On Gupta's Co-density Conjecture

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Abstract

Let $G = (V, E)$ be a multigraph. The *cover index* $\xi(G)$ of G is the greatest integer k for which there is a coloring of E with k colors such that each vertex of G is incident with at least one edge of each color. Let $\delta(G)$ be the minimum degree of G and let $\Phi(G)$ be the *co-density* of G, defined by

$$\Phi(G) = \min\left\{\frac{2|E^+(U)|}{|U|+1} : \ U \subseteq V, \ |U| \geq 3 \ \text{and} \ \text{odd}\right\},$$

where $E^+(U)$ is the set of all edges of G with at least one end in U. It is easy to see that $\xi(G) \leq \min\{\delta(G), \lfloor \Phi(G) \rfloor\}$. In 1978 Gupta proposed the following co-density conjecture: Every multigraph G satisfies $\xi(G) \geq \min\{\delta(G)-1, \ \lfloor \Phi(G) \rfloor\}$, which is the dual version of the Goldberg-Seymour conjecture on edge-colorings of multigraphs. In this note we prove that $\xi(G) \geq \min\{\delta(G)-1, \ \lfloor \Phi(G) \rfloor\}$ if $\Phi(G)$ is not integral and $\xi(G) \geq \min\{\delta(G)-2, \ \lfloor \Phi(G) \rfloor - 1\}$ otherwise. We also show that this co-density conjecture implies another conjecture concerning cover index made by Gupta in 1967.

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1 Introduction

In this note we consider multigraphs, which may have parallel edges but contain no loops. Let $G = (V, E)$ be a multigraph. The *chromatic index* $\chi'(G)$ of G is the least integer k for which there is a coloring of E with k colors such that each vertex of G is incident with at most one edge of each color. Let $\Delta(G)$ be the maximum degree of G and let $\Gamma(G)$ be the *density* of G, defined by

$$\Gamma(G) = \max\left\{\frac{2|E(U)|}{|U|-1} :\ U \subseteq V,\ |U| \geq 3 \text{ and odd}\right\},$$

where $E(U)$ is the set of all edges of G with both ends in U. Clearly, $\chi'(G) \geq \max\{\Delta(G),\, \Gamma(G)\}$; this lower bound, as shown by Seymour [13] using Edmonds' matching polytope theorem [5], is precisely the *fractional chromatic index* of G, which is the optimal value of the *fractional edge-coloring problem*:

$$\begin{array}{ll} \text{Minimize} & \mathbf{1}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \mathbf{1} \\ & \quad \boldsymbol{x} \geq \mathbf{0}, \end{array}$$

where A is the edge-matching incidence matrix of G. In the early 1970s Goldberg [6] and Seymour [13] independently made the following conjecture.

Conjecture 1.1. *Every multigraph G satisfies $\chi'(G) \leq \max\{\Delta(G) + 1,\ \lceil\Gamma(G)\rceil\}$.*

Over the past five decades this conjecture has been a subject of extensive research, and has stimulated an important body of work, with contributions from many researchers; see McDonald [10] for a survey on this conjecture and Stiebitz *et al.* [14] for a comprehensive account of edge-colorings. In [4] three of the authors, Chen, Jing, and Zang, have announced a complete proof of Conjecture 1.1.

The present note is devoted to the study of the dual version of the classical edge-coloring problem (ECP), which asks for a coloring of the edges of G using the maximum number of colors in such a way that at each vertex all colors occur. It is easy to see that each color class induces an edge cover of G. (Recall that an *edge cover* is a subset F of E such that each vertex of G is incident to at least one edge in F.) So this problem is actually the *edge cover packing problem* (ECPP). Let $\xi(G)$ denote the optimal value of ECPP, which we call the *cover index* of G. As it is NP-hard [9] in general to determine the chromatic index $\chi'(G)$ of a simple cubic graph G, determining the cover index $\xi(G)$ is also NP-hard.

Let $\delta(G)$ be the minimum degree of G, let $E^+(U)$ be the set of all edges of G with at least one end in U for each $U \subseteq V$, and let $\Phi(G)$ be the *co-density* of G, defined by

$$\Phi(G) = \min\left\{\frac{2|E^+(U)|}{|U|+1} :\ U \subseteq V,\ |U| \geq 3 \text{ and odd}\right\}.$$

Obviously, $\xi(G) \leq \delta(G)$. Since each edge cover contains at least $(|U| + 1)/2$ edges in $E^+(U)$ for any $U \subseteq V$ with $|U| \geq 3$ and odd, $\Phi(G)$ provides another upper bound for $\xi(G)$. So

$\xi(G) \leq \min\{\delta(G), \Phi(G)\}$. Based on a polyhedral description of edge covers (see Theorem 27.3 in Schrijver [12]), Zhao, Chen, and Sang [15] observed that the parameter $\min\{\delta(G), \, \Phi(G)\}$ is exactly the *fractional cover index* of G, the optimal value of the *fractional edge cover packing problem* (FECPP):

$$\begin{array}{ll} \text{Maximize} & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} & B\boldsymbol{x} = \mathbf{1} \\ & \ \ \boldsymbol{x} \geq \mathbf{0}, \end{array}$$

where B is the edge-edge cover incidence matrix of G. They [15] also devised a combinatorial polynomial-time algorithm for finding the co-density $\Phi(G)$ of any multigraph G.

In 1978 Gupta [8] proposed the following co-density conjecture, which is the counterpart of Conjecture 1.1 on ECPP.

Conjecture 1.2. *Every multigraph G satisfies $\xi(G) \geq \min\{\delta(G) - 1, \, \lfloor \Phi(G) \rfloor\}$.*

This conjecture was listed among the Twenty Pretty Edge Coloring Conjectures by Stiebitz *et al.* [14]. Its validity would imply that, first, there are only two possible values for the cover index $\xi(G)$ of a multigraph G: $\min\{\delta(G) - 1, \, \lfloor \Phi(G) \rfloor\}$ and $\min\{\delta(G), \, \lfloor \Phi(G) \rfloor\}$; second, any multigraph has a cover index within one of its fractional cover index, so FECPP also has a fascinating integer rounding property (see Schrijver [11, 12]); third, even if $P \neq NP$, the NP-hardness of ECPP does not preclude the possibility of designing an efficient algorithm for finding at least $\min\{\delta(G) - 1, \, \lfloor \Phi(G) \rfloor\}$ disjoint edge covers in any multigraph G.

To our knowledge, the bound $\xi(G) \geq \min\{\lfloor \frac{7\delta(G)+1}{8} \rfloor, \lfloor \Phi(G) \rfloor\}$ established by Gupta [8] in 1978 remains to be the best approximate version of Conjecture 1.2.

As is well known, the inequality $\chi'(G) \leq \Delta(G) + \mu(G)$ holds for any multigraph G, where $\mu(G)$ is the maximum multiplicity of an edge in G. This result has been successfully dualized by Gupta [7] to packing edge covers: $\xi(G) \geq \delta(G) - \mu(G)$. It is worthwhile pointing out that this dual version follows from Conjecture 1.2 as a corollary, because $\Phi(G) \geq \delta(G) - \mu(G)$. To see this, let U be a subset of V with $|U| \geq 3$ and odd, let $F(U)$ be the set of all edges of G with precisely one end in U, and let $G[U]$ be the subgraph of G induced by U. Since each vertex in U is adjacent to at most $(|U|-1)\mu(G)$ edges in $G[U]$ and at most $|F(U)|$ edges outside $G[U]$, we have $\delta(G) \leq (|U|-1)\mu(G)+|F(U)|$, which implies that $\delta(G)|U|+|F(U)| \geq (\delta(G)-\mu(G))(|U|+1)$. As $2|E^+(U)| = 2|E(U)|+2|F(U)| \geq \delta(G)|U|+|F(U)|$, we obtain $2|E^+(U)| \geq (\delta(G)-\mu(G))(|U|+1)$ and hence $\Phi(G) \geq \delta(G)-\mu(G)$, as desired. Various attempts have been made to strengthen and generalize this dual result; see, for instance, Andersen [1]. When restricted to a simple graph, it can be found in both Bondy and Murty [2] (see Exercise 6.2.8) and Bondy and Murty [3] (see Exercise 17.2.14).

Gupta [7] demonstrated that the lower bound $\delta(G) - \mu(G)$ for $\xi(G)$ is sharp when $\mu(G) \geq 1$ and $\delta(G) = 2p\mu(G) - q$, where p and q are two integers satisfying $q \geq 0$ and $p > \mu(G) + \lfloor (q-1)/2 \rfloor$. This led Gupta [7] to suggest the following conjecture, which aims to give a complete characterization of all values of $\delta(G)$ and $\mu(G)$ for which no multigraph G with $\xi(G) = \delta(G) - \mu(G)$ exists.

Conjecture 1.3. *Let G be a multigraph such that $\delta(G)$ cannot be expressed in the form $2p\mu(G) - q$, for any two integers p and q satisfying $q \geq 0$ and $p > \mu(G) + \lfloor (q-1)/2 \rfloor$. Then $\xi(G) \geq \delta(G) - \mu(G) + 1$.*

As edge covers are more difficult to manipulate than matchings, it is no surprise that a direct proof of conjecture 1.2 would be more complicated and sophisticated than that of Conjecture 1.1 (see [4], which is under review). One purpose of this note is to establish a slightly weaker version of conjecture 1.2 by using Conjecture 1.1.

Theorem 1.1. *(Assuming Conjecture 1.1) Let G be a multigraph. Then $\xi(G) \geq \min\{\delta(G) - 1, \lfloor \Phi(G) \rfloor\}$ if $\Phi(G)$ is not integral and $\xi(G) \geq \min\{\delta(G) - 2, \lfloor \Phi(G) \rfloor - 1\}$ otherwise.*

Remark. Suppose $\Phi(G) < \delta(G)$. By this theorem, we obtain $\xi(G) = \lfloor \Phi(G) \rfloor$ if $\Phi(G)$ is not integral and $\Phi(G) - 1 \leq \xi(G) \leq \Phi(G)$ otherwise, because $\xi(G) \leq \min\{\delta(G), \lfloor \Phi(G) \rfloor\}$.

In this note we also show that Conjecture 1.3 is contained in Conjecture 1.2 as a special case.

Theorem 1.2. *Conjecture 1.2 implies Conjecture 1.3.*

Throughout this note we shall repeatedly use the following terminology and notation. Let $G = (V, E)$ be a multigraph. A subset U of V is called an *odd set* if $|U|$ is odd and $|U| \geq 3$. For each $v \in V$, let $d_G(v)$ be the degree of v in G. For each $U \subseteq V$, let $E_G(U)$ be the set of all edges of G with both ends in U, let $E_G^+(U)$ be the set of all edges of G with at least one end in U, and let $F_G(U)$ be the set of all edges of G with exactly one end in U. For any two subsets X and Y of V, let $E_G(X, Y)$ be the set of all edges of G with one end in X and the other end in Y. We write $E_G(x, y)$ for $E_G(X, Y)$ if $X = \{x\}$ and $Y = \{y\}$. We shall drop the subscript G if there is no danger of confusion.

2 Approximate Version

We present a proof of Theorem 1.1 in this section. Let $G = (V, E)$ be a multigraph and let $Z \subseteq V$. A set $C \subseteq E$ is called a *Z-cover* if every vertex of Z is incident with at least one edge of C. Note that if $Z = V$, then Z-covers are precisely edge covers of G. Let $e \in E(x, y)$ and let G' be obtained from G by adding a new vertex x' and making e incident with x' instead of x (yet still incident with y); we say that G' arises from G by *splitting off* e from x. To prove the theorem, we shall actually establish the following variant.

Theorem 2.1. *Let $G = (V, E)$ be a multigraph, let $Z \subseteq V$, let k be a positive integer, and let ϵ be 0 or 1. If $d(z) \geq k + 1$ for all $z \in Z$ and $|E^+(U)| \geq \frac{|U|+1}{2}k + \epsilon$ for all odd sets $U \subseteq Z$, then G contains $k - 1 + \epsilon$ disjoint Z-covers.*

To see that Theorem 1.1 follows from Theorem 2.1, set $Z = V$, $k = \min\{\delta(G) - 1, \lfloor \Phi(G) \rfloor\}$, $\epsilon = 1$ if $\Phi(G)$ is not an integer, and $\epsilon = 0$ otherwise.

Proof of Theorem 2.1. Splitting off edges from vertices outside Z if necessary, we may assume that all vertices outside Z have degree one. Suppose for a contradiction that Theorem 2.1 is false. We reserve the triple (G, Z, k) for a counterexample with the minimum $\sum_{z \in Z} d(z)$. For convenience, we call an odd set $U \subseteq Z$ *optimal* if $|E^+(U)| = \frac{|U|+1}{2}k + \epsilon$.

By hypothesis, $d(z) \geq k + 1$ for all $z \in Z$, which can be strengthened as follows.

Claim. $d(z) = k + 1$ for all $z \in Z$.

Otherwise, $d(z) \geq k + 2$ for some $z \in Z$. If z is contained in no optimal odd set $U \subseteq Z$, letting H be obtained from G by splitting off an edge from z, then (H, Z, k) would be a smaller counterexample than (G, Z, k), a contradiction. Hence

(1) there exists an optimal odd set $U_1 \subseteq Z$ containing z; subject to this, we assume that $|U_1|$ is minimum.

Since $(|U_1| + 1)k + 2\epsilon = 2|E^+(U_1)| = 2|E(U_1)| + 2|F(U_1)| \geq (k + 1)|U_1| + |F(U_1)|$, we have $|F(U_1)| \leq k - |U_1| + 2\epsilon \leq k < d(z)$. So z is adjacent to some vertex $y \in U_1$. Let H be arising from G by splitting off one edge $e \in E(y, z)$ from z. We propose to show that

(2) (H, Z, k) is a smaller counterexample than (G, Z, k).

Assume the contrary. Then $|E^+_H(U_2)| < \frac{|U_2|+1}{2}k+\epsilon$ for some odd set $U_2 \subseteq Z$ by the hypothesis of this theorem. Thus

(3) $z \in U_2$, $y \notin U_2$, and $|E^+(U_2)| = \frac{|U_2|+1}{2}k + \epsilon$.

Let $T_1 = U_1 \backslash U_2$ and $T_2 = U_2 \backslash U_1$. By (3), we have $y \in U_1 \backslash U_2$, so $T_1 \neq \emptyset$. By the minimality assumption on $|U_1|$ (see (1)), U_2 is not a proper subset of U_1, which implies $T_2 \neq \emptyset$. Since $z \in U_1 \cap U_2$, we obtain $|U_1 \cap U_2| \geq 1$. Let us consider two cases, according to the parity of $|U_1 \cap U_2|$.

Case 1. $|U_1 \cap U_2|$ is odd.

It is a routine matter to check that

(4) $|E^+(U_1 \cup U_2)| + |E^+(U_1 \cap U_2)| = |E^+(U_1)| + |E^+(U_2)| - 2|E(T_1, T_2)|$.

In this case, $U_1 \cup U_2$ is an odd set. So $|E^+(U_1 \cup U_2)| \geq \frac{|U_1 \cup U_2|+1}{2}k + \epsilon$ by the hypothesis of this theorem.

(5) $|E^+(U_1 \cap U_2)| \geq \frac{|U_1 \cap U_2|+1}{2}k + \epsilon + 1$.

To justify this, note that if $|U_1 \cap U_2| = 1$, then $|E^+(U_1 \cap U_2)| = d(z) \geq k + 2$. So (5) holds. If $|U_1 \cap U_2| \geq 3$, then $U_1 \cap U_2$ is not an optimal odd set by the minimality assumption on $|U_1|$ (see (1)). Thus (5) is also true.

From (4) and (5) we deduce that $\frac{|U_1 \cup U_2|+1}{2}k + \epsilon \leq |E^+(U_1 \cup U_2)| \leq |E^+(U_1)| + |E^+(U_2)| - |E^+(U_1 \cap U_2)| \leq \frac{|U_1|+1}{2}k + \epsilon + \frac{|U_2|+1}{2}k + \epsilon - \frac{|U_1 \cap U_2|+1}{2}k - \epsilon - 1 = \frac{|U_1 \cup U_2|+1}{2}k + \epsilon - 1$, a contradiction.

Case 2. $|U_1 \cap U_2|$ is even.

It is easy to see that

(6) $|E^+(U_1)| + |E^+(U_2)| \geq |E^+(T_1)| + |E^+(T_2)| + 2|E(U_1 \cap U_2)| + |F(U_1 \cap U_2)|$.

(More precisely, $|E^+(U_1)| + |E^+(U_2)| = |E^+(T_1)| + |E^+(T_2)| + 2|E(U_1 \cap U_2)| + |E(U_1 \cap U_2, T_1 \cup T_2)| + 2|E(U_1 \cap U_2, \overline{U_1 \cup U_2})|$, where $\overline{U_1 \cup U_2} = V - (U_1 \cup U_2)$. But we do not need this statement in our proof.) In this case, $|T_i|$ is odd, so $|E^+(T_i)| \geq \frac{|T_i|+1}{2}k + \epsilon$ for $i = 1, 2$ by the hypothesis of this theorem on odd sets (resp. degrees) if $|T_i| \geq 3$ (resp. if $|T_i| = 1$). It follows from (3) and (6) that $\frac{|U_1|+1}{2}k + \epsilon + \frac{|U_2|+1}{2}k + \epsilon \geq \frac{|T_1|+1}{2}k + \epsilon + \frac{|T_2|+1}{2}k + \epsilon + 2|E(U_1 \cap U_2)| + |F(U_1 \cap U_2)| \geq \frac{|T_1|+1}{2}k + \epsilon + \frac{|T_2|+1}{2}k + \epsilon + |U_1 \cap U_2|(k+1) = \frac{|U_1|+1}{2}k + \epsilon + \frac{|U_2|+1}{2}k + \epsilon + |U_1 \cap U_2|$, a contradiction.

Combining the above two cases, we obtain (2). This contradiction justifies the claim.

For each odd set $U \subseteq Z$, by the above claim, we obtain $|U|(k+1) = 2|E(U)| + |F(U)| = |E(U)| + |E^+(U)| \geq |E(U)| + \frac{|U|+1}{2}k + \epsilon$. Thus $|E(U)| \leq \frac{|U|-1}{2}(k+2) + 1 - \epsilon$. Hence $\frac{2|E(U)|}{|U|-1} \leq k+3$ if $\epsilon = 0$ and $\frac{2|E(U)|}{|U|-1} \leq k+2$ if $\epsilon = 1$. By Conjecture 1.1, the chromatic index of $G[Z]$ is at most $k+3-\epsilon$. Since all vertices outside Z have degree one, we further obtain $\chi'(G) \leq k+3-\epsilon$. So E can be partitioned into $k+3-\epsilon$ matchings $M_1, M_2, \ldots, M_{k+3-\epsilon}$.

Let us first consider the case when $\epsilon = 0$. By the above claim,

(7) each vertex $z \in Z$ is disjoint from precisely two of $M_1, M_2, \ldots, M_{k+3}$ (as $d(z) = k+1$).

Let H be the subgraph of G induced by edges in $M_k \cup M_{k+1} \cup M_{k+2} \cup M_{k+3}$, and let N be an orientation of H such that $|d_N^+(v) - d_N^-(v)| \leq 1$ for each vertex v. (It is well known that every multigraph admits such an orientation.) From (7) and this orientation we see that

(8) if a vertex $z \in Z$ is disjoint from precisely one of $M_1, M_2, \ldots, M_{k-1}$, then $d_H(z) = 3$ and $d_N^-(z) \geq 1$; if z is disjoint from precisely two of $M_1, M_2, \ldots, M_{k-1}$, then $d_H(z) = 4$ and $d_N^-(z) = 2$.

For each $i = 1, 2, ..., k-1$, let C_i be obtained from M_i as follows: for each $z \in Z$, if z is not covered by M_i, add an edge from N that is directed to z and has not yet been used in $C_1 \cup C_2 \cup \ldots \cup C_{i-1}$, where $C_0 = \emptyset$. From this construction and (8) we deduce that $C_1, C_2, ..., C_{k-1}$ are pairwise disjoint and each of them is a Z-cover in G.

It remains to consider the case when $\epsilon = 1$. Now

(9) each vertex $z \in Z$ is disjoint from precisely one of $M_1, M_2, \ldots, M_{k+2}$.

Let H be the subgraph of G induced by edges in $M_{k+1} \cup M_{k+2}$, and let N be an orientation of H such that $|d_N^+(v) - d_N^-(v)| \leq 1$ for each vertex v. From (9) and this orientation we see that

(10) if a vertex $z \in Z$ is disjoint from precisely one of $M_1, M_2, \ldots, M_k$, then $d_H(z) = 2$ and $d_N^-(z) = 1$.

For each $i = 1, 2, ..., k$, let C_i be obtained from M_i as follows: for each $z \in Z$, if z is not covered by M_i, add an edge from N that is directed to z. From this construction and (10) we deduce that $C_1, C_2, ..., C_k$ are pairwise disjoint and each of them is a Z-cover in G. ∎

3 Implication

The purpose of this section is to show that Conjecture 1.3 can be deduced from Conjecture 1.2.

Proof of Theorem 1.2. We may assume that

(1) G is connected.

To see this, let $G_1, G_2, \ldots, G_k$ be all the components of G. For each $i = 1, 2, \ldots, k$, we aim to establish the inequality $\xi(G_i) > \delta(G) - \mu(G)$. If $\delta(G_i) - \mu(G_i) > \delta(G) - \mu(G)$, then the desired inequality holds, because $\xi(G_i) \geq \delta(G_i) - \mu(G_i)$. So we assume that $\delta(G_i) - \mu(G_i) \leq \delta(G) - \mu(G)$. Since $\delta(G_i) \geq \delta(G)$ and $\mu(G_i) \leq \mu(G)$, from this assumption we deduce that $\delta(G_i) = \delta(G)$ and $\mu(G_i) = \mu(G)$. Thus G_i satisfies the hypothesis of Conjecture 1.3. Hence we may assume that G is connected, otherwise we consider its components separately.

By hypothesis, $\delta(G)$ cannot be expressed in the form $2p\mu(G) - q$, for any two integers p and q satisfying $q \geq 0$ and $p > \mu(G) + \lfloor (q-1)/2 \rfloor$; these two inequalities are equivalent to $0 \leq q \leq 2p - 2\mu(G)$. Setting $q = 0, 1, \ldots, 2p - 2\mu(G)$ respectively, we see that $\delta(G)$ does not belong to the set

$$\Omega_p = \{2(p+1)\mu(G) - 2p, 2(p+1)\mu(G) - 2p + 1, \ldots, 2p\mu(G)\},$$

where $p \geq \mu(G)$. Note that $2\mu(G)^2$ is the only member of $\Omega_{\mu(G)}$ and that the gap between the largest member of Ω_p and the smallest member of Ω_{p+1} consists of all integers i with $2p\mu(G) + 1 \leq i \leq 2(p+2)\mu(G) - (2p+3)$. So

(2) either $\delta(G) \leq 2\mu(G)^2 - 1$ or $2p\mu(G) + 1 \leq \delta(G) \leq 2(p+2)\mu(G) - (2p+3)$ for some $p \geq \mu(G)$.

We may assume that $\delta(G) \geq 1$, for otherwise, G contains only one vertex by (1) and hence $\delta(G) = 2p\mu(G) - q$ for $p = q = 0$, contradicting the hypothesis of Conjecture 1.3. Thus $\mu(G) \geq 1$.

To prove the theorem, it suffices to show that for any odd set U of G, we have $\frac{2|E^+(U)|}{|U|+1} \geq \delta(G) - \mu(G) + 1$, or equivalently,

(3) $2|E(U)| + 2|F(U)| \geq (|U| + 1)(\delta(G) - \mu(G) + 1)$.

Set $k = \mu(G)$ if $\delta(G) \leq 2\mu(G)^2 - 1$ and set $k = p+1$ if $2p\mu(G) + 1 \leq \delta(G) \leq 2(p+2)\mu(G) - (2p+3)$ for some $p \geq \mu(G)$. We consider two cases according to the size of U.

Case 1. $|U| \geq 2k + 1$.

We divide the present case into two subcases.

Subcase 1.1. Either $U \subsetneq V$ or $U = V$ and $\delta(G)$ is odd. In this subcase,

(4) $2|E(U)| + 2|F(U)| \geq |U|\delta(G) + 1$.

Indeed, if $U \subsetneq V$, then $|F(U)| \geq 1$ by (1). If $U = V$ and $\delta(G)$ is odd, then G contains at least one vertex of degree at least $\delta(G) + 1$, because $|V| = |U|$ is odd and the total number of vertices with odd degree is even. Hence (4) is true.

(5) $|U|\delta(G) + 1 \geq (|U| + 1)(\delta(G) - \mu(G) + 1)$.

Note that (5) amounts to saying that $\delta(G) \leq (|U| + 1)(\mu(G) - 1) + 1$. If $\delta(G) \leq 2\mu(G)^2 - 1$, then $\delta(G) \leq (2\mu(G) + 2)(\mu(G) - 1) + 1 = (2k + 2)(\mu(G) - 1) + 1 \leq (|U| + 1)(\mu(G) - 1) + 1$. If $\delta(G) \leq 2(p+2)\mu(G) - (2p+3)$, then $\delta(G) \leq 2(k+1)\mu(G) - (2k+1) = (2k+2)(\mu(G) - 1) + 1 \leq (|U| + 1)(\mu(G) - 1) + 1$. So (5) is established.

The desired statement (3) follows instantly from (4) and (5).

Subcase 1.2. $U = V$ and $\delta(G)$ is even. In this subcase, we have $\delta(G) \leq 2\mu(G)^2 - 2$ if $\delta(G) \leq 2\mu(G)^2 - 1$ and $\delta(G) \leq 2(p+2)\mu(G) - (2p+4)$ if $\delta(G) \leq 2(p+2)\mu(G) - (2p+3)$. So $\delta(G) \leq (2k+2)(\mu(G)-1)$ by the definition of k and hence

(6) $\delta(G) \leq (|U|+1)(\mu(G)-1)$.

From (6) we deduce that $|U|\delta(G) \geq (|U|+1)(\delta(G)-\mu(G)+1)$. Therefore (3) holds, because $2|E(U)| + 2|F(U)| \geq |U|\delta(G)$.

Case 2. $|U| \leq 2k-1$. (So $k \geq 2$ as $|U| \geq 3$.)

By the Pigeonhole Principle, some vertex $v \in U$ is incident with at most $\frac{|F(U)|}{|U|}$ edges in $F(U)$. Note that v is incident with at most $(|U|-1)\mu(G)$ edges in $G[U]$, so $d(v) \leq (|U|-1)\mu(G) + \frac{|F(U)|}{|U|}$. Hence

(7) $\delta(G) \leq (|U|-1)\mu(G) + \frac{|F(U)|}{|U|}$.

We proceed by considering two subcases.

Subcase 2.1. $2p\mu(G)+1 \leq \delta(G) \leq 2(p+2)\mu(G) - (2p+3)$, where $p \geq \mu(G)$.

From (7) and the hypothesis of the present subcase, we deduce that $2p\mu(G)+1 \leq (|U|-1)\mu(G) + \frac{|F(U)|}{|U|}$. Thus $|F(U)| \geq |U|(2p+1-|U|)\mu(G) + |U|$. So

(8) $|U|\delta(G) + |F(U)| \geq |U|\delta(G) + |U|(2p+1-|U|)\mu(G) + |U|$.

Let us show that

(9) $|U|\delta(G) + |U|(2p+1-|U|)\mu(G) + |U| \geq (|U|+1)(\delta(G)-\mu(G)+1)$.

To justify this, note that (9) is equivalent to

(10) $\delta(G) \leq \{|U|(2p+2-|U|)+1\}\mu(G) - 1$.

By the hypothesis of the present subcase, $\delta(G) \leq 2(p+2)\mu(G) - (2p+3)$. To establish (10), we turn to proving that $2(p+2)\mu(G) - (2p+3) \leq \{|U|(2p+2-|U|)+1\}\mu(G) - 1$, or equivalently

(11) $\{-|U|^2 + 2(p+1)|U| - (2p+3)\}\mu(G) \geq -(2p+2)$.

Let $f(x) = -x^2 + 2(p+1)x - (2p+3)$. Then $f(x)$ is a concave function on $\mathbb{R}$. So on any interval $[a,b]$, $f(x)$ achieves the minimum at a or b. By the hypothesis of the present case, $|U| \leq 2k-1 = 2p+1$, so $3 \leq |U| \leq 2p+1$. By direct computation, we obtain $f(3) = 4p-6 \geq -2$ and $f(2p+1) = -2$. Thus $f(|U|) \geq -2$ for $3 \leq |U| \leq 2p+1$, which implies that the LHS of (11) $\geq -2\mu(G) \geq -(2p+2) =$ RHS of (11), because $p \geq \mu(G)$. This proves (11) and hence (10) and (9).

Since $2|E(U)|+2|F(U)| \geq |U|\delta(G)+|F(U)|$, the desired statement (3) follows instantly from (8) and (9).

Subcase 2.2. $\delta(G) \leq 2\mu(G)^2 - 1$.

We may assume that

(12) $\delta(G) \geq (|U|+1)(\mu(G)-1)+1$, for otherwise, $|U|\delta(G) \geq (|U|+1)(\delta(G)-\mu(G)+1)$. So (3) holds, because $2|E(U)| + 2|F(U)| \geq |U|\delta(G)$.

By (12) and the hypothesis of the present subcase, either $2t(\mu(G)-1)+1 \leq \delta(G) \leq 2(t+1)(\mu(G)-1)$ for some t with $\frac{|U|+1}{2} \leq t \leq \mu(G)$ or $\delta(G) = 2t(\mu(G)-1)+1$ for $t = \mu(G)+1$.

By (7), we have $2t(\mu(G)-1)+1 \leq (|U|-1)\mu(G)+\frac{|F(U)|}{|U|}$. So $\frac{|F(U)|}{|U|} \geq (2t-|U|+1)\mu(G)-2t+1$, and hence

(13) $|U|\delta(G) + |F(U)| \geq |U|\{\delta(G) + (2t - |U| + 1)\mu(G) - 2t + 1\}$.

We propose to show that

(14) $|U|\{\delta(G) + (2t - |U| + 1)\mu(G) - 2t + 1\} \geq (|U| + 1)(\delta(G) - \mu(G) + 1)$.

To justify this, note that (14) is equivalent to

(15) $\delta(G) \leq \{|U|(2t + 2 - |U|) + 1\}\mu(G) - |U|2t - 1$.

Suppose $\delta(G) = 2\mu(G)^2 - 1$. Then $t = \mu(G)+1$. So (15) says that $2\mu(G)^2-1 \leq \{|U|(2\mu(G)+4-|U|)+1\}\mu(G) - |U|(2\mu(G)+2)-1$, or equivalently, $\{|U|(2\mu(G)+4-|U|)+1\}\mu(G) - |U|(2\mu(G)+2) \geq 2\mu(G)^2$. Let $g(x) = \{x(2\mu(G)+4-x)+1\}\mu(G) - x(2\mu(G)+2)$. Then $g(x)$ is a concave function on $\mathbb{R}$. So on any interval $[a, b]$, $g(x)$ achieves the minimum at a or b. By direct computation, we obtain $g(3) = 6\mu(G)^2 - 2\mu(G) - 6$ and $g(2\mu(G) - 1) = 6\mu(G)^2 - 6\mu(G) + 2$. It is easy to see that $\min\{g(3), g(2\mu(G)-1)\} \geq 2\mu(G)^2$, because $\mu(G) = k \geq 2$ (see the hypothesis of Case 2). Hence $g(|U|) \geq 2\mu(G)^2$ for $3 \leq |U| \leq 2\mu(G) - 1 = 2k - 1$. This proves (15) and hence (14) and (13).

So we assume that $\delta(G) \leq 2(t + 1)(\mu(G) - 1)$ for some t with $\frac{|U|+1}{2} \leq t \leq \mu(G)$. We prove (15) by showing that $2(t+1)(\mu(G)-1) \leq \{|U|(2t+2-|U|)+1\}\mu(G) - |U|2t-1$, or equivalently, $\{|U|(2t+2-|U|)-2t-1\}\mu(G)-|U|2t \geq -2t-1$. Let $h(x) = \{x(2t+2-x)-2t-1\}\mu(G)-2tx$. Then $h(x)$ is a concave function on $\mathbb{R}$. So on any interval $[a, b]$, $h(x)$ achieves the minimum at a or b. By direct computation, we obtain $h(3) = 4(t-1)\mu(G)-6t$ and $h(2t-1) = 4(t-1)\mu(G)-2t(2t-1)$. It is easy to see that $\min\{h(3), h(2t-1)\} \geq -2t-1$, because $\mu(G) \geq t \geq \frac{|U|+1}{2} \geq 2$. Hence $h(|U|) \geq -2t-1$ for $3 \leq |U| \leq 2t-1$. This proves (15) and hence (14) and (13).

Since $2|E(U)|+2|F(U)| \geq |U|\delta(G)+|F(U)|$, the desired statement (3) follows instantly from (13) and (14), completing the proof of Theorem 1.2. ∎

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