Journal of Combinatorial Theory, Series B 166 (2024) 154–182

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

journal homepage: www.elsevier.com/locate/jctb

The core conjecture of Hilton and Zhao

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A R T I C L E I N F O

Article history:
Received 12 August 2021
Available online 25 January 2024

Keywords:
Overfull graph
Multifan
Kierstead path
Pseudo-multifan
Lollipop

A B S T R A C T

A simple graph G with maximum degree Δ is *overfull* if $|E(G)| > \Delta\lfloor |V(G)|/2\rfloor$. The *core* of G, denoted G_Δ, is the subgraph of G induced by its vertices of degree Δ. Clearly, the chromatic index of G equals $\Delta + 1$ if G is overfull. Conversely, Hilton and Zhao in 1996 conjectured that if G is a simple connected graph with $\Delta \geq 3$ and $\Delta(G_\Delta) \leq 2$, then $\chi'(G) = \Delta + 1$ implies that G is overfull or $G = P^*$, where P^* is obtained from the Petersen graph by deleting a vertex. Cariolaro and Cariolaro settled the base case $\Delta = 3$ in 2003, and Cranston and Rabern proved the next case, $\Delta = 4$, in 2019. In this paper, we give a proof of this conjecture for all $\Delta \geq 4$.

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https://doi.org/10.1016/j.jctb.2024.01.004
0095-8956/

1. Introduction

For two integers p and q, let $[p,q] = \{i \in \mathbb{Z} : p \le i \le q\}$. Let $k \ge 0$ be an integer and G be a simple graph with maximum degree Δ. An *edge k-coloring* of G is a mapping φ from $E(G)$ to $[1,k]$, called *colors*, such that no two adjacent edges receive the same color with respect to φ. The *chromatic index* of G, denoted $\chi'(G)$, is the smallest k so that G has an edge k-coloring. In 1960's, Gupta [8] and, independently, Vizing [13] proved that for all graphs G, $\Delta \le \chi'(G) \le \Delta + 1$. This leads to a natural classification of simple graphs. Following Fiorini and Wilson [6], a graph G is of *class 1* if $\chi'(G) = \Delta$ and of *class 2* if $\chi'(G) = \Delta+1$. Holyer [11] showed that it is NP-complete to determine whether an arbitrary graph is of class 1.

Following terminologies from [12], the *core* of G, denoted G_Δ, is the subgraph of G induced by its vertices of degree Δ, and we call G *overfull* if $|E(G)| > \Delta\lfloor |V(G)|/2 \rfloor$. Overfullness of graphs is closely related to the *fractional chromatic index*, $\chi'_f(G)$. In the area of edge coloring, the *density* of G, denoted $\rho(G)$, is defined as the maximum of $|E(G[U])|/((|U|-1)/2)$ ranging over all $U \subseteq V(G)$ with $|U|$ odd and at least 3. In fact, $\chi'_f(G) = \max\{\Delta(G), \rho(G)\}$, and $\chi'_f(G)$ can be computed in polynomial time [4]. From the definition, we know that G contains an overfull subgraph of the same maximum degree if and only if $\chi'_f(G) > \Delta(G)$.

Working on graphs whose core has a simple structure (see [12, Sect. 4.2]), Vizing [13] proved that if G_Δ has at most two vertices then G is class 1; Fournier [7] and also independently Ehrenfeucht, Faber, and Kierstead [5, Lemma 3] generalized Vizing's result by showing that if G_Δ contains no cycles then G is class 1. Thus a necessary condition for a graph to be class 2 is to have a core that contains cycles. In 1996, Hilton and Zhao [10] proposed the following conjecture.

Conjecture 1.1 *(Core Conjecture). Let G be a simple connected graph with maximum degree $\Delta \ge 3$ and $\Delta(G_\Delta) \le 2$. Then G is class 2 implies that G is overfull or $G = P^*$.*

As a class 2 graph of maximum degree 2 is an odd cycle and odd cycles are overfull, if true, the Core Conjecture implies that for connected graphs G with $\Delta(G_\Delta) \le 2$, determining whether G is class 2 can be done by checking whether $|E(G)| > \Delta\lfloor |V(G)|/2 \rfloor$ if $G \ne P^*$. We call a connected class 2 graph G with $\Delta(G_\Delta) \le 2$ an *HZ-graph*. A first breakthrough of the Core Conjecture was achieved in 2003, when Cariolaro and Cariolaro [2] settled the base case $\Delta = 3$. They proved that P^* is the only HZ-graph with maximum degree $\Delta = 3$, an alternative proof was given later by Král', Sereny, and Stiebitz (see [12, pp. 67–63]). The next case, $\Delta = 4$, was recently solved by Cranston and Rabern [3]: they proved that the only HZ-graph with maximum degree $\Delta = 4$ is the graph obtained from K_5 with one edge removed. In this paper, we confirm the Core Conjecture for all HZ-graphs G with $\Delta \ge 4$. It worth mentioning that our proof implies a polynomial-time algorithm that, given G with maximum degree $\Delta \ge 4$ and $\Delta(G_\Delta) \le 2$, finds an optimal edge coloring of G.

Theorem 1.2. *Let G be a connected graph with maximum degree $\Delta \geq 4$ and $\Delta(G_\Delta) \leq 2$. Then G is class 2 if and only if G is overfull.*

Since every overfull graph is class 2, we will only prove the "only if" statement in Theorem 1.2. The remainder of the paper is organized as follows. In the next section, we prove Theorem 1.2 by applying Theorems 2.3 to 2.5. In Section 3, we give necessary definitions and list results from [1]. Theorems 2.3 to 2.5 will be proved in Sections 4, 5, and 6, respectively.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by applying Theorems 2.3 to 2.5. We start with some concepts. For an integer $k \geq 0$, we denote by $\mathcal{C}^k(G)$ the set of all edge k-colorings of G. The symbol Δ is reserved for $\Delta(G)$, the maximum degree of G throughout this paper.

Let G be a graph, $v \in V(G)$, and $i \geq 0$ be an integer. An *i-vertex* is a vertex of degree i in G, and an i-vertex from the neighborhood of v is called an *i-neighbor* of v. Define

$$V_i = \{w \in V(G) : d_G(w) = i\}, \qquad N_i(v) = N_G(v) \cap V_i, \quad \text{and} \quad N_i[v] = N_i(v) \cup \{v\}.$$

For $X \subseteq V(G)$, let $N_G(X) = \bigcup_{x \in X} N_G(x)$ and $N_i(X) = N_G(X) \cap V_i$.

Let $e \in E(G)$ and $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$. The set of colors *present* at v is $\varphi(v) = \{\varphi(f) : f \text{ is incident to } v\}$, and the set of colors *missing* at v is $\overline{\varphi}(v) = [1, k] \setminus \varphi(v)$. If $\overline{\varphi}(v) = \{\alpha\}$ is a singleton for some $\alpha \in [1, k]$, we also write $\overline{\varphi}(v) = \alpha$. For $X \subseteq V(G)$, let $\overline{\varphi}(X) = \bigcup_{x \in X} \overline{\varphi}(x)$. The set X is *φ-elementary* if $\overline{\varphi}(x) \cap \overline{\varphi}(y) = \emptyset$ for any distinct $x, y \in X$.

An edge $e \in E(G)$ is a *critical edge* of G if $\chi'(G - e) < \chi'(G)$, and G is *edge Δ-critical* or simply *Δ-critical* if G is connected, $\chi'(G) = \Delta + 1$, and each of its edges is critical. The following result by Hilton and Zhao in [9] reveals certain properties of an HZ graph.

Lemma 2.1. *If G is an HZ-graph with maximum degree Δ, then the following holds.*

(a) G is Δ-critical and G_Δ is 2-regular.
(b) $\delta(G) = \Delta - 1$, or $\Delta = 2$ and G is an odd cycle.
(c) Every vertex of G has at least two neighbors in G_Δ.

Let $\Delta \geq 4$ and let $\mathcal{O}_\Delta$ be the set of all graphs obtained from two graphs H_1 and H_2 by adding all edges between $V(H_1)$ and $V(H_2)$, where H_1 is any 2-regular graph on n_1 vertices, H_2 is any $(\Delta - 1 - n_1)$-regular graph on $(\Delta - 2)$ vertices, and $n_1 \in [3, \Delta - 1]$ such that $n_1 + (\Delta - 2)$ is odd. Stiebitz et al. showed that Conjecture 1.1 is equivalent to the conjecture below.

Conjecture 2.2 *([12, Conjecture 4.10]). If G is an HZ-graph with maximum degree Δ, then either $G \in \mathcal{O}_\Delta$, or $\Delta = 2$ and G is an odd cycle, or $\Delta = 3$ and $G = P^*$.*

We will prove this equivalent form of the Core Conjecture for $\Delta \geq 4$ by applying the following results.

Theorem 2.3. *If G is an HZ-graph with maximum degree $\Delta \geq 4$, then the following two statements hold.*

(i) For any two adjacent vertices $u, v \in V_\Delta$, $N_{\Delta-1}(u) = N_{\Delta-1}(v)$.
(ii) For any $r \in V_\Delta$, there exist $s \in N_{\Delta-1}(r)$ and $\varphi \in \mathcal{C}^\Delta(G - rs)$ such that $N_{\Delta-1}[r]$ is φ-elementary.

For an HZ-graph G with maximum degree $\Delta \geq 4$, each component of G_Δ is a cycle by Lemma 2.1. So Theorem 2.3 (i) implies that $N_{\Delta-1}(x) = N_{\Delta-1}(y)$ for any two vertices x, y from the same cycle of G_Δ.

Theorem 2.4. *If G is an HZ-graph with maximum degree $\Delta \geq 4$, then for any two adjacent vertices $x, y \in V_{\Delta-1}$, $N_\Delta(x) = N_\Delta(y)$.*

Theorem 2.5. *Let G be an HZ-graph with maximum degree $\Delta \geq 7$ and $u, r \in V_\Delta$. If $N_{\Delta-1}(u) \neq N_{\Delta-1}(r)$ and $N_{\Delta-1}(u) \cap N_{\Delta-1}(r) \neq \emptyset$, then $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| = \Delta - 3$, i.e. $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| = |N_{\Delta-1}(r) \setminus N_{\Delta-1}(u)| = 1$.*

Corollary 2.6. *If G is an HZ-graph with maximum degree $\Delta \geq 7$ and there exist $u, v \in V_\Delta$ such that $N_{\Delta-1}(u) \neq N_{\Delta-1}(v)$, then $V_{\Delta-1}$ is an independent set in G.*

Proof. Assume to the contrary that there exist $x, y \in V_{\Delta-1}$ such that $xy \in E(G)$. By Lemma 2.1, there exists $w \in N_\Delta(x)$. By the assumption that there exist $u, v \in V_\Delta$ such that $N_{\Delta-1}(u) \neq N_{\Delta-1}(v)$, there exists some $w' \in V_\Delta$ such that $N_{\Delta-1}(w) \neq N_{\Delta-1}(w')$. We may further assume that the distance between w and w' in G is shortest among all pairs of vertices w_1 and w_1' such that $w_1 \in N_\Delta(x)$ and $N_{\Delta-1}(w_1) \neq N_{\Delta-1}(w_1')$. We claim that $N_{\Delta-1}(w) \cap N_{\Delta-1}(w') \neq \emptyset$. Let P be a shortest path connecting w and w' in G. By the choice of w and w', $(V(P) \cap V_\Delta) \setminus \{w, w'\}$ contains no vertex w^* such that $N_{\Delta-1}(w^*) = N_{\Delta-1}(w)$. Consequently, $V_\Delta \cap V(P) = \{w, w'\}$. Since $N_{\Delta-1}(w) \neq N_{\Delta-1}(w')$, it follows that w and w' are not on the same cycle of G_Δ and so $ww' \notin E(G)$ by Theorem 2.3 (i). Thus $P - \{w, w'\}$ has at least one vertex. By Theorem 2.4, all vertices of $P - \{w, w'\}$ have in G the same set of neighbors from V_Δ. Thus, both w and w' are Δ-neighbors of each vertex from $P - \{w, w'\}$ and so $N_{\Delta-1}(w) \cap N_{\Delta-1}(w') \neq \emptyset$. By Theorem 2.5, we have $|N_{\Delta-1}(w) \cap N_{\Delta-1}(w')| = \Delta - 3$, which together with Theorem 2.4 implies $x, y \in N_{\Delta-1}(w) \cap N_{\Delta-1}(w')$.

Let $N_{\Delta-1}(w') \setminus N_{\Delta-1}(w) = \{z\}$. We claim that $N_{\Delta-1}(z) = \emptyset$. For otherwise, let $z' \in N_{\Delta-1}(z)$. Clearly $z' \neq z$. By Theorem 2.4, $z' \in N_{\Delta-1}(w') \setminus N_{\Delta-1}(w)$, giving a contradiction to $N_{\Delta-1}(w') \setminus N_{\Delta-1}(w) = \{z\}$. We then claim that $N_\Delta(z) \subseteq N_\Delta(x)$. For otherwise let $w^* \in N_\Delta(z) \setminus N_\Delta(x)$. As $x \in N_{\Delta-1}(w')$ and $x \notin N_{\Delta-1}(w^*)$, it follows that $w^* \neq w'$. Since $z \in N_{\Delta-1}(w^*) \cap N_{\Delta-1}(w')$, it follows that $|N_{\Delta-1}(w^*) \cap N_{\Delta-1}(w')| \geq \Delta - 3$ by Theorem 2.5 (it can happen that $N_{\Delta-1}(w^*) = N_{\Delta-1}(w')$). Thus $N_{\Delta-1}(w^*) \cap N_{\Delta-1}(w')$ contains at least one of x and y as $x, y \in N_{\Delta-1}(w')$. As $xy \in E(G)$, we have $x, y \in N_{\Delta-1}(w^*) \cap N_{\Delta-1}(w')$ by Theorem 2.4. This gives a contradiction to the choice of w^*. Therefore we have $N_{\Delta-1}(z) = \emptyset$ and $N_\Delta(z) \subseteq N_\Delta(x)$. However, $d_G(z) \leq |N_\Delta(x)| < |N_\Delta(x) \cup \{y\}| \leq d_G(x)$, contradicting $d_G(x) = d_G(z) = \Delta - 1$. This completes the proof. □

We now prove Conjecture 2.2 for $\Delta \geq 4$ as below.

Theorem 2.7. *If G is an HZ-graph with maximum degree $\Delta \geq 4$, then $G \in \mathcal{O}_\Delta$.*

Proof. Let G be an HZ-graph with maximum degree $\Delta \geq 4$. Then G_Δ is 2-regular and $V(G) = V_\Delta \cup V_{\Delta-1}$ by Lemma 2.1 (a) and (b). First assume that $N_{\Delta-1}(u) = N_{\Delta-1}(v)$ for every pair $u, v \in V_\Delta$. Then V_Δ, $V_{\Delta-1}$ and the edges between them form a complete bipartite graph. As a consequence, G_Δ is $(\Delta - |V_{\Delta-1}|)$-regular and $G[V_{\Delta-1}]$ is $(\Delta - 1 - |V_\Delta|)$-regular. Then G_Δ is both 2-regular and $(\Delta - |V_{\Delta-1}|)$-regular implies $|V_{\Delta-1}| = \Delta - 2$. Let $r \in V_\Delta$. The assumption that $N_{\Delta-1}(u) = N_{\Delta-1}(v)$ for every pair $u, v \in V_\Delta$ also implies that $N_{\Delta-1}(r) = V_{\Delta-1}$. By Theorem 2.3 (ii), there exist $s \in N_{\Delta-1}(r) = V_{\Delta-1}$ and $\varphi \in \mathcal{C}^\Delta(G - rs)$ such that $N_{\Delta-1}[r] = V_{\Delta-1} \cup \{r\}$ is φ-elementary, which thereby implies that $V(G)$ is φ-elementary. Thus each color class of φ is a matching that uncovers exactly one vertex of G, showing that $|V(G)|$ is odd. Therefore $G \in \mathcal{O}_\Delta$.

We now assume that there exist $u, v \in V_\Delta$ such that $N_{\Delta-1}(u) \neq N_{\Delta-1}(v)$. We further assume that $N_{\Delta-1}(u) \cap N_{\Delta-1}(v) \neq \emptyset$ (using the same argument to find u and v as for finding w and w' in the proof of Corollary 2.6). By Theorem 2.3 (i), the cycle C_u containing u and the cycle C_v containing v from G_Δ are distinct. Let $w \in N_{\Delta-1}(u) \cap N_{\Delta-1}(v)$. Then $d_G(w) \geq |V(C_u)| + |V(C_v)| \geq 6$ by Theorem 2.3 (i). Thus $\Delta = d_G(w) + 1 \geq 7$. Applying Corollary 2.6, it follows that $V_{\Delta-1}$ is an independent set of G.

Let $A \subseteq V_\Delta$ be the set of all vertices a satisfying $N_{\Delta-1}(a) = N_{\Delta-1}(u)$, and let $B \subseteq V_\Delta$ be the set of all vertices b satisfying $N_{\Delta-1}(b) \neq N_{\Delta-1}(u)$ and $N_{\Delta-1}(b) \cap N_{\Delta-1}(u) \neq \emptyset$. Clearly $u \in A$ and $v \in B$, so A and B are non-empty. Partition B into non-empty subsets $B_1, B_2, \ldots, B_t$ such that for each $i \in [1, t]$, all vertices in B_i have the same neighborhood in $V_{\Delta-1}$. By Theorem 2.3 (i), each of $A, B_1, B_2, \ldots, B_t$ induces a union of disjoint cycles in G_Δ. So $|A| \geq 3$ and $|B_i| \geq 3$ for each $i \in [1, t]$.

Now we claim $t \geq \Delta - 2$. Assume otherwise $t \leq \Delta - 3$. Since for each $i \in [1, t]$, $|N_{\Delta-1}(A) \setminus N_{\Delta-1}(B_i)| = 1$ by Theorem 2.5 and $|N_{\Delta-1}(A)| = \Delta - 2$, there exists $z \in N_{\Delta-1}(A)$ such that $z \notin N_{\Delta-1}(A) \setminus N_{\Delta-1}(B_i)$ for each $i \in [1, t]$, or equivalently, $z \in N_{\Delta-1}(A) \cap \left(\bigcap_{i=1}^t N_{\Delta-1}(B_i)\right)$. Let $z' \in N_{\Delta-1}(A) \setminus N_{\Delta-1}(B_1)$. Then

$$|A| + \sum_{1\le i\le t} |B_i| = d_G(z) = d_G(z') \le |A| + \sum_{2\le i\le t} |B_i|,$$

achieving a contradiction. Hence $t \ge \Delta - 2$.

We now achieve a contradiction to the assumption $\Delta \ge 7$ by counting the number of edges in G between $N_{\Delta-1}(A)$ and $A \cup B$. Note that $|N_{\Delta-1}(A)| = \Delta - 2$. Since each vertex in B has exactly $\Delta - 3$ neighbors in $N_{\Delta-1}(A)$ and $|B_i| \ge 3$ for each $i \in [1, t]$, we have

$$|E_G(A\cup B, N_{\Delta-1}(A))| = |A|(\Delta-2)+|\cup_{i=1}^t B_i|(\Delta-3) \ge 3(\Delta-2)+3t(\Delta-3) \ge 3(\Delta-2)^2.$$

On the other hand, since $N_{\Delta-1}(A)$ is an independent set and every vertex in it has degree $\Delta - 1$ in G, we have

$$|E_G(A \cap B, N_{\Delta-1}(A))| = (\Delta - 1)(\Delta - 2).$$

Since $\Delta \ge 2$, solving Δ in $(\Delta - 1)(\Delta - 2) \ge 3(\Delta - 2)^2$ gives $\Delta \le 2.5$, achieving a desired contradiction. □

3. Definitions and previous results

In this section, we recall essential concepts from [1] and list a number of results that we will use as lemmas in the proof of Theorems 2.3 to 2.5.

Let G be a graph, $e \in E(G)$, $\varphi \in \mathcal{C}^k(G - e)$ for some $k \ge 0$, and let $\alpha, \beta \in [1, k]$. Each component of $G - e$ induced on edges colored by α or β is either a path or an even cycle, which is called an (α, β)-*chain* of $G - e$ with respect to φ. Interchanging α and β on an (α, β)-chain C of G gives a new edge k-coloring, which is denoted by φ/C. This operation is called a *Kempe change*.

For $x, y \in V(G)$, if x and y are contained in the same (α, β)-chain, we say x and y are (α, β)-*linked* with respect to φ. Otherwise, they are (α, β)-*unlinked*. If an (α, β)-chain P is a path with one endvertex as x, we also denote it by $P_x(\alpha, \beta, \varphi)$ and just write $P_x(\alpha, \beta)$ if φ is understood. For a vertex u and an edge uv contained in $P_x(\alpha, \beta, \varphi)$, we write $u \in P_x(\alpha, \beta, \varphi)$ and $uv \in P_x(\alpha, \beta, \varphi)$. If $u, v \in P_x(\alpha, \beta, \varphi)$ such that u lies between x and v on P, then we say that $P_x(\alpha, \beta, \varphi)$ *meets* u *before* v.

Let T be an alternating sequence of vertices and edges of G. We denote by $V(T)$ the set of vertices contained in T, and by $E(T)$ the set of edges contained in T. We simply write $\overline{\varphi}(T)$ for $\overline{\varphi}(V(T))$. If $V(T)$ is φ-elementary and $\overline{\varphi}(T) \ne \emptyset$, then for a color $\tau \in \overline{\varphi}(T)$, we denote by $\overline{\varphi}_T^{-1}(\tau)$ the unique vertex in $V(T)$ at which τ is missed. A coloring $\varphi' \in \mathcal{C}^k(G-e)$ is (T, φ)-*stable* if for every $x \in V(T)$ and every $f \in E(T)$, it holds that $\overline{\varphi}'(x) = \overline{\varphi}(x)$ and $\varphi'(f) = \varphi(f)$. Clearly, φ is (T, φ)-stable, and if $\varphi_1 \in \mathcal{C}^k(G - e)$ is (T, φ)-stable, and $\varphi_2 \in \mathcal{C}^k(G - e)$ is (T, φ_1)-stable, then φ_2 is also (T, φ)-stable.

3.1. Multifan

Let G be a graph, $rs_1 \in E(G)$ and $\varphi \in \mathcal{C}^k(G - rs_1)$ for some $k \geq 0$. A *multifan* centered at r with respect to rs_1 and φ is a sequence

$$F_\varphi(r, s_1 : s_p) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_p, s_p)$$

with $p \geq 1$ consisting of distinct vertices and edges such that for every edge rs_i with $i \in [2, p]$, there is a vertex s_j with $j \in [1, i-1]$ satisfying $\varphi(rs_i) \in \overline{\varphi}(s_j)$. The following result can be found in [12, Theorem 2.1].

Lemma 3.1. *Let G be a class 2 graph and $F_\varphi(r, s_1 : s_p)$ be a multifan with respect to rs_1 and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. Then the following statements hold.*

(a) $V(F)$ is φ-elementary.
(b) For any $\alpha \in \overline{\varphi}(r)$ and any $\beta \in \overline{\varphi}(s_i)$ with $i \in [1, p]$, r and s_i are (α, β)-linked with respect to φ.

Let $F_\varphi(r, s_1 : s_p)$ be a multifan. We call $s_{\ell_1}, s_{\ell_2}, \ldots, s_{\ell_h}$, a subsequence of $s_2, \ldots, s_p$, an *α-inducing sequence* for some $\alpha \in [1, k]$ with respect to φ and F if $\varphi(rs_{\ell_1}) = \alpha \in \overline{\varphi}(s_1)$ and $\varphi(rs_{\ell_i}) \in \overline{\varphi}(s_{\ell_{i-1}})$ for each $i \in [2, h]$. (By this definition, $(r, rs_1, s_1, rs_{\ell_1}, s_{\ell_1}, \ldots, rs_{\ell_h}, s_{\ell_h})$ is also a multifan with respect to rs_1 and φ.) A color in $\overline{\varphi}(s_{\ell_i})$ for any $i \in [1, h]$ is an *α-inducing color* and is *induced by α*. For $\alpha_i \in \overline{\varphi}(s_{\ell_i})$ and $\alpha_j \in \overline{\varphi}(s_{\ell_j})$ with $i < j$ and $i, j \in [1, h]$, we write $\alpha_i \prec \alpha_j$. For convenience, α itself is an α-inducing color and is induced by α, and $\alpha \prec \beta$ for any $\beta \in \overline{\varphi}(s_{\ell_i})$ and any $i \in [1, h]$. An α-inducing color β is called a *last α-inducing color* if there does not exist any α-inducing color δ such that $\beta \prec \delta$.

By Lemma 3.1 (a), each color in $\overline{\varphi}(F) \backslash \overline{\varphi}(r)$ is induced by a unique color in $\overline{\varphi}(s_1)$. Also if α_1 and α_2 are two distinct colors in $\overline{\varphi}(s_1)$, then an α_1-inducing sequence is disjoint with an α_2-inducing sequence. The following result is a consequence of Lemma 3.1 (a).

Lemma 3.2 *([1, Lemma 3.2]). Let G be a class 2 graph and $F_\varphi(r, s_1 : s_p)$ be a multifan with respect to rs_1 and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. For any two colors δ, λ with $\delta \in \overline{\varphi}(s_i)$ and $\lambda \in \overline{\varphi}(s_j)$ for some distinct $i, j \in [1, p]$, the following statements hold.*

(a) If δ and λ are induced by different colors from $\overline{\varphi}(s_1)$, then s_i and s_j are (δ, λ)-linked with respect to φ.
(b) If δ and λ are induced by the same color from $\overline{\varphi}(s_1)$ such that $\delta \prec \lambda$ and s_i and s_j are (δ, λ)-unlinked with respect to φ, then $r \in P_{s_j}(\lambda, \delta, \varphi)$.

By Lemma 2.1 (a), every edge of an HZ graph is critical. For an HZ-graph G with maximum degree $\Delta \geq 3$, we let $rs_1 \in E(G)$ with $r \in V_\Delta$ and $s_1 \in N_{\Delta-1}(r) :=$

$\{s_1, s_2, \ldots, s_{\Delta-2}\}$, and $\varphi \in \mathcal{C}^{\Delta}(G - rs_1)$. Then we call (G, rs_1, φ) a *coloring-triple*. As Δ-degree vertices in a multifan do not miss any color, for multifans in HZ-graphs, we add a further requirement in its definition as follows and we use this new definition in the remainder of this paper.

Assumption. *For multifans in an HZ-graph, all of its vertices except the center have degree $\Delta - 1$.*

Let (G, rs_1, φ) be a coloring-triple and $F := F_\varphi(r, s_1 : s_p)$ be a multifan. By its definition, $|\overline{\varphi}(s_1)| = 2$, $|\overline{\varphi}(s_i)| = 1$ for each $i \in [2, p]$, and so every color in $\overline{\varphi}(F) \setminus \overline{\varphi}(r)$ is induced by one of the two colors in $\overline{\varphi}(s_1)$. We call F a *typical multifan*, denoted $F_\varphi(r, s_1 : s_\alpha : s_\beta) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_\alpha, s_\alpha, rs_{\alpha+1}, s_{\alpha+1}, \ldots, rs_\beta, s_\beta)$, where $\beta := p$, if $\overline{\varphi}(r) = 1$ (recall we denote $\overline{\varphi}(v)$ by a number if $|\overline{\varphi}(v)| = 1$), $\overline{\varphi}(s_1) = \{2, \Delta\}$, and if $|V(F)| \geq 3$, then $\varphi(rs_{\alpha+1}) = \Delta$ and $\overline{\varphi}(s_{\alpha+1}) = \alpha + 2$ (if $\beta > \alpha$), and for each $i \in [2, \beta]$ with $i \neq \alpha + 1$, $\varphi(rs_i) = i$ and $\overline{\varphi}(s_i) = i + 1$. It is clear that $s_2, \ldots, s_\alpha$ is the longest 2-inducing sequence and $s_{\alpha+1}, \ldots, s_\beta$ is the longest Δ-inducing sequence of $F_\varphi(r, s_1 : s_\alpha : s_\beta)$. By relabeling vertices and colors if necessary, any multifan in an HZ-graph can be assumed to be a typical multifan, see Fig. 1 (a) for a depiction. If $\alpha = \beta$, then we write $F_\varphi(r, s_1 : s_\alpha)$ for $F_\varphi(r, s_1 : s_\alpha : s_\beta)$, and call it a *typical 2-inducing multifan.*

3.2. Kierstead path

Let G be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in \mathcal{C}^k(G - e)$ for some integer $k \geq 0$. A *Kierstead path* with respect to e and φ is a sequence $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices and edges such that for every edge v_iv_{i+1} with $i \in [1, p-1]$, there exists $j \in [0, i-1]$ satisfying $\varphi(v_iv_{i+1}) \in \overline{\varphi}(v_j)$.

A Kierstead path with at most 3 vertices is a multifan. We consider Kierstead paths with 4 vertices. Statement (a) below was proved in Theorem 3.3 from [12] and statement (b) is a consequence of (a).

Lemma 3.3. *Let G be a class 2 graph, $v_0v_1 \in E(G)$, and $\varphi \in \mathcal{C}^{\Delta}(G - v_0v_1)$. If $K = (v_0, v_0v_1, v_1, v_1v_2, v_2, v_2v_3, v_3)$ is a Kierstead path with respect to v_0v_1 and φ, then the following statements hold.*

(a) If $\min\{d_G(v_1), d_G(v_2)\} < \Delta$, then $V(K)$ is φ-elementary.
(b) For any two colors α, δ with $\alpha \in \overline{\varphi}(v_0)$ and $\delta \in \overline{\varphi}(v_3)$, if $\min\{d_G(v_1), d_G(v_2)\} < \Delta$ and $\alpha \notin \{\varphi(v_1v_2), \varphi(v_2v_3)\}$, then v_3 and v_0 are (α, δ)-linked with respect to φ.

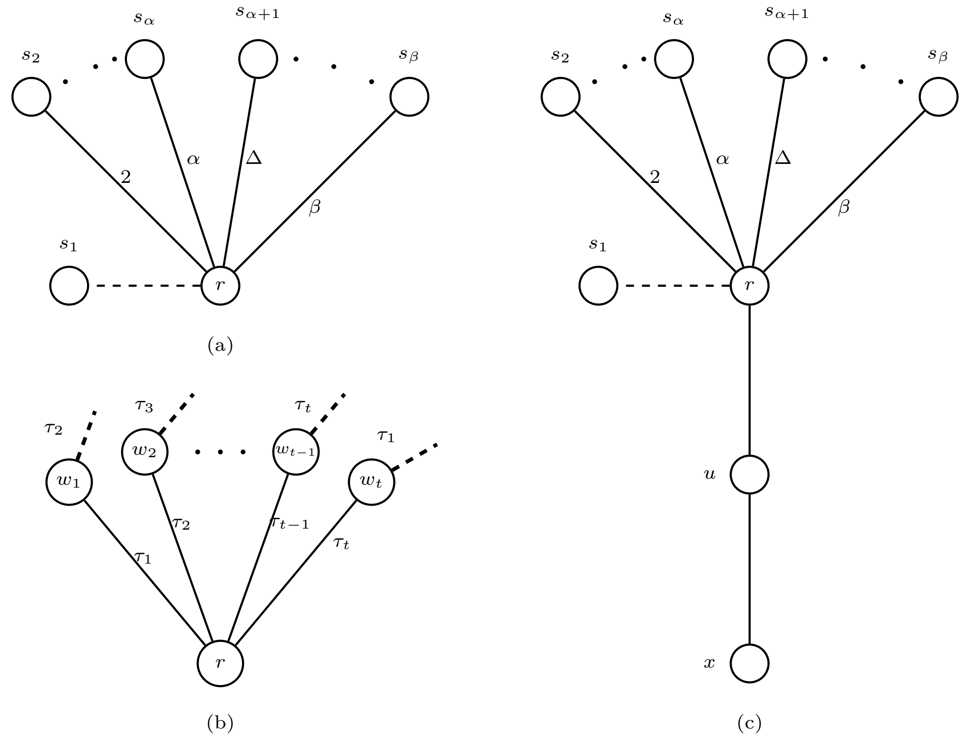

Fig. 1. (a) A typical multifan $F_\varphi(r, s_1 : s_\alpha : s_\beta)$, where $\overline{\varphi}(r) = 1$ and $\overline{\varphi}(s_1) = \{2, \Delta\}$; (b) A rotation centered at r, where a dashed line at a vertex indicates a color missing at the vertex; (c) A lollipop centered at r, where x can be the same as some s_ℓ for $\ell \in [\beta + 1, \Delta - 2]$.

3.3. Pseudo-multifan

Let G be a graph, $rs_1 \in E(G)$ and $\varphi \in \mathcal{C}^k(G - rs_1)$ for some $k \geq 0$. A multifan $F_\varphi(r, s_1 : s_p)$ is *maximum* at r if $|V(F)|$ is maximum among all multifans at r. A *pseudo-multifan* with respect to rs_1 and φ is an alternating sequence $S := S_\varphi(r, s_1 : s_t : s_p) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_t, s_t, rs_{t+1}, s_{t+1}, \ldots, s_{p-1}, rs_p, s_p)$ with $t, p \geq 1$ of distinct vertices and edges satisfying the following conditions:

(P1) the subsequence $F := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_t, s_t)$ is a maximum multifan at r.
(P2) $V(S)$ is φ'-elementary for every (F, φ)-stable $\varphi' \in \mathcal{C}^k(G - rs_1)$.

Every maximum multifan is a pseudo-multifan, and if S is a pseudo-multifan with respect to φ and a multifan F, then by the definition above, S is a pseudo-multifan under every (F, φ)-stable coloring φ'. We call a pseudo-multifan S *typical* (resp. *typical 2-inducing*) if the maximum multifan that is contained in S is typical (resp. typical 2-inducing).

Let (G, rs_1, φ) be a coloring-triple. A sequence of distinct vertices $w_1, \dots, w_t \in N_{\Delta-1}(r)$ form a *rotation* if $\{w_1, \dots, w_t\}$ is φ-elementary, and for each ℓ with $\ell \in [1, t]$, it holds that $\varphi(rw_\ell) = \overline{\varphi}(w_{\ell-1})$, where $w_0 := w_t$. An example of a rotation is given in Fig. 1 (b). Let $i, j \in [2, \Delta - 2]$. The *shift from s_i to s_j* is an operation that, for each ℓ with $\ell \in [i, j]$, recolor rs_ℓ by the color in $\overline{\varphi}(s_\ell)$. We will apply a shift either on a sequence of vertices from a multifan or on a rotation.

Lemma 3.4. *Let (G, rs_1, φ) be a coloring-triple. Then for every typical pseudo-multifan $S := S_\varphi(r, s_1 : s_t : s_p)$, there exists a coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_t)$ and a pseudo-multifan S^* centered at r with respect to rs_t and φ' such that $V(S^*) = V(S)$ and S^* is typical 2-inducing.*

Proof. Let $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be the typical multifan contained in S, where $s_\beta = s_t$. If $\beta = \alpha$, then we are done. Thus we assume $\beta \geq \alpha + 1 \geq 3$. Let φ' be obtained from φ by uncoloring rs_β, shift from $s_{\alpha+1}$ to $s_{\beta-1}$ and coloring rs_1 by Δ. Now $\overline{\varphi}'(s_\beta) = \{\beta, \beta+1\}$, $F^* = (r, rs_\beta, s_\beta, rs_{\beta-1}, s_{\beta-1}, \dots, rs_{\alpha+1}, s_{\alpha+1}, rs_1, s_1, \dots, rs_\alpha, s_\alpha)$ is a β-inducing multifan with respect to rs_β and φ'.

We next show that $S^* = (F^*, rs_{t+1}, s_{t+1}, \dots, rs_p, s_p)$ is a pseudo-multifan with respect to rs_β and φ'. Since $|V(F^*)| = |V(F)|$, F^* is also a maximum multifan at r. Thus it suffices to show that for any (F^*, φ')-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_\beta)$, $V(S^*)$ is φ''-elementary. Suppose to the contrary that there exists (F^*, φ')-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_\beta)$ but $V(S^*)$ is not φ''-elementary. As φ'' is (F^*, φ')-stable, we can undo the operations we did before. Specifically, let φ''' be the coloring obtained from φ'' by uncoloring rs_1, shift from $s_{\alpha+1}$ to $s_{\beta-1}$ and coloring rs_β by β. Then φ''' is (F, φ)-stable and $\overline{\varphi}'''(S^*) = \overline{\varphi}''(S^*)$. Thus, $V(S^*)$ is not φ''-elementary implies that $V(S^*)$ is not φ'''-elementary. Since $V(S^*) = V(S)$, this contradicts the assumption that $V(S)$ is elementary under any (F, φ)-stable coloring. Therefore, S^* is a pseudo-multifan with respect to rs_β and φ'. By renaming colors and vertices, we can assume that F^* is typical 2-inducing and so S^* is typical 2-inducing. □

Lemma 3.5 *([1, Theorem 2.5]). Let (G, rs_1, φ) be a coloring-triple, $S := S_\varphi(r, s_1 : s_t : s_{\Delta-2})$ be a pseudo-multifan with $F := F_\varphi(r, s_1 : s_t)$ being the maximum multifan contained in it. Let $j \in [t + 1, \Delta - 2]$ and $\delta \in \overline{\varphi}(s_j)$. Then*

(a) $\{s_{t+1}, \dots, s_{\Delta-2}\}$ can be partitioned into rotations with respect to φ.
(b) s_j and r are $(1, \delta)$-linked with respect to φ.
(c) For every color $\gamma \in \overline{\varphi}(F)$ with $\gamma \neq 1$, it holds that $r \in P_y(\gamma, \delta) = P_{s_j}(\gamma, \delta)$, where $y = \overline{\varphi}_F^{-1}(\gamma)$. Furthermore, for $z \in N_G(r)$ such that $\varphi(rz) = \gamma$, $P_y(\gamma, \delta)$ meets z before r.
(d) For every $\delta^* \in \overline{\varphi}(S) \setminus \overline{\varphi}(F)$ with $\delta^* \neq \delta$, it holds that $P_y(\delta, \delta^*) = P_{s_j}(\delta, \delta^*)$, where $y = \overline{\varphi}_S^{-1}(\delta^*)$. Furthermore, either $r \in P_{s_j}(\delta, \delta^*)$ or $P_r(\delta, \delta^*)$ is an even cycle.

3.4. Lollipop

If $F = (a_1, \ldots, a_t)$ is a sequence, then for a new entry b, (F, b) denotes the sequence $(a_1, \ldots, a_t, b)$. Let (G, rs_1, φ) be a coloring-triple. A *lollipop* centered at r is a sequence $L = (F, ru, u, ux, x)$ of distinct vertices and edges such that $F = F_\varphi(r, s_1 : s_\alpha : s_\beta)$ is a typical multifan, $u \in N_\Delta(r)$ and $x \in N_{\Delta-1}(u)$ with $x \notin \{s_1, \ldots, s_\beta\}$ (see Fig. 1 (c) for a depiction).

Lemma 3.6 *([1, Lemma 5.1]). Let (G, rs_1, φ) be a coloring-triple, $F := F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be a typical multifan, and $L := (F, ru, u, ux, x)$ be a lollipop centered at r such that $\varphi(ru) = \alpha + 1$ and $\overline{\varphi}(x) = \alpha + 1$. Then*

(a) $\varphi(ux) \neq 1$ and $ux \in P_r(1, \varphi(ux))$.

If $\varphi(ux) = \tau$ is a 2-inducing color with respect to φ and F, then the following holds.

(b) Let $P_x(1, \tau)$ be the $(1, \tau)$-chain starting at x in $G - rs_1 - ux$. Then $P_x(1, \tau)$ ends at r.

(c) For any 2-inducing color δ of F with $\tau \prec \delta$, we have $r \in P_{s_1}(\delta, \Delta) = P_{s_{\delta-1}}(\delta, \Delta)$.

(d) For any Δ-inducing color δ of F, we have $r \in P_{s_{\delta-1}}(\alpha+1, \delta) = P_{s_\alpha}(\alpha+1, \delta)$, where $s_{\Delta-1} = s_1$ if $\delta = \Delta$.

(e) For any 2-inducing color δ of F with $\delta \prec \tau$, we have $r \in P_{s_\alpha}(\delta, \alpha+1) = P_{s_{\delta-1}}(\delta, \alpha+1)$.

Let (G, rs_1, φ) be a coloring-triple. For a color $\alpha \in [1, \Delta]$, a sequence of *Kempe* $(\alpha, *)$-*changes* is a sequence of Kempe changes that each involve the exchanging of the color α with another color from $[1, \Delta]$.

Lemma 3.7 *([1, Lemma 5.2]). Let (G, rs_1, φ) be a coloring-triple, $F := F_\varphi(r, s_1 : s_\alpha : s_\beta)$ be a typical multifan, and $L := (F, ru, u, ux, x)$ be a lollipop centered at r such that $\varphi(ru) = \alpha + 1$. Then for $w_1 \in \{s_{\beta+1}, \ldots, s_{\Delta-2}\}$ with $\varphi(rw_1) = \tau_1 \in [\beta + 2, \Delta - 1]$, the following statements hold.*

*(1) If exists a vertex $w \in V(G) \setminus (V(F) \cup \{w_1\})$ such that $w \in P_r(1, \tau_1, \varphi')$ for every (F, φ)-**stable** $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ with $\varphi'(ru) = \alpha + 1$, then there exists a sequence of distinct vertices $w_1, \ldots, w_t \in \{s_{\beta+1}, \ldots, s_{\Delta-2}\}$ satisfying the following conditions:*

(a) $\varphi(rw_{i+1}) = \overline{\varphi}(w_i) \in [\beta + 2, \Delta - 1]$ for each $i \in [1, t - 1]$;

(b) r and w_i are $(1, \overline{\varphi}(w_i))$-linked with respect to φ for each $i \in [1, t]$;

(c) $\overline{\varphi}(w_t) = \tau_1$.

*(2) If $\overline{\varphi}(x) = \alpha + 1$ and there exists a vertex $w \in V(G) \setminus (V(F) \cup \{w_1\})$ such that $w \in P_r(1, \tau_1, \varphi')$ for every (L, φ)-**stable** $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ obtained from φ through a*

sequence of Kempe $(1,*)$-changes not using r or x as endvertices, then there exists a sequence of distinct vertices $w_1,\dots,w_t \in \{s_{\beta+1},\dots,s_{\Delta-2}\}$ satisfying the following conditions:

(a) $\varphi(rw_{i+1}) = \overline{\varphi}(rw_i) \in [\beta+2, \Delta-1]$ *for each* $i \in [1, t-1]$*;*
(b) r *and* w_i *are* $(1, \overline{\varphi}(w_i))$*-linked with respect to* φ *for each* $i \in [1, t-1]$*;*
(c) $\overline{\varphi}(w_t) = \tau_1$ *or* $\overline{\varphi}(w_t) = \alpha+1$*. If* $\overline{\varphi}(w_t) = \tau_1$*, then* w_t *and* r *are* $(1,\tau_1)$*-linked with respect to* φ*.*

By the definition, the sequence $w_1,\dots,w_t$ in Lemma 3.7 (1) and in the case of Lemma 3.7 (2) when $\overline{\varphi}(w_t) = \tau_1$ form a rotation with the additional property that $\overline{\varphi}(w_i) \in [\beta+2, \Delta-1]$ and r and w_i are $(1, \overline{\varphi}(w_i))$-linked for each $i \in [1,t]$. We call such a rotation a *stable rotation*. In the case of Lemma 3.7 (2) when $\overline{\varphi}(w_t) = \alpha+1$, we call $w_1,\dots,w_t$ a *near stable rotation*. For $u, v \in V(G)$, we write $u \sim v$ if u and v are adjacent in G, and write $u \not\sim v$ otherwise.

Lemma 3.8 *([1, Corollary 2.7]). Let* (G, rs_1, φ) *be a coloring-triple,* $F := F_\varphi(r, s_1 : s_\alpha)$ *be a typical 2-inducing multifan, and* $L := (F, ru, u, ux, x)$ *be a lollipop centered at* r*. If* $\varphi(ru) = \alpha+1$*,* $\overline{\varphi}(x) = \alpha+1$*, and* $\varphi(ux) = \Delta$*, then* $u \not\sim s_1$ *and* $u \not\sim s_\alpha$*.*

Lemma 3.9 *([1, Theorem 2.8]). Let* (G, rs_1, φ) *be a coloring-triple,* $F := F_\varphi(r, s_1 : s_\alpha)$ *be a typical 2-inducing multifan, and* $L := (F, ru, u, ux, x)$ *be a lollipop centered at* r*. If* $\varphi(ru) = \alpha+1$*,* $\overline{\varphi}(x) = \alpha+1$*, and* $\varphi(ux) = \mu \in \overline{\varphi}(F)$ *is a 2-inducing color of* F*, then* $u \not\sim s_{\mu-1}$ *and* $u \not\sim s_\mu$*.*

Let G be a graph, $rs_1 \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$. Let $\alpha, \beta, \gamma, \tau \in [1, \Delta]$ and $x, y \in V(G)$. If P is an (α,β)-chain containing both x and y such that P is a path, we denote by $P_{[x,y]}(\alpha,\beta,\varphi)$ the subchain of P that has endvertices x and y.

Suppose $|\overline{\varphi}(x) \cap \{\alpha,\beta\}| = 1$. Then an (α,β)-*swap* at x is just the Kempe change on $P_x(\alpha,\beta,\varphi)$. By convention, an (α,α)-swap at x does nothing at x. If also $|\overline{\varphi}(y) \cap \{\alpha,\beta\}| = 1$, then an (α,β)-swap *at both* x *and* y is the Kempe change on $P_x(\alpha,\beta,\varphi)$ if x and y are (α,β)-linked with respect to φ, and is obtained from φ by first doing an (α,β)-swap at x and then doing an (α,β)-swap at y if x and y are (α,β)-unlinked with respect to φ. Suppose $\beta_0 \in \overline{\varphi}(x)$ and $\beta_1,\dots\beta_t \in \varphi(x)$ for colors $\beta_0,\dots,\beta_t \in [1,\Delta]$ for some integer $t \geq 1$. Then a

$$(\beta_0, \beta_1) - (\beta_1, \beta_2) - \ldots - (\beta_{t-1}, \beta_t) - \text{swap}$$

at x consists of t Kempe changes: let $\varphi_0 = \varphi$, then $\varphi_i = \varphi_{i-1}/P_x(\beta_{i-1}, \beta_i, \varphi_{i-1})$ for each $i \in [1,t]$. Suppose the current color of an edge uv of G is α, the notation $uv : \alpha \to \beta$ means to recolor the edge uv using the color β.

We will use a matrix with two rows to denote a sequence of coloring operations taken based on φ. For example, the matrix below indicates three operations taken on the graph:

$$\begin{bmatrix} P_{[a,b]}(\alpha,\beta,\varphi) & s_c : s_d & rs \\ \alpha/\beta & \text{shift} & \gamma \to \tau \end{bmatrix}.$$

Step 1 Exchange α and β on the (α, β)-subchain $P_{[a,b]}(\alpha, \beta, \varphi)$.
Step 2 Based on the coloring obtained from Step 1, shift from s_c to s_d for vertices $s_c, \dots, s_d$.
Step 3 Based on the coloring obtained from Step 2, do $rs : \gamma \to \tau$.

In the remainder, for simpler description, we may skip the phrase "with respect to φ" in related notation, which then needs to be understood with respect to the current edge coloring.

4. Proof of Theorem 2.3

We prove the following version of Theorem 2.3.

Theorem 4.1. *If G is an HZ-graph with maximum degree $\Delta \geq 4$, then for every vertex $r \in V_\Delta$, the following two statements hold.*

(i) For every $u \in N_\Delta(r)$, $N_{\Delta-1}(r) = N_{\Delta-1}(u)$.
(ii) There exist $s_1 \in N_{\Delta-1}(r)$ and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ such that $N_{\Delta-1}[r]$ is the vertex set of a typical 2-inducing pseudo-multifan with respect to rs_1 and φ. Consequently $N_{\Delta-1}[r]$ is φ-elementary.

Proof. Let $N_{\Delta-1}(r) = \{s_1, \dots, s_{\Delta-2}\}$. We choose a vertex in $N_{\Delta-1}(r)$, say s_1, a coloring $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ and a multifan F with respect to rs_1 and φ such that F is maximum at r. That is, $|V(F)|$ is maximum among all multifans with respect to rs_i for any $i \in [1, \Delta - 2]$ and any $\varphi' \in \mathcal{C}^\Delta(G - rs_i)$. Assume that $\overline{\varphi}(r) = 1$ and $\overline{\varphi}(s_1) = \{2, \Delta\}$, and $F = F_\varphi(r, s_1 : s_p)$ is such a multifan. Furthermore, by relabeling vertices and colors, we assume that F is typical. As a maximum multifan at r is itself a pseudo-multifan, by Lemma 3.4, we assume that $F_\varphi(r, s_1 : s_p) = F_\varphi(r, s_1 : s_\alpha)$ is a typical 2-inducing multifan, where $\alpha = p$.

Let $u \in N_\Delta(r)$ and assume $N_{\Delta-1}(r) \neq N_{\Delta-1}(u)$. Roughly speaking, the main proof idea is the following. By assuming $\varphi(ru) = \alpha + 1$ and $\overline{\varphi}(x) = \alpha + 1$ for $x \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$, we will apply Lemmas 3.8 and 3.9 to show that u has at least two $(\Delta - 1)$-neighbors outside of $N_{\Delta-1}(r)$. By further applying Lemmas 3.8 and 3.9, we can even find three $(\Delta - 1)$-neighbors of u outside of $N_{\Delta-1}(r)$. A contradiction is then deduced at that point.

Claim 4.1. *We may assume that $\varphi(ru) = \alpha + 1$, which is the last 2-inducing color of $F_\varphi(r, s_1 : s_\alpha)$.*

Proof of Claim 4.1. Since $F_\varphi(r, s_1 : s_\alpha)$ is a maximum typical 2-inducing multifan, $\varphi(ru) \in \{\alpha+1, \Delta\}$. Assume instead that $\varphi(ru) = \Delta$. If $\alpha = 1$, then we are done by exchanging the roles of 2 and Δ. Thus we assume that $\alpha \geq 2$. Shift from s_2 to $s_{\alpha-1}$, color rs_1 by 2 and uncolor rs_α. Then $F^* = (r, rs_\alpha, s_\alpha, rs_{\alpha-1}, s_{\alpha-1}, \ldots, rs_1, s_1)$ is an α-inducing multifan such that Δ is the last α-inducing color. Now, relabeling colors and vertices in F^* by making F^* typical 2-inducing yields the desired assumption. $\diamond$

Claim 4.2. *For any $z \in N_{\Delta-1}(u) \setminus V(F)$ and any (F,φ)-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$, if $\varphi'(ru) = \alpha+1$ and $\overline{\varphi}'(z) = \alpha+1$, then $\varphi'(uz) \in \overline{\varphi}'(F) \setminus \{1\}$.*

Proof of Claim 4.2. Assume to the contrary that $\varphi'(uz) \in \{1, \alpha+2, \ldots, \Delta-1\}$. We first claim that $\varphi'(uz) \neq 1$. As otherwise, $P_r(1, \alpha+1, \varphi') = ruz$, contradicting Lemma 3.1 (b) that r and s_α are $(1, \alpha+1)$-linked with respect to φ'. Let $\varphi'(uz) = \tau \in [\alpha+2, \Delta-1]$, and $w_1 \in N_{\Delta-1}(r)$ such that $\varphi'(rw_1) = \tau$. By Lemma 3.6 (a), $uz \in P_r(1, \tau, \varphi'')$ for every (L, φ')-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$, where $L = (F, ru, u, uz, z)$ is a lollipop. Applying Lemma 3.7 (2) on L with u playing the role of w, we find a sequence of distinct vertices $w_1, \ldots, w_t \in \{s_{\alpha+1}, \ldots, s_{\Delta-2}\}$ that forms either a stable rotation or a near stable rotation.

Assume first that $w_1, \ldots, w_t$ is a stable rotation, which in particular gives $P_r(1, \tau, \varphi') = P_{w_t}(1, \tau, \varphi')$. By Lemma 3.6 (a), $uz \in P_r(1, \tau, \varphi')$. If $P_r(1, \tau, \varphi')$ meets z before u, or equivalently, $P_{w_t}(1, \tau, \varphi')$ meets u before z, we do the following operations:

$$\begin{bmatrix} P_{[r,z]}(1, \tau, \varphi') & ru & uz \\ 1/\tau & \alpha+1 \to \tau & \tau \to \alpha+1 \end{bmatrix}.$$

Denote the new coloring by φ''. Now $(r, rs_1, s_1, \ldots, s_\alpha)$ is a multifan, but $\overline{\varphi}''(s_\alpha) = \overline{\varphi}''(r) = \alpha+1$, giving a contradiction to Lemma 3.1 (a). Thus $P_r(1, \tau, \varphi')$ meets u before z, or equivalently, $P_{w_t}(1, \tau, \varphi')$ meets z before u. Shift from w_1 to w_t to get φ''. Then $P_r(1, \tau, \varphi'')$ meets z before u, giving back to the previous case as φ'' is (F, φ')-stable.

Assume now that $w_1, \ldots, w_t$ is a near stable rotation, i.e., $\overline{\varphi}'(w_t) = \alpha+1$. If $z \neq w_t$, then we shift from w_1 to w_t, and do $ru : \alpha+1 \to \tau$, $uz : \tau \to \alpha+1$. Denote the new coloring by φ''. As φ'' is (F, φ')-stable and so is (F, φ)-stable, we see that $F^* = (F, rw_t, w_t, rw_{t-1}, w_{t-1}, \ldots, rw_1, w_1)$ is a multifan that contains more vertices than F does, showing a contradiction to the choice of F.

Thus we assume that $z = w_t$. Since $\varphi'(rz) \neq \varphi'(uz) = \tau$, we have $t \geq 2$. Note that $uz \in P_r(1, \tau, \varphi') = P_w(1, \tau, \varphi')$ for some vertex $w \in V(G) \setminus (V(F) \cup \{w_1, \ldots, w_t\})$. If $P_w(1, \tau, \varphi')$ meets u before z, we do the following operations:

$$\begin{bmatrix} P_{[w,u]}(1, \tau, \varphi') & ru & uz \\ 1/\tau & \alpha+1 \to 1 & \tau \to \alpha+1 \end{bmatrix}.$$

Denote the new coloring by φ''. Now $(r, rs_1, s_1, \dots, s_\alpha)$ is a multifan, but $\overline{\varphi}''(s_\alpha) = \overline{\varphi}''(r) = \alpha + 1$, giving a contradiction to Lemma 3.1 (a). If $P_w(1, \tau, \varphi')$ meets z before u, we do the following operations:

$$\begin{bmatrix} P_{[w,z]}(1,\tau,\varphi') & w_1 : w_t & rw_t = rz & ru & uz \\ 1/\tau & \text{shift} & \varphi'(rz) \to 1 & \alpha+1 \to \tau & \tau \to \alpha+1 \end{bmatrix}.$$

Denote the new coloring by φ''. Now $(r, rs_1, s_1, \dots, s_\alpha)$ is a multifan, but $\overline{\varphi}''(s_\alpha) = \overline{\varphi}''(r) = \alpha + 1$, giving a contradiction to Lemma 3.1 (a). $\diamond$

By Claim 4.2, $\tau \in \{2, \dots, \alpha + 1, \Delta\}$. Applying Lemmas 3.8 and 3.9, we have the following claim.

Claim 4.3. *Let $z \in N_{\Delta-1}(u) \setminus V(F)$ and any (F, φ)-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $\varphi'(ru) = \alpha + 1$ and $\overline{\varphi}'(z) = \alpha + 1$, and let $\varphi'(uz) = \tau$. Then $\tau \in \overline{\varphi}'(F) \setminus \{1\}$, and $u \not\sim s_1, s_\alpha$ if $\tau = \Delta$; and $u \not\sim s_{\tau-1}, s_\tau$ if $\tau \in [2, \alpha + 1]$.*

Claim 4.4. *Suppose that $N_{\Delta-1}(r) = N_{\Delta-1}(u)$ for every $u \in N_\Delta(r)$. Then for every (F, φ)-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$, $N_{\Delta-1}[r]$ is φ'-elementary. In particular, $N_{\Delta-1}[r]$ is the vertex set of a typical 2-inducing pseudo-multifan with respect to rs^* and $\varphi^* \in \mathcal{C}^\Delta(G - rs^*)$ for some $s^* \in N_{\Delta-1}(r)$.*

Proof of Claim 4.4. Assume to the contrary that there exists an (F, φ)-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $N_{\Delta-1}[r]$ is not φ'-elementary. Since $V(F)$ is φ'-elementary, there exists $z \in N_{\Delta-1}[r] \setminus V(F)$ such that $\overline{\varphi}'(z) \in \overline{\varphi}'(F)$ or there exists $z^* \neq z$ with $z^* \in N_{\Delta-1}[r] \setminus V(F)$ such that $\overline{\varphi}'(z) = \overline{\varphi}'(z^*)$. Let $\overline{\varphi}'(z) = \delta$. If $\delta \in \overline{\varphi}'(F)$, then z and r are $(1, \delta)$-unlinked, so we do $(\delta, 1)-(1, \alpha+1)$-swaps at z; if $\overline{\varphi}'(z) = \overline{\varphi}'(z^*)$, we may assume, without loss of generality, that z and r are $(1, \delta)$-unlinked, we again do $(\delta, 1) - (1, \alpha + 1)$-swaps at z. In either case, we find an (F, φ')-stable coloring $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$ such that $\overline{\varphi}''(z) = \alpha + 1$. Since for any $u \in N_\Delta(r)$, it holds that $N_{\Delta-1}(r) = N_{\Delta-1}(u)$, we can choose $u \in N_\Delta(r)$ such that $\varphi''(ur) = \alpha + 1$, where $\alpha + 1$ is the last 2-inducing color of $F_{\varphi''}(r, s_1 : s_\alpha)$. Since $N_{\Delta-1}(r) = N_{\Delta-1}(u)$, we have $uz \in E(G)$ and so $L = (F_{\varphi''}(r, s_1 : s_\alpha), ru, u, uz, z)$ is a lollipop with respect to φ''. By Claim 4.3, u is not adjacent to at least one vertex in $N_{\Delta-1}(r)$, which in turn shows $N_{\Delta-1}(r) \neq N_{\Delta-1}(u)$, giving a contradiction.

Therefore, for every (F, φ)-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$, it holds that $N_{\Delta-1}[r]$ is φ'-elementary. Consequently, there is a pseudo-multifan with vertex set $N_{\Delta-1}[r]$. By renaming colors and vertices from $N_{\Delta-1}(r)$, we can assume the pseudo-multifan with vertex set $N_{\Delta-1}[r]$ is typical. By Lemma 3.4, we can further assume that the pseudo-multifan is typical 2-inducing. $\diamond$

By Claim 4.4, it suffices to only show Theorem 4.1 (i). Assume to the contrary that there exists $u \in N_\Delta(r)$ such that $N_{\Delta-1}(u) \setminus N_{\Delta-1}(r) \neq \emptyset$.

Claim 4.5. *For every $z \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$, there is an (F, φ)-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $\varphi'(ru) = \alpha + 1$ and $\overline{\varphi}'(z) = \alpha + 1$.*

Proof of Claim 4.5. By Claim 4.1, we assume $\varphi(ru) = \alpha + 1$. Let $\overline{\varphi}(z) = \delta$. If $\delta = \alpha + 1$, we simply let $\varphi' = \varphi$. So $\delta \neq \alpha + 1$. If $\delta \in \overline{\varphi}(F)$, we let φ' be obtained from φ by doing $(\delta, 1) - (1, \alpha + 1)$-swaps at z. This gives that $\overline{\varphi}'(z) = \alpha + 1$. By Lemma 3.1 (b), φ' is (F, φ)-stable and $\varphi'(ru) = \varphi(ru) = \alpha + 1$. Thus φ' is a desired coloring.

Assume now that $\delta \in [\alpha + 2, \Delta - 1]$. If there is an (F, φ)-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$ with $\varphi''(ru) = \alpha + 1$ such that $z \notin P_r(1, \delta, \varphi'')$ (so z and r are $(1, \delta)$-unlinked), let φ' be obtained from φ'' by doing $(\delta, 1) - (1, \alpha + 1)$-swaps at z. Since $\varphi''(ru) = \alpha + 1$ and r and s_α are $(1, \alpha + 1)$-linked with respect to φ'' by Lemma 3.1 (b), it holds that φ' is (F, φ'')-stable and so (F, φ)-stable with $\varphi'(ru) = \varphi''(ru) = \alpha + 1$. Thus, φ' is a desired coloring and we are done. Therefore every (F, φ)-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$ with $\varphi''(ru) = \alpha + 1$ satisfies $z \in P_r(1, \delta, \varphi'')$. Applying Lemma 3.7 (1) with z playing the role of w, there exists $w_t \in N_{\Delta-1}(r) \setminus V(F)$ such that $\overline{\varphi}(w_t) = \delta$ and w_t and r are $(1, \delta)$-linked with respect to φ. This is a contradiction by noting $w_t \neq z$, since φ is (F, φ)-stable but $z \notin P_r(1, \delta, \varphi)$. $\diamond$

Claim 4.6. $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| \geq 2$.

Proof of Claim 4.6. Let $x \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$. By Claim 4.5, we choose an (F, φ)-stable coloring from $\mathcal{C}^\Delta(G - rs_1)$ and call it still φ such that $\varphi(ru) = \alpha + 1$ and $\overline{\varphi}(x) = \alpha + 1$. By Claim 4.3, $\varphi(ux) \in \{2, \dots, \alpha + 1, \Delta\}$. If $|V(F)| \geq 3$, then Claim 4.3 gives that $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| \geq 2$. Thus we have $V(F) = \{r, s_1\}$. Consequently, $\alpha + 1 = 2$, and $\varphi(ux) = \Delta$ by the fact that $\varphi(ux) \in \{2, \Delta\}$. We may assume further that $N_{\Delta-1}(u) \setminus N_{\Delta-1}(r) = \{x\}$.

By Claim 4.3, $u \not\sim s_1$. We consider two cases. Assume first that there exists an (F, φ)-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $N_{\Delta-1}[r]$ is not φ'-elementary. By exchanging the roles of 2 and Δ if necessary, we may assume $\varphi'(ru) = 2$. Since $V(F)$ is φ'-elementary, there exists $z \in N_{\Delta-1}(r) \setminus V(F)$ such that $\overline{\varphi}'(z) \in \overline{\varphi}'(F)$ or there exists $z^* \neq z$ with $z^* \in N_{\Delta-1}(r) \setminus V(F)$ such that $\overline{\varphi}'(z) = \overline{\varphi}'(z^*)$. Let $\overline{\varphi}'(z) = \delta$. If $\delta \in \overline{\varphi}'(F)$, then as r and z are $(1, \delta)$-unlinked, we do $(\delta, 1) - (1, 2)$-swaps at z; if $\overline{\varphi}'(z) = \overline{\varphi}'(z^*)$, we may assume, without loss of generality, that z and r are $(1, \delta)$-unlinked, we again do $(\delta, 1) - (1, 2)$-swaps at z. In either case, we find an (F, φ')-stable coloring $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$ with $\varphi''(ru) = \varphi'(ru) = 2$ and $\overline{\varphi}''(z) = 2$. Note that $z \in N_{\Delta-1}(u)$ since $u \not\sim s_1$, $s_1 \neq z$, and $N_{\Delta-1}(u) \setminus N_{\Delta-1}(r) = \{x\}$. By Claim 4.2, $\varphi''(uz) \in \{2, \Delta\}$, which implies $\varphi''(uz) = \Delta$ by noting $\varphi''(ru) = 2$. Furthermore, we assume $uz \in P_{s_1}(1, \Delta, \varphi'') = P_r(1, \Delta, \varphi'')$ (otherwise, after a $(1, \Delta)$-swap on the chain containing uz, we obtain a contradiction to Claim 4.2). Since $\varphi''(ru) = 2$ and $\varphi''(uz) = \Delta$, $\varphi''(ux) \neq 2, \Delta$. Thus $\varphi''(ux) \in \{1, 3, 4, \dots, \Delta - 1\}$, which implies $\overline{\varphi}''(x) \neq 2$ by Claim 4.2. Let $\overline{\varphi}''(x) = \tau$ and $\varphi''(ux) = \lambda$. Note that if $\tau = \Delta$ then $\lambda \neq 1$, as $uz \in P_{s_1}(1, \Delta, \varphi'') = P_r(1, \Delta, \varphi'')$. Thus if $\tau = \Delta$ or 1, we do $(\tau, 1) - (1, 2)$-swaps at x. As the color of ux is not Δ after these swaps, we get a contradiction to Claim 4.2. Thus, we assume that $\tau \in [3, \Delta - 1]$, and that $P_x(1, \tau, \varphi''') = P_r(1, \tau, \varphi''')$ for any (L, φ'')-stable coloring φ''', where $L = (F, ru, u, uz, z)$ is a lollipop. Let $w_1 \in N_{\Delta-1}(r)$ such that $\varphi''(rw_1) = \tau$. Applying Lemma 3.7 (2) on L with x playing the role of w, we find a sequence of distinct vertices $w_1, \dots, w_t \in \{s_{\alpha+1}, \dots, s_{\Delta-2}\}$ that forms either a stable rotation or a near stable rotation. As x and r are $(1,\tau)$-linked, we conclude that $w_1,\dots,w_t$ form a near stable rotation and so $\overline{\varphi}''(w_t)=2$. As $\varphi''(uz)=\Delta$, $\varphi''(ur)=2$, if $w_t\neq z$, then $\varphi''(uw_t)\in\{1,3,4,\dots,\Delta-1\}$. This gives a contradiction to Claim 4.2. Thus we assume that $w_t=z$. Notice that $r\in P_{s_1}(2,\tau,\varphi'')$ by the maximality of $|V(F)|$. Since $r\in P_x(2,\tau,\varphi'')$ by Claim 4.2, we have $r\in P_x(2,\tau,\varphi'')=P_{s_1}(2,\tau,\varphi'')$. So w_t is $(2,\tau)$-unlinked with any of s_1,x and r with respect to φ''. We do a $(2,\tau)$-swap at w_t and then shift from w_1 to w_t. This gives a coloring such that s_1 and x are $(2,\tau)$-unlinked with respect to the coloring. Again, with respect to the current coloring, r and s_1 are $(2,\tau)$-linked by the maximality of $|V(F)|$. We do a $(2,\tau)$-swap at x to get a coloring φ'''. Note that $\varphi'''(ru)=\varphi''(ru)=2$, $\varphi'''(ux)=\varphi''(ux)=\lambda$, $\overline{\varphi}'''(x)=2$, and $\varphi'''(uz)=\varphi''(uz)=\Delta$. Therefore, $\varphi'''(ux)=\lambda\in\{1,3,4,\dots,\Delta-1\}$, showing a contradiction to Claim 4.2.

Thus we assume that $N_{\Delta-1}[r]$ is φ'-elementary for every (F,φ)-stable $\varphi'\in\mathcal{C}^\Delta(G-rs_1)$. In particular, $N_{\Delta-1}[r]$ is φ-elementary, and as $|V(F)|=2$ and F is maximum at r, we know that $N_{\Delta-1}[r]$ is contained in a pseudo-multifan $S=S_\varphi(r,s_1:s_1:s_{\Delta-2})$. Let $\delta\in\overline{\varphi}(S)\setminus\overline{\varphi}(F)$. By Lemma 3.5 (c), $\overline{\varphi}_S^{-1}(\delta)$ is $(2,\delta)$- and (δ,Δ)-linked with s_1 and the corresponding chains contain the vertex r with respect to φ. Recall that $\varphi(ru)=2,\varphi(ux)=\Delta$, and $\overline{\varphi}(x)=2$. Let φ' be obtained from φ by doing a $(2,\delta)-(\delta,\Delta)-(\Delta,1)-(1,2)$-swap at x. Since φ' is (F,φ)-stable, $\varphi'(ru)=2$, $\varphi'(ux)=\delta$, and $\overline{\varphi}'(x)=2$, we get a contradiction to Claim 4.2. $\diamond$

Claim 4.7. *Let $x,y\in N_{\Delta-1}(u)\setminus N_{\Delta-1}(r)$ be distinct, and $\varphi'\in\mathcal{C}^\Delta(G-rs_1)$ be any (F,φ)-stable coloring with $\varphi'(ru)=\alpha+1$. Suppose $\overline{\varphi}'(x)\in\overline{\varphi}'(F)$ and $\overline{\varphi}'(x)\neq 1$. Then $\overline{\varphi}'(y)\notin\overline{\varphi}'(F)$. Furthermore, y and r are $(1,\overline{\varphi}'(y))$-linked with respect to φ'.*

Proof of Claim 4.7. The second part of the claim follows easily from the first part. Since otherwise, a $(1,\overline{\varphi}'(y))$-swap at y implies that 1 is missing at y, contradicting the first part.

Assume to the contrary that $\overline{\varphi}'(x)\in\overline{\varphi}'(F)$ and $\overline{\varphi}'(y)\in\overline{\varphi}'(F)$. We claim that we may assume $\overline{\varphi}'(x)=\overline{\varphi}'(y)=\alpha+1$ or $\overline{\varphi}'(x)=\alpha+1$ and $\overline{\varphi}'(y)=1$. By doing $(\overline{\varphi}'(x),1)-(1,\alpha+1)$-swaps at x, we assume that $\overline{\varphi}'(x)=\alpha+1$. Since $1,\alpha+1\in\overline{\varphi}'(F)$, we still have $\overline{\varphi}'(y)\in\overline{\varphi}'(F)$. If $\overline{\varphi}'(y)=\alpha+1$, then we are done. Otherwise, doing a $(1,\overline{\varphi}'(y))$-swap at y gives a desired coloring. Let $\varphi'(ux)=\tau$ and $\varphi'(uy)=\lambda$. We consider now two cases to finish the proof of Claim 4.7.

Case A. $\overline{\varphi}'(x)=\overline{\varphi}'(y)=\alpha+1$.

By Claim 4.2, $\tau,\lambda\in\overline{\varphi}'(F)\setminus\{1\}$. Assume, without loss of generality, that $\tau\neq\Delta$. Then $\tau\in\{2,\dots,\alpha+1\}$ is a 2-inducing color of F, since F is assumed to be typical 2-inducing. By Lemma 3.6 (d) that $r\in P_{s_\alpha}(\alpha+1,\Delta)=P_{s_1}(\alpha+1,\Delta)$, we know $\lambda\neq\Delta$. Thus $\lambda\in\{2,\dots,\alpha+1\}$ is also a 2-inducing color. By symmetry between x and y, we assume $\lambda\prec\tau$. Shift from s_2 to $s_{\lambda-1}$, uncolor rs_λ, then color rs_1 by 2. Denote the resulting coloring by φ''. Now $F^* = (r, rs_\lambda, s_\lambda, rs_{\lambda+1}, s_{\lambda+1}, \dots, rs_\alpha, s_\alpha, rs_{\lambda-1}, s_{\lambda-1}, \dots, rs_1, s_1)$ is a new multifan with respect to φ'' that has the same vertex set as $F_{\varphi'}(r, s_1 : s_\alpha)$. In this new multifan F^*, λ is itself a λ-inducing color, τ is a $(\lambda+1)$-inducing color, and $\alpha+1$ is the last $(\lambda+1)$-inducing color. We can further assume that F^* is typical by relabeling colors and vertices. However, $r \in P_y(\alpha+1, \lambda, \varphi'')$, shows a contradiction to Lemma 3.6 (d) that $r \in P_{s_\alpha}(\alpha+1, \lambda, \varphi'') = P_{s_\lambda}(\alpha+1, \lambda, \varphi'')$.

Case B. $\overline{\varphi}'(x) = \alpha+1$ **and** $\overline{\varphi}'(y) = 1$**.**

We assume that x and y are $(1, \alpha+1)$-linked with respect to φ'. For otherwise, a $(1, \alpha+1)$-swap at y reduces the problem to Case A.

We show that $\tau, \lambda \neq \Delta$. If this is not the case, then by swapping colors along $P_{[x,y]}(1, \alpha+1, \varphi')$ and exchanging the roles of x and y if necessary, we assume that $\tau \neq \Delta$ and $\lambda = \Delta$. Let φ'' be obtained from φ' by a $(1, \Delta)$-swap at y. By Lemma 3.6 (d), $r \in P_{s_1}(\alpha+1, \Delta, \varphi'') = P_{s_\alpha}(\alpha+1, \Delta, \varphi'')$. Thus, we can do an $(\alpha+1, \Delta)$-swap at y without affecting the coloring of $F_{\varphi''}(r, s_1 : s_\alpha)$ and $\varphi''(ru)$. Thus, let $\varphi^* = \varphi''/P_y(\alpha+1, \Delta, \varphi'')$. We see that $P_r(1, \alpha+1, \varphi^*) = ruy$, showing a contradiction to Lemma 3.1 (b) that r and s_α are $(1, \alpha+1)$-linked with respect to φ^*.

Since $\tau, \lambda \neq \Delta$, both τ and λ are 2-inducing colors of F by Claim 4.2. By swapping colors along $P_{[x,y]}(1, \alpha+1, \varphi')$ and exchanging the roles of x and y if necessary, we assume $\tau \prec \lambda$. Note that $r \in P_{s_1}(\lambda, \Delta, \varphi') = P_{s_{\lambda-1}}(\lambda, \Delta, \varphi')$ and $r \in P_{s_1}(\alpha+1, \Delta, \varphi') = P_{s_\alpha}(\alpha+1, \Delta, \varphi')$ by Lemma 3.6 (c) and (d), respectively. Let φ'' be obtained from φ' by doing a $(1, \Delta) - (\Delta, \lambda) - (\lambda, 1) - (1, \Delta) - (\Delta, \alpha+1)$-swap at y. Note that φ'' is (F, φ')-stable, and that $P_r(1, \alpha+1, \varphi'') = ruy$, showing a contradiction to Lemma 3.1 (b) that r and s_α are $(1, \alpha+1)$-linked with respect to φ''. $\diamond$

By Claim 4.5 and Claim 4.6, we let $x, y \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$ with $x \neq y$, and assume that $\varphi(ru) = \alpha+1$ and $\overline{\varphi}(x) = \alpha+1$. By Claim 4.7, we also assume that $\overline{\varphi}(y) = \delta \in [\alpha+2, \Delta-1]$ and y and r are $(1, \delta)$-linked with respect to such a coloring φ. Let $w_1 \in N_{\Delta-1}(r)$ such that $\varphi(rw_1) = \delta$ and $L = (F, ru, u, ux, x)$. By Claim 4.7, for any L-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$, it holds that $y \in P_r(1, \delta, \varphi')$. Applying Lemma 3.7 (2) on L with y playing the role of w, we find a sequence of distinct vertices $w_1, \dots, w_t \in \{s_{\alpha+1}, \dots, s_{\Delta-2}\}$ that forms either a stable rotation or a near stable rotation. Since y and r are $(1, \delta)$-linked with respect to φ, $w_1, \dots, w_t$ is a near stable rotation, i.e., $\overline{\varphi}(w_t) = \alpha+1$.

Claim 4.8. $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| \geq 3$.

Proof of Claim 4.8. Let $\varphi(ux) = \tau$ and $\varphi(uy) = \lambda$. Since $\varphi(ru) = \alpha+1$, we have $\alpha+1 \notin \{\tau, \lambda, \delta\}$. By Claim 4.3, $\tau \in \overline{\varphi}(F) \setminus \{1\}$, and

$$\begin{cases} s_1, s_\alpha \notin N_{\Delta-1}(u) & \text{if } \tau = \Delta, \\ s_{\tau-1}, s_\tau \notin N_{\Delta-1}(u) & \text{if } \tau \neq \Delta. \end{cases} \tag{1}$$

We then show that

$$\begin{cases} s_1, s_\alpha \notin N_{\Delta-1}(u) & \text{if } \lambda = \Delta, \\ s_{\lambda-1}, s_\lambda \notin N_{\Delta-1}(u) & \text{if } \lambda \neq \Delta. \end{cases} \tag{2}$$

To see this, let φ' be obtained from φ by first doing a $(1, \alpha+1)$-swap at both x and w_t, and then shift from w_1 to w_t. Now, $\overline{\varphi}'(r) = \delta$ and $\varphi'(ux) = \varphi(ux) = \tau$. Let $\varphi'' = \varphi'/P_y(\alpha+1, \delta, \varphi')$. Note that $\varphi''(ux) = \varphi'(ux) = \tau$ and $\varphi''(uy) = \varphi'(uy) = \lambda$. Applying Claim 4.2 to the coloring φ'', we get $\varphi''(uy) = \lambda \in \overline{\varphi}''(F) \setminus \{\delta\}$. As $\tau, \lambda, \delta, \alpha+1 \in \overline{\varphi}''(F)$ and they are all distinct, $|V(F)| \geq |\{\delta, \tau, \lambda, \alpha+1\}| - 1 = 3$. Then (2) follow from Claim 4.3. This fact, together with (1), implies that either $s_1, s_\alpha, s_{\lambda-1}, s_\lambda \notin N_{\Delta-1}(u)$, or $s_1, s_\alpha, s_{\tau-1}, s_\tau \notin N_{\Delta-1}(u)$. Note that $s_1 \neq s_\alpha$ by $|V(F)| \geq 3$. We obtain $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| \geq 3$ from the above unless either $\lambda = \alpha = 2$ or $\tau = \alpha = 2$.

Therefore we assume $\alpha = 2$ and $\{\lambda, \tau\} = \{2, \Delta\}$. Furthermore, we may assume that $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| = 2$, since (1) and (2) imply that $s_1, s_2 \notin N_{\Delta-1}(u)$. Therefore $N_{\Delta-1}(r) \setminus \{s_1, s_2\} \subseteq N_{\Delta-1}(u)$. In particular, $w_t \in N_{\Delta-1}(u)$. Since r and s_α are $(1, \alpha+1)$-linked with respect to φ and $\overline{\varphi}(w_t) = \alpha+1$, it follows that $\varphi(uw_t) \neq 1$. This, together with the facts that $\overline{\varphi}(F) = \{1, 2, 3, \Delta\}$, $\varphi(ru) = 3$, and $\{\lambda, \tau\} = \{2, \Delta\}$, implies that $\varphi(uw_t) \in [4, \Delta-1]$, showing a contradiction to Claim 4.2. ◇

By Claim 4.8, let $z \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$ with $z \neq x, y$. By Claim 4.7, we assume $\overline{\varphi}(z) = \lambda \in [\alpha+2, \Delta-1]$, and z and r are $(1, \lambda)$-linked with respect to φ. Since $\overline{\varphi}(y) = \delta$, and also y and r are $(1, \delta)$-linked with respect to φ, we have $\lambda \neq \delta$.

Recall $w_1, \ldots, w_t$ is a near stable rotation at r with $\varphi(rw_1) = \delta =: \delta_1$. Let $\overline{\varphi}(w_i) = \delta_{i+1}$ for each $i \in [1, t-1]$. As z and r are $(1, \lambda)$-linked with respect to φ and w_i and r are $(1, \delta_{i+1})$-linked for each $i \in [1, t-1]$, $\lambda \neq \delta_i$ for each $i \in [2, t]$. Let $\lambda_1 = \lambda$ and w_1^* be the neighbor of r such that $\varphi(rw_1^*) = \lambda_1$. For any (L, φ)-stable coloring $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$, $z \in P_r(1, \lambda, \varphi')$. Applying Lemma 3.7 (2) on $L = (F, ru, u, ux, x)$ and z, we find a sequence of distinct vertices $w_1^*, \ldots, w_k^* \in \{s_{\alpha+1}, \ldots, s_{\Delta-2}\}$ that forms either a stable rotation or a near stable rotation. If $\overline{\varphi}(w_k^*) = \lambda_1$, then since w_k^* and r are $(1, \lambda_1)$-linked, a $(1, \lambda_1)$-swap at z gives a contradiction to Claim 4.7. Thus $\overline{\varphi}(w_k^*) = \alpha+1$. Let $\overline{\varphi}(w_i^*) = \lambda_{i+1}$ for each $i \in [1, k-1]$.

Recall that $w_1^* \neq w_i$ for each $i \in [1, t]$. Furthermore, as w_i and r are $(1, \delta_{i+1})$-linked for each $i \in [1, t-1]$ and w_j^* and r are $(1, \lambda_{j+1})$-linked for each $j \in [1, k-1]$, $w_1^* \neq w_i$ for each $i \in [1, t]$ implies that $\lambda_2 \notin \{\delta_1, \ldots, \delta_t\}$. Consequently, $w_2^* \neq w_i$ for each $i \in [1, t]$. Repeating the same process, we get $w_j^* \neq w_i$ for each $j \in [1, k]$ and each $i \in [1, t]$.

We claim that w_t and x are $(1, \alpha+1)$-linked with respect to φ. For otherwise, first doing a $(1, \alpha+1)$-swap at w_t, then shift from w_1 to w_t gives a coloring φ' such that $\varphi'(ru) = \varphi(ru) = \alpha+1$, $\overline{\varphi}'(y) = \overline{\varphi}'(r) = \delta_1$, while $\overline{\varphi}'(x) = \alpha+1$. Based on φ', after doing a $(1, \delta_1)$-swap on all $(1, \delta_1)$-chains in $G - rs_1$, we obtain an (F, φ)-stable coloring φ''. However, $\overline{\varphi}''(x) = \alpha+1$ and $\overline{\varphi}''(y) = 1$, showing a contradiction to Claim 4.7. As w_t and x are $(1, \alpha+1)$-linked, we do a sequence of Kempe changes around r from w_k^* to w_1^* as below: let $\varphi_0 = \varphi$ and $\lambda_{k+1} = \alpha+1$,

$$\varphi_j = \varphi_{j-1}/P_{w^*_{k-(j-1)}}(1, \lambda_{k+1-(j-1)}, \varphi_{j-1}) \quad \text{for each } j \in [1, k].$$

Note that

$$P_r(1, \lambda_{k-(j-1)}, \varphi_j) = rw^*_{k-(j-1)} \quad \text{for each } j \in [1, k],$$

and that φ_k is (F, φ)-stable, $\varphi_k(ru) = \varphi(ru)$, $\varphi_k(ux) = \varphi(ux)$, and $\overline{\varphi}_k(x) = \overline{\varphi}(x) = \alpha+1$, but z and r are $(1, \lambda)$-unlinked with respect to φ_k. Now doing a $(1, \lambda)$-swap at z gives a contradiction to Claim 4.7. This finishes the proof of Theorem 2.3. $\square$

5. Proof of Theorem 2.4

Theorem 2.4. *If G is an HZ-graph with maximum degree $\Delta \geq 4$, then for any two adjacent vertices $x, y \in V_{\Delta-1}$, $N_\Delta(x) = N_\Delta(y)$.*

Proof. Assume to the contrary that $N_\Delta(x) \neq N_\Delta(y)$. Then there exists a vertex $r \in N_\Delta(x) \setminus N_\Delta(y)$. Equivalently, $x \in N_{\Delta-1}(r)$ and $y \notin N_{\Delta-1}(r)$. By Theorem 4.1 (ii), let $s_1 \in N_{\Delta-1}(r)$ and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$, and $F = F_\varphi(r, s_1 : s_\alpha)$ be the typical 2-inducing multifan such that either $V(F) = N_{\Delta-1}[r]$ or F is contained in a pseudo-multifan S with $V(S) = N_{\Delta-1}[r]$. Let $N_{\Delta-1}(r) = \{s_1, \ldots, s_{\Delta-2}\}$. We consider two cases according to if $x \in V(F)$ to finish the proof.

Assume first that $x \notin V(F)$. This implies that $V(F) \neq N_{\Delta-1}[r]$. Applying Theorem 4.1 (ii), it then follows that $N_{\Delta-1}[r]$ is the vertex set of a typical 2-inducing pseudo-multifan. Let $\overline{\varphi}(x) = \delta$ and $\overline{\varphi}(y) = \lambda$. Since $V(S) = N_{\Delta-1}[r]$ is φ-elementary, $\delta, \lambda \in \overline{\varphi}(S)$. By Lemma 3.1 (b) or Lemma 3.5 (b), we know that $\overline{\varphi}_S^{-1}(\lambda)$ and r are $(1, \lambda)$-linked and x and r are $(1, \delta)$-linked. By doing a $(\lambda, 1) - (1, \delta)$-swap at y, we find (S, φ)-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $\overline{\varphi}'(y) = \delta$. Let $\varphi'(xy) = \tau$. Then $P_x(\delta, \tau, \varphi') = xy$, showing a contradiction to Lemma 3.5 (c) or (d) depending on $\tau \in \overline{\varphi}'(F)$ or $\tau \in \overline{\varphi}'(S) \setminus \overline{\varphi}'(F)$.

Assume then that $x \in V(F)$. We claim that we may assume $x = s_1$. Let $x = s_i$ for some $i \in [1, \alpha]$, and φ' be obtained from φ by shift from s_2 to s_{i-1}, uncoloring rs_i, and coloring rs_1 by 2. The sequence $F^* = (r, rs_i, s_i, rs_{i+1}, s_{i+1}, \ldots, rs_\alpha, s_\alpha, rs_{i-1}, s_{i-1}, \ldots, rs_1, s_1)$ is a multifan with respect to φ'. Since the shift and "changing" the uncolored edge operation like above is reversible, and $V(S)$ and $\overline{\varphi}(S)$ are kept unchanged under such an operation, we conclude that $N_{\Delta-1}[r]$ is still the vertex set of a pseudo-multifan. By permuting the names of the colors and the labels of the vertices in $N_{\Delta-1}(r)$, we may assume that $x = s_1$. Still denote the current coloring by φ, the multifan by F, and the pseudo-multifan by S.

By doing a $(1, \overline{\varphi}(y))$-swap at y, we assume $\overline{\varphi}(y) = 1$. Let $\varphi(s_1 y) = \tau$. By exchanging the roles of the color 2 and Δ if necessary, we may assume that $\varphi(s_1 y)$ is either a 2-inducing color of F or is a color from $\overline{\varphi}(S) \setminus \overline{\varphi}(F)$. Let $\varphi' = \varphi/P_y(1, \Delta, \varphi)$. Now $P_{s_1}(\tau, \Delta, \varphi') = s_1 y$. This gives a contradiction to Lemma 3.2 (b) that s_1 and $\overline{\varphi}'^{-1}_F(\tau)$ are (τ, Δ)-linked if τ is 2-inducing, and gives a contradiction to Lemma 3.5 (c) that s_1 and $\overline{\varphi}'^{-1}_S(\tau)$ are (τ, Δ)-linked if $\tau \in \overline{\varphi}(S) \setminus \overline{\varphi}(F)$. $\square$

6. Proof of Theorem 2.5

Theorem 2.5. *Let G be an HZ-graph with maximum degree $\Delta \geq 7$ and $u, r \in V_\Delta$. If $N_{\Delta-1}(u) \neq N_{\Delta-1}(r)$ and $N_{\Delta-1}(u) \cap N_{\Delta-1}(r) \neq \emptyset$, then $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| = \Delta - 3$, i.e. $|N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)| = |N_{\Delta-1}(r) \setminus N_{\Delta-1}(u)| = 1$.*

Proof. Assume to the contrary that there exist $u, r \in N_\Delta$ such that $1 \leq |N_{\Delta-1}(r) \cap N_{\Delta-1}(u)| \leq \Delta - 4$. By Theorem 4.1 (ii), there exist $s_1 \in N_{\Delta-1}(r)$ and $\varphi \in \mathcal{C}^\Delta(G - rs_1)$ such that $N_{\Delta-1}[r]$ is the vertex set of a typical 2-inducing pseudo-multifan. By this assumption of being typical, we have $N_{\Delta-1}(r) = \{s_1, \ldots, s_{\Delta-2}\}$, $\overline{\varphi}(r) = 1$, and $\overline{\varphi}(s_1) = \{2, \Delta\}$. Let $x, y \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$ be two distinct vertices, and $S := S_\varphi(r, s_1 : s_\alpha : s_{\Delta-2})$ be this pseudo-multifan with $F_\varphi(r, s_1 : s_\alpha)$ being the typical 2-inducing multifan contained in S. Since $V(S) = N_{\Delta-1}[r]$ and $V(S)$ is φ-elementary, it follows that $\overline{\varphi}(S) = [1, \Delta]$. We consider two cases.

Case 1. $V(S) \neq V(F)$.

In this case, we will repeatedly apply Lemma 3.5 (b), (c) or (d). Assume first that for each $i \in [1, \alpha]$, $s_i \notin N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$. Then by $N_{\Delta-1}(r) \cap N_{\Delta-1}(u) \neq \emptyset$, there exists $w_1 \in \{s_{\alpha+1}, \ldots, s_{\Delta-2}\}$ such that $w_1 \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$. Let $\varphi(rw_1) = \delta_1$ and $\overline{\varphi}(w_1) = \delta_2$. Note that $\delta_1, \delta_2 \in \overline{\varphi}(S) \setminus \overline{\varphi}(F)$. We claim that we may assume $\varphi(ux) = \delta_2$. Otherwise, let $\varphi(ux) = \delta^* \neq \delta_2$. By Lemma 3.5 (b), (c) or (d) depending on what $\overline{\varphi}(x)$ is, we can do a $(\overline{\varphi}(x), \delta_2) - (\delta_2, \delta^*)$-swap at x in getting an (S, φ)-stable coloring, still call it φ such that $\varphi(ux) = \delta_2$. Let $\varphi(w_1 u) = \tau$ and φ' be obtained from φ by doing a $(\overline{\varphi}(x), \delta_1) - (\delta_1, \tau)$-swap at x. By Lemma 3.5 (b), (c) or (d), φ' is (S, φ)-stable such that $\varphi'(w_1 u) = \varphi(w_1 u) = \tau$ and $\overline{\varphi}'(x) = \tau$. However, $P_{w_1}(\delta_2, \tau, \varphi') = w_1 u x = P_x(\delta_2, \tau, \varphi')$, showing a contradiction to Lemma 3.5 (b), (c) or (d) (depending on if $\tau = 1$, $\tau \in \overline{\varphi}(F) \setminus \{1\}$ or $\tau \in \overline{\varphi}(S) \setminus \overline{\varphi}(F)$) that w_1 and $\overline{\varphi}_S^{-1}(\tau)$ are (δ_2, τ)-linked with respect to φ'.

Assume now that there exists $s_i \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$ for some $i \in [1, \alpha]$. By shift from s_2 to s_{i-1}, uncoloring rs_i, and coloring rs_1 by 2, we obtain a new multifan $F^* = (r, rs_i, s_i, rs_{i+1}, s_{i+1}, \ldots, rs_\alpha, s_\alpha, rs_{i-1}, s_{i-1}, \ldots, rs_1, s_1)$. By permuting the names of the colors and the labels of the vertices in $N_{\Delta-1}(r)$ such that $i + 1$ is permuted to 2 and s_i is renamed as s_1, we assume that $s_1 \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$ and F^* is a typical multifan.

Recall that $x \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$. Let $\overline{\varphi}(x) = \lambda$. By Lemma 3.1 (b) or Lemma 3.5 (b), we know that $\overline{\varphi}_S^{-1}(\lambda)$ and r are $(1, \lambda)$-linked. By doing a $(1, \lambda)$-swap at x if necessary, we assume $\overline{\varphi}(x) = 1$. By exchanging the roles of the colors 2 and Δ, we assume that $\varphi(s_1 u)$ equals 1, or is a 2-inducing color of F, or is a color from $\overline{\varphi}(S) \setminus \overline{\varphi}(F)$. Note that by Lemma 3.5 (c), for a color $\delta \in \overline{\varphi}(S) \setminus \overline{\varphi}(F)$, and for any color $\tau \in \overline{\varphi}(F)$, $\overline{\varphi}_S^{-1}(\delta)$ and $\overline{\varphi}_S^{-1}(\tau)$ are (δ, τ)-linked and $r \in P_{\overline{\varphi}_S^{-1}(\delta)}(\delta, \tau, \varphi)$.

Let $\varphi(ux) = \tau$. If τ is a 2-inducing color of F or is from $\overline{\varphi}(S) \setminus \overline{\varphi}(F)$, we do $(1, \Delta) - (\Delta, \tau) - (\tau, 1)$-swaps at x. If τ is a Δ-inducing color of F, let $\delta \in \overline{\varphi}(S) \setminus \overline{\varphi}(F)$, we do $(1, \delta) - (\delta, \tau) - (\tau, 1) - (1, \Delta) - (\Delta, \delta) - (\delta, 1)$-swaps at x. In both cases, we let φ' be the resulting coloring. We have $\varphi'(ux) = \Delta$ and $\overline{\varphi}'(x) = 1$. Since $\varphi(s_1 u) \neq \Delta, \tau$, still $\varphi'(s_1 u)$ equals 1, or is a 2-inducing color of F, or is from $\overline{\varphi}(S) \setminus \overline{\varphi}(F)$.

Let $\varphi'(s_1u) = \gamma$. Since s_1 and r are $(1, \Delta)$-linked with respect to φ', $\gamma \neq 1$. Thus, γ is a 2-inducing color of F, or is from $\overline{\varphi}(S) \setminus \overline{\varphi}(F)$. By Lemma 3.2 (a) or Lemma 3.5 (c), $u \in P_x(1, \gamma, \varphi')$ (otherwise, s_1 and x are (γ, Δ)-linked after a $(1, \gamma)$-swap at x). Let $\varphi'' = \varphi'/P_x(1, \gamma, \varphi')$. Now $\varphi''(s_1u) = 1$, $\overline{\varphi}''(x) = \gamma$, and $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path with respect to rs_1 and φ''. Let $\delta \in \overline{\varphi}''(S) \setminus \overline{\varphi}''(F)$. If $\gamma \in \overline{\varphi}''(S) \setminus \overline{\varphi}''(F)$, we do nothing. Otherwise, we do a (γ, δ)-swap at x (by Lemma 3.5 (c), this swap does not end at any vertex of S). Denote by φ''' the resulting coloring. Since $d_G(s_1) = \Delta - 1$, in both cases, by Lemma 3.3 (b), x and s_1 are $(2, \overline{\varphi}'''(x))$-linked. Since $\overline{\varphi}'''(x) \in \overline{\varphi}'''(S) \setminus \overline{\varphi}''(F)$, we achieve a contradiction to Lemma 3.5 (c).

Case 2. $V(S) = V(F)$.

We claim that we may choose s_1 such that $s_1 \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$. If $s_1 \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$, then we are done. Otherwise, let $s_i \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$. We shift from s_2 to s_{i-1}, uncolor rs_i and color rs_1 by 2. By relabeling colors and vertices, we may assume that $s_1 \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$ and $F^* = (r, rs_i, s_i, rs_{i+1}, s_{i+1}, \ldots, rs_\alpha, s_\alpha, rs_{i-1}, s_{i-1}, \ldots, rs_1, s_1)$ is a typical multifan. We let $F_\varphi(r, s_1 : s_\alpha : s_{\Delta-2})$ be such a typical multifan.

For a coloring $\psi \in \mathcal{C}^\Delta(G - rs_1)$, if $\overline{\varphi}(s_1) = \overline{\psi}(s_1)$, $\overline{\varphi}(F) = \overline{\psi}(F)$, and some permutation of F is still a multifan with respect to rs_1 and ψ, we call ψ a *near* (F, φ)*-stable* coloring. As only colors in $\overline{\varphi}(s_1)$ will be essential for the proof, we will not distinguish between φ and any near (F, φ)-stable coloring. As the vertex set of all the resulting multifans is always $N_{\Delta-1}[r]$, for a color $\alpha \in [1, \Delta]$, we use $\overline{\psi}^{-1}(\alpha)$ to denote the vertex from $N_{\Delta-1}[r]$ that misses α with respect to ψ.

Let $\psi \in \mathcal{C}^\Delta(G - rs_1)$ be near (F, φ)-stable and F^* be the corresponding multifan. The following two facts will be used frequently in the proof without being mentioned.

- Fact 1 For any $i \in [2, \Delta]$, since r and $\overline{\psi}^{-1}(i)$ are $(1, i)$-linked by Lemma 3.1 (b), doing a $(1, i)$-swap at vertices outside of $V(F^*)$ gives an (F^*, ψ)-stable and so a near (F, φ)-stable coloring.
- Fact 2 For any 2-inducing color τ and Δ-inducing color δ of F^*, $\overline{\psi}^{-1}(\tau)$ and $\overline{\psi}^{-1}(\delta)$ are (τ, δ)-linked by Lemma 3.2 (a). Thus doing a (τ, δ)-swap at a vertex outside of $V(F^*)$ or, when $\tau \neq 2$ and $\delta \neq \Delta$, doing a (τ, δ)-swap at $\overline{\psi}^{-1}(\tau)$ if $r \notin P_{\overline{\psi}^{-1}(\tau)}(\tau, \delta, \psi)$ gives a near (F^*, ψ)-stable and so a near (F, φ)-stable coloring.

We denote by $S(u; s_1, x, y)$ the star subgraph of G that is centered at u consisting of edges us_1, ux, and uy. Recall that $x, y \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$ are distinct vertices.

Claim 6.1. *We may assume that* $\overline{\varphi}(x) = 2$ *and* $\overline{\varphi}(y) = \Delta$ *or* $\overline{\varphi}(x) = \overline{\varphi}(y) = \Delta$.

Proof of Claim 6.1. By doing $(\overline{\varphi}(x), 1) - (1, 2)$-swaps at x, we find (F, φ)-stable $\varphi' \in \mathcal{C}^\Delta(G - rs_1)$ such that $\overline{\varphi}'(x) = 2$. Now, let $\overline{\varphi}'(y) = \lambda$. If $\lambda = 2$, then doing $(2, 1) - (1, \Delta)$-swaps at both x and y, we find (F, φ')-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$ such that $\overline{\varphi}''(x) = \overline{\varphi}''(y) = \Delta$. If $\lambda \neq 2$, by doing $(\lambda, 1) - (1, \Delta)$-swaps at y, we find (F, φ')-stable $\varphi'' \in \mathcal{C}^\Delta(G - rs_1)$

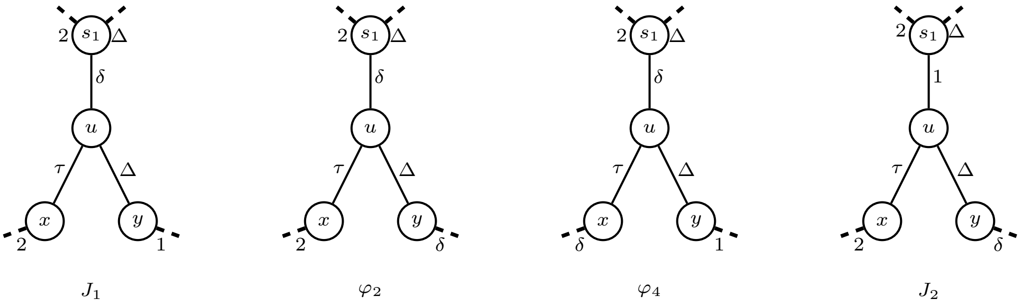

Fig. 2. Coloring of $S(u; s_1, x, y)$.

such that $\overline{\varphi}''(x) = 2$ and $\overline{\varphi}''(y) = \Delta$. As φ'' is (F, φ')-stable and φ' is (F, φ)-stable, it follows that φ'' is (F, φ)-stable. So we can take φ'' to be φ. $\diamond$

By Claim 6.1, we assume $\overline{\varphi}(x) = 2$ and $\overline{\varphi}(y) = \Delta$ or $\overline{\varphi}(x) = \overline{\varphi}(y) = \Delta$ and so consider two cases below.

Subcase 2.1. $\overline{\varphi}(x) = 2$ **and** $\overline{\varphi}(y) = \Delta$.

Note that by doing first a $(2, 1)$-swap at x, then a $(1, \Delta)$-swap at both x and y, and finally a $(1, 2)$-swap at y, we can always identify this current case with the case that $\overline{\varphi}(x) = \Delta$ and $\overline{\varphi}(y) = 2$. Let $\varphi(ux) = \tau$ and $\varphi(uy) = \lambda$. By exchanging the roles of the two colors 2 and Δ, we consider two cases below: (A) $\varphi(uy) = \lambda = 1$; and (B) $\varphi(uy) = \lambda$ is 2-inducing. (When $\varphi(uy)$ is Δ-inducing, by assuming $\overline{\varphi}(x) = \Delta$ and $\overline{\varphi}(y) = 2$, the argument will be symmetric to the argument for case (B) above.)

In both cases of (A) and (B), we do $(\Delta, \lambda) - (\lambda, 1)$-swaps at y and call the resulting coloring φ_1 and the resulting multifan F_1. Note that φ_1 is near (F, φ)-stable. Let $\varphi_1(s_1 u) = \delta$. The current coloring on $S(u; s_1, x, y)$ is as shown in J_1 of Fig. 2.

Claim 6.2. *The color φ_1 can be modified into a near (F_1, φ_1)-stable coloring such that the color on $S(u; s_1, x, y)$ is as in J_2 of Fig. 2.*

Proof of Claim 6.2. Since s_1 and r are $(1, \Delta)$-linked by Lemma 3.1 (b), we know $\varphi(s_1 u) = \delta \neq 1$. If $u \in P_y(1, \delta, \varphi_1)$, then a $(1, \delta)$-swap at y gives J_2. Thus, we assume $u \notin P_y(1, \delta, \varphi_1)$. This implies that δ is Δ-inducing with respect to F_1 and φ_1. (Otherwise, after a $(1, \delta)$-swap at y, s_1 and $\overline{\varphi}_1^{-1}(\delta)$ are (δ, Δ)-unlinked, showing a contradiction to Lemma 3.2 (a).)

Let $\varphi_2 = \varphi_1 / P_y(1, \delta, \varphi_1)$ (see Fig. 2). We claim that τ is 2-inducing with respect to F_1 and φ_2. Otherwise τ is 1 or is Δ-inducing with respect to F_1 and φ_2. We do $(2, \tau) - (\tau, 1)$-swaps at x and call the resulting coloring φ_2' and the resulting multifan F_1'. Again, as $P_{s_1}(\delta, \Delta, \varphi_2') = s_1 u y$, δ is still a Δ-inducing color of F_1' with respect to φ_2' by Lemma 3.2 (a). Since $\varphi_2'(ux) = 2$, we must have $u \in P_x(1, \delta, \varphi_2')$: otherwise, after a $(1, \delta)$-swap at x, s_1 and x are $(2, \delta)$-linked with respect to the current coloring, contradicting

Lemma 3.2 (a). Now, let φ_2^* be obtained from φ_2' by doing a $(1,\delta)$-swap at both x and y. We get $P_{s_1}(1,\Delta,\varphi_2^*) = s_1uy$, showing a contradiction to Lemma 3.1 (b) that s_1 and r are $(1,\Delta)$-linked with respect to φ_2^*.

Thus τ is 2-inducing with respect to F_1 and φ_2. First, we let $\varphi_3 = \varphi_2/P_x(1,2,\varphi_2)$. Note that $u \notin P_x(1,\delta,\varphi_3)$ and $u \notin P_y(1,\delta,\varphi_3)$. Since otherwise, after a $(1,\delta)$-swap at both x and y, s_1 and y are $(1,\Delta)$-linked with respect to the current coloring, showing a contradiction to Lemma 3.1 (b) that s_1 and r are $(1,\Delta)$-linked. Since δ is Δ-inducing and τ is 2-inducing with respect to F_1 and φ_3, $\overline{\varphi}_3^{-1}(\delta)$ and $\overline{\varphi}_3^{-1}(\tau)$ are (δ,τ)-linked by Lemma 3.2 (a). Then we let φ_4 be obtained from φ_3 by doing a $(1,\delta)$-swap at both x and y (see Fig. 2), and doing a (τ,δ)-swap at $\overline{\varphi}_3^{-1}(\delta)$ (and so also at $\overline{\varphi}_3^{-1}(\tau)$). Since φ_4 is near (F_1,φ_3) stable, we let F_2 be the resulting multifan. Note that δ is a 2-inducing color and τ is a Δ-inducing color of F_2 with respect to φ_4. As a consequence, $u \in P_y(1,\delta,\varphi_4)$. Since otherwise, after a $(1,\delta)$-swap at y, s_1 and y are (δ,Δ)-linked, contradicting Lemma 3.2 (a). We then let φ_5 be obtained from φ_4 by doing a $(1,\delta)$-swap at both x, y (and so also u), and then a $(1,2)$-swap at x. We obtain the desired coloring on $S(u;s_1,x,y)$. $\diamond$

By Claim 6.2, we let $\varphi_2 \in \mathcal{C}^\Delta(G - rs_1)$ be a near (F_1,φ_1)-stable coloring and F_2 be a corresponding multifan such that under φ_2, the color on $S(u;s_1,x,y)$ is as in J_2 of Fig. 2. Now $K = (r, rs_1, s_1, s_1u, u, uy, y)$ is a Kierstead path with respect to rs_1 and φ_2. Since $d_G(s_1) = \Delta - 1$, by Lemma 3.3 (b), y and s_1 are $(2,\delta)$-linked. This implies that δ must be a 2-inducing color of F_2, as otherwise, s_1 and $\overline{\varphi}_2^{-1}(\delta)$ should be $(2,\delta)$-linked. If τ is Δ-inducing of F_2, then as $\overline{\varphi}_2^{-1}(\delta)$ and $\overline{\varphi}_2^{-1}(\tau)$ are (δ,τ)-linked by Lemma 3.2 (a), we do a (δ,τ)-swap at y. Again by Lemma 3.3 (b), y and s_1 are $(2,\tau)$-linked, showing a contradiction to Lemma 3.2 (a) that $\overline{\varphi}_2^{-1}(\tau)$ and s_1 are $(2,\tau)$-linked. Therefore, τ is a 2-inducing color of F_2. We first do $(2,1)-(1,\Delta)$ swaps at x, and let φ_3 be the resulting coloring (see Fig. 3). At this step, $\varphi_3(s_1u) = 1$, $\varphi_3(uy) = \Delta$, $\overline{\varphi}_3(y) = \delta$, and y and s_1 are $(2,\delta)$-linked with respect to φ_3 by Lemma 3.3 (b). Call this fact $(*)$.

Let $\varphi_4 = \varphi_3/P_x(\tau,\Delta,\varphi_3)$ (see Fig. 3) and F_3 be the resulting multifan. Since $\varphi_3^{-1}(\tau)$ appears before the edge with color τ in F_2, τ is still a 2-inducing color of F_3. As s_1 and r are $(1,\Delta)$-linked by Lemma 3.1 (b), we have $u \in P_x(1,\tau,\varphi_4)$. Let $\varphi_5 = \varphi_4/P_x(1,\tau,\varphi_4)$. The coloring of $S(u;s_1,x,y)$ is now as in J_3 of Fig. 3. Since $2,\delta \notin \{1,\tau,\Delta\}$, y and s_1 are still $(2,\delta)$-linked with respect to φ_5 by fact $(*)$, which further implies that δ is a 2-inducing color of F_3 with respect to φ_5. Since φ_5 is (F_3,φ_4)-stable, τ is still a 2-inducing colors of F_3 with respect to φ_5. We consider two cases to finish the remaining part of the proof.

Subcase 2.1.1. $\tau \prec \delta$ **in** F_3 **with respect to** φ_5.

Let $s_i \in N_{\Delta-1}(r)$ such that $\overline{\varphi}_5(s_i) = \delta$. Since y and s_1 are still $(2,\delta)$-linked with respect to φ_5 and δ is 2-inducing of F_3, by Lemma 3.2 (b), $r \in P_{s_i}(2,\delta,\varphi_5)$. We reach a contradiction through the following operations:

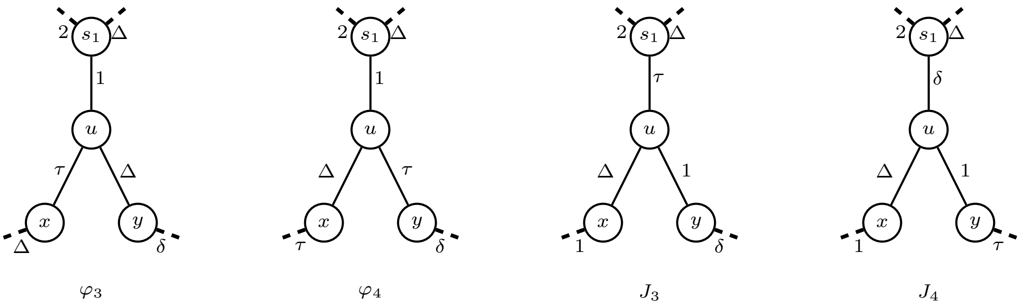

Fig. 3. Coloring of $S(u; s_1, x, y)$.

$$\begin{bmatrix} s_i \text{ (also at } r) & x \text{ and } s_i & s_j : s_i \\ (2,\delta)\text{-swap} & (1,2)\text{-swap} & \text{shift} \end{bmatrix},$$

where we assume that $\varphi_5(rs_j) = \tau$ for some $j \in [2, \Delta - 2]$ and $s_j, s_{j+1}, \ldots, s_i$ is the 2-inducing sequence of F_3 starting at s_j and ending at s_i. Denote the new coloring by φ_6. Now, $\overline{\varphi}_6(r) = \tau$, $\varphi_6(s_1u) = \tau$, $\varphi_6(ux) = \Delta$, and $\overline{\varphi}_6(x) = 2$, and $K = (r, rs_1, s_1, s_1u, u, ux, x)$ is a Kierstead path with respect to rs_1 and φ_6. Since $d_G(s_1) = \Delta - 1$, we get a contradiction to Lemma 3.3 (a) that $\{r, s_1, u, x\}$ is φ_6-elementary.

Subcase 2.1.2. $\delta \prec \tau$ **in** F_3 **with respect to** φ_5.

We only show that by performing Kempe changes, we can find an (F_3, φ_5)-stable coloring such that the color on $S(u; s_1, x, y)$ with respect to it is as given in J_4 of Fig. 3. Then the proof follows the same ideas as in Subcase 2.1.1 by exchanging the roles of τ and δ. Based on the coloring in J_3 of Fig. 3, do a $(1, \delta)$-swap at both x and y and denote the resulting coloring by φ_6.

Claim 6.3. $u \in P_y(1, \tau, \varphi_6)$.

Proof of Claim 6.3. Assume to the contrary that $u \notin P_y(1, \tau, \varphi_6)$. Under this assumption, it must be the case that $u \in P_r(1, \tau, \varphi_6)$ (otherwise, performing a (δ, Δ)-swap at x and a $(1, \tau)$-swap at u shows that s_1 and y are $(1, \Delta)$-linked, showing a contradiction to Lemma 3.1 (b) that s_1 and r are $(1, \Delta)$-linked). Since $u \in P_r(1, \tau, \varphi_6)$, let φ_7 be obtained by doing a $(1, \tau)$-swap at y and (δ, Δ)-swap at x, and let F_3^* be the resulting multifan. Then $P_{s_1}(\tau, \Delta, \varphi_7) = s_1uy$, implying that τ is a Δ-inducing color of F_3^* by Lemma 3.2 (a). Note that δ is still a 2-inducing color of F_3^* as the only operation that changes the color sequence of F_3 was the (δ, Δ)-swap we did to get φ_7 from φ_6. Thus, $\overline{\varphi}_7^{-1}(\delta)$ and $\overline{\varphi}_7^{-1}(\tau)$ are (δ, τ)-linked by Lemma 3.2 (a). Also, since τ is Δ-inducing and δ is 2-inducing of F_3^*, we know $u \notin P_y(\tau, \delta, \varphi_7)$. Since otherwise, after a (τ, δ)-swap at y, s_1 and y are (δ, Δ)-linked, showing a contradiction to Lemma 3.2 (a).

Let $\varphi_8 = \varphi_7/P_y(\tau, \delta, \varphi_7)$. Now $P_y(\delta, \Delta, \varphi_8) = yux$. Note also that $u \in P_{\overline{\varphi}_8^{-1}(\delta)}(\delta, \tau, \varphi_8) = P_{\overline{\varphi}_8^{-1}(\tau)}(\delta, \tau, \varphi_8)$. For otherwise, after a (τ, δ)-swap at u, s_1 and y are (δ, Δ)-linked, showing a contradiction to the fact that δ is still a 2-inducing color of the resulting multifan. Note that $P_x(\delta, \Delta, \varphi_8) = xuy$. Let $\varphi_9 = \varphi_8/P_x(\delta, \Delta, \varphi_8)$. Now $\overline{\varphi}_9^{-1}(\delta)$ and $\overline{\varphi}_9^{-1}(\tau)$ are (δ, τ)-unlinked. However, since φ_9 is (F_3^*, φ_8)-stable, τ is still a Δ-inducing color and δ is 2-inducing of F_3^* with respect to φ_9, we get a contradiction to Lemma 3.2 (a). $\diamond$

Thus by Claim 6.3, $u \in P_y(1, \tau, \varphi_6)$. Do a $(1, \tau)$-swap at y (and u), and denote the resulting coloring by φ_7. Note that $u \in P_x(1, \delta, \varphi_7)$ (as otherwise, after a $(1, \delta)$-swap at x, s_1 and x are $(1, \Delta)$-linked, showing a contradiction to Lemma 3.1 (b) that s_1 and r are $(1, \Delta)$-linked). Let $\varphi_8 = \varphi_7/P_x(1, \delta, \varphi_7)$. Now with respect to φ_8, we have the coloring in J_4 of Fig. 3. By the definition, φ_8 is (F_3, φ_5)-stable so we still have $\delta \prec \tau$ in F_3 with respect to φ_8. The remaining proof follows the same ideas as in Subcase 2.1.1.

Subcase 2.2. $\overline{\varphi}(x) = \Delta$ **and** $\overline{\varphi}(y) = \Delta$**.**

Claim 6.4. *We may assume that* $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| = \Delta - 4$.

Proof of Claim 6.4. We may assume that x and y are $(1, \Delta)$-linked. For otherwise, performing $(\Delta, 1) - (1, 2)$-swaps at x reduces the problem to Subcase 2.1. Since $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| > \Delta - 4$ implies $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| = \Delta - 3$ already, we assume to the contrary that $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| \leq \Delta - 5$. Then there exists $z \in N_{\Delta-1}(u) \setminus N_{\Delta-1}(r)$ such that $z \neq x, y$. Let $\overline{\varphi}(z) = \lambda$. If $\lambda = 2$, by exchanging the roles of x and z, we reduce the problem to Subcase 2.1. Thus, $\lambda \neq 2$. Doing $(\lambda, 1) - (1, 2)$-swaps at z and exchanging the roles of x and z reduces the problem to Subcase 2.1. $\diamond$

Claim 6.5. *We may assume that* $F(r, s_1 : s_\alpha : s_{\Delta-2})$ *is a typical multifan with two sequences. That is,* F *contains both* 2*-inducing sequence and* Δ*-inducing sequence.*

Proof of Claim 6.5. Recall that $F_\varphi(r, s_1 : s_\alpha : s_{\Delta-2})$ is a typical multifan. As $\Delta \geq 7$, $|N_{\Delta-1}(u) \cap N_{\Delta-1}(r)| = \Delta - 4 \geq 3$ by Claim 6.4. If F is a typical 2-inducing multifan, then let $s_i \in N_{\Delta-1}(u) \cap N_{\Delta-1}(r)$ such that $s_i \neq s_1$ and that $\overline{\varphi}(s_i)$ is not the last 2-inducing color of F. Then we shift from s_2 to s_{i-1}, uncolor rs_i, and color rs_1 by 2. Now $F^* = (r, rs_i, s_i, rs_{i+1}, s_{i+1}, \ldots, rs_{\Delta-2}, s_{\Delta-2}, rs_{i-1}, s_{i-1}, \ldots, rs_1, s_1)$ is a multifan with two sequences. By permuting the names of the colors and the labels of vertices in $\{s_1, \ldots, s_{\Delta-2}\}$, we can assume that $F = F^*$ is a typical multifan with two sequences. $\diamond$

Let $\varphi(s_1u) = \delta$, $\varphi(ux) = \tau$, and $\varphi(uy) = \lambda$. By exchanging the roles of the two colors 2 and Δ, we have two possibilities for $\varphi(uy)$: (A) $\varphi(uy) = \lambda = 1$; and (B) $\varphi(uy) = \lambda$ is 2-inducing. (When $\varphi(uy)$ is Δ-inducing, we will first assume that $\overline{\varphi}(x) = 2$ and $\overline{\varphi}(y) = 2$ (by performing $(\Delta, 1) - (1, 2)$-swaps at both x and y). Then all the argument will be symmetric to the argument for the case (B) above.) We now consider two cases to finish the proof.

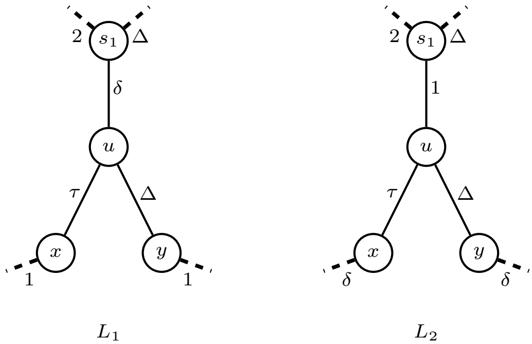

Fig. 4. Coloring of $S(u; s_1, x, y)$.

Subcase 2.2.1. λ **is not the last 2-inducing color of** F**.**

We first perform $(\Delta, \lambda) - (\lambda, 1)$-swaps at both x and y. Denote by φ_1 the resulting coloring and F_1 the corresponding multifan. Since λ is not the last 2-inducing color of F, F_1 still has two sequences with respect to φ_1. The current coloring of $S(u; s_1, x, y)$ is given in L_1 of Fig. 4. Since s_1 and r are $(1, \Delta)$-linked by Lemma 3.1 (b), $\delta \neq 1$. We next show $u \in P_y(1, \delta, \varphi_1)$ that will lead to the coloring in L_2 of Fig. 4 after a $(1, \delta)$-swap at both x and y.

Claim 6.6. $u \in P_y(1, \delta, \varphi_1)$.

Proof of Claim 6.6. Assume to the contrary that $u \notin P_y(1, \delta, \varphi_1)$. This implies that δ is a Δ-inducing color of F_1 (since after doing a $(1, \delta)$-swap at y, s_1 and y are (δ, Δ)-linked). If τ is a Δ-inducing of F_1, then we let φ_2 be obtained by performing $(1, 2) - (2, \tau) - (1, \tau)$-swaps at both x and y based on the coloring of L_1 in Fig. 4. Now, we must have that $u \in P_x(1, \delta, \varphi_2)$ or $u \in P_y(1, \delta, \varphi_2)$ since δ is either 2-inducing or Δ-inducing with respect to F_1. Let φ_3 be obtained from φ_2 by performing a $(1, \delta)$-swap at both x and y. Then both $K_1 = (r, rs_1, s_1, s_u, u, ux, x)$ and $K_2 = (r, rs_1, s_1, s_u, u, uy, y)$ are Kierstead paths with respect to rs_1 and φ_3. Since $d_G(s_1) = \Delta - 1$, applying Lemma 3.3 (b), x and s_1 are (δ, Δ)-linked and y and s_1 are $(2, \delta)$-linked. However, by Lemma 3.2 (a), s_1 and $\overline{\varphi}_3^{-1}(\delta)$ are either $(\delta, 2)$ or (δ, Δ)-linked, showing a contradiction.

Thus we assume that τ is a 2-inducing color of F_1. Based on the coloring of $S(u; s_1, x, y)$ as given in L_1 of Fig. 4, we perform $(1, \tau) - (\tau, \delta)$-swaps at both x and y and let φ_2 be the resulting coloring. Note that either $\varphi_2(s_1u) = \delta$ or $\varphi_2(s_1u) = \tau$. If $\varphi_2(s_1u) = \delta$, then after doing a $(1, \delta)$-swap at both x and y, s_1 and y are $(1, \Delta)$-linked, which gives a contradiction to Lemma 3.1 (b) that s_1 and r are $(1, \Delta)$-linked. Thus $\varphi_2(s_1u) = \tau$. We first do a $(1, \delta)$-swap at both x and y. Then since τ is a 2-inducing color of F_1, $u \in P_y(1, \tau, \varphi_2)$ (since otherwise, after doing a $(1, \tau)$-swap at y, s_1 and y are (τ, Δ)-linked, showing a contradiction to Lemma 3.2 (a)). Thus we do a $(1, \tau)$-swap at both x and y and let φ_3 be the new coloring. Note that δ is still Δ-inducing and τ is 2-inducing with respect to F_1 and φ_3. Thus $\overline{\varphi}_3^{-1}(\delta)$ and $\overline{\varphi}_3^{-1}(\tau)$ are (δ, τ)-linked by

Lemma 3.2 (a). Let φ_4 be obtained from φ_3 by doing a (δ, τ)-swap at y, and let F_1^* be the resulting multifan. Then $K = (r, rs_1, s_1, s_1u, u, uy, y)$ is a Kierstead path with respect to rs_1 and φ_4. Since $d_G(s_1) = \Delta - 1$, applying Lemma 3.3 (b), y and s_1 are $(2, \delta)$-linked. Since δ is still Δ-inducing and τ is 2-inducing with respect to F_1^* and φ_4, we achieve a contradiction to the fact that s_1 and $\overline{\varphi}_4^{-1}(\delta)$ are $(2, \delta)$-linked by Lemma 3.2 (a). Therefore it must be the case that $u \in P_y(1, \delta, \varphi_1)$. $\diamond$

Since $u \in P_y(1, \delta, \varphi_1)$, we perform a $(1, \delta)$-swap at both x and y gives L_2 in Fig. 4. Call the resulting coloring φ_2. Now $K = (r, rs_1, s_1, s_u, u, uy, y)$ is a Kierstead path with respect to rs_1 and φ_2. Since $d_G(s_1) = \Delta - 1$, by Lemma 3.3 (b), y and s_1 are $(2, \delta)$-linked. It deduces that δ must be a 2-inducing color of F_1 with respect to φ_2. Recall that F_1 still has two sequences with respect to φ_2. Let γ be a Δ-inducing color of F_1. Since $\overline{\varphi}_2^{-1}(\delta)$ and $\overline{\varphi}_2^{-1}(\gamma)$ are (δ, γ)-linked by Lemma 3.2 (a), we do a (δ, γ)-swap at y to get φ_3. Still, δ is a 2-inducing color and γ is a Δ-inducing color of the resulting multifan. By Lemma 3.3 (b), s_1 and y are $(2, \gamma)$-linked, showing a contradiction to the fact that s_1 and $\overline{\varphi}_3^{-1}(\gamma)$ are $(2, \gamma)$-linked.

Subcase 2.2.2. λ is the last 2-inducing color of F.

If τ is 2-inducing, then $\tau \prec \lambda$. This gives back to the previous case by exchanging the roles of τ and λ. If τ is Δ-inducing and τ is not the last Δ-inducing color, then by doing $(\Delta, 1) - (1, 2)$-swaps at x and y, a similar proof follows as in the previous case by exchanging the roles of 2 and Δ. Thus τ is the last Δ-inducing color of F.

Let C_u be the cycle in G_Δ containing u. By Theorem 4.1 (i), for every vertex on C_u, its $(\Delta - 1)$-neighborhood is $N_{\Delta-1}(u)$. As $|V(C_u)| \geq 3$, there exist $u^*, u' \in V(C_u) \setminus \{u\}$ such that one of $\varphi(u^*y)$ and $\varphi(u'y)$ is neither τ nor λ. Assume that $\varphi(u^*y) \notin \{\tau, \lambda\}$. Letting u^* play the role of u, we reduce the problem to the previous case, finishing the proof of Theorem 2.5. $\square$

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors are very grateful to the two anonymous referees for their valuable comments. Yan Cao was supported by NSFC grant 12301433; Guantao Chen was supported by NSF grants DMS-1855716 and DMS-2154331; Guangming Jing was supported by NSF grant DMS-2246292; and Songling Shan was supported by NSF grant DMS-2345869.

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