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Fermi Isospectrality of Discrete Periodic Schrödinger Operators with Separable Potentials on $\mathbb{Z}^2$

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Abstract: Given two coprime numbers q_1 and q_2, let $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z}$. Let $\Delta + X$ be the discrete periodic Schrödinger operator on $\mathbb{Z}^2$, where Δ is the discrete Laplacian and $X : \mathbb{Z}^2 \to \mathbb{C}$ is Γ-periodic. In this paper, we develop tools from complex analysis to study the isospectrality of discrete periodic Schrödinger operators. We prove that if two Γ-periodic potentials X and Y are Fermi isospectral and both $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ are separable functions, then, up to a constant, one dimensional potentials X_j and Y_j are Floquet isospectral, $j = 1, 2$. This allows us to prove that for any non-constant separable real-valued Γ-periodic potential, the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ is irreducible for any $\lambda \in \mathbb{C}$, which partially confirms a conjecture of Gieseker, Knörrer and Trubowitz in the early 1990s.

1. Introduction and Main Results

Given $q_j \in \mathbb{Z}_+$, $j = 1, 2, \cdots, d$, let $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$. We say that a function $V : \mathbb{Z}^d \to \mathbb{C}$ is Γ-periodic (or just periodic) if for any $\gamma \in \Gamma$ and $n \in \mathbb{Z}^d$, $V(n + \gamma) = V(n)$. For $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$, denote by $||n||_1 = \sum_{j=1}^d |n_j|$. Let Δ be the discrete Laplacian on the lattice $\mathbb{Z}^d$, namely

$$(\Delta u)(n) = \sum_{n' \in \mathbb{Z}^d, ||n'-n||_1=1} u(n').$$

In the following, we always assume that q_j, $j = 1, 2, \cdots, d$, are pairwise coprime and V is Γ-periodic.

In this article we are interested in the isospectrality problem and irreducibility of Fermi varieties of discrete periodic Schrödinger operators $\Delta + V$. We refer readers to two survey articles [21,27] for background and recent developments about the two topics.

Let $\{\mathbf{e}_j\}$, $j = 1, 2, \cdots d$, be the standard basis in $\mathbb{Z}^d$:

$$\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_d = (0, 0, \cdots, 0, 1).$$

Definition 1. The *Bloch variety* $B(V)$ of $\Delta + V$ consists of all pairs $(k, \lambda) \in \mathbb{C}^{d+1}$ for which there exists a non-zero solution of the equation

$$(\Delta u)(n) + V(n)u(n) = \lambda u(n), n \in \mathbb{Z}^d, \tag{1}$$

satisfying the so called Floquet–Bloch boundary condition

$$u(n + q_j \mathbf{e}_j) = e^{2\pi i k_j} u(n),\ j = 1, 2, \cdots, d,\ \text{and } n \in \mathbb{Z}^d, \tag{2}$$

where $k = (k_1, k_2, \cdots, k_d) \in \mathbb{C}^d$.

Given $\lambda \in \mathbb{C}$, the Fermi surface (variety) $F_\lambda(V)$ is defined as the level set of the Bloch variety:

$$F_\lambda(V) = \{k : (k, \lambda) \in B(V)\}.$$

We call $k = (k_1, k_2, \cdots, k_d)$ that appears in (2) quasi-momentum. One can see that both Fermi and Bloch varieties are analytic sets, in fact algebraic sets with respect to variables $z = (z_1, z_2, \cdots, z_d)$, where $z_j = e^{2\pi i k_j}$, $j = 1, 2, \cdots, d$ [21,26,27].

Our first interest is the isospectrality problems.

Let $D_V(k)$ be the periodic operator $\Delta + V$ with the Floquet–Bloch boundary condition (2) (for fixed k, $D_V(k)$ is a matrix. See Sect. 2 for the precise description of $D_V(k)$). Denote by $\sigma(D_V(k))$ the (counting the algebraic multiplicity) spectrum/eigenvalues of $D_V(k)$. Two Γ-periodic potentials X and Y are called Floquet isospectral if

$$\sigma(D_X(k)) = \sigma(D_Y(k)),\ \text{for all } k \in \mathbb{R}^d. \tag{3}$$

Two Γ-periodic potentials X and Y are called k-isospectral, $k \in \mathbb{C}^d$, if

$$\sigma(D_X(k)) = \sigma(D_Y(k)). \tag{4}$$

Understanding when two periodic potentials X and Y are Floquet isospectral or k-isospectral is a fascinating subject and has been extensively studied [5–7,13–15,17–19,21,28,31].

In [25], the author introduced a new type of isospectrality: Fermi isospectrality.

Definition 2. [25] Let X and Y be two Γ-periodic functions. We say X and Y are Fermi isospectral at energy level λ_0 ($\lambda_0 \in \mathbb{C}$) if $F_{\lambda_0}(X) = F_{\lambda_0}(Y)$. We say X and Y are Fermi isospectral if there exists some $\lambda_0 \in \mathbb{C}$ such that $F_{\lambda_0}(X) = F_{\lambda_0}(Y)$.

It is not difficult to see that two periodic functions X and Y are Floquet isospectral if and only if Bloch varieties of X and Y are the same (or Fermi varieties of X and Y are the same for every $\lambda \in \mathbb{C}$) [25]. So Fermi isospectrality is a “hyperplane" version of Floquet isospectrality.

In [25], the author proved several rigidity theorems of discrete periodic Schrödinger operators about separable functions. We say that a function V on $\mathbb{Z}^d$ is $(d_1, d_2, \cdots, d_r)$ separable (or simply separable, denote it by $V = \bigoplus_{j=1}^r V_j$), where $\sum_{j=1}^r d_j = d$ with $r \geq 2$, if there exist functions V_j on $\mathbb{Z}^{d_j}$, $j = 1, 2, \cdots, r$, such that for any $(n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$,

$$\begin{aligned} V(n_1, n_2, \cdots, n_d) =& V_1(n_1, \cdots, n_{d_1}) + V_2(n_{d_1+1}, n_{d_1+2}, \cdots, n_{d_1+d_2}) \\ &+ \cdots + V_r(n_{d_1+d_2+\cdots+d_{r-1}+1}, \cdots, n_{d_1+d_2+\cdots+d_r}). \end{aligned} \tag{5}$$

One of the rigidity theorems in [25] states

Theorem 1.1. [25] *Let $d \geq 3$. Assume that two separable Γ-periodic potentials $X = \bigoplus_{j=1}^{r} X_j$ and $Y = \bigoplus_{j=1}^{r} Y_j$ are Fermi isospectral. Then, up to a constant, lower dimensional decompositions X_j and Y_j are Floquet isospectral, $j = 1, 2, \cdots, r$.*

In the present work, we prove that the statement in Theorem 1.1 holds for dimension $d = 2$. Namely,

Theorem 1.2. *Let $d = 2$. Assume that q_1 and q_2 are coprime. Assume that two Γ-periodic potentials X and Y are Fermi isospectral and both $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ are separable. Then, up to a constant, the one dimensional functions X_j and Y_j are Floquet isospectral, $j = 1, 2$.*

Remark 1. In Theorems 1.1 and 1.2, potentials are allowed to be complex-valued.

Our second interest of this paper is the irreducibility of Fermi varieties. Irreducibility of Fermi varieties (also Bloch varieties) and related applications such as embedded eigenvalues and spectral band edges have seen continuous progress in the past 30 years [1–4,9–12,20,22–24,29,30].

Recently, the author introduced an algebraic method and provided more general proofs of irreducibility of Fermi varieties [26].

Denote by $[V]$ the average of V over one periodicity cell, namely

$$[V] = \frac{1}{q_1 q_2 \cdots q_d} \sum_{\substack{1 \leq n_j \leq q_j \\ 1 \leq j \leq d}} V(n_1, n_2, \cdots, n_d).$$

Theorem 1.3. [26] *For any $d \geq 3$, the Fermi variety $F_\lambda(V)/\mathbb{Z}^d$ is irreducible for any $\lambda \in \mathbb{C}$. For $d = 2$, the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ is irreducible for any $\lambda \in \mathbb{C}$ except maybe for $\lambda = [V]$ and $F_{[V]}(V)/\mathbb{Z}^2$ has at most two irreducible components.*

Before [26], the irreducibility of Fermi varieties at all energy levels for $d = 3$ and at all energy levels but finitely many λ for $d = 2$ was proved in [2,12] by an different approach (compactification).

Let $d = 2$. When the potential V is a constant function, direct computation (e.g., see [27]) implies that $F_{[V]}(V)/\mathbb{Z}^2$ has exactly two irreducible components. When complex-valued functions are allowed, there exist non-constant complex valued functions V such that the Fermi variety is reducible at the energy level $[V]$ (e.g. [8]).

However, for real-valued potentials, people believe the constant potential is the only case that the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ is reducible at some energy level, which has been formulated as a conjecture by Gieseker, Knörrer and Trubowitz in the early 1990s [12].

Conjecture 1. [12, p.43] Assume that V is a non-constant real-valued periodic potential on $\mathbb{Z}^2$. Then the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ is irreducible for any $\lambda \in \mathbb{C}$.

Theorem 1.2 allows us to confirm the Conjecture 1 for separable potentials.

Theorem 1.4. *Assume that q_1 and q_2 are coprime. Assume that V is a non-constant separable real-valued periodic potential on $\mathbb{Z}^2$. Then the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ is irreducible for any $\lambda \in \mathbb{C}$.*

The irreducibility of Fermi variety and Fermi isospectrality of discrete periodic Schrödinger operators (dimension $d \geq 3$) are well understood in two recent papers [25,26]. Besides Theorem 1.1, there are other Fermi isospectrality results in [25] for dimension $d \geq 3$. However, approaches in [25] can not be extended to dimension $d = 2$

since there are not enough free variables available. For irreducibility results of the Fermi variety in Theorem 1.3, the proof for $d = 2$ is more difficult than that for $d \geq 3$. For continuous periodic Schrödinger operators, Bättig, Knörrer and Trubowitz [3] proved the irreducibility of Fermi varieties and a rigidity theorem of separable functions in dimension three. However, the proof in [3] does not work for dimension $d = 2$. For discrete periodic Schrödinger operators on $\mathbb{Z}^d$ with $d \geq 3$, the Fermi variety $F_\lambda(V)/\mathbb{Z}^d$ for any complex-valued potential is irreducible at any energy level λ (see Theorem 1.3). For $d = 2$, there are many complex-valued potentials V such that the Fermi variety $F_\lambda(V)/\mathbb{Z}^2$ has two irreducible components at the average energy level $[V]$ [8].

Finally, we want to comment that dimension two is the transition of Fermi isospectrality problems of periodic Shcrödinger operators. For $d = 1$, it does not make sense to study Fermi isospectrality since for any periodic potential V, $F_{\lambda_0}(V)$ contains at most two points. For $d = 2$ and any periodic potential Y, all periodic potentials X such that X and Y are Fermi isospectral at λ_0 (namely $F_{\lambda_0}(X) = F_{\lambda_0}(Y)$) is an algebraic set with at least one dimension [8]. For $d \geq 3$ and any periodic potential Y, all periodic potentials X such that X and Y are Fermi isospectral at λ_0 could be an algebraic set with zero dimension [8].

All evidence above seems to indicate that when $d = 2$, problems related to Fermi varieties are special (often more challenging).

In this paper, we present a new approach to study the Fermi isospectrality of discrete periodic Schrödinger operators. As in [25], we focus on the study of a family of Laurent polynomials whose zero sets are Fermi varieties after changing variables ($z_j = e^{2\pi i k_j}$, $j = 1, 2, \cdots, d$). Our strategy is to develop tools from complex analysis to study the eigenvalue problems of (1) and (2) (or (6) and (7)) with complexified quasi-momenta. One needs to relabel spectral band functions of one dimensional periodic Schrödinger operators based on asymptotics of eigenvalues and show that those functions are analytic with respect to quasi-momenta in an appropriate domain. Applying Rouché's Theorem, one sees that for any two one-dimensional q-periodic potentials with the same average, there exist q quasi-momenta such that for those quasi-momenta, labelled eigenvalues of two potentials only differ by a (same) constant. This enables us to show that separable components of Fermi isospectrality potentials with respect to one coordinate are Floquet isospectral and hence remaining separable components are Floquet isospectral as well.

The rest of this paper is organized as follows. In Sect. 2, we recall some basics for Fermi varieties. In Sect. 3, we study one dimensional periodic Schrödinger operators. Section 4 is devoted to proving Theorems 1.2 and 1.4.

2. Basics of Fermi Varieties

Let $\mathbb{C}^\star = \mathbb{C}\backslash\{0\}$ and $z = (z_1, z_2, \cdots, z_d)$. For any $z \in (\mathbb{C}^\star)^d$, consider the equation

$$(\Delta + V)u = \lambda u \tag{6}$$

with the boundary condition

$$u(n + q_j \mathbf{e}_j) = z_j u(n),\ j = 1, 2, \cdots, d, \text{ and } n \in \mathbb{Z}^d, \tag{7}$$

Introduce a fundamental domain W for Γ:

$$W = \{n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d : 0 \leq n_j \leq q_j - 1, j = 1, 2, \cdots, d\}.$$

By writing out $\Delta+V$ as acting on the $Q = q_1 q_2 \cdots q_d$ dimensional space $\{u(n), n \in W\}$, the equation (6) with boundary condition (7) ((1) and (2)) translates into the eigenvalue problem for a $Q \times Q$ matrix $\mathcal{D}_V(z)$ ($D_V(k)$).

Let

$$\mathcal{P}_V(z,\lambda) = \det(\mathcal{D}_V(z) - \lambda I),\ P_V(k,\lambda) = \det(D_V(k) - \lambda I). \tag{8}$$

We remark that $\mathcal{D}_V(z)$ and $D_V(k)$ ($\mathcal{P}_V(z,\lambda)$ and $P_V(k,\lambda)$) are the same under the relations $z_j = e^{2\pi i k_j}$, $j = 1, 2, \cdots, d$.

Example 1. When $d = 1$, the equation $(\Delta+V)u = \lambda u$ with the Floquet–Bloch boundary condition $u(n+q) = zu(n)$, $z \in \mathbb{C}^\star$, can be reduced to an eigenvalue problem of a $q \times q$ matrix:

$$\mathcal{D}_V(z) = \begin{pmatrix} V(1) & 1 & 0 & \cdots & 0 & z^{-1} \\ 1 & V(2) & 1 & \cdots & 0 & 0 \\ 0 & 1 & V(3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & V(q-1) & 1 \\ z & 0 & 0 & \cdots & 1 & V(q) \end{pmatrix}, \quad q \geq 3; \tag{9}$$

$$\mathcal{D}_V(z) = \begin{pmatrix} V(1) & 1+z^{-1} \\ 1+z & V(2) \end{pmatrix}, \quad q = 2; \tag{10}$$

and

$$\mathcal{D}_V(z) = z + z^{-1} + V(1), \quad q = 1. \tag{11}$$

We have the following **Basic Facts**:

(1) The Fermi variety of V at energy level λ is given by

$$F_\lambda(V) = \{k \in \mathbb{C}^d : P_V(k,\lambda) = 0\}. \tag{12}$$

(2) Two periodic functions X and Y are Floquet isospectral[1] if and only if for all $(z,\lambda) \in (\mathbb{C}^\star)^d \times \mathbb{C}$,

$$\mathcal{P}_X(z,\lambda) = \mathcal{P}_Y(z,\lambda). \tag{13}$$

3. One Dimensional Discrete Periodic Schrödinger Operators

In this section, we study one dimensional discrete periodic Schrödinger operators $\Delta+V$:

$$((\Delta+V)u)(n) = u(n+1) + u(n-1) + V(n)u(n), n \in \mathbb{Z},$$

where V is a periodic function on $\mathbb{Z}$, namely, $V(n+q) = V(n)$, $n \in \mathbb{Z}$ for some positive integer q.

In the following, we say z is large if $|z|$ is large.

[1] Although the definition of Floquet isospectrality only involves real k, it extends to all complex k because of the analyticity of $P_V(k,\lambda)$ (algebraicity of $\mathcal{P}_V(z,\lambda)$) .

By (9)–(11) in Example 1, $\mathcal{P}_V(z,\lambda)-(-1)^{q+1}z-(-1)^{q+1}z^{-1}$ is independent of variable z. So let $\hat{\mathcal{P}}_V(\lambda)$ be such that

$$\mathcal{P}_V(z,\lambda)=\hat{\mathcal{P}}_V(\lambda)+(-1)^{q+1}z+(-1)^{q+1}z^{-1}. \tag{14}$$

By (9)–(11), one has that (recall that $[V]=\frac{1}{q}(\sum_{j=1}^{q}V(j))$)

$$\hat{\mathcal{P}}_V(\lambda)=(-1)^q\lambda^q-(-1)^q q[V]\lambda^{q-1}+\text{ lower order terms of }\lambda. \tag{15}$$

Fixing $z\in\mathbb{C}$, solve the algebraic equation

$$\mathcal{P}_V(z^q,\lambda)=0. \tag{16}$$

By (14), (15) and the standard asymptotic analysis, there exist (exact q) solutions $\lambda_V^l(z)$ of equation (16), $l=0,1,2,\cdots,q-1$ such that $\lambda^l(z)$ is analytic in $\Omega=\{z\in\mathbb{C}:|z|>R\}$ with large R (the largeness only depends on the potential V). Moreover, $\lambda_V^l(z)$, $l=0,1,2,\cdots,q-1$ have the following representations in Laurent series,

$$\lambda_V^l(z)=e^{2\pi\frac{l}{q}i}z+[V]+\sum_{k=1}^{\infty}\frac{a_k(V)}{z^k}, \tag{17}$$

where the coefficient $a_k(V)$ depends on V. For readers' convenience, we include a proof of (17) (along with the existence of $\lambda^l(z)$) in the "Appendix".

Clearly, for any large enough z, $\lambda_V^l(z)$, $l=0,1,2,\cdots,q-1$, are eigenvalues of $\mathcal{D}(z^q)$, and hence

$$\mathcal{P}_V(z^q,\lambda)=\det(\mathcal{D}(z^q)-\lambda I)=\prod_{l=0}^{q-1}(\lambda_V^l(z)-\lambda). \tag{18}$$

In the following, when z is large enough, we always let $\lambda_V^l(z)$ (the labeled eigenvalues of $\mathcal{D}_V(z^q)$), $l=0,1,2,\cdots,q-1$, be given by (17).

Lemma 3.1. *Let V and $\tilde{V}$ be two q-periodic functions on $\mathbb{Z}$. Assume that $[V]=[\tilde{V}]$, and V and $\tilde{V}$ are not Floquet isospectral. Then there exist sufficiently large $R>0$ and small $\epsilon>0$ such that for any $\eta\in\mathbb{C}$ with $0<|\eta|<\epsilon$ and $l=0,1,2,\cdots,q-1$, the equation $\lambda_V^l(z)=\lambda_{\tilde{V}}^l(z)+\eta$ has at least one solution in $\{z\in\mathbb{C}:|z|\geq R\}$.*

Proof. Fix any $l\in\{0,1,\cdots,q-1\}$. By (17), one has that for any large enough z,

$$\lambda_V^l(z)=e^{2\pi\frac{l}{q}i}z+[V]+\sum_{k=1}^{\infty}\frac{a_k(V)}{z^k}, \tag{19}$$

and

$$\lambda_{\tilde{V}}^l(z)=e^{2\pi\frac{l}{q}i}z+[\tilde{V}]+\sum_{k=1}^{\infty}\frac{a_k(\tilde{V})}{z^k}, \tag{20}$$

If $\lambda_V^l(z)=\lambda_{\tilde{V}}^l(z)$ for every large z, then V and $\tilde{V}$ must be Flqouet isospectral, which contradicts the assumption. So functions $\lambda_V^l(z)$ and $\lambda_{\tilde{V}}^l(z)$ are not identical.

When $a_1(V) \neq a_1(\tilde{V})$, let $k_0 = 1$. Otherwise, let $k_0 \in \mathbb{Z}_+$ be such that $a_k(V) = a_k(\tilde{V})$ for any $k < k_0$ and $a_{k_0}(V) \neq a_{k_0}(\tilde{V})$. Consider a ball $B_{\epsilon_1} = \{z \in \mathbb{C} : |z| \leq \epsilon_1\}$ with a small $\epsilon_1 > 0$. Define an analytic function $f(z)$ in B_{ϵ_1} by

$$f(z) = \sum_{k=k_0}^{\infty} (a_k(V)z^k - a_k(\tilde{V})z^k). \tag{21}$$

Then for any large enough z,

$$f(z^{-1}) = \lambda_V^l(z) - \lambda_{\tilde{V}}^l(z). \tag{22}$$

Let

$$f_1(z) = a_{k_0}(V)z^{k_0} - a_{k_0}(\tilde{V})z^{k_0},$$

and

$$f_2(z) = \sum_{k=k_0+1}^{\infty} (a_k(V)z^k - a_k(\tilde{V})z^k)$$

For any $z \in \partial B_{\epsilon_1} = \{z \in \mathbb{C} : |z| = \epsilon_1\}$, one has that for small ϵ_1,

$$|f_1(z)| \geq |f_2(z)| + \frac{|a_{k_0}(V) - a_{k_0}(\tilde{V})|}{2} \epsilon_1^{k_0}. \tag{23}$$

Let ϵ be sufficiently small (depending on ϵ_1). Choose any η with $0 < |\eta| \leq \epsilon$. By (23), one has that for any $z \in \partial B_{\epsilon_1}$,

$$|f_1(z) - \eta| > |f_2(z)|. \tag{24}$$

By Rouché's theorem, $f_1(z) - \eta = 0$ and $f(z) - \eta = f_1(z) + f_2(z) - \eta = 0$ have the same number of zeros (counting multiplicity) in $\{z \in \mathbb{C} : |z| < \epsilon_1\}$. This particularly implies that $f(z) = \eta$ has at least one non-zero solution in $\{z \in \mathbb{C} : |z| < \epsilon_1\}$. Now Lemma 3.1 follows from (22). □

4. Proof of Theorems 1.2 and 1.4

Lemma 4.1. [25, Lemma 2.3] *Assume $F_{\lambda_0}(X) = F_{\lambda_0}(Y)$. Then for any $z \in (\mathbb{C}^\star)^2$,*

$$\mathcal{P}_X(z, \lambda_0) = \mathcal{P}_Y(z, \lambda_0). \tag{25}$$

Lemma 4.2. *Assume that separable functions $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ are Fermi isospectral. Assume that X_2 and Y_2 are Floquet isospectral. Then X_1 and Y_1 are Floquet isospectral.*

Proof. Recall that $z = (z_1, z_2)$. By the assumption that $F_{\lambda_0}(X) = F_{\lambda_0}(Y)$ and Lemma 4.1, one has that

$$\mathcal{P}_X(z, \lambda_0) = \mathcal{P}_Y(z, \lambda_0),$$

and hence

$$\mathcal{P}_X(z_1, z_2^{q_2}, \lambda_0) = \mathcal{P}_Y(z_1, z_2^{q_2}, \lambda_0). \tag{26}$$

Using the fact that both X and Y are separable, one has that for any large enough z_2,

$$\begin{aligned}\mathcal{P}_X(z_1, z_2^{q_2}, \lambda_0) &= \prod_{l=0}^{q_2-1} \mathcal{P}_{X_1}(z_1, -\lambda_{X_2}^l(z_2) + \lambda_0) \\ &= \prod_{l=0}^{q_2-1} (\hat{\mathcal{P}}_{X_1}(-\lambda_{X_2}^l(z_2) + \lambda_0) + (-1)^{q_1+1} z_1 + (-1)^{q_1+1} z_1^{-1}),\end{aligned} \tag{27}$$

and

$$\begin{aligned}\mathcal{P}_Y(z_1, z_2^{q_2}, \lambda_0) &= \prod_{l=0}^{q_2-1} \mathcal{P}_{Y_1}(z_1, -\lambda_{Y_2}^l(z_2) + \lambda_0) \\ &= \prod_{l=0}^{q_2-1} (\hat{\mathcal{P}}_{Y_1}(-\lambda_{Y_2}^l(z_2) + \lambda_0) + (-1)^{q_1+1} z_1 + (-1)^{q_1+1} z_1^{-1}).\end{aligned} \tag{28}$$

By (17), (26), (27), (28) and the unique factorization theorem (using $(-1)^{q_1+1} z_1 + (-1)^{q_1+1} z_1^{-1}$ as a variable), one has that for any $l = 0, 1, \cdots, q_2 - 1$,

$$\hat{\mathcal{P}}_{X_1}(-\lambda_{X_2}^l(z_2) + \lambda_0) = \hat{\mathcal{P}}_{Y_1}(-\lambda_{Y_2}^l(z_2) + \lambda_0), \tag{29}$$

and

$$\mathcal{P}_{X_1}(z_1, -\lambda_{X_2}^l(z_2) + \lambda_0) = \mathcal{P}_{Y_1}(z_1, -\lambda_{Y_2}^l(z_2) + \lambda_0). \tag{30}$$

Since X_2 and Y_2 are Floquet isospectral, we know that for any large enough z_2,

$$\lambda_{X_2}^l(z_2) = \lambda_{Y_2}^l(z_2), l = 0, 1, 2, \cdots, q_2 - 1. \tag{31}$$

By (30) and (31), one has that for any $z_1 \in \mathbb{C}^\star$ and $\lambda \in \mathbb{C}$,

$$\mathcal{P}_{X_1}(z_1, \lambda) = \mathcal{P}_{Y_1}(z_1, \lambda). \tag{32}$$

By (32) and basic fact (2) appearing at the end of Sect. 2, we conclude that X_1 and Y_1 are Floquet isospectral. □

Proof of Theorem 1.2. Without loss of generality, assume that $q_2 > q_1$ and $[X_2] = [Y_2] = 0$. If X_2 and Y_2 are Floquet isospectral, then Theorem 1.2 follows from Lemma 4.2. So we assume X_2 and Y_2 are not Floquet isospectral.

Applying Lemma 3.1 with $V = X_2$, $\tilde{V} = Y_2$ and $q = q_2$, there exist η and large x_l, $l = 0, 1, 2, \cdots, q_2 - 1$, such that

$$\lambda^l_{X_2}(x_l) = \lambda^l_{Y_2}(x_l) + \eta. \tag{33}$$

By (29) and (33), we have that for any $l = 0, 1, 2, \cdots, q_2 - 1$,

$$\hat{\mathcal{P}}_{X_1}(-\lambda^l_{Y_2}(x_l) - \eta + \lambda_0) = \hat{\mathcal{P}}_{Y_1}(-\lambda^l_{Y_2}(x_l) + \lambda_0). \tag{34}$$

Since both $\hat{\mathcal{P}}_{X_1}(\lambda - \eta)$ and $\hat{\mathcal{P}}_{Y_1}(\lambda)$ are polynomials of λ with degree q_1, by (34) and the fact that $q_2 > q_1$, one has that

$$\hat{\mathcal{P}}_{X_1}(\lambda - \eta) = \hat{\mathcal{P}}_{X_1}(\lambda), \lambda \in \mathbb{C}. \tag{35}$$

This implies that X_1 and Y_1 are Floquet isospectral up to a constant (by letting $\eta \to 0$ in (35), we can indeed show that X_1 and Y_1 are Floquet isospectral. This is because we have already shifted the constant by setting $[X_2] = [Y_2] = 0$). Now Theorem 1.2 follows from Lemma 4.2 (exchange X_1 and X_2, and Y_1 and Y_2). □

Denote by $\mathbf{0}$ the zero function on $\mathbb{Z}^2$.

Proof of Theorem 1.4. Assume that $F_\lambda(V)$ is reducible at some $\lambda = \lambda_0$. By Remark 4 in [26] (also Theorem 2.5 in [25]), $\lambda_0 = [V]$ and $\mathcal{P}_V(z, \lambda_0) = \mathcal{P}_{\mathbf{0}}(z, 0)$. Therefore $\mathcal{P}_{V-[V]}(z, 0) = \mathcal{P}_{\mathbf{0}}(z, 0)$. By Theorem 1.2, we have that V and a constant potential are Floquet isospectral. Therefore, Ambarzumian-type theorem (e.g. [16]) implies V is constant. This contradicts the assumption. □

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Declarations

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest The author declares no conflicts of interest.

Appendix A. Proof of (17)

Proof. When $q = 1$, (17) (along with the existence of $\lambda^l(z)$, $l = 0$) follows from (11). So we assume $q \geq 2$. For any fixed $l \in \{0, 1, \cdots, q-1\}$, introduce a new variable $\tilde{\lambda}^l$:

$$\lambda = \tilde{\lambda}^l + e^{2\pi \frac{l}{q} i} z + [V]. \tag{36}$$

Substituting (36) into equation (16), simple computations (do the expansion with respect to z variable) imply that

$$\begin{aligned}\mathcal{P}_V(z^q, \lambda) =& (-1)^{q+1} z^q + (-1)^{q+1} z^{-q} + (-1)^q (e^{2\pi \frac{l}{q} i} z)^q \\ &+ (-1)^q q (e^{2\pi \frac{l}{q} i} z)^{q-1} (\tilde{\lambda}^l + [V]) - (-1)^q q [V] (e^{2\pi \frac{l}{q} i} z)^{q-1} \\ &+ \text{ lower order terms of } z \\ =& (-1)^{q+1} z^{-q} + (-1)^q q (e^{2\pi \frac{l}{q} i} z)^{q-1} \tilde{\lambda}^l + \text{ lower order terms of } z. \end{aligned} \tag{37}$$

We remark that lower order terms of z are polynomials of z with degree at most $q-2$ (coefficients depend on $\tilde{\lambda}^l$).
Let $G(z, \tilde{\lambda}^l) = \mathcal{P}_V(z^{-q}, \lambda) z^{q-1}$. By (37), one has that

$$G(z, \tilde{\lambda}^l) = (-1)^q q e^{2\pi \frac{(q-1)l}{q} i} \tilde{\lambda}^l + z G_1(z, \tilde{\lambda}^l), \tag{38}$$

where $G_1(z, \tilde{\lambda}^l)$ is a polynomial of z and $\tilde{\lambda}^l$. Since $G(0, 0) = 0$ and $\partial_{\tilde{\lambda}^l} G(0, 0) = (-1)^q q e^{2\pi \frac{(q-1)l}{q} i} \neq 0$, the implicit function theorem implies that the equation $G(z, \tilde{\lambda}^l) = 0$ with the initial condition $\tilde{\lambda}^l(0) = 0$ has an analytic solution in a small neighborhood of $z = 0$. Now (17) follows from (36) and the definition of $G(z, \tilde{\lambda}^l)$. □

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