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RESEARCH ARTICLE

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Minimal $W^{s,\frac{n}{s}}$-harmonic maps in homotopy classes

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Abstract

Let Σ a closed n-dimensional manifold, $\mathcal{N} \subset \mathbb{R}^M$ be a closed manifold, and let $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ for $s \in (0,1)$. We extend the monumental work of Sacks and Uhlenbeck by proving that if $\pi_n(\mathcal{N}) = \{0\}$, then there exists a minimizing $W^{s,\frac{n}{s}}$-harmonic map homotopic to u. If $\pi_n(\mathcal{N}) \neq \{0\}$, then we prove that there exists a $W^{s,\frac{n}{s}}$-harmonic map from $\mathbb{S}^n$ to $\mathcal{N}$ in a generating set of $\pi_n(\mathcal{N})$. Since several techniques, especially Pohozaev-type arguments, are unknown in the fractional framework (in particular, when $\frac{n}{s} \neq 2$, one cannot argue via an extension method), we develop crucial new tools that are interesting on their own: such as a removability result for point singularities and a balanced energy estimate for nonscaling invariant energies. Moreover, we prove the regularity theory for minimizing $W^{s,\frac{n}{s}}$-maps into manifolds.

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Contents

1 | INTRODUCTION

In the geometric calculus of variations, it is of utmost importance to find and classify not only absolute minimizers, but also one would like to understand the more subtle structure of critical points (local minimizers, saddle points, etc.) within topological classes — with questions ranging from the Willmore conjecture recently solved by Marques and Neves [74] to open questions on existence of critical points for knot energies by Freedman–He–Wang [38] as well as Kusner and Sullivan [58]. In this paper, we study the existence theory of minimal $W^{s,\frac{n}{s}}$-harmonic maps in homotopy between two manifolds Σ and $\mathcal{N}$.

Throughout the paper, we assume that Σ is a smooth compact n-dimensional Riemannian manifold without boundary, and $\mathcal{N}\subset\mathbb{R}^M$ is a connected smooth compact Riemannian manifold isometrically embedded into $\mathbb{R}^M$.

The most fundamental result in existence theory for harmonic maps in homotopy classes is due to Sacks and Uhlenbeck, [96, 97]. Harmonic maps are critical points of the Dirichlet energy

$$\int_\Sigma |\nabla u|^2 \quad \text{such that} \quad u\in W^{1,2}(\Sigma,\mathcal{N}). \tag{1.1}$$

We summarize the results of Sacks and Uhlenbeck, [96, Theorem 5.1] and [96, Theorem 5.5], as follows.

Theorem 1.1 (Sacks and Uhlenbeck). *Let Σ be a two-dimensional manifold.*

(1) *If $\pi_2(\mathcal{N})=\{0\}$, then there exists a minimizing harmonic map in every homotopy class $C^0(\Sigma,\mathcal{N})$.*
(2) *If $\Sigma=\mathbb{S}^2$ and $\pi_1(\mathcal{N})=\{0\}$, then there exists a generating set of homotopy classes in $C^0(\mathbb{S}^2,\mathcal{N})$ in which minimizing harmonic maps exist.*

Theorem 1.1 (1) was originally obtained independently by Lemaire [67] and Schoen and Yau [109]. Also let us remark that the condition $\pi_1(\mathcal{N})=\{0\}$ in Theorem 1.1(2) is for pure commodity of this introduction, for $\pi_1(\mathcal{N})\neq\{0\}$ the same result holds up to the action of $\pi_1(\mathcal{N})$ on $\pi_2(\mathcal{N})$, see Theorem 7.1.

Theorem 1.1 (2) is sharp in the case $\pi_2(\mathcal{N})\neq\{0\}$ in the following sense: harmonic maps may not exist in every homotopy class of $\pi_2(\mathcal{N})$, and a counterexample was provided by Futaki [39].

The motivation to study harmonic maps under topological assumptions is at least twofold. On the one hand, there is the geometric interest as the image of a harmonic map from $\mathbb{S}^2$ to $\mathcal{N}$ is a conformal branched immersion (which seems to have been the main motivation for Sacks and Uhlenbeck). On the other hand, there is an interest from the applications point of view, as the

harmonic map energy can be interpreted as a model case for the Oseen–Frank theory of nematic liquid crystals, see, for example, [3, 70].

In this work, we develop an existence theory for $W^{s,\frac{n}{s}}$-harmonic maps in homotopy classes. For $s \in (0,1)$, such maps are defined to be minimizers or critical points of the energy

$$E_{s,\frac{n}{s}}(u) := \int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \quad \text{such that} \quad u \in W^{s,\frac{n}{s}}(\Sigma,\mathcal{N}). \tag{1.2}$$

Our main result is the counterpart to Theorem 1.1 for the energy $E_{s,\frac{n}{s}}$.

Theorem 1.2. *Let Σ be a closed n-dimensional manifold, $n \geqslant 1$, $s \in (0,1)$, $\frac{n}{s} \geqslant 2$.*[†]

(1) *If $\pi_n(\mathcal{N}) = \{0\}$, then there exists a minimizing $W^{s,\frac{n}{s}}$-harmonic map in every connected component of $C^0(\Sigma,\mathcal{N})$.*
(2) *If $\Sigma = \mathbb{S}^n$, $n \geqslant 2$, and $\pi_1(\mathcal{N}) = \{0\}$, then there exists a generating set of homotopy classes in $\pi_n(\mathcal{N})$ in which minimizing $W^{s,\frac{n}{s}}$-harmonic maps exist.*
(3) *If $\Sigma = \mathbb{S}^1$, then there exists a generating set of homotopy classes in $C^0(\mathbb{S}^n,\mathcal{N})$ in which minimizing $W^{s,\frac{n}{s}}$-harmonic maps exist.*

In particular, we obtain the existence of nontrivial $W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})$-harmonic maps whenever $\pi_n(\mathcal{N}) \neq \{0\}$, see Corollary 7.2. Theorem 1.2 sheds some light on a question raised by Mironescu [81] on the existence of minimizing $W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)$-maps in homotopy groups (see Corollary 7.3).

Here, we also remark that in Theorem 1.2 (2), the condition $\pi_1(\mathcal{N}) = \{0\}$ is not necessary: the same result for $\pi_1(\mathcal{N}) \neq \{0\}$ holds up to the action of $\pi_1(\mathcal{N})$ on $\pi_n(\mathcal{N})$, see Theorem 7.1.

Theorem 1.2 (1) is proven in Section 6, see Theorem 6.1, and Theorem 1.2 (2) is proven in Section 7, see Theorem 7.1.

Similarly as in the case of harmonic maps, there are at least two motivations for studying $W^{s,\frac{n}{s}}$-maps, one coming from geometry and the other one from applications.

First, as an example of a geometric motivation, the $W^{\frac{1}{2},2}$-energy appears as trace energy and one can model the *free boundary* of minimal surfaces with such energies, cf. Moser [86], Roberts [93], Millot–Sire [79], and Pigati–Da Lio [91].[‡]

Second, since the 1990's nonlocal energies have been used by applied topologists to define self-repulsive curvature energies for curves and surfaces. Self-repulsiveness is a property that is desirable for models of cells, DNA, and so on, and one natural way to include this feature is a nonlocal energy. For example, O'Hara's knot energies [88, 89], one of which is the famous Möbius energy [38]; or the tangent points energies proposed by Banavar et al. [4]; or the Menger curvature suggested by Gonzalez and Maddocks [44]. We refer to [2, 114–116] for further details.

There is a close connection of these nonlocal repulsive curvature energies to $W^{s,\frac{n}{s}}$-harmonic maps, as was discovered in [8, 9] for the O'Hara energies: one can construct an energy $\tilde{E}_{s,1/s}$ reminiscent of $E_{s,1/s}$ such that critical knots γ (with respect to their knot energy) induce via their

[†] The condition $n/s \geqslant 2$ is trivially satisfied if $n \geqslant 2$. For $n = 1$, it is mostly a technical assumption that plays only a role in the regularity theory, Section 3. It should not be too much work to extend this theorem to the full case $s \in (0,1)$ for $n = 1$ as well but we will not develop this point here.

[‡] As a curious sidenote, let us mention that to a certain extent, this was actually used in Douglas' proof of the Plateau problem in 1932 [31].

derivative γ' an $\tilde{E}_{s,1/s}$-critical map (as a map into $\mathbb{S}^2$). This link between self-repulsive curvature energies and fractional harmonic map energies (at least formally) seems to extend to existence theory; a famous theorem by Freedman–He–Wang [38] states that minimizers for the Möbius energy exist in prime knot classes (which are generators of the ambient isotopy classes), whereas Kusner–Sullivan [58] conjectured that minimizers may not exist in composite knot classes. This is a very similar statement to Theorem 1.1(2) and Theorem 1.2(2) — minimizers exist in a generating subset of the homotopy class. Moreover, as mentioned above, for harmonic maps, it is known that minimizers may not exist in some elements of the homotopy group: there is an example due to Futaki [39].

The techniques by Freedman–He–Wang [38] are very geometric in nature, and it is completely unclear how to extend them to other topological curvature energies (especially for scale-invariant self-repulsive surfaces energies. There are very few techniques available, see [105, 117]). One of the underlying motivations of the present work is to develop an analytic foundation for techniques that hopefully will be applicable for the wide range of scale-invariant self-repulsive critical knot and surface energies proposed by the applied topology community.

1.1 | Outline, strategy of the proof, and main results

If we take a generic minimizing sequence in some homotopy class of the Dirichlet energy or the $E_{s,\frac{n}{s}}$ energy, then there is no reason that it (even weakly) converges to a minimizer in the same homotopy class. Indeed, for example, if $\Sigma = \mathbb{S}^n$ and u is a minimizer in some nontrivial homotopy class (suppose it exists), then we can conformally rescale without changing the energy. Namely, for any $\lambda > 0$, $u_\lambda(x) := u(\tau(\lambda\tau^{-1}(x)))$, where $\tau : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$ is the inverse stereographic projection, satisfies $E_{s,\frac{n}{s}}(u_\lambda) = E_{s,\frac{n}{s}}(u)$, see Section 5. Then $(u_\lambda)_{\lambda>0}$ is a minimizing sequence, but u_λ weakly converges to a constant map as $\lambda \to 0$. In other words, the set of $W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$-maps belonging to one (nontrivial) homotopy class is not compact under the topology induced by $E_{s,\frac{n}{s}}$.

Sacks and Uhlenbeck mitigated this lack of compactness by introducing a special minimizing sequence. Namely, they defined the minimizing sequence $(u_\alpha)_{\alpha>1}$ as the minimizers of the approximate energy

$$E_\alpha(u) := \int_\Sigma (1 + |\nabla u|^2)^\alpha.$$

As $\alpha \to 1^+$, one hopes that the sequence $(u_\alpha)_{\alpha>1}$ converges to a minimizer of the Dirichlet energy E_1. In the case, when $\Sigma = \mathbb{S}^2$, since the energy E_α is not conformally invariant, Sacks and Uhlenbeck were able to obtain some control over the energy concentration that is likely to happen. Crucially, they showed that in this case, energy concentration cannot happen at only one point, but either happens nowhere or at least at two points.

We follow a similar philosophy, but we have to develop several novel arguments to overcome the problem of nonlocality of the energies $E_{s,n/s}$. Specifically, there are only few available Pohozaev-type arguments and they seem not to be working in our case (in contrast to the case of local equations be it harmonic or n-harmonic maps). Indeed, the only case where such arguments (and consequently monotonicity estimates, etc.) are known is the case $n/s = 2$, cf. Millot–Sire [79].

Following the Sacks–Uhlenbeck approach, we will first construct the minimizing sequence for $E_{s,n/s}$ via minimizers u_t, $t > s$, of the energy

$$E_{t,\frac{n}{s}}(u,\Sigma) := \int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.$$

That is, we try to approximate $W^{s,\frac{n}{s}}$-minimizing maps by $W^{t,\frac{n}{s}}$-minimizing maps and let $t \to s^+$.[†]

In Section 3, we develop a regularity theory for minimizers of $E_{t,\frac{n}{s}}$. More precisely, we show that on balls where the $E_{s,\frac{n}{s}}$-energy of u_t is not concentrating, we have regularity estimates in $W^{s_0,\frac{n}{s}}$ for some $s_0 > s$, which is independent of $t \to s$, see Theorem 3.1.

Then, from a standard covering argument, one obtains that the minimizing sequence u_t converges strongly to u_s outside of a singular set consisting of finitely many points. The crucial next result we need is that u_s is regular. Sacks and Uhlenbeck remove the point singularities by using a Pohozaev-type argument. In the nonlocal situation, Pohozaev-type arguments are still under development, see [25, 40, 94]. Another option is to show that u_s is a critical point of the $W^{s,\frac{n}{s}}$-harmonic map equation (which is easy, see Proposition 4.7) and use regularity theory for critical points. Also, this approach is not feasible for us, because the regularity theory for critical points into general target manifolds for the $E_{s,\frac{n}{s}}$-energy (and also for the local analog n-harmonic maps) is a major open problem since the 1990s, cf. [106]. Our approach is to show in Section 4 that the limit u_s is actually still a minimizer (but in its own homotopy class, which might be different from the homotopy class of u_t), see Theorem 4.1, which is (vaguely) reminiscent of Lions' concentration compactness principle [72]. With the regularity theory from Section 3 for *minimizers*, we then get the desired regularity for the limit map u_s, see Theorem 4.6.

The probably most crucial novelty of our work is contained in Section 5: we essentially show that if $\Sigma = \mathbb{S}^n$, the minimizers of the nonscaling invariant energy $E_{t,\frac{n}{s}}$, $t > s$, will not have energy concentration in only one point as $t \to s$ (it has to be either no point or at least two points of energy concentration). We establish this statement by showing in Theorem 5.1 that the energy of a $E_{t,\frac{n}{s}}$-minimizer on a small ball is controlled by the energy of the complement of that ball. We are not aware of such a statement in the literature even in the local case of p-harmonic maps, see Theorem 5.2. However, see [63], where the authors seem to use a similar effect to show that not all harmonic maps can be obtained from a Sacks–Uhlenbeck approximation. In our case, we use Theorem 5.1 to replace the role of Sacks and Uhlenbeck's [96, Lemma 5.3], which is based on a rather explicit computation of the Euler–Lagrange equation which we could not reproduce in our nonlocal setting.

The remaining outline is as follows: in Section 2, we recall the basic notion of homotopy for Sobolev maps. In Section 6, we prove the analog of Theorem 1.1(1), and in Section 7, we prove the analog of Theorem 1.1(2).

As corollaries we obtain in Corollary 7.2, the existence of nontrivial $W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})$-harmonic maps whenever $\pi_n(\mathcal{N}) \neq 0$ and in Corollary 7.4 existence of minimizers in any nontrivial homotopy class Γ that has small minimal energy $\inf_\Gamma E_{s,\frac{n}{s}}$.

[†] It would be more in line with the original approach of Sacks–Uhlenbeck if we chose $W^{s,\frac{n}{s}\alpha}$-minimizers, $\alpha \to 1^+$. However, that would have the technical drawback that $W^{s,\frac{n}{s}\alpha} \not\hookrightarrow W^{s,\frac{n}{s}}_{loc}$ for $\alpha > 1$ and $s \in (0,1)$, see [83]. But we do have the embedding $W^{t,\frac{n}{s}} \hookrightarrow W^{s,\frac{n}{s}}_{loc}$ for $t > s$ [95].

1.2 | Remarks on earlier extensions of Sacks–Uhlenbeck theory

The work by Sacks and Uhlenbeck has been extended to finding n-harmonic maps in $\pi_n(\mathcal{N})$; a version of [96, Theorem 5.1] follows from White's [122], for a version of [96, Theorem 5.5] see Kawai–Nakauchi–Takeuchi [56]. [96, Theorem 5.5] uses a removability theorem for n-harmonic maps, see, for example, [87] or [32]. Recently, some of those results were also generalized to the polyharmonic case, see [53]. See also [92] for viscosity methods for minimal surfaces. In [61], Kuwert analyzed the topological decomposition of weak limits of maps in $W^{\frac{1}{2},2}(\mathbb{R}/\mathbb{Z},\mathcal{N})$, see also Duzaar–Kuwert [33] for the $W^{1,n}$-version.

There also has been a tremendous amount of work dedicated to the analysis of bubbles forming in the process of the minimization procedure — for harmonic maps [37, 62, 66, 69, 90], for H-surfaces see [18], for Willmore surfaces [5], for n-harmonic maps [33], for biharmonic maps [65], for Dirac-harmonic maps [14, 55], and for fractional harmonic maps [24, 64]. Let us also mention the flow-technique developed for harmonic maps by Struwe [113] which he used to show the existence of nontrivial minimal harmonic maps (cf. Corollary 7.2). For results concerning 1-harmonic maps, we refer to [42].

1.3 | A brief history of fractional harmonic maps

The theory of fractional harmonic maps, that is, critical points and minimizers of the energy in (1.2), can be traced back to the 1930s when Douglas [31] used them (implicitly) to solve the Plateau problem and win the Fields price. Analytically, they were introduced in the pioneering work by Da Lio and Rivière [26, 27] who coined the notion of fractional harmonic maps and developed the regularity theory for critical (i.e., not necessarily minimizing) $W^{\frac{1}{2},2}$-harmonic maps on lines into manifolds. This regularity theory for critical points was extended to various variations of the energy functional [23, 28, 29, 75, 99, 102, 103] — in particular, the notion of (critical) $W^{s,p}$-harmonic maps and their regularity theory into spheres was introduced in [101] (see also [76]). The question of existence of minimizing $W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^n)$-maps of degree one was raised earlier, see Mironescu [81]. Degree one maps were proved to exist in the case $n=1$ and $s\in(\frac{1}{2},\frac{1}{2-\varepsilon}]$ for some $\varepsilon>0$, see [82].

Moser [86] and Roberts [93] developed a theory of *intrinsic* fractional harmonic maps and their regularity theory. Moser [86], Roberts [93], and Millot–Sire [79] used the technique of harmonic extension to the upper half-plane to characterize $W^{\frac{1}{2},2}$-harmonic maps as a partial free boundary problem of a classical harmonic map and obtain regularity theory from arguments for free boundary harmonic maps due to Scheven [98] — see also [91]. Millot–Sire [79] obtained from this approach a monotonicity formula for fractional harmonic maps that lead to the partial regularity theory of stationary harmonic maps. Sadly, the harmonic extension technique is as of now only available for L^2-type functionals, that is, $W^{s,2}$-harmonic maps, thus not applicable in our case.

The singular set of stationary and minimizing $W^{s,2}$-harmonic maps (in the supercritical dimension) was analyzed in [77, 78, 80].

One challenge that keeps appearing when analyzing fractional harmonic maps (e.g., with respect to monotonicity formulas) is the lack of understanding of the fine estimates known for local equations — such as Pohozaev identities. There has been some important progress in this direction [25, 40, 94], but in many cases, the techniques available are bound to some form of the harmonic extension technique introduced for fractional harmonic maps by [79, 86] — which is

not available for $W^{s,\frac{n}{s}}$-harmonic maps we consider here (unless $n/s = 2$). This is very different to the situation of n-harmonic maps, where Pohozaev-type estimates are readily available.

1.4 | Notation

Throughout the paper, we assume that Σ is a closed Riemannian n-manifold embedded into $\mathbb{R}^L$ and $\mathcal{N}$ is a closed Riemannian manifold embedded into $\mathbb{R}^M$.

For the fractional Gagliardo seminorm, we use the standard notation

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}},$$

for any open $\Omega \subseteq \Sigma$.

We will also write

$$E_{s,p}(u, \Omega) := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y.$$

We will write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $B(x, r)$ the geodesic ball about x of radius r in Σ. When the center of the ball will not play any role, we will simply write $B(r)$.

For simplicity of notation, we write $\lesssim$ if there exists a constant C (not depending on any crucial quantity) such that $A \leqslant C\,B$. We use $\gtrsim$ in a similar way. Finally, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 | PRELIMINARIES ON HOMOTOPY THEORY FOR SOBOLEV MAPS

The purpose of this section is to recall the definition for homotopy classes of maps $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$. Here and henceforth, we always assume that Σ is a smooth n-dimensional compact manifold without boundary, and $\mathcal{N} \subset \mathbb{R}^M$ is a smooth embedded manifold, also without boundary.

Let us stress that all of the definitions and statements in this section are well known and we claim no originality whatsoever.

We make no effort to give the most general notion (e.g., considering Σ with boundary), but concentrate on what is needed for our purposes. For more detailed exposition, we refer, for example, to [51, Section 4]. Maps in $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ may not be continuous, so one needs to define homotopy classes $W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ via approximation.

We first recall the usual notion of a homotopy for continuous maps. Two maps, $u, v \in C^0(\Sigma, \mathcal{N})$, are *homotopic*, in symbols $u \sim v$, if there exists a *homotopy* $H \in C^0([0,1], C^0(\Sigma, \mathcal{N}))$, namely, an H that satisfies

$$H(0) = u, \quad H(1) = v.$$

Since Σ and $\mathcal{N}$ are smooth Riemannian manifolds, there is no difference between continuous homotopy and smooth homotopy — and one does not distinguish between them. This is the content of the following lemma.

Lemma 2.1. *Let $u, v \in C^\infty(\Sigma, \mathcal{N})$. The following relations are equivalent:*

- *$u \sim v$ in C^0, that is, there exists $H \in C^0([0,1], C^0(\Sigma, \mathcal{N}))$ such that $H(0) = u$ and $H(1) = v$;*
- *$u \sim v$ in C^∞, that is, there exists $H \in C^\infty([0,1], C^\infty(\Sigma, \mathcal{N}))$ such that $H(0) = u$ and $H(1) = v$.*

The proof of Lemma 2.1 is by approximation (using a standard mollification argument in $[0,1] \times \Sigma$ by constant extension to $(-1, 2) \times \Sigma$).

Remark 2.2. As a sidenote let us remark, that Lemma 2.1 may not be true on non-Riemannian manifolds, for example, sub-Riemannian manifolds, Carnot groups, or more general metric spaces. See, for example, [47–50, 111, 121].

We can make sense of the concept of homotopy for $W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$-maps, even if these maps are not necessarily continuous. This is possible because these maps can be approximated by smooth maps in $C^\infty(\Sigma, \mathcal{N})$.

Indeed, the following is going to be the definition of homotopy that we are going to use from now on.

Definition 2.3. Let $u, v \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$.

(1) We say $u \sim v$ (u is homotopic to v) if the following holds: for any smooth approximation u_ε of u and v_ε of v in $W^{s,\frac{n}{s}}$, we find an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, we have $u_\varepsilon \sim v_\varepsilon$ in C^∞.
(2) We define the homotopy class $[u]$ as

$$[u] := \left\{ v \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N}) : \quad v \sim u \right\}.$$

Remark 2.4. Let us remark that one can define equivalently the relation

$$u \sim v \quad \text{in } W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$$

for $u, v \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ as: there exists a path $H(t) \in C^0([0,1], W^{s,\frac{n}{s}}(\Sigma, \mathcal{N}))$ such that $H(0) = u$ and $H(1) = v$, cf. [17, Section 4].

The justification for Definition 2.3 is contained in the following proposition. In particular, it implies that we do not need to distinguish between $W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$-homotopies and $C^0(\Sigma, \mathcal{N})$-homotopy classes.

Proposition 2.5. *Definition 2.3 is well defined in the following sense:*

(1) *Any map $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ can be approximated by maps $u_k \in C^\infty(\Sigma, \mathcal{N})$ with respect to the $W^{s,\frac{n}{s}}$-norm.*
(2) *For any map $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$, there exists a small number $\varepsilon = \varepsilon(u) > 0$ such that any map $v \in W^{s,\frac{n}{s}} \cap C^0(\Sigma, \mathcal{N})$ with $\|u - v\|_{W^{s,\frac{n}{s}}(\Sigma)} < \varepsilon$ is of the same C^0-homotopy type.*
(3) *If $u, v \in C^0 \cap W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$, then $u \sim v$ in the continuous sense, if and only if $u \sim v$ in the $W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$-sense.*

For the convenience of the reader, we give the proof of Proposition 2.5 below.

We begin by recalling the fact that the manifolds we work with have a tubular neighborhood on which there exists a smooth nearest point projection. For a proof, we refer to [110, Section 2.12.3].

Lemma 2.6. *Let $\mathcal{N} \subset \mathbb{R}^M$ be a smooth, compact manifold without boundary. There exists $\delta = \delta(\mathcal{N}) > 0$ such that on the tubular neighborhood,*

$$B_\delta(\mathcal{N}) := \big\{ p \in \mathbb{R}^M \,:\, \operatorname{dist}(p, \mathcal{N}) < \delta \big\},$$

there exists the nearest point projection $\pi_{\mathcal{N}} \in C^\infty(B_\delta(\mathcal{N}), \mathcal{N})$ such that

$$|\pi_{\mathcal{N}}(p) - p| = \operatorname{dist}(p, \mathcal{N}) \quad \forall p \in B_\delta(\mathcal{N}).$$

Moreover, for $p \in \mathcal{N}$, $\Pi(p) := D\pi_{\mathcal{N}}(p) \in \mathbb{R}^{M\times M}$ is the tangential projection that maps a vector $v \in \mathbb{R}^M$ orthogonally into the tangent plane $T_p\mathcal{N}$.

Proposition 2.5 (1) is a consequence of the following lemma, which was observed by Schoen and Uhlenbeck in their celebrated paper [107, Section 3]. We remark that as shown by Schoen and Uhlenbeck [108, Section 4], an approximation as in Lemma 2.7 may not be possible if $u \in W^{s,p}$ if $sp < n$. We refer the interested reader to [6, 7, 13, 20, 51] for the theory of the approximation of manifold valued Sobolev maps by smooth maps.

Lemma 2.7. *Let $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ then u can be approximated by smooth maps $u_\varepsilon \in C^\infty(\Sigma, \mathcal{N})$ in the $W^{s,\frac{n}{s}}(\Sigma, \mathbb{R}^M)$-norm.*

Proof. For clarity of the proof, we assume that $\Sigma = \mathbb{R}^n$. Let $\varepsilon > 0$ and let us first approximate u by unconstrained smooth maps. To do so, we mollify the function $u \in W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathcal{N})$ by considering

$$\tilde{u}_\varepsilon(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \eta(y)u(x-\varepsilon y)\,\mathrm{d}y,$$

where $\eta \in C_c^\infty(\mathbb{R}^n, [0,1])$, $\operatorname{supp}\eta \subset B(0,1)$, $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(\frac{x}{\varepsilon})$, and $\int_{\mathbb{R}^n}\eta = 1$. Then, $\tilde{u}_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^M)$ and

$$\tilde{u}_\varepsilon \xrightarrow{\varepsilon\to 0} u \quad \text{strongly in } W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{R}^M).$$

The smooth map $\tilde{u}_\varepsilon$ may not map into $\mathcal{N}$, but we can ensure that the image of $\tilde{u}_\varepsilon$ is close to the manifold $\mathcal{N}$.

Let $B_\delta(\mathcal{N})$ be the tubular neighborhood of $\mathcal{N}$ from Lemma 2.6, on which the nearest point projection $\pi_{\mathcal{N}} : B_\delta(\mathcal{N}) \to \mathcal{N}$ is well defined.

Let $z \in \mathbb{R}^n$ be an arbitrary point, then we estimate for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
\operatorname{dist}(\tilde{u}_\varepsilon(x), \mathcal{N}) \leqslant |\tilde{u}_\varepsilon(x) - u(x-\varepsilon z)| &= \left| \int_{\mathbb{R}^n} \eta(y)u(x-\varepsilon y)\,\mathrm{d}y - u(x-\varepsilon z) \right| \\
&= \left| \int_{\mathbb{R}^n} \eta(y)(u(x-\varepsilon y) - u(x-\varepsilon z))\,\mathrm{d}y \right| \\
&\leqslant \int_{\mathbb{R}^n} \eta(y)\big|u(x-\varepsilon y) - u(x-\varepsilon z)\big|\,\mathrm{d}y.
\end{aligned}$$

Thus, multiplying both sides by $\eta(z)$ and integrating over $\mathbb{R}^n$ with respect to the variable z, we obtain

$$\operatorname{dist}(\tilde{u}_\varepsilon(x), \mathcal{N}) \leqslant \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(y)\eta(z)\big|u(x-\varepsilon y) - u(x-\varepsilon z)\big| \,\mathrm{d}y \,\mathrm{d}z,$$

which combined with the support of η leads to the estimate

$$\operatorname{dist}(\tilde{u}_\varepsilon(x), \mathcal{N}) \lesssim \fint_{B(x,\varepsilon)} \fint_{B(x,\varepsilon)} |u(y) - u(z)| \,\mathrm{d}y \,\mathrm{d}z.$$

Applying Hölder's inequality, we get

$$\begin{aligned}
\operatorname{dist}(\tilde{u}_\varepsilon(x), \mathcal{N}) &\lesssim \fint_{B(x,\varepsilon)} \fint_{B(x,\varepsilon)} |u(y) - u(z)| \,\mathrm{d}y \,\mathrm{d}z \\
&\lesssim \left(\fint_{B(x,\varepsilon)} \fint_{B(x,\varepsilon)} |y-z|^{\frac{2n}{\frac{n}{s}-1}} \,\mathrm{d}y \,\mathrm{d}z \right)^{1-\frac{s}{n}} \left(\fint_{B(x,\varepsilon)} \fint_{B(x,\varepsilon)} \frac{|u(y)-u(z)|^{\frac{n}{s}}}{|y-z|^{2n}} \,\mathrm{d}y \,\mathrm{d}z \right)^{\frac{s}{n}} \\
&\lesssim \left(\int_{B(x,\varepsilon)} \int_{B(x,\varepsilon)} \frac{|u(y)-u(z)|^{\frac{n}{s}}}{|y-z|^{2n}} \,\mathrm{d}y \,\mathrm{d}z \right)^{\frac{s}{n}} \xrightarrow{\varepsilon \to 0} 0,
\end{aligned}$$

where the last convergence is a consequence of the absolute continuity of the integral, and holds for a.e. $x \in \mathbb{R}^n$.

Thus, for ε sufficiently close to 0, we know that $\tilde{u}_\varepsilon \in B_\delta(\mathcal{N})$. This implies that the maps $u_\varepsilon := \pi_{\mathcal{N}} \circ \tilde{u}_\varepsilon \in C^\infty(\mathbb{R}^n, \mathcal{N})$ are well defined, Lemma 2.6. Moreover, since $\pi_{\mathcal{N}}$ is smooth, we also have

$$u_\varepsilon = \pi_{\mathcal{N}} \circ \tilde{u}_\varepsilon \to \pi_n \circ u = u \quad \text{strongly in } W^{s,\frac{n}{s}}(\mathbb{R}^n, \mathbb{R}^M) \text{ as } \varepsilon \to 0. \qquad \square$$

Next, we state a helpful lemma that says that if two maps are uniformly close, then they are homotopic.

Lemma 2.8. *Let $\mathcal{N}$ be a smooth manifold without boundary embedded into $\mathbb{R}^M$. Then there exists an $\varepsilon = \varepsilon(\mathcal{N}) > 0$ such that if $f, g \in C^0(\Sigma, \mathcal{N})$ and $\|f - g\|_{L^\infty} < \varepsilon$, then f is homotopic to g.*

Proof. Let $\pi_n : B_\varepsilon(\mathcal{N}) \to \mathcal{N}$ be the nearest point projection into the manifold that must exist for some $\varepsilon > 0$, Lemma 2.6. If $\|f - g\|_{L^\infty} < \varepsilon$, then

$$\operatorname{dist}((1-t)f(x) + tg(x), \mathcal{N}) \leqslant \|f - g\|_{L^\infty} < \varepsilon \quad \forall t \in [0,1], x \in \Sigma,$$

that is, $H(t,x) = \pi_{\mathcal{N}}((1-t)f(x) + tg(x))$ is well defined for all $t \in [0,1]$. It is easy to check that H is a homotopy between f and g. $\square$

Proposition 2.5(2) is a consequence of the following.

Lemma 2.9. *Let $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$, then there exists $\varepsilon = \varepsilon(u) > 0$ such that whenever $g_1, g_2 \in C^0 \cap W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ with*

$$\|u - g_i\|_{L^1(\Sigma)} + [u - g_i]_{W^{s,\frac{n}{s}}(\Sigma)} \leqslant \varepsilon \quad \text{for } i = 1, 2, \tag{2.1}$$

then g_1 and g_2 are homotopic.

Proof. Again, for simplicity of notation, we assume that $\Sigma = \mathbb{R}^n$.

By Lemma 2.6, there exists $\gamma = \gamma(\mathcal{N})$ and a smooth nearest point projection from a γ-neighborhood of $\mathcal{N}$ into $\mathcal{N}$ which we denote by $\pi_{\mathcal{N}} : B_\gamma(\mathcal{N}) \to \mathcal{N}$.

By the absolute continuity of the integral, for any $\theta > 0$, there exists a $\delta_0 = \delta_0(u, \theta)$ such that

$$\sup_{B(2\delta_0)\subset\Sigma} [u]_{W^{s,\frac{n}{s}}(B(2\delta_0))} \leqslant \theta. \tag{2.2}$$

For $i = 1, 2$, let $g_{i,\delta} := \eta_\delta * g_i$, $\delta < \delta_0$ where $\eta \in C_c^\infty(B(0,1))$, $\int_{\mathbb{R}^n} \eta = 1$ is the usual mollifier.

As in the proof of Lemma 2.7,

$$\begin{aligned}
\operatorname{dist}(g_{i,\delta}(x), \mathcal{N}) &\leqslant C(n) \fint_{B(x,\delta)} \fint_{B(x,\delta)} |g_i(y) - g_i(z)| \, \mathrm{d}y \, \mathrm{d}z \\
&\leqslant C(n,s) [g_i]_{W^{s,\frac{n}{s}}(B(2\delta))} \\
&\leqslant C(n,s) \Big([u - g_i]_{W^{s,\frac{n}{s}}(B(2\delta))} + [u]_{W^{s,\frac{n}{s}}(B(2\delta))} \Big) \\
&\leqslant C(n,s)(\varepsilon + \theta),
\end{aligned}$$

where in the last inequality, we used (2.2) and (2.1).

So, if ε and θ are small enough so that $C(n,s)(\varepsilon + \theta) < \gamma$, we have $\tilde{g}_{i,\delta} := \pi_{\mathcal{N}} \circ g_{i,\delta}$ is well defined for any $i = 1, 2$ and any $\delta < \delta_0$.

Observe that $\delta \ni [0, \delta_0] \mapsto \tilde{g}_{i,\delta}$ is a homotopy, so g_i and $\tilde{g}_{i,\delta_0}$ are homotopic for each $i = 1, 2$.

Moreover,

$$\|g_{1,\delta_0} - g_{2,\delta_0}\|_{L^\infty} \leqslant C(n) \frac{1}{\delta_0^n} \|g_1 - g_2\|_{L^1} \leqslant C(n) \frac{\varepsilon}{\delta_0^n}.$$

So if we choose $\varepsilon = \varepsilon(u) > 0$ possibly even smaller so that $C(n)\frac{\varepsilon}{\delta_0^n} < \varepsilon(\mathcal{N})$, where $\varepsilon(\mathcal{N})$ is from Lemma 2.8, then we know that g_{1,δ_0} is homotopic to g_{2,δ_0}. That is, we have shown

$$g_1 \sim g_{1,\delta_0} \sim g_{2,\delta_0} \sim g_2.$$

This concludes the proof. □

Proof of Proposition 2.5(3). Let $u, v \in C^0 \cap W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ and assume $u \sim v$ with respect to continuous homotopy. Denote the usual convolution of u and v with the standard mollifier respectively by u_δ and v_δ. Then u_δ converges uniformly to u. In particular, for all small δ, we have that $\pi_{\mathcal{N}} \circ u_\delta$ is C^0-homotopic to u, by Lemma 2.8. Similarly, $\pi_{\mathcal{N}} \circ v_\delta$ is C^0-homotopic to v. Since v and u are

C^0-homotopic, this implies that $\pi_{\mathcal{N}} \circ u_\delta$ and $\pi_{\mathcal{N}} \circ v_\delta$ are C^0-homotopic to each other for all small δ. But $\pi_{\mathcal{N}} \circ u_\delta$ is a smooth approximation of u with respect to the $W^{s,\frac{n}{s}}$-norm, and similarly, $\pi_{\mathcal{N}} \circ v_\delta$ of v. By Lemma 2.9, this means that any other smooth approximation of v and u, respectively, is also eventually C^0-homotopic to each other. By Definition 2.3, this means that u and v are $W^{s,\frac{n}{s}}$-homotopic.

For the converse, we argue similarly. If u and v are $W^{s,\frac{n}{s}}$-homotopic as defined in Definition 2.3, $\pi_{\mathcal{N}} \circ u_\delta$ and $\pi_{\mathcal{N}} \circ v_\delta$ must be homotopic for all small δ. For small δ, we have $\pi_{\mathcal{N}} \circ u_\delta$ is C^0-homotopic to u (by uniform convergence and Lemma 2.8) and likewise $\pi_{\mathcal{N}} \circ v_\delta$ is C^0-homotopic to v. This implies that u is C^0-homotopic to v. □

Similar to Lemma 2.9, we also obtain the following.

Lemma 2.10. *For any manifold $\Sigma, \mathcal{N}$ as above and $s \in (0,1)$, there exist $\varepsilon = \varepsilon(\mathcal{N}, \Sigma)$ such that the following holds.*

If $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ and

$$[u]_{W^{s,\frac{n}{s}}(\Sigma)} < \varepsilon, \tag{2.3}$$

then u is homotopic to a constant map in the sense of Definition 2.3.

Proof. Let $(u)_\Sigma := |\Sigma|^{-1} \int_\Sigma u$. From (2.3), we obtain as in the proof of Lemma 2.7

$$\operatorname{dist}((u)_\Sigma, \mathcal{N}) \lesssim \varepsilon.$$

If ε is small enough, this implies that $v := \pi_{\mathcal{N}}((u)_\Sigma)$ is well defined by Lemma 2.6.

Also, denoting by $u_\delta := \eta_\delta * u$ the usual mollification, we have

$$\operatorname{dist}(u_\delta, \mathcal{N}) \lesssim \varepsilon \quad \forall \delta \leqslant 1.$$

Setting $w_\delta := \pi_{\mathcal{N}}(u_\delta)$, we have that w_1 is homotopic to u in the sense of Definition 2.3. Moreover, we have

$$\|w_1 - v\|_{L^\infty} \leqslant C(\Sigma, \mathcal{N}) \|u - (u)_\Sigma\|_{L^1(\Sigma)} \lesssim [u]_{W^{s,\frac{n}{s}}(\Sigma)} < \varepsilon.$$

So, choosing ε small enough, we have from Lemma 2.8 that w_1 and v are homotopic. This implies that u and v are homotopic, and v is a constant map. □

3 | REGULARITY THEORY FOR MINIMIZERS IN HOMOTOPY

The main result of this section is the following regularity theory for minimizers.

Theorem 3.1. *Let Σ, $\mathcal{N}$ be as above. If $n = 1$, then assume that $s \leqslant \frac{1}{2}$, if $n \geqslant 2$, then assume that $s \in (0,1)$. There exists $\varepsilon > 0$ and $s_0 > s$ such that the following holds for any $t \in [s, s_0]$.*

Assume that $u \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ and that for a geodesic ball $B(R) \subset \Sigma$, the following holds:

- *u is a minimizing $W^{t,\frac{n}{s}}$-harmonic map in $B(R)$, that is,*

$$E_{t,\frac{n}{s}}(u, \Sigma) \leqslant E_{t,\frac{n}{s}}(v, \Sigma)$$

holds for all $v \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ such that
  - *$u \equiv v$ in $\Sigma \setminus B(R)$, and*
  - *$u \sim v$ in homotopy (as defined in Definition 2.3).*
- $[u]_{W^{s,\frac{n}{s}}(B(R))} < \varepsilon$.

Then $u \in W^{s_0,\frac{n}{s}}(B(R/2)) \cap C^{s_0-s}(B(R/2))$ and we have the estimate

$$[u]_{C^{s_0-s}(B(R/2))} + [u]_{W^{s_0,\frac{n}{s}}(B(R/2))} \leqslant C\, R^{s-s_0} [u]_{W^{s,\frac{n}{s}}(B(R))} \left([u]^{1-\frac{s}{n}}_{W^{s,\frac{n}{s}}(\Sigma)} + [u]_{W^{s,\frac{n}{s}}(\Sigma)} \right). \tag{3.1}$$

The important feature of Theorem 3.1 is that the regularity estimate is uniform as $t \to s^+$. By Morrey–Sobolev embedding, any map $u \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ is C^{t-s}-continuous if $t > s$, but it may not be C^{s_0-s}-continuous.

Clearly, global minimizers (without any assumptions on homotopy type) also fall under the realm of Theorem 3.1, and we record the following.

Corollary 3.2. *Let $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ be a minimizing harmonic map (without restriction to any homotopy class) in an open set $\Omega \subset \Sigma$, that is, assume that*

$$E_{s,\frac{n}{s}}(u, \Sigma) \leqslant E_{s,\frac{n}{s}}(v, \Sigma)$$

for all $v \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ with $u \equiv v$ on Ω^c. Then u is Hölder continuous in Ω.

Remark 3.3. While the initial step in the proof of Theorem 3.1, namely, Theorem 3.5, relies on the minimizing property, this is probably only really necessary in the case $t = s$. Most likely, for $t > s$, one could test the Euler–Lagrange equations to obtain a similar result (but due to the necessity for uniform Hölder exponents, we did not attempt to do this).

That is, most likely Theorem 3.1 holds for *critical* $W^{t,\frac{n}{s}}$-harmonic maps as long as $t > s$. In particular, it seems that a similar statement as in Theorem 3.1 could be made, for example, for maps $u \in W^{s,p}(\Sigma, \mathbb{R}^M)$, $s - \frac{n}{p} > 0$, satisfying

$$\left| \int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \right| \lesssim \int_\Sigma \int_\Sigma |\varphi(x)| \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y.$$

This is not necessary for our purposes, so we do not follow this direction.

Remark 3.4. For $t = s$ and "round" target spaces $\mathcal{N} = \mathbb{S}^{n-1}$ or $\mathcal{N}$, a compact Lie group Theorem 3.1 holds also for (possibly nonminimizing, but only) *critical* $W^{s,\frac{n}{s}}$-harmonic maps [76, 101]. For nonround targets, this is a major open question even for the p-harmonic map case $s = 1$,

$p \neq 2$, and only partial results are known under nongeometric assumptions [34, 57, 100], see also the survey [106].

The proof of Theorem 3.1 consists of three steps:

Step 1. We first prove in Theorem 3.5 local C^α-regularity of the solution. We do not require a precise estimate of $[u]_{C^\alpha}$, but we crucially get that α is independent of t.

Step 2. In Theorem 3.7, we show that the C^α-regularity from above translates into a $W^{s+\beta,\frac{n}{s}}$-regularity for $\beta = \beta(\alpha)$ using a technique developed by Brasco and Lindgren, [15, 16]. We then choose $s_0 := s + \beta$.

Step 3. The *estimate* in Theorem 3.1 is a consequence of a priori estimates of the Euler–Lagrange equations, that is, of the respective harmonic map equation, under the assumption that the solution already belongs to $W^{s_0,\frac{n}{s}}$. This will be done in Theorem 3.10, and is based on a stability estimate for the fractional p-Laplacian, see [104].

3.1 | Step 1: Uniform Hölder continuity

We begin the proof of Theorem 3.1 by first proving Hölder continuity of minimizers — with a uniform Hölder exponent $\alpha > 0$, which does not change as $t \to s^+$. Namely, we obtain the following theorem.

Theorem 3.5. *Let $s \in (0,1)$ and Σ, $\mathcal{N}$ be as above. There exists $\alpha > 0$ and $s_1 > s$ such that the following holds for any $t \in [s, s_1)$.*

Assume that $u \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ and for a geodesic ball $B(R) \subset \Sigma$ u is a minimizing $W^{t,\frac{n}{s}}$-harmonic map in $B(R)$, that is,

$$E_{t,\frac{n}{s}}(u, \Sigma) \leqslant E_{t,\frac{n}{s}}(v, \Sigma)$$

holds for all $v \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ such that

- *$u \equiv v$ in $\Sigma \setminus B(R)$, and*
- *$u \sim v$ in homotopy (as defined in Section 2).*

Then $u \in C^\alpha_{loc}(B(R))$.

Let us remark that Millot–Sire–Yu [80] already obtained partial regularity for $E_{s,2}$-minimizers for $n = 1$, $s < \frac{1}{2}$.

The proof of Theorem 3.5 follows from a Cacciopoli-type estimate (and the technique probably can be traced back to Morrey [85]).

The first step is to construct a suitable competitor map. For $t > s$, this is simply the interpolation between u and the mean value $(u)_{B(r)\setminus B(r/2)}$. For $t = s$, we have to be more careful, and apply an argument similar to Luckhaus' lemma. Namely, we have the following.

Lemma 3.6. *Let Σ, $\mathcal{N}$ be as above, $s \in (0,1)$ and $s_1 \in (s,1)$. There exists a constant $C > 0$ such that the following holds for any $t \in [s, s_1]$.*

Let $u \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$. There exists an $\varepsilon > 0$ (possibly depending on u) such that the following holds.

Assume that for some $r \in (0,1)$ and some ball $B(10r) \subset \Sigma$,

If $t = s$,

$$\int_{B(10r)} \int_{\Sigma} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy < \varepsilon^{\frac{n}{s}}. \tag{3.2}$$

If $t > s$,

$$|u(x)-u(y)| < \varepsilon \quad \forall x, y \in B(10r). \tag{3.3}$$

Then, there exists a $v \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ such that

(1) *$v \equiv u$ in $\Sigma \setminus B(r)$,*
(2) *v is homotopic to u, and*
(3) *we have*

$$\int_{B(r)} \int_{\Sigma} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \, dx \, dy \leqslant C \int_{B(2r)\setminus B(r/2)} \int_{\Sigma} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \, dx \, dy. \tag{3.4}$$

Proof of Lemma 3.6 for $t > s$. For simplicity of the presentation, we assume that $\Sigma = \mathbb{R}^n$; moreover, without loss of generality, we can assume $B(r) = B(0,r)$.

Since $\mathcal{N}$ is a smooth compact manifold without boundary, there exists a tubular neighborhood $B_\varepsilon(\mathcal{N}) \subset \mathbb{R}^M$ and a smooth projection $\pi_{\mathcal{N}} : B_\varepsilon(\mathcal{N}) \to \mathcal{N}$, Lemma 2.6. We choose ε possibly even smaller so that $\varepsilon \|D\pi_{\mathcal{N}}\|_{L^\infty} < \tilde{\varepsilon}(\mathcal{N})$, where $\tilde{\varepsilon}(\mathcal{N})$ is the small quantity from Lemma 2.8.

Set $\eta \in C_c^\infty(B(\frac{9}{10}r), [0,1])$ and $\eta \equiv 1$ in $B(\frac{8}{10}r)$, $|\nabla \eta| \lesssim r^{-1}$. Set

$$w := (1-\eta)u + \eta (u)_{A(r)}.$$

Here $A(r) = B(r) \setminus B(r/2)$ and $(u)_{A(r)}$ denotes the mean value on that set. Observe that

$$\begin{aligned} \operatorname{dist}((1-\eta(x))u(x) + \eta(x)(u)_{A(r)}, \mathcal{N}) &\leqslant |(1-\eta)u(x) + \eta (u)_{A(r)} - u(x)| \\ &\leqslant \eta(x)|u(x) - (u)_{A(r)}| \\ &< \varepsilon, \end{aligned}$$

we used (3.3) in the last inequality.

Thus, we can compose w with $\pi_{\mathcal{N}}$ and set

$$v := \pi_{\mathcal{N}} \circ w.$$

Observe,

$$|u(x) - v(x)| \leqslant \|D\pi_{\mathcal{N}}\|_{L^\infty} |u(x) - (u)_{A(r)}| \chi_{B(r)}(x) \leqslant \|D\pi_{\mathcal{N}}\|_{L^\infty} \frac{\varepsilon}{100} < \tilde{\varepsilon}(\mathcal{N}).$$

From Lemma 2.8, we obtain that u and v are homotopic.

It remains to prove the estimate (3.4). By Lipschitz continuity of $\pi_{\mathcal{N}}$, we have

$$\int_{B(r)} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \, dx \, dy \leqslant C(\pi_{\mathcal{N}}) \int_{B(r)} \int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}.$$

Now we have

$$\begin{aligned} w(x)-w(y) =&(1-\eta(x))u(x)+\eta(x)(u)_{A(r)}-\big((1-\eta(y))u(y)+\eta(y)(u)_{A(r)}\big)\\ =&(1-\eta(x))(u(x)-u(y))-(\eta(x)-\eta(y))\big(u(y)-(u)_{A(r)}\big). \end{aligned}$$

So, we have, decomposing $\mathbb{R}^n = B(r) \cup \mathbb{R}^n \setminus B(r)$ (it is important to observe that we can choose all of the constants to be independent of t as long as $t \in [s, s_1]$),

$$\begin{aligned} \int_{B(r)}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y \lesssim& \int_{B(r)}\int_{B(r)}(1-\eta(x))^{\frac{n}{s}}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{B(r)}\int_{B(r)}\frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(r)}(1-\eta(x))^{\frac{n}{s}}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(r)}\frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

In view of the support of η and $1-\eta$, we find

$$\begin{aligned} \int_{B(r)}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y \lesssim& \int_{B(r)}\int_{B(r)\setminus B(r/2)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+r^{-\frac{n}{s}}\int_{B(r)}\int_{B(r)}\frac{|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+(t-1)\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(r)}\frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Integrating in x and then using Jensen's inequality, we have

$$\begin{aligned} r^{-\frac{n}{s}}\int_{B(r)}\int_{B(r)}\frac{|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+(t-1)\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y \lesssim& r^{-t\frac{n}{s}}\int_{B(r)}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}\,\mathrm{d}y\\ \lesssim&\int_{B(r)}\int_{B(r)/B(r/2)}\frac{|u(y)-u(z)|^{\frac{n}{s}}}{|y-z|^{n+t\frac{n}{s}}}\,\mathrm{d}z\,\mathrm{d}y. \end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_{B(r)}\int_{\mathbb{R}^n\setminus B(r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\int_{B(r)}\int_{B(2r)\setminus B(r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(2r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim\int_{B(2r)\setminus B(r)}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(2r)}\frac{|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(2r)}\frac{|u(x)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim\int_{B(2r)\setminus B(r)}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+r^{-t\frac{n}{s}}\int_{B(r)}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}\,\mathrm{d}y+r^{-n}\int_{B(r)}\int_{B(r)/B(r/2)}\int_{\mathbb{R}^n\setminus B(2r)}\frac{|u(x)-u(z)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\\
&\quad\lesssim\int_{B(2r)\setminus B(r/2)}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

In the last step, we used that if $x\in\mathbb{R}^n\setminus B(2r)$, $y\in B(r)$, and $z\in B(r)\setminus B(r/2)$, then $|x-y|\geqslant|r-|x||$ and

$$|x-z|\leqslant\operatorname{dist}(x,B(r))+2r+\operatorname{dist}(z,B(r))\leqslant 4\operatorname{dist}(x,B(r))\leqslant 4|r-|x||.$$

Similarly,

$$\begin{aligned}
&\int_{B(r)}\int_{\mathbb{R}^n\setminus B(r)}\frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim\int_{B(r)}\int_{B(2r)\setminus B(r)}\frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(2r)}\frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim r^{-\frac{n}{s}}\int_{B(r)}\int_{B(2r)\setminus B(r)}\frac{|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+(t-1)\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{B(r)}\int_{\mathbb{R}^n\setminus B(2r)}\frac{|u(y)-(u)_{A(r)}|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim\int_{B(2r)\setminus B(r/2)}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

This establishes (3.4) and the proof is complete (in the case $t > s$). □

Proof of Lemma 3.6 for $t = s$. In the following, we restrict for simplicity to the case $n \geqslant 2$, but the statement remains true for $n = 1$ with easy modifications. Again, for the simplicity of the presentation, we assume that $\Sigma = \mathbb{R}^n$.

We claim that there exists an radius $\rho \in (\frac{3}{4}r, \frac{5}{6}r)$ such that

$$\begin{aligned} & r\int_{\partial B(\rho)}\int_{B(2r)\setminus B(r/2)} \frac{|u(\theta)-u(\omega)|^{\frac{n}{s}}}{|\theta-\omega|^{2n}}\,\mathrm{d}\theta\,\mathrm{d}\omega + r\int_{\partial B(\rho)}\int_{\partial B(\rho)} \frac{|u(\theta)-u(\omega)|^{\frac{n}{s}}}{|\theta-\omega|^{2n-1}}\,\mathrm{d}\theta\,\mathrm{d}\omega \\ & \precsim \int_{B(r)\setminus B(r/2)}\int_{B(2r)\setminus B(r/2)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.5}$$

Indeed, by Fubini's theorem for any $\kappa \in (0, \frac{1}{2})$, there exists a set $A_\kappa \subset (\frac{3}{4}r, \frac{5}{6}r)$, with $\mathcal{L}^1((\frac{3}{4}r, \frac{5}{6}r) \setminus A_\kappa) \leqslant \kappa r$ such that for any $\tau \in A_\kappa$,

$$r\int_{\partial B(\tau)}\int_{B(2r)\setminus B(r/2)} \frac{|u(x)-u(\omega)|^{\frac{n}{s}}}{|x-\omega|^{2n}}\,\mathrm{d}x\,\mathrm{d}\omega \leqslant \kappa^{-1}\int_{B(r)\setminus B(r/2)}\int_{B(2r)\setminus B(r/2)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6}$$

Indeed, if (3.6) was not true on a set $\bar{A} \subset (\frac{3}{4}r, \frac{5}{6}r)$ of $\mathcal{L}^1$-measure $> \kappa r$, we integrate both sides over that set and get

$$\int_{B\left(\frac{5}{6}r\right)\setminus B\left(\frac{3}{4}r\right)}\int_{B(2r)\setminus B(r/2)} \frac{|u(x)-u(\omega)|^{\frac{n}{s}}}{|x-\omega|^{2n}}\,\mathrm{d}x\,\mathrm{d}\omega > \int_{B(r)\setminus B(r/2)}\int_{B(2r)\setminus B(r/2)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y,$$

a contradiction to the monotonicity of the integral with respect to its integration domain.

Also, by Fubini's theorem for any $\sigma \in (0, \frac{1}{2})$, there exists a set $B_\sigma \subset (\frac{3}{4}r, \frac{5}{6}r)$, with $\mathcal{L}^1((\frac{3}{4}r, \frac{5}{6}r) \setminus B_\sigma) \precsim \sigma r$ such that for any $\tau \in A_\sigma$,

$$\begin{aligned} & r\int_{\partial B(\tau)}\int_{\partial B(\tau)} \frac{|u(\omega)-u(\theta)|^{\frac{n}{s}}}{|\omega-\theta|^{2n-1}}\,\mathrm{d}\omega\,\mathrm{d}\theta \\ & \qquad \leqslant \sigma^{-1}\int_{B(r)\setminus B(r/2)}\int_{B(r)\setminus B(r/2)} \frac{|u(y)-u(x)|^{\frac{n}{s}}}{|y-x|^{2n}}\,\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{3.7}$$

for every $\tau \in B_\sigma$. The arguments to obtain (3.7) are a bit more complicated, although well known to experts. For simplicity, let $r = 1$, the right power for the factor involving r follows from scaling arguments. One argument for (3.7) goes via the Gagliardo extension,[†] we have

$$\int_{B(1)\setminus B(1/2)}\int_{B(1)\setminus B(1/2)} \frac{|u(y)-u(x)|^{\frac{n}{s}}}{|y-x|^{2n}}\,\mathrm{d}y\,\mathrm{d}x \approx \inf_U \int_{B(1)\setminus B(1/2)\times[0,\infty)} t^{\frac{n}{s}-1-n}|\nabla U|^{\frac{n}{s}},$$

[†] This was popularized in the PDE community by [22], see also the harmonic analysis side in [21, 112] or, for an collection of identifications, [68, Proposition 10.2].

where the infimum is taken over all smooth maps $U : B(1)\backslash B(1/2)\times[0,\infty)\to\mathbb{R}^M$ with $U = u$ in the trace sense on $B(1)\backslash B(1/2)\times\{0\}$. Now we can apply the argument via Fubini's theorem in $B(1)\backslash B(1/2)\times[0,\infty)$ and find a large set B_σ so that for each slice $\rho\in B_\sigma$,

$$\int_{\partial B(\rho)\times[0,\infty)} t^{\frac{n}{s}-1-n}|\nabla U|^{\frac{n}{s}} \lesssim \sigma^{-1}\int_{B(1)\backslash B(1/2)\times[0,\infty)} t^{\frac{n}{s}-1-n}|\nabla U|^{\frac{n}{s}}.$$

By the trace theorem, we have

$$[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\partial B(\rho))} \lesssim \int_{\partial B(\rho)\times[0,\infty)} t^{\frac{n}{s}-1-n}|\nabla U|^{\frac{n}{s}}.$$

Suitably scaling this argument, we obtain (3.7) for any $r\in(0,1)$.

Combining (3.6) and (3.7), taking σ and $\varkappa$ small enough, we can ensure that $A_\varkappa\cap B_\sigma\neq\emptyset$ and for $\rho\in A_\varkappa\cap B_\sigma$, (3.5) holds.

From now on we fix such "good slice," that is, a $\rho\in(\frac{3}{4}r,\frac{5}{6}r)$ such that (3.5) holds.

We know from Morrey–Sobolev embedding that $W^{s,\frac{n}{s}}(\partial B(\rho))\subset C^{\frac{s}{n}}(\partial B(\rho))$ with

$$\begin{aligned}|u(\theta)-u(\omega)| &\lesssim |\theta-\omega|^{\frac{s}{n}}[u]_{W^{s,\frac{n}{s}}(\partial B(\rho))}\\ &\lesssim |\theta-\omega|^{\frac{s}{n}} r^{-\frac{s}{n}}\left(\int_{B(r)\backslash B(r/2)}\int_{B(2r)\backslash B(r/2)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s}{n}} \qquad (3.8)\\ &\lesssim \left(\int_{B(r)\backslash B(r/2)}\int_{B(2r)\backslash B(r/2)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s}{n}} \quad \text{for all } \theta,\,\omega\in\partial B(\rho).\end{aligned}$$

Set for a $\delta\in(0,\frac{1}{4})$,

$$w(x) := \begin{cases} u(x) & |x|>\rho\\ (1-\eta(|x|))u(\theta)+\eta(|x|)(u)_{\partial B(\rho)} & \theta=\rho\frac{x}{|x|},\ |x|\in((1-\delta)\rho,\rho)\\ (u)_{\partial B(\rho)} & |x|<(1-\delta)\rho,\end{cases}$$

where $\eta:\mathbb{R}_+\to[0,1]$ is smooth with $\eta(t)=0$ for $t\geqslant(1-\frac{\delta}{2})\rho$ and $\eta\equiv1$ on $(0,(1-\frac{3}{4}\delta)\rho]$, $|\eta'|\leqslant\frac{100}{\delta\rho}$.

We apply a lemma reminiscent of Luckhaus' lemma, namely, Lemma C.1. Observe from (3.8) and (3.2), we obtain

$$\operatorname{dist}((u)_{\partial B(r)},\mathcal{N})\lesssim\varepsilon.$$

From Lemma C.1 and again (3.2), we obtain

$$\operatorname{dist}(w,\mathcal{N})\lesssim\varepsilon.$$

We choose the ε in the assumptions of Lemma 3.6 small enough so that the map w lies in the tubular neighborhood of the manifold $\mathcal{N}$, cf. Lemma 2.6.

We set

$$v := \pi_{\mathcal{N}} \circ w.$$

We need to show (3.4).
Let $A(\rho) = B(\frac{3}{2}\rho) \setminus B(\rho)$ and denote by $(u)_{A(\rho)} = (v)_{A(\rho)}$ the mean value of $u = v$ on $A(\rho)$.

$$\begin{aligned}\int_{B(r)}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \lesssim & \int_{B(2\rho)}\int_{B(2\rho)} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \\ & + \int_{B\left(\frac{4}{3}\rho\right)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|u(x)-(u)_{A(\rho)}|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \\ & + \int_{B\left(\frac{4}{3}\rho\right)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|(v)_{A(\rho)}-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{3.9}$$

Now observe that if $z \in A(\rho)$, $x \in \mathbb{R}^n \setminus B(2\rho)$ and $y \in B(\frac{4}{3}\rho)$, then

$$|x-z| \leqslant \operatorname{dist}(x, B(4/3\rho)) + \operatorname{dist}(z, B(4/3\rho)) \leqslant 2\operatorname{dist}(x, B(4/3\rho)) \leqslant 2\left|\frac{4}{3} - |x|\right| \leqslant |x-y|.$$

Consequently,

$$\begin{aligned}\int_{B\left(\frac{4}{3}\rho\right)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|u(x)-(u)_{A(\rho)}|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y &\lesssim \rho^{-n}\int_{B\left(\frac{4}{3}\rho\right)}\int_{A(\rho)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|u(x)-u(z)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\ &\approx \rho^{-n}\int_{B\left(\frac{4}{3}\rho\right)}\int_{A(\rho)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|u(x)-u(z)|^{\frac{n}{s}}}{|x-z|^{2n}}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y \\ &\approx \int_{A(\rho)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|u(x)-u(z)|^{\frac{n}{s}}}{|x-z|^{2n}}\,\mathrm{d}x\,\mathrm{d}z \\ &\leqslant \int_{B(2r)\setminus B(r/2)}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^{\frac{n}{s}}}{|x-z|^{2n}}\,\mathrm{d}x\,\mathrm{d}z.\end{aligned} \tag{3.10}$$

Also, integrating in x, we have

$$\begin{aligned}\int_{B\left(\frac{4}{3}\rho\right)}\int_{\mathbb{R}^n\setminus B(2\rho)} \frac{|(v)_{A(\rho)}-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y &\approx \rho^{-n}\int_{B\left(\frac{4}{3}\rho\right)} |(v)_{A(\rho)}-v(y)|^{\frac{n}{s}}\,\mathrm{d}y \\ &\lesssim \int_{B(2\rho)}\int_{B(2\rho)} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{3.11}$$

Lastly, observe that since $v = \pi_{\mathcal{N}} \circ w$ and $\pi_{\mathcal{N}}$ is Lipschitz

$$\int_{B(2\rho)} \int_{B(2\rho)} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy \leqslant C(\pi_{\mathcal{N}}) \int_{B(2\rho)} \int_{B(2\rho)} \frac{|w(x)-w(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy. \tag{3.12}$$

Plugging (3.11), (3.10), and (3.12) into (3.9), we arrive at

$$\begin{aligned} \int_{B(r)} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy \lesssim & \int_{B(2r)\setminus B(r/2)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy \\ & + \int_{B(2\rho)} \int_{B(2\rho)} \frac{|w(x)-w(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy. \end{aligned} \tag{3.13}$$

From the estimate of Lemma C.1, namely, from (C.2), combined with (3.5) and (3.8), we have

$$[w]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(2\rho))} \lesssim [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(2\rho)\setminus B(\rho))} + \int_{B(r)\setminus B(r/2)} \int_{B(2r)\setminus B(r/2)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy. \tag{3.14}$$

Plugging (3.14) into (3.13), we obtain

$$\int_{B(r)} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy \lesssim \int_{B(2r)\setminus B(r/2)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy. \tag{3.15}$$

In particular, (3.4) is established.

It remains to show that u and v are homotopic.

Since $u = v$ outside of $B(r)$, we have in particular from (3.15) and (3.2)

$$[u-v]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \lesssim \varepsilon.$$

From Poincaré inequality, we obtain

$$\|u-v\|_{L^1(\mathbb{R}^n)} + [u-v]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \lesssim \varepsilon.$$

Choosing ε small enough, we can conclude that for $\varepsilon(u)$ from Lemma 2.9, we have

$$\|u-v\|_{L^1(\mathbb{R}^n)} + [u-v]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} < \frac{\varepsilon(u)}{4}.$$

In view of Lemma 2.9, we have that u and v are homotopic in the sense of Definition 2.3. This finishes the proof of Lemma 3.6. □

Proof of Theorem 3.5. Hölder continuity is a local property, and since we are not interested in any sort of estimate at this point, it suffices to prove Hölder continuity around any point $x_0 \in B(R)$. Without loss of generality, we may assume that $x_0 = 0$.

Let ε be from Lemma 3.6.

If $t > s$, by Sobolev embedding any map $u \in W^{t,\frac{n}{s}}$ is uniformly continuous in $B(R)$, so there exists $r_0 > 0$ such $B(10r_0) \subset B(R)$ and $|u(x) - u(y)| < \varepsilon$ for all $x, y \in B(10r_0)$, that is, (3.3) is satisfied for any $r < r_0$.

If $t = s$, by absolute continuity of the integral, there exists $r_0 > 0$ such that (3.2) is satisfied for any $r < r_0$.

For any $r < r_0$ and any ball $B(10r) \subset B(R)$, we apply Lemma 3.6 and obtain a competitor v for the minimizer u, that is,

$$E_{t,\frac{n}{s}}(u) \leqslant E_{t,\frac{n}{s}}(v).$$

Since $u \equiv v$ in $\mathbb{R}^n \setminus B(r)$, this implies

$$\int \int_{(\mathbb{R}^n)^2 \setminus (B(r)^c)^2} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \leqslant \int \int_{(\mathbb{R}^n)^2 \setminus (B(r)^c)^2} \frac{|v(x) - v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.$$

Here we denote

$$\begin{aligned}(\mathbb{R}^n)^2 \setminus (B(r)^c)^2 =& (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathbb{R}^n \setminus B(r) \times \mathbb{R}^n \setminus B(r)) \\ =& (B(r) \times B(r)) \cup (\mathbb{R}^n \setminus B(r) \times B(r)) \cup (B(r) \times \mathbb{R}^n \setminus B(r)) \\ =& (\mathbb{R}^n \times B(r)) \cup (B(r) \times \mathbb{R}^n \setminus B(r)).\end{aligned} \tag{3.16}$$

In particular, we have

$$\begin{aligned}\int_{B(r)} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y &\leqslant \int \int_{(\mathbb{R}^n)^2 \setminus (B(r)^c)^2} \frac{|v(x) - v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \int_{B(r)} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Applying (3.4) we find

$$\int_{B(r)} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \leqslant C \int_{B(2r)\setminus B(r/2)} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y, \tag{3.17}$$

where C is a constant depending only on s and s_1, not on $t \in [s, s_1]$.

We now perform the hole filling trick by adding

$$C \int_{B(r/2)} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y$$

to both sides of (3.17), and find for $\tau := \frac{C}{C+1} \leqslant 1$

$$\int_{B(r/2)} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \leqslant \tau \int_{B(2r)} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.$$

This inequality holds for any $r < r_0$. Applying it to $r = 4^{-k} r_0$, we have

$$\int_{B(4^{-k-1}r_0)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \leqslant \tau^k \int_{B(r_0)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.$$

Setting $\beta := \log_4 \tau$, this implies

$$\int_{B(4^{-k-1}r_0)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \lesssim 4^{-k\beta} \int_{B(r_0)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.$$

Since for any $r \in (0, r_0)$, we can find $k \in \mathbb{N}_0$ such that $\frac{r}{r_0} \approx 4^{-k}$, we conclude for any $r \in (0, r_0)$

$$\int_{B(r)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \lesssim \left(\frac{r}{r_0}\right)^{\beta} \int_{B(r_0)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.$$

Observe that β only depends on τ (and thus on s and s_1, but not on $t \in [s, s_1]$).

Since $s \leqslant t$, we have in particular for any $r < r_0$,

$$\int_{B(r)} \int_{B(r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \,\mathrm{d}x\,\mathrm{d}y \lesssim r^{(t-s)\frac{n}{s}} \left(\frac{r}{r_0}\right)^{\beta} \int_{B(r_0)} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.$$

This, in turn, readily implies

$$⨍_{B(r)} |u - (u)_{B(r)}| \leqslant C(r_0, u)\, r^{(t-s)+\beta\frac{s}{n}}.$$

If $t \in [s, s_1]$, then

$$⨍_{B(r)} |u - (u)_{B(r)}| \leqslant C(r_0, u)\, r^{\beta\frac{s}{n}}.$$

So, u belongs to a Campanato space in $B(r)$ and this implies that $u \in C^{\beta\frac{s}{n}}_{loc}(B(r))$, see [43, III, Theorem 1.2]. Setting $\alpha := \beta\frac{s}{n} > 0$, we conclude. □

3.2 | Step 2: Higher differentiability

We show that the Euler–Lagrange equation for harmonic maps combined with the Hölder continuity from Theorem 3.5 implies higher differentiability. The following theorem is strongly inspired by the techniques for the fractional p-Laplacian due to Brasco and Lindgren [15, 16].

Theorem 3.7. *Let $p \geqslant 2$. Assume that $u \in W^{s,p}(\mathbb{R}^n)$, $f \in L^1(\mathbb{R}^n)$ solve*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n} f\varphi \quad \forall \varphi \in C^\infty_c(B(R)). \tag{3.18}$$

If $u \in L^\infty \cap C^\alpha(B(R))$ for some $\alpha > 0$, then $u \in W^{s+\gamma,p}(B(R/2))$ for any $\gamma < \min\{\frac{\alpha}{p}, 1\}$.

Proof. Without loss of generality, we can assume that $\frac{\alpha}{p} < 1$ because if $u \in C^\alpha(B)$, then $u \in C^\beta(B)$ for any $\beta < \alpha$, and we could simply work with β instead of α.

While this is not their statement, the proof of Theorem 3.7 is strongly motivated by the argument in [15], in particular, we take inspiration in [15, Proposition 3.1]. This is why we also follow the notation in [15].

For $h \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^M$, we set $f_h(x) := f(x+h)$, $\delta_h f(x) := f(x+h) - f(x)$. Also, for $v \in \mathbb{R}^M$, set

$$J_p(v) := |v|^{p-2}v.$$

Set $\delta := \frac{1}{100}R$. Let $|h| < \delta$, and let $\varphi \in C_c^\infty(B(R-2\delta))$. Then $\varphi_h \in C_c^\infty(B(R))$, and thus, we have by substitution and with the help of (3.18)

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{J_p(u_h(x)-u_h(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{J_p(u(x)-u(y))(\varphi_h(x)-\varphi_h(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n} f\varphi_h.\end{aligned}$$

Subtracting (3.18) from the above equality, we find for any $\varphi \in C_c^\infty(B(R-2\delta))$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(J_p(u_h(x)-u_h(y)) - J_p(u(x)-u(y))\big)(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n} f(\varphi_h - \varphi).$$

Let $\eta \in C_c^\infty(B(R-20\delta),[0,1])$ with $\eta \equiv 1$ in $B(R-30\delta)$ and $|\nabla\eta| \lesssim \frac{1}{\delta}$. By a density argument, we may choose

$$\varphi := \eta\delta_h u$$

and obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{\big(J_p(u_h(x)-u_h(y)) - J_p(u(x)-u(y))\big)(\eta(x)\delta_h u(x) - \eta(y)\delta_h u(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n} f\delta_h(\eta\delta_h u).\end{aligned}$$

By assumption $u \in C^\alpha(B(R))$, so

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{\big(J_p(u_h(x)-u_h(y)) - J_p(u(x)-u(y))\big)(\eta(x)\delta_h u(x) - \eta(y)\delta_h u(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &\lesssim [u]_{C^\alpha(B(R))}\|f\|_{L^1(\mathbb{R}^n)}\,|h|^\alpha.\end{aligned}$$

Similarly as in [15, p. 320], by analyzing the support of η and using the symmetry of the integral, we split the integral on the left-hand side into two pieces:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big)(\eta(x)\delta_h u(x)-\eta(y)\delta_h u(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \geqslant \mathcal{I}_1 - 2\mathcal{I}_2,$$

where

$$\mathcal{I}_1 = \int_{B(R)}\int_{B(R)}\frac{\big(J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big)(\eta(x)\delta_h u(x)-\eta(y)\delta_h u(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\mathcal{I}_2 = \left|\int_{\mathbb{R}^n\setminus B(R)}\int_{B(R-20\delta)}\frac{\big(J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big)\eta(x)\delta_h u(x)}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right|.$$

We first estimate $\mathcal{I}_2$. Observe that in the integrand $|x-y|\gtrsim\delta$, so there is no singularity.

$$\begin{aligned}
\mathcal{I}_2 &\leqslant |h|^\alpha [u]_{C^\alpha}\int_{\mathbb{R}^n\setminus B(R)}\int_{B(R-20\delta)}\frac{|u_h(x)-u_h(y)|^{p-1}+|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant 2|h|^\alpha [u]_{C^\alpha}\int_{\mathbb{R}^n\setminus B(R-5\delta)}\int_{B(R-15\delta)}\frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant 2|h|^\alpha [u]_{C^\alpha}[u]^{p-1}_{W^{s,p}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus B(R-5\delta)}\int_{B(R-15\delta)}\frac{1}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}\\
&\approx 2|h|^\alpha [u]_{C^\alpha}[u]^{p-1}_{W^{s,p}(\mathbb{R}^n)}\,\delta^{-s}R^{\frac{n}{p}}.
\end{aligned}$$

We now estimate $\mathcal{I}_1$.

$$\begin{aligned}
\mathcal{I}_1 &\geqslant \int_{B(R)}\int_{B(R)}\eta(x)\frac{\big(J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big)(\delta_h u(x)-\delta_h u(y))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad-\int_{B(R)}\int_{B(R)}|\delta_h u(y)|\frac{\big|J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big||\eta(x)-\eta(y)|}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{B(R)}\int_{B(R)}\eta(x)\frac{\big(J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big)(u_h(x)-u_h(y)-(u(x)-u(y)))}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad-\int_{B(R)}\int_{B(R)}|\delta_h u(y)|\frac{\big|J_p(u_h(x)-u_h(y))-J_p(u(x)-u(y))\big||\eta(x)-\eta(y)|}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

For the first term, we use the famous p-Laplace inequality which holds for $p \geqslant 2$ and $v, w \in \mathbb{R}^M$,

$$(J_p(v) - J_p(w))(v - w) \gtrsim |v - w|^p,$$

see [71, Section 12 (I)] or [15, Lemma B.3, (B.4)]. For the second term, we use Lipschitz continuity of η. Then we have (recall $\eta \geqslant 0$ everywhere)

$$\begin{aligned}
\mathcal{I}_1 &\geqslant \int_{B(R-30\delta)} \int_{B(R-30\delta)} \frac{\left|u_h(x) - u_h(y) - (u(x) - u(y))\right|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad - C|h|^\alpha [u]_{C^\alpha} \delta^{-1} \int_{B(R)} \int_{B(R)} \frac{|u_h(x) - u_h(y)|^{p-1} + |u(x) - u(y)|^{p-1}}{|x-y|^{n+sp-1}} \,\mathrm{d}x\,\mathrm{d}y \\
&\geqslant [\delta_h u]^p_{W^{s,p}(B(R-30\delta))} - 2|h|^\alpha [u]_{C^\alpha} \delta^{-1} \int_{B(R+\delta)} \int_{B(R+\delta)} \frac{|u(x) - u(y)|^{p-1}}{|x-y|^{n+s(p-1)-(1-s)}} \,\mathrm{d}x\,\mathrm{d}y \\
&\geqslant [\delta_h u]^p_{W^{s,p}(B(R-30\delta))} - 2|h|^\alpha [u]_{C^\alpha} \delta^{-1} [u]^{p-1}_{W^{s,p}(B(R+\delta))} \left(\int_{B(R+\delta)} \int_{B(R+\delta)} \frac{1}{|x-y|^{n-(1-s)p}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} \\
&\approx [\delta_h u]^p_{W^{s,p}(B(R-30\delta))} - 2|h|^\alpha [u]_{C^\alpha} \delta^{-1} [u]^{p-1}_{W^{s,p}(B(R+\delta))} R^{\frac{n}{p}+(1-s)}.
\end{aligned}$$

That is, we have shown that for any $|h| \leqslant \delta$,

$$\left[|h|^{-\frac{\alpha}{p}} \delta_h u \right]^p_{W^{s,p}(B(R-30\delta))} \leqslant C(u, \delta, R, f).$$

From Lemma 3.8, we obtain that $u \in W^{s+\gamma,p}$ for any $\gamma < \frac{\alpha}{p}$. □

Above we used the following difference quotient estimate for fractional Sobolev spaces. For $W^{1,p}$, a statement like the one below is well known, see [36, §5.8.2, Theorem 3], for Sobolev spaces, it can be argued via the characterization of Besov–Nikol'skii spaces $B^\gamma_{p,\infty}$.

Lemma 3.8. *Let $s \in (0,1)$, $p \in (1,\infty)$, $\alpha > 0$ and $\gamma < \min\{s+\alpha, 1\}$. Assume that $u \in W^{s,p}(\Omega)$ and that for some $\Omega' \subset \Omega$, we have*

$$\sup_{|h| < \operatorname{dist}(\Omega', \partial\Omega)} [|h|^{-\alpha} \delta_h u]_{W^{s,p}(\Omega')} < \infty,$$

then $u \in W^{\gamma,p}_{loc}(\Omega')$.

Proof. In view of [15, Lemma 3.3], we find that $u \in B^{s+\alpha}_{p,\infty}(\Omega')$, where $B^s_{p,\infty}$ denotes the Besov–Nikol'skii space.

Combining [15, (3.20)] with [15, Proposition 2.7], we obtain that $u \in W^{s+\gamma,p}_{loc}(\Omega')$. □

Remark 3.9. Let us conclude this subsection by remarking that in the proof of Theorem 3.1, it seems likely that one could construct a different argument for higher differentiability that is based on the fractional Gehring's lemma, developed in [60], see also [59].

3.3 | Step 3: A priori estimates

Theorem 3.10 (A priori estimates). *Let $s \in (0,1)$ (if $n = 1$ let $s \leqslant \frac{1}{2}$). There exists an $\bar{s} \in (s,1)$ such that for any $s_0 \in (s,\bar{s})$, there exists an $s_1 \in (s,s_0)$ such that the following holds.*[†]

Let Σ be an n-dimensional compact manifold without boundary, and $\mathcal{N} \subset \mathbb{R}^M$ a compact manifold without boundary.

There exists an $\varepsilon = \varepsilon(\mathcal{N},\Sigma,s,s_0,\bar{s}) > 0$ depending on the choices above such that the following holds for any $t \in [s,s_1]$.

Assume that $u \in W^{s,\frac{n}{s}}(\Sigma,\mathcal{N})$ and for a geodesic ball $B(R) \subset \Sigma$

- $u \in W^{s_0,\frac{n}{s}}(B(R))$;
- *u is a critical $W^{t,\frac{n}{s}}$-harmonic map in $B(R)$ for some $t \in [s,s_1]$;*
- $[u]_{W^{s,\frac{n}{s}}(B(R))} < \varepsilon$.

Then we have the estimate

$$[u]_{W^{s_0,\frac{n}{s}}(B(R/2))} \leqslant C\,R^{-(s_0-s)}[u]^{\frac{s}{n}}_{W^{s,\frac{n}{s}}(B(R))}\left([u]^{1-\frac{s}{n}}_{W^{s,\frac{n}{s}}(\Sigma)} + [u]_{W^{s,\frac{n}{s}}(\Sigma)}\right). \tag{3.19}$$

The proof of Theorem 3.10 is based on estimates of the Euler–Lagrange equations, and the stability estimates for the fractional p-Laplacian in [104].

We will also need the following iteration lemma, see [43, Chapter V, Lemma 3.1].

Lemma 3.11 (Iteration lemma). *Let $0 < a < b < \infty$ and $f : [a,b] \to [0,\infty)$ be a bounded function. Suppose that there are constants $\theta \in [0,1)$, K_1, K_2, $\alpha > 0$ such that*

$$f(r) \leqslant \theta f(\rho) + \frac{K_1}{(\rho - r)^\alpha} + K_2 \quad \text{for all } a \leqslant r < \rho \leqslant b. \tag{3.20}$$

Then we obtain the bound

$$f(r) \leqslant C\left(\frac{K_1}{(b-r)^\alpha} + K_2\right) \quad \text{for all } a \leqslant r \leqslant b$$

for a constant $C = C(\theta,\alpha) > 0$.

Proof of Theorem 3.10. For simplicity of notation, we assume[‡] that $\Sigma = \mathbb{R}^n$.

Below we will establish the following estimate for any r,ρ with $\frac{R}{4} < r < \rho < \frac{3}{4}R$,

$$\begin{aligned}[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))} \leqslant{}& \frac{1}{2}[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} \\ &+ C\,(\rho - r)^{-(s_0-s)\frac{n}{s}}\left(\frac{R}{\rho - r}\right)^{\frac{n}{s}}\left([u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} + [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}\right)[u]_{W^{s,\frac{n}{s}}(B(R))}.\end{aligned} \tag{3.21}$$

[†] The relation between those numbers is $0 < s < s_1 < s_0 < \bar{s} < 1$.

[‡] Since Σ is compact, the manifold has a bounded geometry, so it is locally comparable to $\mathbb{R}^n$ and by an extension theorem, this assumption changes mainly the notation.

Once (3.21) is established, we apply Lemma 3.11 to $f(r) := [u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))}$, which gives

$$[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(R/2))} \leqslant C\, R^{-(s_0-s)\frac{n}{s}} \left([u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} + [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \right) [u]_{W^{s,\frac{n}{s}}(B(R))}.$$

This readily implies (3.19).

We now need to establish (3.21). From now on, we fix some r, ρ such that $\frac{R}{4} < r < \rho < \frac{3}{4}R$. We denote $\delta := \frac{\rho - r}{100} \in (0, R)$.

As a $W^{t,\frac{n}{s}}$-harmonic map u solves the following inequality for any $\varphi \in C^\infty_c(B(R))$, cf. [76, Lemma 5.1].

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}y\,\mathrm{d}x \right| \\
&\quad \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(x)| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}\,|\varphi(x)-\varphi(y)|}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}y\,\mathrm{d}x \qquad (3.22) \\
&\quad \leqslant 3 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(x)| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Pick $\bar\eta \in C^\infty_c(B(\rho - 10\delta))$ such that $\bar\eta \equiv 1$ in $B(r + 10\delta)$, and $|\nabla\bar\eta| \lesssim \frac{1}{\delta}$. Set $\eta := \bar\eta^2$. Then $|\nabla\eta| + |\nabla\sqrt{\eta}| \lesssim \frac{1}{\delta}$. We define the test function

$$\varphi := \eta(u - (u)_{B(R)}).$$

We collect the main *estimates of* φ:

First,

$$[\varphi]_{W^{s_0,\frac{n}{s}}(\mathbb{R}^n)} \lesssim [u]_{W^{s_0,\frac{n}{s}}(B(\rho))} + \frac{R^{1-s_0+s}}{\delta} [u]_{W^{s,\frac{n}{s}}(B(R))}. \qquad (3.23)$$

Indeed,

$$\begin{aligned}
[\varphi]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(\mathbb{R}^n)} &\lesssim \int_{B(\rho-5\delta)} \int_{B(\rho-5\delta)} \frac{|\varphi(x)-\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+s_0\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y + \int_{B(\rho-10\delta)} \int_{B(\rho-5\delta)^c} \frac{|\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+s_0\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{B(\rho-5\delta)} \int_{B(\rho-5\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+s_0\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{B(\rho-5\delta)} \int_{B(\rho-5\delta)} \frac{|\eta(x)-\eta(y)|^{\frac{n}{s}}\,|u(x)-(u)_{B(R)}|^{\frac{n}{s}}}{|x-y|^{n+s_0\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{B(\rho-10\delta)} \int_{B(\rho-5\delta)^c} \frac{|u(y)-(u)_{B(R)}|^{\frac{n}{s}}}{|x-y|^{n+s_0\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

That is,

$$
\begin{aligned}
[\varphi]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(\mathbb{R}^n)} &\lesssim \int_{B(\rho)}\int_{B(\rho)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+s_0\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y + \frac{\rho^{(1-s_0)\frac{n}{s}}}{\delta^{\frac{n}{s}}}\int_{B(R)} |u(x)-(u)_{B(R)}|^{\frac{n}{s}}\,\mathrm{d}x \\
&\quad + \frac{1}{\delta^{s_0\frac{n}{s}}}\int_{B(R)} |u(y)-(u)_{B(R)}|^{\frac{n}{s}}\,\mathrm{d}y \\
&\lesssim [u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \frac{R^{(1-s_0+s)\frac{n}{s}}}{\delta^{\frac{n}{s}}}[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(R))}.
\end{aligned}
$$

This establishes (3.23).

Recall that the fractional $\tilde{s}$-Laplacian for $\tilde{s}\in(0,1)$ is defined as

$$
(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(x) := c\int_{\mathbb{R}^n}\frac{\varphi(x)-\varphi(y)}{|x-y|^{n+\tilde{s}}}\,\mathrm{d}y,
$$

or — equivalently — via the Fourier transform $\mathcal{F}((-\Delta)^{\frac{\tilde{s}}{2}}\varphi)(\xi) = c|\xi|^s\mathcal{F}(\varphi)(\xi)$, cf. [30, 41].

With this notation, for any $0\leqslant\tilde{s}<s$,

$$
[(-\Delta)^{\frac{\tilde{s}}{2}}\varphi]_{W^{s-\tilde{s},\frac{n}{s}}(\mathbb{R}^n)} \lesssim \frac{R}{\delta}[u]_{W^{s,\frac{n}{s}}(B(R))}. \tag{3.24}
$$

Indeed, from the theory of Triebel–Lizorkin spaces $F^s_{p,p}\approx W^{s,p}$, [95], we have

$$
[(-\Delta)^{\frac{\tilde{s}}{2}}\varphi]_{W^{s-\tilde{s},\frac{n}{s}}(\mathbb{R}^n)} \approx [\varphi]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}.
$$

Now the estimate (3.24) follows as (3.23).

Also, for any $0\leqslant\tilde{s}\leqslant s$ and any $\gamma\in(0,1)$,

$$
\left(\int_{\mathbb{R}^n\setminus B(R)}\int_{\mathbb{R}^n}\frac{|(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(x)-(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s}{n}} \lesssim R^{-\gamma-\tilde{s}+s}[u]_{W^{s,\frac{n}{s}}(B(R))}. \tag{3.25}
$$

Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(R)}\int_{\mathbb{R}^n}\frac{|(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(x)-(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^n\setminus B(R)}\int_{B(y,\frac{1}{2}R)}\frac{|(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(x)-(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n\setminus B(R)}\int_{\mathbb{R}^n\setminus B(y,\frac{1}{2}R)}\frac{|(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(x)-(-\Delta)^{\frac{\tilde{s}}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

Now we employ the estimate

$$|f(x)-f(y)| \lesssim |x-y|\,(\mathcal{M}\nabla f(x)+\mathcal{M}\nabla f(y)),$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function, cf. [11, 46]. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(R)}\int_{\mathbb{R}^n}\frac{|(-\Delta)^{\frac{\bar s}{2}}\varphi(x)-(-\Delta)^{\frac{\bar s}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim \int_{\mathbb{R}^n\setminus B(R)}|\mathcal{M}\nabla(-\Delta)^{\frac{\bar s}{2}}\varphi(x)|^{\frac{n}{s}}\int_{|x-y|\lesssim R}\frac{1}{|x-y|^{n+(\gamma-1)\frac{n}{s}}}\,\mathrm{d}y\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^n\setminus B(R)}|(-\Delta)^{\frac{\bar s}{2}}\varphi(x)|^{\frac{n}{s}}\int_{|x-y|\gtrsim R}\frac{1}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim R^{(1-\gamma)\frac{n}{s}}\,\|\varphi\|_{L^1(\mathbb{R}^n)}^{\frac{n}{s}}\int_{\mathbb{R}^n\setminus B(R)}|x|^{(-n-1-\bar s)\frac{n}{s}}\,\mathrm{d}x+R^{-\gamma\frac{n}{s}}\,\|\varphi\|_{L^1(\mathbb{R}^n)}^{\frac{n}{s}}\int_{\mathbb{R}^n\setminus B(R)}|x|^{(-n-\bar s)\frac{n}{s}}\,\mathrm{d}x\\
&\lesssim R^{(-\gamma-n-\bar s+s)\frac{n}{s}}\,\|\varphi\|_{L^1(\mathbb{R}^n)}^{\frac{n}{s}}\\
&\lesssim R^{(-\gamma-n-\bar s+s)\frac{n}{s}}\,R^{n\frac{n}{s}}[u]_{W^{s,\frac{n}{s}}(B(R))}^{\frac{n}{s}}\\
&= R^{(-\gamma-\bar s+s)\frac{n}{s}}[u]_{W^{s,\frac{n}{s}}(B(R))}^{\frac{n}{s}}.
\end{aligned}$$

Similarly, for any $0\leqslant\bar s\leqslant s$ and any $\gamma\in(0,1)$,

$$\left(\int_{\mathbb{R}^n\setminus B(\rho-4\delta)}\int_{\mathbb{R}^n}\frac{|(-\Delta)^{\frac{\bar s}{2}}\varphi(x)-(-\Delta)^{\frac{\bar s}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\gamma\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s}{n}}\lesssim\delta^{-(\gamma+\bar s)}R^s[u]_{W^{s,\frac{n}{s}}(B(R))}.\tag{3.26}$$

Also, for any $\tilde\eta\in C_c^\infty(B(\rho),[0,1])$ with $\tilde\eta\equiv1$ in a δ-neighborhood of $\operatorname{supp}\varphi$ and with $|\nabla\tilde\eta|\lesssim\delta^{-1}$, we have for any $\gamma\in(0,1)$ and $\tilde s\in[0,1)$ such that $\gamma+\tilde s<1$,

$$[(1-\tilde\eta)(-\Delta)^{\frac{\bar s}{2}}\varphi]_{W^{\gamma,\frac{n}{s}}(\mathbb{R}^n)}\lesssim\delta^{-(\gamma+\bar s)}R^s[u]_{W^{s,\frac{n}{s}}(B(R))}.\tag{3.27}$$

Indeed, observe that by the disjoint support of $1-\tilde\eta$ and φ, we have

$$(1-\tilde\eta)(-\Delta)^{\frac{\bar s}{2}}\varphi(x)\approx\int_{|x-z|\gtrsim\delta}|x-z|^{-n-\bar s}\,\varphi(z)\,\mathrm{d}z.$$

In particular, we have from Young's convolution inequality

$$\|(1-\tilde\eta)(-\Delta)^{\frac{\bar s}{2}}\varphi\|_{L^p(\mathbb{R}^n)}\lesssim\delta^{-\bar s}\|\varphi\|_{L^p(\mathbb{R}^n)}.$$

In a similar way,

$$\left|\nabla\left((1-\tilde\eta)(-\Delta)^{\frac{\bar s}{2}}\varphi\right)(x)\right|\lesssim\left|\nabla\tilde\eta(-\Delta)^{\frac{\bar s}{2}}\varphi(x)\right|+\int_{|x-z|\gtrsim\delta}|x-z|^{-n-\bar s-1}\,|\varphi(z)|\,\mathrm{d}z,$$

so that

$$\left\|\nabla\left((1-\tilde{\eta})(-\Delta)^{\frac{\bar{s}}{2}}\varphi\right)\right\|_{L^p(\mathbb{R}^n)} \lesssim \delta^{-\bar{s}-1}\|\varphi\|_{L^p(\mathbb{R}^n)}.$$

From interpolation [118], we then have

$$[(1-\tilde{\eta})(-\Delta)^{\frac{\bar{s}}{2}}\varphi]_{W^{\gamma,p}(\mathbb{R}^n)} \lesssim \delta^{-\bar{s}-\gamma}\|\varphi\|_{L^p(\mathbb{R}^n)}.$$

Applying Poincaré inequality for $p=\frac{n}{s}$, this leads to

$$[(1-\tilde{\eta})(-\Delta)^{\frac{\bar{s}}{2}}\varphi]_{W^{\gamma,\frac{n}{s}}(\mathbb{R}^n)} \lesssim \delta^{-\bar{s}-\gamma}R^s[u]_{W^{s,\frac{n}{s}}(B(R))}.$$

Inequality (3.27) is established.

Now we begin to *estimate* u:

For $\tilde{\varphi} := \sqrt{\eta}(u-(u)_{B(R)})$,

$$[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))} \lesssim \int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}|\tilde{\varphi}(x)-\tilde{\varphi}(y)|^2}{|x-y|^{n+s_0\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.$$

Now

$$\tilde{\varphi}(x)-\tilde{\varphi}(y) = (\sqrt{\eta}(x)-\sqrt{\eta}(y))(u(x)-(u)_{B(R)}) + \sqrt{\eta}(y)(u(x)-u(y)).$$

So that (recall $\varphi=\sqrt{\eta}\tilde{\varphi}$),

$$\begin{aligned}|\tilde{\varphi}(x)-\tilde{\varphi}(y)|^2 = {}& (\sqrt{\eta}(x)-\sqrt{\eta}(y))(u(x)-(u)_{B(R)})\cdot(\tilde{\varphi}(x)-\tilde{\varphi}(y))\\ &+(u(x)-u(y))(\sqrt{\eta}(y)-\sqrt{\eta}(x))\tilde{\varphi}(x)\\ &+(u(x)-u(y))(\varphi(x)-\varphi(y)).\end{aligned}$$

That is,

$$\begin{aligned}|\tilde{\varphi}(x)-\tilde{\varphi}(y)|^2 \lesssim {}& \frac{|x-y|}{\delta}|u(x)-(u)_{B(R)}|(|\tilde{\varphi}(x)-\tilde{\varphi}(y)|+|u(x)-u(y)|)\\ &+(u(x)-u(y))(\varphi(x)-\varphi(y)).\end{aligned}$$

Then,

$$\begin{aligned}[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))} \lesssim {}& \int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+s_0\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\delta^{-1}\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-1}|u(x)-(u)_{B(R)}|}{|x-y|^{n+s_0\frac{n}{s}-1}}\,\mathrm{d}x\,\mathrm{d}y\\ &+\delta^{-1}\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}|\tilde{\varphi}(x)-\tilde{\varphi}(y)||u(x)-(u)_{B(R)}|}{|x-y|^{n+s_0\frac{n}{s}-1}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{3.28}$$

Now observe that for any $w : \mathbb{R}^n \to \mathbb{R}^M$, we have

$$\begin{aligned}
&\delta^{-1}\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}|w(x)-w(y)||u(x)-(u)_{B(R)}|}{|x-y|^{n+(s_0-\frac{s}{n})\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\delta^{-1}\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}}{|x-y|^{s\left(\frac{n}{s}-2\right)}}\frac{|w(x)-w(y)|}{|x-y|^{s}}\frac{|u(x)-(u)_{B(R)}|}{|x-y|^{s+(s_0-s)\frac{n}{s}-1}}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n}}\\
&\precsim\delta^{-1}[u]^{\frac{n}{s}-2}_{W^{s,\frac{n}{s}}(B(R))}[w]_{W^{s,\frac{n}{s}}(B(R))}\left(\int_{B(R)}\int_{B(R)}\frac{|u(x)-(u)_{B(R)}|^{\frac{n}{s}}}{|x-y|^{n+(s+(s_0-s)\frac{n}{s}-1)\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s}{n}}\\
&\precsim\delta^{-1}[u]^{\frac{n}{s}-2}_{W^{s,\frac{n}{s}}(B(R))}[w]_{W^{s,\frac{n}{s}}(B(R))}\left(\int_{B(R)}\frac{|u(x)-(u)_{B(R)}|^{\frac{n}{s}}}{R^{(s+(s_0-s)\frac{n}{s}-1)\frac{n}{s}}}\,\mathrm{d}x\right)^{\frac{s}{n}}\\
&\precsim\delta^{-1}R^{1-(s_0-s)\frac{n}{s}}[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(B(R))}[w]_{W^{s,\frac{n}{s}}(B(R))}.
\end{aligned}\tag{3.29}$$

Here we ensure that s_0 is close enough to s so that $s+(s_0-s)\frac{n}{s}-1<0$.

Applying (3.29) to the last two terms in (3.28) first for $w=u$ and then for $w=\tilde{\varphi}$, we obtain in view of (3.24) with $\tilde{s}=0$

$$\begin{aligned}
[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))}\precsim&\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+s_0\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&+\delta^{-1}R^{1-(s_0-s)\frac{n}{s}}\left(1+\frac{R}{\delta}\right)[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(R))}.
\end{aligned}\tag{3.30}$$

For the remaining term, we observe

$$\begin{aligned}
&\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+s_0\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,\frac{\varphi(x)-\varphi(y)}{|x-y|^{(s_0-t)\frac{n}{s}}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

We now follow the ideas in [104]. We use the following identity that holds for any $n>\beta>\alpha\geqslant 0$,

$$(-\Delta)^{\frac{\alpha}{2}}\varphi(x)=c\int_{\mathbb{R}^n}|x-z|^{\beta-\alpha-n}(-\Delta)^{\frac{\beta}{2}}\varphi(z)\,\mathrm{d}z,\tag{3.31}$$

for a constant $c>0$, which from now on changes from line to line.

Let $\gamma\geqslant 0$, $0\leqslant t\leqslant s_1$ such that $\gamma+(s_0-t)\frac{n}{s}\in(0,n)$, and set

$$k(x,y,z)=\left(\frac{|x-z|^{\gamma+(s_0-t)\frac{n}{s}-n}-|y-z|^{\gamma+(s_0-t)\frac{n}{s}-n}}{|x-y|^{(s_0-t)\frac{n}{s}}}\right)-(|x-z|^{\gamma-n}-|y-z|^{\gamma-n}).$$

Then (3.31) implies

$$\int_{\mathbb{R}^n} k(x,y,z)(-\Delta)^{\frac{\gamma+(s_0-t)\frac{n}{s}}{2}}\varphi(z)\,dz = c\frac{\varphi(x)-\varphi(y)}{|x-y|^{(s_0-t)\frac{n}{s}}} - c\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x) - (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right).$$

$$\begin{aligned}
&\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+s_0\frac{n}{s}}}\,dx\,dy\\
&= c\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x) - (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,dx\,dy\\
&\quad + cR(u,\varphi),
\end{aligned} \tag{3.32}$$

where

$$R(u,\varphi) = \int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\,k(x,y,z)(-\Delta)^{\frac{\gamma+(s_0-t)\frac{n}{s}}{2}}\varphi(z)}{|x-y|^{n+t\frac{n}{s}}}\,dz\,dx\,dy.$$

The main observation is that for $s_0 - t = 0$, we have $k \equiv 0$ and thus $R \equiv 0$. In [104, Theorem 1.1], an estimate of R for small $|s_0 - t|$ was obtained. Namely,[†]

$$\begin{aligned}
R(u,\varphi) &\lesssim (s_0-t)[u]^{\frac{n}{s}-1}_{W^{s_0,\frac{n}{s}}(B(\rho))}[\varphi]_{W^{s_0,\frac{n}{s}}(\mathbb{R}^n)}\\
&\lesssim |s_0-s|\,[u]^{\frac{n}{s}-1}_{W^{s_0,\frac{n}{s}}(B(\rho))}[\varphi]_{W^{s_0,\frac{n}{s}}(\mathbb{R}^n)}\\
&\lesssim |s_0-s|\left([u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \frac{R^{1-s_0+s}}{\delta}[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(R))}\right),
\end{aligned} \tag{3.33}$$

in the last inequality we used (3.23).

Next, we begin the estimate of the first term of the right-hand side of (3.32)

$$\begin{aligned}
&\int_{B(\rho-3\delta)}\int_{B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x) - (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,dx\,dy\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x) - (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,dx\,dy\\
&\quad - \iint_{(\mathbb{R}^n)^2\setminus(B(\rho-3\delta))^c)^2} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x) - (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,dx\,dy.
\end{aligned} \tag{3.34}$$

[†] While this follows from the statement of [104, Theorem 1.1], it might be more instructive at first to look into [104, Proof of Theorem 1.1], where the same notation is used as we use here.

We will estimate now the last term in the above inequality. We observe

$$\iint_{(\mathbb{R}^n)^2\setminus(B(\rho-3\delta)^c)^2} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(\rho-3\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\mathbb{R}^n\times\mathbb{R}^n\setminus(B(\rho-3\delta)^c\times B(\rho-3\delta)^c)$$
$$\subseteq (B(\rho-4\delta)\times\mathbb{R}^n\setminus B(\rho-3\delta))\cup(\mathbb{R}^n\setminus B(\rho-4\delta)\times\mathbb{R}^n\setminus B(\rho-4\delta)).$$

Thus, choosing s_0 so close to s so that $(s_0-t)\frac{n}{s}<s$ for all $t\in[s,s_0]$, we have

$$\iint_{(\mathbb{R}^n)^2\setminus(B(\rho-3\delta)^c)^2} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\lesssim \iint_{|x-y|>\delta} \frac{1}{|x-y|^{(s_0-s)\frac{n}{s}}}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+s\left(\frac{n}{s}-1\right)+\left(s-(s_0-t)\frac{n}{s}\right)}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+\int_{\mathbb{R}^n\setminus B(\rho-4\delta)}\int_{\mathbb{R}^n\setminus B(\rho-4\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.35}$$

As for the first term on the right-hand side of (3.35), we first use that $|x-y|>\delta$ and then apply Hölder's inequality to get

$$\iint_{|x-y|>\delta} \frac{1}{|x-y|^{(s_0-s)\frac{n}{s}}}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+s\left(\frac{n}{s}-1\right)+\left(s-(s_0-t)\frac{n}{s}\right)}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\lesssim \delta^{-(s_0-s)\frac{n}{s}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}-1}}{|x-y|^{s\left(\frac{n}{s}-1\right)}}\frac{\left|(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right|}{|x-y|^{s-(s_0-t)\frac{n}{s}}}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^n}$$
$$\lesssim \delta^{-(s_0-s)\frac{n}{s}}[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi]_{W^{s-(s_0-t)\frac{n}{s},\frac{n}{s}}(\mathbb{R}^n)}$$
$$\lesssim \delta^{-(s_0-s)\frac{n}{s}}\frac{R}{\delta}[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(B(R))}[u]_{W^{s,\frac{n}{s}}(B(R))}, \tag{3.36}$$

where in the last estimate, we used (3.24).

To estimate the second term on the right-hand side of (3.35), we again apply Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(\rho-4\delta)}\int_{\mathbb{R}^n\setminus B(\rho-4\delta)} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \int_{\mathbb{R}^n\setminus B(\rho-4\delta)}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-1}}{|x-y|^{n-s}}\frac{\left|(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right|}{|x-y|^{(t-s)\frac{n}{s}+s}}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^n}\\
&\leqslant [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n\setminus B(\rho-4\delta)}\int_{\mathbb{R}^n}\frac{|(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)|^{\frac{n}{s}}}{|x-y|^{n+\left((t-s)\frac{n}{s}+s\right)\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s}{n}}\\
&\lesssim \delta^{-s-(s_0-s)\frac{n}{s}}R^s\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[u]_{W^{s,\frac{n}{s}}(B(R))},
\end{aligned}\tag{3.37}$$

in the last estimate, we applied (3.26) with $\gamma=(t-s)\frac{n}{s}+s<1$ (the latter condition can be satisfied by a good choice of s_1).

We observe that

$$\left(\delta^{-(s_0-s)\frac{n}{s}}\frac{R}{\delta}+\delta^{-s-(s_0-s)\frac{n}{s}}R^s\right)[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[u]_{W^{s,\frac{n}{s}}(B(R))}\lesssim \delta^{-1-(s_0-s)\frac{n}{s}}R\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[u]_{W^{s,\frac{n}{s}}(B(R))}.\tag{3.38}$$

Thus, combining (3.35) with (3.36)–(3.38), we obtain

$$\begin{aligned}
&\iint_{(\mathbb{R}^n)^2\setminus(B(\rho-3\delta)^c)^2}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim \delta^{-1-(s_0-s)\frac{n}{s}}R\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[u]_{W^{s,\frac{n}{s}}(B(R))}.
\end{aligned}\tag{3.39}$$

Bringing together the estimates (3.30), (3.32), (3.33), (3.34), and (3.39), we have shown

$$\begin{aligned}
&[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))}\\
&\lesssim |s_0-s|\,[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))}+\delta^{-1-(s_0-s)\frac{n}{s}}R\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[u]_{W^{s,\frac{n}{s}}(B(R))}\\
&\quad+\delta^{-1}R^{1-(s_0-s)\frac{n}{s}}\left(1+\frac{R}{\delta}\right)[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(R))}\\
&\quad+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{3.40}$$

Let us now estimate the last term of the inequality above. Take $\tilde\eta \in C_c^\infty(B(\rho-3\delta))$ and $\tilde\eta \equiv 1$ in $B(\rho-4\delta)$, with $|\nabla\tilde\eta| \lesssim \delta^{-1}$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left(\tilde\eta(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-\tilde\eta(y)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((1-\tilde\eta(x))(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(1-\tilde\eta(y))(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.41}$$

As for the second term of (3.41), we use, similarly as in (3.37), Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left((1-\tilde\eta(x))(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-(1-\tilde\eta(y))(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant [u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}\left[(1-\tilde\eta)(-\Delta)^{\frac{(s_0-t)\frac{n}{s})}{2}}\varphi\right]_{W^{(t-s)\frac{n}{s}+s,\frac{n}{s}}(\mathbb{R}^n)}\\
&\lesssim \delta^{-(s_0-s)\frac{n}{s}-s}R^s\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}[u]_{W^{s,\frac{n}{s}}(B(R))},
\end{aligned} \tag{3.42}$$

in the last inequality, we used (3.27) with $\gamma=(t-s)\frac{n}{s}+s$ and $\tilde s=(s_0-t)\frac{n}{s}$.

As for the first term of (3.41), we use the PDE (3.22) with test function $\eta(-\Delta)^{\frac{(s_0-t)\frac{n}{s}\varphi}{2}}$ and arrive at

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\left(\tilde\eta(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)-\tilde\eta(y)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(y)\right)}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}x\,\mathrm{d}y\right|\\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\tilde\eta(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right|\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}y\,\mathrm{d}x\\
&\leqslant \int_{B(\rho)}\int_{B(\rho)}\left|\tilde\eta(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right|\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}y\,\mathrm{d}x\\
&\quad+\int_{B(\rho-\delta)}\int_{\mathbb{R}^n\setminus B(\rho)}\left|\tilde\eta(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right|\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned} \tag{3.43}$$

We estimate the first term of the last inequality. We will use $t_0 < s_1 < s_0$ (here we choose s_1 so that $s + (s_1 - s)\frac{s}{n-s} < s_0$ and $s_1 < s + (s_0 - s)(1 - \frac{s}{n})$. Using Hölder's inequality twice, we get

$$\begin{aligned}
&\int_{B(\rho)}\int_{B(\rho)} \left|\tilde{\eta}(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{B(\rho)} \left|\tilde{\eta}(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right| \int_{B(\rho)} \frac{|u(x)-u(y)|^{\frac{n}{s}-1}}{|x-y|^{t\frac{n}{s}-s}} \frac{|u(x)-u(y)|}{|x-y|^{s}} \frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^{n}} \\
&\leqslant \int_{B(\rho)} \left|\tilde{\eta}(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right| \left(\int_{B(\rho)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\left(\left(t\frac{n}{s}-s\right)\frac{s}{n-s}\right)\frac{n}{s}}}\,\mathrm{d}y\right)^{1-\frac{s}{n}} \left(\int_{B(\rho)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}y\right)^{\frac{s}{n}} \mathrm{d}x \\
&\leqslant \left\|(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi\right\|_{L^{\frac{n}{(s_0-s)\left(\frac{n}{s}-1\right)-(t-s)\frac{n}{s}}}(\mathbb{R}^n)} [u]_{W^{s,\frac{n}{s}}(B(\rho))} \\
&\quad \left(\int_{B(\rho)} \left(\int_{B(\rho)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+(t\frac{n}{n-s}-s\frac{s}{n-s})\frac{n}{s}}}\,\mathrm{d}y\right)^{\frac{s}{n}\frac{n}{2s-s_0+(t-s)\frac{n}{(n-s)}}} \mathrm{d}x\right)^{\frac{2s-s_0+(t-s)\frac{n}{(n-s)}}{n}\frac{n-s}{s}}.
\end{aligned} \tag{3.44}$$

In the last inequality, we applied (generalized) Hölder's inequality with exponents $\frac{n}{(s_0-s)(\frac{n}{s}-1)-(t-s)\frac{n}{s}}$, $\frac{n}{2s-s_0+(t-s)\frac{n}{n-s}}\frac{s}{n-s}$, $\frac{n}{s}$.

By Sobolev embedding, Theorem B.1, (B.5) (applied for $s := t\frac{n}{n-s} - s\frac{s}{n-s}$, $p := \frac{n}{s}$, $p^* := \frac{n}{2s-s_0+(t-s)\frac{n}{n-s}}$, $t := s_0$), we obtain

$$\left(\int_{B(\rho)} \left(\int_{B(\rho)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+(t\frac{n}{n-s}-s\frac{s}{n-s})\frac{n}{s}}}\,\mathrm{d}y\right)^{\frac{s}{n}\frac{n}{2s-s_0+(t-s)\frac{n}{(n-s)}}} \mathrm{d}x\right)^{\frac{2s-s_0+(t-s)\frac{n}{(n-s)}}{n}} \lesssim [u]_{W^{s_0,\frac{n}{s}}(B(\rho))}. \tag{3.45}$$

Moreover, also by Sobolev embedding, Theorem B.1, (B.3) (applied for $s := (s_0 - t)\frac{n}{s}$, $p^* := \frac{n}{(s_0-s)(\frac{n}{s}-1)-(t-s)\frac{n}{s}}$, $p = \frac{n}{s}$, $t := s_0$), we have

$$\begin{aligned}
&\|(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi\|_{L^{\frac{n}{(s_0-s)\left(\frac{n}{s}-1\right)-(t-s)\frac{n}{s}}}(\mathbb{R}^n)} \\
&\lesssim [\varphi]_{W^{s_0,\frac{n}{s}}(\mathbb{R}^n)} \lesssim [u]_{W^{s_0,\frac{n}{s}}(B(\rho))} + \frac{R^{1-s_0+s}}{\delta}[u]_{W^{s,\frac{n}{s}}(B(R))},
\end{aligned} \tag{3.46}$$

we used (3.23) in the last estimate.

Thus, collecting the estimates (3.44)–(3.46) and recalling that by assumption $[u]_{W^{s,\frac{n}{s}}(B(\rho))} \leqslant \varepsilon$, we have

$$\begin{aligned}
&\int_{B(\rho)}\int_{B(\rho)} \left|\tilde{\eta}(x)(-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}}\varphi(x)\right| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}}\,\mathrm{d}y\,\mathrm{d}x \\
&\lesssim [u]_{W^{s,\frac{n}{s}}(B(\rho))} [u]^{\frac{n}{s}-1}_{W^{s_0,\frac{n}{s}}(B(\rho))} \left([u]_{W^{s_0,\frac{n}{s}}(B(\rho))} + \frac{R^{1-s_0+s}}{\delta}[u]_{W^{s,\frac{n}{s}}(B(R))}\right)
\end{aligned}$$

$$\begin{aligned}
&\precsim \varepsilon [u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \varepsilon [u]^{\frac{n}{s}-1}_{W^{s_0,\frac{n}{s}}(B(\rho))} \frac{R^{1-s_0+s}}{\delta} [u]_{W^{s,\frac{n}{s}}(B(R))} \\
&\precsim \varepsilon [u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \varepsilon^{\frac{n}{n-s}} [u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \left(\frac{R^{1-s_0+s}}{\delta} [u]_{W^{s,\frac{n}{s}}(B(R))} \right)^{\frac{n}{s}},
\end{aligned}$$

we used Young's inequality in the last estimate.

It remains to treat the second term of the right-hand side of (3.43).

$$\begin{aligned}
&\int_{B(\rho-\delta)} \int_{\mathbb{R}^n \setminus B(\rho)} \left| \tilde{\eta}(x) (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}} \varphi(x) \right| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \, \mathrm{d}y \, \mathrm{d}x \\
&\precsim \iint_{|x-y|\geqslant \delta} \frac{1}{|x-y|^{(s_0-s)\frac{n}{s}}} \left| (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}} \varphi(x) \right| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+(s+t-s_0)\frac{n}{s}}} \, \mathrm{d}y \, \mathrm{d}x \\
&\precsim \delta^{-(s_0-s)\frac{n}{s}} \left\| (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}} \varphi \right\|_{L^{\frac{s}{s_0-t}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+(s+t-s_0))\frac{n}{s}}} \, \mathrm{d}y \right)^{\frac{s}{n}\frac{n}{s+t-s_0}} \mathrm{d}x \right)^{\frac{s+t-s_0}{n}\frac{n}{s}},
\end{aligned} \tag{3.47}$$

in the last estimate, we have used Hölder's inequality with exponents $\frac{s}{s_0-t}$ and $\frac{s}{s+t-s_0}$.

Again applying the Sobolev embedding, Theorem B.1, we obtain,

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+(s+t-s_0))\frac{n}{s}}} \, \mathrm{d}y \right)^{\frac{s}{n}\frac{n}{s+t-s_0}} \mathrm{d}x \right)^{\frac{s+t-s_0}{n}} \leqslant [u]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}.$$

and

$$\left\| (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}} \varphi \right\|_{L^{\frac{s}{s_0-t}}(\mathbb{R}^n)} \precsim [\varphi]_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \precsim \frac{R}{\delta} [u]_{W^{s,\frac{n}{s}}(B(R))},$$

were in the second inequality we used (3.24).

Thus,

$$\begin{aligned}
&\int_{B(\rho-\delta)} \int_{\mathbb{R}^n \setminus B(\rho)} \left| \tilde{\eta}(x) (-\Delta)^{\frac{(s_0-t)\frac{n}{s}}{2}} \varphi(x) \right| \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+t\frac{n}{s}}} \, \mathrm{d}y \, \mathrm{d}x \\
&\precsim \delta^{-(s_0-s)\frac{n}{s}} \frac{R}{\delta} [u]_{W^{s,\frac{n}{s}}(B(R))} [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \\
&\precsim \delta^{-(s_0-s)\frac{n}{s}} \left(\frac{R}{\delta} \right)^{\frac{n}{s}} [u]_{W^{s,\frac{n}{s}}(B(R))} [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}.
\end{aligned} \tag{3.48}$$

Finally, combining (3.40) with (3.41), (3.42), (3.43), (3.44), and (3.48), we obtain

$$
\begin{aligned}
[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(r))} &\lesssim |s_0-s|\,[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \delta^{-1-(s_0-s)\frac{n}{s}} R\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}\,[u]_{W^{s,\frac{n}{s}}(B(R))} \\
&\quad + \delta^{-1} R^{1-(s_0-s)\frac{n}{s}}\Big(1+\frac{R}{\delta}\Big)[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(B(R))} \\
&\quad + \delta^{-(s_0-s)\frac{n}{s}-s} R^s\,[u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}\,[u]_{W^{s,\frac{n}{s}}(B(R))} \\
&\quad + \varepsilon[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \varepsilon^{\frac{n}{n-s}}[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} + \left(\frac{R^{1-s_0+s}}{\delta}[u]_{W^{s,\frac{n}{s}}(B(R))}\right)^{\frac{n}{s}} \\
&\quad + \delta^{-(s_0-s)\frac{n}{s}}\left(\frac{R}{\delta}\right)^{\frac{n}{s}}[u]_{W^{s,\frac{n}{s}}(B(R))}\,[u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} \\
&\lesssim \Big(|s_0-s|+\varepsilon+\varepsilon^{\frac{n}{n-s}}\Big)[u]^{\frac{n}{s}}_{W^{s_0,\frac{n}{s}}(B(\rho))} \\
&\quad + \delta^{-(s_0-s)\frac{n}{s}}\left(\frac{R}{\delta}\right)^{\frac{n}{s}}\left([u]^{\frac{n}{s}-1}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)} + [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{R}^n)}\right)[u]_{W^{s,\frac{n}{s}}(B(R))}.
\end{aligned}
$$

For $|s_0-s|$ and ε small enough, we thus have shown (3.21).

This concludes the proof of Theorem 3.10. □

3.4 | Proof of Theorem 3.1

Proof of Theorem 3.1. Let $s_0 := \min\{\frac{s+\alpha/p}{2}, \frac{s+\bar{s}}{2}\}$, where α is taken from Theorem 3.5 (without loss of generality, we may assume that $\alpha/p<1$) and $\bar{s}$ is taken from Theorem 3.10. Take s_1 and ε from Theorem 3.10.

Assuming that u is a $W^{t,\frac{n}{s}}$-minimizer in $B(R)$, we get from Theorem 3.5 that $u\in C^{\alpha}_{loc}(B(R))$. Since u is a minimizer, it satisfies the Euler–Lagrange equations, cf. (3.22). So, we can apply Theorem 3.7 and obtain that $u\in W^{s+\beta,\frac{n}{s}}_{loc}(B(R),\mathbb{R}^M)$ for any $\beta<s+\frac{\alpha}{p}$. In particular, $u\in W^{s_0,\frac{n}{s}}_{loc}(B(R),\mathbb{R}^M)$. From Theorem 3.10, we obtain for all $t\in[s,s_1]$,

$$
[u]_{C^{s_0-s}(B(R/2))} + [u]_{W^{s_0,\frac{n}{s}}(B(R/2))} \leqslant C\,R^{s-s_0}[u]_{W^{s,\frac{n}{s}}(B(R))}\left([u]^{1-\frac{s}{n}}_{W^{s,\frac{n}{s}}(\Sigma)} + [u]_{W^{s,\frac{n}{s}}(\Sigma)}\right).
$$

In particular, we have

$$
[u]_{C^{s_1-s}(B(R/2))} + [u]_{W^{s_1,\frac{n}{s}}(B(R/2))} \leqslant C\,R^{s-s_1}[u]_{W^{s,\frac{n}{s}}(B(R))}\left([u]^{1-\frac{s}{n}}_{W^{s,\frac{n}{s}}(\Sigma)} + [u]_{W^{s,\frac{n}{s}}(\Sigma)}\right).
$$

So, Theorem 3.1 is proven taking s_0 in the statement of the theorem to be s_1. □

3.5 | Consequences

We will need the following generalization of [96, Lemma 4.3].

Corollary 3.12. *Let Σ and $\mathcal{N}$ be as above. There exists $\varepsilon > 0$ and $s_0 \in (s,1)$ such that the following holds.*

Let $\{u_t\}_{t>s}$ be a sequence of $W^{t,\frac{n}{s}}(\Sigma,\mathcal{N})$-harmonic maps minimizing in a fixed homotopy class. Let us assume that $u_t \rightharpoonup u_s$ converges weakly in $W^{s,\frac{n}{s}}(\Sigma,\mathcal{N})$. If

$$E_{s,\frac{n}{s}}(u_t, B(x_0,\rho)) < \varepsilon$$

for some ball $B(x_0,\rho)$, then $u_t \to u_s$ strongly in $W^{s_0,\frac{n}{s}}(B(x_0,\rho/2),\mathcal{N}))$. The number $s_0 > s$ is taken from Proposition 3.1.

Proof. Let s_1 be the "s_0" from Theorem 3.1 and set $s_0 := \frac{s+s_1}{2}$.

From Theorem 3.1, we obtain

$$\sup_{t\in(s,s_1]} [u_t]_{W^{s_1,\frac{n}{s}}(B(x_0,\rho/2))} < \infty.$$

Thus, u_t converges weakly to u_s in $W^{s_1,\frac{n}{s}}(B(x_0,\rho/2))$. By Rellich–Kondrachov theorem, we obtain strong convergence in $W^{s_0,\frac{n}{s}}(B(x_0,\rho/2))$. □

The following theorem combines Corollary 3.12 with a covering argument, and is a generalization of [96, Proposition 4.3 & Theorem 4.4].

Theorem 3.13. *For any $s \in (0,1)$, there exists $s_0 > s$ such that the following holds. For $t \in (s,s_0]$, let $u_t : \Sigma \to \mathcal{N}$ be a sequence of minimizing $W^{t,\frac{n}{s}}$-harmonic maps in a fixed homotopy class of $C^0(\Sigma,\mathcal{N})$. Then, there is a decreasing sequence $(t_j)_{j\in\mathbb{N}} \subset (s,s_0]$ such that $t_j \xrightarrow{j\to\infty} s$ and a finite number of points $A := \{x_1,\dots,x_K\}$, such that*

$$u_{t_j} \xrightarrow{j\to\infty} u_s \quad \text{locally strongly in } W^{s_0,\frac{n}{s}}(\Sigma\setminus A).$$

Moreover, u_s is an $E_{s,\frac{n}{s}}$-minimizer within its homotopy class in $\Sigma\setminus A$, that is,

$$E_{s,\frac{n}{s}}(u_s,\Sigma) \leqslant E_{s,\frac{n}{s}}(v,\Sigma) \quad \text{if } u \equiv v \text{ in a neighborhood of } A \text{ and } u \sim v.$$

Proof. We can assume $E_{s,\frac{n}{s}}(u_t,\Sigma) < \Lambda$ for all $t \in [s,s_0]$.

Indeed, since Σ is compact and by minimality of u_t,

$$\begin{aligned}
\sup_{t\in[s,s_0]} E_{s,\frac{n}{s}}(u_t) &= \sup_{t\in[s,s_0]} \int_\Sigma\int_\Sigma \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \frac{|x-y|^{n\left(\frac{t-s}{s}\right)}}{|x-y|^{n\left(\frac{t-s}{s}\right)}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \sup_{t\in[s,s_0]} E_{t,\frac{n}{s}}(u_t) \\
&\lesssim \sup_{t\in[s,s_0]} E_{t,\frac{n}{s}}(u_{s_0}) \\
&\lesssim E_{s_0,\frac{n}{s}}(u_{s_0}) < \infty.
\end{aligned}$$

Thus, $E_{S,\frac{n}{s}}(u_t)$ is uniformly bounded.

Let $\alpha \in \mathbb{N}$, we define

$$\mathcal{B}_\alpha := \{B(x_{i,\alpha}, 2^{-\alpha}) : x_{i,\alpha} \in \Sigma\},$$

where the centers $x_{i,\alpha}$ are chosen so that $\bigcup B_\alpha$ covers Σ such that each point $x \in \Sigma$ is covered at most h-times, and for which, the balls with twice smaller radius still cover Σ, that is, $\Sigma \subset \bigcup_i B(x_{i,\alpha}, 2^{-\alpha-1})$. Then,

$$\sum_i \int_{B(x_{i,\alpha},2^{-\alpha})} \int_\Sigma \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy < \Lambda h.$$

Let $\varepsilon > 0$ be taken from Corollary 3.12, then for each $t \in [s, s_0]$, there exists at most $\frac{\Lambda h}{\varepsilon}$ balls in $\mathcal{B}_\alpha$ on which

$$\int_{B(x_{i,\alpha},2^{-\alpha})} \int_\Sigma \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy > \varepsilon. \tag{3.49}$$

Now, we claim that there exists a decreasing sequence $\{t_{K,\alpha}\} \subset \{t\} \subset (s, s_0]$, $t_{K,\alpha} \to s$, for which

$$u_{t_{K,\alpha}} \xrightarrow{t_{K,\alpha} \to s} u_s \quad \text{strongly in } W^{s_0,\frac{n}{s}}(B(x_{i,\alpha}, 2^{-\alpha-1}), \mathcal{N})$$

except for K balls from $\mathcal{B}_\alpha$, where $K < \frac{h\Lambda}{\varepsilon} + 1$.

Indeed, suppose that we have already shown that we have a subsequence $\{t_{k,\alpha}\} \subset \{t\}$ for which

$$u_{t_{k,\alpha}} \xrightarrow{t_{k,\alpha} \to s} u_s \quad \text{strongly in } W^{s_0,\frac{n}{s}}(B(x_{i,\alpha}, 2^{-\alpha-1}), \mathcal{N})$$

for $i = 1, \dots, k$ and that there are more than $\frac{h\Lambda}{\varepsilon}$ balls remaining in $\mathcal{B}_\alpha \setminus \{B(x_{1,\alpha}, 2^{-\alpha}), \dots, B(x_{k,\alpha}, 2^{-\alpha})\}$. Then, by (3.49), there is at least one $x_{j,\alpha}$ for $j \notin \{1, \dots, k\}$ for which on the ball $B(x_{j,\alpha}, 2^{-\alpha}) \in \mathcal{B}_\alpha$ we have

$$\int_{B(x_{j,\alpha},2^{-\alpha})} \int_\Sigma \frac{|u_{t_{k,\alpha}}(x) - u_{t_{k,\alpha}}(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \, dx \, dy < \varepsilon.$$

By Corollary 3.12, we know that there is a subsequence $\{t_{k+1,\alpha}\} \subset \{t_{k,\alpha}\}$ such that on the smaller ball, we have

$$u_{t_{k+1,\alpha}} \xrightarrow{t_{k+1,\alpha} \to s} u_s \quad \text{strongly in } W^{s_0,\frac{n}{s}}(B(x_{j,\alpha}, 2^{-\alpha-1}), \mathcal{N}).$$

We repeat this construction until there are $K < \frac{h\Lambda}{\varepsilon} + 1$ balls left.

Thus, we have shown that for some $\{y_{1,\alpha}, \dots, y_{K,\alpha}\}$, we have

$$u_{t_{K,\alpha}} \xrightarrow{t_{K,\alpha} \to s} u_s \quad \text{strongly in } W^{s_0,\frac{n}{s}}\left(\Sigma \setminus \bigcup_{i \leqslant K} (B(y_{i,\alpha}, 2^{-\alpha-1}))), \mathcal{N}\right).$$

Moreover, we have $\{t_{K,\alpha}\} \subset \{t_{K,\alpha-1}\} \subset \{t\}$. Finally, we choose a diagonal subsequence $\tilde{t}$ of the sequences $\{t_{K,\alpha}\}$, then $u_{\tilde{t}} \to u_s$ in $W^{s_0,\frac{n}{s}}$ on

$$\bigcup_{\alpha\in\mathbb{N}}\left(\Sigma\setminus\bigcup_{i\leqslant K}B(y_{i,\alpha},2^{-\alpha-1})\right)=\Sigma\setminus\bigcap_{\alpha\in\mathbb{N}}\bigcup_{i\leqslant K}B(y_{i,\alpha},2^{-\alpha-1})=\Sigma\setminus\{x_1,\dots,x_K\}.$$

□

4 | REMOVABILITY OF SINGULARITIES

In this section, we show that in the case when limits of minimizing $W^{t,\frac{n}{s}}$-harmonic maps have isolated singularities, then those singularities can be removed.

Theorem 4.1. *Let $\Sigma, \mathcal{N}$ be manifolds as above. Let $B = B(x_0, R) \subset \Sigma$ be a geodesic ball centered at a point $x_0 \in \Sigma$, then the following holds.*

Let $u \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ be a minimizing map in $B(x_0, R)$ in homotopy away from the point x_0. That is, assume for any $\varepsilon > 0$ and any $w \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ satisfying

- *$u \equiv w$ on $B(x_0, \varepsilon) \cup (\Sigma\setminus B(x_0, R))$ and*
- *$u \sim w$,*

we have

$$E_{s,\frac{n}{s}}(u, \Sigma) \leqslant E_{s,\frac{n}{s}}(w, \Sigma). \tag{4.1}$$

Then, u is minimizing in all of $B(x_0, R)$, that is, for any $v \in W^{s,\frac{n}{s}}(\Sigma)$ such that

- *$u \equiv v$ on $\Sigma\setminus B(x_0, R)$ and*
- *$u \sim v$,*

we have

$$E_{s,\frac{n}{s}}(u, \Sigma) \leqslant E_{s,\frac{n}{s}}(v, \Sigma).$$

In particular, we obtain regularity theory for maps as in Theorem 4.1, see Theorem 4.6.

To prove Theorem 4.1, we will construct a comparison map, the construction will be very similar to the one in the paper by Monteil–Van Schaftingen [84, Proof of Theorem 3.1]. We will be using the following lemmata from [84]. The first lemma is called *the opening of maps* in the sense of Brezis–Li [19], and the purpose of it is to connect a given map continuously to a constant within the Sobolev space.

Lemma 4.2 [84, Lemma 2.1]. *Let $0 < s \leqslant 1$, $p \geqslant 1$, $\lambda > 1$, and $\eta \in (0, \lambda)$. Then, there is a constant $C > 0$ such that for any $\rho > 0$, any measurable $u : B(\lambda\rho) \to \mathcal{N}$, and every Lipschitz continuous map $\phi : B((1+\eta)\rho) \to B((\lambda-\eta)\rho)$, there exists a point $a \in B(\eta\rho)$ such that*

$$E_{s,p}(u\circ(\phi(\cdot - a) + a), B(\rho)) \leqslant C\mathrm{Lip}\,(\phi)^{sp}E_{s,p}(u, B(\lambda\rho)).$$

The next lemma allows to glue two maps along a "buffering zone."

Lemma 4.3 [84, Lemma 2.2]. *Let $0 < s \leqslant 1$, $p \geqslant 1$. There exists a constant $C > 0$ such that for every $\eta \in (0,1)$, $A \subset \Sigma$ open, every measurable $u : B(\lambda\rho) \to \mathcal{N}$, and every $\rho > 0$ such that $B_\rho \setminus \overline{B(\eta\rho)} \subset A$ we have*

$$E_{s,p}(u,A) \leqslant \left(1 + \frac{C}{(1-\eta)^{sp+1}}\right) E_{s,p}(u, A \cap B(\rho)) + \left(1 + \frac{C\eta^n}{1-\eta}\right) E_{s,p}(u, A \setminus \overline{B(\eta\rho)}),$$

where the constant $C = C(n,s,p)$ does not depend on the set A nor on the radii ρ, η.

The next lemma says that a Sobolev map on a ball taking values in a manifold can be extended to a larger ball. This can, for example, be proven by an inversion, setting $v(x) := u(\rho^2 x/|x|^2)$ for $|x| > \rho$.

Lemma 4.4 [84, Lemma 2.4]. *Let $s \in (0,1]$, $p \geqslant 1$, $\lambda \geqslant 1$. There exists a constant $C > 0$ such that if $\rho > 0$, $u : B(\rho) \to \mathcal{N}$ is measurable, then there exists $v : B(\lambda\rho) \to \mathcal{N}$ such that $v = u$ on $B(\rho)$ and*

$$\|v\|^p_{L^p(B(\lambda\rho))} \leqslant C\|u\|^p_{L^p(B(\rho))}, \quad E_{s,p}(v, B(\lambda\rho)) \leqslant C E_{s,p}(u, B(\rho)).$$

Finally, the last lemma is also well known and is often used to remove singularities in critical Sobolev spaces (not necessarily of fractional order). The lemma basically says that a point in the critical Sobolev space has zero capacity. For the proof we refer, for example, to [1, Theorem 5.1.9], compare also with [84, Lemma 3.2]. The proof is based on the existence of unbounded functions in the critical Sobolev space and truncation.

Lemma 4.5. *For any $s \in (0,1)$, $n \geqslant 1$, there exist $\{\zeta_\ell\}_{\ell\in\mathbb{N}} \subset C^\infty_c(\Sigma, [0,1])$ such that for all $\ell \in \mathbb{N}$,*

$$\zeta_\ell \equiv 1 \quad \text{on } B(\rho_\ell), \qquad \zeta_\ell \equiv 0 \quad \text{outside } B(R_\ell)$$

for some $0 < \rho_\ell < R_\ell \to 0$ as $\ell \to \infty$ and

$$\lim_{\ell\to\infty} E_{s,\frac{n}{s}}(\zeta_\ell, \Sigma) = 0.$$

We are ready to prove our Theorem.

Proof of Theorem 4.1. We will construct a comparison map, and we begin with a modification of u. We will simply write here $B(r)$ for $B(x_0, r)$.

Step 1. Let us take $B(\rho_\ell)$ from Lemma 4.5 and extend $u|_{B(\rho_\ell)} : B(\rho_\ell) \to \mathcal{N}$ with the help of Lemma 4.4. We know that there exists $u_1 : B(3\rho_\ell) \to \mathcal{N}$ such that $u_1 = u$ on $B(\rho_\ell)$ and

$$E_{s,\frac{n}{s}}(u_1, B(3\rho_\ell)) \lesssim E_{s,\frac{n}{s}}(u, B(\rho_\ell)). \tag{4.2}$$

Step 2. Next, we again modify the map u_1, in such a way that we obtain a map that is constant outside the ball $B(4\rho_\ell)$. Take $\phi_1 : B(6\rho_\ell) \to B(2\rho_\ell)$ Lipschitz continuous such that

$$\phi_1(x) = x \quad \text{if } |x| \leqslant 2\rho_\ell,$$
$$\phi_1(x) = 0 \quad \text{if } |x| \geqslant 3\rho_\ell.$$

Then, by Lemma 4.2, there exists an $a_1 \in B(\rho_\ell)$ such that

$$E_{s,\frac{n}{s}}(u_1\circ(\phi_1(\cdot - a_1) + a_1), B(5\rho_\ell)) \leqslant C\mathrm{Lip}\,(\phi_1)^n E_{s,\frac{n}{s}}(u_1, B(3\rho_\ell)) \lesssim E_{s,\frac{n}{s}}(u_1, B(3\rho_\ell)), \tag{4.3}$$

and we have

(1) if $|x| \leqslant \rho_\ell$, then $|x - a_1| \leqslant 2\rho_\ell$,
(2) if $|x| \geqslant 4\rho_\ell$, then $|x - a| \geqslant 3\rho_\ell$.

Thus,

$$\phi_1(x - a_1) + a_1 = \begin{cases} x & \text{if } |x| \leqslant \rho_\ell, \\ a_1 & \text{if } |x| \geqslant 4\rho_\ell, \end{cases}$$

and

$$u_1\circ(\phi_1(\cdot - a_1) + a_1)(x) = \begin{cases} u_1(x) = u(x) & \text{if } |x| \leqslant \rho_\ell, \\ b_1 & \text{if } |x| \geqslant 4\rho_\ell, \end{cases}$$

where $b_1 := u_1(a_1) \in \mathcal{N}$. We define

$$u_2(x) := \begin{cases} u_1\circ(\phi_1(x - a_1) + a_1 & \text{if } |x| \leqslant 4\rho_\ell \\ b_1 & \text{if } |x| \geqslant 4\rho_\ell. \end{cases}$$

Combining Lemma 4.3 (applied with $A = \Sigma$, $\rho = 5\rho_\ell$, and $\eta = \frac{4}{5}$) with (4.3) and (4.2), we get

$$\begin{aligned} E_{s,\frac{n}{s}}(u_2, \Sigma) &\lesssim E_{s,\frac{n}{s}}(u_1\circ(\phi_1(\cdot - a_1) + a_1), B(5\rho_\ell)) \\ &\lesssim E_{s,\frac{n}{s}}(u, B(\rho_\ell)). \end{aligned} \tag{4.4}$$

Step 3. Now we modify the map u_2 in such a way that it connects on an annulus the constant $b_1 \in \mathcal{N}$ with another constant $b_2 \in \mathcal{N}$. The newly obtained map is again constant outside a bigger ball $B(R_\ell)$.

Since $\mathcal{N}$ is connected, we know that there is a Lipschitz continuous map such that $\gamma : [0,1] \to \mathcal{N}$, $\gamma(0) = b_2$, where $b_2 \in \mathcal{N}$ is point that will be chosen later, $\gamma(1) = b_1$, and the Lipschitz constant satisfies $\mathrm{Lip}\,(\gamma) \leqslant 2d_{\mathcal{N}}(b_1, b_2)$. Then,

$$\gamma\circ\zeta_\ell : \Sigma \to \mathcal{N},$$

where ζ_ℓ is taken from Lemma 4.5 (we just replaced ρ_ℓ by $6\rho_\ell$). For this function, we have

$$\gamma\circ\zeta_\ell = b_1 \text{ on } B(6\rho_\ell), \quad \gamma\circ\zeta_\ell(x) = b_2 \text{ on } \Sigma \setminus B(R_\ell),$$

and

$$E_{s,\frac{n}{s}}(\gamma\circ\zeta_\ell, \Sigma) \leqslant \mathrm{Lip}\,(\gamma)^{\frac{n}{s}} E_{s,\frac{n}{s}}(\zeta_\ell, \Sigma) \leqslant 2d_{\mathcal{N}}(b_1, b_2)^{\frac{n}{s}} E_{s,\frac{n}{s}}(\zeta_\ell, \Sigma) \lesssim E_{s,\frac{n}{s}}(\zeta_\ell, \Sigma), \tag{4.5}$$

where the last constant depends only on the manifold $\mathcal{N}$.

We note that for sufficiently large ℓ, we have $B(6\rho_\ell) \subset B(R_\ell)$. We define $u_3 : \Sigma \to \mathcal{N}$ by

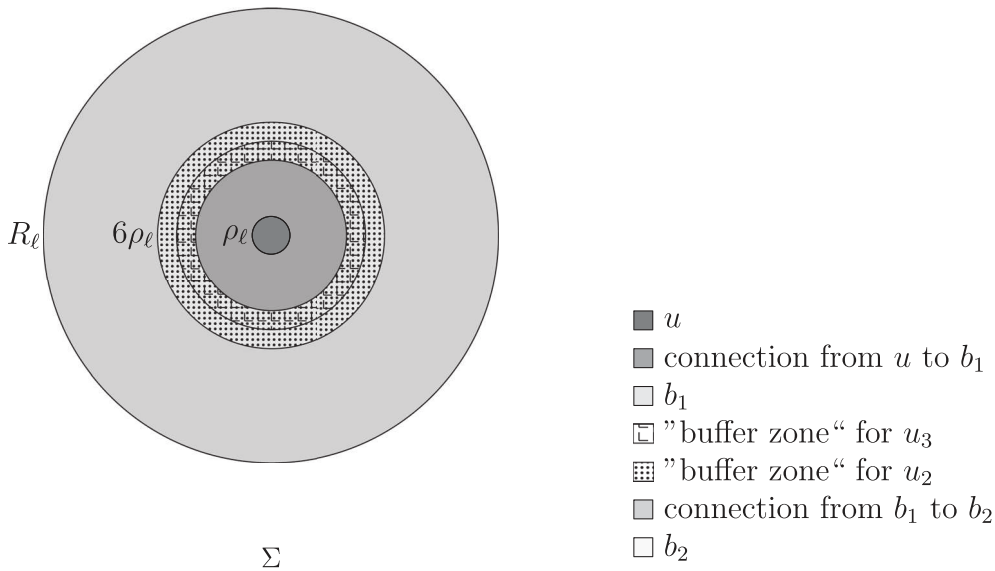
$$u_3(x) := \begin{cases} u_2(x) & \text{if } x \in B(5\rho_\ell) \\ \gamma \circ \zeta_\ell(x) & \text{if } x \in \Sigma \setminus B(5\rho_\ell). \end{cases}$$

Then, by Lemma 4.3 (applied with $A = \Sigma$, $\rho = 6\rho_\ell$, $\eta = \frac{5}{6}$),

$$E_{s,\frac{n}{s}}(u_3, \Sigma) \lesssim E_{s,\frac{n}{s}}(u_2, B(6\rho_\ell)) + E_{s,\frac{n}{s}}(\gamma \circ \zeta_\ell, \Sigma).$$

Which, combined with (4.5) and (4.4), gives

$$E_{s,\frac{n}{s}}(u_3, \Sigma) \lesssim E_{s,\frac{n}{s}}(u, B(\rho_\ell)) + E_{s,\frac{n}{s}}(\zeta_\ell, \Sigma). \tag{4.6}$$



The domain of the map u_3

Step 4. Let $v \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ be any map such that $v \sim u$. We will modify v in such a way that we will be able to compare the energy of the modified v with u.

Let $\phi_2 : B(9R_\ell) \to B(9R_\ell)$ be a Lipschitz continuous function, such that $\phi_2(x) = x$ if $|x| \geqslant 5R_\ell$, $\phi_2(x) = 0$ if $|x| \leqslant 3R_\ell$. Then by Lemma 4.2 with $\rho = 8R_\ell$, $\lambda = \frac{10}{8}$, $\eta = \frac{1}{8}$, we obtain an existence of a point $a_2 \in B_{R_\ell}$ such that

$$E_{s,\frac{n}{s}}(v \circ (\phi_2(\cdot - a_2) + a_2), B_{8R_\ell}) \leqslant C E_{s,\frac{n}{s}}(v, B_{10R_\ell}).$$

We also have

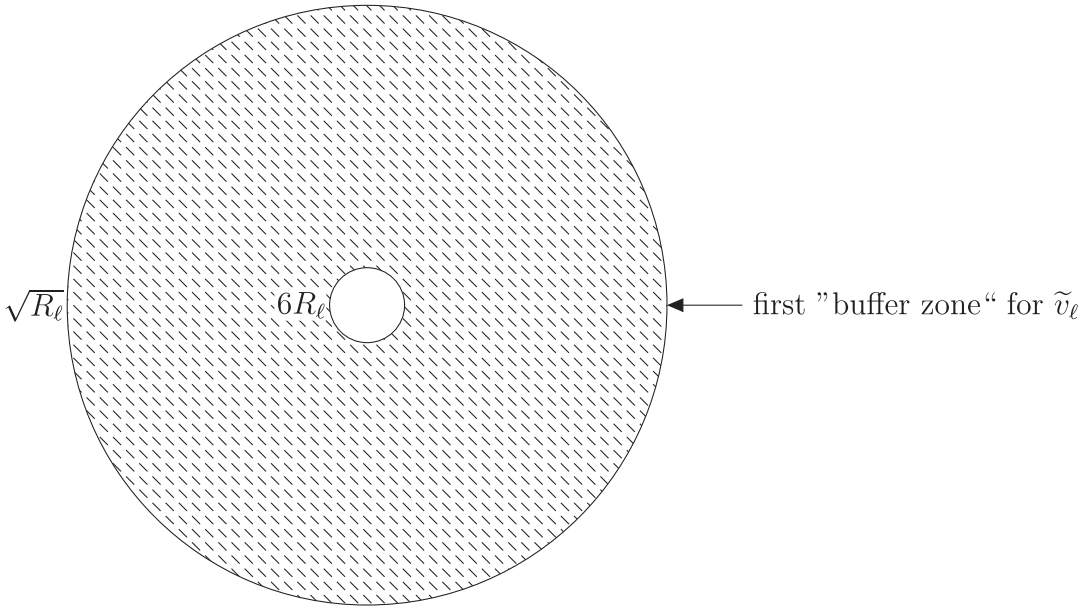
$$\phi_2(x - a_2) + a_2 = \begin{cases} x & \text{if } |x| \geqslant 6R_\ell, \\ a_2 & \text{if } |x| \leqslant 2R_\ell. \end{cases}$$

Now we choose the point b_2 from Step 3 to be $b_2 := v(a_2)$. Thus,

$$v(\phi_2(x-a_2)+a_2) = \begin{cases} v(x) & \text{if } |x| \geqslant 6R_\ell, \\ b_2 & \text{if } |x| \leqslant 2R_\ell. \end{cases}$$

Finally, we define

$$\widetilde{v}_\ell(x) = \begin{cases} v(x) & |x| \geqslant 8R_\ell, \\ v\circ(\phi_2(\cdot - a_2)+a_2) & 2R_\ell \leqslant |x| \leqslant 8R_\ell, \\ u_3(x) & |x| \leqslant 2R_\ell. \end{cases}$$



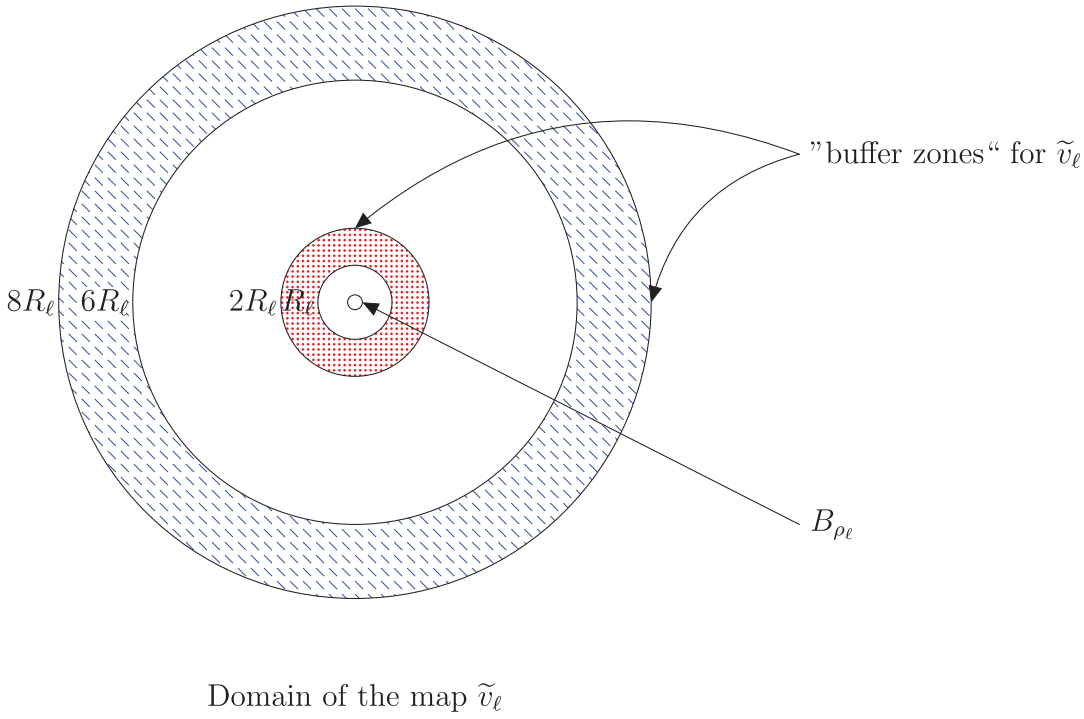
We apply Lemma 4.3 with $A = \Sigma$, $\rho = \sqrt{R_\ell}$, and $\eta = 6\sqrt{R_\ell}$ (for sufficiently large ℓ, we know that $B(6R_\ell) \subset B(\sqrt{R_\ell})$) in order to obtain

$$\begin{aligned} E_{s,\frac{n}{s}}(\widetilde{v}_\ell, \Sigma) \leqslant & \left(1 + \frac{C_1}{(1-6\sqrt{R_\ell})^{n+1}}\right) E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(\sqrt{R_\ell})) \\ & + \left(1 + \frac{C_1(6\sqrt{R_\ell})^n}{1-6\sqrt{R_\ell}}\right) E_{s,\frac{n}{s}}(\widetilde{v}_\ell, \Sigma \setminus B(6R_\ell)). \end{aligned} \tag{4.7}$$

We note that $\widetilde{v}_\ell = v$ for $x \in \Sigma \setminus B(6R_\ell)$, so $E_{s,\frac{n}{s}}(\widetilde{v}_\ell, \Sigma \setminus B(6R_\ell)) = E_{s,\frac{n}{s}}(v, \Sigma \setminus B(6R_\ell))$.

Next, we apply twice again Lemma 4.3 to deal with the term $E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(\sqrt{R_\ell}))$. For the first application, we take $A = B(\sqrt{R_\ell})$, $\rho = 8R_\ell$, $\eta = \frac{3}{4}$ and for the second application, we take $A = B(8R_\ell)$, $\rho = 2R_\ell$, $\eta = \frac{1}{2}$. This way we obtain

$$E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(\sqrt{R_\ell})) \lesssim E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(8R_\ell)) + E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(\sqrt{R_\ell}) \setminus B(6R_\ell))$$
$$\lesssim E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(2R_\ell)) + E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(8R_\ell) \setminus B(R_\ell)) + E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(\sqrt{R_\ell}) \setminus B(6R_\ell)). \tag{4.8}$$



Now, we note that $\widetilde{v}_\ell = v$ on $B(\sqrt{R_\ell}) \setminus B(6R_\ell)$, $\widetilde{v}_\ell = v \circ (\phi_2(\cdot - a_2) + a_2)$ on $B(8R_\ell) \setminus B(R_\ell)$, and $\widetilde{v}_\ell = u_3$ on $B(R_\ell)$. Thus,

$$E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(2R_\ell)) = E_{s,\frac{n}{s}}(u_3, B(2R_\ell)),$$
$$E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(8R_\ell) \setminus B(R_\ell)) = E_{s,\frac{n}{s}}(v \circ (\phi_2(\cdot - a_2) + a_2), B(8R_\ell) \setminus B(R_\ell)),$$
$$E_{s,\frac{n}{s}}(\widetilde{v}_\ell, B(\sqrt{R_\ell}) \setminus B(6R_\ell)) = E_{s,\frac{n}{s}}(v, B(\sqrt{R_\ell}) \setminus B(6R_\ell)).$$

Recall that from Step 3, inequality (4.6), we know that

$$E_{s,\frac{n}{s}}(u_3, B(2R_\ell)) \leqslant E_{s,\frac{n}{s}}(u_3, \Sigma) \lesssim E_{s,\frac{n}{s}}(u, B(\rho_\ell)) + E_{s,\frac{n}{s}}(\zeta_\ell, \Sigma). \tag{4.9}$$

We also have

$$E_{s,\frac{n}{s}}(v \circ (\phi_2(\cdot - a_2) + a_2) \lesssim E_{s,\frac{n}{s}}(v, B(10R_\ell)). \tag{4.10}$$

Combining (4.7), (4.8), (4.9), and (4.10), we get

$$E_{s,\frac{n}{s}}(\widetilde{v}_\ell,\Sigma)\leqslant\left(1+\frac{C_1(6\sqrt{R_\ell})^n}{1-6\sqrt{R_\ell}}\right)E_{s,\frac{n}{s}}(v,\Sigma\setminus B(6R_\ell))$$
$$+C_2\Big(E_{s,\frac{n}{s}}(u,B_{\rho_\ell})+E_{s,\frac{n}{s}}(\zeta_\ell,\Sigma)+E_{s,\frac{n}{s}}(v,B(10R_\ell))+E_{s,\frac{n}{s}}(v,B_{\sqrt{R_\ell}}\setminus B(6R_\ell))\Big) \tag{4.11}$$

for a constant C_2 independent of v, u, ℓ.

Step 5. The only thing left to prove is that the map $\widetilde{v}_\ell$ is a good comparison map. We immediately verify that $\widetilde{v}_\ell\equiv u$ on $B(\rho_\ell)$. Finally, to show that $\widetilde{v}_\ell\sim u$, we recall that $v\sim u$, and thus, it is enough to show that $\widetilde{v}_\ell\sim v$. We have

$$(v-\widetilde{v})(x)=\begin{cases}0 & \text{if } |x|\geqslant 8R_\ell\\ (v-\widetilde{v})(x) & \text{if } |x|\leqslant 8R_\ell.\end{cases}$$

Thus, by Lemma 4.3, we get

$$E_{s,\frac{n}{s}}(v-\widetilde{v}_\ell,\Sigma)\lesssim E_{s,\frac{n}{s}}(v-\widetilde{v}_\ell,B(8R_\ell))\lesssim E_{s,\frac{n}{s}}(v,B(8R_\ell))+E_{s,\frac{n}{s}}(\widetilde{v}_\ell,B(8R_\ell)).$$

By taking ℓ large enough we can ensure, by the absolute continuity of the integral that the latter one is smaller than ε, where ε is taken from Lemma 2.9. Similarly, since v and $\widetilde{v}$ differ only on a small set, we verify that $\|v-\widetilde{v}_\ell\|_{L^1(\Sigma)}\ll\varepsilon$ for sufficiently large ℓ. Thus, from Lemma 2.9, we deduce that $\widetilde{v}_\ell\sim v$.

Combining the minimality outside of a point of u with (4.11), we get

$$E_{s,\frac{n}{s}}(u,\Sigma)\leqslant E_{s,\frac{n}{s}}(\widetilde{v}_\ell,\Sigma)$$
$$\leqslant\left(1+\frac{C_1(6\sqrt{R_\ell})^n}{1-6\sqrt{R_\ell}}\right)E_{s,\frac{n}{s}}(v,\Sigma\setminus B(6R_\ell))$$
$$+C_2\Big(E_{s,\frac{n}{s}}(u,B(\rho_\ell))+E_{s,\frac{n}{s}}(\zeta_\ell,\Sigma)+E_{s,\frac{n}{s}}(v,B(10R_\ell))+E_{s,\frac{n}{s}}(v,B(\sqrt{R_\ell})\setminus B(6R_\ell))\Big). \tag{4.12}$$

We observe that as $\ell\to\infty$, we get $(1+\frac{C(6\sqrt{R_\ell})^n}{1-6\sqrt{R_\ell}})\to 1$ and by the absolute continuity of the integral, since $B(\rho_\ell)$, $B(10R_\ell)$, $B(\sqrt{R_\ell})\setminus B(6R_\ell)$ shrink to $\{0\}$, we get as $\ell\to\infty$

$$E_{s,\frac{n}{s}}(u,B(\rho_\ell))+E_{s,\frac{n}{s}}(v,B(10R_\ell))+E_{s,\frac{n}{s}}(v,B(\sqrt{R_\ell})\setminus B(6R_\ell))\to 0.$$

Finally, by Lemma 4.5, we have

$$E_{s,\frac{n}{s}}(\zeta_\ell,\Sigma)\to 0.$$

Thus, passing with $\ell\to\infty$ in (4.12), we get

$$E_{s,\frac{n}{s}}(u,\Sigma)\leqslant E_{s,\frac{n}{s}}(v,\Sigma).$$

Thus, we can conclude that u is minimizing in all of Σ among all maps in the same homotopy class. □

As a corollary of Theorem 3.13, Theorem 4.1, and Theorem 3.1, we obtain the following.

Theorem 4.6. *There exists $s_0 > s$ such that the following holds.*

Assume that $u_t \in W^{t,\frac{n}{s}}(\Sigma, \mathcal{N})$ is a sequence of minimizers in a homotopy class X that converges weakly to $u_s \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ in the $W^{s,\frac{n}{s}}$-topology. Then $u_s \in W^{s_0,\frac{n}{s}}(\Sigma, \mathcal{N})$.

We finish this section with a remark. We can remove discrete points in the equation, that is, once we know that a map satisfies the equation of $W^{s,\frac{n}{s}}$-harmonic maps in $\Sigma \setminus A$, where A is a set consisting of finitely many points, we know that the equation is satisfied in Σ. Unfortunately, the lack of regularity theory in general does not allow us to conclude that the map is regular everywhere. But in view of [76, 101, 104] if we have $W^{s,\frac{n}{s}}$-harmonic maps in $\Sigma \setminus A$ that maps into a sphere or a compact Lie group, or in view of [26] if we have $W^{\frac{1}{2},2}$-harmonic maps on a line, we have regularity in all of Σ.

Proposition 4.7. *Let A be a finite set in Σ, and let $u \in W^{s,\frac{n}{s}}$ be a $W^{s,\frac{n}{s}}$-harmonic map outside of A, that is,*

$$\int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,(\Pi(u(x))\varphi(x)-\Pi(u(y))\varphi(y))}{|x-y|^{2n}}\,\mathrm{d}y\,\mathrm{d}x = 0 \quad \forall \varphi \in C_c^\infty(\Sigma\setminus A), \tag{4.13}$$

where $\Pi(u)$ is the orthogonal projection onto the tangent space of $T_u\mathcal{N}$ for $u \in \mathcal{N}$, see Lemma 2.6.

Then u is a $W^{s,\frac{n}{s}}$-harmonic map in all of Σ, that is,

$$\int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,(\Pi(u(x))\varphi(x)-\Pi(u(y))\varphi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \forall \varphi \in C_c^\infty(\Sigma).$$

Proof. For simplicity, assume that $A = \{x_0\}$. Let $\phi \in C_c^\infty(\Sigma)$ and let $\zeta_\ell \in C_c^\infty(B_{R_\ell})$ be as in Lemma 4.5, that is,

$$\zeta_\ell \equiv 1 \text{ on } B_{\rho_\ell}(x_0) \quad \text{and} \quad [\zeta_\ell]_{W^{s,\frac{n}{s}}(\Sigma)} \to 0 \text{ as } \ell \to \infty,$$

for a sequence $0 < \rho_\ell < R_\ell \to 0$ as $\ell \to \infty$.

Thus, $\phi_\ell = \phi(1-\zeta_\ell) \in C_c^\infty(\Sigma \setminus \{x_0\})$ is an admissible test function and from (4.13), we get

$$\begin{aligned}
&\int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,(\Pi(u(x))\phi(x)-\Pi(u(y))\phi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_\Sigma \int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,\big(\Pi(u(x))\phi(x)\zeta_\ell(x)-\Pi(u(y))\phi(y)\zeta_\ell(y)\big)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The latter one can be estimated in the following way.

$$\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\left(\Pi(u(x))\phi(x)\zeta_\ell(x)-\Pi(u(y))\phi(y)\zeta_\ell(y)\right)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,\Pi(u(x))\phi(x)\left(\zeta_\ell(x)-\zeta_\ell(y)\right)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,(\Pi(u(x))\phi(x)-\Pi(u(y))\phi(y))\zeta_\ell(y)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y.$$

As for the first term, we have by Hölder's inequality

$$\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,\Pi(u(x))\phi(x)\left(\zeta_\ell(x)-\zeta_\ell(y)\right)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \|\Pi\|_{L^\infty}\|\phi\|_{L^\infty}\left(\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{n-s}{n}}[\zeta_\ell]_{W^{s,\frac{n}{s}}(\Sigma)}\xrightarrow{\ell\to\infty}0.$$

As for the second term, we have

$$\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,(\Pi(u(x))\phi(x)-\Pi(u(y))\phi(y))\zeta_\ell(y)}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \int_{B_{R_\ell}}\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-1}\left|\Pi(u(x))\phi(x)-\Pi(u(y))\phi(y)\right|}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\xrightarrow{\ell\to\infty}0,$$

by the absolute continuity of the integral.

Hence, u is a $W^{s,\frac{n}{s}}$-harmonic map, as

$$\int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^{\frac{n}{s}-2}((u(x)-u(y))\,(\Pi(u(x))\phi(x)-\Pi(u(y))\phi(y))}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y=0$$

for any $\phi\in C_c^\infty(\Sigma)$. □

5 | BALANCED ENERGY ESTIMATE FOR THE NONSCALING INVARIANT NORMS

In this section, we show the main advantage of approximating $W^{s,\frac{n}{s}}$-minimizers by $W^{t,\frac{n}{s}}$-minimizers. It does not avoid energy concentration in a single point, but energy cannot concentrate only in one point and vanish everywhere else. In a sense, the energy needs to be balanced.

We will use Theorem 5.1 in place of [96, Lemma 5.3] in our argument.

Theorem 5.1. *Let $0<s<s_0<1$ and $\rho_0\in(0,\sqrt{\frac{4}{5}})$. There exists a constant $C=C(s,s_0,\rho_0)$ such that the following holds.*

For any $t \in (s, s_0]$, let $u_t \in W^{t,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$ be a minimizing map in its own homotopy class. Then for any $y_0 \in \mathbb{S}^n$

$$\int_{D(y_0,\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}} \,\mathrm{d}x\,\mathrm{d}y \leqslant C\,\rho^{-n\left(\frac{t}{s}-1\right)} \int_{\mathbb{S}^n \setminus D(y_0,\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}} \,\mathrm{d}x\,\mathrm{d}y. \quad (5.1)$$

Here, $D(a,r) := B(a,r) \cap \mathbb{S}^n$ is the intersection of a ball centered at $a \in \mathbb{S}^n$ of radius r intersected with the sphere.

Before we give the proof of Theorem 5.1 in Section 5.1, let us digress a little bit by stating some extensions and consequences.

First, we remark that the arguments for Theorem 5.1 carry over to the $W^{1,p}$-case, and for future reference, we record the following theorem (the changes in the proof are sketched in Section 5.2).

Theorem 5.2. *Let $p_0 \in (n, \infty)$ and $\rho_0 \in (0, \sqrt{\frac{4}{5}})$. There exists a constant $C = C(n, p_0, \rho_0)$ such that the following holds.*

For any $p \in (n, p_0]$, let $u_p \in W^{1,p}(\mathbb{S}^n, \mathcal{N})$ be a $\int_{\mathbb{S}^n} |\nabla u|^p$-minimizing map in its own homotopy group. Then, for any $y_0 \in \mathbb{S}^n$,

$$\int_{D(y_0,\rho)} |\nabla u_p|^p \,\mathrm{d}x \leqslant C\,\rho^{-(p-n)} \int_{\mathbb{S}^n \setminus D(y_0,\rho)} |\nabla u_p|^p \,\mathrm{d}x. \quad (5.2)$$

Here, $D(a,r) := B(a,r) \cap \mathbb{S}^n$ is the intersection of a ball centered at $a \in \mathbb{S}^n$ of radius r intersected with the sphere.

Remark 5.3. We are not aware of results similar to Theorem 5.2 or Theorem 5.1 in the literature. However, it seems that a somewhat similar effect is underlying the arguments in the recent work by Lamm–Malchiodi–Micallef [63].

Also — though we make no direct use of it — we state that as a consequence, the limits of sequence of minimizers u_t "distribute" their energy in the domain. More precisely, from Theorem 5.1 and Theorem 5.2, respectively, we have the following.

Corollary 5.4. *Let $\mathcal{N}$ be as above, $n \geqslant 1$, $s \in (0,1)$ and $s_0 \in (s,1)$. There exists $\tau = \tau(n, \mathcal{N}, s, s_0) \in (0,1)$ such that the following holds.*

Let $u_{t_i} \in W^{t_i,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$, $t_i > s$, $t_i \xrightarrow{i\to\infty} s$, be a sequence of $E_{t_i,\frac{n}{s}}$-minimizers with respect to their own homotopy class.

Assume that u_{t_i} converges weakly in $W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$ to $u \in W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$. Moreover assume that there exists $y_0 \in \mathbb{S}^n$ such that $u_{t_i} \xrightarrow{i\to\infty} u$ weakly in $W^{s_0,\frac{n}{s}}(\mathbb{S}^n \setminus D(y_0, \sqrt{\frac{4}{5}}), \mathcal{N})$ for some $s_0 > s$. Then

$$\int_{D(y_0,\sqrt{\frac{4}{5}})} \int_{\mathbb{S}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \,\mathrm{d}x\,\mathrm{d}y \leqslant \tau \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x-y|^{2n}} \,\mathrm{d}x\,\mathrm{d}y \quad (5.3)$$

and

$$\int_{\mathbb{S}^n\setminus D(y_0,\sqrt{\frac{4}{5}})}\int_{\mathbb{S}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \geqslant (1-\tau)\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.4}$$

Corollary 5.5. *Let $\mathcal{N}$ be as above, $n \geqslant 2$. There exists $\tau = \tau(n,\mathcal{N}) \in (0,1)$ such that the following holds.*

Let $u_{p_i} \in W^{1,p_i}(\mathbb{S}^n,\mathcal{N})$, $p_i > n$, $p_i \xrightarrow{i\to\infty} n$, be a sequence of $\int_{\mathbb{S}^n}|\nabla u|^{p_i}$-minimizers with respect to their own homotopy class.

Assume that u_{p_i} converges weakly in $W^{1,n}(\mathbb{S}^n,\mathcal{N})$ to $u \in W^{1,n}(\mathbb{S}^n,\mathcal{N})$. Moreover, assume that there exists $y_0 \in \mathbb{S}^n$ such that $u_{p_i} \xrightarrow{i\to\infty} u$ strongly in $W^{1,p_0}(\mathbb{S}^n\setminus D(y_0,\sqrt{\frac{4}{5}}),\mathcal{N})$ for some $p_0 > n$. Then

$$\int_{D(y_0,\sqrt{\frac{4}{5}})}|\nabla u|^n\,\mathrm{d}x \leqslant \tau\int_{\mathbb{S}^n}|\nabla u|^n\,dx \tag{5.5}$$

and

$$\int_{\mathbb{S}^n\setminus D(y_0,\sqrt{\frac{4}{5}})}|\nabla u|^n\,\mathrm{d}x \geqslant (1-\tau)\int_{\mathbb{S}^n}|\nabla u|^n\,dx. \tag{5.6}$$

Sketch of the proof of Corollary 5.4 and Corollary 5.5. In (5.2), take the limit using lower semicontinuity of the functional on the left-hand side and strong convergence on the right-hand side. Then apply the hole-filling trick. This implies (5.5). Inequality (5.6) is a direct consequence of (5.5). The details are left to the reader. □

5.1 | Proof of Theorem 5.1

In order to prove Theorem 5.1, we will use the minimizing property of the mapping u_t and compare its energy with a "rescaled" version of u_t. In order to do so, we will first change the coordinates into the spherical coordinates, then we will use the stereographic projection of the sphere and map the n-sphere to the hyperplane. Finally, on the hyperplane, we define the rescaling, which in the polar coordinates (r,ω), for $r > 0$, $\omega \in \mathbb{S}^{n-1}$, on $\mathbb{R}^n$ is given simply by $r \mapsto \lambda r$, with a parameter $\lambda > 1$.

As a quick motivation for using as the comparison map the rescaling we note that in the simple case, when we consider a minimizing map $v \in W^{t,\frac{n}{s}}(\mathbb{R}^n,\mathcal{N})$, we get immediately by comparing with the rescaled map $v_\lambda := v(\lambda\cdot)$ the following:

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v_\lambda(x)-v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \lambda^{n\left(\frac{t-s}{s}\right)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which is possible only if $v \equiv const$. Here, we emphasize that the last equality is true, because the energy is not scaling invariant, and thus, if we would replace $E_{t,\frac{n}{s}}$ by $E_{s,\frac{n}{s}}$, there would not be an extra λ term in front of the integral.

Similarly in the case, when the domain of the minimizing map is $\mathbb{S}^n$, since the $E_{t,\frac{n}{s}}$-energy is not conformally invariant, we already get an "extra" term using the stereographic projection. Then again, an additional term appears after rescaling. Those extra terms can be estimated, accordingly to one of the three cases: integration over two balls $D(\rho)\times D(\rho)$, integration over the complements of the balls $\mathbb{S}^n\setminus D(\rho)\times\mathbb{S}^n\setminus D(\rho)$, and the mixed term $D(\rho)\times\mathbb{S}^n\setminus D(\rho)$. Which after a careful comparison of the energies gives the desired conclusion.

The rescaling is performed in the following proposition that might be of independent interest.

Proposition 5.6. *Let $v:\mathbb{S}^n\to\mathbb{R}^M$, $n\geqslant 1$ and $\lambda>0$.*

If $n=1$, then we let $\tau:\mathbb{R}\to\mathbb{S}^1$ to be the inverse stereographic projection, namely,

$$\tau(r):=\left(\frac{2r}{r^2+1},\frac{r^2-1}{r^2+1}\right)$$

and set $v_\lambda:=v(\tau(\lambda\tau^{-1}(x)))$.

If $n\geqslant 2$, then we write $v=v(r,\omega)$, $r>0$, $\omega\in S^{n-1}$ in terms of the usual stereographic projection (see below) of the punctured sphere $\mathbb{S}^n\setminus\{N\}$, where $N:=(1,0,\dots,0)\in\mathbb{R}^n$ is the north pole. In this case, we set $v_\lambda:=v(\lambda r,\omega)$.

In both cases, for $r_\lambda:=\sqrt{\frac{4}{\lambda^2+1}}$, we have

$$\begin{aligned}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|v_\lambda(x)-v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y&\leqslant(2\lambda)^{n\left(\frac{t}{s}-1\right)}\int_{D(S,r_\lambda)}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad+\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)}\int_{\mathbb{S}^n\setminus D(S,r_\lambda)}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}\tag{5.7}$$

where $S=(-1,0,\dots,0)\in\mathbb{S}^n$ is the south pole.

Remark 5.7. The analog of (5.7) for $W^{1,p}$ is

$$\int_{\mathbb{S}^n}|\nabla v_\lambda|^p\,\mathrm{d}\mathcal{H}^n\leqslant(2\lambda)^{p-n}\int_{D(S,r_\lambda)}|\nabla v|^p\,\mathrm{d}x+\left(\frac{2}{\lambda}\right)^{p-n}\int_{\mathbb{S}^n\setminus D(S,r_\lambda)}|\nabla v|^p\,\mathrm{d}x,\tag{5.8}$$

We will sketch the proof of this in Section 5.2.

The proofs of Proposition 5.6 are only slightly different for $n=1$ and $n\geqslant 2$. However, since they are very technical, we give both of them in full detail.

Proof of Proposition 5.6 for $n=1$. Recall that for $\lambda>0$, we have set

$$v_\lambda(x):=v(\tau(\lambda\tau^{-1}(x))).$$

Here $\tau : \mathbb{R} \to \mathbb{S}^1$ is the inverse stereographic projection, namely,

$$\tau(r) := \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1} \right).$$

Observe that

$$|\tau(r) - \tau(R)|^2 = (R-r)^2 \frac{2}{R^2+1} \frac{2}{r^2+1},$$

from which one obtains

$$|\tau'(r)| = \frac{2}{r^2+1}.$$

Then, changing the variables, we compute

$$\begin{aligned}
[v_\lambda]^{\frac{1}{s}}_{W^{t,\frac{1}{s}}(\mathbb{S}^1)} &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|v_\lambda(x) - v_\lambda(y)|^{\frac{1}{s}}}{|x-y|^{1+\frac{t}{s}}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(\tau(\lambda r)) - v(\tau(\lambda R))|^{\frac{1}{s}}}{|\tau(r) - \tau(R)|^{1+\frac{t}{s}}} |\tau'(R)||\tau'(r)| \,\mathrm{d}R\,\mathrm{d}r \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(\tau(\tilde{r})) - v(\tau(\tilde{R}))|^{\frac{1}{s}}}{|\tau(\lambda^{-1}\tilde{r}) - \tau(\lambda^{-1}\tilde{R})|^{1+\frac{t}{s}}} |\tau'(\lambda^{-1}\tilde{R})||\tau'(\lambda^{-1}\tilde{r})|\lambda^{-2} \,\mathrm{d}\tilde{R}\,\mathrm{d}\tilde{r} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(\tau(\tilde{r})) - v(\tau(\tilde{R}))|^{\frac{1}{s}}}{|\tau(\tilde{r}) - \tau(\tilde{R})|^{1+\frac{t}{s}}} |\tau'(\tilde{R})||\tau'(\tilde{r})| K_\lambda(\tilde{r},\tilde{R}) \,\mathrm{d}\tilde{R}\,\mathrm{d}\tilde{r},
\end{aligned}$$

where

$$\begin{aligned}
K_\lambda(\tilde{r},\tilde{R}) &:= \lambda^{-2} \left(\frac{|\tau(\tilde{r}) - \tau(\tilde{R})|}{|\tau(\lambda^{-1}\tilde{r}) - \tau(\lambda^{-1}\tilde{R})|} \right)^{1+\frac{t}{s}} \frac{|\tau'(\lambda^{-1}\tilde{R})||\tau'(\lambda^{-1}\tilde{r})|}{|\tau'(\tilde{R})||\tau'(\tilde{r})|} \\
&= \lambda^{\frac{t}{s}-1} \left(\frac{((\lambda^{-1}\tilde{R})^2+1)((\lambda^{-1}\tilde{r})^2+1)}{\left(\tilde{R}^2+1\right)\left(\tilde{r}^2+1\right)} \right)^{\frac{\frac{t}{s}-1}{2}} \\
&= \left(\frac{\tilde{r}^2+\lambda^2}{\lambda\left(\tilde{r}^2+1\right)} \right)^{\frac{\frac{t}{s}-1}{2}} \left(\frac{\tilde{R}^2+\lambda^2}{\lambda\left(\tilde{R}^2+1\right)} \right)^{\frac{\frac{t}{s}-1}{2}}.
\end{aligned}$$

Observe that for $|\tilde{r}| \leqslant \lambda$, we have

$$\frac{\tilde{r}^2+\lambda^2}{\lambda\left(\tilde{r}^2+1\right)} \leqslant 2\lambda$$

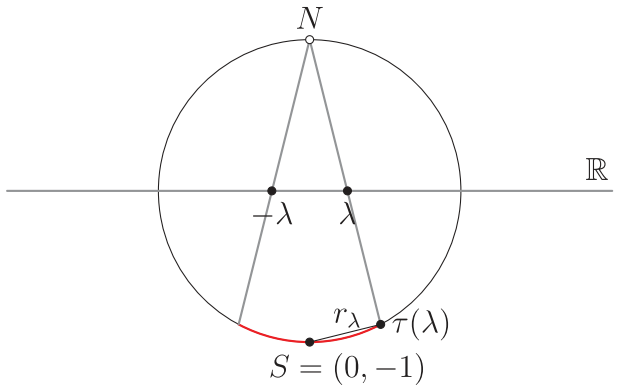
and for $|\tilde{r}| \geqslant \lambda$,

$$\frac{\tilde{r}^2+\lambda^2}{\lambda\left(\tilde{r}^2+1\right)} \leqslant \frac{2}{\lambda}.$$

Thus,

$$K_\lambda(\tilde r,\tilde R)\leqslant\begin{cases}(2\lambda)^{\frac{t}{s}-1} & |\tilde r|\leqslant\lambda,\ |\tilde R|\leqslant\lambda\\ \frac12(2\lambda)^{\frac{t}{s}-1}+\frac12\left(\frac{2}{\lambda}\right)^{\frac{t}{s}-1} & |\tilde r|\geqslant\lambda,\ |\tilde R|\leqslant\lambda \text{ or } |\tilde r|\leqslant\lambda,\ |\tilde R|\geqslant\lambda\\ \left(\frac{2}{\lambda}\right)^{\frac{t}{s}-1} & |\tilde r|\geqslant\lambda,\ |\tilde R|\geqslant\lambda.\end{cases}$$

In the case where $|\tilde r|\geqslant\lambda$ and $|\tilde R|\leqslant\lambda$, we have used the inequality $2ab\leqslant a^2+b^2$.



Observe that $\tau((-\lambda,\lambda))=D(S,r_\lambda)$ for $r_\lambda:=\sqrt{\frac{4\lambda^2}{\lambda^2+1}}$ and $\tau^{-1}(\mathbb{R}\setminus(-\lambda,\lambda))=D(S,r_\lambda)$, where $S=(0,-1)$ is the south pole of $\mathbb{S}^1$.

We thus conclude

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\frac{|v_\lambda(x)-v_\lambda(y)|^{\frac1s}}{|x-y|^{1+\frac ts}}\,\mathrm{d}x\,\mathrm{d}y\leqslant(2\lambda)^{\frac ts-1}\int_{D(S,r_\lambda)}\int_{\mathbb{S}^1}\frac{|v(x)-v(y)|^{\frac1s}}{|x-y|^{1+\frac ts}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+\left(\frac{2}{\lambda}\right)^{\frac ts-1}\int_{\mathbb{S}^1\setminus D(S,r_\lambda)}\int_{\mathbb{S}^1}\frac{|v(x)-v(y)|^{\frac1s}}{|x-y|^{1+\frac ts}}\,\mathrm{d}x\,\mathrm{d}y.$$

That is, (5.7) is established and the proof of Proposition 5.6 for $n=1$ is finished. □

Proof of Proposition 5.6 for $n\geqslant 2$. We begin with introducing spherical coordinates. Since we are dealing with a double integral, we will need separate coordinates to represent a point $x=(x_1,x_2\dots,x_{n+1})\in\mathbb{S}^n$ and to represent a point $y=(y_1,y_2,\dots y_{n+1})\in\mathbb{S}^n$. For x, we will use the coordinates

$$(\varphi,\omega),\quad \text{where } \varphi\in(0,\pi),\ \omega\in\mathbb{S}^{n-1}\quad \text{or}$$
$$(\varphi,\omega_2,\dots\omega_n),\quad \text{where } \varphi,\ \omega_2,\ \dots,\ \omega_{n-1}\in(0,\pi),\ \omega_n\in(0,2\pi),$$

whereas for y, we will use

$$(\psi,\theta),\quad \text{where } \psi\in(0,\pi),\ \theta\in\mathbb{S}^{n-1}\quad \text{or}$$
$$(\psi,\theta_2,\dots,\theta_n)\quad \text{where } \psi,\ \theta_2,\dots,\theta_{n-1}\in(0,\pi),\ \theta_n\in(0,2\pi).$$

The spherical coordinates are given by

$$\begin{aligned}
x_1 &= \cos\varphi, & y_1 &= \cos\psi,\\
x_2 &= \sin\varphi\cos\omega_2, & y_2 &= \sin\psi\cos\theta_2,\\
x_3 &= \sin\varphi\sin\omega_2\cos\omega_3, & y_3 &= \sin\psi\sin\theta_2\cos\theta_3,\\
&\vdots & &\vdots\\
x_n &= \sin\varphi\sin\omega_2\dots\sin\omega_{n-1}\cos\omega_n, & y_n &= \sin\psi\sin\theta_2\dots\sin\theta_{n-1}\cos\theta_n,\\
x_{n+1} &= \sin\varphi\sin\omega_2\dots\sin\omega_{n-1}\sin\omega_n, & y_{n+1} &= \sin\psi\sin\theta_2\dots\sin\theta_{n-1}\sin\theta_n.
\end{aligned}$$

We recall that the volume element is given by

$$\begin{aligned}
\mathrm{d}x &= \sin^{n-1}(\varphi)\sin^{n-2}(\omega_2)\dots\sin(\omega_{n-1})\,\mathrm{d}\varphi\,\mathrm{d}\omega_2\,\dots\,\mathrm{d}\omega_n = \sin^{n-1}(\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\omega\\
\mathrm{d}y &= \sin^{n-1}(\psi)\sin^{n-2}(\theta_2)\dots\sin(\theta_{n-1})\,\mathrm{d}\psi\,\mathrm{d}\theta_2\,\dots\,\mathrm{d}\theta_n = \sin^{n-1}(\psi)\,\mathrm{d}\psi\,\mathrm{d}\theta.
\end{aligned}$$

Now let us compute the squared distance $|x-y|^2$ in spherical coordinates

$$\begin{aligned}
|x-y|^2 &= \sum_{i=1}^{n+1}(x_i-y_i)^2\\
&= (\cos\varphi-\cos\psi)^2 + (\sin\varphi\cos\omega_2-\sin\psi\cos\theta_2)^2+\cdots\\
&\quad\dots+(\sin\varphi\sin\omega_2\dots\sin\omega_{n-1}\sin\omega_n-\sin\psi\sin\theta_2\dots\sin\theta_{n-1}\sin\theta_n)^2\\
&= 2-2(\cos\varphi\cos\psi+\sin\varphi\sin\psi f(\omega,\theta)),
\end{aligned}$$

where $f(\omega,\theta)$ does not depend on φ and ψ, and is the sum of the remaining elements. We recall that the stereographic projection of the punctured sphere $\mathbb{S}^n\setminus\{N\}$, where $N:=(1,0,\dots,0)\in\mathbb{R}^n$ is the north pole onto $\mathbb{R}^n$, is given by

$$(\varphi,\omega)=\Big(2\arctan\frac{1}{r},\omega\Big),\qquad(\psi,\theta)=\Big(2\arctan\frac{1}{R},\theta\Big),$$

where (r,ω) and (R,θ) are polar coordinates on $\mathbb{R}^n$ with $r,\,R>0$, and $\omega,\,\theta\in\mathbb{S}^{n-1}$. We also recall that

$$\begin{aligned}
\sin\varphi &= \frac{2r}{r^2+1}, \quad \frac{\partial\varphi}{\partial r} = -\frac{2}{r^2+1}, \quad \cos\varphi=\frac{r^2-1}{r^2+1},\\
\sin\psi &= \frac{2R}{R^2+1}, \quad \frac{\partial\psi}{\partial R} = -\frac{2}{R^2+1}, \quad \cos\psi=\frac{R^2-1}{R^2+1}.
\end{aligned}\tag{5.9}$$

Let $v\in W^{t,\frac{n}{s}}(\mathbb{S}^n)$, we will compute its energy

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2\left(\frac{n}{2}\left(1+\frac{t}{s}\right)\right)}}\,\mathrm{d}x\,\mathrm{d}y$$

in polar coordinates. By a change of variable and using (5.9), we get

$$\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2\left(\frac{n}{2}\left(1+\frac{t}{s}\right)\right)}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_{\mathbb{S}^{n-1}}\int_0^{\pi}\int_{\mathbb{S}^{n-1}}\int_0^{\pi}\frac{|v(\varphi,\omega)-v(\psi,\theta)|^{\frac{n}{s}}}{(2-2(\cos\varphi\cos\psi+\sin\varphi\sin\psi f(\omega,\theta)))^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\times\sin^{n-1}(\varphi)\sin^{n-1}(\psi)\,\mathrm{d}\varphi\,\mathrm{d}\omega\,\mathrm{d}\psi\,\mathrm{d}\theta\\
&=\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\frac{|v(r,\omega)-v(R,\theta)|^{\frac{n}{s}}}{\left(\left(\frac{2}{r^2+1}\frac{2}{R^2+1}\right)\left(r^2+R^2+2rR\,f(\omega,\theta)\right)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\qquad\left(\frac{2r}{r^2+1}\right)^{n-1}\left(\frac{2R}{R^2+1}\right)^{n-1}\frac{2}{r^2+1}\frac{2}{R^2+1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}R\,\mathrm{d}\theta\\
&=\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\frac{|v(r,\omega)-v(R,\theta)|^{\frac{n}{s}}}{\left(\left(\frac{2}{r^2+1}\frac{2}{R^2+1}\right)\left(r^2+R^2+2rR\,f(\omega,\theta)\right)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\qquad\left(\frac{2}{r^2+1}\right)^{n}\left(\frac{2}{R^2+1}\right)^{n}r^{n-1}R^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}R\,\mathrm{d}\theta\\
&=\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\frac{|v(r,\omega)-v(R,\theta)|^{\frac{n}{s}}}{\left(r^2+R^2+2rR\,f(\omega,\theta)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\qquad\left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}r^{n-1}R^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}R\,\mathrm{d}\theta.
\end{aligned}$$

In the latter, we denote the integrand by

$$|v(r,R,\omega,\theta)|_{t,\frac{n}{s}}:=\frac{|v(r,\omega)-v(R,\theta)|^{\frac{n}{s}}}{\left(r^2+R^2+2rR\,f(\omega,\theta)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}r^{n-1}R^{n-1}.$$

Thus, with this notation

$$\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{2\left(\frac{n}{2}\left(1+\frac{t}{s}\right)\right)}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\int_{\mathbb{S}^{n-1}}\int_0^{\infty}\int_{\mathbb{S}^{n-1}}\int_0^{\infty}|v(r,R,\omega,\theta)|_{t,\frac{n}{s}}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}R\,\mathrm{d}\theta.
\end{aligned}\tag{5.10}$$

We consider the rescaling $v_\lambda(r,\omega) = v(\lambda r, \omega)$ and compute

$$\begin{aligned}
&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|v_\lambda(x)-v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{2\left(\frac{n}{2}\left(1+\frac{t}{s}\right)\right)}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{\mathbb{S}^{n-1}}\int_0^\infty \int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{|v(\lambda r,\omega)-v(\lambda R,\theta)|^{\frac{n}{s}}}{\left(r^2+R^2+2rR\,f(\omega,\theta)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\qquad\qquad \left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)} r^{n-1}R^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}R\,\mathrm{d}\theta\\
&= \int_{\mathbb{S}^{n-1}}\int_0^\infty \int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{|v(\tilde r,\omega)-v(\tilde R,\theta)|^{\frac{n}{s}}}{\lambda^{-n\left(1+\frac{t}{s}\right)}\left(\tilde r^2+\tilde R^2+2\tilde r\tilde R\,f(\omega,\theta)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\qquad\qquad \left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{\tilde r}{\lambda}\right)^{n-1}\left(\frac{\tilde R}{\lambda}\right)^{n-1}\frac{1}{\lambda^2}\,\mathrm{d}\tilde r\,\mathrm{d}\omega\,\mathrm{d}\tilde R\,\mathrm{d}\theta\\
&= \int_{\mathbb{S}^{n-1}}\int_0^\infty \int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{|v(\tilde r,\omega)-v(\tilde R,\theta)|^{\frac{n}{s}}}{\left(\tilde r^2+\tilde R^2+2\tilde r\tilde R\,f(\omega,\theta)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}}\\
&\qquad\qquad \left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\tilde r^{n-1}\tilde R^{n-1}\lambda^{n\left(\frac{t}{s}-1\right)}\,\mathrm{d}\tilde r\,\mathrm{d}\omega\,\mathrm{d}\tilde R\,\mathrm{d}\theta.
\end{aligned} \tag{5.11}$$

We compute

$$\begin{aligned}
&\left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\lambda^{n\left(\frac{t}{s}-1\right)}\\
&= \left(\frac{2}{\tilde r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{\tilde R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{\tilde r^2+1}\right)^{-\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{\tilde R^2+1}\right)^{-\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\\
&\quad\left(\frac{2}{R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\lambda^{n\left(\frac{t}{s}-1\right)}\\
&= \left(\frac{2}{\tilde r^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{2}{\tilde R^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)}\left(\frac{\tilde r^2+\lambda^2}{\lambda(\tilde r^2+1)}\right)^{\frac{n}{2}\left(\frac{t}{s}-1\right)}\left(\frac{\tilde R^2+\lambda^2}{\lambda(\tilde R^2+1)}\right)^{\frac{n}{2}\left(\frac{t}{s}-1\right)}.
\end{aligned} \tag{5.12}$$

Combining (5.11) with (5.12), we get

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|v_\lambda(x)-v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{2\left(\frac{n}{2}\left(1+\frac{t}{s}\right)\right)}}\,\mathrm{d}x\,dy$$

$$
\begin{aligned}
&= \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{|v(\tilde{r},\omega) - v(\tilde{R},\theta)|^{\frac{n}{s}}}{\left(\tilde{r}^2 + \tilde{R}^2 + 2\tilde{r}\tilde{R}\, f(\omega,\theta)\right)^{\frac{n}{2}\left(1+\frac{t}{s}\right)}} \\
&\times \left(\frac{2}{\tilde{r}^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)} \left(\frac{2}{\tilde{R}^2+1}\right)^{\frac{n}{2}\left(1-\frac{t}{s}\right)} \tilde{r}^{n-1}\tilde{R}^{n-1} \\
&\times \left(\left(\frac{\tilde{r}^2+\lambda^2}{\lambda(\tilde{r}^2+1)}\right)^{\frac{n}{2}\left(\frac{t}{s}-1\right)} \left(\frac{\tilde{R}^2+\lambda^2}{\lambda(\tilde{R}^2+1)}\right)^{\frac{n}{2}\left(\frac{t}{s}-1\right)} \right) \mathrm{d}\tilde{r}\, \mathrm{d}\omega\, \mathrm{d}\tilde{R}\, \mathrm{d}\theta \\
&= \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{\mathbb{S}^{n-1}} \int_0^\infty |v(\tilde{r},\tilde{R},\omega,\theta)|_{t,\frac{n}{s}}\, K_\lambda(\tilde{r},\tilde{R})\, \mathrm{d}\tilde{r}\, \mathrm{d}\omega\, \mathrm{d}\tilde{R}\, \mathrm{d}\theta, \qquad (5.13)
\end{aligned}
$$

where

$$
K_\lambda(\tilde{r},\tilde{R}) = \left(\left(\frac{\tilde{r}^2+\lambda^2}{\lambda(\tilde{r}^2+1)}\right)^{\frac{n}{2}\left(\frac{t}{s}-1\right)} \left(\frac{\tilde{R}^2+\lambda^2}{\lambda(\tilde{R}^2+1)}\right)^{\frac{n}{2}\left(\frac{t}{s}-1\right)} \right).
$$

As in the one-dimensional case, we have:
If $\tilde{r} \leqslant \lambda$, then

$$
\frac{\tilde{r}^2+\lambda^2}{\lambda(\tilde{r}^2+1)} \leqslant 2\lambda.
$$

If $\tilde{r} \geqslant \lambda$, then

$$
\frac{\tilde{r}^2+\lambda^2}{\lambda(\tilde{r}^2+1)} \leqslant \frac{2}{\lambda}.
$$

Thus,

$$
K_\lambda(\tilde{r},\tilde{R}) \leqslant \begin{cases} (2\lambda)^{n\left(\frac{t}{s}-1\right)} & \tilde{r} \leqslant \lambda,\ \tilde{R} \leqslant \lambda \\ \frac{1}{2}(2\lambda)^{n\left(\frac{t}{s}-1\right)} + \frac{1}{2}\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)} & \tilde{r} \geqslant \lambda,\ \tilde{R} \leqslant \lambda \text{ or } \tilde{r} \leqslant \lambda,\ \tilde{R} \geqslant \lambda \\ \left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)} & \tilde{r} \geqslant \lambda,\ \tilde{R} \geqslant \lambda. \end{cases}
$$

This leads to

$$
\begin{aligned}
&\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{|v_\lambda(x) - v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{2\left(\frac{n}{2}\left(1+\frac{t}{s}\right)\right)}}\, \mathrm{d}x\, dy \\
&= \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{\mathbb{S}^{n-1}} \int_0^\infty |v(\tilde{r},\tilde{R},\omega,\theta)|_{t,\frac{n}{s}}\, K_\lambda(\tilde{r},\tilde{R})\, \mathrm{d}\tilde{r}\, \mathrm{d}\omega\, \mathrm{d}\tilde{R}\, \mathrm{d}\theta \\
&\leqslant (2\lambda)^{n\left(\frac{t}{s}-1\right)} \int_{\mathbb{S}^{n-1}} \int_0^\lambda \int_{\mathbb{S}^{n-1}} \int_0^\lambda |v(\tilde{r},\tilde{R},\omega,\theta)|_{t,\frac{n}{s}}\, \mathrm{d}\tilde{r}\, \mathrm{d}\omega\, \mathrm{d}\tilde{R}\, \mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&+\left((2\lambda)^{n\left(\frac{t}{s}-1\right)}+\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)}\right)\int_{\mathbb{S}^{n-1}}\int_0^\lambda\int_{\mathbb{S}^{n-1}}\int_\lambda^\infty |v(\tilde r,\tilde R,\omega,\theta)|_{t,\frac{n}{s}}\,\mathrm{d}\tilde r\,\mathrm{d}\omega\,\mathrm{d}\tilde R\,\mathrm{d}\theta\\
&+\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)}\int_{\mathbb{S}^{n-1}}\int_\lambda^\infty\int_{\mathbb{S}^{n-1}}\int_\lambda^\infty |v(\tilde r,\tilde R,\omega,\theta)|_{t,\frac{n}{s}}\,\mathrm{d}\tilde r\,\mathrm{d}\omega\,\mathrm{d}\tilde R\,\mathrm{d}\theta\\
=\;&(2\lambda)^{n\left(\frac{t}{s}-1\right)}\int_{\mathbb{S}^{n-1}}\int_0^\lambda\int_{\mathbb{S}^{n-1}}\int_0^\infty |v(\tilde r,\tilde R,\omega,\theta)|_{t,\frac{n}{s}}\,\mathrm{d}\tilde r\,\mathrm{d}\omega\,\mathrm{d}\tilde R\,\mathrm{d}\theta\\
&+\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)}\int_{\mathbb{S}^{n-1}}\int_\lambda^\infty\int_{\mathbb{S}^{n-1}}\int_0^\infty |v(\tilde r,\tilde R,\omega,\theta)|_{t,\frac{n}{s}}\,\mathrm{d}\tilde r\,\mathrm{d}\omega\,\mathrm{d}\tilde R\,\mathrm{d}\theta.
\end{aligned}
$$

For $r_\lambda=\sqrt{\frac{4\lambda^2}{\lambda^2+1}}$, this inequality can be rephrased as[†]

$$
\begin{aligned}
\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|v_\lambda(x)-v_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\leqslant (2\lambda)^{n\left(\frac{t}{s}-1\right)}\int_{D(S,r_\lambda)}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)}\int_{\mathbb{S}^n\setminus D(S,r_\lambda)}\int_{\mathbb{S}^n}\frac{|v(x)-v(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
\tag{5.14}
$$

where $S=(-1,0,\dots,0)\in\mathbb{S}^n$ is the south pole. □

Having Proposition 5.6 we are ready to proceed with the main theorem of this section.

Proof of Theorem 5.1. Without loss of generality, we may assume that $y_0=S$, where $S=(-1,0,\dots,0)$ is the south pole. Let $u_t\in W^{t,\frac{n}{s}}(\mathbb{S}^n)$ be the minimizing map from the assumptions of this Theorem. For $\lambda>0$, take $(u_t)_\lambda$ from Proposition 5.6. Observe that $\lambda\mapsto(u_t)_\lambda$ is a homotopy.

Since u_t is a minimizer, we can compare the energies of u_t and $(u_t)_\lambda$.

$$
\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y\leqslant\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{|(u_t)_\lambda(x)-(u_t)_\lambda(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \tag{5.15}
$$

Combining (5.15) and (5.7), we obtain

$$
\begin{aligned}
0\leqslant&((2\lambda)^{n\left(\frac{t}{s}-1\right)}-1)\int_{D(S,r_\lambda)}\int_{\mathbb{S}^n}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\
&+\left(\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)}-1\right)\int_{\mathbb{S}^n\setminus D(S,r_\lambda)}\int_{\mathbb{S}^n}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

[†] The circle centered at the origin of radius λ in polar coordinates corresponds to the circle of radius $r_\lambda=\sqrt{\frac{4\lambda^2}{\lambda^2+1}}$ with center at S in Euclidean coordinates. Indeed, one can compute it from the law of cosines: $r_\lambda^2=2-2\cos(\pi-\varphi_\lambda)$, where $\varphi_\lambda=2\arctan\frac{1}{\lambda}$.

For $\lambda < 2$, the expression

$$C_{\lambda,t} := \frac{\left(\frac{2}{\lambda}\right)^{n\left(\frac{t}{s}-1\right)} - 1}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}}$$

is positive for any $t \in (s, s_0]$. Thus, for any $\lambda < 2$,

$$\int_{D(S,r_\lambda)} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C_{\lambda,t} \int_{\mathbb{S}^n\setminus D(S,r_\lambda)} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y.$$

We need to study the asymptotics of $C_{\lambda,t}$. Let $\lambda \in [0, \lambda_0]$ for some $\lambda_0 < 2$. We have

$$C_{\lambda,t} = \lambda^{-n\left(\frac{t}{s}-1\right)} \frac{2^{n\left(\frac{t}{s}-1\right)} - \lambda^{n\left(\frac{t}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}} \leqslant \lambda^{-n\left(\frac{t}{s}-1\right)} \bar{C}_{\lambda_0,s_0},$$

where we set

$$\bar{C}_{\lambda_0,s_0} := \max_{t\in[s,s_0],\lambda\in[0,\lambda_0]} \frac{2^{n\left(\frac{t}{s}-1\right)} - \lambda^{n\left(\frac{t}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}}.$$

We need to show that $\bar{C}_{\lambda_0,s_0} < \infty$. Since $(t,\lambda) \mapsto \lambda^{n(\frac{t}{s}-1)} C_{\lambda,t}$ is continuous in $[s, s_0] \times [0, \lambda_0]$, we only need to estimate $\lambda^{n(\frac{t}{s}-1)} C_{\lambda,t}$ at the asymptotic boundary of $[s, s_0] \times [0, \lambda_0]$.

First, we observe

$$\sup_{\lambda\in[0,\lambda_0]} \lim_{t\to s^+} \frac{2^{n\left(\frac{t}{s}-1\right)} - \lambda^{n\left(\frac{t}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}} = \sup_{\lambda\in[0,\lambda_0]} \frac{\log\left(\frac{2}{\lambda}\right)}{\log(2\lambda)} \in [0,1].$$

Also,

$$\sup_{\lambda\in[0,\lambda_0]} \lim_{t\to s_0^-} \frac{2^{n\left(\frac{t}{s}-1\right)} - \lambda^{n\left(\frac{t}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}} = \sup_{\lambda\in[0,\lambda_0]} \frac{2^{n\left(\frac{s_0}{s}-1\right)} - \lambda^{n\left(\frac{s_0}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{s_0}{s}-1\right)}} < \infty.$$

Next,

$$\sup_{t\in[s,s_0]} \lim_{\lambda\to 0} \frac{2^{n\left(\frac{t}{s}-1\right)} - \lambda^{n\left(\frac{t}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}} = \sup_{t\in[s,s_0]} 2^{n\left(\frac{t}{s}-1\right)} = 2^{n\left(\frac{s_0}{s}-1\right)},$$

and it is easy to see that

$$\sup_{t\in[s,s_0]} \lim_{\lambda\to\lambda_0} \frac{2^{n\left(\frac{t}{s}-1\right)} - \lambda^{n\left(\frac{t}{s}-1\right)}}{1-(2\lambda)^{n\left(\frac{t}{s}-1\right)}} < \infty.$$

In conclusion, we have shown that for any $\lambda_0 < 2$, $s_0 \in (s,1)$ for any $\lambda \in [0,\lambda_0]$, $t \in (s,s_0]$

$$\int_{D(S,r_\lambda)} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C_{\lambda_0,s_0}\lambda^{-n\left(\frac{t}{s}-1\right)} \int_{\mathbb{S}^n\setminus D(S,r_\lambda)} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y.$$

So for any $\rho_0 < \sqrt{\frac{4}{5}}$, let $\lambda_0 < 2$ be such that $\rho_0 = \frac{2\lambda_0}{\sqrt{\lambda_0^2+1}}$. For any $\rho \in (0,\rho_0)$, there exists $\lambda \in (0,\lambda_0)$ such that $\rho = \frac{2\lambda}{\sqrt{\lambda^2+1}}$. We then have $\lambda = \rho\frac{\sqrt{\lambda^2+1}}{2}$ and $0 \leqslant \frac{\sqrt{\lambda^2+1}}{2} \leqslant \sqrt{\frac{5}{4}}$. This gives

$$\int_{D(S,\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C_{\lambda_0,s_0}\rho^{-n\left(\frac{t}{s}-1\right)} \int_{\mathbb{S}^n\setminus D(S,\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{tn}{s}}}\,\mathrm{d}x\,\mathrm{d}y.$$

This finishes the proof of Theorem 5.1. □

5.2 | Proof of Theorem 5.2

The proof of Theorem 5.2 is very similar to the proof of Theorem 5.1, and a consequence of (5.8). For the convenience of the reader, we state the proof of the latter below.

Proof of (5.8). Let (φ,θ) be the spherical coordinates on $\mathbb{S}^n$, where $\varphi \in (0,\pi)$ and $\theta \in \mathbb{S}^{n-1}$.

Let (R,Θ) be the polar coordinates on $\mathbb{R}^n$, where $R > 0$, $\Theta \in \mathbb{S}^{n-1}$.

The stereographic projection $\Phi : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ from the point N and the inverse stereographic projection $\Phi^{-1} : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$ are given by:

$$\Phi(\varphi,\theta) = (R,\Theta) = \left(\cot\frac{\varphi}{2},\theta\right),$$

$$\Phi^{-1}(R,\Theta) = (\varphi,\theta) = \left(2\arctan\frac{1}{R},\Theta\right).$$

We define the rescaled energy

$$v_\lambda(\varphi,\theta) = v(\varphi_\lambda,\theta),$$

where $\varphi_\lambda = 2\arctan(\frac{1}{\lambda R})$, and $R = \cot(\varphi/2)$. Note that

$$\sin\varphi = \frac{2R}{(R^2+1)}, \quad \frac{\partial\varphi}{\partial R} = -\frac{2}{R^2+1}, \quad v_\varphi = -\frac{(R^2+1)}{2}v_R, \quad v_\theta = v_\Theta.$$

Let $v \in W^{1,p}(\mathbb{S}^n)$ be any map. Its energy in polar coordinates is given by the formula:

$$\begin{aligned}\int_{\mathbb{S}^n} |\nabla v|^p\,\mathrm{d}\mathcal{H}^n &= \int_{\mathbb{S}^{n-1}}\int_0^\pi \left(|v_\varphi|^2 + \sin^{-2}\varphi|v_\theta|^2\right)^{\frac{p}{2}} \sin^{n-1}\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta \\ &= \int_{\mathbb{S}^{n-1}}\int_0^\infty \left(\left||v_R|^2 + R^{-2}|v_\Theta|^2\right|\right)^{\frac{p}{2}} \left(\frac{R^2+1}{2}\right)^{p-n} R^{n-1}\,\mathrm{d}R\,\mathrm{d}\Theta.\end{aligned}$$

Now the energy of the rescaled v is

$$\begin{aligned}\int_{\mathbb{S}^n} |\nabla v_\lambda|^p \,\mathrm{d}\mathcal{H}^n &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \left(\left|(v_\lambda)_R\right|^2 + R^{-2} \left|(v_\lambda)_\Theta\right|^2 \right)^{\frac{p}{2}} \left(\frac{R^2+1}{2} \right)^{p-n} R^{n-1} \,\mathrm{d}R \,\mathrm{d}\Theta \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \left(\left| \lambda \frac{\partial v}{\partial R}(\lambda R, \Theta) \right|^2 + \frac{1}{R^2} \left| \frac{\partial v}{\partial \Theta}(\lambda R, \Theta) \right|^2 \right)^{\frac{p}{2}} \left(\frac{R^2+1}{2} \right)^{p-n} R^{n-1} \,\mathrm{d}R \,\mathrm{d}\Theta \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \left(\left| \frac{\partial v}{\partial \tilde{R}} \right|^2 + \frac{1}{\tilde{R}^2} \left| \frac{\partial v}{\partial \Theta} \right|^2 \right)^{\frac{p}{2}} \left(\frac{\tilde{R}^2+\lambda^2}{(\tilde{R}^2+1)\lambda} \right)^{p-n} \left(\frac{\tilde{R}^2+1}{2} \right)^{p-n} \tilde{R}^{n-1} \,\mathrm{d}\tilde{R} \,\mathrm{d}\Theta. \end{aligned} \tag{5.16}$$

Now we separate this into two parts: $R < \lambda$, $R > \lambda$.

For $R \leqslant \lambda$, we have

$$\frac{\tilde{R}^2+\lambda^2}{\lambda(\tilde{R}^2+1)} \leqslant 2\lambda,$$

and for $R \geqslant \lambda$

$$\frac{\tilde{R}^2+\lambda^2}{\lambda(\tilde{R}^2+1)} \leqslant \frac{2}{\lambda}.$$

This gives us (5.8), namely,

$$\begin{aligned}\int_{\mathbb{S}^n} |\nabla v_\lambda|^p \,\mathrm{d}\mathcal{H}^n &\leqslant (2\lambda)^{p-n} \int_{\mathbb{S}^{n-1}} \int_0^\lambda \left(\left| \frac{\partial v}{\partial \tilde{R}} \right|^2 + \frac{1}{\tilde{R}^2} \left| \frac{\partial v}{\partial \Theta} \right|^2 \right)^{\frac{p}{2}} \left(\frac{\tilde{R}^2+1}{2} \right)^{p-n} \tilde{R}^{n-1} \,\mathrm{d}\tilde{R} \,\mathrm{d}\Theta \\ &\quad + \left(\frac{2}{\lambda} \right)^{p-n} \int_{\mathbb{S}^{n-1}} \int_\lambda^\infty \left(\left| \frac{\partial v}{\partial \tilde{R}} \right|^2 + \frac{1}{\tilde{R}^2} \left| \frac{\partial v}{\partial \Theta} \right|^2 \right)^{\frac{p}{2}} \left(\frac{\tilde{R}^2+1}{2} \right)^{p-n} \tilde{R}^{n-1} \,\mathrm{d}\tilde{R} \,\mathrm{d}\Theta \\ &= (2\lambda)^{p-n} \int_{D(S,r_\lambda)} |\nabla v|^p \,\mathrm{d}x + \left(\frac{2}{\lambda} \right)^{p-n} \int_{\mathbb{S}^n \setminus D(S,r_\lambda)} |\nabla v|^p \,\mathrm{d}x, \end{aligned} \tag{5.17}$$

where $S = (-1, 0, \dots, 0) \in \mathbb{S}^n$ is the south pole and $r_\lambda = \sqrt{\frac{4\lambda^2}{\lambda^2+1}}$. □

6 | EXISTENCE OF $W^{S,\frac{N}{S}}(\Sigma, \mathcal{N})$

The following theorem is a generalization of [96, Theorem 5.1].

Theorem 6.1. *Let $n \geqslant 2$ and $s \in (0,1)$ or $n = 1$ and $s \leqslant \frac{1}{2}$. Let $\mathcal{N}$ be compact, $\pi_n(\mathcal{N}) = 0$, and let Σ be as before. Then there exists a minimizing $W^{t,\frac{n}{s}}$-harmonic map in every homotopy class of $C^0(\Sigma, \mathcal{N})$ for any $t \in [s, 1)$.*

The assumption $\pi_n(\mathcal{N}) = 0$ cannot be dropped as shown in an example by Eells and Wood [35]:

Theorem 6.2. *There exists no harmonic map of degree one from $\mathbb{T}^2$ to $\mathbb{S}^2$.*

For a proof, that the infimum in Theorem 6.2 may not be attained in every homotopy class, see also [67, (9.2) Proposition].

Proof of Theorem 6.1. Fix a homotopy class $X \subset C^0(\Sigma, \mathcal{N})$.

The statement for $t > s$ is clear.

Let u_t be the minimizing harmonic maps within X for $t \in (s, s_0)$. Here, $s_0 > s$ is taken from Theorem 3.1.

We use Theorem 3.13 to infer that there is a map $u_s \in W^{s,\frac{n}{s}}(\Sigma, \mathcal{N})$ for which on a subsequence (denoted the same),

$$u_t \xrightarrow{t\to s} u_s \quad \text{strongly in } W^{s_0,\frac{n}{s}}(\Sigma \setminus \{x_1, \dots, x_K\}).$$

Moreover, by Theorem 4.6, isolated singularities can be removed and we deduce that $u_s \in W^{s_0,\frac{n}{s}}(\Sigma, \mathcal{N})$. In order to conclude, we will show that

$$u_t \xrightarrow{t\to s} u_s \quad \text{strongly in } W^{s_0,\frac{n}{s}}(\Sigma).$$

We denote $A = \{x_1, \dots, x_K\}$. Consider t close to s. Let $x_i \in A$ and take ρ small enough, so that

$$B(x_i, 2\Lambda\rho) \cap A = \{x_i\},$$

where $\Lambda > 1$ is the number taken from Lemma A.1 so that the smallness of $E_{t,\frac{n}{s}}(v_t, B(x_i, 2\Lambda\rho)) < \frac{\varepsilon}{4}$ implies

$$\int_{B(x_i,2\rho)} \int_\Sigma \frac{|v_t(x) - v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{n}{s}}} \,\mathrm{d}x\,\mathrm{d}y < \frac{\varepsilon}{2}.$$

with $2\Lambda\rho = \lambda(2\rho)^{\frac{1}{2}}$ and $\lambda = \lambda(\mathcal{N}, n, s, \varepsilon)$.

We construct a comparison map v_t such that

$$v_t = \begin{cases} u_s & \text{in } B(x_i, \rho) \\ u_t & \text{outside of } B(x_i, 2\rho). \end{cases}$$

In order to define v_t, we let $\eta_{B(x_i,\rho)} \in C_c^\infty(B(x_i, 2\rho))$ be a standard cut-off function, such that $\eta_{B(x_i,\rho)} \equiv 1$ in $B(x_i, \rho)$. We claim that for all $x \in \Sigma$ and t sufficiently close to s, we have

$$\operatorname{dist}\left((1 - \eta_{B(x_i,\rho)})u_t(x) + \eta_{B(x_i,\rho)}u_s(x), \mathcal{N}\right) \ll 1. \tag{6.1}$$

This is true, because for x outside of $B(x_i, 2\rho)$ and for $x \in B(x_i, \rho)$, the distance is zero. On the remaining annulus $B(x_i, 2\rho) \setminus B(x_i, \rho)$, we have $W^{s_0,\frac{n}{s}}$ and uniform convergence of u_t to u_s, and thus taking t sufficiently close to s, we have (6.1). Therefore, the map

$$v_t := \begin{cases} u_s(x) & \text{for } x \in B(x_i, \rho) \\ \pi_{\mathcal{N}}\Big((1 - \eta_{B(x_i,\rho)})u_t + \eta_{B(x_i,\rho)}u_s\Big) & \text{for } x \in B(x_i, 2\rho) \setminus B(x_i, \rho) \\ u_t(x) & \text{for } x \in \Sigma \setminus B(x_i, 2\rho) \end{cases} \tag{6.2}$$

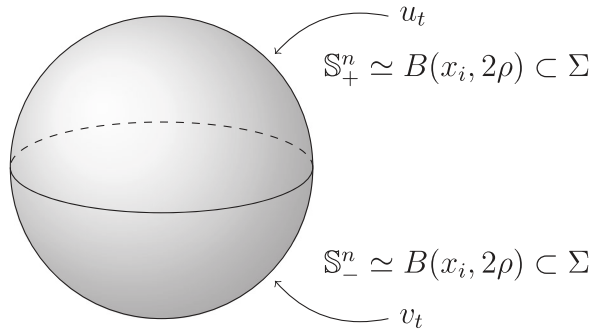
is well defined for t sufficiently close to s. We observe that $v_t \in W^{s_0,\frac{n}{s}} \cap C^0(\Sigma \cap B(x_i, 2\Lambda\rho), \mathcal{N})$. We also have

$$\lim_{t\to s^+} E_{t,\frac{n}{s}}(v_t, B(x_i, 2\Lambda\rho)) = E_{s,\frac{n}{s}}(u_s, B(x_i, 2\Lambda\rho)). \tag{6.3}$$

We observe that as $u_s \in W^{s_0,\frac{n}{s}}$, we have

$$E_{s,\frac{n}{s}}(u_s, B(x_i, 2\Lambda\rho)) \leqslant C\lambda^{n\frac{s_0-s}{s}}\rho^{n\frac{s_0-s}{2s}} E_{s_0,\frac{n}{s}}(u_s, B(x_i, 2\Lambda\rho)) = \mathcal{O}(\rho^{n\frac{s_0-s}{2s}}) \text{ as } \rho \to 0. \tag{6.4}$$

Moreover, since $\pi_n(\mathcal{N}) = \{0\}$, we find that u_t and v_t must be homotopic. Indeed, since they coincide outside of $B(x_i, 2\rho)$, we can glue two copies of $B(x_i, 2\rho)$ to an $\mathbb{S}^n$ with u_t on the upper hemisphere S^n_+ and v_t on the lower hemisphere $\mathbb{S}^n_-$ to construct a continuous map $u : \mathbb{S}^n \to \mathcal{N}$. Since $\pi_n(\mathcal{N})$ is trivial, there exists a continuous extension $U : B^{n+1} \to \mathcal{N}$, which readily leads to a homotopy of u_t and v_t on all of Σ.



As u_t is a minimizer in its homotopy class, we can compare the energies

$$E_{t,\frac{n}{s}}(u_t, \Sigma) \leqslant E_{t,\frac{n}{s}}(v_t, \Sigma). \tag{6.5}$$

Decomposing the integrals into integration over $\Sigma \setminus B(x_i, 2\rho) \times \Sigma \setminus B(x_i, 2\rho)$, $\Sigma \setminus B(x_i, 2\rho)$, $B(x_i, 2\rho) \times B(x_i, 2\rho)$, we obtain

$$\begin{aligned} &\int_{\Sigma\setminus B(x_i,2\rho)} \int_{\Sigma\setminus B(x_i,2\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Sigma \int_{B(x_i,2\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant \int_\Sigma \int_\Sigma \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} &\int_\Sigma \int_\Sigma \frac{|v_t(x) - v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant \int_{\Sigma\setminus B(x_i,2\rho)} \int_{\Sigma\setminus B(x_i,2\rho)} \frac{|v_t(x) - v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_\Sigma \int_{B(x_i,2\rho)} \frac{|v_t(x) - v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{6.7}$$

Thus, combining (6.5)–(6.7), and u_t and v_t coincide outside $B(x_i, 2\rho)$, we get

$$\int_\Sigma \int_{B(x_i,2\rho)} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant 2\int_\Sigma \int_{B(x_i,2\rho)} \frac{|v_t(x)-v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \tag{6.8}$$

From (6.3) and (6.4), we know that for t sufficiently close to s, we have

$$\int_{B(x_i,2\rho)} \int_{B(x_i,2\rho)} \frac{|v_t(x)-v_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \mathcal{O}(\rho^{n\frac{s_0-s}{2s}}) \text{ as } \rho \to 0$$

and thus, choosing ρ sufficiently small, we get from Lemma A.1

$$\int_\Sigma \int_{B(x_i,2\rho)} \frac{|v_t(x)-v_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{\varepsilon}{2}.$$

The latter inequality combined with (6.8) gives for small ρ and t close to s

$$\int_\Sigma \int_{B(x_i,2\rho)} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \varepsilon. \tag{6.9}$$

Therefore, applying the regularity result Corollary 3.12, we get on a smaller disk $u_t \to u_s$ strongly in $W^{s_0,\frac{n}{s}}(B(x_i,\rho))$.

That is, we have found that u_t converges on all of Σ to u_s uniformly, and in $W^{s_0,\frac{n}{s}}$. This readily implies that u_s is a $W^{s,\frac{n}{s}}$-minimizer in X. □

7 | EXISTENCE OF $W^{s,\frac{N}{s}}(\mathbb{S}^n, \mathcal{N})$

In this section, we assume that $\Sigma = \mathbb{S}^n$, $\pi_n(\mathcal{N}) \neq \{0\}$, and look for minimizers in the free homotopy classes of $C^0(\mathbb{S}^n, \mathcal{N})$, which we denote $\pi_0 C^0(\mathbb{S}^n, \mathcal{N})$. We prove the following theorem.

Theorem 7.1. *Let $s \in (0,1)$, $n \geqslant 2$ or $s \leqslant \frac{1}{2}$ and $n = 1$. There exists a set of free homotopy classes $X \subset \pi_0 C^0(\mathbb{S}^n, \mathcal{N})$ with the following properties:*

(1) *Each $\Gamma_i \in X$ contains a minimizing $W^{s,\frac{n}{s}}$-harmonic map.*
(2) *Elements $\Gamma_i \in X$ form a generating set for $\pi_n(\mathcal{N})$ acted on by $\pi_1(\mathcal{N})$.*

We state three corollaries of Theorem 7.1, or rather Lemma 7.7 — Theorem 7.1 main's ingredient.

Corollary 7.2. *Let $\pi_n(\mathcal{N}) \neq \{0\}$. Then there exists a nontrivial $W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})$-harmonic map.*

Proof. From Theorem 7.1, we deduce that if all homotopy classes $\Gamma \in \pi_0 C^0(\mathbb{S}^n, \mathcal{N})$ that have a $W^{s,\frac{n}{s}}$-minimizing harmonic map would be trivial, then we would obtain that the set generated by them would be trivial, thus $\pi_n(\mathcal{N}) = \{0\}$, a contradiction. Thus, there must be a nontrivial homotopy class in which there is a minimizer. □

In particular, we have the following.

Corollary 7.3. *There exists a number $k \in \mathbb{Z}$, $k \neq 0$ such that*

$$\inf \left\{ E_{s,\frac{n}{s}}(u, \mathbb{S}^n) : \ u \in C^0 \cap W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathbb{S}^n), \ \deg u = k \right\}$$

is attained.

Corollary 7.4. *Let $s \in (0,1)$, $n \geqslant 1$ and $\mathcal{N}$ as above. There exists an $\varepsilon = \varepsilon(s, n, \mathcal{N})$ such that the following holds:*

Let

$$\delta := \inf \left\{ E_{s,\frac{n}{s}}(u) : \quad u \in C^\infty(\mathbb{S}^n, \mathcal{N}), \quad u \text{ is not homotopic to a constant} \right\}.$$

Then $\delta > \varepsilon$ and moreover if $\Gamma \in \pi_0 C^0(\mathbb{S}^n, \mathcal{N})$ satisfies

$$\inf_{u \in \Gamma \cap W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})} E_{s,\frac{n}{s}}(u, \mathbb{S}^n) \leqslant \delta + \varepsilon,$$

then Γ contains an $E^{s,\frac{n}{s}}$-harmonic map.

Observe there is no a priori reason that a minimizing nontrivial homotopy class Γ_0 exists, that is, Γ_0 such that

$$\inf_{u \in \Gamma_0 \cap W^{s,\frac{n}{s}}(\mathbb{S}^n, \mathcal{N})} E_{s,\frac{n}{s}}(u, \mathbb{S}^n) = \inf \left\{ E_{s,\frac{n}{s}}(u) : \ u \in C^\infty(\mathbb{S}^n, \mathcal{N}), \ u \text{ is not homotopic to a constant} \right\}.$$

See [120, Proposition 2.4 & Theorem 1.2].

Before we begin the proof of Theorem 7.1, let us recall a few facts about free homotopies and free homotopy decomposition in terms of homotopy groups. For definitions, we refer the reader to the book [52] and for an explanation for an analyst, we refer to [12, III §17] or [120, Section 2.1]. Here, we will adopt the notation of Sacks–Uhlenbeck [96, Section 5].

Each $\gamma \in \pi_n(\mathcal{N})$ determines a free homotopy class of maps from $\mathbb{S}^n$ into $\mathcal{N}$. As free homotopy does not depend on the choice of the base point, two elements $\gamma, \gamma' \in \pi_n(\mathbb{S}^n)$ determine the same free homotopy class if and only if they belong to the same orbit

$$\pi_1(\mathcal{N})\gamma = \pi_1(\mathcal{N})\gamma'$$

under the usual action of $\pi_1(\mathcal{N})$ on $\pi_n(\mathcal{N})$. We denote by $\Gamma \in \pi_0 C^0(\mathbb{S}^n, \mathcal{N})$ the free homotopy class that corresponds to $\pi_1(\mathcal{N})\gamma$. For $\Gamma \in \pi_0 C^0(\mathbb{S}^n, \mathcal{N})$, we will denote by $\gamma \in \pi_n(\mathcal{N})$ any element for which $\pi_1(\mathcal{N})\gamma$ corresponds to Γ, we will write $\gamma \in \Gamma$.

For any $\alpha \in \pi_1(\mathcal{N})$ and $\gamma_1, \gamma_2 \in \pi_n(\mathcal{N})$, we have

$$\alpha(\gamma_1 + \gamma_2) = \alpha\gamma_1 + \alpha\gamma_2.$$

Moreover, for a given $\Gamma_i = \pi_1(\mathcal{N})\gamma_1$, for $i = 1, 2, 3$,

$$\gamma_1 + \gamma_2 = \gamma_3 \quad \Rightarrow \quad \pi_1(\mathcal{N})\gamma_3 \subset \pi_1(\mathcal{N})\gamma_1 + \pi_1(\mathcal{N})\gamma_2, \tag{7.1}$$

because for any $\alpha \in \pi_1(\mathcal{N})$, we have $\alpha\gamma_1 + \alpha\gamma_2 = \alpha\gamma_3$.

We also note that if $\pi_1(\mathcal{N})$ was trivial, then we could drop the action of $\pi_1(\mathcal{N})$ on $\pi_n(\mathcal{N})$.

For a free homotopy class $\Gamma \in \pi_0 C^0(\mathbb{S}^n, \mathcal{N})$, we write

$$\#\Gamma := \inf_{u \in \Gamma \cap W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})} E_{s,\frac{n}{s}}(u, \mathbb{S}^n).$$

The following characterization will be needed in the proof.

Lemma 7.5.

$$\#\Gamma = \lim_{t \to s^+} \inf_{u \in \Gamma \cap W^{t,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})} E_{t,\frac{n}{s}}(u, \mathbb{S}^n).$$

Proof. Let $u_t \in \Gamma$ be a minimizer in Γ for $E_{t,\frac{n}{s}}(\cdot, \mathbb{S}^n)$. Then,

$$\#\Gamma \leqslant E_{s,\frac{n}{s}}(u_t, \mathbb{S}^n) \leqslant \operatorname{diam}(\mathbb{S}^n)^{(t-s)\frac{n}{s}} E_{t,\frac{n}{s}}(u_t, \mathbb{S}^n),$$

which readily leads to

$$\#\Gamma \leqslant \liminf_{t \to s^+} \inf_{u \in \Gamma \cap W^{t,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})} E_{t,\frac{n}{s}}(u, \mathbb{S}^n).$$

On the other hand, by smooth approximation, Lemma 2.7, we can approximate u smoothly in its homotopy class, and thus, combining it with the definition, we get

$$\#\Gamma = \inf_{v \in \Gamma \cap W^{s,\frac{n}{s}} \cap C^\infty(\mathbb{S}^n,\mathcal{N})} E_{s,\frac{n}{s}}(v, \mathbb{S}^n).$$

For such a smooth $v \in \Gamma \cap W^{s,\frac{n}{s}} \cap C^\infty(\mathbb{S}^n, \mathcal{N})$,

$$\inf_{u \in \Gamma \cap W^{t,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})} E_{t,\frac{n}{s}}(u, \mathbb{S}^n) = E_{t,\frac{n}{s}}(u_t, \mathbb{S}^n) \leqslant E_{t,\frac{n}{s}}(v, \mathbb{S}^n).$$

Since $\lim_{t \to s^+} E_{t,\frac{n}{s}}(v, \mathbb{S}^n) = E_{s,\frac{n}{s}}(v, \mathbb{S}^n)$, we conclude that for any smooth $v \in \Gamma \cap W^{s,\frac{n}{s}} \cap C^\infty(\mathbb{S}^n, \mathcal{N})$,

$$\limsup_{t \to s^+} \inf_{u \in \Gamma \cap W^{t,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})} E_{t,\frac{n}{s}}(u, \mathbb{S}^n) \leqslant E_{s,\frac{n}{s}}(v, \mathbb{S}^n).$$

Taking the infimum over all $v \in \Gamma \cap W^{s,\frac{n}{s}} \cap C^\infty(\mathbb{S}^n, \mathcal{N})$,

$$\limsup_{t \to s^+} \inf_{u \in \Gamma \cap W^{t,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})} E_{t,\frac{n}{s}}(u, \mathbb{S}^n) \leqslant \#\Gamma.$$

□

Before we proceed to the proof of Theorem 7.1, we would like to note, as mentioned in the Introduction, that in the case of harmonic maps, the theorem cannot be improved in general. Futaki constructed in [39] a manifold with the following property.

Theorem 7.6. *There is a manifold $\mathcal{N}$ with the following property: there exists a homotopy component of $C^\infty(\mathbb{S}^2, \mathcal{N})$ in which there is no minimizer of the Dirichlet energy.*

Theorem 7.1 follows Lemma 7.7 below as in [96, Proof of Theorem 5.5].

Lemma 7.7. *Let s, n, $\mathcal{N}$ be as in Theorem 7.1. There exists a $\theta = \theta(s,n,\mathcal{N})$ such that the following holds.*

Let $\Gamma_0 \in \pi_0 C^0(\mathbb{S}^n,\mathcal{N})$. Then at least one of the following cases holds:

(1) *There exists a minimizer of $E_{s,\frac{n}{s}}(\cdot,\mathbb{S}^n)$ in Γ_0.*

(2) *For every $\delta > 0$, there exist nontrivial free homotopy classes $\Gamma_1 = \pi_1(\mathcal{N})\gamma_1$ and $\Gamma_2 = \pi_1(\mathcal{N})\gamma_2$, such that*

$$\Gamma_0 = \pi_1(\mathcal{N})\gamma_0 \subset \pi_1(\mathcal{N})\gamma_1 + \pi_1(\mathcal{N})\gamma_2$$

and

$$\#\Gamma_1 + \#\Gamma_2 \leqslant \#\Gamma_0 + \delta, \tag{7.2}$$

$$\theta < \#\Gamma_1 < \#\Gamma_0 - \frac{\theta}{2},$$

$$\theta < \#\Gamma_2 < \#\Gamma_0 - \frac{\theta}{2}.$$

Proof of Lemma 7.7. Let $\{u_t\}$ be a sequence of $W^{t,\frac{n}{s}}$-maps, which minimize $E_{t,\frac{n}{s}}(\cdot,\mathbb{S}^n)$ in $\Gamma_0 \cap W^{t,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})$. Similar to the proof of Theorem 6.1, using Theorem 3.13, we find that the sequence $\{u_t\}$ is uniformly bounded and using Theorem 3.13, we get that on a subsequence u_t converges to u_s strongly in $W^{s_0,\frac{n}{s}}(\mathbb{S}^n \setminus A,\mathcal{N})$, weakly in $W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})$, and locally uniformly in $\mathbb{S}^n \setminus A$, where $A = \{x_1,\dots,x_K\}$ is a set consisting of finite number of points. Moreover, by Theorem 4.6, we obtain that for an $s_0 > s$, we have $u_s \in W^{s_0,\frac{n}{s}}(\Sigma,\mathcal{N})$. Then, we have two possibilities.

Case 1: There are no blowup points. For every point $x_i \in A$ and t sufficiently close to s, there is a ρ such that

$$E_{s,\frac{n}{s}}(u_t, B(x_i,\rho)) \leqslant \varepsilon,$$

where $\varepsilon > 0$ in taken from Corollary 3.12. Then, Corollary 3.12 implies that $u_t \xrightarrow{t\to s} u_s$ in $W^{s_0,\frac{n}{s}}(B(x_i,\frac{\rho}{2}),\mathcal{N})$ and we obtain $u_t \xrightarrow{t\to s} u_s$ strongly in $W^{s_0,\frac{n}{s}}(\mathbb{S}^n,\mathcal{N})$. This implies that $u_s \in \Gamma_0$ and u_s minimizes the energy $E_{s,\frac{n}{s}}(\cdot,\mathbb{S}^n)$.

Case 2: There is a blowup point. We assume that there is a point $x_1 \in A$ such that

$$\lim_{\alpha\to\infty} \limsup_{t\to s^+} \int_{B(x_1,2^{-\alpha})} \int_{\mathbb{S}^n} \frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y \geqslant \varepsilon. \tag{7.3}$$

Similarly as in the proof of Theorem 6.1, we take a small enough radius $\rho \in (0,1)$, so that

$$B(x_1,\lambda\rho^\beta) \cap A = \{x_1\}, \tag{7.4}$$

where $\lambda = \lambda(\mathcal{N},n,s)$ and $\beta > 0$ will be chosen later (in the application of the smallness condition Lemma A.1 and Lemma A.2).

We repeat the construction in (6.2). Let $\eta_{B(x_1,\rho)} \in C_c^\infty(B(x_1,2\rho))$, $0 \leqslant \eta_{B(x_1,\rho)} \leqslant 1$, and $\eta_{B(x_1,\rho)} \equiv 1$ in $B(x_1,\rho)$. For $x \in B(x_1,2\rho)$, we define

$$\tilde{u}_{s;t} := \pi_{\mathcal{N}}((1-\eta_{B(x_1,\rho)})u_t + \eta_{B(x_1,\rho)}u_s).$$

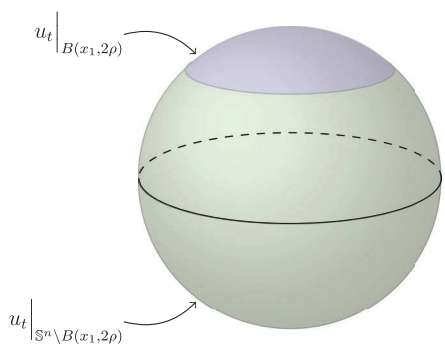
From (6.1), we know that the projection is well defined for t sufficiently close to s. Let

$$v_t := \begin{cases} u_t & \text{in } \mathbb{S}^n \setminus B(x_1, 2\rho) \\ \tilde{u}_{s;t} & \text{in } B(x_1, 2\rho) \end{cases}$$

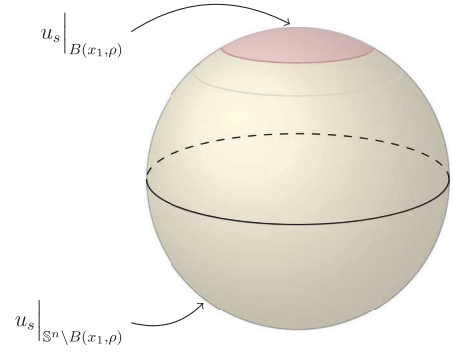
and

$$w_t := \begin{cases} \tilde{u}_{s;t} \circ \tau & \text{in } \mathbb{S}^n \setminus B(x_1, 2\rho) \\ u_t & \text{in } B(x_1, 2\rho), \end{cases}$$

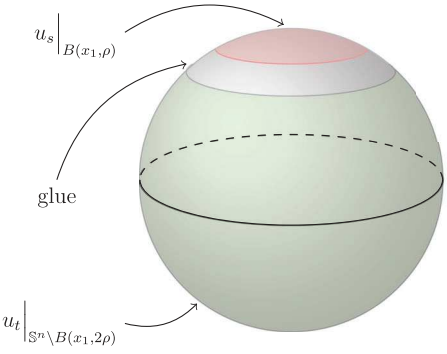
where $\tau : \mathbb{S}^n \setminus B(x_1, 2\rho) \to B(x_1, 2\rho)$ is a diffeomorphism, such that $|\nabla \tau| \simeq \frac{1}{\rho}$.



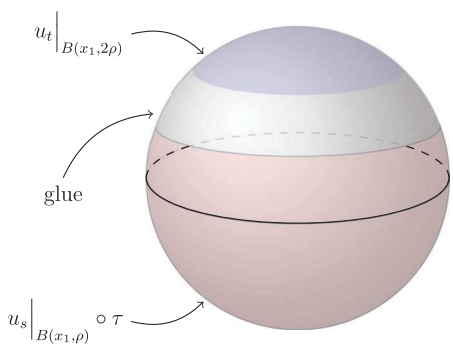
Domain of the map u_t



Domain of the map u_s



Domain of the map v_t



Domain of the map w_t

Let Γ_1, Γ_2 be the free homotopy classes determined respectively by v_t and w_t. Then, we have as in (7.1)

$$\pi_1(\mathcal{N})\gamma_0 \subset \pi_1(\mathcal{N})\gamma_1 + \pi_1(\mathcal{N})\gamma_2.$$

Step 1. We will prove that

$$\lim_{t \to s^+} \left| E_{t,\frac{n}{s}}(v_t, \mathbb{S}^n) + E_{t,\frac{n}{s}}(w_t, \mathbb{S}^n) - E_{t,\frac{n}{s}}(u_t, \mathbb{S}^n) \right| = \mathcal{O}(\rho^{n\frac{s_0 - s}{s_0 + s}}) \text{ as } \rho \to 0.$$

Indeed, to see this, we first decompose $\mathbb{S}^n \times \mathbb{S}^n$ into the compliment of the balls $\mathbb{S}^n \setminus B(x_1, 2\rho) \times \mathbb{S}^n \setminus B(x_1, 2\rho)$, the product of the balls $B(x_1, 2\rho) \times B(x_1, 2\rho)$, and the two mixed terms $B(x_1, 2\rho) \times \mathbb{S}^n \setminus B(x_1, 2\rho)$, $\mathbb{S}^n \setminus B(x_1, 2\rho) \times B(x_1, 2\rho)$. We recall that

$$v_t = u_t \quad \text{on } \mathbb{S}^n \setminus B(x_1, 2\rho),$$
$$w_t = u_t \quad \text{on } B(x_1, 2\rho).$$

Applying those observations, we get

$$\begin{aligned}
&E_{t,\frac{n}{s}}(v_t, \mathbb{S}^n) + E_{t,\frac{n}{s}}(w_t, \mathbb{S}^n) - E_{t,\frac{n}{s}}(u_t, \mathbb{S}^n) \\
&= E_{t,\frac{n}{s}}(v_t, B(x_1, 2\rho)) + 2\int_{\mathbb{S}^n \setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} \frac{|v_t(x) - v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y + E_{t,\frac{n}{s}}(w_t, \mathbb{S}^n \setminus B(x_1, 2\rho)) \\
&\quad + 2\int_{\mathbb{S}^n \setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} \frac{|w_t(x) - w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y - 2\int_{\mathbb{S}^n \setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|E_{t,\frac{n}{s}}(v_t, \mathbb{S}^n) + E_{t,\frac{n}{s}}(w_t, \mathbb{S}^n) - E_{t,\frac{n}{s}}(u_t, \mathbb{S}^n)\right| \\
&\leqslant 2\int_{\mathbb{S}^n} \int_{B(x_1,2\rho)} \frac{|v_t(x) - v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{\mathbb{S}^n} \int_{\mathbb{S}^n \setminus B(x_1,2\rho)} \frac{|w_t(x) - w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\int_{\mathbb{S}^n \setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y. \qquad (7.5)
\end{aligned}$$

We begin with the estimate of the last term in (7.5). To do so, we observe that we can decompose

$$\begin{aligned}
&\int_{\mathbb{S}^n \setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim E_{t,\frac{n}{s}}(u_t, B(x_1, 3\rho) \setminus B(x_1, \rho)) + \int_{\mathbb{S}^n \setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{B(x_1,\rho)} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y =: I_1 + I_2 + I_3. \qquad (7.6)
\end{aligned}$$

The estimate of I_2: We will first estimate the term I_2. We start with noting that for $x \in B(x_1, 2\rho)$, we have $u_t(x) = w_t(x)$ and for $y \in \mathbb{S}^n \setminus B(x_1, 3\rho)$, we have $u_t(y) = v_t(y)$, thus

$$|u_t(x) - u_t(y)|^{\frac{n}{s}} = |w_t(x) - v_t(y)|^{\frac{n}{s}} \lesssim |w_t(x) - w_t(z)|^{\frac{n}{s}} + |w_t(z) - v_t(\tilde{z})|^{\frac{n}{s}}| + |v_t(\tilde{z}) - v_t(y)|^{\frac{n}{s}}. \qquad (7.7)$$

Applying this to I_2 and integrating the inequality over $⨍_{B(x_1,3\rho)\setminus B(x_1,2\rho)}$ with respect to $\mathrm{d}z$ and over $⨍_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})}$ with respect to $\mathrm{d}\tilde{z}$ gives us

$$\begin{aligned}
I_2 \precsim & ⨍_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|w_t(x)-w_t(z)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
& + ⨍_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|v_t(\tilde{z})-v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tilde{z} \\
& + ⨍_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} ⨍_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|w_t(z)-v_t(\tilde{z})|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
=: & \; I_{2,1}+I_{2,2}+I_{2,3}.
\end{aligned}$$

Now we estimate $I_{2,1}$ and note that $|x-y|\geqslant\rho$, thus integrating over the y variable

$$\begin{aligned}
I_{2,1} &= ⨍_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|w_t(x)-w_t(z)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\precsim \rho^{-\frac{nt}{s}} ⨍_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} |w_t(x)-w_t(z)|^{\frac{n}{s}} \frac{|x-z|^{n+\frac{nt}{s}}}{|x-z|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}z \qquad (7.8)\\
&\precsim \int_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{B(x_1,2\rho)} \frac{|w_t(x)-w_t(z)|^{\frac{n}{s}}}{|x-z|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}z,
\end{aligned}$$

in the last inequality we used $|x-z|\precsim\rho$.

As for the term $I_{2,2}$, we observe that $|x-y|\geqslant 2\rho-|y-x_1|$

$$\begin{aligned}
I_{2,2} &= ⨍_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|v_t(\tilde{z})-v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tilde{z} \\
&\precsim \rho^n ⨍_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \frac{|v_t(\tilde{z})-v_t(y)|^{\frac{n}{s}}}{|2\rho-|y-x_1||^{n+\frac{nt}{s}}}\,\mathrm{d}y\,\mathrm{d}\tilde{z}.
\end{aligned}$$

For $y\in\mathbb{S}^n\setminus B(x_1,3\rho)$ and $\tilde{z}\in B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})$, we have

$$|y-\tilde{z}|\leqslant \mathrm{dist}\,(y,B(x_1,2\rho))+4\rho\leqslant 5\mathrm{dist}\,(y,B(x_1,2\rho))\leqslant 5|2\rho-|y-x_1||.$$

Thus,

$$I_{2,2}\precsim \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \frac{|v_t(\tilde{z})-v_t(y)|^{\frac{n}{s}}}{|\tilde{z}-y|^{n+\frac{nt}{s}}}\,\mathrm{d}y\,\mathrm{d}\tilde{z}. \qquad (7.9)$$

Next, we estimate $I_{2,3}$. We begin with the observation that $|x-y| \geqslant \rho$, from which we deduce

$$
\begin{aligned}
I_{2,3} &= \fint_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \fint_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n\setminus B(x_1,3\rho)} \int_{B(x_1,2\rho)} \frac{|w_t(z)-v_t(\tilde{z})|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
&\precsim \rho^{2n-n-\frac{nt}{s}} \fint_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \fint_{B(x_1,3\rho)\setminus B(x_1,2\rho)} |w_t(z)-v_t(\tilde{z})|^{\frac{n}{s}}\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
&\precsim \rho^{-n-\frac{nt}{s}} \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{B(x_1,3\rho)\setminus B(x_1,2\rho)} |\tilde{u}_{s;t}\circ\tau(z)-\tilde{u}_{s;t}(\tilde{z})|^{\frac{n}{s}}\,\mathrm{d}z\,\mathrm{d}\tilde{z} \\
&\precsim \rho^{-\frac{nt}{s}} \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{B(x_1,2\rho)} |\tilde{u}_{s;t}(\bar{z})-\tilde{u}_{s;t}(\tilde{z})|^{\frac{n}{s}} \frac{|\bar{z}-\tilde{z}|^{n+\frac{nt}{s}}}{|\bar{z}-\tilde{z}|^{n+\frac{nt}{s}}}\,\mathrm{d}\bar{z}\,\mathrm{d}\tilde{z} \\
&\precsim \rho^{n} \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{B(x_1,2\rho)} \frac{|\tilde{u}_{s;t}(\bar{z})-\tilde{u}_{s;t}(\tilde{z})|^{\frac{n}{s}}}{|\bar{z}-\tilde{z}|^{n+\frac{nt}{s}}}\,\mathrm{d}\bar{z}\,\mathrm{d}\tilde{z},
\end{aligned}
\tag{7.10}
$$

we have used the estimate $|\nabla\tau| \simeq \frac{1}{\rho}$ and $|\bar{z}-\tilde{z}| \precsim \rho$.

The estimate of I_3: Similarly, we can estimate the term I_3 by noting that for $x \in B(x_1,\rho)$, we also have $u_t(x) = w_t(x)$ and for $y \in B(x_1,3\rho)\setminus B(x_1,2\rho)$, we have $u_t(y) = v_t(y)$. We use again the inequality (7.7) and integrate over $\fint_{B(x_1,3\rho)\setminus B(x_1,2\rho)}$ with respect to $\mathrm{d}z$ and over $\fint_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})}$ with respect to $\mathrm{d}\tilde{z}$ to get

$$
\begin{aligned}
I_3 \precsim{}& \int_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \int_{B(x_1,\rho)} \frac{|w_t(x)-w_t(z)|^{\frac{n}{s}}}{|x-z|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}z \\
&+ \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{B(x_1,3\rho)\setminus B(x_1,2\rho)} \frac{|v_t(\tilde{z})-v_t(y)|^{\frac{n}{s}}}{|\tilde{z}-y|^{n+\frac{nt}{s}}}\,\mathrm{d}y\,\mathrm{d}\tilde{z} \\
&+ \rho^n \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{B(x_1,2\rho)} \frac{|\tilde{u}_{s;t}(\bar{z})-\tilde{u}_{s;t}(\tilde{z})|^{\frac{n}{s}}}{|\bar{z}-\tilde{z}|^{n+\frac{nt}{s}}}\,\mathrm{d}\bar{z}\,\mathrm{d}\tilde{z}.
\end{aligned}
\tag{7.11}
$$

The estimate of I_1: As for the term I_1, we note that on $B(x_1,3\rho)\setminus B(x_1,\rho)$, we have strong convergence of u_t and thus as in (6.4)

$$
\begin{aligned}
\lim_{t\to s^+} E_{t,\frac{n}{s}}(u_t, B(x_1,3\rho)\setminus B(x_1,\rho)) &= E_{s,\frac{n}{s}}(u_s, B(x_1,3\rho)\setminus B(x_1,\rho)) \\
&\leqslant C\rho^{n\frac{s_0-s}{s}} E_{s_0,\frac{n}{s}}(u_s, B(x_1,2\rho)) = \mathcal{O}(\rho^{n\frac{s_0-s}{s}}) \text{ as } \rho\to 0.
\end{aligned}
\tag{7.12}
$$

Finally, combining (7.5) with (7.6), (7.8), (7.9), and (7.10), we obtain

$$
\begin{aligned}
&\left|E_{t,\frac{n}{s}}(v_t,\mathbb{S}^n) + E_{t,\frac{n}{s}}(w_t,\mathbb{S}^n) - E_{t,\frac{n}{s}}(u_t,\mathbb{S}^n)\right| \\
&\precsim \int_{\mathbb{S}^n}\int_{B(x_1,2\rho)} \frac{|v_t(x)-v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)} \frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

$$+\rho^n \int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})} \int_{B(x_1,2\rho)} \frac{|\tilde{u}_{s;t}(x)-\tilde{u}_{s;t}(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y + E_{t,\frac{n}{s}}(u_t, B(x_1,3\rho)\setminus B(x_1,\rho))$$

$$=: II_1 + II_2 + II_3 + II_4. \tag{7.13}$$

The last term II_4 is just I_1 and was estimated in (7.12).

The estimate of II_1: In order to estimate the term $\int_{\mathbb{S}^n}\int_{B(x_1,2\rho)} \frac{|v_t(x)-v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y$, we will use Lemma A.1. Let us assume that $t \leqslant 2s$ and let α, β, and λ be from Lemma A.1. We have assumed in (7.4) that $B(x_1,\lambda\rho^\beta)\cap A = \{x_1\}$, and thus v_t converges to u_s strongly in $W^{s,\frac{n}{s}}$ on this ball and we have

$$\begin{aligned}\lim_{t\to s^+} E_{t,\frac{n}{s}}(v_t, B(x_1,\lambda\rho^\beta)) &= E_{s,\frac{n}{s}}(u_s, B(x_1,\lambda\rho^\beta))\\ &\approx \left(\lambda\rho^\beta\right)^{n\frac{s_0-s}{s}} E_{s_0,\frac{n}{s}}(u_s, B(x_1,\lambda\rho^\beta))\\ &= \mathcal{O}(\rho^\alpha) \text{ as } \rho\to 0,\end{aligned}$$

where $\alpha = \beta n\frac{s_0-s}{s}$, recall from Lemma A.1 that we also have $\beta = \frac{1}{2}(1-\frac{\alpha}{n})$, thus we take

$$\alpha = n\frac{s_0-s}{s_0+s} \quad \text{and} \quad \beta = \frac{1}{2}\left(\frac{s_0+s-n(s_0-s)}{s_0+s}\right),$$

here we can assume without loss of generality that $s_0 < s\frac{n+1}{n-1}$ and thus $\beta > 0$. Therefore, we obtain

$$\lim_{t\to s^+} E_{t,\frac{n}{s}}(v_t, B(x_1,\lambda\rho^\beta)) = \mathcal{O}(\rho^{n\frac{s_0-s}{s_0+s}}) \text{ as } \rho\to 0.$$

This implies, by Lemma A.1,

$$\int_{\mathbb{S}^n}\int_{B(x_1,2\rho)} \frac{|v_t(x)-v_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\rho^{n\frac{s_0-s}{s_0+s}}) \text{ as } \rho\to 0. \tag{7.14}$$

The estimate of II_2: Similarly, in order to estimate the second term on the right-hand side of (7.13), we will use Lemma A.2 for $t \leqslant 2s$ with $\sigma = n\frac{s_0-s}{s}$ and $\theta = 2+\frac{s_0-s}{s} > 1$. We note that ρ can be taken sufficiently small to ensure that $B(x_1,\tilde{\lambda}^{-1}\rho^\theta)\subset B(x_1,2\rho)$ (here $\tilde{\lambda}=\tilde{\lambda}(\mathcal{N},n,s)$ is taken from Lemma A.2). We recall that $B(x_1,2\rho)\subset B(x_1,\lambda\rho^\beta)$ and by (7.4), we know that $B(x_1,2\rho)\setminus B(x_1,\tilde{\lambda}^{-1}\rho^\theta)\cap A = \emptyset$, and thus u_t converges strongly to u_s in $B(x_1,2\rho)\setminus B(x_1,\tilde{\lambda}^{-1}\rho^\theta)$. We have

$$\begin{aligned}&\lim_{t\to s^+} E_{t,\frac{n}{s}}(w_t, \mathbb{S}^n\setminus B(x_1,\tilde{\lambda}^{-1}\rho^\theta))\\ &= \lim_{t\to s^+} E_{t,\frac{n}{s}}(\tilde{u}_{s;t}\circ\tau, \mathbb{S}^n\setminus B(x_1,2\rho)) + \lim_{t\to s^+} E_{t,\frac{n}{s}}(u_t, B(x_1,2\rho)\setminus B(x_1,\tilde{\lambda}^{-1}\rho^\theta))\\ &\lesssim \lim_{t\to s^+}\int_{B(x_1,2\rho}\int_{B(x_1,2\rho)} \frac{|\tilde{u}_{s;t}(\tilde{x})-\tilde{u}_{s;t}(\tilde{y})|^{\frac{n}{s}}}{|\tau^{-1}(\tilde{x})-\tau^{-1}(\tilde{y})|^{n+\frac{nt}{s}}}|\nabla\tau|^{2n}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\\ &\quad + E_{s,\frac{n}{s}}(u_s, B(x_1,2\rho)\setminus B(x_1,\tilde{\lambda}^{-1}\rho^\theta))\end{aligned}$$

$$\lesssim \lim_{t\to s^+}\int_{B(x_1,2\rho)}\int_{B(x_1,2\rho)}\frac{|\tilde{u}_{s;t}(\tilde{x})-\tilde{u}_{s;t}(\tilde{y})|^{\frac{n}{s}}}{|\tilde{x}-\tilde{y}|^{n+\frac{nt}{s}}}|\nabla\tau|^{n\left(\frac{t-s}{s}\right)}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}$$

$$+E_{s,\frac{n}{s}}(u_s,B(x_1,2\rho)\setminus B(x_1,\tilde{\lambda}^{-1}\rho^{\theta}))$$

$$\approx E_{s,\frac{n}{s}}(u_s,B(x_1,2\rho))=\mathcal{O}(\rho^{n\frac{s_0-s}{s}})\text{ as }\rho\to 0. \tag{7.15}$$

By Lemma A.2, this implies the smallness of

$$\lim_{t\to s^+}\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y=\mathcal{O}(\rho^{n\frac{s_0-s}{s}})\text{ as }\rho\to 0. \tag{7.16}$$

The estimate of II_3: We immediately obtain

$$\lim_{t\to s^+}\rho^n\int_{B(x_1,2\rho)\setminus B(x_1,\frac{3\rho}{2})}\int_{B(x_1,2\rho)}\frac{|\tilde{u}_{s;t}(x)-\tilde{u}_{s;t}(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y=\mathcal{O}(\rho^{n\left(\frac{s_0-s}{s}+1\right)})=\mathcal{O}(\rho^{n\frac{s_0-s}{s}})\quad\text{as }\rho\to 0. \tag{7.17}$$

Finally, we note that $\mathcal{O}(\rho^{n\frac{s_0-s}{s}})=\mathcal{O}(\rho^{n\frac{s_0-s}{s_0+s}})$ as $\rho\to 0$. Thus, passing with t to the limit in (7.13) and using (7.14), (7.16), (7.17), and (7.12), we obtain

$$\lim_{t\to s^+}\left|E_{t,\frac{n}{s}}(v_t,\mathbb{S}^n)+E_{t,\frac{n}{s}}(w_t,\mathbb{S}^n)-E_{t,\frac{n}{s}}(u_t,\mathbb{S}^n)\right|=\mathcal{O}(\rho^{n\frac{s_0-s}{s_0+s}})\text{ as }\rho\to 0.$$

Step 2. Here, we verify the inequality (7.2).
From Step 1, we obtain

$$\liminf_{t\to s^+}\left(E_{t,\frac{n}{s}}(v_t,\mathbb{S}^n)+E_{t,\frac{n}{s}}(w_t,\mathbb{S}^n)\right)=\liminf_{t\to s^+}E_{t,\frac{n}{s}}(u_t,\mathbb{S}^n)+o(1)\text{ as }\rho\to 0.$$

In particular, we have by Lemma 7.5,

$$\#\Gamma_1+\#\Gamma_2\leqslant\liminf_{t\to s^+}E_{t,\frac{n}{s}}(v_t,\mathbb{S}^n)+\liminf_{t\to s^+}E_{t,\frac{n}{s}}(w_t,\mathbb{S}^n)$$

$$\leqslant\liminf_{t\to s^+}E_{t,\frac{n}{s}}(u_t,\mathbb{S}^n)+o(1)=\#\Gamma_0+o(1)\text{ as }\rho\to 0.$$

Choosing $\rho\ll\delta$, we obtain

$$\#\Gamma_1+\#\Gamma_2\leqslant\#\Gamma_0+o(1)\leqslant\#\Gamma_0+\delta. \tag{7.18}$$

Step 3. Γ_2 *is nontrivial.* Indeed, if Γ_2 was trivial, then w_t would be homotopic to a constant, and by definition of w_t, this would imply that there is a homotopy between u_t and $\tilde{u}_{s;t}$ in $B(x_1,2\rho)$. But u_t and $\tilde{u}_{s;t}$ coincide outside on $\partial B(x_1,2\rho)$, so we would obtain that $u_t\sim v_t$. Since u_t is a minimizer in its homotopy class, we would get

$$E_{t,\frac{n}{t}}(u_t,\mathbb{S}^n)\leqslant E_{t,\frac{n}{t}}(v_t,\mathbb{S}^n).$$

Similarly as in (6.9), in the proof of Theorem 6.1, for small enough ρ, this would lead to the estimate

$$\lim_{t\to s^+}\int_{\mathbb{S}^n}\int_{B(x_1,2\rho)}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\,\mathrm{d}x\,\mathrm{d}y\leqslant\varepsilon,$$

which is a contradiction to (7.3).

Step 4. Γ_1 *is nontrivial*: Assume that Γ_1 is trivial, then v_t is homotopic to a constant. That gives us a homotopy between then u_t on $\mathbb{S}^n\setminus B(x_1,2\rho)$ and $\tilde{u}_{s;t}$ on $B(x_1,2\rho)$. Thus, we obtain that u_t is homotopically equivalent to $\tilde{u}_{s;t}\circ\tau$ in $\mathbb{S}^n\setminus B(x_1,2\rho)$. Thus, u_t is homotopic to w_t and from the minimality of u_t, we get

$$E_{t,\frac{n}{s}}(u_t,\mathbb{S}^n)\leqslant E_{t,\frac{n}{s}}(w_t,\mathbb{S}^n). \tag{7.19}$$

Noting again, that

$$\mathbb{S}^n\times\mathbb{S}^n=(B(x_1,2\rho)\times B(x_1,2\rho))\cup(\mathbb{S}^n\setminus B(x_1,2\rho)\times B(x_1,2\rho))\cup(\mathbb{S}^n\times\mathbb{S}^n\setminus B(x_1,2\rho)).$$

From (7.19) and $u_t=w_t$ on $B(x_1,2\rho)$, we have

$$\begin{aligned}&\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\int_{B(x_1,2\rho)}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\&\leqslant\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\int_{B(x_1,2\rho)}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

we also have

$$\begin{aligned}&\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\&\leqslant\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\int_{B(x_1,2\rho)}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

and by the symmetry of the integral

$$\begin{aligned}&\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\int_{B(x_1,2\rho)}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\&\leqslant 2\int_{\mathbb{S}^n}\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Thus,

$$\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\int_{\mathbb{S}^n}\frac{|u_t(x)-u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\leqslant 2\int_{\mathbb{S}^n\setminus B(x_1,2\rho)}\int_{\mathbb{S}^n}\frac{|w_t(x)-w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y.$$

In order to estimate the latter one, we will again use Lemma A.2. We recall from *Step 1*, (7.15) that for a $\lambda = \lambda(\mathcal{N}, n, s) > 1$, we have

$$\lim_{t\to s^+} E_{t,\frac{n}{s}}(w_t, \mathbb{S}^n \setminus B(x_1, \lambda^{-1}\rho^{2+\frac{s_0-s}{s}})) = \mathcal{O}(\rho^{n\frac{s_0-s}{s}}) \text{ as } \rho \to 0. \tag{7.20}$$

Now from Lemma A.2, the latter implies

$$\lim_{t\to s^+} \int_{\mathbb{S}^n\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n} \frac{|w_t(x) - w_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\rho^{n\frac{s_0-s}{s}}) \text{ as } \rho \to 0.$$

Thus, passing with t to s in (7.20), we obtain

$$\lim_{t\to s^+} \int_{\mathbb{S}^n\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\rho^{n\frac{s_0-s}{s}}) \text{ as } \rho \to 0.$$

That is, if Γ_1 was trivial, then for all t sufficiently close to s, we would have

$$\int_{\mathbb{S}^n\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C\rho^{n\frac{s_0-s}{s}}. \tag{7.21}$$

Then combining this with Theorem 5.1, we obtain for all t sufficiently close to s

$$\int_{B(x_1,2\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C\rho^{-n\left(\frac{t}{s}-1\right)} \int_{\mathbb{S}^n\setminus B(x_1,2\rho)} \int_{\mathbb{S}^n} \frac{|u_t(x) - u_t(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\lesssim \rho^{n\frac{s_0-t}{s}} \ll \varepsilon.$$

This contradicts (7.3), so Γ_1 has to be also nontrivial.

Step 5. Estimate of $\#\Gamma_1$ *and* $\#\Gamma_2$. Now since both Γ_1 and Γ_2 are nontrivial, we must have $\#\Gamma_1$, $\#\Gamma_2 > \theta$ for some $\theta > 0$, since by Lemma 2.10, we know that very small energy implies trivial homotopy class.

Moreover, choosing $\delta < \frac{\theta}{2}$, we also get from (7.18) that

$$\#\Gamma_1 \leqslant \#\Gamma_0 + \delta - \#\Gamma_2 < \#\Gamma_0 + \delta - \theta \leqslant \#\Gamma_0 - \frac{\theta}{2}$$

and similarly $\#\Gamma_2 < \#\Gamma_0 - \frac{\theta}{2}$. □

The proof of Theorem 7.1 follows now exactly as in [96, Theorem 5.5], but for reader's convenience, we repeat it here.

Proof of Theorem 7.1. Let $\theta > 0$ be the number from Lemma 2.10 such that $E_{s,\frac{n}{s}}(u, \mathbb{S}^n) < \theta$ implies trivial homotopy class. Without loss of generality, we may assume that θ is also the number from Lemma 7.7. Let P be the subgroup generated by the elements $\Gamma_i \in X$. Assume on the contrary that P does not generate the whole $\pi_n(\mathcal{N})$ acted on by $\pi_1(\mathcal{N})$. Then, we would be able to find a class $\tilde{\Gamma} \notin P$, such that for any Γ' with $\#\Gamma' < \#\tilde{\Gamma} - \frac{\theta}{2}$, we have $\Gamma' \in P$.

Since there are no minimizing $W^{s,\frac{n}{s}}$-harmonic maps in $\tilde{\Gamma}$, applying Lemma 7.7 to $\tilde{\Gamma}$, we obtain that there exists two other nontrivial homotopy classes $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ such that

$$\pi_1(\mathcal{N})\tilde{\gamma} \subset \pi_1(\mathcal{N})\tilde{\gamma}_1 + \pi_1(\mathcal{N})\tilde{\gamma}_2, \quad \#\tilde{\Gamma}_1 + \#\tilde{\Gamma}_2 < \#\tilde{\Gamma} + \frac{\theta}{2}, \quad \text{and} \quad \#\tilde{\Gamma}_1, \#\tilde{\Gamma}_2 > \theta.$$

This implies that $\#\tilde{\Gamma}_1, \#\tilde{\Gamma}_2 < \#\tilde{\Gamma} - \frac{\theta}{2}$, so both sets $\pi_1(\mathcal{N})\tilde{\gamma}_1, \pi_1(\mathcal{N})\tilde{\gamma}_2 \in P$. Thus, we also have

$$\pi_1(\mathcal{N})\tilde{\gamma} \subset \pi_1(\mathcal{N})\tilde{\gamma}_1 + \pi_1(\mathcal{N})\tilde{\gamma}_2 \subset P.$$

□

APPENDIX A: OBSERVATIONS ON THE SMALLNESS CONDITION

Let us remark that smallness conditions (that will be needed throughout the paper)

$$\int_{B(r)}\int_{B(r)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y < \varepsilon$$

and

$$\int_{B(r)}\int_{\Sigma} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y < \varepsilon$$

are essentially equivalent. This is due to the following lemma.

Lemma A.1. *Let $u \in W^{s,\frac{n}{t}}(\Sigma,\mathcal{N})$, where $s \in (0,1)$, $t \geqslant s$ then there exists a $\lambda = \lambda(\mathcal{N},n,s,\varepsilon) > 1$ such that*

$$\text{if } \int_{B(\lambda r^{\frac{s}{t}})}\int_{B(\lambda r^{\frac{s}{t}})} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y < \frac{\varepsilon}{2}, \text{ then } \int_{B(r)}\int_{\Sigma} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y < \varepsilon.$$

where $0 < r \leqslant 1$. In particular if $t \leqslant 2s$, then it suffices to take on the left-hand side of the inequality the integration over the ball $B(\lambda r^{\frac{1}{2}})$.

Moreover, there exists a $\lambda = \lambda(\mathcal{N},n,s)$ such that if $\alpha > 0$ and $0 < \beta := \frac{s}{t}(1-\frac{\alpha}{n}) < 1$

$$\int_{B(\lambda r^{\beta})}\int_{B(\lambda r^{\beta})} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(r^{\alpha}) \text{ as } r \to 0,$$

then

$$\int_{B(r)}\int_{\Sigma} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(r^{\alpha}) \text{ as } r \to 0.$$

In particular, when $t \leqslant 2s$, then it suffices to take $\beta = \frac{1}{2}(1-\frac{\alpha}{n})$.

Proof. We begin with the decomposition

$$\begin{aligned}\int_{B(r)}\int_{\Sigma}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{B(r)}\int_{B(\Lambda r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{B(r)}\int_{\Sigma\setminus B(\Lambda r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{A.1}$$

We begin with the estimate of the second term. We have

$$\begin{aligned}\int_{B(r)}\int_{\Sigma\setminus B(\Lambda r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}r^n\int_{|z|\geqslant r(\Lambda-1)}|z|^{-n-\frac{nt}{s}}\,\mathrm{d}z\\ &\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}r^n(r(\Lambda-1))^{-\frac{nt}{s}}\\ &= \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}r^{n-\frac{nt}{s}}\Lambda^{-\frac{nt}{s}}\left(1-\Lambda^{-1}\right)^{-\frac{nt}{s}}.\end{aligned}\tag{A.2}$$

Thus, taking $\Lambda=\lambda r^{\frac{s}{t}-1}$ in (A.2), we get

$$\begin{aligned}\int_{B(r)}\int_{\Sigma\setminus B(\Lambda r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}\lambda^{-\frac{nt}{s}}\left(1-\frac{r^{1-\frac{s}{t}}}{\lambda}\right)^{-\frac{nt}{s}}\\ &\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}\lambda^{-n}\xrightarrow{\lambda\to\infty}0,\end{aligned}$$

where the estimate does not depend on t.

As for the first term of (A.1), with this choice of Λ, we observe that $B(\Lambda r)=B(\lambda r^{\frac{s}{t}})$ and since $r<1$ and $s\leqslant t$, we also have

$$B(r)\subset B(\lambda r^{\frac{s}{t}})$$

and thus,

$$\int_{B(r)}\int_{B(\Lambda r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y\leqslant\int_{B(\lambda r^{\frac{s}{t}})}\int_{B(\lambda r^{\frac{s}{t}})}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y<\frac{\varepsilon}{2}.$$

This finishes the proof of the first part of the lemma.

Similarly to get the second part, we set $\Lambda=\lambda r^{\frac{s}{t}-1-\frac{s}{nt}\alpha}$ in (A.2) and obtain

$$\begin{aligned}\int_{B(r)}\int_{\Sigma\setminus B(\Lambda r)}\frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}\lambda^{-\frac{nt}{s}}r^{\alpha}\left(1-\frac{r^{1-\frac{s}{t}\left(1-\frac{\alpha}{n}\right)}}{\lambda}\right)^{-\frac{nt}{s}}\\ &\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}\lambda^{-n}r^{\alpha}.\end{aligned}$$

Now, it suffices to estimate the first term of (A.1). With this choice of Λ, we have $\Lambda r = \lambda r^{\frac{s}{t}(1-\frac{\alpha}{n})}$ and we observe that for $\alpha > 0$, we have $\beta := \frac{s}{t}(1-\frac{\alpha}{n}) < 1$, thus since $0 < r \leqslant 1$, we have

$$B(r) \subset B(\lambda r^{\beta}),$$

which gives by assumptions

$$\int_{B(r)}\int_{B(\Lambda r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{B(\lambda r^{\beta})}\int_{B(\lambda r^{\beta})} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(r^{\alpha}) \text{ as } r \to 0. \qquad \square$$

Similarly, we also have the following smallness condition.

Lemma A.2. *Let $u \in W^{s,\frac{n}{t}}(\Sigma, \mathcal{N})$, $s \in (0,1)$, and $t \geqslant s$, then there exists a $\lambda = \lambda(\mathcal{N}, n, s) > 1$ such that if for a $\sigma > 0$ and $\theta := \frac{t}{s} + \frac{\sigma}{n} > 1$, we have*

$$\int_{\Sigma\setminus B(\lambda^{-1}r^{\theta})}\int_{\Sigma\setminus B(\Lambda^{-1}r^{\theta})} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(r^{\sigma}) \text{ as } r \to 0$$

then

$$\int_{\Sigma\setminus B(r)}\int_{\Sigma} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(r^{\sigma}) \text{ as } r \to 0.$$

In particular, if $t \leqslant 2s$, it suffices to take $\theta = 2 + \frac{\sigma}{n}$.

Proof. We begin with the decomposition

$$\begin{aligned}\int_{\Sigma\setminus B(r)}\int_{\Sigma} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\Sigma\setminus B(r)}\int_{B(\Lambda^{-1}r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\Sigma\setminus B(r)}\int_{\Sigma\setminus B(\Lambda^{-1}r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{A.3}$$

We begin with the estimate of the first term. We have

$$\begin{aligned}\int_{\Sigma\setminus B(r)}\int_{B(\Lambda^{-1}r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y &\lesssim \|u\|^{\frac{n}{s}}_{L^{\infty}(\Sigma)}(\Lambda^{-1}r)^{n}\int_{|z|\geqslant r(1-\Lambda^{-1})} |z|^{-n-\frac{nt}{s}}\,\mathrm{d}z \\ &\lesssim \|u\|^{\frac{n}{s}}_{L^{\infty}(\Sigma)}(\Lambda^{-1}r)^{n}\big(r(1-\Lambda^{-1})\big)^{-\frac{nt}{s}} \\ &= \|u\|^{\frac{n}{s}}_{L^{\infty}(\Sigma)} r^{n-\frac{nt}{s}}\Lambda^{-n}\big(1-\Lambda^{-1}\big)^{-\frac{nt}{s}}.\end{aligned}$$

Thus taking $\Lambda = \lambda r^{1-\frac{t}{s}-\frac{\sigma}{n}}$, we have

$$\int_{\Sigma\setminus B(r)}\int_{B(\Lambda^{-1}r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}\lambda^{-n}r^{\sigma}\left(1-\frac{r^{\frac{t}{s}+\frac{\sigma}{n}-1}}{\lambda}\right)^{-\frac{nt}{s}}$$
$$\lesssim \|u\|^{\frac{n}{s}}_{L^\infty(\Sigma)}\lambda^{-n}r^{\sigma},$$

where the estimate does not depend on t.

As for the first term of (A.3), for this choice of Λ, we have $\Lambda^{-1}r = \lambda^{-1}r^{\frac{t}{s}+\frac{\sigma}{n}}$ and since $t \geqslant s$, we have $\theta := \frac{t}{s}+\frac{\sigma}{n} > 1$, thus for sufficiently small r,

$$B(\Lambda^{-1}r)\subset B(r), \quad \text{thus } \Sigma\setminus B(r)\subset \Sigma\setminus B(\Lambda^{-1}r).$$

This implies

$$\int_{\Sigma\setminus B(r)}\int_{\Sigma\setminus B(\Lambda^{-1}r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\Sigma\setminus B(\Lambda^{-1}r)}\int_{\Sigma\setminus B(\Lambda^{-1}r)} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{n+\frac{nt}{s}}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \mathcal{O}(r^{\sigma}), \text{ as } r\to 0. \qquad \square$$

APPENDIX B: A SOBOLEV-TYPE ESTIMATE FOR GAGLIARDO-TYPE SPACES

Here we record the Sobolev estimates we are using throughout the paper. All of them essentially follow the theory of Triebel–Lizorkin and Besov spaces, cf. [45, 95, 119] and their Sobolev embedding theory.

Theorem B.1. *Assume that $s\in(0,1)$, $t\in(s,1)$, and $p,p^*\in(1,\infty)$ with*

$$s-\frac{n}{p^*} = t-\frac{n}{p}.$$

Then

(1) *If $f\in \dot{W}^{t,p}(\mathbb{R}^n)$*

$$[f]_{W^{s,p^*}(\mathbb{R}^n)} \lesssim [f]_{W^{t,p}(\mathbb{R}^n)} \tag{B.1}$$

and

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\right)^{\frac{p^*}{p}}\mathrm{d}x\right)^{\frac{1}{p^*}} \lesssim [f]_{W^{t,p}(\mathbb{R}^n)}. \tag{B.2}$$

Moreover,

$$\|(-\Delta)^{\frac{s}{2}}f\|_{L^{p^*}(\mathbb{R}^n)} \lesssim [f]_{W^{t,p}(\mathbb{R}^n)}. \tag{B.3}$$

(2) *If $f\in W^{t,p}(B)$ for some ball $B\subset\mathbb{R}^n$, we have (with a constant independent of the specific ball)*

$$[f]_{W^{s,p^*}(B)} \lesssim [f]_{W^{t,p}(B)}, \tag{B.4}$$

and

$$\left(\int_B \left(\int_B \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\right)^{\frac{p^*}{p}}\right)^{\frac{1}{p^*}} \lesssim [f]_{W^{t,p}(B)}. \tag{B.5}$$

Proof. The statements are consequences of the theory of Besov spaces $\dot{B}^s_{p,q}$ and Triebel–Lizorkin spaces $\dot{F}^s_{p,q}$.

The first estimate (B.1) follows from Sobolev embedding for Triebel–Lizorkin spaces $\dot{F}^t_{p,p}(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p^*,p^*}(\mathbb{R}^n)$, see [54] or [119, Theorem 2.71]. We then have by the characterization of $W^{s,p}$ in terms of Triebel–Lizorkin spaces $F^s_{p,p}$, see [95, §2.6, Proposition 3, p.95] and [95, §2.1.2, Proposition, p.14],

$$[f]_{W^{s,p^*}(\mathbb{R}^n)} \approx \|f\|_{\dot{F}^s_{p^*,p^*}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^t_{p,p}(\mathbb{R}^n)} \approx [f]_{W^{t,p}(\mathbb{R}^n)}.$$

For the second estimate (B.2), we first recall the following well-known integral inequality (which follows from Riesz duality and Fubini's theorem) for any $r \geqslant 1$,

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|G(x,h)|dh\right)^r \mathrm{d}x\right)^{\frac{1}{r}} \leqslant \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|G(x,h)|^r\,\mathrm{d}x\right)^{\frac{1}{r}} dh.$$

Applying this to $G(x,h) := \frac{|f(x)-f(x+h)|^p}{|h|^{n+sp}}$ and $r = \frac{p^*}{p} > 1$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\right)^{\frac{p^*}{p}}\mathrm{d}x\right)^{\frac{1}{p^*}}\\
&=\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(x+h)|^p}{|h|^{n+sp}}\,\mathrm{d}h\right)^{\frac{p^*}{p}}\mathrm{d}x\right)^{\frac{1}{p^*}}\\
&\leqslant\left(\int_{\mathbb{R}^n}\frac{1}{|h|^{n+sp}}\left(\int_{\mathbb{R}^n}|f(x)-f(x+h)|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}dh\right)^{\frac{1}{p}}\\
&\approx \|f\|_{\dot{B}^s_{p^*,p}(\mathbb{R}^n)}.
\end{aligned}$$

In the last step, we used the integral identification of the Besov space $\dot{B}^s_{p^*,p}$, see again [95, §2.6, Proposition 3, p.95] and [95, §2.1.2, Proposition, p.14].

By Sobolev embedding for Besov spaces, [54], we have $\dot{B}^s_{p,p}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p^*,p}(\mathbb{R}^n)$, and moreover, $\dot{B}^s_{p,p}(\mathbb{R}^n) = \dot{F}^s_{p,p}(\mathbb{R}^n)$, see [95, §2.1, Remark 6., p.10]. Again by the characterization of $W^{s,p}$ in terms of Triebel–Lizorkin spaces $F^s_{p,p}$, see [95, §2.6, Proposition 3, p.95] and [95, §2.1.2, Proposition, p.14], we arrive at

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\right)^{\frac{p^*}{p}}\mathrm{d}x\right)^{\frac{1}{p^*}} \lesssim \|f\|_{\dot{F}^t_{p,p}(\mathbb{R}^n)} \approx [f]_{W^{t,p}(\mathbb{R}^n)}.$$

This establishes (B.2).

As for (B.3), by [95, §2.6, Proposition 3, p.95] and [95, §2.1.2, Proposition, p.14], we have

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L^{p^*}(\mathbb{R}^n)} \approx \|f\|_{\dot{F}^s_{p^*,2}(\mathbb{R}^n)}.$$

Sobolev embedding for Triebel–Lizorkin spaces also implies $\dot{F}^t_{p,2}(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p^*,p^*}(\mathbb{R}^n)$ since $t > s$, see [54], so we have

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L^{p^*}(\mathbb{R}^n)} \approx \|f\|_{\dot{F}^s_{p^*,2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^t_{p,p}(\mathbb{R}^n)} \approx [f]_{W^{t,p}(\mathbb{R}^n)}.$$

As for (B.4) and (B.5), by rescaling, we may assume without loss of generality that $B = B(0,1)$.

Let $f \in W^{t,p}(B(0,1))$. We may assume that $(f)_{B(0,1)} = 0$ otherwise we consider $f - (f)_{B(0,1)}$ instead. $B(0,1)$ is an extension domain, so there exists $\tilde{f} \in W^{t,p}(\mathbb{R}^n)$, and

$$[\tilde{f}]_{W^{t,p}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(B(0,1))} + [f]_{W^{t,p}(B(0,1))} \lesssim [f]_{W^{t,p}(B(0,1))}.$$

In the last step, we used Poincaré lemma. Applying (B.1) to $\tilde{f}$, we obtain (B.4), and applying (B.2) to $\tilde{f}$, we obtain (B.5). □

APPENDIX C: A LUCKHAUS-TYPE LEMMA

The Luckhaus' lemma [73, Lemma 1] or [110, Section 2.6, Lemma 1] provides a way to glue together two maps in different regions with a precise estimate on the Sobolev norms. This is an important tool in the theory of harmonic maps — in particular in the supercritical space. In [10], there is a one-dimensional fractional version of this lemma. We extend this here to any dimension, which might be a useful result in its own right. Observe that the estimate is somewhat suboptimal for $W^{s,p}$-spaces with $sp < n-1$ (where Luckhaus' original lemma develops its full force). We also make no effort to obtain an optimal estimate with respect to δ as $\delta \to 0$, since this is not what we need. So, one might argue that the following is only reminiscent of the Luckhaus' lemma.

Lemma C.1. *Let $n \geqslant 1$ $s \in (0,1)$, $p \in (1,\infty)$, $r > 0$, $u, v : \mathbb{R}^n \to \mathbb{R}^M$ such that $u, v|_{\partial B(r)}$ are continuous and*[†]

$$\int_{\partial B(r)} \int_{\mathbb{R}^n} \frac{|u(\theta) - u(y)|^p}{|\theta - y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}\theta + \int_{\partial B(r)} \int_{\partial B(r)} \frac{|u(\theta) - u(\omega)|^p}{|\theta - \omega|^{n-1+sp}} \,\mathrm{d}\theta \,\mathrm{d}\omega < \infty$$

as well as

$$\int_{\partial B(r)} \int_{\mathbb{R}^n} \frac{|v(\theta) - v(y)|^p}{|\theta - y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}\theta < \infty.$$

For any $\delta \in (0, \frac{1}{4})$, we set

$$w(x) := \begin{cases} u(x) & |x| \geqslant r \\ (1 - \eta(|x|))u(\theta) + \eta(|x|)v(\theta) & \theta = r\frac{x}{|x|},\ |x| \in ((1-\delta)r, r) \\ v(x/(1-\delta)) & |x| \leqslant (1-\delta)r, \end{cases}$$

[†] For $n = 1$, the term $\int_{\partial B(r)} \int_{\partial B(r)} \frac{|u(\theta)-u(\omega)|^p}{|\theta-\omega|^{n-1+sp}} \,\mathrm{d}\theta \,\mathrm{d}\omega$ is not needed.

where $\eta : \mathbb{R}_+ \to [0,1]$ *is smooth with* $\eta(t) = 0$ *for* $t \geqslant (1-\frac{\delta}{2})r$ *and* $\eta \equiv 1$ *on* $[0,(1-\frac{3}{4}\delta)r]$, $|\eta'| \leqslant \frac{100}{\delta r}$.

Then

- *For any* $R \geqslant r$, *where* $K := u(B(R)) \cup v(B(R))$,

$$\sup_{x\in B(R)} \operatorname{dist}(w(x),K) \leqslant \sup_{\theta\in\partial B(r)} |u(\theta)-v(\theta)|. \tag{C.1}$$

- *We have for* $\sigma := \max\{p-1, sp\}$

$$\begin{aligned}
&[w]^p_{W^{s,p}(B(2r))}\\
&\quad\leqslant [u]^p_{W^{s,p}(B(2r)\setminus B(r))} + [v]^p_{W^{s,p}(B(r))}\\
&\qquad + \delta^{-sp} r\left(\int_{\partial B(r)}\int_{B(2r)\setminus B(r)} \frac{|u(\theta)-u(y)|^p}{|\theta-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}\theta + \int_{\partial B(r)}\int_{B(r)} \frac{|v(\theta)-v(y)|^p}{|\theta-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}\theta\right)\\
&\qquad + \delta r\int_{\partial B(r)}\int_{\partial B(r)} \frac{|u(\theta)-u(\omega)|^p}{|\theta-\omega|^{n-1+sp}}\,\mathrm{d}\theta\,\mathrm{d}\omega + \delta^{-\sigma} r^{n-sp}\,\|v-u\|^p_{L^\infty(\partial B(r))}.
\end{aligned} \tag{C.2}$$

Proof. Estimate (C.1) is almost obvious, indeed, for $|x| > r$ or $|x| < (1-\delta)r$, we have $w(x) = u(x)$ or $w(x) = v(x/(1-\delta))$ so $\operatorname{dist}(w(x),K) = 0$ unless $|x| \in ((1-\delta)r, r)$.

If $|x| \in ((1-\delta)r, r)$, then

$$\operatorname{dist}(w(x),K) \leqslant |w(x) - u(rx/|x|)| \leqslant |u(rx/|x|) - v(rx/|x|)| \leqslant \sup_{\theta\in\partial B(r)} |u(\theta)-v(\theta)|.$$

This establishes (C.1).

We now provide the estimate (C.2). We assume from now on $n \geqslant 2$, and refer to the (very similar) case $n = 1$ to [10].

We have

$$[w]^p_{W^{s,p}(B(2r))} \leqslant [u]^p_{W^{s,p}(B(2r)\setminus B(r))} + I + II + 2III + 2IV + 2V,$$

where

$$\begin{aligned}
I &:= [v(\cdot/(1-\delta))]^p_{W^{s,p}(B((1-\delta)r))}\\
II &:= \int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
III &:= \int_{B(2r)\setminus B(r)}\int_{B((1-\delta)r)} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
IV &:= \int_{B(2r)\setminus B(r)}\int_{B(r)\setminus B((1-\delta)r)} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
V &:= \int_{B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} \frac{|w(x)-w(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

First, we observe

$$I = (1-\delta)^{n-sp}[v]^p_{W^{s,p}(B(r))} \lesssim [v]^p_{W^{s,p}(B(r))}.$$

Next for II we observe that for $x, y \in B(r) \setminus B((1-\delta)r)$, we have

$$\begin{aligned} &w(x) - w(y) \\ &= (1-\eta(|x|))u(rx/|x|) + \eta(|x|)v(rx/|x|) - ((1-\eta(|y|))u(ry/|y|) + \eta(|y|)v(ry/|y|)) \\ &= (1-\eta(|x|))(u(rx/|x|) - u(ry/|y|)) + \eta(|x|)(v(rx/|x|) - v(ry/|y|)) \\ &\quad + (\eta(|x|) - \eta(|y|))(v(ry/|y|) - u(ry/|y|)). \end{aligned}$$

Thus,

$$\begin{aligned} II \lesssim &\int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|u(rx/|x|) - u(ry/|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|v(rx/|x|) - v(ry/|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \|v-u\|^p_{L^\infty(\partial B(r))} \int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|\eta(|x|) - \eta(|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{C.3}$$

Firstly, we deal with the last term of (C.3) and observe

$$\begin{aligned} &\int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|\eta(|x|) - \eta(|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim (\delta r)^{-p} \int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{1}{|x-y|^{n+(s-1)p}}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim (\delta r)^{-p} r^{-(s-1)p} |B(r)\setminus B((1-\delta)r)| \\ &\lesssim \delta^{1-p} r^{n-sp}. \end{aligned} \tag{C.4}$$

Next, we estimate the first term of (C.3). We have with the help of Lemma C.2

$$\begin{aligned} &\int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|u(rx/|x|) - u(ry/|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |u(r\theta) - u(r\omega)|^p \int_{(1-\delta)r}^{r} \int_{(1-\delta)r}^{r} \frac{1}{|\rho_1\theta - \rho_2\omega|^{n+sp}} \rho_1^{n-1}\rho_2^{n-1}\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2\,\mathrm{d}\omega\,\mathrm{d}\theta \\ &\lesssim \delta r \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(r\theta) - u(r\omega)|^p}{|r\theta - r\omega|^{n-1+sp}}\, r^{n-1}\,\mathrm{d}\omega\, r^{n-1}\,\mathrm{d}\theta \\ &\approx \delta r \int_{\partial B(r)} \int_{\partial B(r)} \frac{|u(\theta) - u(\omega)|^p}{|\theta-\omega|^{n-1+sp}}\,\mathrm{d}\omega\,\mathrm{d}\theta. \end{aligned} \tag{C.5}$$

Moreover, for the second term of (C.3), we observe that if $x, y \in B(r) \setminus B((1-\delta)r)$, then for $z_{x,y} := \frac{x+y}{2}$, we have $|z_{x,y} - x| \approx |x - y| \approx |z_{x,y} - y|$, so

$$\begin{aligned}
&\int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|v(rx/|x|) - v(ry/|y|)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \\
&\precsim 2 \int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|v(rx/|x|) - v(z_{x,y})|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \\
&\approx \int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|v(rx/|x|) - v(z_{x,y})|^p}{|x-z_{x,y}|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \qquad \text{(C.6)} \\
&\precsim \int_{B(r)\setminus B((1-\delta)r)} \int_{B(r)} \frac{|v(rx/|x|) - v(z)|^p}{|x-z|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}z \\
&\precsim \delta r \int_{\partial B(r)} \int_{B(r)} \frac{|v(\theta) - v(z)|^p}{|\theta - z|^{n+sp}} \,\mathrm{d}\theta\,\mathrm{d}z.
\end{aligned}$$

In the second to last step, we used the transformation $y \mapsto z_{x,y}$.

Plugging (C.4), (C.5), and (C.6) into (C.3), we have shown

$$\begin{aligned}
II \precsim\ & \delta r \int_{\partial B(r)} \int_{\partial B(r)} \frac{|u(\theta) - u(\omega)|^p}{|\theta - \omega|^{n-1+sp}} \,\mathrm{d}\omega\,\mathrm{d}\theta \\
&+ \delta r \int_{\partial B(r)} \int_{B(r)} \frac{|v(\theta) - v(z)|^p}{|\theta - z|^{n+sp}} \,\mathrm{d}\theta\,\mathrm{d}z + \delta^{1-p} r^{n-sp} \|v-u\|^p_{L^\infty(\partial B(r))}.
\end{aligned}$$

Next, we estimate III. For any $\theta \in \partial B(r)$, we have

$$\begin{aligned}
III =\ & \int_{B(2r)\setminus B(r)} \int_{B((1-\delta)r)} \frac{|v(x/(1-\delta)) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \\
\precsim\ & \int_{B(2r)\setminus B(r)} \int_{B((1-\delta)r)} \frac{|v(x/(1-\delta)) - v(\theta)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \\
&+ \int_{B(2r)\setminus B(r)} \int_{B((1-\delta)r)} \frac{|u(\theta) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y + \int_{B(2r)\setminus B(r)} \int_{B((1-\delta)r)} \frac{|u(\theta) - v(\theta)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \\
\precsim\ & (\delta r)^{-sp} \int_{B(r)} |v(x) - v(\theta)|^p \,\mathrm{d}x \\
&+ (\delta r)^{-sp} \int_{B(2r)\setminus B(r)} |u(\theta) - u(y)|^p \,\mathrm{d}y + r^n (\delta r)^{-sp} |u(\theta) - v(\theta)|^p.
\end{aligned}$$

Multiplying this inequality by $|\partial B(r)|^{-1} \approx r^{1-n}$ and integrating in θ on $\partial B(r)$, we find

$$\begin{aligned}
III \precsim\ & \delta^{-sp} r \left(\int_{\partial B(r)} \int_{B(r)} \frac{|v(x) - v(\theta)|^p}{|x-\theta|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}\theta + \int_{\partial B(r)} \int_{B(2r)/B(r)} \frac{|u(y) - u(\theta)|^p}{|y-\theta|^{n+sp}} \,\mathrm{d}y\,\mathrm{d}\theta \right) \\
&+ \delta^{-sp} r^{n-sp} \|u - v\|^p_{L^\infty(\partial B(r))}.
\end{aligned}$$

Now we estimate IV

$$\begin{aligned} IV &= \int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{\left|((1-\eta(|x|))u(rx/|x|)+\eta(|x|)v(rx/|x|))-u(y)\right|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{\left|u(rx/|x|)-u(y)\right|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|\eta(|x|)|^p\left|u(rx/|x|)-v(rx/|x|)\right|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Observe that for $y \in B(2r)\setminus B(r)$ and $x \in B(r)\setminus B((1-\delta)r)$, we have $|x-y| \gtrsim |rx/|x|-y|$.

$$\begin{aligned} &\int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{\left|u(rx/|x|)-u(y)\right|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \int_{(1-\delta)r}^{r} \int_{B(2r)\setminus B(r)} \int_{\partial B(r)} \frac{\left|u(\theta)-u(y)\right|^p}{|\theta-y|^{n+sp}}\left(\frac{\rho}{r}\right)^{n-1}\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}\rho \\ &\approx \delta r \int_{B(2r)\setminus B(r)} \int_{\partial B(r)} \frac{\left|u(\theta)-u(y)\right|^p}{|\theta-y|^{n+sp}}\,\mathrm{d}\theta\,\mathrm{d}y. \end{aligned}$$

Also, we know that for $y \in B(2r)\setminus B(r)$, we have $\eta(|y|)=0$, and thus, we estimate

$$\begin{aligned} &\int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|\eta(|x|)|^p\left|u(rx/|x|)-v(rx/|x|)\right|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|\eta(|x|)-\eta(|y|)|^p\left|u(rx/|x|)-v(rx/|x|)\right|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \|u-v\|^p_{L^\infty(\partial B(r))} \int_{B(2r)\setminus B(r)} \int_{B(r)\setminus B((1-\delta)r)} \frac{|\eta(|x|)-\eta(|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \delta^{1-sp} r^{n-p}\|u-v\|^p_{L^\infty(\partial B(r))}. \end{aligned}$$

In the last step, we argued similar to (C.4).

So, we have shown

$$IV \lesssim \delta r \int_{B(2r)\setminus B(r)} \int_{\partial B(r)} \frac{\left|u(\theta)-u(y)\right|^p}{|\theta-y|^{n+sp}}\,\mathrm{d}\theta\,\mathrm{d}y + \delta^{1-p} r^{n-sp}\|u-v\|^p_{L^\infty(\partial B(r))}.$$

With essentially the same argument, we get the estimate of V

$$V \lesssim \delta r \int_{B(r)} \int_{\partial B(r)} \frac{\left|v(\theta)-v(y)\right|^p}{|\theta-y|^{n+sp}}\,\mathrm{d}\theta\,\mathrm{d}y + \delta^{1-p} r^{n-sp}\|u-v\|^p_{L^\infty(\partial B(r))}.$$

Combining the estimates on I, II, III, IV, and V, we obtain inequality (C.2). This concludes the proof of Lemma C.1. □

Above we used the following.

Lemma C.2. *For any $\alpha > 1$, there exists a constant $C(\alpha)$ such that for any $R > 0$, $\lambda \in (0,1)$ and any $\theta, \omega \in \mathbb{S}^{n-1}$,*

$$\int_{\lambda R}^{R}\int_{\lambda R}^{R} |r\theta - \rho\omega|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho \leqslant C(\alpha)(1-\lambda)\lambda^{1-\alpha}\,R\,|R\theta - R\omega|^{1-\alpha}.$$

Proof. Observe that

$$\int_{\lambda R}^{R}\int_{\lambda R}^{R} |r\theta - \rho\sigma|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho = R^{2-\alpha}\int_{\lambda}^{1}\int_{\lambda}^{1} |r\theta - \rho\sigma|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho,$$

so it suffices to prove the claim for $R = 1$, which we will assume from now on.

Furthermore, we observe

$$\begin{aligned}|r\theta - \rho\omega|^2 &= r^2 + \rho^2 - 2r\rho\langle\theta,\omega\rangle\\ &= r^2 + \rho^2 - 2r\rho + 2r\rho(1 - \langle\theta,\omega\rangle)\\ &= (r-\rho)^2 + r\rho|\theta-\omega|^2.\end{aligned}$$

Now observe that for $r, \rho \geqslant \lambda$,

$$|r\theta - \rho\omega| \gtrsim \max\{|r-\rho|, \lambda|\theta-\omega|\},$$

and thus for any $\alpha > 0$,

$$|r\theta - \rho\omega|^{-\alpha} \lesssim \min\{|r-\rho|^{-\alpha}, \lambda^{-\alpha}|\theta-\omega|^{-1}\}.$$

Then we split

$$\begin{aligned}&\int_{\lambda}^{1}\int_{\lambda}^{1} |r\theta - \rho\omega|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho\\ &= \int_{\lambda}^{1}\int_{\lambda}^{1} \chi_{\{|r-\rho|<\lambda|\theta-\omega|\}}|r\theta - \rho\omega|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho + \int_{\lambda}^{1}\int_{\lambda}^{1} \chi_{\{|r-\rho|>\lambda|\theta-\omega|\}}|r\theta - \rho\omega|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho\\ &\lesssim \lambda^{-\alpha}\int_{\lambda}^{1}\int_{\lambda}^{1} \chi_{\{|r-\rho|<\lambda|\theta-\omega|\}}|\theta-\omega|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho + \int_{\lambda}^{1}\int_{\lambda}^{1} \chi_{\{|r-\rho|>\lambda|\theta-\omega|\}}(r-\rho)^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho.\end{aligned}$$

Observe that

$$\begin{aligned}&\lambda^{-\alpha}\int_{\lambda}^{1}\int_{\lambda}^{1} \chi_{|r-\rho|<\lambda|\theta-\omega|}|\theta-\omega|^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho\\ &\leqslant \lambda^{-\alpha}|\theta-\omega|^{-\alpha}\int_{\lambda}^{1}\mathcal{L}^1(\{\rho : |r-\rho| < \lambda|\theta-\omega|\})\\ &= 2(1-\lambda)\lambda^{1-\alpha}|\theta-\omega|^{1-\alpha}.\end{aligned}$$

Moreover, since $\alpha > 1$,

$$\begin{aligned}
&\int_\lambda^1 \int_\lambda^1 \chi_{|r-\rho|>\lambda|\theta-\omega|}(r-\rho)^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho \\
&\leqslant \int_\lambda^1 \int_{|r-\rho|>\lambda|\theta-\omega|} (r-\rho)^{-\alpha}\,\mathrm{d}r\,\mathrm{d}\rho \\
&= \int_\lambda^1 \frac{1}{1-\alpha}\lambda^{1-\alpha}|\theta-\omega|^{1-\alpha}\,\mathrm{d}\rho \\
&= \frac{(1-\lambda)\lambda^{1-\alpha}}{(1-\alpha)}\,|\theta-\omega|^{1-\alpha}.
\end{aligned}$$

We now conclude. □

Remark C.3. While the formulation of Lemma C.1 suffices for our purposes, let us remark that in the assumptions and in the inequality (C.2), the term

$$r\int_{\partial B(r)}\int_{\partial B(r)} \frac{|u(\theta)-u(\sigma)|^p}{|\theta-\sigma|^{n-1+sp}}\,\mathrm{d}\theta\,\mathrm{d}\sigma$$

can be replaced by

$$r\int_{\partial B(r)}\int_{\mathbb{R}^n\setminus B(r)} \frac{|u(\theta)-u(y)|^p}{|\theta-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}\theta.$$

Indeed, the only modification that has to be made is in the estimate of (C.5). This can be done in the following way:

$$\begin{aligned}
&\int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} \frac{|u(rx/|x|)-u(ry/|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} \chi_{|x-y|<\frac{1}{100}r}\frac{|u(rx/|x|)-u(ry/|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + r^{-n-sp}\int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} |u(rx/|x|)-(u)_{A(r)}|^p\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
&r^{-n-sp}\int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} |u(rx/|x|)-(u)_{A(r)}|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim r^{n-sp}|B(r)\setminus B((1-\delta)r)|\int_{B(r)\setminus B((1-\delta)r)} |u(rx/|x|)-(u)_{A(r)}|^p\,\mathrm{d}x \\
&\approx \delta r^{-sp}\int_{(1-\delta r)}^{r}\int_{\partial B(r)} |u(\theta)-(u)_{A(r)}|^p\,\mathrm{d}\theta \\
&\lesssim \delta^2 r^{1-sp}r^{-n}\int_{\partial B(r)}\int_{B(2r)\setminus B(r)} \frac{|u(\theta)-u(z)|^p}{|\theta-z|^{n+sp}}|\theta-z|^{n+sp}\,\mathrm{d}\theta\,\mathrm{d}z \\
&\lesssim \delta^2 r\int_{\partial B(r)}\int_{B(2r)\setminus B(r)} \frac{|u(\theta)-u(z)|^p}{|\theta-z|^{n-1+sp}}\,\mathrm{d}\theta\,\mathrm{d}z.
\end{aligned}$$

For the first term, we observe

$$\begin{aligned}
&\int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} \chi_{\{|x-y|<\frac{1}{100}r\}}\frac{|u(rx/|x|)-u(ry/|y|)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim \int_{B(r)\setminus B((1-\delta)r)}\int_{B(r)\setminus B((1-\delta)r)} \chi_{\{|x-y|<\frac{1}{100}r\}}\frac{|u(rx/|x|)-u(z_{x,y})|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\lesssim \int_{B(r)\setminus B((1-\delta)r)}\int_{B(2r)\setminus B(r)} \frac{|u(rx/|x|)-u(z)|^p}{|rx/|x|-z|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}z\\
&\quad\lesssim \delta r\int_{\partial B(r)}\int_{B(2r)\setminus B(r)} \frac{|u(\theta)-u(z)|^p}{|\theta-z|^{n+sp}}\,\mathrm{d}z\,\mathrm{d}\theta,
\end{aligned}$$

where we have chosen an intermediate point $z_{x,y}$ with the following properties: $z_{x,y}\in B(2r)\setminus B(r)$, $|rx/|x|-z_{x,y}|\approx|ry/|y|-z_{x,y}|\approx|x-y|$, $z_{x,x}=rx/|x|$, and $z_{y,y}=ry/|y|$. This can be done by using a diffeomorphism $\tau:\ C\cap\{|x-y|<\frac{1}{100}r\}\to\mathcal{K}$ that transforms a cone C centered at the origin that contains the ball $\{|x-y|<\frac{1}{100}r\}$ intersected with the annulus $B(2r)\setminus B(r)$, into a convex set $\mathcal{K}$. Then we can take as the intermediate point $z_{x,y}$ the preimage τ^{-1} of the convex combination of the image (under the diffeomorphism τ) of the points $rx/|x|$ and $ry/|y|$. This is quite technical, so for convenience of the authors, we leave the details to the reader.

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