

Article

Scalar QED Model for Polarizable Particles in Thermal Equilibrium or in Hyperbolic Motion in Vacuum

Kanu Sinha ¹  and Peter W. Milonni ^{2,*}

¹ College of Optical Sciences and Department of Physics, University of Arizona, Tucson, AZ 85721, USA; kanu@arizona.edu

² Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

* Correspondence: peter_milonni@comcast.net

Abstract: We consider a scalar QED (quantum electrodynamics) model for the frictional force and the momentum fluctuations of a polarizable particle in thermal equilibrium with radiation or in hyperbolic motion in a vacuum. In the former case the loss of particle kinetic energy due to the frictional force is compensated by the increase in kinetic energy associated with the momentum diffusion, resulting in the Planck distribution when it is assumed that the average kinetic energy satisfies the equipartition theorem. For hyperbolic motion in vacuum the frictional force and the momentum diffusion are similarly consistent with an equilibrium with a Planckian distribution at the temperature $T = \hbar a / 2\pi k_B c$. The quantum fluctuations of the momentum imply that it is only the average acceleration a that is constant when the particle is subject to a constant applied force.

Keywords: scalar quantum electrodynamics; Einstein–Hopf model; hyperbolic motion; Davies-Unruh temperature

1. Introduction

The question of whether there is radiation from a uniformly accelerated oscillator in vacuum continues to be of interest [1] (and references therein). In this paper, we revisit the question within the context of a 2D (2-dimensional) scalar QED (quantum electrodynamics) model [2,3]. Our approach differs from previous study with this model in that we focus on the force on the particle and its momentum fluctuations as it exchanges energy with the field. Since it is well established that the particle will perceive itself to be in a field with a Planck distribution at a temperature $T = \hbar a / 2\pi k_B c$, where a is the acceleration, \hbar is the reduced Planck constant, c is the speed of light, and k_B is the Boltzmann constant [4,5], we first consider, within the 2D scalar QED model, a polarizable particle in equilibrium with thermal radiation in the absence of any external force that would accelerate the particle. We follow the original approach of Einstein and Hopf [6] and later Einstein [7] in which it is shown that the frictional force and the momentum diffusion experienced by the particle are consistent with the Planck distribution for the field. The 2D scalar QED model exhibits the basic physics of the problem while permitting some simplification compared to the complete QED theory [8,9].

Essentially the same approach in the case of a uniformly accelerated particle in vacuum yields the expected result: the particle undergoes a frictional force and momentum fluctuations as if it were in thermal equilibrium at the temperature $T = \hbar a / 2\pi k_B c$. The result for the momentum fluctuations seems particularly interesting in the way that the Bose–Einstein form of the fluctuations emerges. The coupling of the particle to the quantum field results in quantum fluctuations about its classical hyperbolic trajectory. These fluctuations are superposed on an otherwise uniformly accelerated motion.

In the following Section we present the basic equations of motion needed for our analysis. In Section 3 we derive, based on the Lorentz transformation of the spectral energy density of the field, the frictional force on a polarizable particle in a field of blackbody



Citation: Sinha, K.; Milonni, P.W. Scalar QED Model for Polarizable Particles in Thermal Equilibrium or in Hyperbolic Motion in Vacuum. *Physics* **2024**, *6*, 356–367. <https://doi.org/10.3390/physics6010023>

Received: 7 November 2023

Revised: 5 January 2024

Accepted: 15 January 2024

Published: 5 March 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

radiation, and in Section 4 we derive the momentum fluctuations experienced by such a particle. The expressions obtained for the frictional force and the momentum fluctuations are shown to be consistent with a Planck distribution for the spectral energy density of the field. In Section 5 we extend these considerations to the case of hyperbolic motion in a vacuum. Our results are summarized in Section 6.

2. Hamiltonian and Equations of Motion

We consider a point particle interacting with a (1 + 1)D scalar field $\phi(y, t)$, with y the coordinate and t the time, via a harmonically bound charged oscillator, as shown in Figure 1. The Lagrangian of the system is [10]

$$\mathcal{L} = \frac{1}{2}M\dot{Y}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2 + \int dy \left[(\partial_t\phi(y, t))^2 - (\partial_y\phi(y, t))^2 + e\phi(y, t)\dot{x}\delta(y - Y) \right]. \tag{1}$$

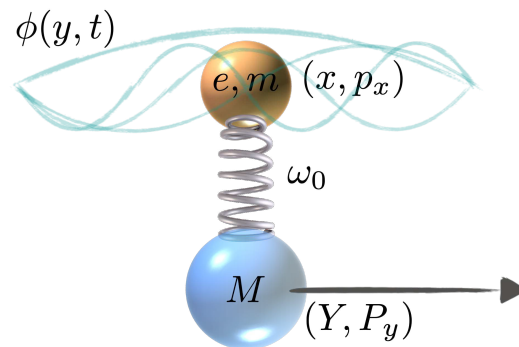


Figure 1. Schematic representation of a point particle of mass M and harmonically bound oscillator with a charge e and mass m . The charged oscillator moves along the x -axis with a frequency ω_0 and interacts with a scalar field $\phi(y, t)$ at $y = Y$, where Y is the center-of-mass position of the point particle. See text for more details.

Here, M denotes the mass of the particle, m and e are the mass and charge of the oscillator, respectively, ω_0 is the resonant frequency of the oscillator, $\partial_a = \partial/\partial a$, $\delta(\cdot)$ is the Dirac delta function, and the dot over a letter denotes the time derivative. We denote the center-of-mass position of the particle by $Y(t)$ and the charged oscillator coordinate by $x(t)$. The charged oscillator interacts with the field and its y coordinate is constrained to be at $Y(t)$. The bilinear coupling term is motivated by the minimal coupling form of interaction, considering that one can draw a correspondence between ϕ and the vector potential, as discussed in similar models used in microscopic descriptions of mirrors and optomechanics [10–12]. Defining the canonical conjugate momenta associated with the various degrees of freedom as $P_y = M\dot{Y}$, $p_x = m\dot{x} + e\phi(Y, t)$ and $\Pi(y, t) = \dot{\phi}(y, t)$, we obtain the corresponding Hamiltonian in the minimal coupling form as

$$H = P_y^2/2M + \frac{1}{2}m\omega_0^2x^2 + \frac{1}{2m}[p_x - e\phi(Y, t)]^2 + \sum_k \hbar\omega_k a_k^\dagger a_k, \tag{2}$$

with

$$\phi(y, t) = \sum_k C_k [a_k(t)e^{iky} + a_k^\dagger(t)e^{-iky}], \quad C_k = \left(\frac{2\pi\hbar}{\omega_k L} \right)^{1/2}. \tag{3}$$

a_k and a_k^\dagger are the bosonic annihilation and creation operators, respectively, following the canonical commutation relations $[a_k, a_{k'}^\dagger] = \delta_{kk'}$, with δ_{ij} the Kronecker delta, and the quantization volume is of length L . (We omit the speed of light c in Equations (1)–(3)) From

the Heisenberg equations of motion that follow from the Hamiltonian (2) we obtain the following equations for the operators x, ϕ_0 , and (see Appendix A)

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = -\frac{e}{m} \frac{\partial\phi_0(Y, t)}{\partial t}, \tag{4}$$

$$\phi_0(y, t) = \sum_k C_k [a_k e^{-i(\omega_k t - ky)} + a_k^\dagger e^{i(\omega_k t - ky)}], \tag{5}$$

$$\dot{P}_y = \frac{e}{2} \left(\dot{x} \frac{\partial\phi_0(Y, t)}{\partial y} + \frac{\partial\phi_0(Y, t)}{\partial y} \dot{x} \right). \tag{6}$$

$\phi_0(y, t)$ is the source-free part of $\phi(y, t)$; the source (“radiation reaction”) part of $\phi(y, t)$ results in the damping force $-2\beta m\dot{x}$ in Equation (4). β is found without approximation to be $\pi e^2/m$, consistent with the commutation relation $[x(t), m\dot{x}(t)] = i\hbar$ maintained by the formal solution of Equation (4); see, for instance, Ref. [13].

Equations (4) and (5), together with the equation for the radiation field $\phi(y, t)$, are the basis for the analyses by Raine, Sciamia and Grove [2] and Ford and O’Connell [3] and their conclusion that an oscillator in hyperbolic motion does not radiate. We focus instead on Equation (6). In other words, we focus not on the radiation field but on the force acting on the particle. In the case of hyperbolic motion, this force results in fluctuations superposed on the otherwise uniform acceleration.

For the linear response of $x(t)$ to $\partial\phi_0(Y, t)/\partial t$ we define the polarizability

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - 2i\beta\omega} \tag{7}$$

such that the steady-state expression for $ex(t)$ for the particle at rest is

$$ex(t) = i \sum_k C_k \omega_k [\alpha(\omega_k) a_k e^{-i\omega_k t} - \alpha^*(\omega_k) a_k^\dagger e^{i\omega_k t}], \tag{8}$$

where we have assumed without loss of generality that $Y = 0$ specifies the particle coordinate.

3. Frictional Force

A polarizable particle moving in a blackbody field experiences a drag force, as shown nonrelativistically by Einstein and Hopf [6] and Einstein [7]. To obtain a relativistic expression for the drag force we can proceed as in Ref. [8] or Ref. [9], for instance. Here we take a different, more heuristic approach based on the Lorentz transformation of the Planck spectrum in the 2D scalar QED model. Our starting point is the expression for the free-space energy density of the 2D massless scalar field:

$$E = \frac{1}{2} \left[\left(\frac{\partial\phi}{\partial t} \right)^2 + \left(\frac{\partial\phi}{\partial y} \right)^2 \right], \tag{9}$$

which is proportional to ω^2 for a mode of frequency ω . In a frame S' moving with velocity v with respect to the frame in which the energy density is E , the energy density $E' \propto \omega'^2$, with

$$\begin{aligned} \omega &= \gamma\omega'(1 + v) \\ E &= \gamma^2(1 + v)^2 E', \quad \gamma = (1 - v^2)^{-1/2}. \end{aligned} \tag{10}$$

We define the spectral energy density $\rho(\omega)$ such that $\rho(\omega)d\omega$ is the energy per unit length (or “volume”) in the frequency interval $[\omega, \omega + d\omega]$. In the moving frame $\rho(\omega, \hat{\mathbf{k}}) \rightarrow \rho'(\omega', \hat{\mathbf{k}})$,

$$\rho'(\omega', \hat{\mathbf{k}})d\omega' = \frac{E'}{E}\rho(\omega, \hat{\mathbf{k}})d\omega = \frac{1}{\gamma^2(1+v)^2}\rho(\omega, \hat{\mathbf{k}})d\omega \tag{11}$$

for a plane wave of frequency ω propagating in the direction of the unit vector $\hat{\mathbf{k}}$, and therefore

$$\rho'(\omega', \hat{\mathbf{k}}) = \frac{1}{\gamma^2(1+v)^2}\rho(\omega, \hat{\mathbf{k}})\frac{d\omega}{d\omega'} = \frac{1}{\gamma(1+v)}\rho(\omega, \hat{\mathbf{k}}). \tag{12}$$

For plane waves propagating in the direction parallel to the velocity \mathbf{v} ,

$$\rho'(\omega') = \frac{\rho[\gamma\omega'(1+v)]}{\gamma(1+v)}. \tag{13}$$

The energy eigenvalues for the free field are ($\omega_k = |k|$)

$$\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(n(\omega_{\mathbf{k}}) + \frac{1}{2} \right) = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk \hbar\omega \left(n(\omega) + \frac{1}{2} \right) \equiv L \int_0^{\infty} d\omega \rho(\omega), \tag{14}$$

where the spectral energy density

$$\rho(\omega) = \frac{\hbar\omega}{\pi} \left(n(\omega) + \frac{1}{2} \right), \tag{15}$$

with $n(\omega)$ the average photon number at frequency ω . From Equations (13) and (15),

$$\omega' \left(n(\omega) + \frac{1}{2} \right) = \omega' \left[n(\gamma\omega'(1+v)) + \frac{1}{2} \right], \tag{16}$$

and so the Lorentz-transformed spectral energy density is

$$\rho'(\omega') = \frac{\hbar\omega'}{\pi} \left[n(\gamma\omega'(1+v)) + \frac{1}{2} \right]. \tag{17}$$

In the case of thermal equilibrium at temperature T ,

$$n(\gamma\omega'(1+v)) + \frac{1}{2} = \frac{1}{2} \coth \left[\frac{\gamma\hbar\omega'}{2k_B T} (1+v) \right] \tag{18}$$

and

$$\rho'(\omega') = \frac{\hbar\omega'}{2\pi} \coth \left[\frac{\gamma\hbar\omega'}{2k_B T} (1+v) \right]. \tag{19}$$

This is the 2D version of the Lorentz-transformed Planck spectrum obtained in earlier work [14,15].

The force on the particle may be expressed as

$$\mathcal{F} = \frac{1}{2} \int_0^{\infty} d\omega R(\omega) \rho'(\omega) = \frac{\hbar}{2\pi} \int_0^{\infty} d\omega R(\omega) \omega \left[n(\gamma\omega(1+v)) + \frac{1}{2} \right], \tag{20}$$

where $R(\omega)$ is the rate at which the field loses energy (and momentum $p = E$) in a given direction ($k > 0$ or $k < 0$), this momentum being taken up by the particle. Perhaps the simplest way to obtain an expression for $R(\omega)$ is to start from the wave equation for ϕ [10]:

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial t^2} = 4\pi e \dot{x} \delta(y - Y), \tag{21}$$

which has the same form as the wave equation for an electric field propagating in one direction in a medium with number density N ($= \delta(y - Y)$ here) of particles with polarizability $\alpha(\omega)$. In that case the energy dissipation rate $R(\omega) = 2k_I(\omega) \cong 4\pi N \omega \alpha_I(\omega)$ in the dilute-medium approximation, where k_I and $\alpha_I(\omega)$ are respectively the imaginary parts of the wave number and the polarizability [16].

In the present case we have one particle in our volume L , implying $R(\omega) = 4\pi \omega \alpha_I(\omega)$ and

$$\mathcal{F} = 2\hbar \int_0^\infty d\omega \omega^2 \alpha_I(\omega) \left[n(\gamma\omega(1+v)) + \frac{1}{2} \right]. \tag{22}$$

This expresses the average force in terms of the linear response of the particle to a broadband field with Doppler-shifted frequencies $\omega \sqrt{(1+v)/(1-v)}$. Including field modes with \mathbf{k} either parallel or anti-parallel to the particle velocity \mathbf{v} , we obtain the net force

$$\begin{aligned} \mathcal{F} &= 2\hbar \int_0^\infty d\omega \omega^2 \alpha_I(\omega) [n(\gamma\omega(1+v)) - n(\gamma\omega(1-v))] \cong 4\hbar v \int_0^\infty d\omega \omega^3 \alpha_I(\omega) \frac{\partial n}{\partial \omega} \\ &= -4\pi v \int_0^\infty d\omega \omega \alpha_I(\omega) \left[\rho - \omega \frac{\partial \rho}{\partial \omega} \right]. \end{aligned} \tag{23}$$

In the full QED theory, in contrast, the nonrelativistic approximation to the friction force is [14,15]

$$\mathcal{F} = -4\pi v \int_0^\infty d\omega \omega \alpha_I(\omega) \left[\rho - \frac{\omega}{3} \frac{\partial \rho}{\partial \omega} \right], \tag{24}$$

with $\rho(\omega) = \hbar \omega^3 n(\omega) / \pi^2 c^3$. It is noteworthy that Einstein and Hopf [6] and Einstein [7] obtain $\rho(\omega) - \frac{\omega}{3} \frac{\partial \rho}{\partial \omega}$ in their formula for the friction force from essentially just kinematic effects involving Doppler shifts and aberration. The factor 1/3 that appears in (24) is due to the vector character of the three-dimensional electromagnetic field, whereas our expression (23) assumes a one-dimensional scalar field.

4. Momentum Fluctuations and Thermal Equilibrium

From Equation (6),

$$\begin{aligned} P_y(t) &= i \sum_K \sum_k C_K C_k \omega_K^2 k [\alpha(\omega_K) a_K a_k g_{Kk}(t) - \alpha(\omega_K) a_K a_k^\dagger f_{Kk}(t) \\ &\quad + \alpha^*(\omega_K) a_K^\dagger a_k f_{Kk}^*(t) - \alpha^*(\omega_K) a_K^\dagger a_k^\dagger g_{Kk}^*(t)], \end{aligned} \tag{25}$$

where

$$f_{Kk}(t) = \int_0^t d\tau e^{-i(\omega_K - \omega_k)\tau} \quad \text{and} \quad g_{Kk}(t) = \int_0^t d\tau e^{-i(\omega_K + \omega_k)\tau}. \tag{26}$$

The calculation of momentum fluctuations $\langle \Delta P_y^2(t) \rangle = \langle P_y^2(t) \rangle$, averaged over the state of the field, is simplified if we assume that ϕ_0 and its derivatives may be regarded as Gaussian random processes. This allows us to treat the K and k modes ($K \neq k$) in Equation (25) as uncorrelated. This property was originally employed in a classical context by Einstein and Hopf [6,17]. A simplified proof of this property for Gaussian random processes X and Y with $\langle X \rangle = \langle Y \rangle = \langle XY \rangle = 0$ is given in Ref. [9]. We further simplify by dropping terms involving $g_{Kk}(t)$, since their nonresonant character results in no contribution to $\langle P_y^2(t) \rangle$. Then

$$\begin{aligned}
 \langle P_y^2(t) \rangle &= \sum_K \sum_k C_K^2 C_k^2 \omega_K^4 k^2 |\alpha(\omega_K)|^2 \left[\langle a_K a_K^\dagger \rangle \langle a_k a_k^\dagger \rangle + \langle a_K^\dagger a_K \rangle \langle a_k^\dagger a_k \rangle \right] |f_{Kk}(t)|^2 \\
 &\rightarrow \left(\frac{L}{2\pi} \right)^2 \int_{-\infty}^{\infty} dK \left(\frac{2\pi\hbar}{\omega_K L} \right) \omega_K^4 |\alpha(\omega_K)|^2 \int_{-\infty}^{\infty} dk \left(\frac{2\pi\hbar}{\omega_k L} \right) k^2 \\
 &\quad \times \left([n(\omega_K) + 1]n(\omega_k) + n(\omega_K)[n(\omega_k) + 1] \right) \frac{\sin^2 \frac{1}{2}(\omega_K - \omega_k)t}{(\omega_K - \omega_k)^2/4} \\
 &\cong 4\hbar^2 \int_0^\infty d\omega \omega^4 |\alpha(\omega)|^2 \int_0^\infty d\omega' [2n(\omega)n(\omega') + n(\omega) + n(\omega')] 2\pi t \delta(\omega - \omega') \\
 &= 16\pi\hbar^2 t \int_0^\infty d\omega \omega^4 |\alpha(\omega)|^2 [n^2(\omega) + n(\omega)]. \tag{27}
 \end{aligned}$$

In the penultimate step we have used the long-time approximation $\int_{-\infty}^{\infty} dK \frac{\sin^2((\omega_K - \omega_k)t/2)}{[(\omega_K - \omega_k)/2]^2} \cong \int_{-\infty}^{\infty} dK 2\pi t \delta(\omega_K - \omega_k)$.

In thermal equilibrium the average rate of increase of the kinetic energy must balance the average rate $\langle \mathcal{F}v \rangle$ at which the particle loses energy because of the frictional force:

$$\left\langle \frac{d}{dt} \frac{P_y^2}{2m} + \mathcal{F}v \right\rangle = 0, \tag{28}$$

or, from (23) and (27), and the energy equipartition assumption, $\langle \frac{1}{2}mv^2 \rangle = \frac{1}{2}k_B T$, for thermal equilibrium at temperature T ,

$$\frac{8\pi\hbar^2}{m} \int_0^\infty d\omega \omega^4 |\alpha(\omega)|^2 [n^2(\omega) + n(\omega)] + \frac{4\hbar k_B T}{m} \int_0^\infty d\omega \omega^3 \alpha_I(\omega) \frac{\partial n}{\partial \omega} = 0. \tag{29}$$

Using the relation $|\alpha(\omega)|^2 = \alpha_I(\omega)/2\pi\omega$ (Equation (7)), and the fact that $\alpha_I(\omega) > 0$ for all $\omega > 0$, we obtain

$$\frac{\partial n}{\partial \omega} = -\frac{\hbar}{k_B T} (n^2 + n), \tag{30}$$

which is satisfied as expected by

$$n(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1}. \tag{31}$$

5. Hyperbolic Motion in Vacuum

We expect the results above to apply to the case of hyperbolic motion in the zero-temperature vacuum if we simply replace $k_B T$ by $\hbar a/2\pi$. In particular, we expect the coupling of the particle to the quantum vacuum field to cause quantum fluctuations about its classical trajectory. A calculation showing this seems nevertheless worthwhile. We now present such a calculation for the momentum fluctuations experienced by a polarizable particle in hyperbolic motion in vacuum.

We start with Equations (5), (6) and (8), but now with

$$t(\tau) = \frac{1}{a} \sinh a\tau \quad \text{and} \quad y(\tau) = \frac{1}{a} \cosh a\tau \tag{32}$$

for hyperbolic motion, and therefore

$$\omega t(\tau) + ky(\tau) = \omega(t(\tau) + y(\tau)) = \frac{\omega}{a} e^{a\tau} \tag{33}$$

for a particle accelerated in the y direction and for modes $k = \omega > 0$, which we assume for now. We will carry out the calculation with (proper) time durations τ registered by a

clock moving with the particle characterized by a polarizability $\alpha(\omega)$. Since $\phi_0(y, t)$ may be expressed as a function of τ , we write it in terms of operators $g(\Omega)$ and $g^\dagger(\Omega)$ as follows:

$$\phi_0(\tau) = \int_0^\infty d\Omega [g(\Omega)e^{-i\Omega\tau} + g^\dagger(\Omega)e^{i\Omega\tau}]. \tag{34}$$

Then

$$e\dot{x} = \int_0^\infty d\Omega \Omega^2 [\alpha(\Omega)g(\Omega)e^{-i\Omega\tau} + \alpha^*(\Omega)g^\dagger(\Omega)e^{i\Omega\tau}], \tag{35}$$

$$\frac{\partial\phi_0}{\partial y} = i \int_0^\infty d\Omega \Omega [g(\Omega)e^{-i\Omega\tau} - g^\dagger(\Omega)e^{i\Omega\tau}], \tag{36}$$

and

$$P_y(\tau) = \frac{i}{2} \int_0^\infty d\Omega_1 \int_0^\infty d\Omega_2 \Omega_1^2 \Omega_2 [\alpha(\Omega_1)g(\Omega_1)g(\Omega_2)G_{12}(\tau) - \alpha(\Omega_1)g(\Omega_1)g^\dagger(\Omega_2)F_{12}(\tau) + \alpha^*(\Omega_1)g^\dagger(\Omega_1)g(\Omega_2)F_{12}^*(\tau) - \alpha^*(\Omega_1)g^\dagger(\Omega_1)g^\dagger(\Omega_2)G_{12}^*(\tau)] + \text{h.c.}, \tag{37}$$

where ‘‘h.c.’’ stands for Hermitian conjugate and we have defined

$$F_{12}(\tau) = 2e^{-i(\Omega_1-\Omega_2)\tau/2} \frac{\sin \frac{1}{2}(\Omega_1 - \Omega_2)\tau}{\Omega_1 - \Omega_2},$$

$$G_{12}(\tau) = 2e^{-i(\Omega_1+\Omega_2)\tau/2} \frac{\sin \frac{1}{2}(\Omega_1 + \Omega_2)\tau}{\Omega_1 + \Omega_2}. \tag{38}$$

Now we proceed as in the calculation for thermal equilibrium, regarding ϕ_0 and its derivatives as Gaussian random processes. Then, in taking expectation values, we can treat the operators depending on Ω_1 and Ω_2 as uncorrelated.

5.1. Momentum Fluctuations

The momentum fluctuations, $\langle \Delta P_y^2(\tau) \rangle = \langle P_y^2(\tau) \rangle$, of the particle can be obtained as:

$$\langle P_y^2(\tau) \rangle = 4\text{Re} \int_0^\infty d\Omega_1 \int_0^\infty d\Omega_2 \int_0^\infty d\Omega_3 \int_0^\infty d\Omega_4 \Omega_1^2 \Omega_2^2 \Omega_3 \Omega_4 \times [\alpha(\Omega_1)\alpha^*(\Omega_3)\langle g(\Omega_1)g^\dagger(\Omega_3) \rangle \langle g^\dagger(\Omega_2)g(\Omega_4) \rangle F_{12}(\tau)F_{34}^*(\tau)]. \tag{39}$$

We have dropped terms involving G_{ij} and G_{ij}^* . Because of their nonresonant character they make no contribution to $\langle P_y^2 \rangle$.

From Equation (34), again for a particle accelerated along the y direction,

$$g(\Omega) = \frac{1}{2\pi} \int_{-\infty}^\infty d\tau e^{i\Omega\tau} \phi_0(\tau) = \frac{1}{2\pi} \int_{-\infty}^\infty d\tau e^{i\Omega\tau} \sum_k C_k [a_k e^{i\frac{\omega_k}{a}e^{a\tau}} + a_k^\dagger e^{-i\frac{\omega_k}{a}e^{a\tau}}]$$

$$= \frac{1}{2\pi} \sum_k C_k [a_k \zeta(\omega_k, \Omega) + a_k^\dagger \eta(\omega_k, \Omega)], \tag{40}$$

where we have defined the functions $\zeta(\omega_k, \Omega)$ and $\eta(\omega_k, \Omega)$, corresponding to the time-dependent Doppler shift observed by the accelerated observer [18], as

$$\zeta(\omega, \Omega) = \int_{-\infty}^\infty d\tau e^{i\Omega\tau} e^{i\frac{\omega}{a}e^{a\tau}} = \frac{1}{a} \Gamma\left(\frac{i\Omega}{a}\right) \left(\frac{\omega}{a}\right)^{-i\Omega/a} e^{-\pi\Omega/2a}, \tag{41}$$

$$\eta(\omega, \Omega) = \int_{-\infty}^\infty d\tau e^{i\Omega\tau} e^{-i\frac{\omega}{a}e^{a\tau}} = \frac{1}{a} \Gamma\left(\frac{i\Omega}{a}\right) \left(\frac{\omega}{a}\right)^{-i\Omega/a} e^{\pi\Omega/2a}, \tag{42}$$

where $\Gamma(x)$ is the Gamma function (see Appendix B for details).

We must now evaluate the correlation functions appearing in (39) that relate to the power spectrum of the vacuum fluctuations as seen by the accelerated particle. Consider first

$$\langle g(\Omega_1)g^\dagger(\Omega_3) \rangle = \left(\frac{1}{2\pi}\right)^2 \sum_k C_k^2 \zeta(\omega_k, \Omega_1) \zeta^*(\omega_k, \Omega_3). \tag{43}$$

We have used the vacuum expectation values $\langle a_k a_{k'} \rangle = \langle a_k^\dagger a_{k'}^\dagger \rangle = \langle a_k^\dagger a_{k'} \rangle = 0$ and $\langle a_k a_{k'}^\dagger \rangle = \delta_{kk'}$. Now, in the usual mode continuum limit,

$$\begin{aligned} \sum_k C_k^2 \zeta(\omega_k, \Omega_1) \zeta^*(\omega_k, \Omega_3) &= \frac{\hbar}{a^2} \Gamma\left(\frac{i\Omega_1}{a}\right) \Gamma^*\left(\frac{i\Omega_3}{a}\right) e^{-\pi(\Omega_1 + \Omega_3)/2a} e^{i(\Omega_1 - \Omega_3)/a} \\ &\times \int_0^\infty d\omega \frac{1}{\omega} \left(\frac{\omega}{a}\right)^{i(\Omega_1 - \Omega_3)/a}. \end{aligned} \tag{44}$$

The integral may be evaluated with the change of variable $x = \ln(\omega/a)$ to obtain $2\pi a \delta(\Omega_1 - \Omega_3)$, and so

$$\langle g(\Omega_1)g^\dagger(\Omega_3) \rangle = \frac{\hbar}{2\pi a} \left| \Gamma\left(\frac{i\Omega_1}{a}\right) \right|^2 e^{-\pi\Omega_1/a} \delta(\Omega_1 - \Omega_3). \tag{45}$$

We find similarly that

$$\langle g^\dagger(\Omega_2)g(\Omega_4) \rangle = \frac{\hbar}{2\pi a} \left| \Gamma\left(\frac{i\Omega_2}{a}\right) \right|^2 e^{\pi\Omega_2/a} \delta(\Omega_2 - \Omega_4) \tag{46}$$

and $\langle g(\Omega_1)g(\Omega_3) \rangle = \langle g^\dagger(\Omega_1)g^\dagger(\Omega_3) \rangle = 0$. After some algebra we obtain

$$\begin{aligned} \langle P_y^2(\tau) \rangle &= 4 \int_0^\infty d\Omega_1 \int_0^\infty d\Omega_2 \int_0^\infty d\Omega_3 \int_0^\infty d\Omega_4 \Omega_1^2 \Omega_3^2 \Omega_2 \Omega_4 |\alpha(\Omega_1)|^2 \\ &\times \left(\frac{\hbar}{2\pi a}\right)^2 \left| \Gamma\left(\frac{i\Omega_1}{a}\right) \right|^2 \left| \Gamma\left(\frac{i\Omega_3}{a}\right) \right|^2 \delta(\Omega_1 - \Omega_3) \delta(\Omega_2 - \Omega_4) \\ &\times e^{-\pi(\Omega_1 - \Omega_2)/a} \frac{\sin^2 \frac{1}{2}(\Omega_1 - \Omega_2)\tau}{\frac{1}{2}(\Omega_1 - \Omega_2)^2} \\ &\rightarrow 4 \left(\frac{\hbar}{2\pi a}\right)^2 (2\pi\tau) \int_0^\infty d\Omega_1 \Omega_1^6 |\alpha(\Omega_1)|^2 \left| \Gamma\left(\frac{i\Omega_1}{a}\right) \right|^4. \end{aligned} \tag{47}$$

Finally we use the identity

$$\left| \Gamma\left(\frac{i\Omega}{a}\right) \right|^4 = \frac{\pi^2 a^2 / \Omega^2}{\sinh^2(\pi\Omega/a)} = \frac{\pi^2 a^2}{\Omega^2} \frac{4e^{2\pi\Omega/a}}{(e^{2\pi\Omega/a} - 1)^2} \tag{48}$$

to obtain

$$\langle P_y^2(\tau) \rangle = 16\pi\hbar^2\tau \int_0^\infty d\omega \omega^4 |\alpha(\omega)|^2 [n^2(\omega) + n(\omega)], \tag{49}$$

where a factor 2 is included to account for modes with $k < 0$ as well as $k > 0$, and have defined

$$n(\omega) = (e^{\hbar\omega/k_B T} - 1)^{-1}, \quad T \equiv \hbar a / 2\pi k_B. \tag{50}$$

The result (49) has the same form as Equation (27), as expected. Interestingly, the Bose–Einstein factor $n^2 + n$ in Equation (27) was obtained from the algebra of annihilation and creation operators, whereas the derivation of Equation (49) depended mainly on the analytical properties of $\zeta(\omega, \Omega)$ and $\eta(\omega, \Omega)$, as well as the vacuum expectation values $\langle a_k a_{k'}^\dagger \rangle = \delta_{kk'}$.

5.2. Frictional Force

Finally we consider briefly the frictional force on the particle's center-of-mass, starting with

$$\mathcal{F} = \left\langle \frac{dP_y(\tau)}{d\tau} \right\rangle. \tag{51}$$

Using Equation (37) and ignoring the non-resonant terms, we obtain the simplified expression

$$\begin{aligned} \mathcal{F} = & -\frac{i}{2} \int_0^\infty d\Omega_1 \int_0^\infty d\Omega_2 \Omega_1^2 \Omega_2 \left[-\alpha(\Omega_1) \langle g(\Omega_1) g^\dagger(\Omega_2) \rangle \frac{dF_{12}(\tau)}{d\tau} \right. \\ & \left. + \alpha^*(\Omega_1) \langle g^\dagger(\Omega_1) g(\Omega_2) \rangle \frac{dF_{12}^*(\tau)}{d\tau} \right] + \text{h.c.} \end{aligned} \tag{52}$$

Noting that $\frac{dF_{12}(\tau)}{d\tau} = e^{-i(\Omega_1 - \Omega_2)\tau}$, and using the expectation values of the correlation functions from Equations (45) and (46), yields

$$\begin{aligned} \mathcal{F} = & -\frac{i\hbar}{4\pi a} \int_0^\infty d\Omega \Omega^3 \left| \Gamma\left(\frac{i\Omega}{a}\right) \right|^2 \left[-\alpha(\Omega) e^{-\pi\Omega/a} + \alpha^*(\Omega) e^{\pi\Omega/a} \right] \\ & + \frac{i\hbar}{4\pi a} \int_0^\infty d\Omega \Omega^3 \left| \Gamma\left(\frac{i\Omega}{a}\right) \right|^2 \left[-\alpha^*(\Omega) e^{-\pi\Omega/a} + \alpha(\Omega) e^{\pi\Omega/a} \right] \\ = & \frac{\hbar}{\pi a} \int_0^\infty d\Omega \Omega^3 \cosh(\pi\Omega/a) \left| \Gamma\left(\frac{i\Omega}{a}\right) \right|^2 \alpha_I(\Omega) \end{aligned} \tag{53}$$

$$\begin{aligned} = & \hbar \int_0^\infty d\Omega \Omega^2 \alpha_I(\Omega) \coth(\pi\Omega/a) \\ = & 2\hbar \int_0^\infty d\omega \omega^2 \alpha_I(\omega) \left[n(\omega) + \frac{1}{2} \right], \end{aligned} \tag{54}$$

where in the last step we have again defined $n(\omega) = (e^{2\pi\omega/a} - 1)^{-1}$ and have expressed $\left| \Gamma\left(\frac{i\Omega}{a}\right) \right|^2$ as $\frac{\pi a/\Omega}{\sinh(\pi\Omega/a)}$. This expression has the form $\mathcal{F} = \frac{1}{2} \int_0^\infty R(\omega) \rho(\omega)$, as in Equation (20) but with $\rho(\omega)$ appearing instead of the Lorentz-transformed $\rho'(\omega)$. In our heuristic approach we now argue, as in Section 3, that if the particle has a velocity v the spectral density $\rho(\omega)$ experienced by it should be $\rho'(\omega)$, just as in Equation (17), such that

$$\mathcal{F} = 2\hbar \int_0^\infty d\omega \omega^2 \alpha_I(\omega) \left[n(\gamma\omega(1+v)) + \frac{1}{2} \right]. \tag{55}$$

The net frictional force is then, exactly as in Section 3,

$$\mathcal{F} = 4\hbar v \int_0^\infty d\omega \omega^3 \alpha_I(\omega) \frac{\partial n}{\partial \omega}. \tag{56}$$

Assuming an equilibrium power balance according to Equation (28) we obtain, from Equations (56) and (49), Equation (30), but now with $k_B T = \hbar a/2\pi$ if we assume the energy equipartition relation $\langle \frac{1}{2} m v^2 \rangle = \hbar a/2\pi$ for thermal equilibrium at the Davies-Unruh temperature.

6. Summary

The 2D scalar QED model was used previously [2,3] to address the question of whether there is radiation from a uniformly accelerated oscillator in vacuum. We have focused here not on the radiation, or lack thereof, but on the force on a polarizable particle in thermal equilibrium (at some temperature $T > 0$) with the field or in uniform acceleration in a vacuum. In both cases the momentum fluctuations of the particle depend on the photon number variance having the Bose–Einstein form $n^2(\omega) + n(\omega)$, where $n(\omega)$ is the mean photon number. In the former case this variance reflects the blackbody photon statistics of real photons undergoing absorption and emission, whereas in the latter case it emerges,

along with the Davies-Unruh temperature, as a consequence of the different vacuum field experienced by the particle in hyperbolic motion [19]. The quantum-mechanical momentum fluctuations of the particle imply that only the *average* acceleration is constant when a constant external force acts on the particle. We also demonstrate that the frictional force and momentum fluctuations obtained for hyperbolic motion in vacuum are related to each other via the power balance (fluctuation-dissipation) relation (28).

Author Contributions: Conceptualization, methodology, writing—original draft and preparation, and writing—review and editing, K.S. and P.W.M.; appendices, K.S. All authors have read and agreed to the published version of the manuscript

Funding: Kanu Sinha’s research was supported in part by the National Science Foundation under Grant No. NSF PHY-2309341 and by the John Templeton Foundation under Award No. 62422.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: Peter Milonni has fond memories of discussions with Franco Persico many years ago in Trieste and Erice and thanks the editors for the opportunity to contribute to this Special Issue in his memory.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Equations of Motion

The equations of motion for the charge at $Y = 0$ are given by:

$$\frac{dx}{dt} = \frac{1}{2m}(p_x - e\phi(0, t)), \tag{A1}$$

$$\frac{dp_x}{dt} = -m\omega_0^2 x, \tag{A2}$$

such that

$$\frac{d^2x}{dt^2} + \omega_0^2 x = -\frac{e}{m} \frac{\partial\phi(0, t)}{\partial t}. \tag{A3}$$

Dividing the field into source-free (ϕ_0) and radiation reaction parts (ϕ_{RR}), we get:

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \frac{e}{m} \frac{\partial\phi_{RR}(0, t)}{\partial t} = -\frac{e}{m} \frac{\partial\phi_0(0, t)}{\partial t}, \tag{A4}$$

where a friction term arises explicitly from the radiation reaction part.

The Heisenberg equation of motion for the field operator a_k is

$$i\hbar \frac{da_k}{dt} = \left[a_k, \sum_{k'} \hbar\omega_k a_{k'}^\dagger a_{k'} - \frac{e}{m} \phi(0, t) p_x + \frac{e^2}{m} \phi^2(0, t) \right] \tag{A5}$$

$$\implies \frac{da_k}{dt} = -i\omega_k a_k + i\frac{e}{\hbar} C_k \left(\frac{p_x - e\phi(0, t)}{m} \right) \tag{A6}$$

Using Equation (A1) in the above and integrating over time, we obtain:

$$a_k(t) = a_k(0)e^{-i\omega_k t} + \frac{ie}{\hbar} C_k \int_0^t d\tau \dot{x}(\tau) e^{i\omega_k(t-\tau)} \tag{A7}$$

Substituting the above in Equation (3), one gets the free and source parts of the field as follows

$$\phi(y, t) = \underbrace{\sum_k C_k \left(a_k(0) e^{-i\omega_k t} e^{iky} + a_k^\dagger(0) e^{i\omega_k t} e^{-iky} \right)}_{\phi_0(y, t) \text{ (Free field)}} + \underbrace{\sum_k \frac{2C_k^2 e}{\hbar} \int_0^t d\tau \dot{x}(\tau) \sin(t - \tau)}_{\text{Radiation reaction}} \quad (\text{A8})$$

The radiation reaction part of the field as it appears in the equation of motion for the charge is

$$\frac{\partial \phi_{RR}(0, t)}{\partial t} = \frac{2e}{\hbar} \sum_k C_k^2 \omega_k \int_0^t d\tau \dot{x}(\tau) \cos(\omega_k(t - \tau)) \rightarrow 2\pi e \dot{x}(t), \quad (\text{A9})$$

where in the last step we have taken the continuum limit of the summation ($\sum_k \rightarrow \frac{L}{2\pi} \int dk$). Substitution of the above in Equation (A4) yields Equation (4).

Appendix B. Derivation of $\zeta(\omega_k, \Omega)$ and $\eta(\omega_k, \Omega)$

In this Appendix, we present a derivation of the functions $\zeta(\omega, \Omega)$ and $\eta(\omega, \Omega)$ that are related to the frequency spectrum seen by the accelerated observer [18]. We define the variable $z = e^{a\tau}$, such that, $dz = ae^{a\tau} d\tau$, yielding:

$$\zeta(\omega, \Omega) = \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} e^{i\frac{\omega}{a} e^{a\tau}} = \frac{1}{a} \int_0^{\infty} dz z^{i\Omega/a - 1} e^{-z/(ia/\omega)} \quad (\text{A10})$$

Substituting $\tilde{z} = z/(ia/\omega)$, we obtain

$$\zeta(\omega, \Omega) = \frac{1}{a} \left(\frac{ia}{\omega} \right)^{i\Omega/a} \int_0^{\infty} d\tilde{z} \tilde{z}^{i\Omega/a - 1} e^{-\tilde{z}} = \frac{1}{a} \Gamma\left(\frac{i\Omega}{a}\right) \left(\frac{\omega}{a}\right)^{-i\Omega/a} \left(e^{i\pi/2}\right)^{i\Omega/a}, \quad (\text{A11})$$

which readily gives Equation (41).

Similarly,

$$\begin{aligned} \eta(\omega, \Omega) &= \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau} e^{-i\frac{\omega}{a} e^{a\tau}} = \frac{1}{a} \Gamma\left(\frac{i\Omega}{a}\right) \left(-\frac{\omega}{a}\right)^{-i\Omega/a} e^{-\pi\Omega/2a} \\ &= \frac{1}{a} \Gamma\left(\frac{i\Omega}{a}\right) \left(e^{i\pi} \frac{\omega}{a}\right)^{-i\Omega/a} e^{-\pi\Omega/2a}, \end{aligned} \quad (\text{A12})$$

which yields Equation (42).

References

- O’Connell, R.F. The demise of Unruh radiation. *Mod. Phys. Lett. A* **2020**, *35*, 2050329. [CrossRef]
- Raine, D.J.; Sciamia, D.W.; Grove, P.G. Does a uniformly accelerated charge radiate? *Proc. R. Soc. Lond. A Math. Phys. Eng. Sci.* **1991**, *435*, 205–215. Available online: <https://www.jstor.org/stable/52011> (accessed on 10 January 2024).
- Ford, G.W.; O’Connell, R.F. Is there Unruh radiation? *Phys. Lett. A* **2006**, *350*, 17–26. [CrossRef]
- Davies, P.C.W. Scalar production in Schwarzschild and Rindler metrics. *J. Phys. A Math. Gen.* **1975**, *8*, 609–616. [CrossRef]
- Unruh, W.G. Notes on black hole evaporation. *Phys. Rev. D* **1976**, *14*, 870–892. [CrossRef]
- Einstein, A.; Hopf, L. Statistische Untersuchung der Bewegung eines Resonators in einem Strahlungsfeld. *Ann. d. Phys.* **1910**, *338*, 1105–1115; [CrossRef] English translation: Statistical investigation of a resonator’s motion in a radiation field. In *The Collected Papers of Albert Einstein. Volume 3: The Swiss Years: Writings 1909–1911 (English Translation Supplement)*; Schulmann, R., Kox, A.J., Renn, J., Eds.; Princeton University Press: Princeton, NJ, USA, 1994; pp. 220–230. Available online: <https://einsteinpapers.press.princeton.edu/vol3-trans/234> (accessed on 10 January 2024).

7. Einstein, A. Zur Quantentheorie der Strahlung. *Phys. Z.* **1917**, *18*, 121–128; English translation: On the quantum theory of radiation. In *The Collected Papers of Albert Einstein. Volume 6: The Berlin Years: Writings, 1914–1917 (English Translation Supplement)*; Kox, A.J., Schulmann, R., Eds.; Princeton University Press: Princeton, NJ, USA, 1997; pp. 220–233. Available online: <https://einsteinpapers.press.princeton.edu/vol6-trans/232> (accessed on 10 January 2024).
8. Milton, K.A.; Day, H.; Li, Y.; Guo, X.; Kennedy, G. Self-force on moving electric and magnetic dipoles: Dipole radiation, Vavilov-Cerenkov radiation, friction with a conducting surface, and the Einstein-Hopf effect. *Phys. Rev. Res.* **2020**, *2*, 043347. [[CrossRef](#)]
9. Sinha, K.; Milonni, P.W. Dipoles in blackbody radiation: Momentum fluctuations, decoherence, and drag force. *J. Phys. B At. Mol. Opt. Phys.* **2022**, *55*, 204002. [[CrossRef](#)]
10. Wang, Q.; Unruh, W.G. Motion of a mirror under infinitely fluctuating quantum vacuum stress. *Phys. Rev. D* **2014**, *89*, 085009. [[CrossRef](#)]
11. Sinha, K.; Lin, S.-Y.; Hu, B.L. Mirror-field entanglement in a microscopic model for quantum optomechanics. *Phys. Rev. A* **2015**, *92*, 023852. [[CrossRef](#)]
12. Sinha, K. A Microscopic Model for Quantum Optomechanics. Ph.D Thesis, University of Maryland, College Park, MD, USA, 2015. [[CrossRef](#)]
13. Milonni, P.W. Radiation reaction and the nonrelativistic theory of the electron. *Phys. Lett. A* **1981**, *82*, 225–226. [[CrossRef](#)]
14. Pauli, W. *Theory of Relativity*; Pergamon Press Ltd.: New York, NY, USA, 1958; Section 49. Available online: [https://archive.org/details/theoryofrelativi00paul/](https://archive.org/details/theoryofrelativi00paul/details/theoryofrelativi00paul/) (accessed on 10 January 2024).
15. Ford, G.W.; O’Connell, R.F. Lorentz transformation of blackbody radiation. *Phys. Rev. E* **2013**, *88*, 044101. [[CrossRef](#)] [[PubMed](#)]
16. Milonni, P.W.; Eberly, J.H. *Laser Physics*; John Wiley & Sons, Inc.: New York, NY, USA, 2010; Section 8.2. [[CrossRef](#)]
17. Einstein, A.; Hopf, L. Über einen Satz der Wahrscheinlichkeitsrechnung und seine Anwendung in der Strahlungstheorie. *Ann. d. Phys.* **1910**, *338*, 1096–1104; [[CrossRef](#)] English translation: On a theorem of the probability calculus and its application in the theory of radiation. In *The Collected Papers of Albert Einstein. Volume 3: The Swiss Years: Writings 1909–1911 (English Translation Supplement)*; Schulmann, R., Kox, A.J., Renn, J., Eds.; Princeton University Press: Princeton, NJ, USA, 1994; pp. 211–219. Available online: <https://einsteinpapers.press.princeton.edu/vol3-trans/225> (accessed on 10 January 2024).
18. Alsing, P.M.; Milonni, P.W., Simplified derivation of the Hawking–Unruh temperature for an accelerated observer in vacuum. *Am. J. Phys.* **2004**, *72*, 1524–1529. [[CrossRef](#)]
19. Cetto, A.M.; de la Peña, L. Real vacuum fluctuations and virtual Unruh radiation. *Fortschr. Phys.* **2017**, *65*, 1600039. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.