mathbb{N}} \cup \left(\bigcup_{i\in\mathbb{N}}\{t_{\text{ref},i_w}\}_{w\in\mathbb{N}}\right)$. The resetting times $\{t_{s,i}\}_{i\in\mathbb{N}}$ are assumed to be measurable by observing, for instance, instantaneous variations in $x(\cdot)$, or, alternatively, by detecting if $(t, x(t)) \in \mathcal{D}_{\sigma(t)}$ for some $t \in [t_0, \infty)$. The resetting times $\bigcup_{i\in\mathbb{N}}\{t_{\text{ref},i_w}\}_{w\in\mathbb{N}}$ are given by (37). Since Carathéodory solutions of (33) are considered, it holds that $\hat{\Theta}(t_k^+) = \hat{\Theta}(t_k)$ for all $k \in \mathbb{N}$. Thus, it follows from (35) that

$$\sum_{j=1}^{k-1}\left[V(j^+, e(t_j^+), \hat{\Theta}(t_j)) - V(t_j, e(t_j), \hat{\Theta}(t_j))\right] = \sum_{j=1}^{k-1}\left[e^{\mathrm{T}}(t_j^+)P_{\sigma(t_j^+)}e(t_j^+) - e^{\mathrm{T}}(t_j)P_{\sigma(t_j)}e(t_j)\right], \tag{40}$$

which can be measured directly assuming that either $e(\cdot)$ can be measured directly and instantaneous variations in $e(\cdot)$ can be detected or assuming that $x(\cdot)$ and instantaneous variations in $x(\cdot)$ can be measured directly, and noting that $x_{\text{ref}}(\cdot)$ is known. Thus, the right-hand side of (37) can be computed in real-time, and the dwell-time of the controlled trajectory tracking error dynamics can be computed explicitly over the time interval $[t_0, t]$ for the current time instant $t \in (t_0, \infty)$. However, the right-hand side of (37) can not be computed for future times, and the dwell time of the controlled trajectory tracking error dynamics can be neither controlled directly nor computed a priori since the resetting events $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$, A_σ and Θ_σ, $\sigma \in \Sigma$, and the mappings $\sigma \mapsto A_\sigma$ and $\sigma \mapsto \Theta_\sigma$ are unknown.

From (37), we deduce that if $\int_{t_0}^{t} W(e(\tau))\mathrm{d}\tau$, $t \geq t_0$, does not increase sufficiently fast over $\bigcup_{j\in\{1,\dots,i\}}(t_{s,j}, t_{s,j+1}]$ for any $i \in \mathbb{N}$, then t_{ref,i_w}, $w \in \mathbb{N}$, can not be defined. However, the collection of positive-definite matrices $\{Q_\sigma\}_{\sigma\in\Sigma}$ is a user-defined parameter, and the smallest eigenvalues of these matrices can be set arbitrarily large for t_{ref,i_w}, $w \in \mathbb{N}$, to be defined. In light of this consideration, in the remainder of this paper, we consider the following assumption verified.

Assumption 3. For each $i \in \mathbb{N}$, the set of reference model's resetting times $\{t_{\text{ref},i_w}\}_{w\in\mathbb{N}}$ is always defined.

In general, large eigenvalues of Q_σ, $\sigma \in \Sigma$, may imply a slower convergence of the trajectory tracking error to zero. This limitation can be overcome in multiple ways. For instance, despite the classical model reference adaptive control framework, wherein the underlying Lyapunov equation is fixed a priori, in the proposed framework, the set of matrices $\{Q_\sigma\}_{\sigma\in\Sigma}$ can be arbitrarily tuned by the user at discrete time instants as the controlled system evolves to induce the switching of the reference model and meet user-defined criteria on the rate of convergence of the trajectory tracking error. An alternative, more systematic approach consists in applying the two-layer model reference control,[26] wherein the trajectory tracking error's rate of convergence is set by the user introducing an additional reference model, which affects the closed-loop system's transient dynamics only; this extension of the present work will be considered in the future.

5.3 | Properties of the tracking error dynamics and of the reference model

In the following, we present some key properties of the trajectory tracking error dynamics (23) and (24) and of the reference model (25) and (26), which, under Assumption 3, are needed to prove the main result of this paper.

Proposition 1. Consider the trajectory tracking error dynamics given by (23) and (24) and the control law (32). If $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, then $\hat{t}_k = t_k$, $k \in \mathbb{N}$.

Proposition 1, whose proof is provided in the Appendix, shows that the resetting times of the reference model are designed so that $\hat{t}_k = t_k$, $k \in \mathbb{N}$. Next, we leverage this proposition to prove that Assumptions 1 and 2 are verified by (23) and (24) with the proposed adaptive control law and by (25) and (26).

Proposition 2. Consider the reference model given by (25) and (26), the trajectory tracking error dynamics given by (23) and (24), and the control law (32). If $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, then Assumptions 1 and 2 are verified by (23) and (24). Additionally, Assumptions 1 and 2 are verified by (25) and (26).

Since Assumptions 1 and 2 are verified by (23) and (24), it follows from Proposition 2, whose proof is provided in the Appendix, that the dwell time of the controlled trajectory tracking error's dynamics is greater than zero. Next, we verify the boundedness of the variations of the Lyapunov function candidate (35) across resetting events. This fact is instrumental to prove the main result of this paper.

Proposition 3. Consider the trajectory tracking error dynamics given by (23) and (24), the control law (32), and the Lyapunov function candidate (35). If $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, then $\sum_{k=1}^{\infty} \left[V(t_k^+, e(t_k^+), \hat{\Theta}(t_k)) - V(t_k, e(t_k), \hat{\Theta}(t_k)) \right]$ exists and is finite.

The proof of Proposition 3 is in the Appendix. The next proposition shows the effect of (39) on the closed-loop system's trajectory tracking error. For the statement of this result, given the symmetric and positive-definite matrix $P \in \mathbb{R}^{n \times n}$, let $P^{\frac{1}{2}} \in \mathbb{R}^{n \times n}$ be symmetric, positive-definite, and such that $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

Proposition 4. Consider the reference model dynamics given by (25) and (26) and the control law (32). If $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, then, for each $i \in \mathbb{N}$ and for all $w \in \mathbb{N}$, $P^{\frac{1}{2}}_{\sigma(t^+_{ref,i_w})} e(t^+_{i_w})$ and $P^{\frac{1}{2}}_{\sigma(t_{ref,i_w})} e(t_{i_w})$ are collinear, and $\left\| P^{\frac{1}{2}}_{\sigma(t^+_{ref,i_w})} e(t^+_{ref,i_w}) \right\| < \left\| P_{\sigma(t_{ref,i_w})} e(t_{ref,i_w}) \right\|$.

It follows from Proposition 4, whose proof is in the Appendix, that if $P_\sigma = P$ for all $\sigma \in \Sigma$, then, for each $k \in \mathbb{N}$, $e(t^+_{i_w})$ and $e(t_{i_w})$ are collinear for all $w \in \mathbb{N}$, that is, the direction of the trajectory tracking error does not change across resetting events produced by (39). Additionally, Proposition 4 shows that, if $P_\sigma = P$ for all $\sigma \in \Sigma$, then the trajectory tracking error decreases across resetting events produced by the reference model.

5.4 | Main result

The following theorem provides the main result of this section, namely a model reference adaptive control law for hybrid dynamical models in the same form as (31) and (24).

Theorem 4. Consider the trajectory tracking error dynamics (31) and (24), the control law (32), the adaptive law (33), and the reference model (25) and (26). If $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, the matching condition (27) is verified, and $\{Q_\sigma\}_{\sigma \in \Sigma}$ are chosen so that Assumption 3 is verified, then both the trajectory tracking error $e(\cdot)$ and the adaptive gain matrix $\hat{\Theta}(\cdot)$ are bounded uniformly in both $t_0 \in [0, \infty)$ and $\{t_k\}_{k \in \mathbb{N}}$, and $e(t) \to 0$ as $t \to \infty$ for all $e_0 \in \mathbb{R}^n$ uniformly in $\{t_k\}_{k \in \overline{\mathbb{N}}}$.

Theorem 4 proves the effectiveness of the proposed model reference adaptive control system by showing that both the trajectory tracking error and the adaptive gains are bounded, and the trajectory tracking error asymptotically converges to zero, uniformly in the initial time and the sequence of resetting times. We remark how the adaptive gains are continuous functions of time. This feature eases the implementation of the proposed system to problems of practical interest.

Note that if $\Delta e(t_k^+) = 0$, $k \in \mathbb{N}$, then Theorem 4 provides a model reference control system for switched dynamical systems leveraging Krasovskii solutions; the model reference adaptive control systems in Reference 7 employ Carathéodory and Filippov solutions. Furthermore, if there is no resetting event in the plant dynamics, that is, if $\mathcal{D}_\sigma = \{\emptyset\}$, $\sigma \in \Sigma$, then $\mathcal{D}_{\mathrm{ref},\sigma} = \{\emptyset\}$ and the proposed model reference adaptive control framework for hybrid dynamical systems captured by Theorem 4 reduces to the classical model reference control framework[8(Ch. 9)].

6 | ILLUSTRATIVE NUMERICAL EXAMPLE

In this section, we present the results of a numerical simulation to show the applicability of the proposed results. This simulation concerns a two-dimensional point mass able to deliver the desired thrust force in any direction and tasked with following a user-defined rectangular reference trajectory. While tracking this reference trajectory, the point mass must remain inside a rectangular constraint set at all times. If this point mass impinges the boundary of the constraint set, then it experiences a perfectly elastic collision, that is, the component of the velocity of the point mass that is locally orthogonal to the boundary of the rectangle at the point of the impact instantaneously changes sign. Applying the model reference adaptive control law outlined in Theorem 4, the point mass must closely follow the reference trajectory despite two challenges. A challenge is given by the fact that the reference trajectory starts within the rectangular constraint set, and violates this constraint for an extended period of time. An additional challenge is given by the fact that the point mass is subject to some aerodynamic drag, and the aerodynamic coefficients are unknown. To verify Assumptions 1 and 2, we impose that, after each collision, the controller remains inactive for 0.05s.

Given a two-dimensional orthonormal reference frame $\mathbb{I} = \{O; X, Y\}$, the position of the point mass is denoted by $p : [t_0, \infty) \to \overline{\mathcal{R}}$, where $\overline{\mathcal{R}} \subset \mathbb{R}^2$ denotes a closed rectangle, whose vertexes are $[0.0, 0.0]^{\mathrm{T}}$m, $[0.0, 5.5]^{\mathrm{T}}$m, $[12.0, 5.5]^{\mathrm{T}}$m, and $[12.0, 0.0]^{\mathrm{T}}$m. Thus, the point mass dynamics are captured by Newton's laws through (28), where $t_0 = 0$, $\mathcal{S} = \mathbb{R}^2$, $\Sigma = \{1, 2\}$, $\sigma = 1$ denotes resetting events due to bounces of the point mass, $\sigma = 2$ denotes resetting events in the reference model, $x(t) = [p^{\mathrm{T}}(t), \dot{p}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $t \geq t_0$, $u(t) \in \mathbb{R}^2$, $A_\sigma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\sigma \in \Sigma$, $B_\sigma = \begin{bmatrix} 0_{2\times 2} \\ m^{-1}\mathbf{1}_2 \end{bmatrix}$, $m > 0$ denotes the mass of the point, $\tilde{\Theta}_\sigma = -\frac{1}{2}\rho S c_{\mathrm{D}} \mathbf{1}_2$, $\rho > 0$ is unknown and denotes the air density, $S > 0$ is unknown and denotes a reference surface, $c_{\mathrm{D}} > 0$ is unknown and denotes an aerodynamic drag coefficient, and $\tilde{\Phi}_\sigma(t, x) = \|\dot{p}\|\dot{p}$.

To capture purely elastic collisions, we introduce a two-dimensional orthonormal reference frame for each side of the rectangular constraint set $\overline{\mathcal{R}}$. These reference frames are denoted by $\mathbb{J}_i = \{O_i; X_i, Y_i\}$, $i \in \{1, \ldots, 4\}$, where X_i is aligned with the corresponding side of the rectangle and Y_i points toward the interior of $\overline{\mathcal{R}}$. If $p(\cdot)$ is expressed in the reference frame $\mathbb{J}_i$, $i \in \{1, \ldots, 4\}$, then it is denoted by $p^{\mathbb{J}_i}(\cdot)$. Thus, elastic collisions are captured by (29) with $g_{\mathrm{d},1}(t, x)$, $(t, x) \in [t_0, \infty) \times \mathcal{S}$, and $\mathcal{D}_1$ defined so that if $\mathbf{e}_{2,2}^{\mathrm{T}} p^{\mathbb{J}_i}(t) = 0$, for some $t \in [t_0, \infty)$ and for some $i \in \{1, \ldots, 4\}$, and $\mathbf{e}_{2,2}^{\mathrm{T}} \dot{p}^{\mathbb{J}_i}(t) < 0$, then $\mathbf{e}_{2,2}^{\mathrm{T}} p^{\mathbb{J}_i}(t^+) = 0$ and $\mathbf{e}_{2,2}^{\mathrm{T}} \dot{p}^{\mathbb{J}_i}(t^+) = -\mathbf{e}_{2,2}^{\mathrm{T}} \dot{p}^{\mathbb{J}_i}(t)$.

The point mass should move clockwise along the rectangle of vertexes $[1.0, 1.0]^{\mathrm{T}}$m, $[1.0, 6.0]^{\mathrm{T}}$m, $[11.0, 6.0]^{\mathrm{T}}$m, and $[11.0, 1.0]^{\mathrm{T}}$m at the constant speed of 0.5m/s in 60s; the position along this user-defined rectangular trajectory is denoted

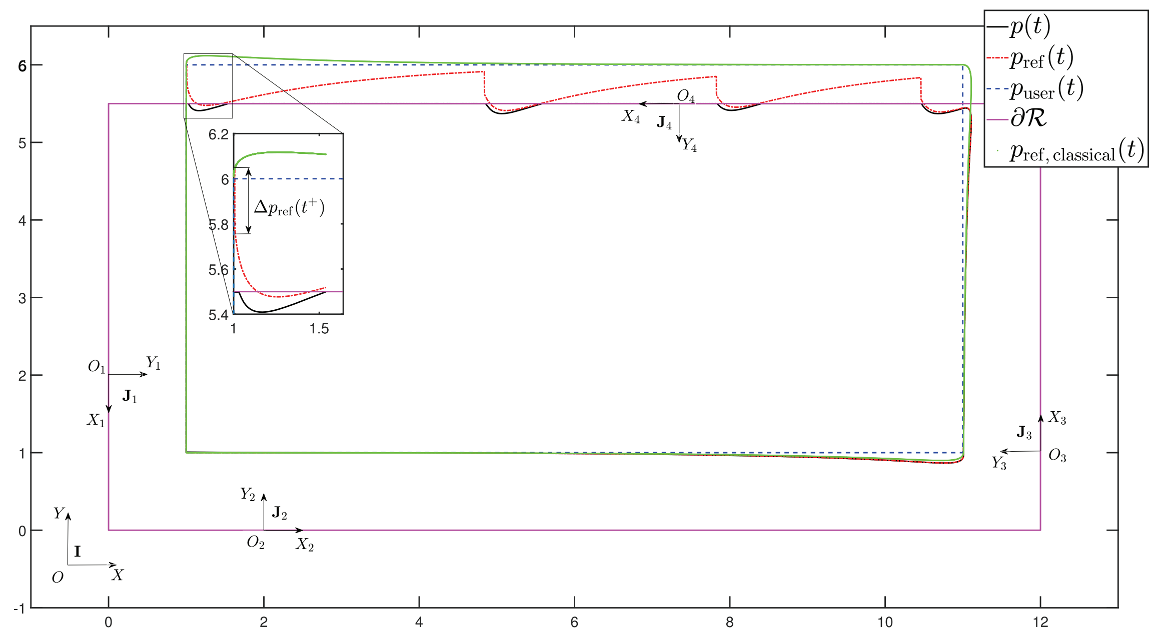

FIGURE 2 Trajectory of the point mass, user-defined reference trajectory, trajectory of the reference model employing (26), and trajectory of the reference model without employing (26). The proposed model reference adaptive control framework allows to modify the reference model's trajectory, while retaining the objective of steering the point mass toward the user-defined reference trajectory.

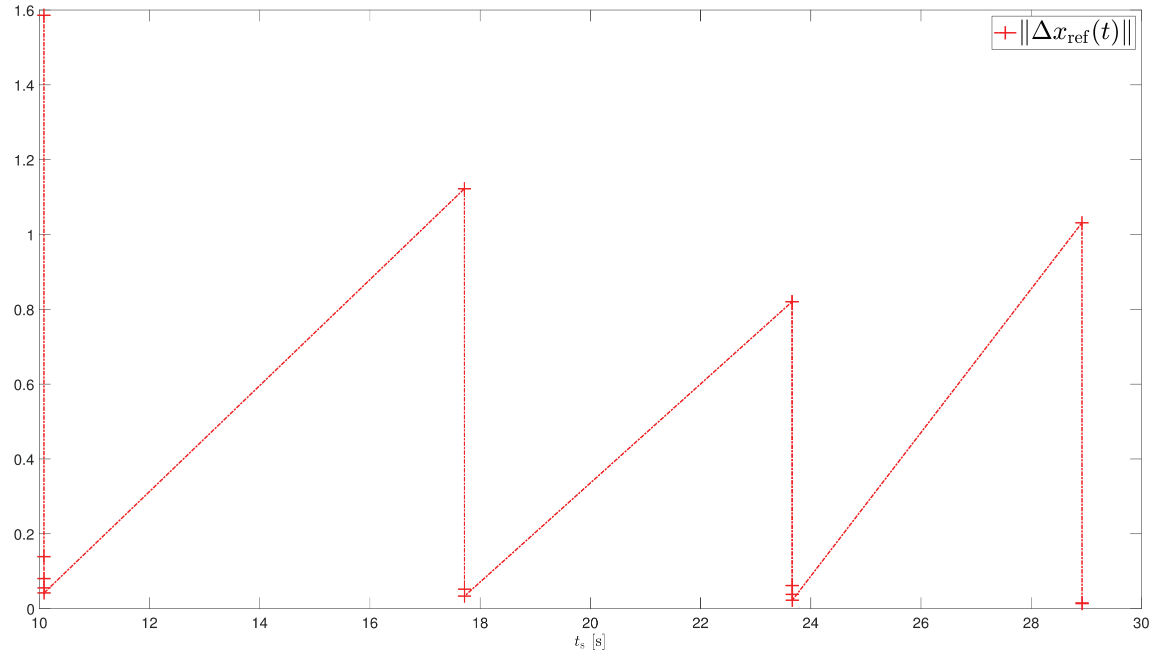

FIGURE 3 Variations in the reference model's trajectory due to (26).

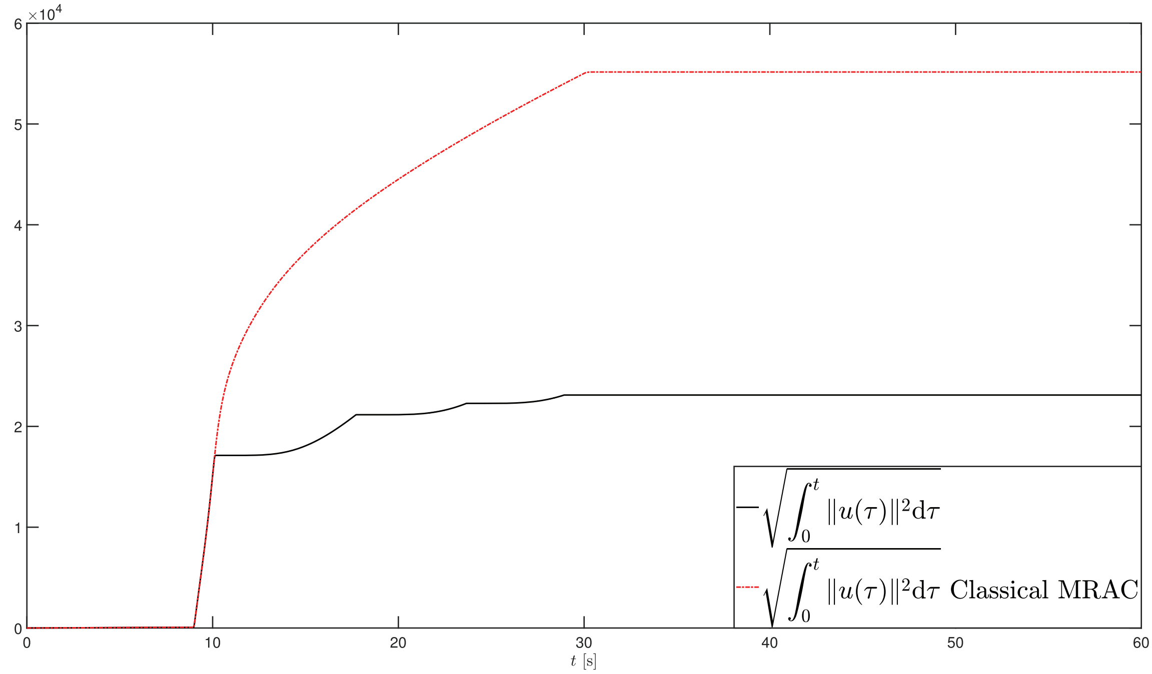

FIGURE 4 $\mathcal{L}_2$-norm of the control input applying Theorem 4 and the classical model reference adaptive control (MRAC) approach. The proposed control framework allows to exert a considerably smaller control effort. Before resetting events are introduced in the reference model, both control frameworks require the same control effort. Condition (39) allows to lower the control effort, while retaining the ability of the reference model's trajectory to follow a user-defined signal.

by $p_{\text{user}} : [t_0, \infty) \to \mathbb{R}^2$. It is impossible for a point mass, whose dynamics verify Newton's laws and that is controlled by non-impulsive forces, to perform 90-degree turns at a constant translational velocity. For this reason, we introduce the reference model (25) with $x_{\text{ref}}(t) = \left[p_{\text{ref}}^{\text{T}}(t), \dot{p}_{\text{ref}}^{\text{T}}(t)\right]^{\text{T}}$, $t \geq t_0$, $r(t) = \omega^2 p_{\text{user}}(t) + 2\zeta\omega\dot{p}_{\text{user}}(t)$, $A_{\text{ref},\sigma} = \begin{bmatrix} 0_{2\times2} & \mathbf{1}_2 \\ -\omega^2\mathbf{1}_2 & -2\zeta\omega\mathbf{1}_2 \end{bmatrix}$, $\sigma \in \Sigma$, $\omega > 0$ and $\zeta > 0$ user-defined, and $B_{\text{ref},\sigma} = \begin{bmatrix} 0_{2\times2} \\ \mathbf{1}_2 \end{bmatrix}$.

Figure 2 shows the trajectory $p(t)$, $t \geq t_0$, of the point mass obtained as a solution of (28) and (29) with control law (32) and adaptive law (33). Figure 2 also shows the position of the reference model $p_{\text{ref}}(t)$, $t \geq t_0$, obtained as a solution of (25) and (26) subject to instantaneous variations captured by (39) with $s_{\text{ref},i_w} = j_{i_w}^{-1.001}$, where $j_{i_w} = \max\{z \in \mathbb{N} : z > j_{i_w} - 1, z^{-1.001} < e^{\text{T}}(t_{i_w})P_{\sigma(t_{i_w})}e(t_{i_w})\}$ and $i_w \in \mathbb{N}$. Finally, Figure 2 shows $p_{\text{user}}(t)$, $t \geq t_0$, and $p_{\text{ref,classical}}(t)$ obtained as a solution of (25) and (26) without accounting for (39). Since $p_{\text{user}}(\cdot)$ violates the constraint set $\overline{\mathcal{R}}$, $p_{\text{ref}}(\cdot)$ is lead outside $\overline{\mathcal{R}}$. Employing (39), $p_{\text{ref}}(\cdot)$ is modified to reduce the tracking error, while allowing $p(\cdot)$ to follow $p_{\text{user}}(\cdot)$ as closely as possible, compatibly with both the constraint whereby $p(t) \in \overline{\mathcal{R}}$, $t \in [t_0, \infty)$, and the elastic collision model.

While coasting the upper horizontal side of $\overline{\mathcal{R}}$, on four instances, both the point mass trajectory and the reference model's trajectory lie in the interior of the constraint set. On these instances, the convergent series $\sum_{i_w=1}^{\infty} s_{\text{ref},i_w}$ is re-initialized, that is, we re-set $i_w = 1$, to maximize the effect of (39) whenever $p_{\text{ref}}(\cdot)$ newly trespasses $\partial\mathcal{R}$ and the point mass newly starts bouncing off the boundary of the constraint set. Figure 3 shows the norm of the variations in the reference model's trajectory due to (39) when the reference path lies outside $\overline{\mathcal{R}}$.

For $t \in [30, 60]$s, the third side and fourth sides of the user-defined reference trajectory lie in $\mathcal{R}$. In these cases, the control law (32) and the adaptive law (33) reduce to the control law and adaptive law of the model reference adaptive control framework for continuous-time dynamical systems[8(Ch. 9)], and both $\left[p^{\text{T}}(\cdot), \dot{p}^{\text{T}}(\cdot)\right]^{\text{T}}$ and $\left[p_{\text{ref}}^{\text{T}}(\cdot), \dot{p}_{\text{ref}}^{\text{T}}(\cdot)\right]^{\text{T}}$ rapidly converge toward $\left[p_{\text{user}}^{\text{T}}(\cdot), \dot{p}_{\text{user}}^{\text{T}}(\cdot)\right]^{\text{T}}$.

Figure 4 compares the $\mathcal{L}_2$-norm of the control input as a function of time obtained applying the proposed framework and the classical model reference adaptive control approach[8(Ch. 9)]. Similarly, Figure 5 compares the $\mathcal{L}_2$-norm of $\tilde{e}(t) \triangleq \left[p^{\text{T}}(t) - p_{\text{user}}^{\text{T}}(t), \dot{p}^{\text{T}}(t) - \dot{p}_{\text{user}}^{\text{T}}(t)\right]^{\text{T}}$, $t \geq t_0$, applying the proposed framework and the classical model reference

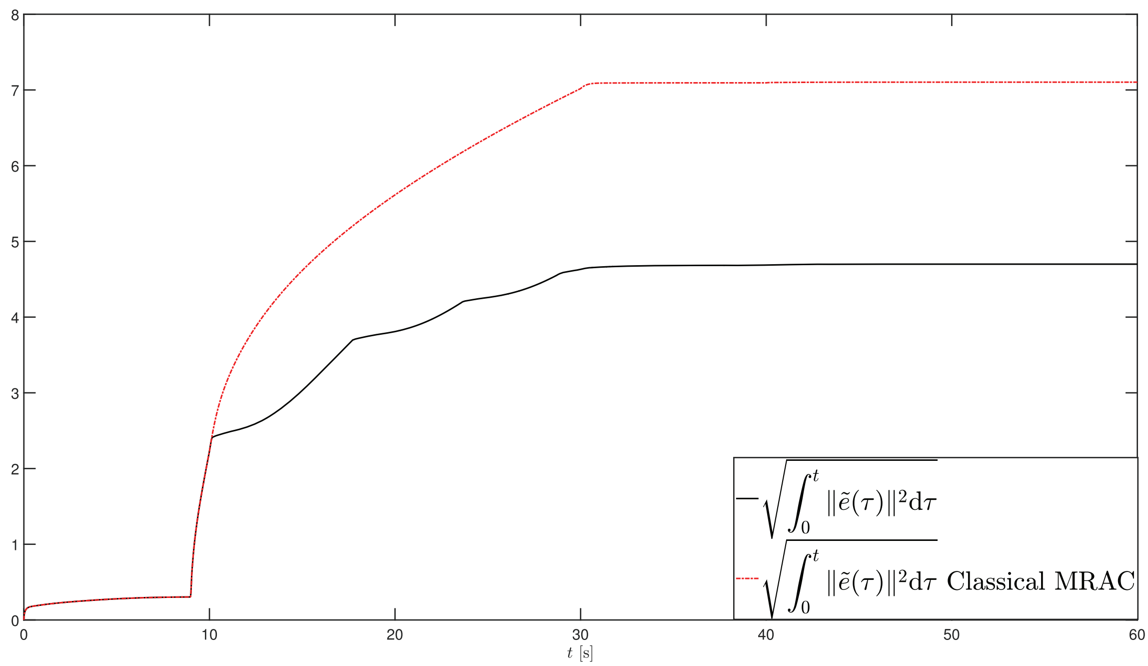

FIGURE 5 $\mathcal{L}_2$-norm of the error in tracking the user-defined trajectory applying Theorem 4 and the classical model reference adaptive control law (MRAC). The proposed control law allows to attain a considerably smaller tracking error than classical model reference adaptive control. Together with the plots in Figure 4, these plots show the advantage of applying the proposed model reference adaptive control framework for hybrid systems.

TABLE 1 $\mathcal{L}_2$-norms of $u(\cdot)$, $\tilde{e}(\cdot)$, and $e(\cdot)$ applying Theorem 4 and the classical model reference adaptive control (MRAC) approach[8(Ch. 9)]. The proposed framework provides smaller tracking errors with smaller control effort.

	Proposed MRAC Law	Classical MRAC Law
$\|u(\cdot)\|_{\mathcal{L}_{2,[0,60]}}$	$2.3109 \cdot 10^4$	$5.5156 \cdot 10^4$
$\|\tilde{e}(\cdot)\|_{\mathcal{L}_{2,[0,60]}}$	4.6993	7.1030
$\|e(\cdot)\|_{\mathcal{L}_{2,[0,60]}}$	4.0364	7.1074

adaptive control approach[8(Ch. 9)]. It is apparent how the proposed approach and, in particular, the ability to impose instantaneous variations in the reference model's trajectory according to (39), considerably reduces both the control effort and the tracking error, while retaining the ability of the reference model's trajectory to follow a user-defined signal. These considerations are supported also by the results shown in Table 1, which compares the $\mathcal{L}_2$-norms of $u(\cdot)$, $\tilde{e}(\cdot)$, and $e(\cdot)$ applying Theorem 4 and the classical model reference adaptive control approach[8(Ch. 9)].

7 | CONCLUSION

This paper presented two novel results. It presented the first extension of the classical LaSalle–Yoshizawa theorem to nonlinear, time-varying, hybrid dynamical systems. Additionally, it provided the first model reference adaptive control system for nonlinear, time-varying, hybrid plants affected by parametric and matched uncertainties as well as uncertainties in the resetting events.

The proposed extension of the LaSalle–Yoshizawa theorem relies on the relationship between the integral of a positive-definite upper bound on the total time derivative of an underlying Lyapunov-like function along the system's trajectories and the cumulative effect of resetting events on the variations of this Lyapunov-like function. The proposed model reference adaptive control system relies both on a control law and an adaptive law, which resemble those of the classical model reference adaptive control framework, and a definition of resetting events for the reference model, which allows meeting the conditions of the proposed extension of the LaSalle–Yoshizawa theorem. Usually, reference models are designed so that the reference model's trajectory converges to a user-defined reference signal. In the proposed adaptive control system, the reference model's resetting events instantaneously reduce the trajectory tracking error, while retaining the ability of the reference model's trajectory to asymptotically converge to the user-defined reference signal. Additionally, the proposed adaptive gains are continuous across resetting events, which eases their implementation in problems of practical interest.

A numerical example, which involves the problem of controlling a point mass tasked with following a user-defined trajectory while remaining in a rectangular constraint set, proves the applicability of the proposed adaptive control system. The proposed system is challenged by the fact that the user-defined trajectory exceeds the constraint set, and the point mass is subject to elastic collisions whenever it impacts the boundary of this set. Additionally, the point mass is subject to an aerodynamic drag force captured by means of unknown coefficients. Compared to a classical model reference adaptive control system, the proposed system guarantees considerably smaller tracking errors and control effort.

Future work directions are multiple. The convergence of the trajectory tracking error to zero can be accelerated using multiple techniques. In the future, we propose to study the implementation of a two-layer model reference adaptive control law[26] within the proposed hybrid systems framework. Additionally, the proposed framework can be extended to account for not complete solutions of the trajectory tracking error dynamics and external disturbances. The problem of not complete solutions of the trajectory tracking error dynamics will be addressed by leveraging, for instance, Theorem 1. External disturbances will be addressed extending, for instance, the e-modification of model reference adaptive control,[27] the projection operator,[28,29] or barrier Lyapunov functions.[26] Finally, the proposed work can be extended to the context of indirect model reference adaptive control and alternative adaptive control techniques that allow estimating the unknown plant parameters; examples of such techniques are provided in References 30-33.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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APPENDIX

Proof of Lemma 2. Let $t \in \mathcal{I}$ and $t \geq t_k$ for some $k \in \overline{\mathbb{N}}$. Since $V(\cdot,\cdot)$ is absolutely continuous in its first argument for each $x \in \mathcal{S}$ and Lipschitz continuous and regular in the second argument for each $t \in \mathcal{I}$ and $x(\cdot)$ is piecewise absolutely continuous, $V(\cdot, x(\cdot))$ is absolutely continuous on compact subsets of $\mathcal{I}$ that do not include resetting times in their interior[34](Lemma 6.1.3). Thus, by the fundamental theorem of calculus for Lebesgue integrals[34](Th. 6.4.2), it holds that

$$V(t,x(t)) - V(t_0,x_0) = \int_{t_0}^{t} \dot{V}(\tau,x(\tau))\mathrm{d}\tau + \sum_{j=1}^{k} \left[V(t_j^+,x(t_j^+)) - V(t_j,x(t_j))\right], \quad t \in \mathcal{I}. \tag{A1}$$

Adding and subtracting $\sum_{j=1}^{k-1} V(t_j,x(t_j))$ from (A1), it follows from (17) and Lemma 4.1.8 of Reference 34 that

$$\begin{aligned} V(t,x(t)) &- V(t_k^+,x(t_k^+)) + \sum_{j=0}^{k-1} \left[V(t_{j+1},x(t_{j+1})) - V(t_j,x(t_j))\right] \\ &= \sum_{j=1}^{k-1} \left[V(t_j^+,x(t_j^+)) - V(t_j,x(t_j))\right] + \int_{t_0}^{t} \dot{V}(\tau,x(\tau))\mathrm{d}\tau \\ &\leq \sum_{j=1}^{k-1} \left[V(t_j^+,x(t_j^+)) - V(t_j,x(t_j))\right] - \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau, \qquad t \in \mathcal{I}. \end{aligned} \tag{A2}$$

Now, suppose *ad absurdum* that

$$V(t,x(t)) - V(t_k^+,x(t_k^+)) + \sum_{j=0}^{k-1} \left[V(t_{j+1},x(t_{j+1})) - V(t_j,x(t_j))\right] > 0 \tag{A3}$$

for some $t \in \mathcal{I}$ such that $t > \hat{t}_k$. In this case, for some $k \in \overline{\mathbb{N}}$ such that $\hat{t}_k \in \mathcal{I}$, it would hold that

$$\sum_{j=1}^{k-1} \left[V(t_j^+,x(t_j^+)) - V(t_j,x(t_j))\right] - \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau > 0, \qquad t \in \left(\hat{t}_k, t_{k+1}\right] \cap \mathcal{I}. \tag{A4}$$

However, per definition of $\hat{t}_k$, $k \in \overline{\mathbb{N}}$, it holds that

$$\sum_{j=1}^{k-1} \left[V(t_j^+,x(t_j^+)) - V(t_j,x(t_j))\right] - \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau \leq 0, \qquad t \in \left(\hat{t}_k, t_{k+1}\right] \cap \mathcal{I}, \tag{A5}$$

which is a contradiction, and (18) is proven. ■

Proof of Theorem 1. The results follows by proceeding similarly to the proof of Theorem 2 below and is omitted for brevity. ■

Proof of Theorem 2. This proof is divided into three parts. Firstly, we find an ordering of the c-sublevel sets of $W_1(\cdot)$, $V(t,\cdot)$, and $W_2(\cdot)$ for all $t \in [t_0, \infty)$. Successively, we leverage this ordering to prove the boundedness of solutions of (1) and (2) such that $x_0 \in \{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_2(x) \leq c\}$. Finally, we deduce that $\lim_{t\to\infty} W(x(t)) = 0$ uniformly in $t_0 \in [0, \infty)$ and $\{t_k\}_{k\in\mathbb{N}}$ from Lemma 1 and the boundedness of $\int_{t_0}^{\infty} W(x(t))\mathrm{d}t$.

The c-sublevel set $\{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_1(x) \leq c\}$ is a proper subset of $\mathcal{B}_r(\overline{\mathcal{A}})$ since $c < \min_{x\in\partial\mathcal{B}_r(\overline{\mathcal{A}})} W_1(x)$. Next, define $\Omega_{t,c} \triangleq \{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : V(t,x) \leq c\}$, $t \geq t_0$. It follows from (19) that if $W_2(x) \leq c$ for some $x \in \mathcal{S}$, then $V(t,x) \leq c$ for all $t \geq t_0$. Therefore, $\{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_2(x) \leq c\} \subset \Omega_{t,c}$, $t \geq t_0$. Similarly, it follows from (19) that if $V(t,x) \leq c$ for some $(t,x) \in [t_0,\infty) \times \mathcal{S}$, then $W_1(x) \leq c$. Therefore, for all $t \in [t_0, \infty)$,

$$\{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_2(x) \leq c\} \subset \Omega_{t,c} \subset \{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_1(x) \leq c\} \subset \mathcal{B}_r(\overline{\mathcal{A}}) \subset \mathcal{S}. \tag{A6}$$

Next, two alternative cases are to be considered, namely $x(t) \notin \overline{\mathcal{A}}$, $t \in [t_0, \infty)$, and $x(T) \in \overline{\mathcal{A}}$ for some $T \in [t_0, \infty)$. Let us start assuming that $x(t) \notin \overline{\mathcal{A}}$, $t \in [t_0, \infty)$. In this case, since $\hat{t}_k \leq t_k$ for all $k \in \overline{\mathbb{N}}$, it follows from Lemma 2 that (18) with $\mathcal{I} = [t_0, \infty)$ can be rewritten as

$$V(t_k^+, x(t_k^+)) - \sum_{j=0}^{k-1} \left[V(t_{j+1}, x(t_{j+1})) - V(t_j, x(t_j))\right] \geq V(t, x(t)), \qquad t \in (t_k, t_{k+1}], \tag{A7}$$

along the trajectories of (1) and (2). It follows from (20) that $\dot{V}(t, x(t)) \leq 0$, $t \in [t_0, \infty) \setminus \{t_k\}_{k\in\mathbb{N}}$, along the trajectories of (1) and (2), and, since $\sum_{j=1}^{\infty} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right]$ exists and is finite, it follows from (A1) that the sequence $\{V(t_{k+1}, x(t_{k+1}))\}_{k\in\mathbb{N}}$ is bounded. Thus,

$$\lim_{k\to\infty} [V(t_k, x(t_k)) - V(t_0, x(t_0))] = \sum_{j=0}^{\infty} \left[V(t_{j+1}, x(t_{j+1})) - V(t_j, x(t_j))\right]$$

exists and is finite, and the sequence of partial sums $\left\{\sum_{j=0}^{k-1} \left[V(t_{j+1}, x(t_{j+1})) - V(t_j, x(t_j))\right]\right\}_{k\in\mathbb{N}}$ is bounded uniformly in $\{t_k\}_{k\in\mathbb{N}}$. Therefore, it follows from (A7) that

$$V(t_{\max}^+, x(t_{\max}^+)) - \Delta V(t_{\max}) \geq V(t, x(t)), \qquad t \in [t_0, \infty), \tag{A8}$$

where $\Delta V(t_k) \triangleq \sum_{j=0}^{k-1} \left[V(t_{j+1}, x(t_{j+1})) - V(t_j, x(t_j))\right]$ and $t_{\max} \triangleq \operatorname{argmax}_{k\in\mathbb{N}}[V(t_k^+, x(t_k^+)) - \Delta V(t_k)]$. Additionally, for any $t_0 \in [0, \infty)$, if $x_0 \in \{x \in \mathcal{B}_r(\overline{\mathcal{A}}) : W_2(x) \leq c\}$, then it follows from (A6) with $t = t_{\max}$ that any solution $x(\cdot)$ of (1) and (2) is such that $x(t) \in \Omega_{t_{\max},c}$, $t \in [t_0, \infty)$, and, consequently, $\|x(t)\|_{\overline{\mathcal{A}}} < r$.

Finally, since $W(\cdot)$ is continuously differentiable, $W(\cdot)$ is absolutely continuous[34(p. 222)]. Thus, it follows from (20), Lemma 4.1.8 of Reference 34, (A1), and (A7) that

$$\begin{aligned}
\int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau &\leq -\int_{t_0}^{t} \dot{V}(\tau, x(\tau))\mathrm{d}\tau \\
&= V(t_0, x_0) - V(t, x(t)) + \sum_{j=1}^{k} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right] \\
&\leq V(t_0, x_0) + \sum_{j=1}^{k} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right], \qquad t \geq t_0.
\end{aligned} \tag{A9}$$

Since $\sum_{j=1}^{\infty} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right]$ exists and is finite by assumption, and $\int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$, $t \geq t_0$, is a monotonically nondecreasing function, $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$ exists and is boudned, that is,

$$\int_{t_0}^{\infty} W(x(\tau))\mathrm{d}\tau \leq V(t_0, x_0) + \sum_{j=1}^{\infty} \left[V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\right], \qquad t \geq t_0. \tag{A10}$$

By assumption, $W(x)$, $x \in \mathcal{S}$, is continuously differentiable. Additionally, $x(t)$, $t \geq t_0$, is bounded. Finally, $f_{c,\sigma}(\cdot,\cdot)$ is locally bounded. Therefore, $W(x(\cdot))$ is bounded and piecewise continuously differentiable uniformly in $\{t_k\}_{k\in\mathbb{N}}$, and $\dot{W}(x(\cdot))$ is bounded on $[t_0,\infty)\setminus\{t_k\}_{k\in\mathbb{N}}$ uniformly in $\{t_k\}_{k\in\mathbb{N}}$. Finally, by Assumptions 1 and 2, it holds that $\inf_{k\in\mathbb{N}}|t_k - t_{k-1}| > 0$. Since the conditions of Lemma 1 are verified, $\lim_{t\to\infty} W(x(t)) = 0$ uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$. This proves the result locally, assuming that $x(t) \notin \overline{\mathcal{A}}$, $t \in [t_0,\infty)$.

Next, assume that $x(T) \in \overline{\mathcal{A}}$ for some $T \in [t_0,\infty)$. Since $\left(\{t\}\times\overline{\mathcal{A}}\right) \not\subset \mathcal{D}$ for any $t \in [t_0,\infty)$ and $\overline{\mathcal{A}}$ is compact, $x(t) \in \overline{\mathcal{A}}$, $t \in [T,\infty)$, is bounded uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$. Furthermore, since $\overline{\mathcal{A}} \subset \{x \in \mathcal{S} : W(x) = 0\}$, it holds that $\lim_{t\to\infty} W(x(t)) = 0$ uniformly in both $t_0 \in [0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$, This proves the result locally, assuming that $x(T) \in \overline{\mathcal{A}}$ for some $T \in [t_0,\infty)$.

If $\mathcal{S} = \mathbb{R}^n$ and both $W_1(\cdot)$ and $W_2(\cdot)$ are radially unbounded, then the result can be proven globally by proceeding in a similar manner; this part of the proof is omitted for brevity. ■

Proof of Theorem 3. The result directly follows from Theorem 2 with $\overline{\mathcal{A}} = \{0\}$. ■

Proof of Proposition 1. The adaptive gain $\hat{\Theta}(t)$ is absolutely continuous for all $t \in [t_0,\infty)$, and, hence, continuous for all t over compact subintervals of $[t_0,\infty)\setminus\{t_k\}_{k\in\mathbb{N}}$[34(Th. 6.3.1)]. Furthermore, $\Phi_{\sigma(\cdot)}(\cdot, x(\cdot))$ is continuous over compact subintervals of $[t_0,\infty)$. Therefore, if $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, then the control input is piecewise continuous over $[t_0,\infty)$. By assumption, the resetting events $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$ are such that if $x_{\rm ref}(t) \equiv 0$, $t \geq t_0$, then Assumptions 1 and 2 are verified by (23) and (24) with any piecewise continuous input function $u(\cdot)$, which implies that $t_{{\rm s},i} < t_{{\rm s},i+1}$ for all $i \in \mathbb{N}$. It follows from (16) that $\hat{t}_k = \max\{t_k, \bar{t}_k\} = \max\{t_{{\rm s},i}, t_{{\rm ref},i_w}\} = t_k$, $k \in \mathbb{N}$, which proves the result. ■

Proof of Proposition 2. As shown in the proof of Proposition 1, if $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$, then $t_{{\rm s},i} < t_{{\rm s},i+1}$ for all $i \in \mathbb{N}$. Furthermore, it follows from (37) that $t_{{\rm ref},i_w} < t_{{\rm ref},i_w+1}$, $(k,w) \in \mathbb{N}\times\mathbb{N}$, and $t_{{\rm s},i} < t_{{\rm ref},i_w}$. Thus, Assumptions 1 and 2 are verified by (23) and (24) with $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t, x(t)))$, $t \geq t_0$.

Next, the resetting events $\{\mathcal{D}_\sigma\}_{\sigma\in\Sigma}$ are so that if $x_{\rm ref}(t) \equiv 0$, $t \geq t_0$, and $u(\cdot)$ is piecewise continuous, then Assumptions 1 and 2 are verified by (23) and (24). Therefore, since $\eta(\hat{\Theta}(\cdot), \overline{\Phi}_\sigma(\cdot)(\cdot, x(\cdot)))$ is piecewise continuous over $[t_0,\infty)$, $t_k < t_{k+1}$, $k \in \overline{\mathbb{N}}$, and $x(t) = e(t) + x_{\rm ref}(t)$ per definition, if $u(t) = \eta(\hat{\Theta}(t), \overline{\Phi}_\sigma(t)(t, x(t)))$, then Assumptions 1 and 2 are verified by (25) and (26). ■

Proof of Proposition 3. Since Carathéodory solutions of (33) are considered, it holds that $\hat{\Theta}(t_k) = \hat{\Theta}(t_k^+)$ for all $k \in \mathbb{N}$. Thus, (40) is verified, and, since

$$\sum_{k=1}^{\infty}\left[e^{\rm T}(t_k^+)P_{\sigma(t_k^+)}e(t_k^+) - e^{\rm T}(t_k)P_{\sigma(t_k)}e(t_k)\right] = -\sum_{i=1}^{\infty}\sum_{w=1}^{\infty} s_{{\rm ref},i_w}, \tag{A11}$$

the result follows from the boundedness of $\sum_{i=1}^{\infty}\sum_{w=1}^{\infty} s_{{\rm ref},i_w}$. ■

Proof of Proposition 4. It follows from (39) that

$$P^{\frac{1}{2}}_{\sigma(t^+_{{\rm ref},i_w})}e(t^+_{{\rm ref},i_w}) = \sqrt{\frac{e^{\rm T}(t_{{\rm ref},i_w})P_{\sigma(t_{{\rm ref},i_w})}e(t_{{\rm ref},i_w}) - s_{{\rm ref},i_w}}{e^{\rm T}(t_{{\rm ref},i_w})P_{\sigma(t_{{\rm ref},i_w})}e(t_{{\rm ref},i_w})}}\,P^{\frac{1}{2}}_{\sigma(t_{{\rm ref},i_w})}e(t_{{\rm ref},i_w}), \qquad (i,w) \in \mathbb{N}\times\mathbb{N}, \tag{A12}$$

which proves that $P^{\frac{1}{2}}_{\sigma(t^+_{{\rm ref},i_w})}e(t^+_{i_w})$ and $P^{\frac{1}{2}}_{\sigma(t_{{\rm ref},i_w})}e(t_{i_w})$ are collinear. Additionally, it holds that $\left\|P^{\frac{1}{2}}_{\sigma(t^+_{{\rm ref},i_w})}e(t^+_{{\rm ref},i_w})\right\| < \left\|P_{\sigma(t_{{\rm ref},i_w})}e(t_{{\rm ref},i_w})\right\|$ for all $(i,w) \in \mathbb{N}\times\mathbb{N}$, since $s_{{\rm ref},i_w} \in \left(0, e^{\rm T}(t_{{\rm ref},i_w})P_{\sigma(t_{{\rm ref},i_w})}e(t_{{\rm ref},i_w})\right)$. ■

Proof of Theorem 4. Consider the Lyapunov function candidate (35) and note that

$$W_1(e,\hat{\Theta}) \leq V(t,e,\hat{\Theta}) \leq W_2(e,\hat{\Theta}), \quad (t,e,\Delta\Theta) \in [t_0,\infty)\times\mathbb{R}^n\times\mathbb{R}^{N\times N}, \tag{A13}$$

where

$$W_1(e,\hat{\Theta}) \triangleq \overline{\lambda}_{\min}\left(\{P_\sigma\}_{\sigma\in\Sigma}\right)\|e\|^2 + \mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}\Gamma^{-1}\Delta\Theta\right),$$
$$W_2(e,\hat{\Theta}) \triangleq \overline{\lambda}_{\max}\left(\{P_\sigma\}_{\sigma\in\Sigma}\right)\|e\|^2 + \mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}\Gamma^{-1}\Delta\Theta\right)$$

are radially unbounded, $\overline{\lambda}_{\max}\left(\{P_\sigma\}_{\sigma\in\Sigma}\right) \triangleq \max\{\lambda_{\max}(P_\sigma), \sigma\in\Sigma\}$, and $\lambda_{\max}(P_\sigma)$, $\sigma\in\Sigma$, denotes the largest eigenvalue of P_σ. Thus, it follows from (35) that

$$\begin{aligned}\dot{V}(t,e(t),\hat{\Theta}(t)) &\leq -W(e(t)) + 2e^{\mathrm{T}}(t)P_{\sigma(t)}B_{\sigma(t)}\Delta\Theta^{\mathrm{T}}(t)\overline{\Phi}_{\sigma(t)}(t,x(t)) \\ &\quad + 2\mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}(t)\Gamma^{-1}\dot{\hat{\Theta}}(t)\right) \\ &= -W(e(t)) + 2\mathrm{tr}\left(\Delta\Theta^{\mathrm{T}}(t)\left[\Gamma^{-1}\dot{\hat{\Theta}}(t) + \overline{\Phi}_{\sigma(t)}(t,x(t))e^{\mathrm{T}}(t)P_{\sigma(t)}B_{\sigma(t)}\right]\right) \\ &= -W(e(t)), \qquad t\in[t_0,\infty) \text{ a.e.,}\end{aligned} \tag{A14}$$

along the trajectories of (31), (24), and (33).

It follows from Proposition 1 that $\hat{t}_k \leq t_k$, $k\in\mathbb{N}$, and it follows from Proposition 3 $\sum_{k=1}^{\infty}\left[V(t_k^+, e(t_k^+), \hat{\Theta}(t_k)) - V(t_k, e(t_k), \hat{\Theta}(t_k))\right]$ exists and finite. Thus, it follows from Theorem 3 that every maximal solution $e(\cdot)$ of (31) and (24) is uniformly bounded in both $t_0\in[0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$ and such that $\lim_{t\to\infty} W(e(t)) = 0$ for all $e_0\in\mathbb{R}^n$ uniformly in $\{t_k\}_{k\in\overline{\mathbb{N}}}$. Since $W(e) = \overline{\lambda}_{\min}\left(\{Q_\sigma\}_{\sigma\in\Sigma}\right)\|e\|^2$, we deduce that $e(t)\to 0$ as $t\to\infty$ uniformly in $t_0\in[0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$. Similarly, boundedness of $\hat{\Theta}(\cdot)$ uniformly in $t_0\in[0,\infty)$ and $\{t_k\}_{k\in\mathbb{N}}$ directly follows from (A13) and (A14) and Theorem 3. ■