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Floquet isospectrality for periodic graph operators

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Abstract

Let $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$ with arbitrary positive integers q_l, $l = 1, 2, \cdots, d$. Let $\Delta_{\text{discrete}} + V$ be the discrete Schrödinger operator on $\mathbb{Z}^d$, where Δ_{discrete} is the discrete Laplacian on $\mathbb{Z}^d$ and the function $V : \mathbb{Z}^d \to \mathbb{C}$ is Γ-periodic. We prove two rigidity theorems for discrete periodic Schrödinger operators:

(1) If for real-valued Γ-periodic functions V and Y, the operators $\Delta_{\text{discrete}} + V$ and $\Delta_{\text{discrete}} + Y$ are Floquet isospectral and Y is separable, then V is separable.
(2) If for complex-valued Γ-periodic functions V and Y, the operators $\Delta_{\text{discrete}} + V$ and $\Delta_{\text{discrete}} + Y$ are Floquet isospectral, and both $V = \bigoplus_{j=1}^{r} V_j$ and $Y = \bigoplus_{j=1}^{r} Y_j$ are separable functions, then, up to a constant, lower dimensional decompositions V_j and Y_j are Floquet isospectral, $j = 1, 2, \cdots, r$.

Our theorems extend the results of Kappeler. Our approach is developed from the author's recent work on Fermi isospectrality and can be applied to studying more general lattices.

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1. Introduction and main results

In this paper, we first study the discrete periodic Schrödinger equation

$$(\Delta_{\text{discrete}}u)(n) + V(n)u(n) = \lambda u(n), n \in \mathbb{Z}^d, \tag{1}$$

with the so called Floquet-Bloch boundary condition

$$u(n + q_j\mathbf{e}_j) = e^{2\pi i k_j}u(n),\ j = 1, 2, \cdots, d,\ \text{and } n \in \mathbb{Z}^d, \tag{2}$$

where Δ_{discrete} is the discrete Laplacian on $\mathbb{Z}^d$, $\{\mathbf{e}_j\}_{j=1}^d$ is the standard basis in $\mathbb{R}^d$ and V is Γ-periodic with $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$.

By standard Floquet theory, the equation (1) with the boundary condition (2) can be realized as the eigen-equation of a matrix $D_V(k)$, where $k = (k_1, k_2, \cdots, k_d)$. Denote by $\sigma(D_V(k))$ the eigenvalues of $D_V(k)$, counted with their algebraic multiplicity. Since until Section 5 we only discuss discrete periodic Schrödinger equations (1) and (2), for simplicity, write Δ for Δ_{discrete}.

Definition 1. We say that two Γ-periodic operators $\Delta + V$ and $\Delta + Y$ (or equivalently two Γ-periodic functions/potentials V and Y) are Floquet isospectral if

$$\sigma(D_V(k)) = \sigma(D_Y(k)),\ \text{for any } k \in \mathbb{R}^d. \tag{3}$$

Studying Floquet isospectrality of periodic potentials started about four decades ago [5–7,12–14,16–18,29,33]. We refer readers to three surveys [20,21,27] for the background and recent development.

Definition 2. We say that a function V on $\mathbb{Z}^d$ is $(d_1, d_2, \cdots, d_r)$-separable (or simply separable), where $\sum_{j=1}^r d_j = d$ with $r \geq 2$, if there exist functions V_j on $\mathbb{Z}^{d_j}$, $j = 1, 2, \cdots, r$, such that for any $(n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$,

$$\begin{aligned} V(n_1, n_2, \cdots, n_d) =& V_1(n_1, \cdots, n_{d_1}) + V_2(n_{d_1+1}, n_{d_1+2}, \cdots, n_{d_1+d_2}) \\ &+ \cdots + V_r(n_{d_1+d_2+\cdots+d_{r-1}+1}, \cdots, n_{d_1+d_2+\cdots+d_r}). \end{aligned} \tag{4}$$

Definition 3. We say $V : \mathbb{Z}^d \to \mathbb{C}$ is completely separable if V is $(1, 1, \cdots, 1)$-separable.

When there is no ambiguity, we write down (4) as $V = \bigoplus_{j=1}^r V_j$.

The paper aims to investigate when periodic potentials are Floquet isospectral to separable potentials. In [6,7,12], Eskin-Ralston-Trubowitz and Gordon-Kappeler asked the following two questions:

Q1. If real-valued functions Y and V are Floquet isospectral, and Y is separable, is V separable?
Q2. Assume that both $Y = \bigoplus_{j=1}^r Y_j$ and $V = \bigoplus_{j=1}^r V_j$ are separable. If V and Y are Floquet isospectral, are the components V_j and Y_j (up to a constant) Floquet isospectral?

We remark that the original questions (Q1 and Q2) in [6,7,12] were asked in the setting of continuous Schrödinger equations with completely separable functions. It is natural to ask the above two questions in the discrete cases with arbitrary separable functions (e.g., [20, Section 5]).

In [18], Kappeler partially answered questions Q1 and Q2:

Theorem 1.1. *[18] Let $q_1 = q_2 = \cdots = q_d$. Then the following statements hold:*

(1) *If real-valued Γ-periodic functions Y and V are Floquet isospectral, and Y is completely separable, then V completely separable.*
(2) *Assume that complex-valued Γ-periodic functions $Y = \bigoplus_{j=1}^{d} Y_j$ and $V = \bigoplus_{j=1}^{d} V_j$ are completely separable. If V and Y are Floquet isospectral, then up to a constant, the one-dimensional functions V_j and Y_j are Floquet isospectral.*

The proof of Theorem 1.1 relies on spectral invariants of discrete Schrödinger equations. We will present a different approach to study the Floquet isospectrality based on the idea introduced by the author in [25].

Definition 4. The *Bloch variety* $B(V)$ of $\Delta + V$ is defined as

$$B(V) = \{(k, \lambda) \in \mathbb{C}^{d+1} : \det(D_V(k) - \lambda I) = 0\}.$$

Given $\lambda \in \mathbb{C}$, the Fermi surface (variety) $F_\lambda(V)$ is defined as the level set of the Bloch variety:

$$F_\lambda(V) = \{k \in \mathbb{C}^d : (k, \lambda) \in B(V)\}.$$

Both Fermi and Bloch varieties (zero sets of $\det(D_V(k) - \lambda I)$) play a crucial role in the study of (inverse) spectral and related problems arising in periodic operators [1,4,11,15,19,22,23,31,32], such as embedded eigenvalue problems and integrated density of states. Clearly, $\det(D_V(k) - \lambda I)$ is algebraic in appropriate coordinates. Recently, the author discovered that the algebraic properties of Fermi and Bloch varieties are related to more spectral problems such as the quantum ergodicity [24,30], spectral edges [26] and isospectrality problems [25,28].

A basic fact of linear algebra allows to reformulate Floquet isospectrality as $\det(D_V(k) - \lambda I) \equiv \det(D_Y(k) - \lambda I)$. Like in the author's recent works [24–26], in this paper we will focus on studying $\det(D_V(k) - \lambda I)$. One of the advantages of using $\det(D_V(k) - \lambda I)$ (Bloch/Fermi varieties) is to enable us to apply various tools from algebraic/analytic geometry and complex analysis.

In [26], the author proved the irreducibility conjectures of Bloch and Fermi varieties (see [8–10] for more recent works of the irreducibility of Bloch and Fermi varieties), where those conjectures were only previously studied for $d = 2, 3$ [2,3,11]. This particularly implies that $B(V) = B(Y)$ iff $\det(D_V(k) - \lambda I) \equiv \det(D_Y(k) - \lambda I)$. Therefore functions V and Y are Floquet isospectral iff $B(V) = B(Y)$. In [25], the author introduced a new type of inverse spectral problem-Fermi isospectrality.

Definition 5. [25] Let V and Y be two Γ-periodic functions, and $\lambda_0 \in \mathbb{C}$. We say that V and Y are Fermi isospectral at the level λ_0 if $F_{\lambda_0}(V) = F_{\lambda_0}(Y)$.

From the definitions, one can see that Fermi isospectrality is a "hyperplane" version (weaker assumption) of Floquet isospectrality. In [25], the author proved the following rigidity statements.

Theorem 1.2. *[25] Assume that $d \geq 3$ and q_j, $j = 1, 2, \cdots, d$ are piecewise co-prime. Then the following statements hold:*

(1) *If real-valued Γ-periodic functions Y and V are Fermi isospectral, and Y is separable, then V separable.*
(2) *Assume that complex-valued Γ-periodic functions $Y = \bigoplus_{j=1}^{r} Y_j$ and $V = \bigoplus_{j=1}^{r} V_j$ are separable. If V and Y are Fermi isospectral, then up to a constant, lower dimensional decompositions V_j and Y_j are Floquet isospectral, $j = 1, 2, \cdots, r$.*

Theorem 1.2 immediately implies

Corollary 1.3. *Assume that $d \geq 3$ and q_j, $j = 1, 2, \cdots, d$ are piecewise co-prime. Then the following statements hold:*

(1) *If real-valued Γ-periodic functions Y and V are Floquet isospectral, and Y is separable, then V separable.*
(2) *Assume that complex-valued Γ-periodic functions $Y = \bigoplus_{j=1}^{r} Y_j$ and $V = \bigoplus_{j=1}^{r} V_j$ are separable. If V and Y are Floquet isospectral, then up to a constant, lower dimensional decompositions V_j and Y_j are Floquet isospectral, $j = 1, 2, \cdots, r$.*

The aim of this paper is twofold. Firstly, we answer Q1 and Q2 for arbitrary positive integers q_l, $l = 1, 2, \cdots, d$ and any types of separable functions which extends Theorem 1.1 and Corollary 1.3. Secondly, we propose a simple and new approach to study the Floquet isospectrality based on the ideas formulated in [25]. It turns out that our approach is quite robust and works for more general lattices (see Section 5).

The main results of the present paper are

Theorem 1.4. *Let real-valued Γ-periodic functions V and Y be Floquet isospectral, and Y be $(d_1, d_2, \cdots, d_r)$-separable, then V is $(d_1, d_2, \cdots, d_r)$-separable.*

Theorem 1.5. *Let complex-valued Γ-periodic functions V and Y be Floquet isospectral, and both $V = \bigoplus_{j=1}^{r} V_j$ and $Y = \bigoplus_{j=1}^{r} Y_j$ be separable functions. Then, up to a constant, lower dimensional decompositions V_j and Y_j are Floquet isospectral, $j = 1, 2, \cdots, r$.*

Theorems 1.4 and 1.5 imply

Theorem 1.6. *Assume that real-valued functions V and Y are Floquet isospectral, and $Y = \bigoplus_{j=1}^{r} Y_j$ is separable. Then $V = \bigoplus_{j=1}^{r} V_j$ is separable. Moreover, up to a constant, Y_j and V_j, $j = 1, 2, \cdots, r$, are Floquet isospectral.*

Although the approach comes from [25], the technical parts are different. In the following, we will present the main ideas of the proof of Theorem 1.4. Let us take $r = 2$ as an example. A basic fact of discrete Fourier transform states that a function V is (d_1, d_2)-separable iff for any nontrivial (non-zero modulo periodicity) $l_1 \in \mathbb{Z}^{d_1}$ and $l_2 \in \mathbb{Z}^{d_2}$, $\hat{V}(l_1, l_2) = 0$, where $\hat{V}$ is the discrete Fourier transform of V. In [25], to prove the separability of functions, the author used a direct way to show that $\hat{V}(l_1,l_2)=0$. In the present work, we use a detour. We first show that for Floquet isospectral functions V and Y, $\sum_{l_1,l_2}|\hat{V}(l_1,l_2)|^2=\sum_{l_1,l_2}|\hat{Y}(l_1,l_2)|^2$, $\sum_{l_1}|\hat{V}(l_1,0)|^2=\sum_{l_1}|\hat{Y}(l_1,0)|^2$ and $\sum_{l_2}|\hat{V}(0,l_2)|^2=\sum_{l_2}|\hat{Y}(0,l_2)|^2$. Therefore, $\hat{Y}(l_1,l_2)=0$ for all nontrivial (non-zero modulo periodicity) l_1 and l_2 implies that for all nontrivial l_1 and l_2 $\hat{V}(l_1,l_2)=0$.

The techniques of this paper can be applied to study the isospectrality problems for periodic operators on more general lattices. In Section 5, we show the generalization for the triangular lattice.

The rest of this paper is organized as follows. In Section 2, we recall some basics about the discrete periodic Schrödinger equations. Sections 3 and 4 are devoted to proving Theorems 1.4 and 1.5. In Section 5, we discuss the Floquet isospectrality of periodic operators on the triangular lattice.

2. Basics

In this section, we first recall some basic facts about the Fermi variety, see, e.g., [20,26,27].

Let W be a fundamental domain for $\Gamma=q_1\mathbb{Z}\oplus q_2\mathbb{Z}\oplus\cdots\oplus q_d\mathbb{Z}$:

$$W=\{n=(n_1,n_2,\cdots,n_d)\in\mathbb{Z}^d:0\leq n_j\leq q_j-1,\ j=1,2,\cdots,d\}.$$

By writing out the equation (1) on the $Q=q_1q_2\cdots q_d$ dimensional space $\{u(n),n\in W\}$, the equation (1) with the boundary condition (2) translates into the eigenvalue problem for a $Q\times Q$ matrix $D_V(k)$.

Let $\mathbb{C}^\star=\mathbb{C}\backslash\{0\}$ and $z=(z_1,z_2,\cdots,z_d)$. Let $z_j=e^{2\pi i k_j}$, $j=1,2,\cdots,d$, $\mathcal{D}_V(z)=D_V(k)$ and $\mathcal{P}_V(z,\lambda)=\det(\mathcal{D}_V(z)-\lambda I)$.

By the basic facts of linear algebra, one has

Proposition 2.1. *Let $d\geq 1$. Two complex-valued Γ-periodic functions V and Y are Floquet isospectral iff $\mathcal{P}_V(z,\lambda)\equiv\mathcal{P}_Y(z,\lambda)$.*

Define the discrete Fourier transform $\hat{V}(l)$ for $l\in W$ by

$$\hat{V}(l)=\frac{1}{Q}\sum_{n\in W}V(n)\exp\left\{-2\pi i\left(\sum_{j=1}^{d}\frac{l_jn_j}{q_j}\right)\right\}.$$

For convenience, we extend $\hat{V}(l)$ to $\mathbb{Z}^d$ periodically, namely, for any $l\equiv m \mod \Gamma$,

$$\hat{V}(l)=\hat{V}(m).$$

Define

$$\tilde{\mathcal{D}}_V(z)=\tilde{\mathcal{D}}_V(z_1,z_2,\cdots,z_d)=\mathcal{D}_V(z_1^{q_1},z_2^{q_2},\cdots,z_d^{q_d}),\tag{5}$$

and

$$\tilde{\mathcal{P}}_V(z,\lambda)=\det(\tilde{\mathcal{D}}_V(z,\lambda)-\lambda I)=\mathcal{P}_V(z_1^{q_1},z_2^{q_2},\cdots,z_d^{q_d},\lambda). \tag{6}$$

Let

$$\rho_{n_j}^j=e^{2\pi\frac{n_j}{q_j}i},$$

where $0\leq n_j\leq q_j-1$, $j=1,2,\cdots,d$.

A straightforward application of the discrete Floquet transform (e.g., [9,20,26]) leads to

Lemma 2.2. *Let $n=(n_1,n_2,\cdots,n_d)\in W$ and $n'=(n_1',n_2',\cdots,n_d')\in W$. Then $\tilde{\mathcal{D}}_V(z)$ is unitarily equivalent to $A+B_V$, where A is a diagonal matrix with entries*

$$A(n;n')=\left(\sum_{j=1}^d\left(\rho_{n_j}^jz_j+\frac{1}{\rho_{n_j}^jz_j}\right)\right)\delta_{n,n'} \tag{7}$$

and

$$B_V(n;n')=\hat{V}\left(n_1-n_1',n_2-n_2',\cdots,n_d-n_d'\right). \tag{8}$$

In particular,

$$\tilde{\mathcal{P}}_V(z,\lambda)=\det(A+B_V-\lambda I).$$

3. Proof of Theorem 1.4

Assume $\sum_{j=1}^r d_j=d$ with $r\geq 2$ and $d_j\geq 1$, $j=1,2,\cdots,r$. For convenience, let $d_0=0$. For any $l=(l_1,l_2,\cdots,l_d)\in\mathbb{Z}^d$ and $j=1,2,\cdots,r$, denote by

$$\tilde{l}_j=(l_{1+\sum_{m=0}^{j-1}d_m},l_{2+\sum_{m=0}^{j-1}d_m},\cdots,l_{d_j+\sum_{m=0}^{j-1}d_m})\in\mathbb{Z}^{d_j}. \tag{9}$$

For $j=1,2,\cdots,r$, denote by

$$W_j=\{n\in\mathbb{Z}^{d_j}:0\leq n_b\leq q_{b+\sum_{m=0}^{j-1}d_m}-1,b=1,2,\cdots,d_j\}.$$

For any $n\in\mathbb{Z}^{d_j}$, $j=1,2,\cdots,r$, denote by

$$n^{j\neg}=(\overbrace{0,0,\cdots,0}^{a_j},n_1,n_2,\cdots,n_{d_j},\overbrace{0,0,\cdots,0}^{b_j})\in\mathbb{Z}^d, \tag{10}$$

where $a_j=\sum_{m=0}^{j-1}d_m$ for $j=1,2,\cdots,r$ and $b_j=\sum_{m=j+1}^r d_m$ for $j=1,2,\cdots,r-1$ (set $b_r=0$).

Lemma 3.1. *(e.g., [25]) A function V is $(d_1,d_2,\cdots,d_r)$-separable if and only if for any $l\in W$ with at least two non-zero $\tilde{l}_j$, $j=1,2,\cdots,r$, $\hat{V}(l)=0$.*

Denote by $[V]$ the average of V:

$$[V] = \frac{1}{Q} \sum_{n \in W} V(n).$$

Lemma 3.2. *Assume that real-valued* Γ*-periodic functions* V *and* Y *are Floquet isospectral. Then*

$$[V] = [Y] \tag{11}$$

and

$$\begin{aligned} &\sum_{\substack{n \in W \\ n' \in W}} \frac{|\hat{V}(n - n')|^2}{(-\lambda + \sum_{j=1}^{d} \rho_{n_j}^{j} z_j)(-\lambda + \sum_{j=1}^{d} \rho_{n'_j}^{j} z_j)} \\ &\equiv \sum_{\substack{n \in W \\ n' \in W}} \frac{|\hat{Y}(n - n')|^2}{(-\lambda + \sum_{j=1}^{d} \rho_{n_j}^{j} z_j)(-\lambda + \sum_{j=1}^{d} \rho_{n'_j}^{j} z_j)}. \end{aligned} \tag{12}$$

The proof of Lemma 3.2 is similar to that of Theorem 4.1 [25]. We include a proof in the appendix for the readers' convenience.

Theorem 3.3. *Assume that real-valued* Γ*-periodic functions* V *and* Y *are Floquet isospectral. Then we have that*

$$\sum_{l \in W} |\hat{V}(l)|^2 = \sum_{l \in W} |\hat{Y}(l)|^2, \tag{13}$$

and for any $j = 1, 2, \cdots, r$,

$$\sum_{l \in W_j} |\hat{V}(l^{j\neg})|^2 = \sum_{l \in W_j} |\hat{Y}(l^{j\neg})|^2. \tag{14}$$

Proof. Letting $z = 0$ in (12), one has (13). Without loss of generality, we only prove (14) for $j = 1$. Rewrite (12) as

$$\begin{aligned} &\sum_{n \in W, l \in W} \frac{|\hat{V}(l)|^2}{(-\lambda + \sum_{j=1}^{d} \rho_{n_j}^{j} z_j)(-\lambda + \sum_{j=1}^{d} \rho_{n_j + l_j}^{j} z_j)} \\ &\equiv \sum_{n \in W, l \in W} \frac{|\hat{Y}(l)|^2}{(-\lambda + \sum_{j=1}^{d} \rho_{n_j}^{j} z_j)(-\lambda + \sum_{j=1}^{d} \rho_{n_j + l_j}^{j} z_j)}. \end{aligned} \tag{15}$$

Let $z_1 = z_2 = \cdots = z_{d_1} = 0$ and consider the plane $-\lambda + \sum_{j=d_1+1}^{d} z_j = 0$. Since for any $(n_{d_1+1}, n_{d_1+2}, \cdots, n_d)$ with $(n_{d_1+1}, n_{d_1+2}, \cdots, n_d) \neq 0$ (modulo periodicity), planes $-\lambda +$

$\sum_{j=d_1+1}^{d} z_j = 0$ and $-\lambda + \sum_{j=d_1+1}^{d} \rho_{n_j}^j z_j = 0$ are not the same. This implies that the leading term in the sum of (15) (double zeros in the denominators) consists of

$$\frac{1}{(-\lambda + \sum_{j=d_1+1}^{d} \rho_{n_j}^j z_j)(-\lambda + \sum_{j=d_1+1}^{d} \rho_{n_j+l_j}^j z_j)}$$

with $n_{d_1+1} = n_{d_1+2} = \cdots = n_d = 0$ and $l_{d_1+1} = l_{d_1+2} = \cdots = l_d = 0$. Therefore, one has

$$\sum_{\substack{l \in W \\ l_m=0, d_1+1 \le m \le d}} |\hat{V}(l)|^2 = \sum_{\substack{l \in W \\ l_m=0, d_1+1 \le m \le d}} |\hat{Y}(l)|^2. \tag{16}$$

This implies (14). □

Proof of Theorem 1.4. Let $S \subset W$ consist of all l with at least two of $\tilde{l}_j$, $j = 1, 2, \cdots, r$ non-zero.

By Lemma 3.1, one has

$$\sum_{l \in S} |\hat{Y}(l)|^2 = 0. \tag{17}$$

By Theorem 3.3 and (17), one has

$$\sum_{l \in S} |\hat{V}(l)|^2 = 0. \tag{18}$$

By Lemma 3.1 again, V is separable. □

4. Proof of Theorem 1.5

Proof of Theorem 1.5. By induction, it suffices to prove the case $r = 2$. Recall (11). Without loss of generality, assume $[V_1] = [V_2] = [Y_1] = [Y_2] = 0$ and thus $[V] = [Y] = 0$. Fix any y. Let

$$\lambda = \lambda(z_1, z_2, \cdots, z_{d_1}, y) = y + \sum_{j=1}^{d_1} \left(z_j + \frac{1}{z_j} \right). \tag{19}$$

Then for any $n \in W_1 \backslash \{(0, 0, \cdots, 0)\}$,

$$-\lambda + \sum_{j=1}^{d_1} \left(\rho_{n_j}^j z_j + \frac{1}{\rho_{n_j}^j z_j} \right) = -y + \sum_{j=1}^{d_1} (-1 + \rho_{-n_j}^j) \frac{1}{z_j} + \sum_{j=1}^{d_1} (-1 + \rho_{n_j}^j) z_j. \tag{20}$$

Comparing the coefficients of highest degree terms (the highest degree is $Q - |W_2|$) of $(z_1, z_2, \cdots, z_{d_1})$ in both $\tilde{\mathcal{P}}_V(z, \lambda)$ and $\tilde{\mathcal{P}}_Y(z, \lambda)$ with λ given by (19), one has that

$$\tilde{\mathcal{P}}_{Y_2}(z_{d_1+1}, z_{d_1+2}, \cdots, z_d, y) \equiv \tilde{\mathcal{P}}_{V_2}(z_{d_1+1}, z_{d_1+2}, \cdots, z_d, y).$$

This implies V_2 and Y_2 are Floquet isospectral. Interchanging, V_1 and V_2, Y_1 and Y_2, we have V_1 and Y_1 are Floquet isospectral. We complete the proof. □

5. Floquet isospectrality for periodic Schrödinger operators on the triangular lattice

We consider discrete periodic Schrödinger equations on the triangular lattice,

$$(\Delta_{\mathrm{Tri}} u)(n) + V(n)u(n) = \lambda u(n), n \in \mathbb{Z}^2, \tag{21}$$

with the boundary condition

$$u(n + q_j \mathbf{e}_j) = e^{2\pi i k_j} u(n),\ j = 1, 2 \text{ and } n \in \mathbb{Z}^2, \tag{22}$$

where $\{\mathbf{e}_1 = (0, 1), \mathbf{e}_2 = (0, 1)\}$ is the standard basis in $\mathbb{R}^2$, V is Γ-periodic with $\Gamma = q_1\mathbb{Z} \oplus q_2\mathbb{Z}$, and Δ_{Tri} is the discrete Laplacian on the triangular lattice, namely

$$\begin{aligned}(\Delta_{\mathrm{Tri}} u)(n_1, n_2) =& u(n_1 + 1, n_2) + u(n_1 - 1, n_2) + u(n_1, n_2 + 1) + u(n_1, n_2 - 1) \\ &+ u(n_1 + 1, n_2 - 1) + u(n_1 - 1, n_2 + 1).\end{aligned}$$

We say $\Delta_{\mathrm{Tri}} + V$ and $\Delta_{\mathrm{Tri}} + Y$ are Floquet isospectral if for any $k \in \mathbb{R}^2$, $\Delta_{\mathrm{Tri}} + V$ and $\Delta_{\mathrm{Tri}} + Y$ with the boundary condition (22) have the same eigenvalues.

Theorem 5.1. *Assume that real-valued functions V and Y on $\mathbb{Z}^2$ satisfy that $\Delta_{\mathrm{Tri}} + V$ and $\Delta_{\mathrm{Tri}} + Y$ are Floquet isospectral, and Y is separable, then V is separable.*

We can transfer all notions for discrete periodic Schrödinger equations on the lattice $\mathbb{Z}^2$ to the triangular lattice. For example, we will use $D_{\mathrm{Tri},V}(k)$ to represent the equation (21) with the boundary condition (22). Similarly, we will use notations $\mathcal{D}_{\mathrm{Tri},V}$, $\tilde{\mathcal{P}}_{\mathrm{Tri},V}(z, \lambda)$, A_{Tri} and $B_{\mathrm{Tri},V}$.

By the standard discrete Floquet transform (e.g., [9,20]), one has

Lemma 5.2. *The matrix $\tilde{\mathcal{D}}_{\mathrm{Tri},V}(z)$ is unitarily equivalent to $A_{\mathrm{Tri}} + B_{\mathrm{Tri},V}$, where A_{Tri} is a diagonal matrix with entries*

$$A_{\mathrm{Tri}}(n; n') = \left(\rho^1_{n_1} z_1 + \frac{1}{\rho^1_{n_1} z_1} + \rho^2_{n_2} z_2 + \frac{1}{\rho^2_{n_2} z_2} + \frac{\rho^2_{n_2} z_2}{\rho^1_{n_1} z_1} + \frac{\rho^1_{n_1} z_1}{\rho^2_{n_2} z_2}\right) \delta_{n,n'} \tag{23}$$

and

$$B_{\mathrm{Tri},V}(n; n') = \hat{V}\left(n_1 - n'_1, n_2 - n'_2\right). \tag{24}$$

In particular,

$$\tilde{\mathcal{P}}_{\mathrm{Tri},V}(z, \lambda) = \det(A_{\mathrm{Tri}} + B_{\mathrm{Tri},V} - \lambda I).$$

Proof of Theorem 5.1 . By Lemma 5.2, we can first show that (11) and (12) hold by a similar argument in the appendix. One can see that the proof of Theorem 1.4 only uses (11) and (12). So Theorem 5.1 holds. □

Data availability

No data was used for the research described in the article.

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Appendix A. Proof of Lemma 3.2

For $a = (a_1, a_2, \cdots, a_d) \in \mathbb{Z}^d$ and $z^a = z_1^{a_1} z_2^{a_2} \cdots z_d^{a_d}$, denote by $|a| = a_1 + a_2 + \cdots + a_d$ the degree of z^a.

Proof. By Proposition 2.1, one has

$$\mathcal{P}_V(z,\lambda) \equiv \mathcal{P}_Y(z,\lambda), \tag{25}$$

and hence

$$\tilde{\mathcal{P}}_V(z,\lambda) \equiv \tilde{\mathcal{P}}_Y(z,\lambda). \tag{26}$$

For any $n = (n_1, n_2, \cdots, n_d) \in W$, let

$$t_n(z,x) = x + \sum_{j=1}^{d} \left(\rho^j_{n_j} z_j + \frac{1}{\rho^j_{n_j} z_j} \right). \tag{27}$$

Clearly, $t_n(z, [V] - \lambda)$ is the n-th diagonal entry of the matrix $A + B_V$. By Lemma 2.2, direct calculations imply that

$$\begin{aligned}\tilde{\mathcal{P}}_V(z,\lambda) &= \det(A + B_V - \lambda I) \\ &= \prod_{n\in W} t_n(z,[V]-\lambda) - \frac{1}{2} \sum_{\substack{n\in W, n'\in W \\ n\neq n'}} \frac{\prod_{n\in W} t_n(z,[V]-\lambda)}{t_{n'}(z,[V]-\lambda)t_n(z,[V]-\lambda)} |B_V(n;n')|^2 \\ &\quad + \text{ products of at most } q_1 q_2 \cdots q_d - 3 \text{ terms of } t_n(z,[V]-\lambda). \end{aligned} \tag{28}$$

Let

$$h(z,\lambda) = \prod_{n\in W} \left(-\lambda + \sum_{j=1}^{d} \rho^j_{n_j} z_j \right),$$

and $h^1_V(z,\lambda)$ be all terms in $\tilde{\mathcal{P}}_V(z,\lambda)$ consisting of $z^a \lambda^b$ with $|a| + b = q_1 q_2 \cdots q_d - 1$.

By (28), one has that $h^1_V(z,\lambda)$ must come from $\prod_{n\in W} t_n(z,[V]-\lambda)$ and hence

$$h^1_V(z,\lambda)=[V]\left(\sum_{n\in W}\frac{h(z,\lambda)}{-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j}\right). \tag{29}$$

By (26) and (29), one has

$$\sum_{n\in W}\frac{[V]}{-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j}\equiv\sum_{n\in W}\frac{[Y]}{-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j}. \tag{30}$$

This implies $[V]=[Y]$.

Let $h^2_V(z,\lambda)$ ($\bar{h}^2_V(z,\lambda)$) be all terms in $\tilde{\mathcal{P}}_V(z,\lambda)$ ($\prod_{n\in W} t_n(z,[V]-\lambda)$) consisting of $z^a\lambda^b$ with $|a|+b=q_1q_2\cdots q_d-2$.

By the fact that $[V]=[Y]$, one has that $\bar{h}^2_V(z,\lambda)\equiv\bar{h}^2_Y(z,\lambda)$. Therefore, by (26), we have that

$$h^2_V(z,\lambda)-\bar{h}^2_V(z,\lambda)\equiv h^2_Y(z,\lambda)-\bar{h}^2_Y(z,\lambda). \tag{31}$$

By (28), one has that

$$h^2_V(z,\lambda)-\bar{h}^2_V(z,\lambda)=\frac{h(z,\lambda)}{2}\sum_{\substack{n\in W,n'\in W\\ n\neq n'}}\frac{-|B_V(n;n')|^2}{(-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j)(-\lambda+\sum_{j=1}^d \rho^j_{n'_j}z_j)}. \tag{32}$$

By (31) and (32), one has

$$\sum_{\substack{n\in W,n'\in W\\ n\neq n'}}\frac{|B_V(n;n')|^2}{(-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j)(-\lambda+\sum_{j=1}^d \rho^j_{n'_j}z_j)}$$
$$\equiv\sum_{\substack{n\in W,n'\in W\\ n\neq n'}}\frac{|B_Y(n;n')|^2}{(-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j)(-\lambda+\sum_{j=1}^d \rho^j_{n'_j}z_j)},$$

and hence

$$\sum_{\substack{n\in W,n'\in W\\ n\neq n'}}\frac{|\hat{V}(n-n')|^2}{(-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j)(-\lambda+\sum_{j=1}^d \rho^j_{n'_j}z_j)}$$
$$\equiv\sum_{\substack{n\in W,n'\in W\\ n\neq n'}}\frac{|\hat{Y}(n-n')|^2}{(-\lambda+\sum_{j=1}^d \rho^j_{n_j}z_j)(-\lambda+\sum_{j=1}^d \rho^j_{n'_j}z_j)}. \tag{33}$$

By (11) and (33), one has (12). □

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