Discrete Mathematics 346 (2023) 113610

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Decomposition of class II graphs into two class I graphs ☆

Yan Cao [a], Guangming Jing [b], Rong Luo [b], Vahan Mkrtchyan [c,*], Cun-Quan Zhang [b], Yue Zhao [d]

[a] School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning, 116024, China
[b] Department of Mathematics, West Virginia University, Morgantown, WV, 26505, USA
[c] Computer Science Department, Boston College, Chestnut Hill, MA, 02467, USA
[d] Department of Mathematics, University of Central Florida, Orlando, FL, 32816, USA

ARTICLE INFO

Article history:
Received 10 November 2022
Received in revised form 27 June 2023
Accepted 8 July 2023
Available online 25 July 2023

Keywords:
Edge coloring
Chromatic index
Partition
k-edge-colorable subgraph
Class I graph

ABSTRACT

Mkrtchyan and Steffen (2012) [3] showed that every class II simple graph can be decomposed into a maximum Δ-edge-colorable subgraph and a matching. They further conjectured that every graph G with chromatic index $\Delta(G)+k$ ($k \geq 1$) can be decomposed into a maximum $\Delta(G)$-edge-colorable subgraph (not necessarily class I) and a k-edge-colorable subgraph. In this paper, we first generalize their result to multigraphs and show that every multigraph G with multiplicity μ can be decomposed into a maximum $\Delta(G)$-edge-colorable subgraph and a subgraph with maximum degree at most μ. Then we prove that every graph G with chromatic index $\Delta(G)+k$ can be decomposed into two class I subgraphs H_1 and H_2 such that $\Delta(H_1) = \Delta(G)$ and $\Delta(H_2) = k$, which is a variation of their conjecture.

1. Introduction

Graphs considered in this paper are finite, undirected and may contain multiple edges, but no loops. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. For a vertex $x \in V$, let $N(x) = \{v \mid xv \in E(G)\}$ and $E(x) = \{e \mid e \text{ is incident with } x\}$. The degree of $x \in V(G)$ is denoted by $d(x) = |E(x)|$. The minimum and maximum degrees of vertices in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is regular if $\delta(G) = \Delta(G)$. For two vertices x and y, let $E(x, y)$ denote the set of edges between x and y and let $\mu_G(x, y) = |E(x, y)|$. $\mu(G) = \max\{\mu_G(x, y)| \ x, y \in V(G)\}$ is called the *multiplicity* of G. If there is no confusion from the context, we denote $\delta(G)$, $\Delta(G)$, $\mu(G)$, $\mu_G(x, y)$ by δ, Δ, μ, and $\mu(x, y)$ respectively.

An *edge coloring* of a graph is a function assigning values (colors) to the edges of the graph in such a way that any two adjacent edges receive different colors. A graph is *k-edge-colorable* if there is an edge coloring of the graph with colors from $C = \{1, \dots, k\}$. The smallest integer k such that G is k-edge-colorable is called the *chromatic index* of G, and is denoted by $\chi'(G)$. Clearly $\chi'(G) \geq \Delta(G)$.

The classical theorems of Shannon and Vizing present upper bounds on $\chi'(G)$ as follows.

☆ The work of the second author has been supported by the National Science Foundation through Grant DMS-2246292. The work of the third author has been supported by the Simons Foundation through the Grant No. 839830. The work of the fourth author has been supported by the National Science Foundation through Grant CCF-2008422.

* Corresponding author.
E-mail addresses: 314169731@qq.com (Y. Cao), Rong.Luo@mail.wvu.edu (R. Luo), vahan.mkrtchyan@bc.edu (V. Mkrtchyan), Cun-quan.Zhang@mail.wvu.edu (C.-Q. Zhang), Yue.Zhao@ucf.edu (Y. Zhao).

https://doi.org/10.1016/j.disc.2023.113610

Theorem 1.1 *(Shannon [5]). For any graph G, $\Delta(G) \le \chi'(G) \le \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$.*

Theorem 1.2 *(Vizing [8]). For any graph G, $\Delta(G) \le \chi'(G) \le \Delta(G) + \mu(G)$.*

Let G be a graph with maximum degree Δ. It is said to be *class I* if $\chi'(G) = \Delta$, otherwise it is *class II*. There are many hard problems related to edge coloring of graphs (see [6]). A subgraph H of G is *maximum Δ-colorable* if it is Δ-edge-colorable and contains as many edges as possible. The maximum Δ-edge-colorable subgraphs have been extensively studied (see [1,4,2,3]). A *decomposition* of a graph G is a set $D = \{H_1, \dots, H_k\}$ of pairwise edge-disjoint subgraphs of G that cover the set of edges of G. Since the decision problem of classifying graphs as class I or class II is NP-complete, and every class II graph contains a subgraph of class I, it is natural to consider how one could decompose a class II graph into subgraphs with certain properties, for example, into subgraphs of class I with degree conditions.

Mkrtchyan and Steffen [3] proved the following result.

Theorem 1.3 *(Mkrtchyan and Steffen [3]). Let G be a simple graph with $\Delta(G) = \Delta$ and $\chi'(G) = \Delta + 1$. Then G can be decomposed into a maximum Δ-edge-colorable subgraph H_1 and a subgraph H_2 with $\chi'(H_2) = \Delta(H_2) = 1$.*

They conjectured that Theorem 1.3 can be generalized to graphs with multiple edges.

Conjecture 1.4 *(Mkrtchyan and Steffen [3]). Let G be a graph with $\Delta(G) = \Delta$ and $\chi'(G) = \Delta + k$ where $k \ge 1$. Then G can be decomposed into a maximum Δ-edge-colorable subgraph H_1 and a subgraph H_2 such that $\chi'(H_2) = k$.*

In this paper, we first generalize Theorem 1.3 as follows.

Theorem 1.5. *Let G be a class II graph with multiplicity μ and maximum degree Δ. Then G can be decomposed into two subgraphs H_1 and H_2 such that H_1 is maximum Δ-edge-colorable and $\Delta(H_2) \le \mu$. Moreover, if $\mu \le 2$ and $\chi'(G) = \Delta + \mu$, then $\chi'(H_2) = \Delta(H_2) = \mu$.*

In Theorem 1.5, if G is simple and class II, then $\mu = 1$ and $\Delta(H_2) = 1 = \mu$. Thus it implies Theorem 1.3.

Note that a graph has chromatic index one if and only if its maximum degree is one, and for simple graphs, it is easy to see that a maximum Δ-edge-colorable subgraph is class I by Vizing's adjacency lemma. However, Vizing's adjacency lemma only works for simple graphs and consequently, as shown in [3], maximum Δ-edge-colorable subgraphs could be class II for multigraphs. Thus in Conjecture 1.4, H_1 or H_2 could be Class II.

Our second main result decomposes an edge colored graph into two Class I subgraphs, which is a variation of Conjecture 1.4.

Theorem 1.6. *Let G be a graph with $\Delta(G) = \Delta$ and $\chi'(G) = \Delta + k$ where $k \ge 1$. Then G can be decomposed into two Class I subgraphs H_1 and H_2 such that $\chi'(H_1) = \Delta(H_1) = \Delta$ and $\chi'(H_2) = \Delta(H_2) = k$.*

The proofs of Theorems 1.5 and 1.6 will be presented in Section 3.

2. Preliminaries and lemmas

In this section, we introduce additional notations and lemmas needed in the proofs of the theorems. Let G be a graph. A *k-vertex*, *k^+-vertex*, or *k^--vertex* is a vertex of degree k, at least k, or at most k, respectively. We denote the set of all *k-vertices*, *k^+-vertices*, or *k^--vertices* in $V(G)$ by $V_k(G)$, $V_{\ge k}(G)$, or $V_{\le k}(G)$, respectively. For integers $r, s, t > 0$, let $T = T(r, s, t)$ be the graph consisting of three vertices x, y, z such that $\mu(x, y) = r$, $\mu(y, z) = s$, and $\mu(x, z) = t$. The graph $S_d = T(\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor, \lfloor \frac{d+1}{2} \rfloor)$ is called a *Shannon graph* of degree d. Vizing [7] proved the following structural result for graphs achieving the upper bound in Theorem 1.1.

Theorem 2.1 *(Vizing [7]). Let G be a graph with $\Delta(G) = \Delta \ge 4$. If $\chi'(G) = \lfloor \frac{3}{2}\Delta \rfloor$, then G contains a Shannon graph S_Δ as a subgraph.*

We denote $\mathcal{C}^k(G)$ to be the set of all k-edge colorings of a graph G. Let $\varphi \in \mathcal{C}^k(G)$ and $C = \{1, 2, \dots, k\}$ be the color set. For a vertex $v \in V$, denote by $\varphi(v) = \{\varphi(e) | e \in E(v)\}$ the set of colors present at v and $\bar{\varphi}(v) = C \setminus \varphi(v)$ the set of colors not assigned to any edge incident with v. A color γ is said to be *missing* at v if $\gamma \in \bar{\varphi}(v)$. For a color α, let $E_\alpha = \{e \in E : \varphi(e) = \alpha\}$. For an edge set $E_0 \subseteq E(G)$, let $\varphi(E_0) = \{c \in C \mid \varphi(e) = c \ for\ some\ e \in E_0\}$. Let us start with the following observation.

Lemma 2.2. *Let G be a graph with $\Delta(G) = \Delta$ and $\chi'(G) = \Delta + k$ where $k \ge 1$. Let x be a vertex in G. If there is a $(\Delta + k)$-edge coloring φ of G and a vertex $v \in V(G)$ such that $\bar{\varphi}(x) \cap \bar{\varphi}(v) = \emptyset$, then G can be decomposed into two Class I subgraphs H_1 and H_2 such that $\Delta(H_1) = d_G(x)$ and $\Delta(H_2) = \Delta + k - d_G(x)$.*

Proof. Assume without loss of generality that $\varphi(x) = \{1, 2, \ldots, d_G(x)\}$. Let

$$H_1 = \bigcup_{i=1}^{d_G(x)} E_i \quad \text{and} \quad H_2 = \bigcup_{i=d_G(x)+1}^{\Delta+k} E_i.$$

Clearly $\chi'(H_1) \le d_G(x)$ and $\chi'(H_2) \le \Delta + k - d_G(x)$. Note $d_{H_1}(x) = d_G(x)$ and $\Delta + k \le \chi'(H_1) + \chi'(H_2)$. Thus $\chi'(H_1) = \Delta(H_1) = d_{H_1}(x) = d_G(x)$. Since $\bar{\varphi}(x) \cap \bar{\varphi}(v) = \emptyset$, we have $d_{H_2}(v) = \Delta + k - d_G(x)$. Therefore $\chi'(H_2) = \Delta(H_2) = d_{H_2}(v) = \Delta + k - d_G(x)$, as desired. □

Corollary 2.3. *If in Lemma 2.2, we have $d_G(x) = \Delta$, then $\Delta(H_1) = \Delta$ and $\Delta(H_2) = k$.*

Let G be a graph with an edge $e \in E(x, y)$, and let φ be an edge coloring of $G - e$. A sequence $F = (e_1, y_1, \ldots, e_p, y_p)$ consisting of vertices and distinct edges is called a *multi-fan* at x with respect to e and φ if $y_1 = y$, $e_1 = e$, and for each $2 \le i \le p$, we have $e_i \in E(x, y_i)$ and $\varphi(e_i) \in \overline{\varphi}(y_j)$ for some $1 \le j \le i-1$. A *linear sequence* from y_1 to y_s, denoted by $L = (e_1, y_1, e_2, y_2, \ldots, e_s, y_s)$, is a sequence consisting of distinct vertices and distinct edges such that $e_i \in E(x, y_i)$ for $1 \le i \le s$ and $\varphi(e_i) \in \overline{\varphi}(y_{i-1})$ for each $2 \le i \le s$. Clearly for any $y_i \in V(F)$, the multi-fan F contains a linear sequence from y to y_i, as we could just pick the shortest multi-fan contained in F ending with y_i.

A *shifting* from y_1 to y_s in a linear sequence $L = (e_1, y_1, \ldots, e_s, y_s)$ is an operation that obtains a new edge coloring of $G - e_s$ from φ by recoloring the edge e_t with the color of e_{t+1} under φ sequentially for each $1 \le t \le s-1$ and uncoloring the edge e_s.

An edge $e \in E(x, y)$ is called a *critical* edge of G if $\chi'(G - e) < \chi'(G)$. Clearly if e is critical, then $\chi'(G - e) = \chi'(G) - 1$. We first have the following result regarding multi-fans.

Lemma 2.4. *Let G be a graph with $\Delta(G) = \Delta$ and multiplicity μ. Suppose that $\chi'(G) \ge \Delta + 1$ and $e \in E(x, y)$ is a critical edge of G. Let $F = (e_1, y_1, \ldots, e_p, y_p)$ be a maximal multi-fan at x with respect to e and $\varphi \in \mathcal{C}^{\chi'(G)-1}(G - e)$. Then we have the following:*
(a) (Stiebitz et al. [6]) $|V(F)| \ge 2$ and $\sum_{z \in V(F)} (d_G(z) + \mu_F(x, z) - (\chi'(G) - 1)) = 2$, where $\mu_F(x, z)$ is the number of edges between x and z in F.
(b) If $d_G(y) \le \chi'(G) - \mu$, then F contains a vertex $z \ne y$ with $d_G(z) \ge \chi'(G) - \mu$.

Proof. (a) is Theorem 2.1 in book [6] due to Stiebitz et al.

(b) Since $\mu \ge \mu_F(x, z)$, by (a) we have

$$2 = \sum_{z \in V(F)} (d_G(z) + \mu_F(x, z) - (\chi'(G) - 1)) \le \sum_{z \in V(F)} (d_G(z) + \mu - (\chi'(G) - 1)).$$

Thus F contains either a vertex z with $d_G(z) + \mu - (\chi'(G) - 1) > 1$, or two vertices z_i with $i = 1, 2$ such that $d_G(z_i) + \mu - (\chi'(G) - 1) = 1$. In the former case we have $d_G(z) > (\chi'(G) - 1) - \mu + 1 = \chi'(G) - \mu$, so $z \ne y$ (because $d_G(y) \le \chi'(G) - \mu$). In the latter case we have $d_G(z_i) \ge (\chi'(G) - 1) - \mu + 1 = \chi'(G) - \mu$ and one of z_1 or z_2 is not y, as desired. □

3. Decomposition of class II graphs

In this section, we prove Theorems 1.5 and 1.6. We first would like to point out that the proof would be much easier if there is no restriction on the maximum degrees of H_1 and H_2.

Proposition 3.1. *Every class II graph can be decomposed into two class I graphs.*

Proof. Let G be a graph with maximum degree Δ and $\chi'(G) = \Delta + k$ where $k \ge 1$. Let $\varphi \in \mathcal{C}^{\Delta+k}(G)$ such that $|E_1|$ is minimum, where E_1 is the set of edges in G colored by the color 1. Let $e \in E(x, y)$ and $e \in E_1$.

By the minimality of $|E_1|$, we have $\bar{\varphi}(x) \cap \bar{\varphi}(y) = \emptyset$. Otherwise one may recolor the edge e with a color $i \in \bar{\varphi}(x) \cap \bar{\varphi}(y)$, which contradicts the minimality of $|E_1|$. By Lemma 2.2, G can be decomposed into two class I subgraphs H_1 and H_2 such that $\Delta(H_1) = d_G(x)$ and $\Delta(H_2) = \Delta + k - d_G(x)$. □

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. We first prove the general case. Take a maximum Δ-edge-colorable subgraph H_1 of G minimizing

$$t(H_2) = \sum_{d_{H_2}(v) > \mu} d_{H_2}(v),$$

where $H_2 = G - E(H_1)$. It is sufficient to show $t(H_2) = 0$. Suppose to the contrary that $t(H_2) \ge 1$. Let y be a vertex with $d_{H_2}(y) > \mu$ and $e \in E(x, y)$ be an edge in H_2.

Since H_1 is maximum Δ-edge-colorable, the edge e is critical in $G' = H_1 + e$ and $\chi'(G') = \Delta + 1$. Thus $d_{H_1}(x) \geq 1$. Let $\varphi \in \mathcal{C}^{\Delta}(H_1)$ and F be a maximal multi-fan at x with respect to e and φ. Since $d_{H_2}(y) > \mu$ and $d_G(y) \leq \Delta$, we have

$$d_{G'}(y) = d_{H_1}(y) + 1 = d_G(y) - d_{H_2}(y) + 1 \leq \Delta - (\mu + 1) + 1 < \Delta - \mu + 1 \leq \chi'(G') - \mu(G').$$

By Lemma 2.4 (b) with $\chi'(G') = \Delta + 1 \geq \Delta(G') + 1$, there is a vertex $z \in V(F) \setminus \{y\}$ such that $d_{H_1}(z) \geq \chi'(G') - \mu(G') \geq \Delta + 1 - \mu$. This implies $d_{H_2}(z) \leq \mu - 1$.

By the definition of a multi-fan, F has a linear sequence L from y to z with the last edge $f \in E(x, z)$. By shifting colors along L, we see that $H_1' = G' - f = H_1 + e - f$ is Δ-edge-colorable and has the same number of edges as H_1. Thus H_1' is also maximum Δ-edge-colorable. Let $H_2' = G - E(H_1')$. Then $d_{H_2}(w) = d_{H_2'}(w)$ for each vertex $w \neq y, z$, $d_{H_2'}(y) = d_{H_2}(y) - 1 \geq \mu$ and $d_{H_2'}(z) = d_{H_2}(z) + 1$. Since $d_{H_2}(z) \leq \mu - 1$, we have $d_{H_2'}(z) \leq \mu$. Therefore

$$t(H_2') = \begin{cases} t(H_2) - 1 & \text{if } d_{H_2}(y) > \mu + 1, \\ t(H_2) - d_{H_2}(y) & \text{if } d_{H_2}(y) = \mu + 1. \end{cases}$$

In either case we have $t(H_2') < t(H_2)$, which contradicts the choice of H_1. This proves the general case.

Now we prove the "moreover" part. Assume that $\mu \leq 2$ and $\chi'(G) = \Delta + \mu$. By the first part, let H_1 be a maximum Δ-edge-colorable subgraph, such that $\Delta(H_2) = \Delta(G - H_1) \leq \mu$ where $H_2 = G - E(H_1)$, and the number of odd cycles in H_2 is smallest. We will show $\chi'(H_2) = \Delta(H_2) = \mu$. Note $1 \leq \Delta(H_2) \leq \mu \leq 2$.

If $\Delta(H_2) = 1$, then $\mu = 1$ and $\chi'(H_2) = \Delta(H_2) = 1$.

Now assume $\Delta(H_2) = 2$. Then $\mu = 2$ and H_2 consists of vertex disjoint cycles and paths. To show $\chi'(H_2) = 2$, it is sufficient to show that H_2 contains no odd cycles. Suppose by contradiction that H_2 contains an odd cycle and $e \in E(x, y)$ is an edge in $E(H_2)$ lying in an odd cycle of H_2.

Similar to the argument in the first part, e is critical in $G' = H_1 + e$ and $\chi'(G') = \Delta + 1$. Let $\varphi \in \mathcal{C}^{\Delta}(H_1)$ and F be a maximal multi-fan at x with respect to e and φ. Since $d_{H_2}(y) = 2$ and $d_G(y) \leq \Delta$, we have $d_{G'}(y) \leq \Delta - 1 = \Delta - \mu + 1 \leq \chi'(G') - \mu(G')$. Then by Lemma 2.4 with $\chi'(G) = \Delta + 1 \geq \Delta(G') + 1$, F has a vertex $z \neq y$ such that $d_{H_1}(z) \geq \chi'(G) - \mu(G') \geq \Delta - 1$. So $d_{H_2}(z) \leq 1$. This implies that z is an endvertex of a path or a 0-vertex in H_2. Again F has a linear sequence L from y to z with last edge $f \in E(x, z)$, and by shifting colors along L, we see that $H_1' = G' - f = H_1 + e - f$ is maximum Δ-edge-colorable. We have that $\Delta(H_2') = 2$, where $H_2' = G - E(H_1') = H_2 - e + f$, and the number of odd cycles in H_2' is one less than in H_2. This contradicts the choice of H_1 and H_2 and thus proves the "moreover" part. □

For the proof of Theorem 1.6, we introduce additional notations. Let $\varphi \in \mathcal{C}^k(G)$. For any two distinct colors α and β, let $G_\varphi(\alpha, \beta)$ be the subgraph of G induced by $E_\alpha \cup E_\beta$. The components of $G_\varphi(\alpha, \beta)$ are called (α, β)-*chains*. Clearly, each (α, β)-chain is either a path or a cycle of edges alternately colored with α and β. For each (α, β)-chain P, let φ/P denote the k-edge-coloring of G obtained from φ by exchanging colors α and β on P.

For any $v \in V(G_\varphi(\alpha, \beta))$, let $P_v(\alpha, \beta, \varphi)$ denote the unique (α, β)-chain containing v. Note that, for any two vertices $u, v \in V(G_\varphi(\alpha, \beta))$, either $P_u(\alpha, \beta, \varphi) = P_v(\alpha, \beta, \varphi)$ or $P_u(\alpha, \beta, \varphi)$ is vertex-disjoint from $P_v(\alpha, \beta, \varphi)$. This fact will be used very often without mentioning. We are ready to prove Theorem 1.6.

Proof of Theorem 1.6. We prove the theorem by contradiction. Let G be a counterexample minimizing $\Delta = \Delta(G)$. Clearly, $\Delta \geq 3$.

Let $x \in V(G)$ be a Δ-vertex. We have the following claim.

Claim 1. *For any $y \in V_{\geq \Delta-1}(G) \setminus \{x\}$, we have $|E(x, y)| \geq d(y) - k + 1$. Consequently $V_{\geq \Delta-1}(G) \subseteq N(x) \cup \{x\}$.*

Proof. Suppose to the contrary that $|E(x, y)| < d(y) - k + 1$. Let $\varphi \in C^{\Delta+k}(G)$ be a coloring such that $|\bar{\varphi}(x) \cap \bar{\varphi}(y)|$ is minimum.

Define:

$$\begin{aligned} C_{00} &= \bar{\varphi}(x) \cap \bar{\varphi}(y) = \overline{\varphi(x) \cup \varphi(y)}, \\ C_{01} &= \bar{\varphi}(x) \setminus \bar{\varphi}(y) = \varphi(y) \setminus \varphi(x), \\ C_{10} &= \bar{\varphi}(y) \setminus \bar{\varphi}(x) = \varphi(x) \setminus \varphi(y), \\ \text{and } \ C_{11} &= \varphi(x) \cap \varphi(y). \end{aligned}$$

It is easy to see that $C_{00}, C_{01}, C_{10}, C_{11}$ are pairwise disjoint and form a partition of the color set $C = \{1, 2, \ldots, \Delta + k\}$. By Lemma 2.2, we have $C_{00} = \bar{\varphi}(x) \cap \bar{\varphi}(y) \neq \emptyset$.

Subclaim 1.1. *$P_x(\alpha, \eta, \varphi) = P_y(\alpha, \eta, \varphi)$ for any two colors $\alpha \in C_{00}$ and $\eta \in C_{11}$.*

Proof. Otherwise let $\varphi' = \varphi/P_y(\alpha, \eta, \varphi)$. Then $\bar{\varphi}'(x) \cap \bar{\varphi}'(y) = (\bar{\varphi}(x) \cap \bar{\varphi}(y)) \setminus \{\alpha\}$, a contradiction to the minimality of $|\bar{\varphi}(x) \cap \bar{\varphi}(y)|$. □

Subclaim 1.2. $\varphi(E(x) \setminus E(x,y)) \cap C_{11} \neq \emptyset$.

Proof. Suppose to the contrary $\varphi(E(x) \setminus E(x,y)) \cap C_{11} = \emptyset$. Since $\varphi(E(x,y)) \subseteq C_{11}$, we have $C_{11} = \varphi(E(x,y))$. This implies that $C_{01} = \varphi(E(y) \setminus E(x,y))$ and $C_{10} = \varphi(E(x) \setminus E(x,y))$. Thus $|C_{11}| = |E(x,y)|$, $|C_{01}| = d(y) - |E(x,y)|$, and $|C_{10}| = d(x) - |E(x,y)| = \Delta - |E(x,y)|$. Therefore

$$\begin{aligned}\Delta + k = |C| &= |C_{00}| + |C_{01}| + |C_{10}| + |C_{11}| \\ &= |C_{00}| + \Delta + d(y) - |E(x,y)| \\ &> 1 + \Delta + d(y) - (d(y) - k + 1) \\ &= \Delta + k.\end{aligned}$$

This contradiction proves the subclaim. □

By Subclaim 1.2, let $z_1 \in N(x) \setminus \{y\}$ and $e_1 \in E(x, z_1)$ such that $\varphi(e_1) = \eta \in C_{11}$.

Subclaim 1.3. $\bar{\varphi}(z_1) \cap (C_{00} \cup C_{11}) = \emptyset$. *Therefore* $\bar{\varphi}(z_1) \subseteq C_{01} \cup C_{10}$.

Proof. Suppose to the contrary that there exists a color $\alpha \in \bar{\varphi}(z_1) \cap (C_{00} \cup C_{11})$.

If $\alpha \in C_{00}$, then $P_x(\alpha, \eta, \varphi) = P_{z_1}(\alpha, \eta, \varphi) = xz_1$, a contradiction to Subclaim 1.1.

Assume $\alpha \in C_{11}$. Let $\beta \in C_{00}$. Then $\beta \in \varphi(z_1)$ and $P_{z_1}(\beta, \alpha, \varphi)$ does not contain x, y by Subclaim 1.1. Let $\varphi' = \varphi / P_{z_1}(\beta, \alpha, \varphi)$. Then $\beta \in \bar{\varphi}'(z_1)$ and $\beta \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y)$. Note $\bar{\varphi}'(x) \cap \bar{\varphi}'(y) = \bar{\varphi}(x) \cap \bar{\varphi}(y)$. Thus we are back to the previous case. □

By Subclaim 1.3, $\bar{\varphi}(z_1) \subseteq C_{01} \cup C_{10}$. Since $|\bar{\varphi}(z_1)| = \Delta + k - d(z_1) \geq k$ and $|C_{01}| = |\bar{\varphi}(x)| - |C_{00}| = k - |C_{00}| < k$, we have $\bar{\varphi}(z_1) \cap C_{10} \neq \emptyset$.

Let γ_1 be a color in $\bar{\varphi}(z_1) \cap C_{10}$. Then there are $z_2 \in N(x)$ and $e_2 \in E(x, z_2)$ such that $\varphi(xz_2) = \gamma_1$. Since $\gamma_1 \in \bar{\varphi}(z_1) \cap \bar{\varphi}(y)$, we have $z_2 \neq z_1$ and $z_2 \neq y$.

Subclaim 1.4. $\bar{\varphi}(z_2) \subseteq C_{01} \cup C_{10} = (\bar{\varphi}(x) \cup \bar{\varphi}(y)) \setminus (\bar{\varphi}(x) \cap \bar{\varphi}(y))$.

Proof. Suppose to the contrary that there is a color $\alpha \in \bar{\varphi}(z_2)$ with $\alpha \notin C_{01} \cup C_{10}$. Then $\alpha \in C_{00} \cup C_{11}$. Thus by Subclaim 1.3, $\alpha \in \varphi(z_1)$.

If $\alpha \in C_{00}$, then $P_x(\alpha, \gamma_1, \varphi) = P_{z_2}(\alpha, \gamma_1, \varphi) = xz_2$. Thus $P_{z_1}(\alpha, \gamma_1, \varphi)$ does not contain x. Note that both α and γ_1 are missing at y. Let $\varphi' = \varphi / P_{z_1}(\alpha, \gamma_1, \varphi)$. Then $\alpha \in \bar{\varphi}'(z_1)$ and $\alpha \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y)$, a contradiction to Subclaim 1.3.

Now assume $\alpha \in C_{11}$. Let β be a color in C_{00}. Then $P_{z_2}(\beta, \alpha, \varphi)$ does not contain x or y by Subclaim 1.1. By the above argument, $\beta \in \varphi(z_2)$. Let $\varphi' = \varphi / P_{z_2}(\beta, \alpha, \varphi)$. Then $\beta \in \bar{\varphi}'(z_2)$ and $\beta \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y) = C_{00}$, where we have reached the case above with β replacing α. □

Note that $|\bar{\varphi}(z_1)| + |\bar{\varphi}(z_2)| = 2(\Delta + k) - d(z_1) - d(z_2) \geq 2k$ and $|C_{01} \cup C_{10}| = |\bar{\varphi}(x)| + |\bar{\varphi}(y)| - 2|C_{00}| \leq k + (k+1) - 2 = 2k - 1$. By Subclaims 1.3 and 1.4, $\bar{\varphi}(z_1) \cup \bar{\varphi}(z_2) \subseteq C_{01} \cup C_{10}$. Thus $\bar{\varphi}(z_1) \cap \bar{\varphi}(z_2) \neq \emptyset$. Let β be a color in $\bar{\varphi}(z_1) \cap \bar{\varphi}(z_2)$. Then $\beta \in C_{01} \cup C_{10}$. Let α be a color in C_{00}. By Subclaims 1.3 and 1.4, $\alpha \in \varphi(z_1)$ and $\alpha \in \varphi(z_2)$.

If $\beta \in C_{01}$, then $\beta, \alpha \in \bar{\varphi}(x)$ and at least one of $P_{z_1}(\beta, \alpha, \varphi), P_{z_2}(\beta, \alpha, \varphi)$ does not contain y.

If $\beta \in C_{10}$, then $\beta, \alpha \in \bar{\varphi}(y)$ and at least one of $P_{z_1}(\beta, \alpha, \varphi)$ and $P_{z_2}(\beta, \alpha, \varphi)$ does not contain x.

In either case, at least one of $P_{z_1}(\beta, \alpha, \varphi), P_{z_2}(\beta, \alpha, \varphi)$ does not contain x or y.

If $P_{z_1}(\beta, \alpha, \varphi)$ does not contain x or y, let $\varphi' = \varphi / P_{z_1}(\beta, \alpha, \varphi)$. Then $\alpha \in \bar{\varphi}'(z_1)$ and $\alpha \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y)$, a contradiction to Subclaim 1.3.

If $P_{z_2}(\beta, \alpha, \varphi)$ does not contain x or y, let $\varphi' = \varphi / P_{z_2}(\beta, \alpha, \varphi)$. Then $\alpha \in \bar{\varphi}'(z_2)$ and $\alpha \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y)$, a contradiction to Subclaim 1.4. This completes the proof of Claim 1. □

By Claim 1, every $(\Delta - 1)^+$-vertex is either x or is adjacent to x. The next claim shows that there are at most two $(\Delta - 1)^+$-vertices in G.

Claim 2. *There is at most one* $(\Delta - 1)^+$*-vertex in* $N(x)$.

Proof. Suppose to the contrary that there exist two $(\Delta - 1)^+$-vertices $y, z \in N(x)$. By Claim 1,

$$\Delta = d(x) \geq |E(x,y)| + |E(x,z)| \geq d(y) - k + 1 + d(z) - k + 1 \geq 2\Delta - 2k.$$

Thus $k \geq \frac{\Delta}{2}$. On the other hand, we know $k \leq \frac{\Delta}{2}$ by Theorem 1.1. This implies that $k = \frac{\Delta}{2}$, and then $\chi'(G) = \frac{3}{2}\Delta$, Δ is even, $d(y) = d(z) = \Delta - 1$, and $N(x) = \{y, z\}$. Thus by Claim 1, we have $V_{\geq \Delta-1}(G) = \{x, y, z\}$. However, Theorem 2.1 implies that G must contain a Shannon graph of degree Δ as a subgraph which contains at least three Δ-vertices, a contradiction. □

Let $y \in N(x)$ such that $d_G(y) = \max\{d_G(z)|z \in N(x)\}$. Define $e \in E(x, y)$. Then, each vertex in $V(G) - \{x, y\}$ has degree at most $\Delta - 2$. Let $\varphi \in \mathcal{C}^{\Delta+k}(G)$ and $\alpha = \varphi(e)$. Let $G' = G - E_\alpha$. Then $\Delta(G') = \Delta - 1$ and $\chi'(G') = \Delta + k - 1$. Thus by the minimality of Δ, $G' = G - E_\alpha$ can be decomposed into two class I subgraphs H'_1 and H'_2 such that $\Delta(H'_1) = \Delta - 1$ and $\Delta(H'_2) = k$. Let $H_1 = H'_1 + E_\alpha$ and $H_2 = H'_2$. Since $V_{\geq \Delta-1}(G) \subseteq \{x, y\}$, we have $\Delta(H_1) = \Delta(H'_1) + 1 = \Delta$. Thus H_1 and H_2 form a desired decomposition. This completes the proof of Theorem 1.6. □

4. Concluding remarks

Theorem 1.6 states that if we have a graph G of chromatic index $\chi'(G) = \Delta + k$, then G can be decomposed into two class I subgraphs H_1 and H_2, such that $\Delta(H_1) = \Delta$ and $\Delta(H_2) = k$. We believe that it can be generalized as follows.

Conjecture 4.1. *Let G be a graph with $\chi'(G) = \Delta + k$ and p, q be two positive integers such that $p + q = \Delta + k$ with $p, q \leq \Delta$. Then G can be decomposed into two class I subgraphs H_1 and H_2 such that $\Delta(H_1) = p$ and $\Delta(H_2) = q$.*

Regarding maximum Δ-edge-colorable subgraphs, Mkrtchyan and Steffen [3] proved the following.

Theorem 4.2 *(Mkrtchyan and Steffen [3]). Let G be a bridgeless cubic graph and M be a perfect matching of G. Then there is a maximum 3-edge-colorable subgraph H such that $M \cup E(H) = E(G)$.*

We believe that Theorem 4.2 can be generalized as follows:

Conjecture 4.3. *Let G be a simple graph. Then for each maximum matching M of G, there is a maximum Δ-colorable subgraph H of G with $M \cup E(H) = E(G)$.*

Conjecture 4.4. *Let G be an r-regular graph with $\chi'(G) = \Delta + 1$. Then for each maximum matching M of G, there is a maximum Δ-colorable subgraph H of G with $M \cup E(H) = E(G)$.*

Declaration of competing interest

The authors declare that they have no known competing financial interests, personal relationships that could have influenced the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] M. Albertson, R. Haas, Parsimonious edge coloring, Discrete Math. 148 (1996) 1–7.
[2] V. Mkrtchyan, S. Petrosyan, G. Vardanyan, On disjoint matchings in cubic graphs, Discrete Math. 310 (2010) 1588–1613.
[3] V.V. Mkrtchyan, E. Steffen, Maximum Δ-edge-colorable subgraphs of class II graphs, J. Graph Theory 70 (4) (2012) 473–482.
[4] R. Rizzi, Approximating the maximum 3-edge-colorable subgraph problem, Discrete Math. 309 (12) (2009) 4166–4170.
[5] C.E. Shannon, A theorem on coloring the lines of a network, J. Math. Phys. 28 (1949) 148–151.
[6] M. Stiebitz, D. Scheide, B. Toft, L.M. Favrholdt, Graph Edge Coloring, John Wiley and Sons, 2012.
[7] V.G. Vizing, The chromatic class of a multigraph, Kibernetika (Kiev) 3 (1965) 29–39 (in Russian), English translation in: Cybern. Syst. Anal. 1 (1965) 32–41.
[8] V. Vizing, On an estimate of the chromatic class of a p-graph, Diskretn. Anal. 3 (1964) 25–30.