

# NONLOCAL BOUNDED VARIATIONS WITH APPLICATIONS\*

HARBIR ANTIL<sup>†</sup>, HUGO DÍAZ<sup>‡</sup>, TIAN JING<sup>§</sup>, AND ARMIN SCHIKORRA<sup>¶</sup>

**Abstract.** Motivated by problems where jumps across lower dimensional subsets and sharp transitions across interfaces are of interest, this paper studies the properties of fractional bounded variation ( $BV$ )-type spaces. Two different natural fractional analogs of classical  $BV$  are considered:  $BV^\alpha$ , a space induced from the Riesz-fractional gradient that has been recently studied by Comi-Stefani; and  $bv^\alpha$ , induced by the Gagliardo-type fractional gradient often used in Dirichlet forms and Peridynamics – this one is naturally related to the Caffarelli-Roquejoffre-Savin fractional perimeter. Our main theoretical result is that the latter  $bv^\alpha$  actually corresponds to the Gagliardo-Slobodeckij space  $W^{\alpha,1}$ . As an application, using the properties of these spaces, novel image denoising models are introduced and their corresponding Fenchel pre-dual formulations are derived. The latter requires density of smooth functions with compact support. We establish this density property for convex domains.

**Key words.** fractional derivatives, fractional BV, image denoising

**MSC codes.** 35R11, 49Jxx, 65R20 65D18

**1. Introduction.** In recent years, fractional calculus and nonlocal operators have emerged as natural tools to study various phenomena in science and engineering. Unlike their classical counterparts, fractional operators have several distinct abilities, for instance, they require less smoothness and they are nonlocal in nature. Such flexibilities have led to multiple successes of fractional derivative-based models in practical applications. For instance, magnetotellurics in geophysics [44], viscoelastic models [33], quantum spin chains and harmonic maps [17, 30, 3], deep neural networks [4], repulsive curves [45], etc.

A fundamental concept in inverse problems, such as image denoising, is the use of regularization. The article [2] introduced the fractional Laplacian as a regularizer in image denoising as an alternative to well-known approaches such as total-variation regularization. Subsequently, this model has been successfully used by various authors in imaging science as it provides a behavior that is closer to total variation based approaches [28], but it is easy to implement in practice. The current paper is motivated by these observations. We also refer to [23] for a different (discrete) nonlocal regularization in imaging.

Fundamental developments are being made in fractional calculus. In fact, now there exist notions of fractional divergence and gradient. The aforementioned fractional Laplacian, for instance, can be obtained by the composition of fractional divergence and fractional gradient. This is similar to the classical integer order setting.

---

\*Submitted to the editors DATE.

**Funding:** HA and HD are partially supported by NSF grants DMS-2110263, DMS-2111315, DMS-1913004, the Air Force Office of Scientific Research under Award NO: FA9550-22-1-0248. TJ and AS are partially supported under NSF Career DMS-2044898. AS is partially supported by Simons foundation grant no 579261

<sup>†</sup>Department of Mathematical Sciences and the Center for Mathematics and Artificial Intelligence (CMAI) George Mason University, Fairfax, VA 22030, USA. ORCID: 0000-0002-6641-1449. (hantil@gmu.edu)

<sup>‡</sup>Department for Mathematical Sciences, University of Delaware, Newark, DE 19716, USA. (hugo-diaz@udel.edu).

<sup>§</sup>Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043 (tnjing@umich.edu).

<sup>¶</sup>Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15261, USA. (armin@pitt.edu).

Such discoveries are not only fueling further developments in analysis but are also leading to new application areas or improving the existing ones. Motivated by image denoising, the goal of this paper is to study fundamental properties of the space of (nonlocal) fractional bounded variation. Based on such fractional order spaces, we introduce novel image denoising models, and we derive Fenchel dual formulations [20, chapter III] for these. Notice that such formulations are critical in deriving efficient numerical methods in the classical setting. The remainder of this section provides a precise discussion on new image denoising models to motivate the analytical tools developed in this paper.

A well-established method to solve image denoising problems is based on *total variation minimization* [1, 37, 38]. Let  $u_N : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  denote a continuous representation of an image (possibly noisy). Given a regularization parameter  $\beta > 0$ , a standard image denoising problem amounts to finding  $u$  solving

$$(1.1) \quad \arg \min_{u \in \mathcal{X}} \left\{ \beta |Du|_{\mathcal{X}} + \frac{1}{2} \|u - u_N\|_{L^2(\Omega)}^2 \right\},$$

where the space  $\mathcal{X}$  is chosen in conjunction with the norm  $|\cdot|_{\mathcal{X}}$  such that  $Du$  is well defined at least in a distributional sense, and  $u$  can be piecewise smooth. In practice, one of the most common spaces used is the space of functions with *bounded variation* (BV) defined by

$$\text{BV}(\Omega) = \{u \in L^1(\Omega) : \text{Var}(u; \Omega) < \infty\}.$$

Namely, a function  $u$  in  $L^1(\Omega, \mathbb{R})$  is said to have bounded variation if and only if

$$\text{Var}(u; \Omega) := \sup \left\{ \int_{\mathbb{R}^n} u(x) \text{Div } \Phi(x) dx : \Phi \in C_c^1(\Omega, \mathbb{R}^n), \|\Phi\|_{L^\infty(\Omega)} \leq 1 \right\} < \infty.$$

If the variation  $\text{Var}(u; \Omega)$  is finite, one can show that its distributional derivative  $Du$  is a Radon measure and  $\text{Var}(u; \Omega) = |Du|(\Omega)$ , see [5, Ch. 10]. It is well-known that  $\text{BV}(\Omega)$  preserves edges, in a noisy image, better than  $W^{1,1}(\Omega)$  while retaining several of its properties. For instance, it is a Banach space, it is lower semi-continuous on  $L^1(\Omega)$ , Sobolev inequalities, etc.

In this work, we are interested in the fractional version of the problem (1.1). For this, we first need to decide on a notion of fractional *BV*. We do so by replacing in the definition above the derivative with some suitable fractional derivative. Alas, there are many different, yet natural, fractional operators that are considered extensions of the usual gradient – and each one induces its own *BV*-space.

We will consider the two most popular notions. Firstly, we will consider the space  $BV^\alpha$ , which we refer to as *Riesz-type*. The study of  $BV^\alpha$  was initiated by Comi-Stefani in [15], see Section 2. It relies on the notion of what is sometimes referred to as Riesz gradient  $D^\alpha$ , which is simply the usual gradient combined with a regularizing Riesz potential.

The other type of fractional *BV* we consider will be denoted by  $bv^\alpha$  and is referred to as *Gagliardo-type*, see Section 3. We are not aware whether this has been considered in the literature prior to this work. The notion of fractional derivative is what we will refer to as the Gagliardo-type derivative considered in various aspects of mathematics, e.g. Dirichlet forms [26], Peridynamics [19] and harmonic analysis [34]. This Gagliardo-type  $bv^\alpha$  is naturally related to the most popular notion of a fractional perimeter defined by Caffarelli–Roquejoffre–Savin [10]. Indeed, we will show in

Theorem 3.4 that  $bv^\alpha$  coincides with the Gagliardo-Sobolev space  $W^{\alpha,1}$  – a maybe surprising feature of the case  $\alpha < 1$ , since this is false for  $\alpha = 1$ : indeed it is well-known that  $W^{1,1} \neq BV$ , see [22]. This is one of the main theoretical contributions of the current paper.

Based on these notions of fractional  $BV$ , we will introduce new types of variational models for image denoising. Namely, we study the fractional versions of (1.1),

$$(1.2) \quad \arg \min_{u \in \mathcal{X}} \left\{ \beta \text{Var}_\alpha(u; \Omega) + \frac{\gamma}{p} \|u - u_N\|_{L^p(\Omega)}^p \right\}.$$

A related model was studied by Bartels and one of the authors in [2] but working in fractional order Hilbert space  $H^s(\Omega)$  instead of  $\mathcal{X} = BV^\alpha(\Omega)$ .

We emphasize that the numerical algorithms for solving problems of type (1.1) make use of the Fenchel dual formulations [6, 12]. However, this requires dealing with the dual space of  $BV(\Omega)$ , whose full characterization is still an unknown [43]. Instead one proceeds by finding a predual problem to (1.1), i.e., a problem whose Fenchel conjugate is (1.1), see for instance [9, 11, 24]. In this case, one does not need to deal with  $BV(\Omega)^*$ , but instead the closure in  $L^p(\Omega)$  of the range of a divergence-like operator, which is the conjugate of  $-D : \mathcal{X} \subset BV(\Omega) \rightarrow \mathcal{M}(\Omega, \mathbb{R}^n)$ . We will derive a pre-dual problem corresponding to (1.2) in Section 4. Derivation of pre-dual requires density of smooth functions with compact support. This is highly non-trivial in general even in the local case. We establish this result provided that the domain  $\Omega$  is convex. Such results are of interest independent of this paper, see Propositions 4.4 and 4.8.

**2. Fractional BV in the Riesz sense.** We begin by recalling the notion of fractional Laplacian and its inverse, the Riesz potential. Denote by  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  the Fourier transform on  $\mathbb{R}^n$ . For  $\alpha > 0$  the fractional Laplacian of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with differential order  $\alpha$ , denoted by  $|D|^\alpha f$ , is given by

$$|D|^\alpha f(x) := \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}f(\xi))(x).$$

The notation  $|D|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$  is common, but we will mostly use the notation  $|D|^\alpha$  in this paper, since it states the order of derivatives more clearly. The definition above makes sense when  $\alpha < 0$ . In that case, we call the operator Riesz potential. More precisely, for all  $\alpha \in (0, n)$  we define

$$I^\alpha f(x) := \mathcal{F}^{-1}(|\xi|^{-\alpha} \mathcal{F}f(\xi))(x).$$

It is then easy to see that  $|D|^\alpha I^\alpha f = I^\alpha |D|^\alpha f = f$ , at least for suitably smooth functions with decay at infinity, i.e. the fractional Laplacian and Riesz potential are inverses to each other. The fractional Laplacian  $|D|^\alpha$  has no gradient structure. It does not converge to the gradient  $D$  when  $\alpha \rightarrow 1$ . Recently, many authors considered a fractional-order operator with a gradient structure. Although this operator can be traced as far back as [27], it has received increased interest in various applications since the works e.g. [15, 39, 40, 42]. It is defined very simply as the usual gradient of the Riesz potential

$$(2.1) \quad D^\alpha f := D I^{1-\alpha} f.$$

From its Fourier transform representation, it is easy to show that  $D^\alpha \rightarrow D$  as  $\alpha \rightarrow 1$ . The fractional divergence  $\text{Div}_\alpha$  is defined as

$$\text{Div}_\alpha f = \text{div } I^{1-\alpha} f.$$

Note that  $\text{Div}_\alpha$  is the adjoint of  $-D^\alpha$ . In fact, the following integration-by-parts formula holds

$$(2.2) \quad \int_{\mathbb{R}^n} F \cdot D^\alpha g dx = - \int_{\mathbb{R}^n} \text{Div}_\alpha F g dx \quad \forall F \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n), \forall g \in C_c^\infty(\mathbb{R}^n),$$

which follows readily from the definition via the Fourier transform and Plancherel's theorem. We remark on the integral definition of the above operators. For any  $\alpha \in (0, 1]$ , we have

$$(2.3) \quad \begin{aligned} |D|^\alpha f(x) &= c_{1,\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy, \\ D^\alpha f(x) &= c_{2,\alpha} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))(x - y)}{|x - y|^{n+\alpha+1}} dy, \\ \text{Div}_\alpha F(x) &= c_{3,\alpha} \int_{\mathbb{R}^n} \frac{(F(x) - F(y)) \cdot (x - y)}{|x - y|^{n+\alpha+1}} dy, \end{aligned}$$

for some constants  $c_{1,\alpha}$ ,  $c_{2,\alpha}$  and  $c_{3,\alpha}$ , which can be found in the literature. Having the notion of fractional gradient, we naturally obtain an associated notion of fractional BV spaces. Our definitions are very similar to [15] and different from other natural approaches as in [8] or an approach via a different type of nonlocal gradient and divergence as in [19, 34], which we will discuss later in Section 3. When using the notion of fractional BV spaces in this paper, most of the required properties will follow similar general principles as in typical BV spaces. We provide a derivation of the results that we could not find in the literature and provide references otherwise. It is possible that some of these results are known to the experts.

To distinguish the resulting space from the one discussed in Section 3, we use the notations  $BV^\alpha$ ,  $\text{Div}_\alpha$  and  $\text{Var}_\alpha$ . In Section 3 we will use  $bv^\alpha$ ,  $\text{div}_\alpha$ , and  $\text{var}_\alpha$  instead.

Let  $\alpha \in (0, 1]$  and  $f \in L^1(\mathbb{R}^n)$ , the variation of  $f$  is defined as

$$(2.4) \quad \text{Var}_\alpha(f; \mathbb{R}^n) := \sup \left\{ \int_{\mathbb{R}^n} f \text{Div}_\alpha \Phi dx : \Phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n), \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq 1 \right\}.$$

Let  $\Omega \subseteq \mathbb{R}^n$ . For any  $f \in L^1(\Omega)$ , we define

$$\text{Var}_\alpha(f; \Omega) := \text{Var}_\alpha(\chi_\Omega f; \mathbb{R}^n),$$

where  $\chi_\Omega f$  is the extension of  $f$  by zero to  $\mathbb{R}^n$ . The integral

$$\int_{\mathbb{R}^n} f \text{Div}_\alpha \Phi dx$$

is well defined for all  $f \in L^1(\mathbb{R}^n)$  and  $\Phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ , which is a consequence of the following result.

LEMMA 2.1. *Let  $\Phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ , then for any  $\alpha \in (0, 1]$  and any  $p \in [1, \infty]$  we have*

$$\text{Div}_\alpha \Phi \in L^p(\mathbb{R}^n).$$

*Proof.* Fix  $\Phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ . For  $\alpha = 1$ , we have  $\text{Div}_\alpha \Phi \in C_c(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$  for all  $p \in [1, \infty]$ . For  $\alpha < 1$ , we have from (2.3) that

$$|\text{Div}_\alpha \Phi(x)| \lesssim_\alpha (2\|\Phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \Phi\|_{L^\infty(\mathbb{R}^n)}) \int_{\mathbb{R}^n} \frac{\min\{1, |x - y|\}}{|x - y|^{n+\alpha}} dy.$$

Here  $\lesssim_\alpha$  implies that the hidden constant depends on  $\alpha$  (and any constant may depend on the dimension  $n$ ). Since  $\alpha < 1$ , the following integral is finite and has the same value for every  $x \in \mathbb{R}^n$ , i.e.

$$\int_{\mathbb{R}^n} \frac{\min\{1, |x-y|\}}{|x-y|^{n+\alpha}} dy \equiv C(n, \alpha) < \infty,$$

which implies that

$$\|\operatorname{Div}_\alpha \Phi\|_{L^\infty(\mathbb{R}^n)} \lesssim_\alpha (\|\Phi\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \Phi\|_{L^\infty(\mathbb{R}^n)}).$$

It remains to prove that  $\operatorname{Div}_\alpha \Phi \in L^1(\mathbb{R}^n)$ . Once this is shown we conclude  $\operatorname{Div}_\alpha \Phi \in L^p(\mathbb{R}^n)$  for any  $p \in [1, \infty]$  by interpolation. Taking  $R \geq 1$  large enough, such that  $\operatorname{supp} \Phi \subset B(0, R/2)$ , then for  $x \in \mathbb{R}^n \setminus B(0, R)$  we have

$$|\operatorname{Div}_\alpha \Phi(x)| \lesssim_\alpha \int_{B(0, R/2)} \frac{|\Phi(y)|}{|x-y|^{n+\alpha}} dy.$$

By Fubini's theorem, we have

$$\begin{aligned} & \|\operatorname{Div}_\alpha \Phi\|_{L^1(\mathbb{R}^n \setminus B(0, R))} \\ & \lesssim \int_{B(0, R/2)} |\Phi(y)| \left( \int_{\mathbb{R}^n \setminus B(0, R)} \frac{1}{|x-y|^{n+\alpha}} dx \right) dy \\ & \lesssim \|\Phi\|_{L^1(\mathbb{R}^n)} \sup_{y \in \mathbb{R}^n \setminus B(0, R/2)} \left( \int_{\{x: |x-y| \geq R/2\}} \frac{1}{|x-y|^{n+\alpha}} dx \right). \end{aligned}$$

Here we hide the constant by using  $\lesssim$ . Using the fact that

$$\int_{\{x: |x-y| \geq R/2\}} \frac{1}{|x-y|^{n+\alpha}} dx \lesssim_\alpha R^{-\alpha} < \infty,$$

we obtain

$$\|\operatorname{Div}_\alpha \Phi\|_{L^1(\mathbb{R}^n \setminus B(0, R))} \lesssim_\alpha \|\Phi\|_{L^1(\mathbb{R}^n)}.$$

On the complement  $B(0, R)$ , we have  $\operatorname{Div}_\alpha \Phi \in L^\infty(B(0, R)) \subset L^1(B(0, R))$ . Thus, we obtain that  $\|\operatorname{Div}_\alpha \Phi\|_{L^1(\mathbb{R}^n)} < \infty$ , which finishes the proof.  $\square$

Now we are ready to define the first fractional BV space of this work, i.e.  $BV^\alpha$ , see also [15, 16, 14] where this space was considered first. This space inherits most of its properties from the gradient structure of the Riesz-derivative  $D^\alpha$ , cf. (2.1).

**DEFINITION 2.2** (Riesz-type fractional BV). *For  $\Omega \subset \mathbb{R}^n$ , we define*

$$(2.5) \quad BV_{00}^\alpha(\Omega) := \{f \in L^1(\mathbb{R}^n) : f \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega, \operatorname{Var}_\alpha(f; \Omega) < \infty\},$$

*endowed with the norm*

$$\|f\|_{BV^\alpha(\Omega)} := \|f\|_{L^1(\Omega)} + \operatorname{Var}_\alpha(f; \Omega).$$

In this paper, we often identify  $f \in L^1(\Omega)$  with its extension by zero  $\chi_\Omega f \in L^1(\mathbb{R}^n)$ . Observe that we do not need to assume any regularity of  $\partial\Omega$  in the above (and below) definitions and results. The regularity of  $\partial\Omega$  only comes into play when we consider whether constant functions in  $\Omega$  belong to  $BV^\alpha(\Omega)$ . Namely  $1 \in L^1(\Omega)$  belongs to  $BV_{00}^\alpha(\Omega)$  (with the usual identification  $1 \in L^1(\Omega)$  corresponds to  $\chi_\Omega \in L^1(\mathbb{R}^n)$ ) if the  $\alpha$ -Cacciopoli-perimeter of  $\partial\Omega$  is finite. We refer to [15] for the definition of this perimeter. Essentially by definition we immediately obtain

PROPOSITION 2.3.  $\text{Per}_\alpha(\partial\Omega) < \infty$ , that is the surface  $\partial\Omega$  has finite  $\alpha$ -Cacciopoli-perimeter, if and only if  $\text{Var}_\alpha(1; \Omega) < \infty$ .

Observe that the Cacciopoli-perimeter above is different from the more commonly used fractional perimeter introduced by Caffarelli-Roquejoffre-Savin [10]. The latter one is related to the fractional version of BV functions defined using the divergence as used in, e.g. [19, 34]. We shall discuss it in Section 3.

Next, we note that one can obtain the existence of the distributional derivative  $D^\alpha f$  (which is a Radon measure) just like for BV, see [22, p.167, Theorem 1, Structure Theorem] If  $f \in BV_{00}^\alpha(\Omega)$ , then the mapping

$$C_c^1(\mathbb{R}^n; \mathbb{R}^n) \ni \Phi \mapsto \int_{\mathbb{R}^n} f \text{Div}_\alpha \Phi dx$$

extends to a linear functional on  $(C_c(\mathbb{R}^n; \mathbb{R}^n), \|\cdot\|_{L^\infty(\mathbb{R}^n)})$ . By the Riesz representation theorem [22, Section 1.8, Theorem 1], there exists a Radon measure  $\mu$  on  $\mathbb{R}^n$  and a  $\mu$ -measurable function  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $|\sigma| = 1$   $\mu$ -a.e. and

$$\int_{\mathbb{R}^n} f \text{Div}_\alpha \Phi dx = - \int_{\mathbb{R}^n} \Phi \cdot \sigma d\mu.$$

Moreover, we have

$$|\mu(\mathbb{R}^n)| \leq \text{Var}_\alpha(f; \Omega).$$

The latter follows by the definition of the norm. By slight abuse of notation we will denote by  $D^\alpha f$  both the distributional derivative and the measure  $D^\alpha f := \sigma \lrcorner \mu$  (where  $\lrcorner$  denotes the concatenation of function and measure), whichever is applicable.

We now consider the approximation of  $BV_{00}^\alpha(\Omega)$  functions by smooth functions. Since  $f$  is compactly supported, the convolution  $f * \eta_\varepsilon$  is in  $C_c^\infty(\mathbb{R}^n)$ . Using the same argument as in [22, Theorem 5.2], we obtain the following result.

PROPOSITION 2.4. *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded. For any  $f \in BV_{00}^\alpha(\Omega)$  there exists  $f_k \in C_c^\infty(\mathbb{R}^n)$  such that*

$$\|f_k - f\|_{L^1(\mathbb{R}^n)} + |\text{Var}_\alpha(f; \mathbb{R}^n) - \text{Var}_\alpha(f_k; \mathbb{R}^n)| \xrightarrow{k \rightarrow \infty} 0.$$

Equivalently, (since  $f$  vanishes outside of  $\Omega$ ),

$$\|f_k - f\|_{L^1(\mathbb{R}^n)} + |\text{Var}_\alpha(f; \Omega) - \text{Var}_\alpha(f_k; \mathbb{R}^n)| \xrightarrow{k \rightarrow \infty} 0.$$

We also have the following embedding theorem.

PROPOSITION 2.5. *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded and  $n \geq 2$ . Then for all  $p \in [1, \frac{n}{n-\alpha}]$  we have  $BV_{00}^\alpha(\Omega) \subseteq L^p(\mathbb{R}^n)$  and*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C(n, p, \alpha) \|f\|_{BV_{00}^\alpha(\Omega)}.$$

If  $n = 1$ , then the same results hold for all  $p \in [1, \frac{1}{1-\alpha})$ .

*Proof.* Let  $f_k$  be the approximation of  $f$  as in Proposition 2.4. By the main result in [41], for all  $p \in [1, \frac{n}{n-\alpha}]$  we have

$$\|f_k\|_{L^p(\mathbb{R}^n)} \leq C \left( \|f_k\|_{L^1(\mathbb{R}^n)} + \|D^\alpha f_k\|_{L^1(\mathbb{R}^n)} \right),$$

228 since  $f_k \in C_c^\infty(\mathbb{R}^n)$ . Observe that by an integration-by-parts formula, since we already  
 229 know  $D^\alpha f_k \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ ,

$$230 \quad \|D^\alpha f_k\|_{L^1(\mathbb{R}^n, \mathbb{R}^n)} = \text{Var}_\alpha(f_k; \mathbb{R}^n).$$

231 Since up to subsequences  $f_k$  converges to  $f$  almost everywhere we conclude from  
 232 Fatou's lemma,

$$233 \quad \begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} &\leq \liminf_{k \rightarrow \infty} \|f_k\|_{L^p(\mathbb{R}^n)} \leq C \liminf_{k \rightarrow \infty} (\|f_k\|_{L^1(\mathbb{R}^n)} + \text{Var}_\alpha(f_k; \mathbb{R}^n)) \\ &= C (\|f\|_{L^1(\mathbb{R}^n)} + \text{Var}_\alpha(f; \mathbb{R}^n)). \end{aligned}$$

234 The proof is complete.  $\square$

235 Using the duality definition of  $\text{Var}_\alpha$  and the same argument as in [22, Theorem 5.2], we  
 236 obtain the lower semicontinuity with respect to the so called intermediate convergence,  
 237 see Definition 10.1.3 and Remark 10.1.3 in [5] for details.

238 **PROPOSITION 2.6** (Lower semicontinuity). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded.*  
 239 *Assume  $\{f_k\}_{k=1}^\infty \subset BV_{00}^\alpha(\Omega)$ , and assume that  $f \in L^1(\mathbb{R}^n)$  such that*

$$240 \quad \|f_k - f\|_{L^1(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0.$$

241 *Then  $f \in BV_{00}^\alpha(\Omega)$  and we have*

$$242 \quad \text{Var}_\alpha(f; \mathbb{R}^n) \leq \liminf_{k \rightarrow \infty} \text{Var}_\alpha(f_k; \mathbb{R}^n).$$

243 *Or, equivalently,*

$$244 \quad \text{Var}_\alpha(f; \Omega) \leq \liminf_{k \rightarrow \infty} \text{Var}_\alpha(f_k; \Omega).$$

245 **COROLLARY 2.7.** *Let  $\Omega \subset \mathbb{R}^n$  be bounded. Then  $(BV_{00}^\alpha(\Omega), \|\cdot\|_{BV^\alpha(\Omega)})$  is a com-*  
 246 *plete space.*

247 *Proof.* Let  $\{f_k\}_{k=1}^\infty$  be a Cauchy sequence in  $BV_{00}^\alpha(\Omega)$ . Since  $f_k$  is Cauchy in  
 248  $L^1(\mathbb{R}^n)$ , there exists  $f \in L^1(\mathbb{R}^n)$  with  $f \equiv 0$  in  $\mathbb{R}^n \setminus \Omega$ , such that  $f_k \rightarrow f$  in  $L^1(\mathbb{R}^n)$ .  
 249 By Proposition 2.6, we find that  $f \in BV_{00}^\alpha(\Omega)$ . Using the lower semicontinuity of the  
 250 variation still from Proposition 2.6, we obtain

$$251 \quad \lim_{k \rightarrow \infty} \text{Var}_\alpha(f - f_k; \Omega) \leq \lim_{k \rightarrow \infty} \liminf_{\ell \rightarrow \infty} \text{Var}_\alpha(f_\ell - f_k; \Omega) = 0,$$

252 which completes the proof.  $\square$

253 Using the weak\*-convergence of Radon measures, and the arguments of the stan-  
 254 dard Rellich-Kondrachov compactness, see [22, Theorem 5.2 & Theorem 5.5], we have  
 255 the following result.

256 **PROPOSITION 2.8** (Weak compactness). *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded. As-*  
 257 *sume  $\{f_k\}_{k=1}^\infty \subset BV_{00}^\alpha(\Omega)$  such that*

$$258 \quad \sup_{k \geq 1} \|f_k\|_{BV^\alpha(\Omega)} < \infty.$$

259 *Then there exists  $f \in BV_{00}^\alpha(\Omega)$  such that*

$$260 \quad \text{Var}_\alpha(f; \mathbb{R}^n) \leq \liminf_{k \rightarrow \infty} \text{Var}_\alpha(f_k; \mathbb{R}^n),$$

or equivalently,

$$\mathrm{Var}_\alpha(f; \Omega) \leq \liminf_{k \rightarrow \infty} \mathrm{Var}_\alpha(f_k; \Omega),$$

and there is a subsequence  $\{f_{k_i}\}_{i=1}^\infty$  such that for all  $p \in \left[1, \frac{n}{n-\alpha}\right)$  we have

$$\|f_{k_i} - f\|_{L^p(\mathbb{R}^n)} \xrightarrow{i \rightarrow \infty} 0.$$

Lastly, as in the local case where we know that  $H^{1,1}(\Omega)$  is a subspace of  $BV(\Omega)$  (where  $H^{1,1}(\Omega)$  is the space of functions  $f \in L^1(\Omega)$  such that  $Df \in L^1(\Omega)$ ), the corresponding result for the fractional situation holds as well.

**LEMMA 2.9.** *Let  $f \in H^{\alpha,1}(\mathbb{R}^n)$ , i.e.  $f \in L^1(\mathbb{R}^n)$  and  $D^\alpha f \in L^1(\mathbb{R}^n; \mathbb{R}^n)$ . Assume additionally that  $f \equiv 0$  in  $\mathbb{R}^n \setminus \Omega$ . Then  $f \in BV_{00}^\alpha(\Omega)$ .*

*Proof.* We only need to show  $\mathrm{Var}_\alpha(\chi_\Omega f; \mathbb{R}^n) < \infty$ . For any  $\Phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$  such that  $\|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq 1$ , we have by Fubini's theorem

$$\int_{\mathbb{R}^n} \chi_\Omega f \mathrm{Div}_\alpha \Phi = - \int_{\mathbb{R}^n} D^\alpha f \cdot \Phi \leq \|\Phi\|_{L^\infty(\mathbb{R}^n)} \|D^\alpha f\|_{L^1(\mathbb{R}^n)} \leq \|D^\alpha f\|_{L^1(\mathbb{R}^n)},$$

which implies that  $\mathrm{Var}_\alpha(\chi_\Omega f; \mathbb{R}^n) < \infty$ .  $\square$

**3. Fractional BV in the Gagliardo sense.** The notion of fractional BV from Section 2 (as in [15]) is very similar to the usual  $BV$ , since it is essentially a lifting by the Riesz potential. In this section, we introduce another natural notion, which is denoted by  $bv^\alpha$ . This notion recovers the fractional perimeter as defined by Caffarelli-Roquejoffre-Savin in [10]. We begin by introducing a different type of fractional divergence as defined in [34]. We stress that related notions were known before [19] and are classically used in the theory of Dirichlet forms, cf. [26].

A (nonlocal) vector-field  $F$  on  $\mathbb{R}^n$  is defined as an  $\mathcal{L}^n \times \mathcal{L}^n$ -measurable map  $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ , which is additionally antisymmetric, i.e.  $F(x, y) = -F(y, x)$ . As in [34] the set of such vector-fields is denoted by  $\mathcal{M}(\wedge_{od} \mathbb{R}^n)$ , where  $od$  stands for off-diagonal and (as in the theory of Dirichlet forms)  $\wedge_{od}$  stands for a sort of one-form (we will not really use this aspect, we recommend the reader to take it as a purely notational choice).

We say that  $F \in L^p(\wedge_{od} \mathbb{R}^n)$  if  $F \in \mathcal{M}(\wedge_{od} \mathbb{R}^n)$  and

$$\|F\|_{L^p(\wedge_{od} \mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|F(x, y)|^p}{|x - y|^n} dx dy \right)^{\frac{1}{p}} < \infty$$

for  $p \in [1, \infty)$ , and

$$\|F\|_{L^\infty(\wedge_{od} \mathbb{R}^n)} := \mathrm{ess\,sup}_{x, y \in \mathbb{R}^n} |F(x, y)| < \infty$$

for  $p = \infty$ . For  $\Omega \subset \mathbb{R}^n$ , we say  $F \in L_{00}^p(\wedge_{od} \Omega)$  if  $F \in L^p(\wedge_{od} \mathbb{R}^n)$  and  $F(x, y) = 0$  for  $\mathcal{L}^n$ -a.e.  $x \in \mathbb{R}^n \setminus \Omega$  (and thus for a.e.  $y \in \mathbb{R}^n \setminus \Omega$ ).

The (Gagliardo sense) fractional derivative  $d_\alpha$ , which has similar properties as the gradient of a function, takes an  $\mathcal{L}^n$ -measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  into a vector-field

$$(d_\alpha f)(x, y) := \frac{f(x) - f(y)}{|x - y|^\alpha}.$$

Let us remark that if one was to consider stability as  $\alpha \rightarrow 1$ , then it would make more sense to set

$$(d_\alpha f)(x, y) := (1 - \alpha) \frac{f(x) - f(y)}{|x - y|^\alpha}.$$

However, we will not use this definition in the paper, for the simplicity of presentation.

The scalar product of two vectorfields  $F$  and  $G$  is given by

$$(F \cdot G)(x) := \int_{\mathbb{R}^n} \frac{F(x, y)G(x, y)}{|x - y|^n} dy.$$

The fractional divergence  $\text{div}_\alpha$  is then the formal adjoint to  $-d_\alpha$  with respect to the  $L^2(\mathbb{R}^n)$  scalar product, i.e. for all  $\varphi \in C_c^\infty(\mathbb{R}^n)$ , we have

$$\int_{\mathbb{R}^n} \text{div}_\alpha F \varphi dx := - \int_{\mathbb{R}^n} F \cdot d_\alpha \varphi dx = - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{F(x, y)(\varphi(x) - \varphi(y))}{|x - y|^{n+\alpha}} dy dx.$$

The multiplication of a scalar function  $f(x)$  and a vector field  $F(x, y)$  is defined as:

$$(fF)(x, y) := \frac{f(x) + f(y)}{2} F(x, y).$$

Using (3.2), we can obtain the integral formula of  $\text{div}_\alpha$ . By antisymmetry  $F(x, y) = -F(y, x)$  and the Fubini's theorem, we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x, y) \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+\alpha}} dy dx = 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{F(x, y)}{|x - y|^{n+\alpha}} dy \varphi(x) dx,$$

which enables us to give the integral definition of  $\text{div}_\alpha F$  by

$$(\text{div}_\alpha F)(x) := -2 \int_{\mathbb{R}^n} \frac{F(x, y)}{|x - y|^{n+\alpha}} dy = - \int_{\mathbb{R}^n} \frac{F(x, y) - F(y, x)}{|x - y|^{n+\alpha}} dy.$$

In what follows, by yet another slight abuse of notation we are going to use this formulation even when  $F(x, y) \neq -F(y, x)$ :

$$(\text{div}_\alpha F)(x) := - \int_{\mathbb{R}^n} \frac{F(x, y) - F(y, x)}{|x - y|^{n+\alpha}} dy.$$

It was shown in [34] how this fractional divergence naturally appears and leads to conservation laws and div-curl type results in the theory of fractional harmonic maps.

With the Fourier transform, one can check that

$$(-\Delta)^\alpha f = -c \text{div}_\alpha(d_\alpha f)$$

for some constant  $c = c(n, \alpha)$ .

Armed with the fractional divergence  $\text{div}_\alpha$ , we can define the fractional bounded variation in the Gagliardo sense.

**DEFINITION 3.1** (Gagliardo-type fractional BV). *Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . For an open set  $\Omega \subset \mathbb{R}^n$ , we define*

$$\text{var}_\alpha(f; \Omega) := \sup \left\{ \int_{\mathbb{R}^n} f \text{div}_\alpha \Phi dx : \Phi \in C_c^\infty(\Omega \times \Omega), \quad \|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq 1 \right\}.$$

Observe that this is equivalent to

$$\text{var}_\alpha(f; \Omega) = \sup \left\{ \int_\Omega f \operatorname{div}_\alpha \Phi \, dx : \Phi \in C_c^\infty(\Omega \times \Omega), \quad \|\Phi\|_{L^\infty(\Omega \times \Omega)} \leq 1 \right\}.$$

We say that  $f \in bv^\alpha(\Omega)$  if

$$\|f\|_{bv^\alpha(\Omega)} := \|f\|_{L^1(\Omega)} + \text{var}_\alpha(f; \Omega) < \infty.$$

The notion  $\text{var}_\alpha(f; \Omega)$  is well-defined by the following observations. First, to have consistency, we observe that

LEMMA 3.2. *Let  $\Phi \in C_c^1(\mathbb{R}^n \times \mathbb{R}^n)$ , then for all  $\alpha \in (0, 1)$  and all  $p \in [1, \infty]$ , we have*

$$\operatorname{div}_\alpha \Phi \in L^p(\mathbb{R}^n).$$

Observe that we exclude the case  $\alpha = 1$  since  $\operatorname{div}_\alpha \Phi$  is not well defined for  $\alpha = 1$ . A multiplication with  $(1 - \alpha)$  would lead to a stable theory as  $\alpha \rightarrow 1$ .

*Proof.* Observe that by differentiability of  $\Phi$ ,

$$|\Phi(x, y) - \Phi(y, x)| \leq |\Phi(x, y) - \Phi(x, x)| + |\Phi(x, x) - \Phi(y, x)| \leq 2\|D\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} |x - y|.$$

Then, using a similar argument as in Lemma 2.1, we have

$$\begin{aligned} |(\operatorname{div}_\alpha \Phi)(x)| &\leq 2(\|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} + \|D\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}) \int_{\mathbb{R}^n} \frac{\min\{1, |x - y|\}}{|x - y|^{n+\alpha}} dy \\ &\lesssim_\alpha (\|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} + \|D\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}), \end{aligned}$$

which implies that  $\|\operatorname{div}_\alpha \Phi\|_{L^\infty(\mathbb{R}^n)} < \infty$ . It remains to prove that  $\|\operatorname{div}_\alpha \Phi\|_{L^1(\mathbb{R}^n)} < \infty$ . Then the required result can be obtained using interpolation. Since  $\Phi$  is compactly supported, we may suppose  $\operatorname{supp} \Phi \subseteq B(0, M) \times B(0, M)$  for some  $M > 0$ . Thus, we obtain

$$\begin{aligned} \|\operatorname{div}_\alpha \Phi\|_{L^1(\mathbb{R}^n)} &= \int_{B(0, M)} \left| \int_{B(0, M)} \frac{\Phi(x, y) - \Phi(y, x)}{|x - y|^{n+\alpha}} dy \right| dx \\ &\lesssim \|D\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \int_{B(0, M)} \int_{B(0, M)} \frac{1}{|x - y|^{n+\alpha-1}} dy dx < \infty, \end{aligned}$$

which finishes the proof.  $\square$

We introduce the definition of space  $W^{\alpha, 1}(\Omega)$ , see [34] for details.

DEFINITION 3.3. *Let  $\Omega \subseteq \mathbb{R}^n$  be an open set. A function  $f$  is in  $W^{\alpha, 1}(\Omega)$  when  $f \in L^1(\Omega)$  and*

$$[f]_{W^{\alpha, 1}(\Omega)} := \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha}} dy dx < \infty.$$

The norm of  $W^{\alpha, 1}(\Omega)$  is defined as

$$\|f\|_{W^{\alpha, 1}(\Omega)} := \|f\|_{L^1(\Omega)} + [f]_{W^{\alpha, 1}(\Omega)}.$$

We now state our main theorem of this section, which is in strong contrast to the Riesz-type fractional  $BV$  functions, cf. Lemma 2.9. The fractional  $BV$  space  $bv^\alpha$  is actually equivalent to  $W^{\alpha, 1}$ , which makes this space more tractable and probably more attainable for numerical purposes.

THEOREM 3.4. Let  $\alpha \in (0, 1)$ . Let  $\Omega \subseteq \mathbb{R}^n$  be any open set. Then  $bv^\alpha(\Omega) = W^{\alpha,1}(\Omega)$ . More precisely, for any  $f \in L^1(\Omega)$  we have

$$\text{var}_\alpha(f; \Omega) = [f]_{W^{\alpha,1}(\Omega)},$$

whenever one of the two sides are finite.

*Remark 3.5.* It is well known that Theorem 3.4 is false for  $\alpha = 1$ : e.g. take any nonempty open and bounded set  $\Omega$  with finite perimeter. Then  $\chi_\Omega \notin W^{1,1}(\mathbb{R}^n)$  (e.g. because it is not continuous on almost all lines). However, we have  $\chi_\Omega \in BV(\mathbb{R}^n)$ . In that sense, Theorem 3.4 may be surprising at first. Let us mention that, although we are not aware of Theorem 3.4 in the literature, intuitively related observations have been made by people working with fractional perimeters.

*Remark 3.6.* An immediate corollary is that the fractional perimeter as defined by Caffarelli-Roquejoffre-Savin, [10],  $\text{Per}_\alpha(\Omega; \mathbb{R}^n) = \text{var}_\alpha(\chi_\Omega; \mathbb{R}^n)$ . Thus, the space  $bv^\alpha$  is the naturally associated notion for a fractional  $BV$  space when working with that perimeter.

We prove several lemmas before proving Theorem 3.4.

LEMMA 3.7. Suppose  $f \in W^{\alpha,1}(\Omega)$ , then we have  $f \in bv^\alpha(\Omega)$  and  $\text{var}_\alpha(f; \Omega) = [f]_{W^{\alpha,1}(\Omega)}$ .

*Proof.* Given any  $\Phi \in C_c^1(\Omega \times \Omega, \mathbb{R})$ . Without loss of generality, we may suppose  $\text{supp } \Phi \subseteq K \times K$ , while  $K$  is a compact subset of  $\Omega$ . Then we obtain that also  $\text{div}_\alpha \Phi = 0$  outside  $K$ .

We have from Fubini's theorem (since  $f \in W^{\alpha,1}(\Omega)$ , both sides converge absolutely)

$$(3.10) \quad \int_{\mathbb{R}^n} f \text{div}_\alpha \Phi dx = - \int_\Omega \int_\Omega \frac{f(x) - f(y)}{|x - y|^\alpha} \Phi(x, y) \frac{dy dx}{|x - y|^n}.$$

Since  $L^\infty(\Omega \times \Omega)$  is the dual of  $L^1(\Omega \times \Omega)$ , from (3.10) we obtain

$$\text{var}_\alpha(f; \Omega) = [f]_{W^{\alpha,1}(\Omega)},$$

which completes the proof.  $\square$

The lemma above has not yet proven Theorem 3.4: if we only know  $f \in bv^\alpha(\Omega)$  we cannot yet apply Lemma 3.7. However, Lemma 3.7 does give us the direction  $\text{var}_\alpha(f; \Omega) \leq [f]_{W^{\alpha,1}(\Omega)}$  whenever the right-hand side is finite (because in that case we can indeed apply Lemma 3.7).

Next, we observe the following lower semi-continuity result.

LEMMA 3.8. Suppose  $f_k \in bv^\alpha(\Omega)$  for all  $k \in \mathbb{N}$  and  $\|f_k - f\|_{L^1(\Omega)} \rightarrow 0$  as  $k \rightarrow \infty$ . Then we have

$$\text{var}_\alpha(f; \Omega) \leq \liminf_{k \rightarrow \infty} \text{var}_\alpha(f_k; \Omega),$$

and in particular  $f \in bv^\alpha(\Omega)$ .

*Proof.* Consider any  $\Phi \in C_c^1(\Omega \times \Omega)$  with  $\|\Phi\|_{L^\infty(\Omega \times \Omega)} \leq 1$ . Since  $f_k \rightarrow f$  in  $L^1(\Omega)$ , and  $\text{div}_\alpha \Phi$  is bounded by Lemma 3.2, we have  $\|f_k \text{div}_\alpha \Phi - f \text{div}_\alpha \Phi\|_{L^1(\Omega)} \rightarrow 0$ . Thus, we have

$$(3.11) \quad \int_\Omega f \text{div}_\alpha \Phi dx = \lim_{k \rightarrow \infty} \int_\Omega f_k \text{div}_\alpha \Phi dx \leq \liminf_{k \rightarrow \infty} \text{var}_\alpha(f_k; \Omega).$$

Taking the supremum over all admissible  $\Phi$ , we have

$$(3.12) \quad \text{var}_\alpha(f; \Omega) \leq \liminf_{k \rightarrow \infty} \text{var}_\alpha(f_k; \Omega),$$

which completes the proof.  $\square$

To prove  $\text{var}_\alpha(f; \Omega) \geq [f]_{W^{\alpha,1}(\Omega)}$  (whenever the left-hand side is finite), the last missing ingredient is the following recovery sequence result. In the following we say that a set  $G$  is compactly contained in a set  $\Omega$ , in symbols  $G \subset\subset \Omega$ , if  $G$  is bounded and  $\overline{G} \subset \Omega$ .

**LEMMA 3.9.** *Let  $\Omega \subseteq \mathbb{R}^n$  be any open set. Assume  $f \in L^1(\Omega)$  with  $\text{var}_\alpha(f; \Omega) < \infty$ . Then for any open  $G \subset\subset \Omega$  there exists  $f_k \in C_c^\infty(\Omega)$ , for all  $k \in \mathbb{N}$ , such that*

$$f_k \rightarrow f \quad \text{in } L^1(G)$$

and

$$\limsup_{k \rightarrow \infty} \text{var}_\alpha(f_k; G) \leq \text{var}_\alpha(f; \Omega).$$

*Proof.* Since  $G \subset\subset \Omega$ , there exist open sets  $U$  and  $V$  such that  $G \subset\subset U \subset\subset V \subset\subset \Omega$ . Pick  $\zeta \in C_c^\infty(V)$  such that  $\zeta = 1$  on  $U$  and  $\zeta \leq 1$  in all of  $\mathbb{R}^n$ . Take  $\varepsilon_0 > 0$  such that  $B_\varepsilon(G) := \{z \in \mathbb{R}^n : \text{dist}(z, G) < \varepsilon\} \subset\subset U$  and  $B_\varepsilon(V) \subset\subset \Omega$  for any  $\varepsilon \in (0, \varepsilon_0)$ . Let  $\eta \in C_c^\infty(B(0, 1))$ ,  $\int \eta = 1$ , be the usual mollifier kernel and set  $\eta_\varepsilon := \varepsilon^{-n} \eta(\cdot/\varepsilon)$ . For  $\varepsilon \in (0, \varepsilon_0)$ , we define  $f_\varepsilon := \eta_\varepsilon * (f\zeta)$ , then  $\text{supp } f_\varepsilon \subseteq \Omega$ . Given any  $\Phi \in C_c^1(G \times G)$  with  $\|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq 1$ . Using (3.4), the Fubini's theorem and the substitution  $x' = x - z$  and  $y' = y - z$ , we obtain

$$\begin{aligned} (3.13) \quad & \int_{\mathbb{R}^n} f_\varepsilon \text{div}_\alpha \Phi dx \\ &= \int_G \eta_\varepsilon * (f\zeta) \text{div}_\alpha \Phi dx \\ &= \int_G \left( \int_{B(0, \varepsilon)} \eta_\varepsilon(z) f(x-z) \zeta(x-z) dz \right) \left( - \int_G \frac{\Phi(x, y) - \Phi(y, x)}{|x-y|^{n+\alpha}} dy \right) dx \\ &= - \int_G \int_G \int_{B(0, \varepsilon)} f(x-z) \zeta(x-z) \eta_\varepsilon(z) \frac{\Phi(x, y) - \Phi(y, x)}{|x-y|^{n+\alpha}} dz dy dx \\ &= - \int_{B_\varepsilon(G)} \int_{B_\varepsilon(G)} \int_{B(0, \varepsilon)} f(x') \zeta(x') \eta_\varepsilon(z) \frac{\Phi(x' + z, y' + z) - \Phi(y' + z, x' + z)}{|x' - y'|^{n+\alpha}} dz dy' dx'. \end{aligned}$$

Notice that since  $\eta_\varepsilon(-z) = \eta_\varepsilon(z)$ , we have

$$(3.14) \quad (\eta_\varepsilon * \Phi)(x', y') := \int_{B(0, \varepsilon)} \eta_\varepsilon(z) (\Phi(x' + z, y' + z) - \Phi(y' + z, x' + z)) dz.$$

Thus, by (3.13) we have

$$\begin{aligned} \int_{\mathbb{R}^n} f_\varepsilon \text{div}_\alpha \Phi dx &= - \int_{B_\varepsilon(G)} \int_{B_\varepsilon(G)} f(x') \zeta(x') \frac{(\eta_\varepsilon * \Phi)(x', y')}{|x' - y'|^{n+\alpha}} dy' dx' \\ &= - \int_{B_\varepsilon(G)} \int_{B_\varepsilon(G)} f(x') \frac{1}{2} (\zeta(x') + \zeta(y')) \frac{(\eta_\varepsilon * \Phi)(x', y')}{|x' - y'|^{n+\alpha}} dy' dx' \\ &\quad - \int_{B_\varepsilon(G)} \int_{B_\varepsilon(G)} f(x') \frac{1}{2} (\zeta(x') - \zeta(y')) \frac{(\eta_\varepsilon * \Phi)(x', y')}{|x' - y'|^{n+\alpha}} dy' dx'. \end{aligned}$$

420 Since  $B_\varepsilon(G) \subset U$  and  $\zeta \equiv 1$  in  $U$ , the second term vanishes. Setting

$$421 \quad \Psi_\varepsilon(x', y') := \frac{1}{2}(\zeta(x') + \zeta(y')) (\eta_\varepsilon * \Phi)(x', y'),$$

422 we see that  $\Psi_\varepsilon \in C_c^\infty(\Omega \times \Omega)$ ,  $\Psi_\varepsilon(x', y') = -\Psi_\varepsilon(y', x')$ , and

$$423 \quad \int_{\mathbb{R}^n} f_\varepsilon \operatorname{div}_\alpha \Phi dx = \int_\Omega f \operatorname{div}_\alpha \Psi_\varepsilon dx'.$$

424 It is easy to check that

$$425 \quad \left| \frac{1}{2}(\zeta(x') + \zeta(y')) (\eta_\varepsilon * \Phi)(x', y') \right| \leq |(\eta_\varepsilon * \Phi)(x', y')| \leq \|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq 1.$$

426 Thus, we have shown that for any  $\Phi \in C_c^\infty(G \times G)$  with  $\|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq 1$ , and any  
427  $\varepsilon < \varepsilon_0$ , there is

$$428 \quad \int_{\mathbb{R}^n} f_\varepsilon \operatorname{div}_\alpha \Phi dx \leq \operatorname{var}_\alpha(f; \Omega).$$

429 Taking the supremum over such test-functions  $\Phi$  we obtain

$$430 \quad \sup_{\varepsilon \in (0, \varepsilon_0)} \operatorname{var}_\alpha(f_\varepsilon; G) \leq \operatorname{var}_\alpha(f; \Omega).$$

431 In particular,

$$432 \quad \limsup_{\varepsilon \rightarrow 0} \operatorname{var}_\alpha(f_\varepsilon; G) \leq \operatorname{var}_\alpha(f; \Omega).$$

433 By usual mollifier arguments we have  $f_\varepsilon \rightarrow \zeta f$  in  $L^1(\mathbb{R}^n)$  as  $\varepsilon \rightarrow 0$ . Since  $\zeta \equiv 1$  in  $G$ ,  
434 we have  $f_\varepsilon \rightarrow f$  in  $L^1(G)$  as  $\varepsilon \rightarrow 0$ .  $\square$

435 We now finish the proof of the main theorem.

*Proof of Theorem 3.4.* Let  $f \in L^1(\Omega)$ . In Lemma 3.7, we have proved

$$[f]_{W^{\alpha,1}(\Omega)} \geq \operatorname{var}_\alpha(f; \Omega),$$

whenever the left-hand side is finite. So we only need to establish

$$[f]_{W^{\alpha,1}(\Omega)} \leq \operatorname{var}_\alpha(f; \Omega),$$

436 whenever the right-hand side is finite.

437 Given any  $G \subset\subset \Omega$ , we can take a sequence  $\{f_k\}_{k=1}^\infty$  as stated in Lemma 3.9.  
438 Since  $f_k \in C_c^\infty(\Omega)$ , we have  $f_k \in W^{\alpha,1}(G)$ , so Lemma 3.7 is applicable. Combining  
439 Lemma 3.7 and Lemma 3.9, we find

$$440 \quad \limsup_{k \rightarrow \infty} [f_k]_{W^{\alpha,1}(G)} \leq \limsup_{k \rightarrow \infty} \operatorname{var}_\alpha(f_k; G) \leq \operatorname{var}_\alpha(f; \Omega).$$

441 Since  $f_k \rightarrow f$  in  $L^1(G)$ , up to passing to a subsequence, we may assume that  $f_k(x)$   
442 converges to  $f(x)$  a.e. in  $G$ . Using Fatou's lemma, we obtain

$$443 \quad (3.15) \quad \int_G \int_G \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha}} dy dx \leq \liminf_{k \rightarrow \infty} \int_G \int_G \frac{|f_k(x) - f_k(y)|}{|x - y|^{n+\alpha}} dy dx.$$

444 Thus, we obtain

$$445 \quad (3.16) \quad [f]_{W^{\alpha,1}(G)} \leq \liminf_{k \rightarrow \infty} [f_k]_{W^{\alpha,1}(G)} \leq \operatorname{var}_\alpha(f; \Omega).$$

Picking an increasing sequence of open sets  $\{G_m\}$ , such that  $G_m \subset \subset \Omega$  and

$$(3.17) \quad \bigcup_{m=1}^{\infty} G_m = \Omega.$$

Applying the above argument to  $G = G_m$ , we have  $[f]_{W^{\alpha,1}(G_m)} \leq \text{var}_{\alpha}(f; \Omega)$  for any  $m \in \mathbb{N}$ . Using Fatou's lemma again, we have

$$(3.18) \quad \begin{aligned} [f]_{W^{\alpha,1}(\Omega)} &\leq \liminf_{m \rightarrow \infty} \int_{G_m} \int_{G_m} \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha}} dy dx \\ &\leq \liminf_{m \rightarrow \infty} [f]_{W^{\alpha,1}(G_m)} \leq \text{var}_{\alpha}(f; \Omega), \end{aligned}$$

which concludes the proof.  $\square$

Using Theorem 3.4, we can easily obtain the following result.

**PROPOSITION 3.10** (Weak compactness). *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set with Lipschitz boundary. Assume that  $\{f_k\}_{k=1}^{\infty} \subset bv^{\alpha}(\Omega)$  such that*

$$\sup_{k \in \mathbb{N}} \|f_k\|_{bv^{\alpha}(\Omega)} < \infty.$$

*Then there exists  $f \in bv^{\alpha}(\Omega)$  such that*

$$\text{var}_{\alpha}(f; \Omega) \leq \liminf_{k \rightarrow \infty} \text{var}_{\alpha}(f_k; \Omega),$$

*and there is a subsequence  $\{f_{k_i}\}_{i=1}^{\infty}$ , such that for all  $p \in \left[1, \frac{n}{n-\alpha}\right)$*

$$\|f_{k_i} - f\|_{L^p(\Omega)} \xrightarrow{i \rightarrow \infty} 0.$$

*Proof.* By Theorem 3.4, we have

$$\text{var}_{\alpha}(f_k, \Omega) = [f_k]_{W^{\alpha,1}(\Omega)}.$$

Since  $\Omega$  is a Lipschitz domain, it is regular in the sense of [46]. Thus, by the main result of [46], we can find an extension  $\tilde{f}_k \in W^{\alpha,1}(\mathbb{R}^n)$  with compact support,  $\tilde{f}_k = f_k$  a.e. in  $\Omega$ , such that

$$[\tilde{f}_k]_{W^{\alpha,1}(\mathbb{R}^n)} \lesssim [f_k]_{W^{\alpha,1}(\Omega)}.$$

From the usual Rellich theorem, we find a subsequence  $(f_{k_i})_{i \in \mathbb{N}}$ , such that for all  $p \in \left[1, \frac{n}{n-\alpha}\right)$

$$\|f_{k_i} - f\|_{L^p(\Omega)} \xrightarrow{i \rightarrow \infty} 0,$$

see [18, Corollary 7.2]. In particular, in view of Lemma 3.8,

$$\text{var}_{\alpha}(f; \Omega) \leq \liminf_{k \rightarrow \infty} \text{var}_{\alpha}(f_k; \Omega). \quad \square$$

Using Theorem 3.4, we also readily obtain the Sobolev embedding theorem, which can be proved using the extension theorem as in Proposition 3.10 above and then [32, Theorem 9].

**PROPOSITION 3.11.** *Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded set with Lipschitz boundary. Then there exists a constant  $C = C(n, \alpha) > 0$  such that for any  $f \in bv^{\alpha}(\Omega)$ ,*

$$\|f\|_{L^{\frac{n}{n-\alpha}}(\Omega)} \leq C \text{var}_{\alpha}(f; \Omega).$$

We also obtain the following density result, which might be known to experts (observe that this density is not true for  $\alpha = 1$ , cf. [22, Theorem 5.3 and remark after]). Using the identification in Theorem 3.4, the extension property in [46], and the usual mollification in [18, Theorem 2.4.] or [35, Lemma 26], we have the following result.

**COROLLARY 3.12.** *Let  $\alpha \in (0, 1)$ . Let  $\Omega \subset \mathbb{R}^n$  be any open and bounded set with Lipschitz boundary, then  $\mathcal{C}^\infty(\overline{\Omega})$  is dense in  $bv^\alpha(\Omega)$ .*

Let us make a last remark about traces. For a classical  $BV$  function there is a trace, [5, Theorem 10.2.1]. However, this will not be true for  $bv^\alpha(\Omega)$ , since  $W^{\alpha,1}(\Omega)$  does not have a reasonably defined trace. The typical approach is then the notion of a fat boundary trace, which we do not pursue in this paper.

**4. Image Denoising and Predual Problem.** Let  $\Omega \subset \mathbb{R}^n$  be a open and bounded set with a Lipschitz continuous boundary,  $\alpha \in (0, 1)$ ,  $p \in (1, \infty)$ ,  $p^\infty := \frac{n}{n-\alpha}$ , and  $u_N \in L^p(\Omega)$ . Based on the two fractional variations considered in this work we consider the (primal) problems for some fixed positive parameters  $\beta$  and  $\gamma$

$$(\mathcal{P}_R) \quad \inf_{u \in L^p(\Omega)} \left\{ \frac{\gamma}{p} \|u - u_N\|_{L^p(\Omega)}^p + \beta \text{Var}_\alpha(\chi_\Omega u; \mathbb{R}^n) \right\},$$

$$(\mathcal{P}_G) \quad \inf_{u \in L^p(\Omega)} \left\{ \frac{\gamma}{p} \|u - u_N\|_{L^p(\Omega)}^p + \beta \text{var}_\alpha(u; \Omega) \right\}.$$

Note that the condition of  $u$  having bounded fractional variation is imposed implicitly, and it is also clear that both problems are strictly convex for  $p > 1$ . Therefore, we use well-known results from *convex analysis*, cf. [20], to study the minimizers of Problems  $(\mathcal{P}_R)$  and  $(\mathcal{P}_G)$ . Regularity theory to a related problem to  $\mathcal{P}_G$  was recently studied in [36, 7].

**Convex Analysis and Optimization.** As usual in convex optimization, we consider the so-called dual problem, which usually gives new insights about the structure of the primal problem. In this work, we consider a different but related approach coined as predual method. Here we mainly follow the approach given in [9, 13, 24]. In order to introduce this method, we need some definitions, cf. [20, Ch. I]. Consider a Banach space  $V$  and its topological dual  $V^*$ , with duality pairing denoted by  $\langle \cdot, \cdot \rangle_{V^*, V}$ . Given  $\mathcal{F} : V \rightarrow \overline{\mathbb{R}}$ , its *Fenchel conjugate* is given by  $\mathcal{F}^* : V^* \rightarrow \overline{\mathbb{R}}$ ,

$$(4.1) \quad u^* \mapsto \mathcal{F}^*(u^*) := \sup_{u \in V} \{ \langle u^*, u \rangle_{V^*, V} - \mathcal{F}(u) \}.$$

We denote by  $\partial \mathcal{F}(u)$  the subdifferential map of  $\mathcal{F}$  at the point  $u \in V$ , see [20, Definition I.5.1]. The following characterization holds,

$$(4.2) \quad \begin{aligned} u^* \in \partial \mathcal{F}(u) \text{ if and only if } \mathcal{F}(u) \text{ is finite and} \\ \langle u^*, v - u \rangle_{V^*, V} + \mathcal{F}(u) \leq \mathcal{F}(v), \quad \forall v \in V. \end{aligned}$$

We now introduce a process known as *dualization* [20, Chs. III-IV], here we will focus on problems of the form:

$$(\mathcal{Q}) \quad \inf_{u \in V} \{ F(u) + G(\Lambda u) \},$$

where  $Y$  is a Hausdorff topological space with dual  $Y^*$ ,  $\Lambda \in \mathcal{L}(V, Y)$ , with transpose  $\Lambda^* \in \mathcal{L}(Y^*, V^*)$ , and  $F : V \rightarrow \overline{\mathbb{R}}$ ,  $G : V \rightarrow \overline{\mathbb{R}}$ . We define the dual problem of  $(\mathcal{Q})$  as

$$(\mathcal{Q}^*) \quad \sup_{v \in Y^*} -\Phi^*(0, v),$$

where  $\Phi^* : V^* \times Y^* \rightarrow \overline{\mathbb{R}}$  is the Fenchel conjugate (dual) of  $\Phi : V \times Y \rightarrow \overline{\mathbb{R}}$ ,  $(u, p) \mapsto \Phi(u, p) := F(u) + G(p + \Lambda u)$ , see (4.1). The next theorem gives conditions for the so-called *Fenchel's duality*, cf. [20, Theorem III.4.1] and [21, Pg. 130].

**THEOREM 4.1.** *Assume  $V$  and  $Y$  are Banach spaces,  $F$  and  $G$  are convex and lower semicontinuous (l.s.c.), and there exists  $v_0 \in V$  such that  $F(v_0) < \infty$ ,  $G(\Lambda v_0) < \infty$ , and  $G$  is continuous at  $\Lambda v_0$ . Then, the problems  $(\mathcal{Q})$  and  $(\mathcal{Q}^*)$  are related by:*

$$\begin{aligned} \inf_{u \in V} \{F(u) + G(\Lambda u)\} &= \sup_{v \in Y^*} -\Phi^*(0, v) \\ &= \sup_{v \in Y^*} \{-F^*(\Lambda^* v) - G^*(-v)\}, \end{aligned}$$

and there exists at least one solution to  $(\mathcal{Q}^*)$ . Moreover, if  $\bar{u}$  and  $\bar{v}$  are solutions for  $(\mathcal{Q})$  and  $(\mathcal{Q}^*)$ , respectively, then

$$\begin{aligned} (4.3) \quad &\Lambda^* \bar{v} \in \partial F(\bar{u}), \\ &-\bar{v} \in \partial G(\Lambda \bar{u}). \end{aligned}$$

In general, there are different choices for  $F, G$  and  $\Lambda$  in order to write a given problem as in  $(\mathcal{Q})$ . Here we consider one that satisfy the hypothesis of Theorem 4.1 in a straightforward manner. We now show existence and characterization for minimizers of problems  $(\mathcal{P}_R)$  and  $(\mathcal{P}_G)$ .

**Riesz-type.** By Proposition 2.5, for  $p \in [1, p^\infty]$ , with  $p^\infty := \frac{n}{n-\alpha}$ , we can consider the problem  $(\mathcal{P}_R)$  defined on  $L^p(\Omega)$  or  $BV_{00}^\alpha(\Omega)$ , cf. (2.5), interchangeably. The next lemma shows that the problem  $(\mathcal{P}_R)$ , related to the Riesz-type of fractional bounded variation, has a solution and for  $p > 1$  it is unique.

**LEMMA 4.2.** *For  $p \in (1, p^\infty)$ , the problem  $(\mathcal{P}_R)$  has a unique solution  $\bar{u} \in BV_{00}^\alpha(\Omega)$*

*Proof.* Let  $p \in [1, \infty)$ , define  $\mathcal{J}_R : (L^p(\Omega), \|\cdot\|_{L^p(\Omega)}) \rightarrow \overline{\mathbb{R}}$ , given by

$$(4.4) \quad \mathcal{J}_R(u) := \frac{\gamma}{p} \|u - u_N\|_{L^p(\Omega)}^p + \beta \text{Var}_\alpha(\chi_\Omega u; \mathbb{R}^n).$$

It is clear that

$$0 \leq \inf_{u \in L^p(\Omega)} \mathcal{J}_R(u) \leq \frac{\gamma}{p} \|u_N\|_{L^p(\Omega)}^p.$$

Now, let  $(u_k)_{k \in \mathbb{N}} \subseteq L^p(\Omega)$  be a minimizing sequence associated to the problem  $(\mathcal{P}_R)$ , then for each  $k \in \mathbb{N}$

$$\begin{aligned} \|u_k\|_{L^p(\Omega)} &\leq \|u_k - u_N\|_{L^p(\Omega)} + \|u_N\|_{L^p(\Omega)} \leq 2\|u_N\|_{L^p(\Omega)}, \text{ and} \\ \text{Var}_\alpha(\chi_\Omega u_k; \mathbb{R}^n) &\leq \frac{\gamma}{p\beta} \|u_N\|_{L^p(\Omega)}. \end{aligned}$$

Then, for  $p \in [1, p^\infty)$ , Propositions 2.5 and 2.8 imply there exist  $\bar{u} \in BV_{00}^\alpha(\Omega) \hookrightarrow L^p(\mathbb{R}^n)$  and a subsequence  $\{u_{k_i}\}_{i \in \mathbb{N}}$  such that

$$\text{Var}_\alpha(\bar{u}; \mathbb{R}^n) \leq \liminf_{i \rightarrow \infty} \text{Var}_\alpha(u_{k_i}; \mathbb{R}^n) \quad \text{and} \quad \|u_{k_i} - u_N\|_{L^p(\Omega)}^p \xrightarrow{i \rightarrow \infty} \|\bar{u} - u_N\|_{L^p(\Omega)}^p.$$

Thus, the existence of a solution for  $(\mathcal{P}_R)$  follows from the fact that  $\bar{u} = \chi_\Omega \bar{u}$ , a.e., for the uniqueness it is enough to notice that  $\mathcal{J}_R$ , cf. (4.4), is a strictly convex

functional for  $p > 1$ . In fact, if  $\bar{u}_1$  and  $\bar{u}_2$  were two different solutions to  $(\mathcal{P}_R)$ , then for  $\lambda \in (0, 1)$ ,

$$\begin{aligned}
& \mathcal{J}_R(\lambda \bar{u}_1 + (1 - \lambda) \bar{u}_2) \\
&= \frac{\gamma}{p} \|\lambda \bar{u}_1 + (1 - \lambda) \bar{u}_2 - u_d\|_{L^p(\Omega)}^p + \beta \text{Var}_\alpha(\chi_\Omega(\lambda \bar{u}_1 + (1 - \lambda) \bar{u}_2); \mathbb{R}^n) \\
&= \frac{\gamma}{p} \|\lambda(\bar{u}_1 - u_d) + (1 - \lambda)(\bar{u}_2 - u_d)\|_{L^p(\Omega)}^p \\
&\quad + \beta \text{Var}_\alpha(\chi_\Omega(\lambda \bar{u}_1 + (1 - \lambda) \bar{u}_2); \mathbb{R}^n) \\
&< \frac{\gamma}{p} \lambda \|\bar{u}_1 - u_d\|_{L^p(\Omega)}^p + \frac{\gamma}{p} (1 - \lambda) \|\bar{u}_2 - u_d\|_{L^p(\Omega)}^p \\
&\quad + \beta \text{Var}_\alpha(\chi_\Omega(\lambda \bar{u}_1 + (1 - \lambda) \bar{u}_2); \mathbb{R}^n) \\
&\leq \lambda \mathcal{J}_R(\bar{u}_1) + (1 - \lambda) \mathcal{J}_R(\bar{u}_2).
\end{aligned}$$

Thus,  $\bar{u}_1 = \bar{u}_2$  a.e. and by the definition of  $\text{Var}_\alpha$ , cf. (2.4), the proof concludes.  $\square$

Next, we will derive an expression of the predual of  $(\mathcal{P}_R)$ . In order to do that, we start with the regularity for the “test functions” in (2.4). It is clear that if  $u \in L^1(\mathbb{R}^n)$  and  $\text{supp } u \subset \Omega$ , then  $\int_{\mathbb{R}^n} u \text{Div}_\alpha \Phi \, dx$  does not depend on  $\text{Div}_\alpha \Phi|_{\Omega^c}$ . This motivates us to define  $\|\Phi\|_{X_{\text{Riesz}}} := \sqrt[q]{\|\Phi\|_{L^q(\mathbb{R}^n, \mathbb{R}^n)}^q + \|\text{Div}_\alpha \Phi\|_{L^q(\Omega)}^q}$  for  $\Phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ , where  $q := \frac{p}{p-1}$ . We consider the space  $X_{\text{Riesz}} := X_{\text{Riesz}}(\Omega, q, \alpha)$ , given by

$$X_{\text{Riesz}} := \overline{C_c^1(\mathbb{R}^n; \mathbb{R}^n)}^{\|\cdot\|_{X_{\text{Riesz}}}}.$$

We also define an auxiliary problem

$$(\mathcal{Q}_R) \quad \inf_{\Phi \in X_{\text{Riesz}}} \left\{ \frac{1}{q} \|\Phi\|_{L^q(\Omega)}^q - \int_{\Omega} u_N (-\text{Div}_\alpha \Phi) + I_\beta(\Phi) \right\},$$

where  $I_\beta$  denotes the convex indicator function defined as

$$I_\beta(\Phi) := \begin{cases} 0 & : \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta, \\ +\infty & : \text{otherwise.} \end{cases}$$

We will establish that  $(\mathcal{Q}_R)$  is the pre-dual problem to  $(\mathcal{P}_R)$ , i.e., dual of  $(\mathcal{Q}_R)$  will be  $(\mathcal{P}_R)$  if  $\Omega$  is convex.

We begin by noticing that  $(\mathcal{Q}_R)$  fits in the abstract framework of  $(\mathcal{Q})$  if we consider the spaces:  $Y := (L^q(\Omega), \|\cdot\|_{L^q(\Omega)})$ ,  $V := (X_{\text{Riesz}}, \|\cdot\|_{X_{\text{Riesz}}})$ , and the operators:

$$\begin{aligned}
(4.5) \quad & G : Y \rightarrow \overline{\mathbb{R}}, \quad G(v) := \frac{1}{q} \|v\|_{L^q(\Omega)}^q - \int_{\Omega} u_N v \, dx, \\
& F : V \rightarrow \overline{\mathbb{R}}, \quad F(\Phi) := I_\beta(\Phi), \\
& \Lambda : V \rightarrow Y, \quad \Lambda(\Phi) := (-\text{Div}_\alpha \Phi)|_{\Omega}.
\end{aligned}$$

To compute the dual problem of  $(\mathcal{Q}_R)$ , we compute the Fenchel conjugate of  $F, G$  and  $\Lambda$ , given in (4.5).

**PROPOSITION 4.3.** *Let  $\Omega \subseteq \mathbb{R}^n$  be open, bounded, convex. Let*

$$Y := (L^q(\Omega), \|\cdot\|_{q, \Omega}), \quad V := (X_{\text{Riesz}}, \|\cdot\|_{X_{\text{Riesz}}}),$$

and operators  $F, G$  and  $\Lambda$  be defined as in (4.5), then

$$\begin{aligned} G^* : Y^* &\rightarrow \overline{\mathbb{R}}, \quad u \mapsto \frac{1}{p} \|u + u_N\|_{L^p(\Omega)}^p, \\ F^* : V^* &\rightarrow \overline{\mathbb{R}}, \quad \Psi^* \mapsto \sup_{\substack{\Phi \in V \\ \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta}} \langle \Psi^*, \Phi \rangle_{V^*, V}, \\ F^* \circ \Lambda^* : Y^* &\rightarrow \overline{\mathbb{R}}, \quad u \mapsto \beta \text{Var}_\alpha(\chi_\Omega u; \mathbb{R}^n). \end{aligned}$$

Proposition 4.3, while in principle looking very similar to the arguments in [24, Section 2], contains a quite serious subtlety. Observe that [24] does not consider test-functions with the natural restriction  $\|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta$ , but resorts to discussing component-wise control  $|\Phi^i| \leq \beta$ ,  $i = 1, \dots, n$  (leading to a nonstandard  $BV$ -space) – which is crucially needed in their argument to compute the predual.

Instead we show in our paper that for bounded, open, convex sets  $\Omega$  we do not need such unnatural restrictions. The main property we use is the controllable distance of rescaled sets, cf. Lemma A.1 and Lemma A.2. The main novelty is contained in the next proposition. Notice that such a result is even critical to prove the result in [24, 25], where  $\alpha = 1$ , for the natural  $BV$ -space.

PROPOSITION 4.4. *If  $\Omega$  is convex, then for all  $\Phi \in X_{\text{Riesz}}$ ,*

$$I_\beta(\Phi) = \tilde{I}_\beta(\Phi),$$

where

$$\tilde{I}_\beta(\Phi) := \begin{cases} 0 & : \text{if there exists } \Psi_k \in C_c^\infty(\mathbb{R}^n), \Psi_k \rightarrow \Phi \text{ in } X_{\text{Riesz}} \\ & \text{such that } \sup_k \|\Psi_k\|_{L^\infty(\mathbb{R}^n)} \leq \beta, \\ +\infty & : \text{otherwise.} \end{cases}$$

*Proof.* We first observe that

$$(4.6) \quad \tilde{I}_\beta(\Phi) = 0 \Rightarrow I_\beta(\Phi) = 0.$$

Indeed, if  $\tilde{I}_\beta(\Phi) = 0$  then there exists a sequence  $\Psi_k \in C_c^\infty(\mathbb{R}^n)$  with  $\|\Psi_k\|_{L^\infty(\mathbb{R}^n)} \leq \beta$ , such that  $\Psi_k \rightarrow \Phi$  in  $X_{\text{Riesz}}$ . In particular, we have  $\|\Psi_k - \Phi\|_{L^q(\mathbb{R}^n)} \xrightarrow{k \rightarrow \infty} 0$ . Then there exists a subsequence, still denoted by  $\Psi_k$ , such that  $\Psi_k$  converges a.e. to  $\Phi$ , which implies that  $|\Phi(x)| \leq \beta$  a.e. in  $\mathbb{R}^n$ , i.e.  $\|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta$ . By the definition of  $I_\beta$ , we have  $I_\beta(\Phi) = 0$ , which proves (4.6).

From (4.6) we conclude  $\tilde{I}_\beta(\Phi) \geq I_\beta(\Phi)$ . It remains to prove  $\tilde{I}_\beta(\Phi) \leq I_\beta(\Phi)$ . If the right-hand side is  $+\infty$  then there is nothing to show. Thus, we only need to show

$$(4.7) \quad I_\beta(\Phi) = 0 \Rightarrow \tilde{I}_\beta(\Phi) = 0.$$

Suppose that  $\Phi \in X_{\text{Riesz}}$  and  $I_\beta(\Phi) = 0$ . In order to establish (4.7), we need to show

$$(4.8) \quad \forall \varepsilon > 0 \exists \Theta \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \quad \|\Theta\|_{L^\infty(\mathbb{R}^n)} \leq \beta, \quad \|\Theta - \Phi\|_{X_{\text{Riesz}}} < \varepsilon.$$

We proceed in several steps.

Step 1: We first show that

$$(4.9) \quad \forall \varepsilon > 0 \exists \Theta_1 \in X_{\text{Riesz}}, \quad \text{supp } \Theta_1 \subset\subset \mathbb{R}^n, \quad \|\Theta_1\|_{L^\infty(\mathbb{R}^n)} \leq \beta, \quad \|\Theta_1 - \Phi\|_{X_{\text{Riesz}}} < \varepsilon.$$

For  $m \in \mathbb{N}$ , we choose a smooth cut-off function  $0 \leq \zeta_m \leq 1$ , such that  $\zeta_m = 1$  when  $|x| < m$ ;  $\zeta_m = 0$  when  $|x| > 2m$ ; and  $|\nabla \zeta_m| \lesssim \frac{1}{m}$ . For sufficiently large  $m$ , we set

$$\Theta_1 := \zeta_m \Phi.$$

It is clear that  $\|\Theta_1\|_{L^\infty(\mathbb{R}^n)} \leq \|\zeta_m \Phi\|_{L^\infty(\mathbb{R}^n)} \leq \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta$ . Thus, we only need to show

$$\|\Phi \zeta_m - \Phi\|_{L^q(\mathbb{R}^n)} + \|\operatorname{Div}_\alpha(\Phi \zeta_m) - \operatorname{Div}_\alpha \Phi\|_{L^q(\Omega)} \rightarrow 0.$$

Since  $|\Phi \zeta_m| \leq |\Phi|$  and  $\zeta_m \rightarrow 1$  almost everywhere on  $\mathbb{R}^n$ , using the Lebesgue dominated convergence theorem, we have  $\|\Phi \zeta_m - \Phi\|_{L^q(\mathbb{R}^n)} \rightarrow 0$ . Since  $m$  is sufficiently large, we may assume without loss of generality that  $\zeta_m(x) \equiv 1$  when  $\operatorname{dist}(x, \Omega) \leq 1$ . Since  $\Phi \in X_{\operatorname{Riesz}}$ , there exists a sequence  $\Phi_k \in C_c^\infty(\mathbb{R}^n)$ , such that  $\|\Phi_k - \Phi\|_{X_{\operatorname{Riesz}}} \xrightarrow{k \rightarrow \infty} 0$  and

$$\int_{\mathbb{R}^n} \Phi_k \cdot D^\alpha \varphi = - \int_{\Omega} \operatorname{Div}_\alpha(\Phi_k) \varphi, \quad \forall \varphi \in C_c^\infty(\Omega).$$

By the definition of  $X_{\operatorname{Riesz}}$ -convergence, we can take the limits of both sides, which implies

$$\int_{\mathbb{R}^n} \Phi \cdot D^\alpha \varphi = - \int_{\Omega} \operatorname{Div}_\alpha(\Phi) \varphi, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Now we claim that  $\operatorname{Div}_\alpha(\zeta_m \Phi) \in L^q(\Omega)$ . Indeed, let  $\varphi \in C_c^\infty(\Omega)$ , then

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi \zeta_m \cdot D^\alpha \varphi &= \int_{\mathbb{R}^n} \Phi \cdot D^\alpha(\zeta_m \varphi) + \int_{\mathbb{R}^n} \Phi \cdot (\zeta_m D^\alpha(\varphi) - D^\alpha(\zeta_m \varphi)) \\ &= \int_{\mathbb{R}^n} \Phi \cdot D^\alpha \varphi + \int_{\mathbb{R}^n} \Phi \cdot (\zeta_m D^\alpha(\varphi) - D^\alpha(\zeta_m \varphi)). \end{aligned}$$

In the last step we used that  $\zeta_m \varphi = \varphi$  by the definition of  $\zeta_m$  and the support of  $\varphi$ . Using e.g. the Coifman-McIntosh-Meyer commutator estimate (e.g., see [31, Theorem 6.1] or [29, Theorem 3.2.1]), we have

$$\|\zeta_m D^\alpha(\varphi) - D^\alpha(\zeta_m \varphi)\|_{L^{q'}(\mathbb{R}^n)} \lesssim [\zeta_m]_{\operatorname{Lip}} \|I^{1-\alpha} \varphi\|_{L^{q'}(\mathbb{R}^n)},$$

where  $I^{1-\alpha}$  denotes the Riesz potential and  $q' = \frac{q}{q-1}$ . Since  $\varphi$  has compact support in the bounded set  $\Omega$ , we have by Sobolev-Poincaré inequality

$$\|I^{1-\alpha} \varphi\|_{L^{q'}(\mathbb{R}^n)} \lesssim_\Omega \|\varphi\|_{L^{q'}(\Omega)},$$

which follows from the usual blow-up argument used for the classical Poincaré inequality. That is, we have shown that for any  $\varphi \in C_c^\infty(\Omega)$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \operatorname{Div}_\alpha(\Phi \zeta_m - \Phi) \varphi \right| &= \left| \int_{\mathbb{R}^n} (\Phi \zeta_m - \Phi) \cdot D^\alpha \varphi \right| \\ &\leq C(\Omega, q) \|\Phi\|_{L^q(\mathbb{R}^n)} \|\varphi\|_{L^{q'}(\mathbb{R}^n)} [\zeta_m]_{\operatorname{Lip}}. \end{aligned}$$

Observe that  $[\zeta_m]_{\operatorname{Lip}} \lesssim \frac{1}{m}$ , so we have shown by duality that

$$\|\operatorname{Div}_\alpha(\Phi \zeta_m - \Phi)\|_{L^q(\Omega)} \lesssim \frac{1}{m} \|\Phi\|_{L^q(\mathbb{R}^n)} \xrightarrow{m \rightarrow \infty} 0,$$

which establishes (4.9)

Step 2: By translation we may assume  $\Omega$  is convex and  $0 \in \Omega$ . For  $\rho > 1$ , we set

$$(4.10) \quad \Omega_\rho := \rho\Omega = \{\rho x : x \in \Omega\}.$$

Then, from Lemma A.1 and Lemma A.2, we have  $\Omega \subset\subset \Omega_\rho$  for  $\rho > 1$ .

In this step, we show that

$$(4.11) \quad \begin{aligned} &\forall \varepsilon > 0 \exists \Theta_2 \in X_{\text{Riesz}}, \rho > 1, \text{supp } \Theta_2 \subset\subset \mathbb{R}^n, \text{Div}_\alpha \Theta_2 \in L^q(\Omega_\rho), \\ &\|\Theta_2\|_{L^\infty(\mathbb{R}^n)} \leq \beta, \quad \|\Theta_2 - \Phi\|_{X_{\text{Riesz}}} < \varepsilon. \end{aligned}$$

By the results in Step 1, we only need to show (4.11) with  $\Phi$  replaced by  $\Theta_1$ . We let  $\Psi := \Theta_1$  for convenience. For  $\rho > 1$ ,

$$\Psi_\rho(x) := \Psi(x/\rho).$$

Then we have

$$\|\Psi_\rho\|_{L^\infty(\mathbb{R}^n)} = \|\Psi\|_{L^\infty(\mathbb{R}^n)} \leq \beta.$$

Moreover, in view of Lemma A.4, we have

$$\|\Psi_\rho - \Psi\|_{L^q(\mathbb{R}^n)} \xrightarrow{\rho \rightarrow 1^+} 0.$$

It remains to show  $\text{Div}_\alpha \Psi_\rho \in L^q(\Omega_\rho)$  and

$$\|\text{Div}_\alpha \Psi_\rho - \text{Div}_\alpha \Psi\|_{L^q(\Omega)} \xrightarrow{\rho \rightarrow 1^+} 0.$$

We first observe that

$$\int_{\mathbb{R}^n} \Psi_\rho \cdot D^\alpha \varphi \, dx = \rho^{n-\alpha} \int_{\mathbb{R}^n} \Psi \cdot D^\alpha (\varphi(\rho \cdot)) \, dx.$$

Thus, from  $\varphi(\cdot) \in C_c^\infty(\Omega_\rho)$  we have  $\varphi(\rho \cdot) \in C_c^\infty(\Omega)$ . Then we conclude

$$\text{Div}_\alpha \Psi_\rho(x) = \rho^{-\alpha} (\text{Div}_\alpha \Psi)(x/\rho) \quad \text{for a.e. } x \in \Omega_\rho.$$

In particular  $\text{Div}_\alpha \Psi_\rho \in L^q(\Omega_\rho)$ . We now have

$$\begin{aligned} &\|\text{Div}_\alpha \Psi_\rho - \text{Div}_\alpha \Psi\|_{L^q(\Omega)} \\ &\leq \|\text{Div}_\alpha \Psi_\rho - \rho^{-\alpha} \text{Div}_\alpha \Psi\|_{L^q(\Omega)} + \|\rho^{-\alpha} \text{Div}_\alpha \Psi - \text{Div}_\alpha \Psi\|_{L^q(\Omega)} \\ &= \rho^{-\alpha} \|(\text{Div}_\alpha \Psi)(\cdot/\rho) - (\text{Div}_\alpha \Psi)(\cdot)\|_{L^q(\Omega)} + (1 - \rho^{-\alpha}) \|\text{Div}_\alpha \Psi\|_{L^q(\Omega)} \\ &\xrightarrow{\rho \rightarrow 1^+} 0, \end{aligned}$$

where we have used Lemma A.4 for the first term and  $\text{Div}_\alpha \Psi \in L^q(\Omega)$  for the second term. This implies that (4.11) is satisfied, by considering  $\Psi_\rho$  for  $\rho > 1$  close enough to 1.

Step 3: Conclusion Given  $\varepsilon > 0$ , we take  $\Psi := \Theta_2$  and pick  $\rho > 1$  such that (4.11) is satisfied for  $\frac{\varepsilon}{2}$  instead of  $\varepsilon$ . Since  $\Omega$  is convex, by Lemma A.1 and Lemma A.2, there exists  $D > 0$  such that  $\text{dist}(\Omega, \partial\Omega_\rho) > D$ . We let  $\delta_0 := \frac{D}{100}$  and choose  $\delta \in (0, \delta_0)$ . Let  $\eta \in C_c^\infty(B(0, 1))$  be the usual symmetric mollifier kernel, and set

$$\Psi_\delta := \eta_\delta * \Psi.$$

Since  $\text{supp } \Psi \subset\subset \mathbb{R}^n$ , we have  $\Psi_\delta \in C_c^\infty(\mathbb{R}^n)$  and

$$\|\Psi_\delta\|_{L^\infty(\mathbb{R}^n)} \leq \|\Psi\|_{L^\infty(\mathbb{R}^n)} \leq \beta$$

We also have by usual mollification

$$\|\Psi_\delta - \Psi\|_{L^q(\mathbb{R}^n)} \xrightarrow{\delta \rightarrow 0} 0.$$

Lastly, for  $\varphi \in C_c^\infty(\Omega)$  we have by Fubini's theorem

$$\int_{\mathbb{R}^n} \Psi_\delta \cdot D^\alpha \varphi = \int_{\mathbb{R}^n} \Psi \cdot D^\alpha (\varphi * \eta_\delta).$$

Observe that  $\varphi \in C_c^\infty(\Omega)$  implies that  $\varphi * \eta_\delta \in C_c^\infty(B(\Omega, \delta)) \subset C_c^\infty(\Omega_\rho)$ . Thus, we have

$$\int_{\mathbb{R}^n} \Psi_\delta \cdot D^\alpha \varphi = \int_{\mathbb{R}^n} \text{Div}_\alpha \Psi (\varphi * \eta_\delta) \quad \forall \varphi \in C_c^\infty(\Omega).$$

Since  $\text{Div}_\alpha \Psi \in L^q(\Omega_\rho)$ , we conclude that

$$\text{Div}_\alpha \Psi_\delta = (\text{Div}_\alpha \Psi) * \eta_\delta \quad \text{in } \Omega.$$

Then, since  $\text{Div}_\alpha \Psi \in L^q(\Omega_\rho)$  we have that  $(\text{Div}_\alpha \Psi) * \eta_\delta$  converges to  $\text{Div}_\alpha \Psi$  in  $L^q(\Omega)$  as  $\delta \rightarrow 0$ , i.e.

$$\|\text{Div}_\alpha \Psi_\delta - \text{Div}_\alpha \Psi\|_{L^q(\Omega)} \xrightarrow{\delta \rightarrow 0^+} 0.$$

Thus, we conclude that

$$\|\Psi_\delta - \Psi\|_{X_{\text{Riesz}}} \xrightarrow{\delta \rightarrow 0^+} 0.$$

Now by choosing  $\delta > 0$  sufficiently small, we have

$$\|\Psi_\delta - \Phi\|_{X_{\text{Riesz}}} \leq \|\Psi_\delta - \Psi\|_{X_{\text{Riesz}}} + \|\Psi - \Phi\|_{X_{\text{Riesz}}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Letting  $\Theta := \Psi_\delta$ , we have shown (4.8), which implies (4.7). Therefore, we have proved  $\tilde{I}_\beta(\Phi) \leq I_\beta(\Phi)$ , which completes the proof.  $\square$

With the help of Proposition 4.4, we can now continue with the optimizing problem.

*Proof of Proposition 4.3.* For  $G^*$  the procedure is standard, cf. [20, Ch. I], and follows from (4.1),

$$\begin{aligned} G^* : Y^* &\rightarrow \overline{\mathbb{R}}, \\ G^*(u) &= \sup_{v \in L^q(\Omega)} \left\{ \int_{\Omega} v u dx - G(v) \right\} = \frac{1}{p} \|u + u_N\|_{L^p(\Omega)}^p. \end{aligned}$$

As for  $F^*$ , we follow [24, Section 2], with the crucial adaptation of using Proposition 4.4 in the last step

$$\begin{aligned} F^* : V^* &\rightarrow \overline{\mathbb{R}}, \\ F^*(\Psi^*) &= \sup_{\Phi \in V} \{ \langle \Psi^*, \Phi \rangle_{V^*, V} - F(\Phi) \} = \sup_{\Phi \in V} \{ \langle \Psi^*, \Phi \rangle_{X^*, X} - I_\beta(\Phi) \} \\ &= \sup_{\Phi \in V} \left\{ \langle \Psi^*, \Phi \rangle_{X^*, X} - \tilde{I}_\beta(\Phi) \right\} = \sup_{\substack{\Phi \in V \cap C_c^\infty(\mathbb{R}^n) \\ \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta}} \langle \Psi^*, \Phi \rangle_{V^*, V}. \end{aligned}$$

The condition in the last line that we can assume  $\Phi \in C_c^\infty(\mathbb{R}^n)$  is the crucial point of Proposition 4.4, and the only place where convexity of  $\Omega$  appears. Finally, by definition we have  $\Lambda^* : Y^* \rightarrow V^*$ . Therefore,

$$\begin{aligned} F^*(\Lambda^*u) &= \sup_{\substack{\Phi \in V \cap C_c^\infty(\mathbb{R}^n) \\ \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta}} \langle \Lambda^*u, \Phi \rangle_{V^*, V} = \sup_{\substack{\Phi \in V \cap C_c^\infty(\mathbb{R}^n) \\ \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq \beta}} \langle u, \Lambda\Phi \rangle_{Y^*, Y} \\ &= \beta \sup_{\substack{\Phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \\ \|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq 1}} \int_{\mathbb{R}^n} \chi_\Omega u (-\operatorname{Div}_\alpha \Phi) dx = \beta \operatorname{Var}_\alpha(\chi_\Omega u; \mathbb{R}^n), \end{aligned}$$

which concludes the proof.  $\square$

From Theorem 4.1, we have the following result.

COROLLARY 4.5. *If  $\Omega$  is convex, the problems  $(\mathcal{P}_R)$  and  $(\mathcal{Q}_R)$  are related by*

$$\begin{aligned} \min_{\Phi \in X_{\text{Riesz}}} \left\{ \frac{1}{q} \|\cdot - \operatorname{Div}_\alpha \Phi\|_{L^q(\Omega)}^q - \int_{\Omega} u_N (-\operatorname{Div}_\alpha \Phi) dx + I_\beta(\Phi) \right\} \\ = - \min_{u \in L^p(\Omega)} \left\{ \frac{\gamma}{p} \|u - u_N\|_{L^p(\Omega)}^p + \beta \operatorname{Var}_\alpha(\chi_\Omega u; \mathbb{R}^n) \right\}. \end{aligned}$$

It is important to mention that the (predual) problem  $(\mathcal{Q}_R)$  has at least one solution. Moreover, we have the following results for the optimality conditions, cf. (4.3),

LEMMA 4.6. *Let  $\bar{u}$  be the unique solution for  $(\mathcal{P}_R)$  and let  $\bar{\Phi}$  be any solution for  $(\mathcal{Q}_R)$ . Then we have*

$$(4.12) \quad \Lambda^* \bar{u} \in \partial F(\bar{\Phi}) \Leftrightarrow \langle \Lambda^* \bar{u}, \mathbf{v} - \bar{\Phi} \rangle \leq 0 \quad \forall \mathbf{v} \in X_{\text{Riesz}},$$

$$(4.13) \quad -\bar{u} \in \partial G(\Lambda \bar{\Phi}) \Leftrightarrow -\bar{u} = -|\operatorname{Div}_\alpha \bar{\Phi}|^{q-2} \operatorname{Div}_\alpha \bar{\Phi} - u_N.$$

*Proof.* It is clear that (4.12) follows from (4.2). On the other hand, if  $G$  is *Gâteaux differentiable* at  $u \in Y$ , then  $\partial G(u) = \{G'(u)\}$ , cf. [20, Prop. I.5.3]. In turn, the following property about the *duality map*, it is also well known:

$$\begin{aligned} \partial \|\cdot\|_{L^q(\Omega)}^q : L^q(\Omega) &\rightarrow L^p(\Omega) \\ u &\mapsto \{q|u|^{q-2}u\}, \end{aligned}$$

which proves (4.13) and finishes the proof.  $\square$

**Gagliardo-Type.** Next, we focus on the Gagliardo case. We refer to [36] where they studied a related problem. As in case of Riesz, we begin by establishing the existence and uniqueness of solution to  $(\mathcal{P}_G)$ .

LEMMA 4.7. *For  $p \in (1, p^\infty)$ , the problem  $(\mathcal{P}_G)$  has a unique solution  $\bar{u} \in bv_\alpha(\Omega) \cap L^p(\Omega)$ .*

*Proof.* The proof is similar to the Riesz case in Lemma 4.2, after using Proposition 3.10.  $\square$

Now, we characterize the minimizers of  $(\mathcal{P}_G)$  using the predual strategy as discussed in the Riesz case. Note that  $u$  does not need to be extended by zero outside  $\Omega$ . As a result, our approach is largely motivated by [9, Section 2]. We now study the predual

problem associated to  $(\mathcal{P}_G)$ . In a similar way as in the Riesz case, we consider the spaces

$$(4.14) \quad X_{\text{Gagliardo}} = \overline{\{\Phi : \Phi \in C_c^1(\Omega \times \Omega), \Phi(x, y) = -\Phi(y, x)\}}^{\|\cdot\|_{X_{\text{Gagliardo}}}},$$

where

$$\|\Phi\|_{X_{\text{Gagliardo}}} := \sqrt[q]{\|\Phi\|_{L^q(\Lambda_{od} \Omega)}^q + \|\operatorname{div}_\alpha \Phi\|_{L^q(\Omega)}^q},$$

which is well defined because of Lemma 3.2. Observe that we can equivalently assume  $\Phi \equiv 0$  in  $(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega \times \Omega)$  and set

$$\|\Phi\|_{X_{\text{Gagliardo}}} := \sqrt[q]{\|\Phi\|_{L^q(\Lambda_{od} \mathbb{R}^n)}^q + \|\operatorname{div}_\alpha \Phi\|_{L^q(\mathbb{R}^n)}^q}.$$

As in the Riesz case, we will use the indicator function  $I_\beta$  for some  $\beta > 0$ . For  $\Phi \in X_{\text{Gagliardo}}$ , we define

$$I_\beta(\Phi) := \begin{cases} 0 & : \|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta, \\ +\infty & : \text{otherwise.} \end{cases}$$

As in the Riesz case, our main novelty is that we are able to pass from  $I_\beta$  to a new  $\tilde{I}_\beta$  which has better approximation properties.

PROPOSITION 4.8. *If  $\Omega$  is convex then for all  $\Phi \in X_{\text{Gagliardo}}$ ,*

$$I_\beta(\Phi) = \tilde{I}_\beta(\Phi),$$

where

$$\tilde{I}_\beta(\Phi) := \begin{cases} 0 & : \text{if there exists } \Psi_k \in C_c^\infty(\Omega \times \Omega), \\ & \Psi_k \rightarrow \Phi \text{ in } X_{\text{Gagliardo}} \text{ such that } \sup_k \|\Psi_k\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta, \\ +\infty & : \text{otherwise.} \end{cases}$$

*Proof.* We may assume without loss of generality that  $\Omega$  is convex and  $0 \in \Omega$ . First, we establish that  $I_\beta(\Phi) \leq \tilde{I}_\beta(\Phi)$  for all  $\Phi \in X_{\text{Gagliardo}}$ . The case  $\tilde{I}_\beta(\Phi) = \infty$  is trivial. Suppose that  $\tilde{I}_\beta(\Phi) = 0$ , then there exist  $\Psi_k \in C_c^\infty(\Omega \times \Omega)$  with  $\Psi_k(x, y) = -\Psi_k(y, x)$  and  $\sup_k \|\Psi_k\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta$ , such that  $\Psi_k \rightarrow \Phi$  in  $X_{\text{Gagliardo}}$ . From the  $L^q(\Lambda_{od} \mathbb{R}^n)$ -convergence of  $\Psi_k$ , we can find a subsequence, still denoted by  $\Psi_k$ , such that

$$\frac{|\Psi_k(x, y) - \Phi(x, y)|}{|x - y|^{\frac{n}{q} + s}} \xrightarrow{k \rightarrow \infty} 0 \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } (x, y) \in \mathbb{R}^{2n},$$

which in particular implies

$$|\Psi_k(x, y) - \Phi(x, y)| \xrightarrow{k \rightarrow \infty} 0 \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } (x, y) \in \mathbb{R}^{2n}.$$

Thus, we have

$$|\Phi(x, y)| \leq \beta \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } (x, y) \in \mathbb{R}^{2n},$$

which implies that  $I_\beta(\Phi) = 0$  and proves that  $I_\beta(\Phi) \leq \tilde{I}_\beta(\Phi)$  for all  $\Phi \in X_{\text{Gagliardo}}$ .

Now we to prove the opposite direction, i.e.  $I_\beta(\Phi) \geq \tilde{I}_\beta(\Phi)$  for all  $\Phi \in X_{\text{Gagliardo}}$ . If  $I_\beta(\Phi) = \infty$  there is nothing to show, so we actually need to show

$$I_\beta(\Phi) = 0 \quad \Rightarrow \quad \tilde{I}_\beta(\Phi) = 0.$$

Assuming that  $\Phi \in X_{\text{Gagliardo}}$  satisfies  $\|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta$ , we prove that

$$(4.15) \quad \forall \varepsilon > 0 \exists \Theta \in C_c^\infty(\Omega \times \Omega), \quad \|\Theta\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta, \quad \|\Theta - \Phi\|_{X_{\text{Gagliardo}}} < \varepsilon.$$

Step 1: In contrast to the Riesz case, we scale the functions inwards for the Gagliardo case, which ensures that the mollification produces a function still in  $C_c^\infty(\Omega)$ . Using again the notation

$$\Omega_\rho := \rho\Omega = \{\rho x : x \in \Omega\}$$

in (4.10) with  $\rho < 1$ . Since  $\Omega$  is convex and  $0 \in \Omega$ , we have that  $\Omega_\rho \subset \subset \Omega$  for any  $\rho < 1$ . We prove that

$$(4.16) \quad \forall \varepsilon > 0 \exists \Theta_1 \in X_{\text{Gagliardo}}, \exists \rho < 1, \text{ supp } \Theta_1 \subset \Omega_\rho \times \Omega_\rho, \|\Theta_1\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta, \text{ and } \|\Theta_1 - \Phi\|_{X_{\text{Gagliardo}}} < \varepsilon.$$

For  $\rho < 1$ , we define

$$\Phi_\rho(x, y) := \Phi\left(\frac{x}{\rho}, \frac{y}{\rho}\right).$$

Then we have  $\text{supp } \Phi_\rho \subset \Omega_\rho \times \Omega_\rho$  and  $\|\Phi_\rho\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} = \|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta$ . So in order to establish (4.16) we need to show

$$(4.17) \quad \|\Phi_\rho - \Phi\|_{X_{\text{Gagliardo}}} \xrightarrow{\rho \rightarrow 0} 0.$$

We first observe that

$$\frac{\Phi_\rho(x, y)}{|x - y|^{\frac{n}{q}}} = \rho^{-\frac{n}{q}} \frac{\Phi(x/\rho, y/\rho)}{|x/\rho - y/\rho|^{\frac{n}{q}}}.$$

So we have

$$\begin{aligned} & \|\Phi_\rho - \Phi\|_{L^q(\Lambda_{od} \mathbb{R}^n)} \\ & \leq \left| \rho^{-\frac{n}{q}} - 1 \right| \left\| \frac{\Phi(x/\rho, y/\rho)}{|x/\rho - y/\rho|^{\frac{n}{q}}} \right\|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)} + \left\| \frac{\Phi(x/\rho, y/\rho)}{|x/\rho - y/\rho|^{\frac{n}{q}}} - \frac{\Phi(x, y)}{|x - y|^{\frac{n}{q}}} \right\|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)} \\ & = \left| \rho^{-\frac{n}{q}} - 1 \right| \rho^{\frac{2n}{q}} \left\| \frac{\Phi(x, y)}{|x - y|^{\frac{n}{q}}} \right\|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)} + \left\| \frac{\Phi(x/\rho, y/\rho)}{|x/\rho - y/\rho|^{\frac{n}{q}}} - \frac{\Phi(x, y)}{|x - y|^{\frac{n}{q}}} \right\|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)} \\ & = \left| \rho^{-\frac{n}{q}} - 1 \right| \rho^{\frac{2n}{q}} \|\Phi\|_{L^q(\Lambda_{od} \mathbb{R}^n)} + \left\| \frac{\Phi(x/\rho, y/\rho)}{|x/\rho - y/\rho|^{\frac{n}{q}}} - \frac{\Phi(x, y)}{|x - y|^{\frac{n}{q}}} \right\|_{L^q(\mathbb{R}^n \times \mathbb{R}^n)} \\ & \xrightarrow{\rho \rightarrow 1} 0, \end{aligned}$$

where for the first term we use that  $\|\Phi\|_{L^q(\Lambda_{od} \mathbb{R}^n)} = \|\Phi\|_{L^q(\Lambda_{od} \Omega)} < \infty$ , for the second term we use Lemma A.4 in  $\mathbb{R}^n \times \mathbb{R}^n$ . Moreover, a direct computation from (3.4) yields

$$\text{div}_\alpha \Phi_\rho(x) = \rho^{-\alpha} (\text{div}_\alpha \Phi)(x/\rho) \quad \text{a.e. } x \in \mathbb{R}^n.$$

So again with Lemma A.4 we have

$$\begin{aligned} & \|\text{div}_\alpha \Phi_\rho - \text{div}_\alpha \Phi\|_{L^q(\mathbb{R}^n)} \\ & \leq \left| \rho^{-\alpha} - 1 \right| \|(\text{div}_\alpha \Phi)(\cdot/\rho)\|_{L^q(\mathbb{R}^n)} + \|(\text{div}_\alpha \Phi)(\cdot/\rho) - \text{div}_\alpha \Phi\|_{L^q(\mathbb{R}^n)} \\ & = \left| \rho^{-\alpha} - 1 \right| \rho^{\frac{n}{q}} \|(\text{div}_\alpha \Phi)\|_{L^q(\mathbb{R}^n)} + \|(\text{div}_\alpha \Phi)(\cdot/\rho) - \text{div}_\alpha \Phi\|_{L^q(\mathbb{R}^n)} \\ & \xrightarrow{\rho \rightarrow 1} 0 \end{aligned}$$

where we used crucially that by the support of  $\Phi$  in  $\Omega \times \Omega$  we have

$$\|\operatorname{div}_\alpha \Phi\|_{L^q(\mathbb{R}^n)} = \|\operatorname{div}_\alpha \Phi\|_{L^q(\Omega)} < \infty.$$

This establishes (4.17) and thus (4.16) is proven.

Step 2: Let  $\Theta_1$  and  $\rho \in (0, 1)$  be from Step 1. Set  $\delta_0 := \frac{D}{100}$  where  $D := \operatorname{dist}(\overline{\Omega_\rho}, \partial\Omega) > 0$ . Let  $\eta \in C_c^\infty(B(0, 1))$  be the usual symmetric mollifier, i.e.  $\eta \geq 0$  and  $\int \eta = 1$ . For  $\delta \in (0, \delta_0)$ , we define  $\eta_\delta(x) := \eta(x/\delta)/\delta^n$ . Using the notation from (3.14), we define

$$\Psi_\delta(x, y) := (\eta_\delta * \Theta_1)(x, y).$$

Then  $\Psi_\delta \in C_c^\infty(B(\Omega_\rho \times \Omega_\rho, \delta)) \subset C_c^\infty(\Omega \times \Omega)$  and  $\|\Psi_\delta\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|\Theta_1\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \beta$ . Notice that

$$(4.18) \quad \frac{(\eta_\delta * \Theta_1)(x, y)}{|x - y|^{n/q}} = (\eta_\delta * \Xi)(x, y)$$

where  $\Xi(x', y') := \Theta_1(x', y')/|x' - y'|^{n/q}$ . By the definition of  $\Theta_1 \in L^q(\wedge_{od} \mathbb{R}^n)$ , we have  $\Xi \in L^q(\mathbb{R}^n \times \mathbb{R}^n)$ . Thus, we have

$$(4.19) \quad \|\Psi_\delta - \Theta_1\|_{L^q(\wedge_{od}^1 \mathbb{R}^n)} \xrightarrow{\delta \rightarrow 0} 0.$$

For any  $x \in \mathbb{R}^n$ , by letting  $y' = y - z$ , we obtain

$$\begin{aligned} (\operatorname{div}_\alpha \Psi_\delta)(x) &= (\operatorname{div}_\alpha (\eta_\delta * \Theta_1))(x) = - \int_{\mathbb{R}^n} \frac{(\eta_\delta * \Theta_1)(x, y) - (\eta_\delta * \Theta_1)(y, x)}{|x - y|^{n+\alpha}} dy \\ &= - \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \eta_\delta(z) (\Theta_1(x - z, y - z) - \Theta_1(y - z, x - z)) dz \right) \frac{dy}{|x - y|^{n+\alpha}} \\ &= \int_{\mathbb{R}^n} \eta_\delta(z) \left( - \int_{\mathbb{R}^n} \frac{\Theta_1(x - z, y') - \Theta_1(y', x - z)}{|(x - z) - y'|^{n+\alpha}} dy' \right) dz \\ &= \int_{\mathbb{R}^n} \eta_\delta(z) (\operatorname{div}_\alpha \Theta_1)(x - z) dz = (\eta_\delta * (\operatorname{div}_\alpha \Theta_1))(x). \end{aligned}$$

Thus, we have

$$(4.21) \quad \|\operatorname{div}_\alpha \Psi_\delta - \operatorname{div}_\alpha \Theta_1\|_{L^q(\mathbb{R}^n)} = \|\eta_\delta * (\operatorname{div}_\alpha \Theta_1) - \operatorname{div}_\alpha \Theta_1\|_{L^q(\mathbb{R}^n)} \rightarrow 0$$

as  $\delta \rightarrow 0^+$ . Using (4.19) and (4.21), for a sufficiently small  $\delta$ , the function  $\Theta := \Psi_\delta$  satisfies the requirements in (4.15), which completes the proof.  $\square$

Now we continue with the optimizing problem. We set

$$V = (X_{\text{Gagliardo}}, \|\cdot\|_{X_{\text{Gagliardo}}}), \text{ and } Y = (L^q(\Omega), \|\cdot\|_{L^q(\Omega)})$$

and the operators

$$(4.22) \quad \begin{aligned} G : Y &\rightarrow \overline{\mathbb{R}}, \quad G(v) := \frac{1}{q} \|v\|_{L^q(\Omega)}^q - \int_{\Omega} u_N v dx, \\ F : V &\rightarrow \overline{\mathbb{R}}, \quad F(\Phi) := I_\beta(\Phi), \end{aligned}$$

$$\Lambda : V \rightarrow Y, \quad \Lambda(\Phi) := -\operatorname{div}_\alpha \Phi.$$

Similarly as in Lemma 4.3, we have

COROLLARY 4.9. *Let  $\Omega$  be open, bounded, and convex set,*

$$V = (X_{Gagliardo}, \|\cdot\|_{X_{Gagliardo}}), \text{ and } Y = (L^q(\Omega), \|\cdot\|_{L^q(\Omega)})$$

, and let  $F, G$  and  $\Lambda$  defined as in (4.22), then for all  $u \in L^p(\Omega)$

$$G^*(-u) = \frac{1}{p} \|u - u_N\|_{L^p(\Omega)}^p, \quad \text{and}$$

$$F^*(\Lambda^*u) = \beta \text{var}_\alpha(u; \Omega).$$

This motivates us to consider the problem

$$(\mathcal{Q}_G) \quad \inf_{\Phi \in X_{Gagliardo}} \left\{ \frac{1}{q} \|\text{div}_\alpha \Phi\|_{L^q(\Omega)}^q - \int_\Omega u_N (-\text{div}_\alpha \Phi) dx + I_\beta(\Phi) \right\}.$$

We have the following result (see, Corollary 4.5 for the Riesz case).

COROLLARY 4.10. *If  $\Omega$  is an open, bounded, and convex set, the problems  $(\mathcal{P}_G)$  and  $(\mathcal{Q}_G)$  are related by*

$$\begin{aligned} & \min_{\Phi \in X_{Gagliardo}} \left\{ \frac{1}{q} \|\text{div}_\alpha \Phi\|_{L^q(\Omega)}^q - \int_\Omega u_N (-\text{div}_\alpha \Phi) dx + I_\beta(\Phi) \right\} \\ &= - \min_{u \in L^p(\Omega)} \left\{ \frac{\gamma}{p} \|u - u_N\|_{L^p(\Omega)}^p + \beta \text{var}_\alpha(u; \Omega) \right\}. \end{aligned}$$

Finally, we have the following optimality conditions as consequences of Theorem 4.1.

COROLLARY 4.11. *Let  $\bar{u}$  be the unique solution to  $(\mathcal{P}_G)$  and let  $\bar{\Phi}$  be any solution to  $(\mathcal{Q}_G)$ , then*

$$(4.23) \quad \Lambda^* \bar{u} \in \partial F(\bar{\Phi}) \Leftrightarrow \langle \Lambda^* \bar{u}, \Psi - \bar{\Phi} \rangle \leq 0 \quad \forall \Psi \in X_{Gagliardo}, \text{ and}$$

$$(4.24) \quad -\bar{u} \in \partial G(\Lambda \bar{\Phi}) \Leftrightarrow -\bar{u} = -|\text{div}_\alpha \bar{\Phi}|^{q-2} \text{div}_\alpha \bar{\Phi} - u_N.$$

**Appendix A. Scaling in  $L^p$ -norms and star-shaped domains.** In this appendix we state and prove for the convenience of the reader some facts about star-shaped domains that are most likely well-known to experts.

Denote the  $n - 1$ -dimensional unit sphere by  $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ . For  $x \in \mathbb{S}^{n-1}$ .

LEMMA A.1. *Assume  $\lambda : \mathbb{S}^{n-1} \rightarrow (0, \infty)$  is continuous and consider*

$$\Omega = \left\{ x \in \mathbb{R}^n \setminus \{0\} : |x| < \lambda \left( \frac{x}{|x|} \right) \right\} \cup \{0\}.$$

For  $\rho > 0$  set

$$\Omega_\rho := \{\rho x : x \in \Omega\}$$

then we have for any  $\rho_1 < \rho_2$

$$\text{dist}(\Omega_{\rho_1}, \mathbb{R}^n \setminus \Omega_{\rho_2}) > 0.$$

*Proof.* We first observe

$$(A.1) \quad \partial \Omega = \left\{ x \in \mathbb{R}^n \setminus \{0\} : |x| = \lambda \left( \frac{x}{|x|} \right) \right\}.$$

Indeed  $\bar{x} \in \partial\Omega$ . Since  $0 \in \Omega$  and  $\Omega$  is open by continuity of  $\lambda$ , we have  $\bar{x} \neq 0$ . Then there exists  $0 \neq x_k \in \Omega$ ,  $0 \neq y_k \in \mathbb{R}^n \setminus \Omega$  such that  $\lim_k |x_k - \bar{x}| = \lim_k |y_k - \bar{x}| = 0$ . We have

$$|x_k| < \lambda\left(\frac{x_k}{|x_k|}\right), \quad |y_k| \geq \lambda\left(\frac{y_k}{|y_k|}\right) \quad \forall k.$$

Since  $x_k, y_k, \bar{x} \neq 0$  these expressions are continuous and passing to the limit as  $k \rightarrow \infty$ ,

$$|\bar{x}| \leq \lambda\left(\frac{\bar{x}}{|\bar{x}|}\right), \quad |\bar{x}| \geq \lambda\left(\frac{\bar{x}}{|\bar{x}|}\right) \quad \forall k.$$

This implies  $|\bar{x}| = \lambda(\frac{\bar{x}}{|\bar{x}|})$  and thus we have established

$$\partial\Omega \subseteq \left\{x \in \mathbb{R}^n \setminus \{0\} : |x| = \lambda\left(\frac{x}{|x|}\right)\right\}.$$

Now assume  $\bar{x} \in \mathbb{R}^n \setminus \{0\}$  with  $|\bar{x}| = \lambda\left(\frac{\bar{x}}{|\bar{x}|}\right)$ . Then for  $\mu > 0$  we have

$$|\mu\bar{x}| = \mu\lambda\left(\frac{\mu\bar{x}}{|\mu\bar{x}|}\right).$$

Thus, if  $\mu > 1$  we have  $\mu\bar{x} \notin \Omega$  and if  $\mu < 1$  we have  $\mu\bar{x} \in \Omega$ . In particular,

$$x_k := \left(1 - \frac{1}{k}\right)\bar{x} \in \Omega, \quad y_k := \left(1 + \frac{1}{k}\right)\bar{x} \notin \Omega,$$

and  $\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} y_k = \bar{x}$ , so  $\bar{x} \in \overline{\Omega} \cap \overline{\mathbb{R}^n \setminus \Omega} = \partial\Omega$ . This implies

$$\partial\Omega \supseteq \left\{x \in \mathbb{R}^n \setminus \{0\} : |x| = \lambda\left(\frac{x}{|x|}\right)\right\}.$$

So (A.1) is established.

Next we observe

$$\Omega_\rho = \left\{x \in \mathbb{R}^n \setminus \{0\} : |x| < \rho\lambda\left(\frac{x}{|x|}\right)\right\} \cup \{0\}.$$

In particular if  $\rho_1 < \rho_2$  we have that

$$\Omega_{\rho_1} \cap (\mathbb{R}^n \setminus \Omega_{\rho_2}) = \{x : |x| < \rho_1\lambda\left(\frac{x}{|x|}\right), \quad \text{and} \quad |x| \geq \rho_2\lambda\left(\frac{x}{|x|}\right)\} = \emptyset.$$

Since  $\Omega_{\rho_1}$  and  $(\mathbb{R}^n \setminus \Omega_{\rho_2})$  are disjoint, and  $\Omega_{\rho_1}$  is bounded we conclude that

$$\begin{aligned} \text{dist}(\Omega_{\rho_1}, \mathbb{R}^n \setminus \Omega_{\rho_2}) &= \text{dist}(\partial\Omega_{\rho_1}, \partial\Omega_{\rho_2}) \\ &= \inf_{x, y \in \mathbb{R}^n} \left| \rho_1 \frac{x}{|x|} \lambda\left(\frac{x}{|x|}\right) - \rho_2 \frac{y}{|y|} \lambda\left(\frac{y}{|y|}\right) \right| \\ &= \inf_{x, y \in \mathbb{S}^{n-1}} |\rho_1 x \lambda(x) - \rho_2 y \lambda(y)|. \end{aligned}$$

Since  $\lambda(\cdot)$  is continuous and  $\mathbb{S}^{n-1}$  is compact, this infimum is attained at some  $\bar{x}, \bar{y} \in \mathbb{S}^{n-1}$ ,

$$\text{dist}(\Omega_{\rho_1}, \mathbb{R}^n \setminus \Omega_{\rho_2}) = |\rho_1 \bar{x} \lambda(\bar{x}) - \rho_2 \bar{y} \lambda(\bar{y})|$$

We claim that  $|\rho_1 \bar{x} \lambda(\bar{x}) - \rho_2 \bar{y} \lambda(\bar{y})| > 0$ . Indeed if this was not the case we would have

$$\rho_1 \bar{x} \lambda(\bar{x}) = \rho_2 \bar{y} \lambda(\bar{y})$$

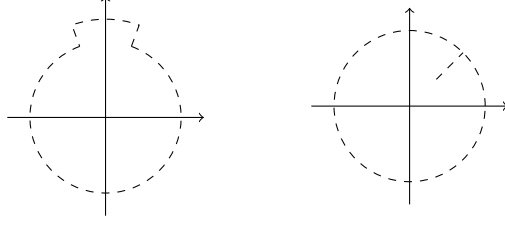


FIG. 1. Examples of star-shaped sets with discontinuous  $\lambda$ . Both sets are star-shaped with respect to the origin, and the first has even Lipschitz continuous boundary – however the conclusions of Lemma A.1 are not true.

Since the scalar factors  $\rho_1, \rho_2, \lambda(\bar{x}), \lambda(\bar{y})$  are all positive – and  $|\bar{x}| = |\bar{y}| = 1$  this implies that  $\bar{x} = \bar{y}$ . Whence we would find

$$\rho_1 \lambda(\bar{x}) = \rho_2 \lambda(\bar{x}),$$

and thus – since  $\lambda(\bar{x}) \in (0, \infty)$ ,  $\rho_1 = \rho_2$  – a contradiction to  $\rho_1 < \rho_2$ . Thus we have established

$$\text{dist}(\Omega_{\rho_1}, \mathbb{R}^n \setminus \Omega_{\rho_2}) > 0. \quad \square$$

In Lemma A.2, the continuity of  $\lambda$  is not guaranteed for generic star-shaped domain – even if their boundaries are Lipschitz. We provide two examples in Figure 1. The first example is the union of an open disk and an open sector. The second is an open unit disk with a slit, i.e. the ray  $\{(c + 1/2, c + 1/2) : c \geq 0\}$  is excluded from the disk.

However, the assumptions of the set  $\Omega$  in Lemma A.1 are satisfied if  $\Omega$  is star-shaped w.r.t to an *open neighborhood* of the origin – this can be obtained by a careful inspection of the proof below. We will focus on convexity here.

LEMMA A.2. *Let  $\Omega$  be an open, bounded, convex set with  $0 \in \Omega$ , then there exists continuous  $\lambda : \mathbb{S}^{n-1} \rightarrow (0, \infty)$  such that*

$$(A.2) \quad \Omega = \left\{ x \in \mathbb{R}^n \setminus \{0\} : |x| < \lambda\left(\frac{x}{|x|}\right) \right\} \cup \{0\}.$$

In particular the results of Lemma A.1 are true.

*Proof.* For  $x \in \mathbb{S}^{n-1}$ , we define

$$(A.3) \quad \lambda(x) := \sup \{ r \geq 0 : rx \in \Omega \}.$$

Since  $\Omega$  is open and  $0 \in \Omega$  there exists a ball  $B(0, a) \subset \Omega$ , and thus  $\lambda(x) \geq r$  for all  $x \in \mathbb{S}^{n-1}$ . Since  $\Omega$  is bounded there must be some  $b > 0$  such that  $\lambda(x) \leq b$  for all  $x \in \mathbb{S}^{n-1}$ .

We first establish (A.2). If  $x \in \Omega$  then  $|x| \frac{x}{|x|} \in \Omega$  and since  $0 \in \Omega$  we have that  $r \frac{x}{|x|} \in \Omega$  for all  $r \in [0, |x|]$ . Since  $\Omega$  is open, there actually must be some  $\delta > 0$  such that  $r \frac{x}{|x|} \in \Omega$  for all  $r \in [0, |x| + \delta]$ . Thus  $\lambda(x/|x|) \geq |x| + \delta > |x|$ .

On the other hand if  $x \in \mathbb{R}^n \setminus \{0\}$  and  $|x| < \lambda(x/|x|)$ , then by definition of  $\lambda(\cdot)$  there must be some  $r > |x|$  such that  $rx/|x| \in \Omega$ . Since  $0 \in \Omega$  and  $\Omega$  is convex we conclude that  $x = |x|x/|x| \in \Omega$ . Thus (A.2) is established.

It remains to prove the continuity of  $\lambda$  on  $\mathbb{S}^{n-1}$ . Given any  $\bar{x} \in \mathbb{S}^{n-1}$ , we let  $\{x_k\}_{k=1}^\infty \subseteq \mathbb{S}^{n-1}$  be a sequence such that  $x_k \xrightarrow{k \rightarrow \infty} \bar{x}$ .

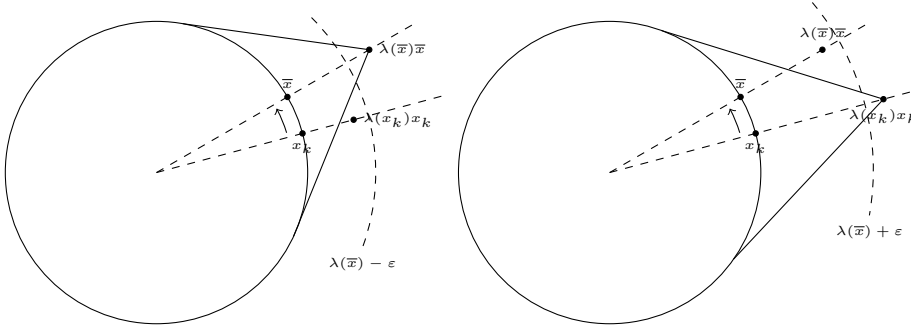


FIG. 2. Assuming that the ball  $B(0, a)$  in the proof is actually equal to  $B(0, 1)$  (which can always be obtained by scaling) the above figure explains the proof of Lemma A.2. First picture: if  $\lambda(x_k) < \lambda(\bar{x}) - \varepsilon$  and  $x_k$  is sufficiently close to  $\bar{x}$  then  $\lambda(x_k)x_k$  must belong to the cone  $A$ . Second picture: if  $\lambda(x_k) > \lambda(\bar{x}) + \varepsilon$  and  $x_k$  is sufficiently close to  $\bar{x}$  then  $\bar{x}$  must belong to  $A_k$  (using that the cone  $A_k$  has a minimal aperture that does not change and is determined by  $B(0, a)$  as  $k$  changes)

Recall that the open ball  $B(0, a) \subset \Omega$ . We denote the open cone from  $\lambda(\bar{x})\bar{x}$  to  $B(0, a)$  as

$$(A.4) \quad A := \{\theta\lambda(\bar{x})\bar{x} + (1 - \theta)z : z \in B(0, a), \theta \in [0, 1]\}.$$

Clearly,  $A$  is an open set. Also, whenever  $\theta \in [0, 1)$  we have that  $\theta\lambda(\bar{x})\bar{x} \in \Omega$ , by convexity of  $\Omega$  and definition of  $\lambda(\cdot)$ . Since  $z$  is taken from an open ball  $B(0, a) \subset \Omega$  we conclude that  $\theta\bar{x} + (1 - \theta)z \in \Omega$ . That is we have  $A \subset \Omega$ .

Similarly, we define the open sets  $A_k$  by

$$(A.5) \quad A_k := \{\theta\lambda(x_k)x_k + (1 - \theta)z : z \in B(0, a), \theta \in [0, 1]\} \subset \Omega$$

Now we assume that there exists  $\varepsilon > 0$  and a sequence  $x_k \in \mathbb{S}^{n-1}$  converging to  $\bar{x} \in \mathbb{S}^{n-1}$  such that  $\lambda(x_k) \leq \lambda(\bar{x}) - \varepsilon$ . Then  $x_k\lambda(x_k) \in A$  when  $k$  is sufficiently large, see Figure 2. Thus we have lower semicontinuity of  $\lambda$ :

$$\lambda(\bar{x}) \leq \liminf_{\mathbb{S}^{n-1} \ni x \rightarrow \bar{x}} \lambda(x).$$

On the other hand, if there exists  $\varepsilon > 0$  and a sequence  $x_k \in \mathbb{S}^{n-1}$  converging to  $\bar{x} \in \mathbb{S}^{n-1}$  such that  $\lambda(x_k) \geq \lambda(\bar{x}) + \varepsilon$ . Then we have that  $\bar{x}\lambda(\bar{x}) \in A_k$  for all large  $k$ , see Figure 2. Thus we have established upper semicontinuity of  $\lambda$

$$\lambda(\bar{x}) \geq \limsup_{\mathbb{S}^{n-1} \ni x \rightarrow \bar{x}} \lambda(x).$$

□

Therefore, we have proved the continuity of  $\lambda$ .

*Remark A.3.* We leave the technical details to the reader, but observe that the lower semicontinuity of  $\lambda$  holds under the assumption that  $\Omega$  is open and star-shaped. It is the upper semicontinuity of  $\lambda$  that requires the center of  $\Omega$  containing an open neighborhood of the origin  $B(0, a)$  (which in particular is a consequence of convexity and openness).

LEMMA A.4. Let  $\Omega \subseteq \mathbb{R}^n$  be an open domain star-shaped with respect to the origin. Fix  $p \in [1, \infty)$ , let  $f \in L^p(\Omega)$  and set for  $\rho > 1$

$$f_\rho := f(\cdot/\rho).$$

Then

$$\|f_\rho - f\|_{L^p(\Omega)} \xrightarrow{\rho \rightarrow 1^+} 0.$$

*Proof.* Let  $\varepsilon > 0$ . Since  $p \in [1, \infty)$  we have  $C^0(\overline{\Omega})$  is dense in  $L^p(\Omega)$ , and thus there exists  $g \in C_c^0(\mathbb{R}^n)$  with

$$\|f - g\|_{L^p(\Omega)} \leq \varepsilon.$$

Set for some  $\rho \in (1, 2)$ ,

$$g_\rho := g(\cdot/\rho).$$

Since  $\Omega$  is star-shaped with respect to the origin,

$$\|f_\rho - g_\rho\|_{L^p(\Omega)} = \rho^{+\frac{n}{p}} \|f - g\|_{L^p(\frac{1}{\rho}\Omega)} \stackrel{\rho < 2}{\leq} 2^{\frac{n}{p}} \|f - g\|_{L^p(\Omega)} \leq 2^{\frac{n}{p}} \varepsilon.$$

Then we have for any  $\rho > 1$ ,

(A.6)

$$\|f_\rho - f\|_{L^p(\Omega)} \leq \|f_\rho - g_\rho\|_{L^p(\Omega)} + \|f - g\|_{L^p(\Omega)} + \|g - g_\rho\|_{L^p(\Omega)} \leq 2^{\frac{n}{p}+1} \varepsilon + \|g - g_\rho\|_{L^p(\Omega)}.$$

Take  $R > 0$  such that

$$\text{supp } g \subset B(0, R/4).$$

Since  $g$  has compact support we can find such an  $R$ . Then  $g$  is uniformly continuous on  $\overline{B(0, R)}$  and thus there exists some  $\rho_0 \in (1, 2)$  such that

$$|g(x) - g(x/\rho)| \leq \frac{\varepsilon}{|B(0, R)|^{\frac{1}{p}}} \quad \forall x \in \overline{B(0, R)}, \forall \rho \in [1, \rho_0].$$

On the other hand if  $x \notin B(0, R)$  then

$$g(x) = g(x/\rho) = 0 \quad \forall \rho \in [1, 2].$$

Thus we have

$$\|g - g_\rho\|_{L^\infty(\mathbb{R}^n)} < \frac{\varepsilon}{|B(0, R)|^{\frac{1}{p}}} \quad \forall \rho \in [1, \rho_0]$$

and thus

$$\|g - g_\rho\|_{L^p(\Omega)} = \|g - g_\rho\|_{L^p(B(0, R))} \leq |B(0, R)|^{\frac{1}{p}} \|g - g_\rho\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon.$$

Combining this with (A.6), we have shown

$$\|f_\rho - f\|_{L^p(\Omega)} \leq (2^{\frac{n}{p}+1} + 1)\varepsilon,$$

which holds for any  $\rho \in [1, \rho_0)$ . We can conclude.  $\square$

## REFERENCES

- [1] L. ÁLVAREZ, P.-L. LIONS, AND J.-M. MOREL, *Image selective smoothing and edge detection by nonlinear diffusion. ii*, SIAM Journal on Numerical Analysis, 29 (1992), pp. 845–866.
- [2] H. ANTIL AND S. BARTELS, *Spectral Approximation of Fractional PDEs in Image Processing and Phase Field Modeling*, Comput. Methods Appl. Math., 17 (2017), pp. 661–678, <https://doi.org/10.1515/cmam-2017-0039>.
- [3] H. ANTIL, S. BARTELS, AND A. SCHIKORRA, *Approximation of fractional harmonic maps*, IMA J. Numer. Anal., 43 (2023), pp. 1291–1323, <https://doi.org/10.1093/imanum/drac029>, <https://doi.org/10.1093/imanum/drac029>.

- [4] H. ANTIL, R. KHATRI, R. LÖHNER, AND D. VERMA, *Fractional deep neural network via constrained optimization*, Machine Learning: Science and Technology, 2 (2020), p. 015003, <https://doi.org/10.1088/2632-2153/aba8e7>, <https://doi.org/10.1088/2632-2153/aba8e7>.
- [5] H. ATTOUCH, G. BUTTAZZO, AND G. MICHAILLE, *Variational Analysis in Sobolev and BV Spaces*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2014, <https://doi.org/10.1137/1.9781611973488>, <https://epubs.siam.org/doi/abs/10.1137/1.9781611973488>, <https://arxiv.org/abs/https://epubs.siam.org/doi/pdf/10.1137/1.9781611973488>.
- [6] S. BARTELS, *Numerical methods for nonlinear partial differential equations*, vol. 47 of Springer Series in Computational Mathematics, Springer, Cham, 2015, <https://doi.org/10.1007/978-3-319-13797-1>, <https://doi.org/10.1007/978-3-319-13797-1>.
- [7] K. BESSAS, *Fractional total variation denoising model with  $L^1$  fidelity*, Nonlinear Anal., 222 (2022), pp. Paper No. 112926, 20, <https://doi.org/10.1016/j.na.2022.112926>, <https://doi.org/10.1016/j.na.2022.112926>.
- [8] C. BOURDARIAS, M. GISCLON, AND S. JUNCA, *Fractional BV spaces and applications to scalar conservation laws*, J. Hyperbolic Differ. Equ., 11 (2014), pp. 655–677, <https://doi.org/10.1142/S0219891614500209>, <https://doi.org/10.1142/S0219891614500209>.
- [9] M. BURGER, K. FRICK, S. OSHER, AND O. SCHERZER, *Inverse total variation flow*, Multiscale Model. Simul., 6 (2007), pp. 365–395, <https://doi.org/10.1137/060660564>, <https://doi.org/10.1137/060660564>.
- [10] L. CAFFARELLI, J.-M. ROQUEJOFFRE, AND O. SAVIN, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math., 63 (2010), pp. 1111–1144, <https://doi.org/10.1002/cpa.20331>, <https://doi.org/10.1002/cpa.20331>.
- [11] A. CHAMBOLLE, *An algorithm for total variation minimization and applications*, J. Math. Imaging Vision, 20 (2004), pp. 89–97, <https://doi.org/10.1023/B:JMIV.0000011320.81911.38>, <http://dx.doi.org/10.1023/B:JMIV.0000011320.81911.38>. Special issue on mathematics and image analysis.
- [12] A. CHAMBOLLE AND T. POCK, *A first-order primal-dual algorithm for convex problems with applications to imaging*, J. Math. Imaging Vision, 40 (2011), pp. 120–145, <https://doi.org/10.1007/s10851-010-0251-1>, <https://doi.org/10.1007/s10851-010-0251-1>.
- [13] G. CHAVENT AND K. KUNISCH, *Regularization of linear least squares problems by total bounded variation*, ESAIM Control Optim. Calc. Var., 2 (1997), pp. 359–376, <https://doi.org/10.1051/cocv:1997113>, <https://doi.org/10.1051/cocv:1997113>.
- [14] G. E. COMI, D. SPECTOR, AND G. STEFANI, *The fractional variation and the precise representative of  $BV^{\alpha,p}$  functions*, Fractional Calculus and Applied Analysis, 25 (2022), pp. 520–558, <https://doi.org/10.1007/s13540-022-00036-0>, <https://doi.org/10.1007/s13540-022-00036-0>.
- [15] G. E. COMI AND G. STEFANI, *A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up*, J. Funct. Anal., 277 (2019), pp. 3373–3435, <https://doi.org/10.1016/j.jfa.2019.03.011>, <https://doi.org/10.1016/j.jfa.2019.03.011>.
- [16] G. E. COMI AND G. STEFANI, *A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics I*, Rev. Mat. Complut., 36 (2023), pp. 491–569, <https://doi.org/10.1007/s13163-022-00429-y>, <https://doi.org/10.1007/s13163-022-00429-y>.
- [17] F. DA LIO AND T. RIVIÈRE, *Three-term commutator estimates and the regularity of  $\frac{1}{2}$ -harmonic maps into spheres*, Anal. PDE, 4 (2011), pp. 149–190, <https://doi.org/10.2140/apde.2011.4.149>, <https://doi.org/10.2140/apde.2011.4.149>.
- [18] E. DI NEZZA, G. PALATUCCI, AND E. VALDINOCI, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), pp. 521–573, <https://doi.org/10.1016/j.bulsci.2011.12.004>, <https://doi.org/10.1016/j.bulsci.2011.12.004>.
- [19] Q. DU, M. GUNZBURGER, R. B. LEHOUCQ, AND K. ZHOU, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Math. Models Methods Appl. Sci., 23 (2013), pp. 493–540, <https://doi.org/10.1142/S0218202512500546>, <https://doi.org/10.1142/S0218202512500546>.
- [20] I. EKELAND AND R. TÉMAM, *Convex Analysis and Variational Problems*, Society for Industrial and Applied Mathematics, 1999, <https://doi.org/10.1137/1.9781611971088>, <https://epubs.siam.org/doi/abs/10.1137/1.9781611971088>, <https://arxiv.org/abs/https://epubs.siam.org/doi/pdf/10.1137/1.9781611971088>.
- [21] I. EKELAND AND T. TURNBULL, *Infinite-dimensional optimization and convexity*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1983.
- [22] L. C. EVANS AND R. F. GARIEPY, *Measure theory and fine properties of functions*, Textbooks in Mathematics, CRC Press, Boca Raton, FL, revised ed., 2015.
- [23] G. GILBOA AND S. OSHER, *Nonlocal linear image regularization and supervised segmentation*, Multiscale Model. Simul., 6 (2007), pp. 595–630, <https://doi.org/10.1137/060669358>, <https://doi.org/10.1137/060669358>.

- 1051 [//doi.org/10.1137/060669358](https://doi.org/10.1137/060669358).
- 1052 [24] M. HINTERMÜLLER AND K. KUNISCH, *Total bounded variation regularization as a bilaterally*  
 1053 *constrained optimization problem*, SIAM J. Appl. Math., 64 (2004), pp. 1311–1333, <https://doi.org/10.1137/S0036139903422784>, <https://doi.org/10.1137/S0036139903422784>.
- 1054 [25] M. HINTERMÜLLER, C. N. RAUTENBERG, AND S. RÖSEL, *Density of convex intersections and*  
 1055 *applications*, Proc. A., 473 (2017), pp. 20160919, 28, <https://doi.org/10.1098/rspa.2016.0919>, <https://doi.org/10.1098/rspa.2016.0919>.
- 1056 [26] M. HINZ, *Magnetic energies and Feynman-Kac-Itô formulas for symmetric Markov processes*,  
 1057 Stoch. Anal. Appl., 33 (2015), pp. 1020–1049, <https://doi.org/10.1080/07362994.2015.1077715>, <https://doi.org/10.1080/07362994.2015.1077715>.
- 1058 [27] J. HORVÁTH, *On some composition formulas*, Proc. Amer. Math. Soc., 10 (1959), pp. 433–437,  
 1059 <https://doi.org/10.2307/2032862>, <https://doi.org/10.2307/2032862>.
- 1060 [28] J. A. IGLESIAS AND G. MERCIER, *Convergence of level sets in fractional Laplacian regulariza-*  
 1061 *tion*, Inverse Problems, 38 (2022), pp. Paper No. 124003, 35.
- 1062 [29] J. INGMANN, *Estimates for commutators of fractional differential operators via harmonic ex-*  
 1063 *tension*, Master Thesis, arXiv:2012.12072, (2020), <https://doi.org/10.48550/ARXIV.2012.12072>, <https://arxiv.org/abs/2012.12072>.
- 1064 [30] E. LENZMANN AND A. SCHIKORRA, *On energy-critical half-wave maps into  $\mathbb{S}^2$* , Invent.  
 1065 Math., 213 (2018), pp. 1–82, <https://doi.org/10.1007/s00222-018-0785-1>, <https://doi.org/10.1007/s00222-018-0785-1>.
- 1066 [31] E. LENZMANN AND A. SCHIKORRA, *Sharp commutator estimates via harmonic extensions*,  
 1067 Nonlinear Anal., 193 (2020), pp. 111375, 37, <https://doi.org/10.1016/j.na.2018.10.017>, <https://doi.org/10.1016/j.na.2018.10.017>.
- 1068 [32] M. LUDWIG, *Anisotropic fractional perimeters*, J. Differential Geom., 96 (2014), pp. 77–93,  
 1069 <http://projecteuclid.org/euclid.jdg/1391192693>.
- 1070 [33] F. MAINARDI, *Fractional calculus and waves in linear viscoelasticity*, Imperial College  
 1071 Press, London, 2010, <https://doi.org/10.1142/9781848163300>, <https://doi.org/10.1142/9781848163300>. An introduction to mathematical models.
- 1072 [34] K. MAZOWIECKA AND A. SCHIKORRA, *Fractional div-curl quantities and applications to nonlocal*  
 1073 *geometric equations*, J. Funct. Anal., 275 (2018), pp. 1–44, <https://doi.org/10.1016/j.jfa.2018.03.016>, <https://doi.org/10.1016/j.jfa.2018.03.016>.
- 1074 [35] P. MIRONESCU, *Fine properties of functions: an introduction*. Lecture, June 2005, <https://cel.archives-ouvertes.fr/cel-00747696>.
- 1075 [36] M. NOVAGA AND F. ONOUE, *Local Hölder regularity of minimizers for nonlocal variational*  
 1076 *problems*, Commun. Contemp. Math., 25 (2023), p. Paper No. 2250058, <https://doi.org/10.1142/S0219199722500584>, <https://doi.org/10.1142/S0219199722500584>.
- 1077 [37] L. I. RUDIN, *Images, numerical analysis of singularities and shock filters*, PhD thesis, California  
 1078 Institute of Technology, 1987.
- 1079 [38] L. I. RUDIN, S. OSHER, AND E. FATEMI, *Nonlinear total variation based noise removal al-*  
 1080 *gorithms*, Physica D: Nonlinear Phenomena, 60 (1992), pp. 259–268, [https://doi.org/10.1016/0167-2789\(92\)90242-F](https://doi.org/10.1016/0167-2789(92)90242-F), <https://www.sciencedirect.com/science/article/pii/016727899290242F>.
- 1081 [39] A. SCHIKORRA,  *$L^p$ -gradient harmonic maps into spheres and  $SO(N)$* , Differential Integral  
 1082 Equations, 28 (2015), pp. 383–408, <http://projecteuclid.org/euclid.die/1423055234>.
- 1083 [40] A. SCHIKORRA,  *$\varepsilon$ -regularity for systems involving non-local, antisymmetric operators*, Calc.  
 1084 Var. Partial Differential Equations, 54 (2015), pp. 3531–3570, <https://doi.org/10.1007/s00526-015-0913-3>, <https://doi.org/10.1007/s00526-015-0913-3>.
- 1085 [41] A. SCHIKORRA, D. SPECTOR, AND J. VAN SCHAFTINGEN, *An  $L^1$ -type estimate for Riesz po-*  
 1086 *tentials*, Rev. Mat. Iberoam., 33 (2017), pp. 291–303, <https://doi.org/10.4171/RMI/937>, <https://doi.org/10.4171/RMI/937>.
- 1087 [42] T.-T. SHIEH AND D. E. SPECTOR, *On a new class of fractional partial differential equations*,  
 1088 Adv. Calc. Var., 8 (2015), pp. 321–336, <https://doi.org/10.1515/acv-2014-0009>, <https://doi.org/10.1515/acv-2014-0009>.
- 1089 [43] M. TORRES, *On the dual of BV*, in Contributions of Mexican mathematicians abroad in pure  
 1090 and applied mathematics. Second meeting “Matemáticos Mexicanos en el Mundo”, Cen-  
 1091 tro de Investigación en Matemáticas, Guanajuato, Mexico, December 15–19, 2014, Provi-  
 1092 dence, RI: American Mathematical Society (AMS); México: Sociedad Matemática Mexi-  
 1093 cana, 2018, pp. 115–129, <https://doi.org/10.1090/conm/709/14296>.
- 1094 [44] C. J. WEISS, B. G. VAN BLOEMEN WAANDERS, AND H. ANTIL, *Fractional operators applied*  
 1095 *to geophysical electromagnetics*, Geophysical Journal International, 220 (2020), pp. 1242–  
 1096 1259.
- 1097 [45] C. YU, H. SCHUMACHER, AND K. CRANE, *Repulsive curves*, ACM Trans. Graph., 40 (2021).

- 1113 [46] Y. ZHOU, *Fractional Sobolev extension and imbedding*, Trans. Amer. Math. Soc., 367 (2015),  
1114 pp. 959–979, <https://doi.org/10.1090/S0002-9947-2014-06088-1>, <https://doi.org/10.1090/S0002-9947-2014-06088-1>.  
1115