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A TRAJECTORIAL APPROACH TO THE GRADIENT FLOW PROPERTIES OF LANGEVIN–SMOLUCHOWSKI DIFFUSIONS*

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Abstract. We revisit the variational characterization of conservative diffusion as entropic gradient flow and provide for it a probabilistic interpretation based on stochastic calculus. It was shown by Jordan, Kinderlehrer, and Otto that, for diffusions of Langevin–Smoluchowski type, the Fokker–Planck probability density flow maximizes the rate of relative entropy dissipation, as measured by the distance traveled in the ambient space of probability measures with finite second moments, in terms of the quadratic Wasserstein metric. We obtain novel, stochastic-process versions of these features, valid along almost every trajectory of the diffusive motion in the backwards direction of time, using a very direct perturbation analysis. By averaging our trajectorial results with respect to the underlying measure on path space, we establish the maximal rate of entropy dissipation along the Fokker–Planck flow and measure exactly the deviation from this maximum that corresponds to any given perturbation. A bonus of our trajectorial approach is that it derives the HWI inequality relating relative entropy (H), Wasserstein distance (W), and relative Fisher information (I).

Key words. relative entropy, Wasserstein distance, Fisher information, optimal transport, gradient flow, diffusion processes, time reversal, functional inequalities

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1. Introduction.

We provide a trajectorial interpretation of a seminal result by Jordan, Kinderlehrer, and Otto [31], and present a proof based on stochastic analysis. The basic theme of our approach could be described epigrammatically as "applying Itô calculus to Otto calculus." More precisely, we follow a stochastic analysis approach to the characterization of diffusions of Langevin–Smoluchowski type as entropic gradient flows in Wasserstein space, as in [31]. We provide stronger, trajectorial versions of these results. For consistency and readability we adopt the setting and notation of [31], sometimes quoting almost verbatim from this paper in the remainder of this section.

Along the lines of [31], we consider thus a Fokker–Planck or forward Kolmogorov [36] equation of the form

$$\partial_t p(t,x) = \operatorname{div}\big(\nabla\Psi(x)\, p(t,x)\big) + \frac{1}{2}\Delta p(t,x), \quad (t,x) \in (0,\infty)\times \mathbf{R}^n, \tag{1.1}$$

with initial condition

$$p(0,x) = p^0(x), \qquad x \in \mathbf{R}^n. \tag{1.2}$$

Here, p is a real-valued function defined for $(t,x) \in [0,\infty)\times\mathbf{R}^n$, the function $\Psi\colon \mathbf{R}^n \to [0,\infty)$ is smooth and plays the role of a potential, and p^0 is a probability

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density on $\mathbf{R}^n$. The solution $p(t,x)$ of (1.1) with initial condition (1.2) stays nonnegative and conserves its mass, which means that the spatial integral $\int_{\mathbf{R}^n} p(t,x)\,\mathrm{d}x$ is independent of the time parameter $t \geqslant 0$ and is thus equal to $\int p^0\,\mathrm{d}x = 1$. Therefore, $p(t,\,\cdot\,)$ must be a probability density on $\mathbf{R}^n$ for every fixed time $t \geqslant 0$.

As in [31] we note that the Fokker–Planck equation (1.1) with initial condition (1.2) is inherently related to the stochastic differential equation of Langevin–Smoluchowski type [23], [24], [48], [52]

$$\mathrm{d}X(t) = -\nabla\Psi\big(X(t)\big)\,\mathrm{d}t + \mathrm{d}W(t), \qquad t \geqslant 0. \tag{1.3}$$

In the equation above, $(W(t))_{t\geqslant 0}$ is an n-dimensional Brownian motion started at the origin, and the $\mathbf{R}^n$-valued random variable (r.v.) $X(0)$ is independent of the process $(W(t))_{t\geqslant 0}$. The probability distribution of $X(0)$ has density p^0, and, unless specified otherwise, the reference measure will always be the Lebesgue measure on $\mathbf{R}^n$. Then $p(t,\,\cdot\,)$, the solution of (1.1) with initial condition (1.2), gives at any given time $t \geqslant 0$ the probability density function of the r.v. $X(t)$ from (1.3).

If the potential Ψ grows rapidly enough so that $\mathrm{e}^{-2\Psi} \in L^1(\mathbf{R}^n)$, then the partition constant

$$Z = \int_{\mathbf{R}^n} \mathrm{e}^{-2\Psi(x)}\,\mathrm{d}x \tag{1.4}$$

is finite, and there exists a unique stationary solution of the Fokker–Planck equation (1.1), namely, the probability density q_Z of the *Gibbs distribution* given by (see [24], [30], [48])

$$q_Z(x) = Z^{-1}\,\mathrm{e}^{-2\Psi(x)} \tag{1.5}$$

for $x \in \mathbf{R}^n$. When it exists, the probability measure on $\mathbf{R}^n$ with density function q_Z is called the Gibbs distribution and is the unique invariant measure for the Markov process $(X(t))_{t\geqslant 0}$ defined by the stochastic differential equation (1.3); see, e.g., [35, Exercise 5.6.18, p. 361].

In [30] it is shown that the stationary probability density q_Z satisfies the following variational principle: it minimizes the *free energy functional*

$$\mathscr{F}(p) = \mathcal{E}(p) + \frac{1}{2}\,\mathcal{S}(p) \tag{1.6}$$

over all probability densities p on $\mathbf{R}^n$. Here, the functionals

$$\mathcal{E}(p) := \int_{\mathbf{R}^n} \Psi(x)p(x)\,\mathrm{d}x, \qquad \mathcal{S}(p) := \int_{\mathbf{R}^n} p(x)\ln p(x)\,\mathrm{d}x \tag{1.7}$$

model, respectively, the potential energy and the internal energy (given by the negative of the Gibbs–Boltzmann entropy functional).

Preview. We set up in section 2 the model for the Langevin–Smoluchowski diffusion and introduce its fundamental quantities: the current and the invariant distributions of particles, the resulting likelihood ratio process, the associated concepts of free energy, relative entropy, and relative Fisher information. In subsection 2.1 we discuss the regularity assumptions imposed in the present paper.

Sections 3 and 4 present the basic results. Foremost among these is Theorem 3.1, which computes in terms of the relative Fisher information the rate of decay for the relative entropy in the ambient Wasserstein space of probability density functions with finite second moment, and its "perturbed" counterpart, Theorem 3.2. We compute explicitly the difference between these perturbed and unperturbed rates and show that it is always nonnegative — in fact strictly positive, unless the perturbation and the gradient of the log-likelihood ratio function are collinear. This way, the Langevin–Smoluchowski diffusion emerges as the *steepest descent* (or "gradient flow") of the relative entropy functional with respect to the Wasserstein metric — and the celebrated result of [31] receives a crisp, direct probabilistic treatment via perturbation analysis. This is the main contribution of the present work.

Certain aspects of Theorems 3.1 and 3.2 are well known, and the special case $\Psi(x) = |x|^2/2$ of Ornstein–Uhlenbeck dynamics goes as far back as the 1950s. Our novel contribution here is that Theorems 3.1 and 3.2 are simple consequences of their stronger, trajectorial versions, Theorems 4.1 and 4.2, respectively. These results provide very detailed descriptions for the semimartingale dynamics of the relative entropy process in both its "pure" and "perturbed" forms and are most transparent when time is *reversed*. Theorems 3.1 and 3.2 then follow from Theorems 4.1 and 4.2 simply by taking expectations.

Several consequences and ramifications of Theorems 4.1 and 4.2 are developed in subsections 4.1 and 4.2, including a derivation of the famous HWI inequality of Otto and Villani [45], [56], [57], [14] that relates relative entropy (H) to Wasserstein distance (W) and to relative Fisher information (I). Detailed arguments and proofs are collected in section 5. The limiting behavior of the Wasserstein distance along the Langevin–Smoluchowski diffusion is analyzed in section 6; here, most of the effort goes into showing that relative entropy and Wasserstein distance have exactly the same exceptional sets of zero Lebesgue measure for their temporal rate of change. This, seemingly purely technical, point is of paramount importance for the rigorous justification of the perturbation analysis deployed in Theorem 3.2; it turns out also to involve a rather delicate analysis. We leave the probabilistic derivation of the Wasserstein limits as an interesting open problem.

2. The stochastic approach. In section 1 we primarily quoted from the paper [31]. We adopt now a more probabilistic point of view and translate our setting into the language of stochastic processes and probability measures.

Let $P(0)$ be a probability measure on the Borel sets of $\mathbf{R}^n$ with density function $p^0 = p(0, \cdot)$. This measure induces a probability measure $\mathbf{P}$ on the path space $\Omega = \mathcal{C}(\mathbf{R}_+; \mathbf{R}^n)$ of $\mathbf{R}^n$-valued continuous functions on $\mathbf{R}_+ = [0,\infty)$, under which the canonical coordinate process $(X(t,\omega))_{t\geqslant 0} = (\omega(t))_{t\geqslant 0}$ satisfies the stochastic differential equation (1.3) with initial probability distribution $P(0)$. We denote by $P(t)$ the probability distribution of the random vector $X(t)$ under $\mathbf{P}$, and by $p(t) \equiv p(t, \cdot)$ the corresponding probability density function, at each time $t \geqslant 0$. This function solves the equation (1.1) with initial condition (1.2).

An important role will be played by the *Radon–Nikodým derivative*, or *likelihood ratio process*,

$$\frac{\mathrm{d}P(t)}{\mathrm{dQ}}\big(X(t)\big) = \ell\big(t, X(t)\big), \quad \text{where} \quad \ell(t,x) := \frac{p(t,x)}{q(x)} = p(t,x)\mathrm{e}^{2\Psi(x)} \tag{2.1}$$

for $t \geqslant 0$ and $x \in \mathbf{R}^n$. Here and throughout, we denote by Q the σ-finite measure on the Borel sets of $\mathbf{R}^n$, whose density with respect to Lebesgue measure is

$$q(x) := \mathrm{e}^{-2\Psi(x)}, \qquad x \in \mathbf{R}^n. \tag{2.2}$$

The *relative entropy* and the *relative Fisher information* (see, e.g., [45], [15]) of $P(t)$ with respect to this measure Q, are defined, respectively, as

$$H\big(P(t)\,|\,\mathrm{Q}\big) := \mathbf{E_P}\big[\ln \ell\big(t, X(t)\big)\big] = \int_{\mathbf{R}^n} \ln\bigg(\frac{p(t,x)}{q(x)}\bigg) p(t,x)\,\mathrm{d}x, \qquad t \geqslant 0, \tag{2.3}$$

$$I\big(P(t)\,|\,\mathrm{Q}\big) := \mathbf{E_P}\Big[\big|\nabla \ln \ell\big(t, X(t)\big)\big|^2\Big] = \int_{\mathbf{R}^n} |\nabla \ln \ell(t,x)|^2 p(t,x)\,\mathrm{d}x, \qquad t \geqslant 0. \tag{2.4}$$

It follows from section 3 in [38] (see also Appendix C in [34]) that the relative entropy $H(P\,|\,\mathrm{Q})$ is well defined and takes values in $(-\infty,\infty]$ if the probability measure P has finite second moment. The latter is always the case in our paper.

Direct computation reveals that, along the curve of probability measures $(P(t))_{t\geqslant 0}$, the free energy functional (1.6) and the relative entropy (2.3) are related for each $t \geqslant 0$ through the equation

$$2\mathscr{F}\big(p(t,\,\cdot\,)\big) = H\big(P(t)\;\big|\;\mathrm{Q}\big). \tag{2.5}$$

This shows that studying the decay of the free energy $t \mapsto \mathscr{F}(p(t,\,\cdot\,))$ is equivalent to studying the decay of the relative entropy $t \mapsto H(P(t)\,|\,\mathrm{Q})$, a key aspect of thermodynamics. In light of condition (ii) in Assumptions 2.1 below, the identity (2.5) implies that $H(P(0)\,|\,\mathrm{Q})$ is finite, so the quantity in (2.3) is finite for $t = 0$; thus, on account of (4.13) below, it is finite also for $t > 0$.

2.1. Regularity assumptions. In order to provide mathematically precise formulations of subsequent results, we have to specify convenient regularity assumptions. These issues are of a rather technical nature, and subsection 2.1 may be skipped on a first reading of this paper.

By analogy with [31, Theorem 5.1] we consider the following assumptions.

Assumptions 2.1. (i) The potential $\Psi\colon \mathbf{R}^n \to [0,\infty)$ is of the class $\mathcal{C}^\infty(\mathbf{R}^n;[0,\infty))$.

(ii) The distribution $P(0)$ of $X(0)$ in (1.3) has probability density function $p^0 = p(0,\,\cdot\,)$ with respect to Lebesgue measure on $\mathbf{R}^n$, with finite second moment and free energy, i.e.,

$$\int_{\mathbf{R}^n} p^0(x)|x|^2\,\mathrm{d}x < \infty \quad \text{and} \quad \mathscr{F}(p^0) = \frac{1}{2}\,H\big(P(0)\,|\,\mathrm{Q}\big) \in (-\infty,\infty). \tag{2.6}$$

In [31] it is also assumed that the potential Ψ satisfies, for some real constant $C > 0$, the bound $|\nabla\Psi| \leqslant C\,(\Psi + 1)$, which we do not need here. Instead of this requirement, we impose the following rather weak assumptions.

Assumptions 2.2 (regularity assumptions for the trajectorial results of the present paper). In addition to conditions (i) and (ii) of Assumptions 2.1, we also impose the following:

(iii) The potential Ψ satisfies, for some real constants $c \geqslant 0$ and $R \geqslant 0$, the drift (or coercivity) condition

$$\forall\, x \in \mathbf{R}^n, \quad |x| \geqslant R\colon \quad \langle x, \nabla\Psi(x)\rangle \geqslant -c\,|x|^2. \tag{2.7}$$

(iv) The potential Ψ is sufficiently well-behaved to guarantee that the solution $(t,x) \mapsto p(t,x)$ of (1.1) with initial condition (1.2) is continuous and strictly positive on $(0,\infty)\times\mathbf{R}^n$, differentiable with respect to the time variable t for each $x \in \mathbf{R}^n$, and smooth in the space variable x for each $t > 0$. We also assume that the logarithmic derivative $(t,x) \mapsto \nabla \ln p(t,x)$ is continuous on $(0,\infty) \times \mathbf{R}^n$.[1]

For the formulation of Theorem 3.2 we will need a vector field $\beta\colon \mathbf{R}^n \to \mathbf{R}^n$, which is the gradient of a potential $B\colon \mathbf{R}^n \to \mathbf{R}$ satisfying the following regularity assumption:

(v) The potential $B\colon \mathbf{R}^n \to \mathbf{R}$ is of class $\mathcal{C}^\infty(\mathbf{R}^n;\mathbf{R})$ and has compact support. Consequently, its gradient $\beta := \nabla B\colon \mathbf{R}^n \to \mathbf{R}^n$ is of class $\mathcal{C}^\infty(\mathbf{R}^n;\mathbf{R}^n)$ and again compactly supported. We also assume that, for every such β, the perturbed potential $\Psi + B$ satisfies condition (iv).

Assumptions 2.2 are satisfied by typical convex potentials Ψ. They also accommodate examples such as double-well potentials of the form $\Psi(x) = (x^2 - \alpha^2)^2$ on the real line for real constants $\alpha > 0$. It is important to point out that these assumptions do not rule out the case when the constant Z in (1.4) is infinite; thus, they allow for cases (such as $\Psi \equiv 0$) in which the stationary probability density function q_Z in (1.5) does not exist. In fact, in [31] the authors point out explicitly that, even when the stationary probability density q_Z is not defined, the free energy (1.6) of a density $p(t,x)$ satisfying the Fokker–Planck equation (1.1) with initial condition (1.2) can be defined, provided that the free energy $\mathscr{F}(p^0)$ is finite. Furthermore, we note that Assumptions 2.2 are designed in such a way that they are invariant when passing from the potential Ψ to $\Psi + B$ if B satisfies condition (v).

Under Assumptions 2.2, the Langevin–Smoluchowski diffusion equation (1.3) with initial distribution $P(0)$ admits a pathwise unique strong solution satisfying $P(t) \in \mathscr{P}_2(\mathbf{R}^n)$ for all $t \geqslant 0$; here $\mathscr{P}_2(\mathbf{R}^n)$ is the set of probability measures on the Borel sets of $\mathbf{R}^n$ with finite second moment. Indeed, the drift condition (2.7) guarantees that the second-moment condition in (2.6) propagates in time, i.e.,

$$\forall\, t \geqslant 0\colon \quad \int_{\mathbf{R}^n} p(t,x)|x|^2 \,\mathrm{d}x < \infty; \tag{2.8}$$

see [23, Theorem 2.2] and the first problem on page 125 of [23], as well as Appendix B in [34], for a solution to this problem.

Assumptions 2.3 (regularity assumptions regarding the Wasserstein distance). In addition to conditions (i)–(v) of Assumptions 2.2, we require the following:

(vi) For every $t \geqslant 0$, there exists a sequence of functions $(\varphi_m(t,\cdot\,))_{m\geqslant 1} \subseteq \mathcal{C}_c^\infty(\mathbf{R}^n;\mathbf{R})$, whose gradients $(\nabla\varphi_m(t,\cdot\,))_{m\geqslant 1}$ converge in $L^2(P(t))$ to the velocity field $v(t,\cdot\,) = \nabla\varphi(t,\cdot\,)$ of gradient type as in (6.1) with $\varphi(t,x) = -\Psi(x) - (1/2)\ln p(t,x)$, as $m \to \infty$.

This last requirement guarantees, for every $t \geqslant 0$, that the velocity field $v(t,\cdot\,)$ is an element of the tangent space of $\mathscr{P}_2(\mathbf{R}^n)$ at the point $P(t) \in \mathscr{P}_2(\mathbf{R}^n)$ in the sense of [4, Definition 8.4.1]. For details we refer the reader to section 6 below, in particular, the display (6.2). We do not know whether this condition (vi) in Assumptions 2.3 is actually an additional requirement or whether it is automatically satisfied in our setting. But as this issue only affects the Wasserstein distance, and has no relevance for our trajectorial results Theorems 4.1 and 4.2, we will not pursue this issue further here.

[1]For example, by requiring that all derivatives of Ψ grow at most exponentially as $|x|$ tends to infinity, one may adapt the arguments from [49] showing that this is indeed the case.

The condition (vi) in Assumptions 2.3 is satisfied by simple potentials such as, for example, $\Psi \equiv 0$ or $\Psi(x) = |x|^2/2$. More generally, this condition is satisfied by the potentials with a curvature lower bound $\mathrm{Hess}(\Psi) \geqslant \kappa\, I_n$ for some $\kappa \in \mathbf{R}$ (as in (4.45) below), for instance, the double-well potential $\Psi(x) = (x^2 - \alpha^2)^2$ on the real line; more on this can be found in [4, Theorem 10.4.13], as was kindly pointed out to us by Luigi Ambrosio.

3. The main theorems in aggregate form. In light of (2.5), the goal of [31] is to relate the decay of the *relative entropy functional*

$$\mathscr{P}_2(\mathbf{R}^n) \ni P \longmapsto H(P\,|\,\mathrm{Q}) \in (-\infty, \infty], \tag{3.1}$$

along the curve $(P(t))_{t \geqslant 0}$, to the quadratic Wasserstein distance

$$W_2(\mu,\nu) = \Big(\inf_{Y\sim\mu,\, Z\sim\nu} \mathbf{E}|Y - Z|^2\Big)^{1/2}, \qquad \mu,\nu \in \mathscr{P}_2(\mathbf{R}^n), \tag{3.2}$$

on $\mathscr{P}_2(\mathbf{R}^n)$ (cf. [56], [4], [3]). We resume the remarkable relation between these two quantities in the following two theorems; these quantify the relationship between displacement in the ambient space (the denominator in (3.5)) and fluctuations of the free energy or, equivalently, of the relative entropy (the numerator in (3.5)). The proofs will be given in subsection 4.1 below.

THEOREM 3.1. *Under Assumptions* 2.3, *the relative Fisher information* $I(P(t_0)\,|\,\mathrm{Q})$ *is finite for Lebesgue-a.e.* $t_0 \geqslant 0$, *and we have the* generalized de Bruijn identity

$$\lim_{t\to t_0} \frac{H\big(P(t)\,|\,\mathrm{Q}\big) - H\big(P(t_0)\,|\,\mathrm{Q}\big)}{t - t_0} = -\frac{1}{2}\, I\big(P(t_0)\,|\,\mathrm{Q}\big), \tag{3.3}$$

as well as the limiting behavior of the quadratic Wasserstein distance

$$\lim_{t\to t_0} \frac{W_2\big(P(t), P(t_0)\big)}{|t - t_0|} = \frac{1}{2}\sqrt{I\big(P(t_0)\,|\,\mathrm{Q}\big)}, \tag{3.4}$$

so that

$$\lim_{t\to t_0} \left(\mathrm{sgn}(t - t_0) \cdot \frac{H\big(P(t)\,|\,\mathrm{Q}\big) - H\big(P(t_0)\,|\,\mathrm{Q}\big)}{W_2\big(P(t), P(t_0)\big)} \right) = -\sqrt{I\big(P(t_0)\,|\,\mathrm{Q}\big)}. \tag{3.5}$$

Furthermore, if $t_0 \geqslant 0$ *is chosen so that the generalized de Bruijn identity* (3.3) *does hold, then the limiting assertions* (3.4) *and* (3.5) *are also valid.*

The ratio on the left-hand side of (3.5) can be interpreted as the rate of decay for the relative entropy functional (3.1) at $P = P(t_0)$ along the curve $(P(t))_{t\geqslant 0}$ if distances in the ambient space $\mathscr{P}_2(\mathbf{R}^n)$ are measured by the quadratic Wasserstein distance W_2. The quantity appearing on the right-hand side of (3.5) is the square root of the relative Fisher information in (2.4), written more explicitly in terms of the "score function" $\nabla\ell(t,\,\cdot\,)/\ell(t,\,\cdot\,)$ as

$$I\big(P(t)\,|\,\mathrm{Q}\big) = \mathbf{E_P}\left[\frac{\big|\nabla\ell\big(t, X(t)\big)\big|^2}{\ell\big(t, X(t)\big)^2}\right] = \int_{\mathbf{R}^n} \left|\frac{\nabla p(t,x)}{p(t,x)} + 2\nabla\Psi(x)\right|^2 p(t,x)\,\mathrm{d}x. \tag{3.6}$$

For future reference, we denote by N the set of exceptional points $t_0 \geqslant 0$ for which the right-hand side version of the limit in (3.3), i.e., the limiting assertion

$$\lim_{t \downarrow t_0} \frac{H\big(P(t)\,|\,\mathrm{Q}\big) - H\big(P(t_0)\,|\,\mathrm{Q}\big)}{t - t_0} = -\frac{1}{2}\, I\big(P(t_0)\,|\,\mathrm{Q}\big), \tag{3.7}$$

fails. According to Theorem 3.1, this exceptional set N has zero Lebesgue measure.

The remarkable insight of [31] states that the rate of entropy decay (3.5) along the curve $(P(t))_{t \geqslant 0}$ is, in fact, the slope of *steepest descent* for the relative entropy functional (3.1) with respect to the Wasserstein distance W_2 at the point $P = P(t_0)$ on the curve. To formalize this assertion, we fix a time $t_0 \geqslant 0$ and let the vector field $\beta = \nabla B \colon \mathbf{R}^n \to \mathbf{R}^n$ be the gradient of a potential B, as in condition (v) of Assumptions 2.2. This gradient vector field β will serve as a perturbation in

$$\partial_t p^\beta(t,x) = \operatorname{div}\big(\big(\nabla\Psi(x) + \beta(x)\big)\, p^\beta(t,x)\big) + \frac{1}{2}\Delta p^\beta(t,x), \qquad (t,x) \in (t_0,\infty) \times \mathbf{R}^n, \tag{3.8}$$

and thus the perturbed Fokker–Planck equation with initial condition

$$p^\beta(t_0,x) = p(t_0,x), \qquad x \in \mathbf{R}^n. \tag{3.9}$$

We denote by $\mathbf{P}^\beta$ the probability measure on the path space $\Omega = \mathcal{C}([t_0,\infty);\mathbf{R}^n)$, under which the canonical coordinate process $(X(t))_{t \geqslant t_0}$ satisfies the stochastic differential equation

$$\mathrm{d}X(t) = -\big(\nabla\Psi\big(X(t)\big) + \beta\big(X(t)\big)\big)\,\mathrm{d}t + \mathrm{d}W^\beta(t), \qquad t \geqslant t_0, \tag{3.10}$$

with initial probability distribution $P(t_0)$. Here, the process $(W^\beta(t))_{t \geqslant t_0}$ is a Brownian motion under $\mathbf{P}^\beta$. The probability distribution of $X(t)$ under $\mathbf{P}^\beta$ on $\mathbf{R}^n$ will be denoted by $P^\beta(t)$ for $t \geqslant t_0$; as before, the corresponding probability density function $p^\beta(t) \equiv p^\beta(t,\,\cdot\,)$ solves (3.8) subject to the initial condition (3.9).

After these preparations we can state the result formalizing the gradient flow, or *steepest descent*, property of the curve $(P(t))_{t \geqslant 0}$ generated by the Langevin–Smoluchowski diffusion (1.3) in the ambient space of probability measures $\mathscr{P}_2(\mathbf{R}^n)$ endowed with the quadratic Wasserstein metric.

THEOREM 3.2. *Under Assumptions* 2.3, *the following assertions hold for every point* $t_0 \in \mathbf{R}_+ \setminus N$ (*at which the right-hand side limiting identity* (3.7) *is valid*): *the* $\mathbf{R}^n$*-valued random vectors*

$$a := \nabla \ln \ell\big(t_0, X(t_0)\big) = \nabla \ln p\big(t_0, X(t_0)\big) + 2\,\nabla\Psi\big(X(t_0)\big), \qquad b := \beta\big(X(t_0)\big), \tag{3.11}$$

are elements of the Hilbert space $L^2(\mathbf{P})$, *and the perturbed version of the generalized de Bruijn identity* (3.3) *reads*

$$\begin{aligned} \lim_{t \downarrow t_0} \frac{H\big(P^\beta(t)\,|\,\mathrm{Q}\big) - H\big(P^\beta(t_0)\,|\,\mathrm{Q}\big)}{t - t_0} &= -\frac{1}{2}\, I\big(P(t_0)\,|\,\mathrm{Q}\big) - \langle a, b\rangle_{L^2(\mathbf{P})} \\ &= -\frac{1}{2}\,\langle a, a + 2b\rangle_{L^2(\mathbf{P})}. \end{aligned} \tag{3.12}$$

The limiting behavior of the quadratic Wasserstein distance (3.4) *in this perturbed context is given by*

$$\lim_{t \downarrow t_0} \frac{W_2\big(P^\beta(t), P^\beta(t_0)\big)}{t - t_0} = \frac{1}{2}\,\|a + 2b\|_{L^2(\mathbf{P})}. \tag{3.13}$$

Combining (3.12) *with* (3.13), *and assuming* $a+2b \neq 0$, *we have*

$$\lim_{t \downarrow t_0} \frac{H\big(P^\beta(t) \,|\, \mathrm{Q}\big) - H\big(P^\beta(t_0) \,|\, \mathrm{Q}\big)}{W_2\big(P^\beta(t), P^\beta(t_0)\big)} = -\Big\langle a,\, \frac{a+2b}{\|a+2b\|_{L^2(\mathbf{P})}} \Big\rangle_{L^2(\mathbf{P})}, \tag{3.14}$$

and therefore

$$\lim_{t \downarrow t_0} \bigg(\frac{H\big(P^\beta(t) \,|\, \mathrm{Q}\big) - H\big(P^\beta(t_0) \,|\, \mathrm{Q}\big)}{W_2\big(P^\beta(t), P^\beta(t_0)\big)} - \frac{H\big(P(t) \,|\, \mathrm{Q}\big) - H\big(P(t_0) \,|\, \mathrm{Q}\big)}{W_2\big(P(t), P(t_0)\big)} \bigg) \tag{3.15}$$

$$= \|a\|_{L^2(\mathbf{P})} - \Big\langle a,\, \frac{a+2b}{\|a+2b\|_{L^2(\mathbf{P})}} \Big\rangle_{L^2(\mathbf{P})}. \tag{3.16}$$

In view of the Cauchy–Schwarz inequality, the expression in (3.16) is nonnegative and vanishes if and only if $a+2b$ is a positive multiple of a. Consequently, when the vector field β is not a scalar multiple of $\nabla \ln \ell(t_0,\,\cdot\,)$, the difference between the two slopes in (3.15) is strictly positive. In other words, the slope quantified by the first term of the difference (3.15) is then strictly bigger than the (negative) slope expressed by the second term of (3.15).

These two theorems are essentially well known. They build upon a vast amount of previous work. In the quadratic case $\Psi(x) = |x|^2/2$, i.e., when the process $(X(t))_{t \geqslant 0}$ in (1.3) is Ornstein–Uhlenbeck, with invariant measure in (1.5) being standard Gaussian, the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} H\big(P(t) \,|\, \mathrm{Q}\big) = -\frac{1}{2}\, I\big(P(t) \,|\, \mathrm{Q}\big) \tag{3.17}$$

has been known since [53] as the *de Bruijn identity.* This relationship between the two fundamental information measures, due to Shannon and Fisher, respectively, is a dominant theme in many aspects of information theory and probability. We refer the reader to the book [15] by Cover and Thomas for an account of the results by Barron, Blachman, Brown, Linnik, Rényi, Shannon, Stam, and many others; in a similar vein, see also the seminal work [7] by Bakry and Émery, as well as the paper [40] by Markowich and Villani and the book [56] by Villani. Consult also Carlen and Soffer [13] and Johnson [29] on the relation of (3.17) to the central limit theorem. For connections with large deviations we refer the reader to [2] and [18]. In [21] related pathwise results are obtained for the case $\Psi \equiv 0$, i.e., when the process $(X(t))_{t \geqslant 0}$ is Brownian motion.

The paper [31] broke new ground in this respect, as it considered a general potential Ψ and established the relation to the quadratic Wasserstein distance, culminating with the characterization of the curve $(P(t))_{t \geqslant 0}$ as a gradient flow. This relation was further investigated by Otto in the paper [44], where the theory now known as *Otto calculus* was developed. For a recent application of Otto calculus to the Schrödinger problem, see [25].

The statements of our Theorems 3.1 and 3.2 complement the existing results in some details, e.g., the precise form (3.16), measuring the difference of the two slopes appearing in (3.15). The main novelty of our approach, however, will only become apparent below with the formulation of Theorems 4.1 and 4.2, the trajectorial versions of Theorems 3.1 and 3.2.

4. The main theorems in trajectorial form. Our main goal is to investigate Theorems 3.1 and 3.2 in a trajectorial fashion by considering the *relative entropy process*

$$\ln \ell\big(t, X(t)\big) = \ln\left(\frac{p\big(t, X(t)\big)}{q\big(X(t)\big)}\right) = \ln p\big(t, X(t)\big) + 2\Psi\big(X(t)\big), \qquad t \geqslant 0, \tag{4.1}$$

along each trajectory of the canonical coordinate process $(X(t))_{t\geqslant 0}$, and calculating its dynamics (stochastic differential) under the probability measure $\mathbf{P}$. The $\mathbf{P}$-expectation of this quantity is, of course, the relative entropy in (2.3). A decisive tool in the analysis of the relative entropy process (4.1) is to *reverse time*, and use a remarkable insight due to Pavon [47] and Fontbona and Jourdain [22]. These authors consider the canonical coordinate process $(X(t))_{0\leqslant t\leqslant T}$ on the path space $\Omega = \mathcal{C}([0,T]; \mathbf{R}^n)$ in the reverse direction of time, i.e., they work with the time-reversed process $(X(T-s))_{0\leqslant s\leqslant T}$; it is then notationally convenient to consider a finite time interval $[0,T]$ rather than $\mathbf{R}_+$. For another application of time reversal in a similar context, see [39].

At this stage it becomes important to specify the relevant filtrations. So, we denote by $(\mathcal{F}(t))_{t\geqslant 0}$ the smallest continuous filtration to which the canonical coordinate process $(X(t))_{t\geqslant 0}$ is adapted. That is, modulo $\mathbf{P}$-augmentation, we have

$$\mathcal{F}(t) = \sigma\big(X(u) \colon 0 \leqslant u \leqslant t\big), \qquad t \geqslant 0; \tag{4.2}$$

$(\mathcal{F}(t))_{t\geqslant 0}$ is called the *filtration generated by* $(X(t))_{t\geqslant 0}$. Likewise, we let $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T}$ be the "filtration generated by the time-reversed canonical coordinate process $(X(T-s))_{0\leqslant s\leqslant T}$" in the same sense as before. In other words,

$$\mathcal{G}(T-s) = \sigma\big(X(T-u) \colon 0 \leqslant u \leqslant s\big), \qquad 0 \leqslant s \leqslant T, \tag{4.3}$$

modulo $\mathbf{P}$-augmentation. For the necessary measure-theoretic operations that ensure the continuity (from both the left and right) of filtrations associated with continuous processes, the reader may consult section 2.7 in [35] — in particular, Problems 7.1–7.6 and Proposition 7.7.

Theorems 4.1 and 4.2 are the main new results of this paper. They can be regarded as trajectorial versions of Theorems 3.1 and 3.2, whose proofs will follow from Theorems 4.1 and 4.2 simply by taking expectations. Similar trajectorial approaches have already been applied to the temporal dissipation of relative entropy and Fisher information [16], [47], [22], to the theory of optimal stopping [17], to the Doob martingale inequalities [1], and to the Burkholder–Davis–Gundy inequality [8]. An analogue of Theorem 4.1 in a different setting is provided by Theorem 1.4 in [22]: it is formulated under a probability measure $\mathbf{Q}$ on the path space, induced by a *probability measure* Q on $\mathbf{R}^n$. In our context, it is crucial to allow for an invariant Gibbs measure Q which has possibly infinite mass. Our emphasis then lies in establishing the square-integrability of the process (4.5) and in proving (4.7) under the mild assumption that the initial relative entropy $H(P(0)\,|\,\mathrm{Q})$ is finite.

The significance of Theorem 4.1 below is that the trade-off between the temporal decay of relative entropy, and the temporal growth of the quadratic Wasserstein distance along the curve of probability measures $(P(t))_{t\geqslant 0}$, both of which are characterized in terms of the cumulative relative Fisher information process, is valid not only in expectation but also along (almost) every trajectory, provided we run time in the reverse direction.[2]

[2]As David Kinderlehrer kindly pointed out to the second author, the implicit Euler scheme used in [31] also reflects the idea of going back in time at each step of the discretization.

THEOREM 4.1. *Under Assumptions* 2.2, *we fix* $T \in (0,\infty)$ *and define the cumulative relative Fisher information process, accumulated from the right, as*

$$F(T-s) := \int_0^s \frac{1}{2} \frac{\big|\nabla \ell\big(T-u, X(T-u)\big)\big|^2}{\ell\big(T-u, X(T-u)\big)^2}\,\mathrm{d}u$$

$$= \int_0^s \frac{1}{2} \left| \frac{\nabla p\big(T-u, X(T-u)\big)}{p\big(T-u, X(T-u)\big)} + 2\nabla\Psi\big(X(T-u)\big)\right|^2 \mathrm{d}u \tag{4.4}$$

for $0 \leqslant s \leqslant T$. *Then the process*

$$M(T-s) := \Big(\ln \ell\big(T-s, X(T-s)\big) - \ln \ell\big(T, X(T)\big)\Big) - F(T-s), \qquad 0 \leqslant s \leqslant T, \tag{4.5}$$

is a square-integrable martingale of the backwards filtration $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T}$ *under the probability measure* $\mathbf{P}$. *More explicitly, the martingale of* (4.5) *can be represented as*

$$M(T-s) = \int_0^s \left\langle \frac{\nabla \ell\big(T-u, X(T-u)\big)}{\ell\big(T-u, X(T-u)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}}(T-u) \right\rangle, \qquad 0 \leqslant s \leqslant T, \tag{4.6}$$

for a $\mathbf{P}$*-Brownian motion* $\big(\overline{W}^{\mathbf{P}}(T-s)\big)_{0\leqslant s\leqslant T}$ *of the backwards filtration* $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T}$. *In particular, the quadratic variation of the martingale of* (4.5) *is given by the nondecreasing process in* (4.4), *up to the multiplicative factor of* $1/2$, *and we have*

$$H\big(P(0)\,|\,\mathrm{Q}\big) - H\big(P(T)\,|\,\mathrm{Q}\big) = \mathbf{E}_{\mathbf{P}}[F(0)] = \frac{1}{2}\int_0^T I\big(P(t)\,|\,\mathrm{Q}\big)\,\mathrm{d}t < \infty. \tag{4.7}$$

Next, we state the trajectorial version of Theorem 3.2 or, equivalently, the "perturbed" analogue of Theorem 4.1. As we did in Theorem 3.2, in particular in the preceding equations (3.8)–(3.10), we consider the perturbation $\beta\colon \mathbf{R}^n \to \mathbf{R}^n$ and denote the *perturbed likelihood ratio function* by

$$\ell^\beta(t,x) := \frac{p^\beta(t,x)}{q(x)} = p^\beta(t,x)\,\mathrm{e}^{2\Psi(x)}, \qquad (t,x) \in [t_0,\infty)\times\mathbf{R}^n. \tag{4.8}$$

The stochastic analogue of this quantity is the *perturbed likelihood ratio process* $\ell^\beta(t, X(t))$, $t \geqslant t_0$. The logarithm of this process is the *perturbed relative entropy process*

$$\ln \ell^\beta\big(t, X(t)\big) = \ln\left(\frac{p^\beta\big(t, X(t)\big)}{q\big(X(t)\big)}\right) = \ln p^\beta\big(t, X(t)\big) + 2\Psi\big(X(t)\big), \qquad t \geqslant t_0. \tag{4.9}$$

THEOREM 4.2. *Under Assumptions* 2.2, *let* $t_0 \geqslant 0$ *and* $T > t_0$. *We define the perturbed cumulative relative Fisher information process, accumulated from the right, as*

$$F^\beta(T-s) := \int_0^s \left(\frac{1}{2}\frac{\big|\nabla \ell^\beta\big(T-u, X(T-u)\big)\big|^2}{\ell^\beta\big(T-u, X(T-u)\big)^2} + \big(\langle \beta, 2\nabla\Psi\rangle - \operatorname{div}\beta\big)\big(X(T-u)\big)\right)\mathrm{d}u \tag{4.10}$$

for $0 \leqslant s \leqslant T - t_0$. *Then* $\mathbf{E}_{\mathbf{P}^\beta}\big[F^\beta(t_0)\big] < \infty$, *and, for* $0 \leqslant s \leqslant T-t_0$, *the process*

$$M^\beta(T-s) := \Big(\ln \ell^\beta\big(T-s, X(T-s)\big) - \ln \ell^\beta\big(T, X(T)\big)\Big) - F^\beta(T-s) \tag{4.11}$$

is a square-integrable martingale of the backwards filtration $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T-t_0}$ *under the probability measure* $\mathbf{P}^\beta$. *More explicitly, the martingale* (4.11) *can be represented as*

$$M^\beta(T-s) = \int_0^s \left\langle \frac{\nabla \ell^\beta\big(T-u, X(T-u)\big)}{\ell^\beta\big(T-u, X(T-u)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}^\beta}(T-u)\right\rangle, \quad 0 \leqslant s \leqslant T-t_0, \tag{4.12}$$

for a $\mathbf{P}^\beta$*-Brownian motion* $\big(\overline{W}^{\mathbf{P}^\beta}(T-s)\big)_{0\leqslant s\leqslant T-t_0}$ *of the filtration* $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T-t_0}$.

4.1. Consequences of the trajectorial results. Before tackling the proofs of Theorems 4.1 and 4.2, we state several important consequences of these two basic results. In particular, we indicate how the corresponding assertions in the earlier Theorems 3.1 and 3.2 follow directly from these results by taking expectations.

Corollary 4.3 (dissipation of relative entropy). *Under Assumptions* 2.3, *we have for all* $t, t_0 \geqslant 0$ *the relative entropy identity*

$$\begin{aligned} H\big(P(t)\,|\,\mathrm{Q}\big) - H\big(P(t_0)\,|\,\mathrm{Q}\big) &= \mathbf{E}_\mathbf{P}\left[\ln\left(\frac{\ell\big(t, X(t)\big)}{\ell\big(t_0, X(t_0)\big)}\right)\right] \\ &= \mathbf{E}_\mathbf{P}\left[\int_{t_0}^t \left(-\frac{1}{2}\frac{\big|\nabla\ell\big(u, X(u)\big)\big|^2}{\ell\big(u, X(u)\big)^2}\right)\mathrm{d}u\right]. \end{aligned} \tag{4.13}$$

Furthermore, we have for Lebesgue-a.e. $t_0 \geqslant 0$ *the generalized de Bruijn identity*

$$\lim_{t\to t_0} \frac{H\big(P(t)\,|\,\mathrm{Q}\big) - H\big(P(t_0)\,|\,\mathrm{Q}\big)}{t-t_0} = -\frac{1}{2}\,\mathbf{E}_\mathbf{P}\left[\frac{\big|\nabla\ell\big(t_0, X(t_0)\big)\big|^2}{\ell\big(t_0, X(t_0)\big)^2}\right], \tag{4.14}$$

as well as the limiting behavior of the quadratic Wasserstein distance

$$\lim_{t\to t_0} \frac{W_2\big(P(t), P(t_0)\big)}{|t-t_0|} = \frac{1}{2}\left(\mathbf{E}_\mathbf{P}\left[\frac{\big|\nabla\ell\big(t_0, X(t_0)\big)\big|^2}{\ell\big(t_0, X(t_0)\big)^2}\right]\right)^{1/2}. \tag{4.15}$$

If $t_0 \geqslant 0$ *is chosen so that the generalized de Bruijn identity* (4.14) *holds, then the limiting assertion* (4.15) *pertaining to the Wasserstein distance is also valid.*

Proof of Corollary 4.3 *from Theorem* 4.1. The identity (4.13) follows by taking expectations in (4.6) with respect to the probability measure $\mathbf{P}$, recalling the definitions (4.4), (4.5), and invoking the martingale property of the process in (4.5) for $T \geqslant \max\{t_0, t\}$. In particular, (4.13) shows that the relative entropy function $t \mapsto H(P(t)\,|\,\mathrm{Q})$ from (2.3), and thus also the free energy function $t \mapsto \mathscr{F}(p(t, \cdot\,))$ from (2.5), are strictly decreasing provided $\ell(t, \cdot\,)$ is not constant.

By the Lebesgue differentiation theorem, the monotone function $t \mapsto H(P(t)\,|\,\mathrm{Q})$ is differentiable for Lebesgue-a.e. $t_0 \geqslant 0$, in which case (4.13) leads to the identity (4.14).

The limiting behavior (4.15) of the Wasserstein distance, for Lebesgue-a.e. $t_0 \geqslant 0$, is well known and worked out in [4]; section 6 below provides details. Theorem 6.1 establishes the important, novel aspect of Corollary 4.3, namely, its last assertion that the validity of (4.14) for some $t_0 \geqslant 0$ implies that the limiting assertion (4.15) also holds for the same point t_0. This seemingly harmless issue is actually quite delicate and will be of crucial importance for our gradient flow analysis; it is here that we have to rely on condition (vi) of Assumptions 2.3. Corollary 4.3 is proved.

Proof of Theorem 3.1 *from Theorem* 4.1. This result is a direct consequence of Corollary 4.3.

In a manner similar to the derivation of Corollary 4.3 from Theorem 4.1, we deduce now from Theorem 4.2 the following Corollary 4.4. Its first identity (4.16) shows, in particular, that the relative entropy $H(P^\beta(t)\,|\,\mathrm{Q})$ is finite for all $t \geqslant t_0$.

COROLLARY 4.4 (dissipation of relative entropy under perturbations). *Under Assumptions* 2.3, *we have, for all* $t \geqslant t_0 \geqslant 0$, *the relative entropy identity*

$$H\big(P^\beta(t)\,|\,\mathrm{Q}\big) - H\big(P^\beta(t_0)\,|\,\mathrm{Q}\big) = \mathbf{E}_{\mathbf{P}^\beta}\bigg[\ln\bigg(\frac{\ell^\beta\big(t, X(t)\big)}{\ell^\beta\big(t_0, X(t_0)\big)}\bigg)\bigg]$$

$$(4.16)\qquad = \mathbf{E}_{\mathbf{P}^\beta}\bigg[\int_{t_0}^t \bigg(-\frac{1}{2}\frac{\big|\nabla\ell^\beta\big(u, X(u)\big)\big|^2}{\ell^\beta\big(u, X(u)\big)^2} + \big(\operatorname{div}\beta - \langle\beta, 2\nabla\Psi\rangle\big)\big(X(u)\big)\bigg)\mathrm{d}u\bigg].$$

Furthermore, for every point $t_0 \in \mathbf{R}_+ \setminus N$ (*at which the right-sided limiting assertion* (3.7) *is valid*), *we have also the limiting identities*

$$(4.17)\qquad \lim_{t\downarrow t_0}\frac{H\big(P^\beta(t)\,|\,\mathrm{Q}\big) - H\big(P^\beta(t_0)\,|\,\mathrm{Q}\big)}{t - t_0} = \mathbf{E}_{\mathbf{P}}\bigg[-\frac{1}{2}\frac{\big|\nabla\ell\big(t_0, X(t_0)\big)\big|^2}{\ell\big(t_0, X(t_0)\big)^2} + \big(\operatorname{div}\beta - \langle\beta, 2\nabla\Psi\rangle\big)\big(X(t_0)\big)\bigg],$$

$$(4.18)\qquad \lim_{t\downarrow t_0}\frac{W_2\big(P^\beta(t), P^\beta(t_0)\big)}{t - t_0} = \frac{1}{2}\Bigg(\mathbf{E}_{\mathbf{P}}\bigg[\bigg|\frac{\nabla\ell\big(t_0, X(t_0)\big)}{\ell\big(t_0, X(t_0)\big)} + 2\beta\big(X(t_0)\big)\bigg|^2\bigg]\Bigg)^{1/2}.$$

Proof of Corollary 4.4 *from Theorem* 4.2. Taking expectations in (4.12) under the probability measure $\mathbf{P}^\beta$, recalling the definitions (4.10) and (4.11), and using the martingale property of the process in (4.11) for $T \geqslant t \geqslant t_0$ leads to the identity (4.16). In order to derive from (4.16) the limiting identity (4.17), extra care is needed to show that (4.17) is valid for every time $t_0 \in \mathbf{R}_+ \setminus N$.

We will verify in Lemma 5.9 of subsection 5.3 the following estimates on the ratio between the probability density function $p(t,\cdot)$ and its perturbed version $p^\beta(t,\cdot)$: For every $t_0 \geqslant 0$ and $T > t_0$, there is a constant $C > 0$ such that

$$(4.19)\qquad \bigg|\frac{\ell^\beta(t,x)}{\ell(t,x)} - 1\bigg| = \bigg|\frac{p^\beta(t,x)}{p(t,x)} - 1\bigg| \leqslant C(t - t_0), \quad (t,x) \in [t_0, T] \times \mathbf{R}^n,$$

as well as

$$(4.20)\qquad \mathbf{E}_{\mathbf{P}}\bigg[\int_{t_0}^t \bigg|\nabla\ln\bigg(\frac{\ell^\beta\big(u, X(u)\big)}{\ell\big(u, X(u)\big)}\bigg)\bigg|^2 \mathrm{d}u\bigg] \leqslant C(t - t_0)^2, \qquad t_0 \leqslant t \leqslant T.$$

We turn now to the derivation of (4.17) from (4.16). First, since the perturbation β is smooth and compactly supported, and the paths of the canonical coordinate process $(X(t))_{t\geqslant 0}$ are continuous, we have

$$(4.21)\qquad \lim_{t\downarrow t_0}\frac{1}{t - t_0}\,\mathbf{E}_{\mathbf{P}^\beta}\bigg[\int_{t_0}^t \big(\operatorname{div}\beta - \langle\beta, 2\nabla\Psi\rangle\big)\big(X(u)\big)\,\mathrm{d}u\bigg] = \mathbf{E}_{\mathbf{P}^\beta}\big[\big(\operatorname{div}\beta - \langle\beta, 2\nabla\Psi\rangle\big)\big(X(t_0)\big)\big]$$

for every $t_0 \geqslant 0$. Second, the r.v. $X(t_0)$ has the same distribution under $\mathbf{P}$, as it does under $\mathbf{P}^\beta$, so it is immaterial whether we express the expectation on the right-hand side of (4.21) with respect to the probability measure $\mathbf{P}$ or $\mathbf{P}^\beta$. Hence this expression equals the corresponding term on the right-hand side of (4.17).

Regarding the remaining term on the right-hand side of (4.17), the equality

$$\begin{aligned} &\lim_{t \downarrow t_0} \frac{1}{t-t_0} \mathbf{E}_{\mathbf{P}^\beta}\left[\int_{t_0}^t \left(-\frac{1}{2} \frac{\big|\nabla \ell^\beta\big(u, X(u)\big)\big|^2}{\ell^\beta\big(u, X(u)\big)^2}\right) \mathrm{d}u\right] \\ &\qquad = \lim_{t \downarrow t_0} \frac{1}{t-t_0} \mathbf{E}_{\mathbf{P}}\left[\int_{t_0}^t \left(-\frac{1}{2} \frac{\big|\nabla \ell\big(u, X(u)\big)\big|^2}{\ell\big(u, X(u)\big)^2}\right) \mathrm{d}u\right] \end{aligned} \tag{4.22}$$

holds as long as $t_0 \geqslant 0$ is chosen so that one of the limits exists. Indeed, the equality

$$\begin{aligned} &\lim_{t \downarrow t_0} \frac{1}{t-t_0} \mathbf{E}_{\mathbf{P}}\left[\int_{t_0}^t \left(-\frac{1}{2} \frac{\big|\nabla \ell^\beta\big(u, X(u)\big)\big|^2}{\ell^\beta\big(u, X(u)\big)^2}\right) \mathrm{d}u\right] \\ &\qquad = \lim_{t \downarrow t_0} \frac{1}{t-t_0} \mathbf{E}_{\mathbf{P}}\left[\int_{t_0}^t \left(-\frac{1}{2} \frac{\big|\nabla \ell\big(u, X(u)\big)\big|^2}{\ell\big(u, X(u)\big)^2}\right) \mathrm{d}u\right] \end{aligned} \tag{4.23}$$

follows from (4.20), and (4.19) implies that it is immaterial whether we take expectations with respect to $\mathbf{P}$ or $\mathbf{P}^\beta$ in the two limits appearing in (4.23). Summing up, the existence and the equality of the limits in (4.22) are guaranteed if and only if $t_0 \in \mathbf{R}_+ \setminus N$. It develops that both limits in (4.22) exist if $t_0 \geqslant 0$ is not in the exceptional set N of zero Lebesgue measure, and their common value is

$$-\frac{1}{2} I\big(P(t_0)\,|\,\mathrm{Q}\big) = -\frac{1}{2} \mathbf{E}_{\mathbf{P}}\left[\frac{\big|\nabla \ell\big(t_0, X(t_0)\big)\big|^2}{\ell\big(t_0, X(t_0)\big)^2}\right]. \tag{4.24}$$

In conjunction with (4.21), which is valid for every $t_0 \geqslant 0$, this establishes the limiting identity (4.17) for every $t_0 \in \mathbf{R}_+ \setminus N$. Therefore, the right-sided limiting assertion (3.7), and the similar perturbed limiting assertion in (4.17), fail on precisely the same set of exceptional points N.

As regards the final assertion, we note that, by analogy with (4.15), the limiting behavior of the Wasserstein distance (4.18), for Lebesgue-a.e. $t_0 \geqslant 0$, is well known [4]; details are in section 6 below. More precisely, Theorem 6.2 establishes the novel and very crucial aspect that the limiting assertion

$$\lim_{t \downarrow t_0} \frac{W_2\big(P(t), P(t_0)\big)}{t-t_0} = \frac{1}{2} \sqrt{I\big(P(t_0)\,|\,\mathrm{Q}\big)} \tag{4.25}$$

is valid for every $t_0 \in \mathbf{R}_+ \setminus N$. Once again, concerning the relation between the limits in (4.25) and (4.18) pertaining to the Wasserstein distance, we discern a similar pattern as in the case of the generalized de Bruijn identity. In fact, Theorem 6.2 will tell us that the perturbed Wasserstein limit (4.18) also holds for every $t_0 \in \mathbf{R}_+ \setminus N$. Corollary 4.4 is proved.

Proof of Theorem 3.2 *from Theorems* 4.1 *and* 4.2 *and Corollaries* 4.3 *and* 4.4. Let $t_0 \in \mathbf{R}_+ \setminus N$, so that the limiting identities (4.17) and (4.18) from Corollary 4.4 are valid. Recalling the abbreviations in (3.11), we summarize now the identities just mentioned as

$$\lim_{t\downarrow t_0} \frac{H\big(P(t)\,|\,\mathrm{Q}\big) - H\big(P(t_0)\,|\,\mathrm{Q}\big)}{t-t_0} = -\frac{1}{2}\,\|a\|^2_{L^2(\mathbf{P})}, \tag{4.26}$$

$$\lim_{t\downarrow t_0} \frac{W_2\big(P(t), P(t_0)\big)}{t-t_0} = \frac{1}{2}\,\|a\|_{L^2(\mathbf{P})}, \tag{4.27}$$

$$\lim_{t\downarrow t_0} \frac{H\big(P^\beta(t)\,|\,\mathrm{Q}\big) - H\big(P^\beta(t_0)\,|\,\mathrm{Q}\big)}{t-t_0} = -\frac{1}{2}\,\langle a, a+2b\rangle_{L^2(\mathbf{P})}, \tag{4.28}$$

$$\lim_{t\downarrow t_0} \frac{W_2\big(P^\beta(t), P^\beta(t_0)\big)}{t-t_0} = \frac{1}{2}\,\|a+2b\|_{L^2(\mathbf{P})}. \tag{4.29}$$

Indeed, the equations (4.26), (4.27), and (4.29) correspond to (3.7), (4.25), and (4.18), respectively. As for (4.28), we note that, according to equation (4.17) of Corollary 4.4, the limit in (4.28) equals

$$-\frac{1}{2}\,\|a\|^2_{L^2(\mathbf{P})} + \mathbf{E_P}\big[(\operatorname{div}\beta - 2\langle\beta,\nabla\Psi\rangle)\big(X(t_0)\big)\big]. \tag{4.30}$$

Therefore, in view of the right-hand side of (4.28), we have to show the identity

$$\mathbf{E_P}\big[(\operatorname{div}\beta - \langle\beta, 2\nabla\Psi\rangle)\big(X(t_0)\big)\big] = -\langle a, b\rangle_{L^2(\mathbf{P})}. \tag{4.31}$$

In order to do this, we write the left-hand side of (4.31) as

$$\int_{\mathbf{R}^n} (\operatorname{div}\beta(x) - \langle\beta(x), 2\nabla\Psi(x)\rangle) p(t_0,x)\,\mathrm{d}x. \tag{4.32}$$

Using — for the first time, and only in order to show the identity (4.31) — integration by parts, and the fact that the perturbation β is assumed to be smooth and compactly supported, we see that the expression (4.32) becomes

$$-\int_{\mathbf{R}^n} \langle\beta(x), \nabla\ln p(t_0,x) + 2\nabla\Psi(x)\rangle p(t_0,x)\,\mathrm{d}x, \tag{4.33}$$

which is the same as $-\langle\beta(X(t_0)), \nabla\ln\ell(t_0,X(t_0))\rangle_{L^2(\mathbf{P})} = -\langle b, a\rangle_{L^2(\mathbf{P})}$.

The limiting identities (4.26)–(4.29) now imply the assertions of Theorem 3.2.

The following Propositions 4.5 and 4.7 are trajectorial versions of Corollaries 4.3 and 4.4, respectively. They compute the rate of temporal change of relative entropy for (1.3) and for its perturbed version (3.10), respectively, in the more precise trajectorial manner of Theorems 4.1 and 4.2.

PROPOSITION 4.5 (trajectorial rate of relative entropy dissipation). *Under Assumptions* 2.2, *we let* $t_0 \in \mathbf{R}_+ \setminus N$ *and* $T > t_0$. *Then the relative entropy process* (4.1) *satisfies the trajectorial relation*

$$\lim_{s\uparrow T-t_0} \frac{\mathbf{E_P}\big[\ln\ell\big(t_0,X(t_0)\big)\,|\,\mathcal{G}(T-s)\big] - \ln\ell\big(T-s,X(T-s)\big)}{T-t_0-s} = \frac{1}{2}\,\frac{\big|\nabla\ell\big(t_0,X(t_0)\big)\big|^2}{\ell\big(t_0,X(t_0)\big)^2}, \tag{4.34}$$

where the limit exists in $L^1(\mathbf{P})$.

Remark 4.6. The limiting assertion (4.34) of Proposition 4.5 is the conditional trajectorial version of the generalized de Bruijn identity (4.14).

Proof of Proposition 4.5 *from Theorem* 4.1. Let $t_0 \in \mathbf{R}_+ \setminus N$, i.e., so that the right-sided limiting assertion (3.7) is valid, and select $T > t_0$. The martingale property of the process in (4.5) allows us to write the numerator in (4.34) as

$$\mathbf{E}_{\mathbf{P}}\big[F(t_0) - F(T-s) \,|\, \mathcal{G}(T-s)\big], \qquad 0 \leqslant s \leqslant T - t_0, \tag{4.35}$$

in the notation of (4.4), which expresses the process $(F(T-s))_{0\leqslant s\leqslant T}$ as the primitive of

$$B(u) = \frac{1}{2}\,\frac{\big|\nabla\ell\big(T-u, X(T-u)\big)\big|^2}{\ell\big(T-u, X(T-u)\big)^2}, \qquad 0 \leqslant u \leqslant T. \tag{4.36}$$

By analogy with the derivation of (4.14) from (4.13), where we calculated real-valued expectations, we rely on the Lebesgue differentiation theorem to obtain the corresponding result (4.34) for conditional expectations. Using the left-continuity of the backwards filtration $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T}$, we can invoke the measure-theoretic result in Proposition A.2 of Appendix A, with the choice of the process B as in (4.36) and $C \equiv 0$. This establishes the claim (4.34). Proposition 4.5 is proved.

PROPOSITION 4.7 (trajectorial rate of relative entropy dissipation under perturbations). *Under Assumptions* 2.2, *we let* $t_0 \in \mathbf{R}_+ \setminus N$ *and* $T > t_0$. *Then the perturbed relative entropy process* (4.9) *satisfies the trajectorial relations*

$$\begin{aligned}
&\lim_{s\uparrow T-t_0} \frac{\mathbf{E}_{\mathbf{P}^\beta}\big[\ln \ell^\beta\big(t_0, X(t_0)\big) \,|\, \mathcal{G}(T-s)\big] - \ln \ell^\beta\big(T-s, X(T-s)\big)}{T-t_0-s} \\
&\qquad = \frac{1}{2}\,\frac{\big|\nabla\ell\big(t_0, X(t_0)\big)\big|^2}{\ell\big(t_0, X(t_0)\big)^2} - \operatorname{div}\beta\big(X(t_0)\big) + \big\langle \beta\big(X(t_0)\big), 2\nabla\Psi\big(X(t_0)\big)\big\rangle,
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
&\lim_{s\uparrow T-t_0} \frac{\mathbf{E}_{\mathbf{P}}\big[\ln \ell^\beta\big(t_0, X(t_0)\big) \,|\, \mathcal{G}(T-s)\big] - \ln \ell^\beta\big(T-s, X(T-s)\big)}{T-t_0-s} \\
&\qquad = \frac{1}{2}\,\frac{\big|\nabla\ell\big(t_0, X(t_0)\big)\big|^2}{\ell\big(t_0, X(t_0)\big)^2} - \operatorname{div}\beta\big(X(t_0)\big) - \big\langle \beta\big(X(t_0)\big), \nabla\ln p\big(t_0, X(t_0)\big)\big\rangle,
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
&\lim_{s\uparrow T-t_0} \frac{\ln \ell^\beta\big(T-s, X(T-s)\big) - \ln \ell\big(T-s, X(T-s)\big)}{T-t_0-s} \\
&\qquad = \operatorname{div}\beta\big(X(t_0)\big) + \big\langle \beta\big(X(t_0)\big), \nabla\ln p\big(t_0, X(t_0)\big)\big\rangle,
\end{aligned} \tag{4.39}$$

where the limits in (4.37)–(4.39) *exist in both* $L^1(\mathbf{P})$ *and* $L^1(\mathbf{P}^\beta)$.

Remark 4.8. It is noteworthy that the three limiting expressions in (4.37), (4.38), and (4.39) are quite different from one another. The first limiting assertion (4.37) of Proposition 4.7 is the conditional trajectorial version of the perturbed de Bruijn identity (4.17). We also note that in fact the third limiting assertion (4.39) is valid for all $t_0 > 0$.

Proof of (4.37) *from Theorem* 4.2. Let $t_0 \in \mathbf{R}_+ \setminus N$, i.e., so that the right-sided limiting assertion (3.7) is valid, and select $T > t_0$. In (4.22) from Corollary 4.4 of Theorem 4.2 we have seen that the limits in (3.7) and (4.17) have the same exceptional sets; hence the limiting identity (4.17) also holds. Now, for such $t_0 \in \mathbf{R}_+ \setminus N$, we derive the limiting assertion (4.37) in the same way as the assertion (4.34) in the proof of Proposition 4.5 above. Indeed, this time we invoke the $\mathbf{P}^\beta$-martingale property of the process in (4.11) and write the numerator on the first line of (4.37) as

$\mathbf{E}_{\mathbf{P}^\beta}[F^\beta(t_0) - F^\beta(T-s) \,|\, \mathcal{G}(T-s)]$, $0 \leqslant s \leqslant T - t_0$, in the notation of (4.10), which expresses the process $(F^\beta(T-s))_{0\leqslant s\leqslant T-t_0}$ as the primitive of $(B(u)+C(u))_{0\leqslant s\leqslant T-t_0}$, with

$$\begin{aligned} B(u) &= \frac{1}{2}\,\frac{\big|\nabla \ell^\beta\big(T-u, X(T-u)\big)\big|^2}{\ell^\beta\big(T-u, X(T-u)\big)^2}, \\ C(u) &= \big(\langle \beta, 2\nabla\Psi\rangle - \operatorname{div}\beta\big)\big(X(T-u)\big). \end{aligned} \tag{4.40}$$

An application of Proposition A.2 of Appendix A in this situation proves the limiting identity (4.37) in $L^1(\mathbf{P}^\beta)$. As we will see in Lemma 5.8 of subsection 5.3 below, the probability measures $\mathbf{P}$ and $\mathbf{P}^\beta$ are equivalent, and the mutual Radon–Nikodým derivatives $\mathrm{d}\mathbf{P}^\beta/\mathrm{d}\mathbf{P}$ and $\mathrm{d}\mathbf{P}/\mathrm{d}\mathbf{P}^\beta$ are bounded on the σ-algebra $\mathcal{F}(T) = \mathcal{G}(0)$ (recall, in this vein, the claims of (4.19)). Hence, the convergence in $L^1(\mathbf{P})$ is equivalent to that in $L^1(\mathbf{P}^\beta)$. This proves assertion (4.37).

The proofs of the limiting assertions (4.38) and (4.39) are postponed to subsection 5.4.

4.2. A trajectorial proof of the HWI inequality. The aim of this section is to provide a proof of the celebrated *HWI inequality* due to Otto and Villani [45] by applying trajectorial arguments similar to those in Theorem 4.1 (in fact quite easier). We thus obtain an intuitive geometric picture and deduce the sharpened form of the HWI inequality; see also [14], [45], and [57, p. 650].

The goal is to compare the relative entropies $H(P_0 \,|\, \mathrm{Q})$ and $H(P_1 \,|\, \mathrm{Q})$ for arbitrary probability measures $P_0, P_1 \in \mathscr{P}_2(\mathbf{R}^n)$. Using *Brenier's theorem* [12], we first define the constant speed geodesic $(P_t)_{0\leqslant t\leqslant 1}$ between P_0 and P_1 with respect to the Wasserstein distance W_2 (details are given below). We remark that we have chosen the subscript notation for P_t in order to avoid confusion with the probability measure $P(t)$ from our section 2 here. With $p_t(\,\cdot\,)$ the density function of the probability measure P_t, we define the likelihood ratio function

$$\ell_t(x) := \frac{p_t(x)}{q(x)}, \qquad (t,x) \in [0,1] \times \mathbf{R}^n. \tag{4.41}$$

We will investigate the behavior of the relative entropy function $t \mapsto f(t) := H(P_t \,|\, \mathrm{Q})$ along the constant speed geodesic $(P_t)_{0\leqslant t\leqslant 1}$ by estimating two quantities. First, we need a lower bound on the first derivative $f'(0^+)$. Second, we need a lower bound on the second derivative $(f''(t))_{0\leqslant t\leqslant 1}$. It should be geometrically obvious (and will be spelled out in the proof of Theorem 4.11 below) that information on these two lower bounds leads to a lower bound on $f(1) - f(0)$. The latter is the content of the HWI inequality. As regards the second derivative $(f''(t))_{0\leqslant t\leqslant 1}$, we rely on a fundamental result on displacement convexity due to McCann [41] and have no novel contribution. As regards $f'(0^+)$, however, we will obtain a sharp estimate for this quantity by applying a trajectorial reasoning similar to that employed in the proof of Theorem 4.1.

We will define an $\mathbf{R}^n$-valued stochastic process $(X_t)_{0\leqslant t\leqslant 1}$, with marginal distributions $(P_t)_{0\leqslant t\leqslant 1}$ moving along straight lines in $\mathbf{R}^n$, and calculate the relevant quantities of this finite variation process along every trajectory, by analogy with the proof of Theorem 4.1. This gives us the desired bound (and actually an equality) for the derivative $f'(0^+)$.

We now cast these ideas into formal terms. The first step is to calculate the decay of the relative entropy function $t \mapsto H(P_t \,|\, \mathrm{Q})$ along the "straight line" $(P_t)_{0 \leqslant t \leqslant 1}$ joining the elements P_0 and P_1 in $\mathscr{P}_2(\mathbf{R}^n)$. To this end, we impose temporarily the following strong regularity conditions. In the proof of Theorem 4.11 we will see that these will not restrict the generality of the argument.

Assumptions 4.9 (regularity assumptions of Lemma 4.10). We impose that P_0 and P_1 are probability measures in $\mathscr{P}_2(\mathbf{R}^n)$ with smooth densities, which are compactly supported and strictly positive in the interior of their respective supports. Hence there exists a map $\gamma \colon \mathbf{R}^n \to \mathbf{R}^n$ of the form $\gamma(x) = \nabla(G(x) - |x|^2/2)$ for some convex function $G \colon \mathbf{R}^n \to \mathbf{R}$, uniquely defined on and supported by the support of P_0, and smooth in the interior of this set, such that γ induces the optimal quadratic Wasserstein transport from P_0 to P_1 via

$$T_t^{\gamma}(x) := x + t\gamma(x) = (1-t)x + t\nabla G(x) \quad \text{and} \quad P_t := (T_t^{\gamma})_{\#}(P_0) = P_0 \circ (T_t^{\gamma})^{-1} \tag{4.42}$$

for $0 \leqslant t \leqslant 1$; to wit, the curve $(P_t)_{0 \leqslant t \leqslant 1}$ is the displacement interpolation (constant speed geodesic) between P_0 and P_1, and we have along it the linear growth of the quadratic Wasserstein distance

$$W_2(P_0, P_t) = t \sqrt{\int_{\mathbf{R}^n} |x - \nabla G(x)|^2 \, \mathrm{d}P_0(x)} = t \|\gamma\|_{L^2(P_0)}, \qquad 0 \leqslant t \leqslant 1. \tag{4.43}$$

For the existence and uniqueness of the optimal transport map γ we refer the reader to [56, Theorem 2.12], and for its smoothness, to [56, Theorem 4.14] as well as to [56, Remarks 4.15]. These results are known collectively under the rubric of Brenier's theorem [12].

Next, we compute the slope of the function $t \mapsto H(P_t \,|\, \mathrm{Q})$ along the straight line $(P_t)_{0 \leqslant t \leqslant 1}$.

LEMMA 4.10. *Under Assumptions* 4.9, *let $X_0 \colon S \to \mathbf{R}^n$ be an r.v. with probability distribution $P_0 \in \mathscr{P}_2(\mathbf{R}^n)$ defined on some probability space $(S, \mathcal{S}, \nu)$. Then*

$$\lim_{t \downarrow 0} \frac{H(P_t \,|\, \mathrm{Q}) - H(P_0 \,|\, \mathrm{Q})}{t} = \langle \nabla \ln \ell_0(X_0), \gamma(X_0) \rangle_{L^2(\nu)}. \tag{4.44}$$

We relegate to Appendix B the proof of Lemma 4.10, which follows a similar (but considerably simpler) trajectorial line of reasoning as the proof of Theorem 3.2. Combining Lemma 4.10 with well-known arguments, in particular, with a fundamental result on displacement convexity due to McCann [41], we derive now the HWI inequality of Otto and Villani [45].

THEOREM 4.11 (HWI inequality [45]). *We fix $P_0, P_1 \in \mathscr{P}_2(\mathbf{R}^n)$, assume that the relative entropy $H(P_1 \,|\, \mathrm{Q})$ is finite, and suppose, in addition, that the potential $\Psi \in \mathcal{C}^{\infty}(\mathbf{R}^n; [0, \infty))$ satisfies a curvature lower bound*

$$\mathrm{Hess}(\Psi) \geqslant \kappa I_n \tag{4.45}$$

for some $\kappa \in \mathbf{R}$. Then

$$H(P_0 \,|\, \mathrm{Q}) - H(P_1 \,|\, \mathrm{Q}) \leqslant -\langle \nabla \ln \ell_0(X_0), \gamma(X_0) \rangle_{L^2(\nu)} - \frac{\kappa}{2} W_2^2(P_0, P_1), \tag{4.46}$$

where the likelihood ratio function ℓ_0, the r.v. X_0, the optimal transport map γ, and the probability measure ν are as in Lemma 4.10.

We stress that Theorem 4.11 does not require the measure Q with density $q(x) = \mathrm{e}^{-2\Psi(x)}$ to be a finite measure in the formulation of the HWI inequality (4.46).

On the strength of the Cauchy–Schwarz inequality, we have

$$-\langle \nabla \ln \ell_0(X_0), \gamma(X_0)\rangle_{L^2(\nu)} \leqslant \|\nabla \ln \ell_0(X_0)\|_{L^2(\nu)} \|\gamma(X_0)\|_{L^2(\nu)}, \tag{4.47}$$

with equality if and only if the functions $\nabla \ln \ell_0(\,\cdot\,)$ and $\gamma(\,\cdot\,)$ are negatively collinear. The relative Fisher information of P_0 with respect to Q equals

$$I(P_0 \,|\, \mathrm{Q}) = \mathbf{E}_\nu[|\nabla \ln \ell_0(X_0)|^2] = \|\nabla \ln \ell_0(X_0)\|^2_{L^2(\nu)}, \tag{4.48}$$

and by Brenier's theorem [56, Theorem 2.12] we deduce

$$\|\gamma(X_0)\|_{L^2(\nu)} = W_2(P_0, P_1) \tag{4.49}$$

as in (4.43), along with the inequality

$$-\langle \nabla \ln \ell_0(X_0), \gamma(X_0)\rangle_{L^2(\nu)} \leqslant \sqrt{I(P_0 \,|\, \mathrm{Q})}\; W_2(P_0, P_1). \tag{4.50}$$

Inserting (4.50) into (4.46) we obtain the usual form of the HWI inequality

$$H(P_0 \,|\, \mathrm{Q}) - H(P_1 \,|\, \mathrm{Q}) \leqslant W_2(P_0, P_1)\sqrt{I(P_0 \,|\, \mathrm{Q})} - \frac{\kappa}{2}\, W_2^2(P_0, P_1). \tag{4.51}$$

When there is a nontrivial angle between $-\nabla \ln \ell_0(X_0)$ and $\gamma(X_0)$ in $L^2(\nu)$, the inequality (4.46) gives a sharper bound than (4.51). We refer the reader to the original paper [45], as well as to [14], [56, Chap. 5], [57, p. 650], and the recent papers [26], [33] for detailed discussions of the HWI inequality in several contexts. For a good survey on transport inequalities, see [27].

Proof of Theorem 4.11. As elaborated in [56, section 9.4] we may assume without loss of generality that P_0 and P_1 satisfy the strong regularity Assumptions 4.9, which guarantees the existence and smoothness of the optimal transport map γ.

We consider now the relative entropy with respect to Q along the constant-speed geodesic $(P_t)_{0 \leqslant t \leqslant 1}$, namely, the function $f(t) := H(P_t \,|\, \mathrm{Q})$, for $0 \leqslant t \leqslant 1$. The displacement convexity results of McCann [41] imply

$$f''(t) \geqslant \kappa W_2^2(P_0, P_1), \qquad 0 \leqslant t \leqslant 1. \tag{4.52}$$

Indeed, under the condition (4.45), the potential Ψ is κ-uniformly convex. Consequently, by items (i) and (ii) of [56, Theorem 5.15], the internal and potential energies

$$g(t) := \int_{\mathbf{R}^n} p_t(x) \ln p_t(x)\, \mathrm{d}x, \quad h(t) := 2\int_{\mathbf{R}^n} \Psi(x) p_t(x)\, \mathrm{d}x, \qquad 0 \leqslant t \leqslant 1, \tag{4.53}$$

are displacement convex and κ-uniformly displacement convex, respectively; i.e.,

$$g''(t) \geqslant 0, \quad h''(t) \geqslant \kappa W_2^2(P_0, P_1), \qquad 0 \leqslant t \leqslant 1. \tag{4.54}$$

As we have $f = g + h$, we conclude that the relative entropy function f is κ-uniformly displacement convex, i.e., its second derivative satisfies (4.52). We appeal now to Lemma 4.10, according to which

$$f'(0^+) = \lim_{t \downarrow 0} \frac{f(t) - f(0)}{t} = \langle \nabla \ln \ell_0(X_0), \gamma(X_0)\rangle_{L^2(\nu)}. \tag{4.55}$$

Now (4.46) follows from the Taylor formula $f(1) = f(0) + f'(0^+) + \int_0^1 (1-t) f''(t)\, \mathrm{d}t$ in conjunction with (4.52) and (4.55). Theorem 4.11 is proved.

5. Details and proofs.

In this section, we complete the proofs of Corollary 4.4 and Proposition 4.7, and provide the proofs of the crucial results, Theorems 4.1 and 4.2. For this, we apply Itô's formula so as to calculate the dynamics, i.e., the stochastic differentials, of the "pure" and "perturbed" relative entropy processes of (4.1) and (4.9) under the measures $\mathbf{P}$ and $\mathbf{P}^{\beta}$, respectively. As already discussed, we will do this in the backwards direction of time.

5.1. The proof of Theorem 4.1.

We start by calculating the stochastic differential of the time-reversed canonical coordinate process $(X(T-s))_{0\leqslant s\leqslant T}$ under $\mathbf{P}$, a well-known and classical theme; see, e.g., [19], [20], [28], [42], [43], and [46]. The reader may consult Appendix G of [34] for an extensive presentation of the relevant facts regarding the theory of time reversal for diffusion processes. The idea of time reversal goes back to Boltzmann [9], [10], [11] and Schrödinger [50], [51], as well as Kolmogorov [37]. In fact, the relation between time reversal of a Brownian motion and the quadratic Wasserstein distance may *in nuce* be traced back to an insight of Bachelier in his thesis [5], [6] from 1900. This theme is discussed in Appendix A of [34].

Recall that the probability measure $\mathbf{P}$ was defined on the path space $\Omega = \mathcal{C}(\mathbf{R}_+;\mathbf{R}^n)$ so that the canonical coordinate process $(X(t,\omega))_{t\geqslant 0} = (\omega(t))_{t\geqslant 0}$ satisfies the stochastic differential equation (1.3) with initial probability distribution $P(0)$ for $X(0)$ under $\mathbf{P}$. In other words, the process

$$W(t) = X(t) - X(0) + \int_0^t \nabla\Psi\big(X(u)\big)\,\mathrm{d}u, \qquad t \geqslant 0, \tag{5.1}$$

is a Brownian motion of the forward filtration $(\mathcal{F}(t))_{t\geqslant 0}$ under the probability measure $\mathbf{P}$. Passing to the reverse direction of time, the following classical result is well known to hold under the present assumptions. For proof and references we refer the reader to Theorems G.2 and G.5 of Appendix G in [34].

PROPOSITION 5.1. *Under Assumptions* 2.2, *let $T>0$ be fixed. The process*

$$\overline{W}^{\mathbf{P}}(T-s) := W(T-s) - W(T) - \int_0^s \nabla \ln p\big(T-u, X(T-u)\big)\,\mathrm{d}u, \qquad 0\leqslant s\leqslant T, \tag{5.2}$$

is a Brownian motion of the backwards filtration $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T}$ under the probability measure $\mathbf{P}$. Moreover, the time-reversed canonical coordinate process $(X(T-s))_{0\leqslant s\leqslant T}$ satisfies the stochastic differential equation

$$\mathrm{d}X(T-s) = \Big(\nabla \ln p\big(T-s, X(T-s)\big) + \nabla\Psi\big(X(T-s)\big)\Big)\,\mathrm{d}s + \mathrm{d}\overline{W}^{\mathbf{P}}(T-s) \tag{5.3}$$

$$\qquad = \Big(\nabla \ln \ell\big(T-s, X(T-s)\big) - \nabla\Psi\big(X(T-s)\big)\Big)\,\mathrm{d}s + \mathrm{d}\overline{W}^{\mathbf{P}}(T-s) \tag{5.4}$$

for $0\leqslant s\leqslant T$, with respect to the backwards filtration $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T}$.

The following result computes the forward dynamics of the likelihood ratio process $(\ell(t,X(t)))_{t\geqslant 0}$ of (2.1) and compares it with the stochastic differential of the time-reversed likelihood ratio process

$$\ell\big(T-s, X(T-s)\big) = \frac{p\big(T-s, X(T-s)\big)}{q\big(X(T-s)\big)}, \qquad 0\leqslant s\leqslant T, \tag{5.5}$$

as well as its logarithmic differential.

PROPOSITION 5.2. *Under Assumptions* 2.2, *the likelihood ratio process* (2.1) *is a continuous semimartingale with respect to the forward filtration* $(\mathcal{F}(t))_{t \geqslant 0}$ *and satisfies, for* $t \geqslant 0$, *the stochastic differential equation*

$$\mathrm{d}\ell\big(t, X(t)\big) = \big\langle \nabla\ell\big(t, X(t)\big), \mathrm{d}W(t)\big\rangle + \Big(\Delta\ell\big(t, X(t)\big) - \big\langle \nabla\ell\big(t, X(t)\big), 2\nabla\Psi\big(X(t)\big)\big\rangle\Big)\,\mathrm{d}t. \tag{5.6}$$

Furthermore, the time-reversed likelihood ratio process (5.5) *is a continuous semimartingale with respect to the backwards filtration* $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T}$ *and satisfies, for* $0 \leqslant s \leqslant T$, *the stochastic differential equations*

$$\begin{aligned} \mathrm{d}\ell\big(T-s, X(T-s)\big) &= \big\langle \nabla\ell\big(T-s, X(T-s)\big), \mathrm{d}\overline{W}^{\mathbf{P}}(T-s)\big\rangle \\ &\quad + \frac{\big|\nabla\ell\big(T-s, X(T-s)\big)\big|^2}{\ell\big(T-s, X(T-s)\big)}\,\mathrm{d}s, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \frac{\mathrm{d}\ell\big(T-s, X(T-s)\big)}{\ell\big(T-s, X(T-s)\big)} &= \left\langle \frac{\nabla\ell\big(T-s, X(T-s)\big)}{\ell\big(T-s, X(T-s)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}}(T-s)\right\rangle \\ &\quad + \frac{\big|\nabla\ell\big(T-s, X(T-s)\big)\big|^2}{\ell\big(T-s, X(T-s)\big)^2}\,\mathrm{d}s, \end{aligned} \tag{5.8}$$

$$\begin{aligned} \mathrm{d}\ln\ell\big(T-s, X(T-s)\big) &= \left\langle \frac{\nabla\ell\big(T-s, X(T-s)\big)}{\ell\big(T-s, X(T-s)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}}(T-s)\right\rangle \\ &\quad + \frac{1}{2}\,\frac{\big|\nabla\ell\big(T-s, X(T-s)\big)\big|^2}{\ell\big(T-s, X(T-s)\big)^2}\,\mathrm{d}s. \end{aligned} \tag{5.9}$$

Proof. We start with the following observation. Writing the Fokker–Planck equation (1.1) as

$$\partial_t p(t,x) = \frac{1}{2}\Delta p(t,x) + \langle \nabla p(t,x), \nabla\Psi(x)\rangle + p(t,x)\Delta\Psi(x), \qquad t > 0, \tag{5.10}$$

and substituting the expression

$$p(t,x) = \ell(t,x)q(x) = \ell(t,x)\,\mathrm{e}^{-2\Psi(x)}, \qquad t \geqslant 0, \tag{5.11}$$

into this equation, we find that the likelihood ratio function $(t,x) \mapsto \ell(t,x)$ solves the *backwards* Kolmogorov equation

$$\partial_t \ell(t,x) = \frac{1}{2}\Delta\ell(t,x) - \big\langle \nabla\ell(t,x), \nabla\Psi(x)\big\rangle, \qquad t > 0. \tag{5.12}$$

Now we turn to the proofs of (5.6)–(5.9). By Assumptions 2.2, the likelihood ratio function $(t,x) \mapsto \ell(t,x)$ is sufficiently smooth to allow an application of Itô's formula. Together with the Langevin–Smoluchowski dynamics (1.3) and the backwards Kolmogorov equation (5.10), we obtain (5.6) by direct calculation. A similar calculation, this time relying on the backwards dynamics (5.4), shows (5.7). Finally, the equations (5.8) and (5.9) follow from (5.7) and Itô's formula. Proposition 5.2 is proved.

The crucial feature of the stochastic differentials (5.6)–(5.9) is that, after passing to time reversal, the finite-variation term $\Delta\ell - \langle\nabla\ell, 2\nabla\Psi\rangle$ in (5.6), involving the Laplacian $\Delta\ell$, gets replaced by a term involving only the likelihood ratio function ℓ and its gradient $\nabla\ell$.

Proof of Theorem 4.1. On a formal level, the expressions (4.4), (4.6) are just integral versions of the Itô differential (5.9). What remains to check is that the integrals in (4.4) and (4.6) indeed make rigorous sense and satisfy the claimed integrability conditions.

By condition (iv) of Assumptions 2.2 the function $(t,x) \mapsto \nabla \ln \ell(t,x)$ is continuous. Together with the continuity of the paths of the canonical coordinate process $(X(t))_{t \geqslant 0}$, this implies

$$\int_0^{T-\varepsilon} \frac{\big|\nabla \ell\big(T-u, X(T-u)\big)\big|^2}{\ell\big(T-u, X(T-u)\big)^2} \, \mathrm{d}u < \infty \qquad \mathbf{P}\text{-a.s.} \tag{5.13}$$

for every $0 < \varepsilon \leqslant T$. On account of (5.13), the sequence of stopping times (with respect to the backwards filtration)

$$\tau_n := \inf\left\{ t \geqslant 0 \colon \int_0^t \frac{\big|\nabla \ell\big(T-u, X(T-u)\big)\big|^2}{\ell\big(T-u, X(T-u)\big)^2} \, \mathrm{d}u \geqslant n \right\} \wedge T, \qquad n \in \mathbf{N}_0, \tag{5.14}$$

is nondecreasing and converges $\mathbf{P}$-a.s. to T. Defining M via (4.5), each stopped process M^{τ_n} is bounded in $L^2(\mathbf{P})$ and satisfies the stopped version of (4.6), i.e.,

$$\begin{aligned} M^{\tau_n}(T-s) &= M\big(T-(s\wedge\tau_n)\big) \\ &= \int_0^{s\wedge\tau_n} \left\langle \frac{\nabla \ell\big(T-u, X(T-u)\big)}{\ell\big(T-u, X(T-u)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}}(T-u) \right\rangle, \qquad 0 \leqslant s \leqslant T. \end{aligned} \tag{5.15}$$

To show that, in fact, the process M is a true $\mathbf{P}$-martingale, we have to rely on condition (2.6), which asserts that the initial relative entropy $H(P(0)\,|\,\mathrm{Q})$ is finite.

We consider the process

$$\ell^{-1}\big(T-s, X(T-s)\big) = \frac{q\big(X(T-s)\big)}{p\big(T-s, X(T-s)\big)}, \qquad 0 \leqslant s \leqslant T, \tag{5.16}$$

where $\ell^{-1}(t,\,\cdot\,) = 1/\ell(t,\,\cdot\,)$ is the likelihood ratio function of $\frac{\mathrm{dQ}}{\mathrm{d}P(t)}(\,\cdot\,)$. Applying Itô's formula and using (5.7), we find the stochastic differential

$$\mathrm{d}\ell^{-1}\big(T-s, X(T-s)\big) = -\left\langle \frac{\nabla \ell\big(T-s, X(T-s)\big)}{\ell\big(T-s, X(T-s)\big)^2}, \mathrm{d}\overline{W}^{\mathbf{P}}(T-s) \right\rangle, \tag{5.17}$$

revealing that the locally bounded process (5.16) is a local martingale under $\mathbf{P}$. In fact, this result does not come as a surprise: it is a consequence of an eye-opening result by Pavon [47], and Fontbona and Jourdain [22], at least when Q is a finite measure. We refer the reader to subsection 4.2 of [34] for more information on this theme and for a more direct proof of Theorem 4.1 in the case when Q is a finite measure on $\mathbf{R}^n$; see also Theorem 4.2 and Appendix E in [34] for an extensive discussion and a proof of the Pavon–Fontbona–Jourdain theorem.

From (5.17), we deduce the stochastic differential of the logarithm of the process (5.16) and obtain in accordance with (5.9) its form

$$\begin{aligned} \mathrm{d}\ln \ell^{-1}\big(T-s, X(T-s)\big) = &-\left\langle \frac{\nabla \ell\big(T-s, X(T-s)\big)}{\ell\big(T-s, X(T-s)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}}(T-s) \right\rangle \\ &- \frac{1}{2} \frac{\big|\nabla \ell\big(T-s, X(T-s)\big)\big|^2}{\ell\big(T-s, X(T-s)\big)^2} \, \mathrm{d}s. \end{aligned} \tag{5.18}$$

We know that the terminal value $\ln \ell^{-1}(0, X(0))$ is $\mathbf{P}$-integrable, with

$$\mathbf{E_P}\big[\ln \ell^{-1}\big(0, X(0)\big)\big] = -H\big(P(0)\,|\,\mathrm{Q}\big) \in (-\infty, \infty). \tag{5.19}$$

On the other hand, the initial value

$$\mathbf{E_P}\big[\ln \ell^{-1}\big(T, X(T)\big)\big] = -H\big(P(T)\,|\,\mathrm{Q}\big) \in [-\infty, \infty) \tag{5.20}$$

cannot take the value ∞, as mentioned after the definition (2.3) of relative entropy. Hence we can apply Proposition A.3 in Appendix A to the local martingale (5.16) (in the reverse direction of time) and the deterministic stopping time $\tau = T$, to conclude that

$$\begin{aligned} &\mathbf{E_P}\big[\ln \ell^{-1}\big(0, X(0)\big)\big] - \mathbf{E_P}\big[\ln \ell^{-1}\big(T, X(T)\big)\big] \\ &\qquad = -\mathbf{E_P}\bigg[\int_0^T \frac{1}{2}\,\frac{\big|\nabla \ell\big(T-u, X(T-u)\big)\big|^2}{\ell\big(T-u, X(T-u)\big)^2}\,\mathrm{d}u\bigg], \end{aligned} \tag{5.21}$$

where all terms are well defined and finite. This shows that the local martingale M is bounded in $L^2(\mathbf{P})$, with

$$\|M(0)\|^2_{L^2(\mathbf{P})} = H\big(P(0)\,|\,\mathrm{Q}\big) - H\big(P(T)\,|\,\mathrm{Q}\big) = \frac{1}{2}\int_0^T I\big(P(t)\,|\,\mathrm{Q}\big)\,\mathrm{d}t < \infty, \tag{5.22}$$

completing the proof of Theorem 4.1.

5.2. The proof of Theorem 4.2. The first step in the proof of Theorem 4.2 is to compute the stochastic differentials of the time-reversed perturbed likelihood ratio process

$$\ell^\beta\big(T-s, X(T-s)\big) = \frac{p^\beta\big(T-s, X(T-s)\big)}{q\big(X(T-s)\big)}, \qquad 0 \leqslant s \leqslant T - t_0, \tag{5.23}$$

and its logarithm. By analogy with Proposition 5.1, the following result is well known (see, e.g., Theorems G.2 and G.5 in Appendix G of [34]) to hold under suitable regularity conditions, such as Assumptions 2.2. Recall that $(W^\beta(t))_{t \geqslant t_0}$ denotes the $\mathbf{P}^\beta$-Brownian motion (in the forward direction of time) defined in (3.10).

PROPOSITION 5.3. *Under Assumptions* 2.2, *we let* $t_0 \geqslant 0$ *and* $T > t_0$. *The process*

$$\overline{W}^{\mathbf{P}^\beta}(T-s) := W^\beta(T-s) - W^\beta(T) - \int_0^s \nabla \ln p^\beta\big(T-u, X(T-u)\big)\,\mathrm{d}u \tag{5.24}$$

for $0 \leqslant s \leqslant T - t_0$ *is a Brownian motion of the backwards filtration* $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T-t_0}$ *under the probability measure* $\mathbf{P}^\beta$. *Furthermore, the semimartingale decomposition of the time-reversed canonical coordinate process* $(X(T-s))_{0 \leqslant s \leqslant T-t_0}$ *is given by*

$$\mathrm{d}X(T-s) = \Big(\nabla \ln p^\beta\big(T-s, X(T-s)\big) + \big(\nabla\Psi + \beta\big)\big(X(T-s)\big)\Big)\,\mathrm{d}s + \mathrm{d}\overline{W}^{\mathbf{P}^\beta}(T-s) \tag{5.25}$$

$$= \Big(\nabla \ln \ell^\beta\big(T-s, X(T-s)\big) - \big(\nabla\Psi - \beta\big)\big(X(T-s)\big)\Big)\,\mathrm{d}s + \mathrm{d}\overline{W}^{\mathbf{P}^\beta}(T-s) \tag{5.26}$$

for $0 \leqslant s \leqslant T - t_0$, *with respect to the backwards filtration* $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T-t_0}$.

Comparing (5.3) with (5.25), we see that the reverse-time Brownian motions $\overline{W}^{\mathbf{P}^\beta}$ and $\overline{W}^{\mathbf{P}}$ are related as follows.

Lemma 5.4. *Under Assumptions* 2.2, *let* $t_0 \geqslant 0$ *and* $T > t_0$. *Then, for* $0 \leqslant s \leqslant T - t_0$,

$$\mathrm{d}\big(\overline{W}^{\mathbf{P}} - \overline{W}^{\mathbf{P}^\beta}\big)(T-s) = \left(\beta\big(X(T-s)\big) + \nabla \ln\left(\frac{p^\beta\big(T-s, X(T-s)\big)}{p\big(T-s, X(T-s)\big)}\right)\right) \mathrm{d}s \tag{5.27}$$

$$= \left(\beta\big(X(T-s)\big) + \nabla \ln\left(\frac{\ell^\beta\big(T-s, X(T-s)\big)}{\ell\big(T-s, X(T-s)\big)}\right)\right) \mathrm{d}s. \tag{5.28}$$

Remark 5.5. We apply Lemma 5.4 down the road, when s is close to $T - t_0$. In this case the logarithmic gradients in (5.27) and (5.28) will become small in view of $p^\beta(t_0, \cdot\,) = p(t_0, \cdot\,)$, so that these logarithmic gradients will disappear in the limit $s \uparrow T - t_0$; see also Lemma 5.9 below. By contrast, the term $\beta(X(T-s))$ will not go away in the limit $s \uparrow T - t_0$. Rather, it will tend to the r.v. $\beta(X(t_0))$, which plays an important role in distinguishing between (4.37) and (4.38) in Proposition 4.7.

By analogy with the proof of Proposition 5.2, for $t > t_0$, we write now the perturbed Fokker–Planck equation (3.8) as

$$\partial_t p^\beta(t,x) = \frac{1}{2}\,\Delta p^\beta(t,x) + \langle \nabla p^\beta(t,x), \nabla\Psi(x) + \beta(x)\rangle + p^\beta(t,x)\big(\Delta\Psi(x) + \operatorname{div}\beta(x)\big). \tag{5.29}$$

Using the relation

$$p^\beta(t,x) = \ell^\beta(t,x) q(x) = \ell^\beta(t,x)\,\mathrm{e}^{-2\Psi(x)}, \qquad t \geqslant t_0, \tag{5.30}$$

determined computation shows that the perturbed likelihood ratio function $\ell^\beta(t,x)$ satisfies

$$\begin{aligned}\partial_t \ell^\beta(t,x) = {}& \frac{1}{2}\,\Delta \ell^\beta(t,x) + \langle \nabla \ell^\beta(t,x), \beta(x) - \nabla\Psi(x)\rangle \\ & + \ell^\beta(t,x)\big(\operatorname{div}\beta(x) - \langle \beta(x), 2\nabla\Psi(x)\rangle\big), \qquad t > t_0;\end{aligned} \tag{5.31}$$

this is the analogue of the backwards Kolmogorov equation (5.12) in this "perturbed" context, and reduces to (5.12) when $\beta \equiv 0$.

With these preparations, we obtain the following stochastic differentials for our objects of interest.

Lemma 5.6. *Under Assumptions* 2.2, *we let* $t_0 \geqslant 0$ *and* $T > t_0$. *The time-reversed perturbed likelihood ratio process* (5.23) *and its logarithm satisfy the stochastic differential equations*

$$\begin{aligned}\frac{\mathrm{d}\ell^\beta\big(T-s, X(T-s)\big)}{\ell^\beta\big(T-s, X(T-s)\big)} = {}& \big(\langle \beta, 2\nabla\Psi\rangle - \operatorname{div}\beta\big)\big(X(T-s)\big)\,\mathrm{d}s \\ & + \frac{\big|\nabla \ell^\beta\big(T-s, X(T-s)\big)\big|^2}{\ell^\beta\big(T-s, X(T-s)\big)^2}\,\mathrm{d}s + \left\langle \frac{\nabla \ell^\beta\big(T-s, X(T-s)\big)}{\ell^\beta\big(T-s, X(T-s)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}^\beta}(T-s)\right\rangle\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\mathrm{d}\ln \ell^\beta\big(T-s, X(T-s)\big) = {}& \big(\langle \beta, 2\nabla\Psi\rangle - \operatorname{div}\beta\big)\big(X(T-s)\big)\,\mathrm{d}s \\ & + \frac{1}{2}\frac{\big|\nabla \ell^\beta\big(T-s, X(T-s)\big)\big|^2}{\ell^\beta\big(T-s, X(T-s)\big)^2}\,\mathrm{d}s + \left\langle \frac{\nabla \ell^\beta\big(T-s, X(T-s)\big)}{\ell^\beta\big(T-s, X(T-s)\big)}, \mathrm{d}\overline{W}^{\mathbf{P}^\beta}(T-s)\right\rangle,\end{aligned} \tag{5.33}$$

respectively, for $0 \leqslant s \leqslant T - t_0$, *with respect to the backwards filtration* $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T-t_0}$.

Proof. The equations (5.32), (5.33) follow from Itô's formula together with (5.26) and (5.31).

We have assembled now all the ingredients needed for the proof of Theorem 4.2.

Proof of Theorem 4.2. Formally, the stochastic differential in (5.33) amounts to the conclusions (4.10)–(4.12) of Theorem 4.2. But as in the proof of Theorem 4.1, we still have to substantiate the claim that the stochastic process M^β defined in (4.11) with representation (4.12) is indeed a $\mathbf{P}^\beta$-martingale of the backwards filtration $(\mathcal{G}(T-s))_{0 \leqslant s \leqslant T-t_0}$, and is bounded in $L^2(\mathbf{P}^\beta)$.

By (5.33) and the same stopping argument as in the proof of Theorem 4.1, the process M^β is a local $\mathbf{P}^\beta$-martingale. We have to show that $\mathbf{E}_{\mathbf{P}^\beta}[F^\beta(t_0)] < \infty$.

We recall that $\beta = \nabla B$ and define the density

$$q^\beta(x) := \mathrm{e}^{-2(\Psi + B)(x)}, \qquad x \in \mathbf{R}^n. \tag{5.34}$$

This density function solves the stationary version of the perturbed Fokker–Planck equation (3.8). Equivalently, it induces an invariant measure for the stochastic differential equation (3.10). We now consider the "doubly perturbed" likelihood ratio function

$$\ell^\beta_\beta(t,x) := \frac{p^\beta(t,x)}{q^\beta(x)}, \qquad (t,x) \in [t_0,\infty) \times \mathbf{R}^n. \tag{5.35}$$

Assumptions 2.2 are invariant under the passage from the potential Ψ to $\Psi + B$, so we can apply Theorem 4.1 to the potential $\Psi + B$ and obtain that the process (cf. (4.4))

$$F^\beta_\beta(T-s) := \int_0^s \frac{1}{2} \frac{\big|\nabla \ell^\beta_\beta\big(T-u, X(T-u)\big)\big|^2}{\ell^\beta_\beta\big(T-u, X(T-u)\big)^2}\, \mathrm{d}u, \quad 0 \leqslant s \leqslant T - t_0, \tag{5.36}$$

satisfies $\mathbf{E}_{\mathbf{P}^\beta}[F^\beta_\beta(t_0)] < \infty$. This latter condition implies also that $\mathbf{E}_{\mathbf{P}^\beta}[F^\beta(t_0)] < \infty$, where the process F^β is defined as in (4.10). Indeed, the function $\langle \beta, 2\nabla\Psi\rangle - \operatorname{div}\beta$ in (4.10) is bounded, so that

$$\mathbf{E}_{\mathbf{P}^\beta}\left[\int_0^{T-t_0} |\langle \beta, 2\nabla\Psi\rangle - \operatorname{div}\beta|\big(X(T-u)\big)\,\mathrm{d}u\right] < \infty. \tag{5.37}$$

As regards the remaining difference between (5.36) and (4.10), note that $\ell^\beta(t,x)/\ell^\beta_\beta(t,x) = \mathrm{e}^{2B(x)}$ and consequently $\nabla \ln \ell^\beta(t,x) - \nabla \ln \ell^\beta_\beta(t,x) = 2\nabla B(x)$, which again is a bounded function.

In conclusion, we obtain $\mathbf{E}_{\mathbf{P}^\beta}[F^\beta(t_0)] < \infty$, finishing the proof of Theorem 4.2.

5.3. Some useful lemmas. In this subsection we collect some useful results needed to justify the claims (4.19), (4.20) made in the course of the proof of Corollary 4.4, and to complete the proof of Proposition 4.7 in subsection 5.4.

First, let us introduce the "perturbed-to-unperturbed" ratio

$$Y^\beta(t,x) := \frac{\ell^\beta(t,x)}{\ell(t,x)} = \frac{p^\beta(t,x)}{p(t,x)}, \qquad (t,x) \in [t_0,\infty) \times \mathbf{R}^n, \tag{5.38}$$

and recall the backwards Kolmogorov-type equations (5.12), (5.31). These lead to the equation

$$\begin{aligned} \partial_t Y^\beta(t,x) &= \frac{1}{2}\,\Delta Y^\beta(t,x) + \langle \nabla Y^\beta(t,x),\, \beta(x) + \nabla \ln p(t,x) + \nabla\Psi(x)\rangle \\ &\quad + Y^\beta(t,x)\big(\operatorname{div}\beta(x) + \langle \beta(x), \nabla \ln p(t,x)\rangle\big), \qquad t > t_0, \end{aligned} \tag{5.39}$$

with $Y^\beta(t_0,\,\cdot\,) = 1$, for the ratio in (5.38). In conjunction with (5.3), this equation leads by direct calculation to the following backwards dynamics.

LEMMA 5.7. *Under Assumptions* 2.2, *let* $t_0 \geqslant 0$ *and* $T > t_0$. *The time-reversed ratio process* $\big(Y^\beta(T-s, X(T-s))\big)_{0\leqslant s\leqslant T-t_0}$ *and its logarithm satisfy the stochastic differential equations*

$$\begin{aligned} &\frac{\mathrm{d}Y^\beta\big(T-s, X(T-s)\big)}{Y^\beta\big(T-s, X(T-s)\big)} \\ &\quad = \left\langle \frac{\nabla Y^\beta\big(T-s, X(T-s)\big)}{Y^\beta\big(T-s, X(T-s)\big)},\, \mathrm{d}\overline{W}^{\mathbf{P}}(T-s) - \beta\big(X(T-s)\big)\,\mathrm{d}s \right\rangle \\ &\qquad - \Big(\operatorname{div}\beta\big(X(T-s)\big) + \big\langle \beta\big(X(T-s)\big), \nabla \ln p\big(T-s, X(T-s)\big)\big\rangle\Big)\,\mathrm{d}s \end{aligned} \tag{5.40}$$

and

$$\begin{aligned} &\mathrm{d}\ln Y^\beta\big(T-s, X(T-s)\big) \\ &\quad = \left\langle \frac{\nabla Y^\beta\big(T-s, X(T-s)\big)}{Y^\beta\big(T-s, X(T-s)\big)},\, \mathrm{d}\overline{W}^{\mathbf{P}}(T-s) - \beta\big(X(T-s)\big)\,\mathrm{d}s \right\rangle \\ &\qquad - \Big(\operatorname{div}\beta\big(X(T-s)\big) + \big\langle \beta\big(X(T-s)\big), \nabla \ln p\big(T-s, X(T-s)\big)\big\rangle\Big)\,\mathrm{d}s \\ &\qquad - \frac{1}{2}\,\frac{\big|\nabla Y^\beta\big(T-s, X(T-s)\big)\big|^2}{Y^\beta\big(T-s, X(T-s)\big)^2}\,\mathrm{d}s, \end{aligned} \tag{5.41}$$

respectively, for $0 \leqslant s \leqslant T-t_0$, *relative to the filtration* $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T-t_0}$.

We first establish a preliminary control on $Y^\beta(\,\cdot\,,\,\cdot\,)$, which will be refined in Lemma 5.9 below.

LEMMA 5.8. *Under Assumptions* 2.2, *let* $t_0 \geqslant 0$ *and* $T > t_0$. *There is a real constant* $C > 1$ *such that*

$$\frac{1}{C} \leqslant Y^\beta(t,x) \leqslant C, \qquad (t,x) \in [t_0,T] \times \mathbf{R}^n. \tag{5.42}$$

Proof. In the forward direction of time, the canonical coordinate process $(X(t))_{t_0\leqslant t\leqslant T}$ on the path space $\Omega = \mathcal{C}([t_0,T];\mathbf{R}^n)$ satisfies equations (1.3) and (3.10) with initial distribution $P(t_0)$ under the probability measures $\mathbf{P}$ and $\mathbf{P}^\beta$, respectively. Hence the $\mathbf{P}$-Brownian motion $(W(t))_{t_0\leqslant t\leqslant T}$ from (1.3) can be represented as

$$W(t) - W(t_0) = W^\beta(t) - W^\beta(t_0) - \int_{t_0}^{t} \beta\big(X(u)\big)\,\mathrm{d}u, \quad t_0 \leqslant t \leqslant T, \tag{5.43}$$

where $(W^\beta(t))_{t_0\leqslant t\leqslant T}$ is the $\mathbf{P}^\beta$-Brownian motion appearing in (3.10). By the Girsanov theorem, this amounts, for $t_0 \leqslant t \leqslant T$, to the likelihood ratio computation

$$Z(t) := \frac{\mathrm{d}\mathbf{P}^\beta}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}(t)} = \exp\bigg(-\int_{t_0}^{t} \big\langle \beta\big(X(u)\big), \mathrm{d}W(u)\big\rangle - \frac{1}{2}\int_{t_0}^{t} \big|\beta\big(X(u)\big)\big|^2\,\mathrm{d}u\bigg). \tag{5.44}$$

Now, for each $(t,x) \in [t_0,T] \times \mathbf{R}^n$, the ratio $Y^\beta(t,x) = p^\beta(t,x)/p(t,x)$ equals the conditional expectation of the r.v. (5.44) with respect to the probability measure $\mathbf{P}$, where we condition on $X(t)=x$; to wit,

$$Y^\beta(t,x) = \mathbf{E_P}\big[Z(t)\,|\,X(t)=x\big], \qquad (t,x) \in [t_0,T] \times \mathbf{R}^n. \tag{5.45}$$

Therefore, in order to obtain the estimate (5.42), it suffices to show that the log-density process $(\ln Z(t))_{t_0 \leqslant t \leqslant T}$ is uniformly bounded. Since the perturbation β is smooth and has compact support, the Lebesgue integral inside the exponential of (5.44) is uniformly bounded, as required.

In order to handle the stochastic integral with respect to the $\mathbf{P}$-Brownian motion $(W(u))_{t_0 \leqslant u \leqslant t}$ inside the exponential (5.44), we invoke the assumption that the vector field β equals the gradient of a potential $B\colon \mathbf{R}^n \to \mathbf{R}$, which is of class $\mathcal{C}^\infty(\mathbf{R}^n;\mathbf{R})$ and has compact support. According to Itô's formula and (1.3), we can express the stochastic integral appearing in (5.44) as

$$\int_{t_0}^t \big\langle \beta\big(X(u)\big), \mathrm{d}W(u)\big\rangle = B\big(X(t)\big) - B\big(X(t_0)\big) + \int_{t_0}^t \Big(\langle \beta, \nabla\Psi\rangle - \frac{1}{2}\operatorname{div}\beta\Big)\big(X(u)\big)\,\mathrm{d}u \tag{5.46}$$

for $t_0 \leqslant t \leqslant T$. At this stage it becomes obvious that the expression of (5.46) is uniformly bounded. This completes the proof of Lemma 5.8.

The following Lemma 5.9 provides the crucial estimates (4.19) and (4.20) required in the proofs of Corollary 4.4 and Proposition 4.7.

LEMMA 5.9. *Under Assumptions* 2.2, *let* $t_0 \geqslant 0$ *and* $T > t_0$. *There is a constant* $C > 0$ *such that*

$$|Y^\beta(T-s,x) - 1| \leqslant C(T-t_0-s), \tag{5.47}$$

as well as

$$\mathbf{E_P}\bigg[\int_s^{T-t_0} \big|\nabla \ln Y^\beta\big(T-u, X(T-u)\big)\big|^2\,\mathrm{d}u \;\bigg|\; X(T-s)=x\bigg] \leqslant C(T-t_0-s)^2, \tag{5.48}$$

hold for all $0 \leqslant s \leqslant T-t_0$ *and* $x \in \mathbf{R}^n$. *Furthermore, for every* $t_0 > 0$ *and* $x \in \mathbf{R}^n$, *the pointwise limiting assertion*

$$\lim_{s \uparrow T-t_0} \frac{\ln Y^\beta(T-s,x)}{T-t_0-s} = \operatorname{div}\beta(x) + \langle \beta(x), \nabla \ln p(t_0,x)\rangle \tag{5.49}$$

holds, where the fraction on the left of (5.49) *is uniformly bounded on* $[0,T-t_0] \times \mathbf{R}^n$.

Remark 5.10. The pointwise limiting assertion (5.49) is the deterministic analogue of the trajectorial relation (4.39) from Proposition 4.7. In subsection 5.4 below we will prove that the limiting assertion (4.39) holds in L^1 under both $\mathbf{P}$ and $\mathbf{P}^\beta$ and is valid for all $t_0 > 0$.

Proof. As $\ln Y^\beta = \ln \ell^\beta - \ln \ell$, we obtain from Theorems 4.1 and 4.2 and (5.42) that the martingale part of the process in (5.41) is bounded in $L^2(\mathbf{P})$, i.e.,

$$\mathbf{E_P}\bigg[\int_0^{T-t_0} \frac{\big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2}{Y^\beta\big(T-u, X(T-u)\big)^2}\,\mathrm{d}u\bigg] < \infty. \tag{5.50}$$

Once again using (5.42), we compare $\nabla Y^\beta / Y^\beta$ with ∇Y^β to see that (5.50) also implies

$$\mathbf{E}_{\mathbf{P}}\left[\int_0^{T-t_0} \big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2 \,\mathrm{d}u\right] < \infty. \tag{5.51}$$

By (5.40), the time-reversed ratio process $\big(Y^\beta(T-s, X(T-s))\big)_{0\leqslant s\leqslant T-t_0}$ satisfies with respect to the backwards filtration $(\mathcal{G}(T-s))_{0\leqslant s\leqslant T-t_0}$ the stochastic differential equation

$$\begin{aligned}
&\mathrm{d}Y^\beta\big(T-s, X(T-s)\big)\\
&\quad= \big\langle \nabla Y^\beta\big(T-s, X(T-s)\big), \,\mathrm{d}\overline{W}^{\mathbf{P}}(T-s) - \beta\big(X(T-s)\big)\,\mathrm{d}s\big\rangle\\
&\qquad - Y^\beta\big(T-s, X(T-s)\big)\big(\operatorname{div}\beta\big(X(t-s)\big)\\
&\qquad\qquad + \big\langle \beta\big(X(T-s)\big), \nabla \ln p\big(T-s, X(T-s)\big)\big\rangle\big)\,\mathrm{d}s.
\end{aligned} \tag{5.52}$$

In view of (5.51), the martingale part in (5.52) is bounded in $L^2(\mathbf{P})$. As regards the drift term of this equation, we observe that it vanishes when $X(T-s)$ takes values outside the compact support of the smooth vector field β. Consequently, the drift term is bounded, i.e., the constant

$$C_1 := \sup_{\substack{t_0\leqslant t\leqslant T\\ y\in\mathbf{R}^n}} \left| -Y^\beta(t,y)\left(\operatorname{div}\beta(y) + \left\langle \beta(y), \nabla\ln p(t,y) + \frac{\nabla Y^\beta(t,y)}{Y^\beta(t,y)}\right\rangle\right)\right| \tag{5.53}$$

is finite, and the processes

$$Y^\beta\big(T-s, X(T-s)\big) + C_1 s \quad \text{and} \quad Y^\beta\big(T-s, X(T-s)\big) - C_1 s \tag{5.54}$$

for $0\leqslant s\leqslant T-t_0$ are a sub- and a supermartingale, respectively. We conclude that

$$\big|Y^\beta(T-s,x) - \mathbf{E}_{\mathbf{P}}\big[Y^\beta\big(t_0, X(t_0)\big) \;\big|\; X(T-s)=x\big]\big| \leqslant C_1(T-t_0-s) \tag{5.55}$$

holds for all $0\leqslant s\leqslant T-t_0$ and $x\in\mathbf{R}^n$. Since $Y^\beta(t_0,\,\cdot\,)=1$, this establishes the first estimate

$$|Y^\beta(T-s,x) - 1| \leqslant C_1(T-t_0-s). \tag{5.56}$$

Now we turn our attention to the second estimate (5.48). We fix $0\leqslant s\leqslant T-t_0$ and $x\in\mathbf{R}^n$. By means of the stochastic differentials in (5.41) and (5.52), we find that the expression

$$\frac{1}{2}\,\mathbf{E}_{\mathbf{P}}\left[\int_s^{T-t_0} \big|\nabla\ln Y^\beta\big(T-u, X(T-u)\big)\big|^2\,\mathrm{d}u \;\middle|\; X(T-s)=x\right] \tag{5.57}$$

is equal to

$$\ln Y^\beta(T-s,x) - Y^\beta(T-s,x) + 1 + \mathbf{E}_{\mathbf{P}}\left[\int_s^{T-t_0} G\big(T-u, X(T-u)\big)\,\mathrm{d}u \;\middle|\; X(T-s)=x\right], \tag{5.58}$$

where we have set

$$G(t,y) := \big(Y^\beta(t,y) - 1\big)\left(\operatorname{div}\beta(y) + \left\langle \beta(y), \nabla\ln p(t,y) + \frac{\nabla Y^\beta(t,y)}{Y^\beta(t,y)}\right\rangle\right) \tag{5.59}$$

for $t_0 \leqslant t \leqslant T$ and $y \in \mathbf{R}^n$. Introducing the finite constant

$$C_2 := \sup_{\substack{t_0 \leqslant t \leqslant T \\ y \in \mathbf{R}^n}} \left| \operatorname{div} \beta(y) + \left\langle \beta(y), \nabla \ln p(t,y) + \frac{\nabla Y^\beta(t,y)}{Y^\beta(t,y)} \right\rangle \right| \tag{5.60}$$

and using the just proved estimate (5.56), we see that the absolute value of the conditional expectation appearing in (5.58) can be bounded by $C_1 C_2 (T - t_0 - s)^2$. In order to handle the remaining terms of (5.58), we apply the elementary inequality $\ln p \leqslant p - 1$, which is valid for all $p > 0$, and obtain

$$\ln Y^\beta(T-s,x) - Y^\beta(T-s,x) + 1 \leqslant 0. \tag{5.61}$$

This implies that the expression of (5.57) is bounded by $C_1 C_2 (T - t_0 - s)^2$, which establishes the second estimate (5.48). We also note that the elementary inequality (5.61) in conjunction with the estimate (5.56) shows that

$$\ln Y^\beta(T-s,x) \leqslant C_1 (T - t_0 - s) \tag{5.62}$$

for all $0 \leqslant s \leqslant T - t_0$ and $x \in \mathbf{R}^n$; this implies that the fraction on the left-hand side of (5.49) is uniformly bounded on $[0, T - t_0] \times \mathbf{R}^n$.

Regarding the limiting assertion (5.49), we fix $t_0 > 0$, $x \in \mathbf{R}^n$, and $0 \leqslant s \leqslant T - t_0$ and take conditional expectations with respect to $X(T-s) = x$ in the integral version of the stochastic differential (5.41). On account of (5.50), the stochastic integral with respect to the $\mathbf{P}$-Brownian motion $(\overline{W}^{\mathbf{P}}(T-s))_{0 \leqslant s \leqslant T}$ in (5.41) vanishes. Dividing by $T - t_0 - s$ and passing to the limit as $s \uparrow T - t_0$, we can use the estimate (5.48) to deduce that the expression in the fourth line of (5.41) vanishes in the limit. Applying the Cauchy–Schwarz inequality, we see that the normalized integral involving the perturbation β appearing in the first line of (5.41) can be bounded by

$$\frac{1}{T - t_0 - s} \int_s^{T - t_0} \left| \nabla \ln Y^\beta\big(T-u, X(T-u)\big) \right| \cdot \left| \beta\big(X(T-u)\big) \right| \mathrm{d}u. \tag{5.63}$$

By conditions (iv) and (v) of Assumptions 2.2, the function $(t,x) \mapsto \nabla \ln Y^\beta(t,x)$ is continuous on $(0,\infty) \times \mathbf{R}^n$, and thus the expression in (5.63) is uniformly bounded on the rectangle $[0, T - t_0] \times \operatorname{supp} \beta$. As $\ln Y^\beta(t_0, \cdot\,) = 0$, it converges $\mathbf{P}$-a.s. to zero; hence also

$$\begin{aligned} &\lim_{s \uparrow T - t_0} \mathbf{E}_{\mathbf{P}} \left[\frac{1}{T - t_0 - s} \int_s^{T - t_0} \left| \nabla \ln Y^\beta\big(T-u, X(T-u)\big) \right| \right. \\ &\qquad \left. \cdot \left| \beta\big(X(T-u)\big) \right| \mathrm{d}u \,\middle|\, X(T-s) = x \right] = 0. \end{aligned} \tag{5.64}$$

Finally, the continuity and uniform boundedness imply that the conditional expectations of the normalized integrals over the second line of (5.41) converge to the right-hand side of (5.49), as claimed. Lemma 5.9 is proved.

5.4. Completing the proof of Proposition 4.7. With the preparations of subsection 5.3, we are now able to complete the proof of Proposition 4.7 by establishing the remaining limiting assertions (4.39) and (4.38) therein.

Proof of (4.39) *in Proposition* 4.7. Let $t_0 > 0$ and select $T > t_0$. Using the notation of (5.38) above, we have to calculate the limit

$$\lim_{s \uparrow T-t_0} \frac{\ln Y^\beta\big(T-s, X(T-s)\big)}{T-t_0-s}. \tag{5.65}$$

Fix $0 \leqslant s \leqslant T-t_0$. According to the integral version of the stochastic differential (5.41), the fraction in (5.65) is equal to the sum of the following four normalized integral terms (5.66)–(5.68) and (5.70), whose behavior as $s \uparrow T-t_0$ we will study separately below. By conditions (iv) and (v) of Assumptions 2.2, the function $(t,x) \mapsto \nabla \ln Y^\beta(t,x)$ is continuous on $(0,\infty) \times \mathbf{R}^n$; thus the first expression

$$\frac{1}{T-t_0-s} \int_s^{T-t_0} \big(\operatorname{div} \beta(X(T-u)) + \langle \beta(X(T-u)), \nabla \ln p(T-u, X(T-u))\rangle\big)\, \mathrm{d}u \tag{5.66}$$

is uniformly bounded on $[0, T-t_0] \times \operatorname{supp} \beta$. By continuity and uniform boundedness, we conclude that (5.66) converges $\mathbf{P}$-a.s. as well as in $L^1(\mathbf{P})$ to the right-hand side of (4.39), as required. Thus it remains to show that the three remaining terms converge to zero. Using the continuity and uniform boundedness once again, we deduce from $\ln Y^\beta(t_0, \cdot\,) = 0$ that the second integral term

$$\frac{1}{T-t_0-s} \int_s^{T-t_0} \left\langle \frac{\nabla Y^\beta\big(T-u, X(T-u)\big)}{Y^\beta\big(T-u, X(T-u)\big)},\, \beta\big(X(T-u)\big) \right\rangle \mathrm{d}u \tag{5.67}$$

converges to zero $\mathbf{P}$-a.s. and in $L^1(\mathbf{P})$. Since $\ln Y^\beta(t_0, \cdot\,) = 0$ and because the integrand is continuous, we see that the third expression

$$\frac{1}{T-t_0-s} \int_s^{T-t_0} \frac{1}{2} \frac{\big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2}{Y^\beta\big(T-u, X(T-u)\big)^2}\, \mathrm{d}u \tag{5.68}$$

converges $\mathbf{P}$-a.s. to zero. Furthermore, owing to Lemma 5.9, there is a constant $C > 0$ such that

$$\mathbf{E}_\mathbf{P}\left[\frac{1}{T-t_0-s} \int_s^{T-t_0} \frac{\big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2}{Y^\beta\big(T-u, X(T-u)\big)^2}\, \mathrm{d}u\right] \leqslant C(T-t_0-s) \tag{5.69}$$

holds for all $0 \leqslant s \leqslant T-t_0$, which implies that (5.68) converges to zero also in $L^1(\mathbf{P})$.

The fourth and last term is the stochastic integral

$$-\frac{1}{T-t_0-s} \int_s^{T-t_0} \left\langle \frac{\nabla Y^\beta\big(T-u, X(T-u)\big)}{Y^\beta\big(T-u, X(T-u)\big)}, \mathrm{d}\overline{W}^\mathbf{P}(T-u) \right\rangle. \tag{5.70}$$

The expression (5.68) converges to zero $\mathbf{P}$-a.s., and according to (5.69) we have

$$\mathbf{E}_\mathbf{P}\left[\frac{1}{(T-t_0-s)^2} \int_s^{T-t_0} \frac{\big|\nabla Y^\beta\big(T-u, X(T-u)\big)\big|^2}{Y^\beta\big(T-u, X(T-u)\big)^2}\, \mathrm{d}u\right] \leqslant C. \tag{5.71}$$

By means of the Itô isometry, we deduce that the expression

$$\mathbf{E}_\mathbf{P}\left[\left(\frac{1}{T-t_0-s} \int_s^{T-t_0} \left\langle \frac{\nabla Y^\beta\big(T-u, X(T-u)\big)}{Y^\beta\big(T-u, X(T-u)\big)}, \mathrm{d}\overline{W}^\mathbf{P}(T-u) \right\rangle\right)^2\right] \tag{5.72}$$

converges to zero as $s \uparrow T - t_0$. In other words, the normalized stochastic integral of (5.70) converges to zero in $L^2(\mathbf{P})$.

Summing up, we have shown that the limiting assertion (4.39) holds in $L^1(\mathbf{P})$ for every $t_0 > 0$. As we have seen in Lemma 5.8, the probability measures $\mathbf{P}$ and $\mathbf{P}^\beta$ are equivalent, the Radon–Nikodým derivatives $\mathrm{d}\mathbf{P}^\beta/\mathrm{d}\mathbf{P}$ and $\mathrm{d}\mathbf{P}/\mathrm{d}\mathbf{P}^\beta$ are bounded on the σ-algebra $\mathcal{F}(T) = \mathcal{G}(0)$, and therefore convergence in $L^1(\mathbf{P})$ is equivalent to convergence in $L^1(\mathbf{P}^\beta)$. This completes the proof of (4.39).

Proof of (4.38) *in Proposition* 4.7. This is proved in very much the same way, as (4.37) and (4.39). The only novelty here is the use of (5.27) to pass to the $\mathbf{P}$-Brownian motion $\overline{W}^{\mathbf{P}}$ from the $\mathbf{P}^\beta$-Brownian motion $\overline{W}^{\mathbf{P}^\beta}$ and the reliance on $\mathbf{E}_{\mathbf{P}^\beta}\big[F^\beta(t_0)\big] < \infty$ to ensure that the resulting stochastic integral is a (square-integrable) $\mathbf{P}$-martingale. We leave the details to the diligent reader.

6. The rate of growth for the Wasserstein distance. Let us recapitulate the message of Corollaries 4.3 and 4.4: in these results we compare the rate of decay for the relative entropy with the rate of growth for the quadratic Wasserstein distance W_2 along the curves $(P(t))_{t\geqslant 0}$ and $(P^\beta(t))_{t\geqslant t_0}$ in $\mathscr{P}_2(\mathbf{R}^n)$. This is the essence of the gradient flow property formalized in Theorem 3.2.

In order to complete the proofs of Corollaries 4.3 and 4.4, we have to establish the limits (4.15) and (4.18). The limit (4.15) is well known (see [4]) to exist, under suitable regularity assumptions, for *Lebesgue-a.e.* $t_0 \geqslant 0$. A similar remark pertains to the "perturbed" limit (4.18): if we replace t_0 by s_0 in (4.18), it is well known that this limit exists for *Lebesgue-a.e.* $s_0 \geqslant t_0$. But this is not what we need. We have to prove the validity of (4.18) for *the point* t_0 *itself*, in order to calculate the slope of the function $(H(P^\beta(t)\,|\,\mathrm{Q}))_{t\geqslant t_0}$ with respect to the Wasserstein distance *at time* t_0. After all, the deviation of $P^\beta(t)$ from $P(t)$ takes place at time t_0.

This technical aspect turns out to be quite delicate. We already needed a careful analysis (recall the estimates (4.19) and (4.20)) to show that the exceptional set N of (3.7), defined in terms of the decay of entropy of the unperturbed curve $(P(t))_{t\geqslant 0}$, does not change when passing to the perturbed curve $(P^\beta(t))_{t\geqslant t_0}$. In addition, we have to show that this set N also cannot increase when passing from the unperturbed Wasserstein limit (4.15) to its perturbed counterpart (4.18). In order to do this, we have to rely here (and only here) on condition (vi) of Assumptions 2.3.

For a detailed discussion of metric measure spaces and in particular Wasserstein spaces, we refer the reader to [3], [4], [54], and [55]. We also refer to section 5 in [34], where some results on quadratic Wasserstein transport are reviewed for the convenience of the reader.

For fixed $T \in (0,\infty)$, we define now the time-dependent velocity field

$$[0,T]\times\mathbf{R}^n \ni (t,x) \mapsto v(t,x) := -\bigg(\frac{1}{2}\,\frac{\nabla p(t,x)}{p(t,x)} + \nabla\Psi(x)\bigg) = -\frac{1}{2}\,\frac{\nabla \ell(t,x)}{\ell(t,x)} \in \mathbf{R}^n. \tag{6.1}$$

According to condition (vi) in Assumptions 2.3, this gradient vector field $v(t,\,\cdot\,)$ is an element of the tangent space (see Definition 8.4.1 in [4]) of $\mathscr{P}_2(\mathbf{R}^n)$ at the point $P(t) \in \mathscr{P}_2(\mathbf{R}^n)$, i.e.,

$$v(t,\,\cdot\,) \in \mathrm{Tan}_{P(t)}\,\mathscr{P}_2(\mathbf{R}^n) := \overline{\{\nabla\varphi\colon\ \varphi \in \mathcal{C}_c^\infty(\mathbf{R}^n;\mathbf{R})\}}^{\,L^2(P(t))}. \tag{6.2}$$

We can now formulate the "unperturbed" version of our desired result.

Theorem 6.1 (the limiting behavior of the quadratic Wasserstein distance). *Under Assumptions* 2.3, *let* $t_0 \geqslant 0$ *be such that the generalized de Bruijn identity* (3.3), (4.14) *is valid. Then we have the two-sided limit*

$$\lim_{t \to t_0} \frac{W_2\big(P(t), P(t_0)\big)}{|t - t_0|} = \big(\mathbf{E}_{\mathbf{P}}\big[\big|v\big(t_0, X(t_0)\big)\big|^2\big]\big)^{1/2} = \frac{1}{2}\sqrt{I\big(P(t_0)\,|\,\mathrm{Q}\big)}. \tag{6.3}$$

Before dealing with Theorem 6.1, we will prove the more general Theorem 6.2 below which amounts to the perturbed version of Theorem 6.1. For right-derivatives, the latter then simply follows by setting $\beta \equiv 0$ in the statement of Theorem 6.2.

We consider the "perturbed" curve $(P^\beta(t))_{t \geqslant t_0}$ in $\mathscr{P}_2(\mathbf{R}^n)$, as defined in (3.8)–(3.10), and define the time-dependent perturbed velocity field

$$[t_0, T] \times \mathbf{R}^n \ni (t, x) \longmapsto v^\beta(t, x) := -\bigg(\frac{1}{2}\,\frac{\nabla p^\beta(t, x)}{p^\beta(t, x)} + \nabla\Psi(x) + \beta(x)\bigg) \in \mathbf{R}^n. \tag{6.4}$$

At this point, we recall that the perturbation $\beta\colon \mathbf{R}^n \to \mathbf{R}^n$ is a *gradient vector field*, i.e., of the form $\beta = \nabla B$ for some smooth potential $B\colon \mathbf{R}^n \to \mathbf{R}$ with compact support. Since $p(t_0,\,\cdot\,) = p^\beta(t_0,\,\cdot\,)$, at time t_0 the vector fields of (6.1) and (6.4) are related via

$$v^\beta(t_0, x) = v(t_0, x) - \nabla B(x) = -\nabla\bigg(\frac{1}{2}\ln \ell(t_0, x) + B(x)\bigg), \qquad x \in \mathbf{R}^n. \tag{6.5}$$

Using the regularity assumption that the potential B is of class $\mathcal{C}_c^\infty(\mathbf{R}^n; \mathbf{R})$, we conclude from (6.2) and (6.5) that the perturbed vector field $v^\beta(t_0,\,\cdot\,)$ is also an element of the tangent space of $\mathscr{P}_2(\mathbf{R}^n)$ at the point $P^\beta(t_0) = P(t_0) \in \mathscr{P}_2(\mathbf{R}^n)$, i.e.,

$$v^\beta(t_0,\,\cdot\,) \in \mathrm{Tan}_{P^\beta(t_0)}\,\mathscr{P}_2(\mathbf{R}^n) = \overline{\{\nabla\varphi^\beta\colon\ \varphi^\beta \in \mathcal{C}_c^\infty(\mathbf{R}^n; \mathbf{R})\}}^{L^2(P^\beta(t_0))}. \tag{6.6}$$

Theorem 6.2 (the limiting behavior of the quadratic Wasserstein distance under perturbations). *Under Assumptions* 2.3, *for every point* $t_0 \in \mathbf{R}_+ \setminus N$ (*at which the right-sided limiting identity* (3.7) *is valid*), *we have the one-sided limit*

$$\lim_{t \downarrow t_0} \frac{W_2\big(P^\beta(t), P^\beta(t_0)\big)}{t - t_0} = \big(\mathbf{E}_{\mathbf{P}}\big[\big|v^\beta\big(t_0, X(t_0)\big)\big|^2\big]\big)^{1/2} = \frac{1}{2}\,\|a + 2b\|_{L^2(\mathbf{P})}. \tag{6.7}$$

Here $a = \nabla \ln \ell(t_0, X(t_0))$ *and* $b = \beta(X(t_0))$ *as in* (3.11).

Proof of Theorem 6.2. The second equality in (6.7) is apparent from the definition of the time-dependent perturbed velocity field $(v^\beta(t,\,\cdot\,))_{t \geqslant t_0}$ from (6.4) above. The delicate point is to show that the limiting assertion (6.7) is valid for every $t_0 \in \mathbf{R}_+ \setminus N$.

In order to see this, let us fix some $t_0 \in \mathbf{R}_+ \setminus N$ so that the limiting identity (3.7) is valid. In the following steps we prove that the limiting assertion (6.7) also holds.

Step 1. The gradient vector field $v^\beta(t_0,\,\cdot\,)$ induces a family of *linearized transport maps*

$$\mathcal{X}_t^\beta(x) := x + (t - t_0) \cdot v^\beta(t_0, x), \qquad x \in \mathbf{R}^n, \tag{6.8}$$

for $t \geqslant t_0$ in the manner of (4.42), and we denote by $P_{\mathcal{X}}^\beta(t)$ the image measure of $P^\beta(t_0) = P(t_0)$ under the transport map $\mathcal{X}_t^\beta\colon \mathbf{R}^n \to \mathbf{R}^n$, i.e.,

$$P_{\mathcal{X}}^\beta(t) := (\mathcal{X}_t^\beta)_\# P^\beta(t_0), \qquad t \geqslant t_0. \tag{6.9}$$

To motivate the arguments that follow, let us first pretend that, for all $t > t_0$ sufficiently close to t_0, the map $\mathcal{X}_t^\beta$ is the *optimal quadratic Wasserstein transport* from $P^\beta(t_0)$ to $P_{\mathcal{X}}^\beta(t)$, i.e.,

$$W_2^2\big(P_{\mathcal{X}}^\beta(t), P^\beta(t_0)\big) = \mathbf{E}_{\mathbf{P}^\beta}\big[\big|\mathcal{X}_t^\beta\big(X(t_0)\big) - X(t_0)\big|^2\big] = \mathbf{E}_{\mathbf{P}}\big[\big|\mathcal{X}_t^\beta\big(X(t_0)\big) - X(t_0)\big|^2\big], \tag{6.10}$$

where we have used in the last equality the fact that $X(t_0)$ has the same distribution under $\mathbf{P}^\beta$ as it does under $\mathbf{P}$. Then, on account of (6.8), we could conclude that

$$\lim_{t\downarrow t_0} \frac{W_2\big(P_{\mathcal{X}}^\beta(t), P^\beta(t_0)\big)}{t - t_0} = \big(\mathbf{E}_{\mathbf{P}}\big[\big|v^\beta\big(t_0, X(t_0)\big)\big|^2\big]\big)^{1/2} = \frac{1}{2}\,\|a + 2b\|_{L^2(\mathbf{P})}. \tag{6.11}$$

Furthermore, let us suppose that we can show the limiting identity

$$\lim_{t\downarrow t_0} \frac{W_2\big(P^\beta(t), P_{\mathcal{X}}^\beta(t)\big)}{t - t_0} = 0, \tag{6.12}$$

which has the interpretation that "the straight line $(P_{\mathcal{X}}^\beta(t))_{t\geqslant t_0}$ is *tangential* to the curve $(P^\beta(t))_{t\geqslant t_0}$." Using (6.11) and (6.12), we could now derive the desired equality (6.7). Indeed, invoking the triangle inequality for the quadratic Wasserstein distance, we obtain

$$\lim_{t\downarrow t_0} \frac{W_2\big(P_{\mathcal{X}}^\beta(t), P^\beta(t_0)\big)}{t - t_0} \leqslant \lim_{t\downarrow t_0} \frac{W_2\big(P_{\mathcal{X}}^\beta(t), P^\beta(t)\big)}{t - t_0} + \liminf_{t\downarrow t_0} \frac{W_2\big(P^\beta(t), P^\beta(t_0)\big)}{t - t_0}, \tag{6.13}$$

and one more application of the triangle inequality yields

$$\limsup_{t\downarrow t_0} \frac{W_2(P^\beta(t), P^\beta(t_0))}{t - t_0} \leqslant \lim_{t\downarrow t_0} \frac{W_2(P^\beta(t), P_{\mathcal{X}}^\beta(t))}{t - t_0} + \lim_{t\downarrow t_0} \frac{W_2(P_{\mathcal{X}}^\beta(t), P^\beta(t_0))}{t - t_0}. \tag{6.14}$$

Step 2. The bad news at this point is that there is little reason why, for $t > t_0$ sufficiently close to t_0, the map $\mathcal{X}_t^\beta$ defined in (6.8) of Step 1 should be optimal with respect to quadratic Wasserstein transportation costs, i.e., by Brenier's theorem [12], equal to the gradient of a *convex* function. The good news is that we can reduce the general case to the situation of optimal transports $\mathcal{X}_t^\beta$ as in Step 1 by localizing the vector field $v^\beta(t_0, \cdot\,)$ as well as the transport maps $(\mathcal{X}_t^\beta)_{t\geqslant t_0}$ to compact subsets of $\mathbf{R}^n$ (Steps 2–4) and that, after these localizations have been carried out, an analogue of the equality (6.12) also holds, allowing us to complete the argument (Steps 5 and 6).

To this end, we recall that $v^\beta(t_0, \cdot\,)$ from (6.5) is an element of the tangent space $\operatorname{Tan}_{P^\beta(t_0)} \mathscr{P}_2(\mathbf{R}^n)$ of the quadratic Wasserstein space $\mathscr{P}_2(\mathbf{R}^n)$ at the point $P^\beta(t_0) \in \mathscr{P}_2(\mathbf{R}^n)$. Thus, we can choose a sequence of functions $(\varphi_m^\beta(t_0, \cdot\,))_{m\geqslant 1} \subseteq \mathcal{C}_c^\infty(\mathbf{R}^n; \mathbf{R})$ such that

$$\lim_{m\to\infty} \mathbf{E}_{\mathbf{P}}\big[\big|v^\beta\big(t_0, X(t_0)\big) - \nabla\varphi_m^\beta\big(t_0, X(t_0)\big)\big|^2\big] = 0. \tag{6.15}$$

Next, for each $m \in \mathbf{N}$, we define the *localized gradient vector fields*

$$v_m^\beta(t_0, x) := \nabla\varphi_m^\beta(t_0, x), \qquad x \in \mathbf{R}^n. \tag{6.16}$$

These have compact support, approximate the gradient vector field $v^\beta(t_0, \cdot\,)$ in $L^2(P(t_0))$ as in (6.15), and induce a family of *localized linear transports* $(\mathcal{X}_t^{\beta,m})_{t\geqslant t_0}$ defined by analogy with (6.8) via

$$\mathcal{X}_t^{\beta,m}(x) := x + (t - t_0) \cdot v_m^\beta(t_0, x), \qquad x \in \mathbf{R}^n. \tag{6.17}$$

We denote by $P_{\mathcal{X}}^{\beta,m}(t)$ the image measure of $P^{\beta}(t_0) = P(t_0)$ under this localized linear transport map $\mathcal{X}_t^{\beta,m} \colon \mathbf{R}^n \to \mathbf{R}^n$, i.e.,

$$P_{\mathcal{X}}^{\beta,m}(t) := (\mathcal{X}_t^{\beta,m})_{\#} P^{\beta}(t_0), \qquad t \geqslant t_0. \tag{6.18}$$

Step 3. We claim that, for every $m \in \mathbf{N}$, there exists some $\varepsilon_m > 0$ such that for all $t \in (t_0, t_0 + \varepsilon_m)$, the localized linear transport map $\mathcal{X}_t^{\beta,m} \colon \mathbf{R}^n \to \mathbf{R}^n$ constructed in Step 2 defines an optimal Wasserstein transport from $P^{\beta}(t_0)$ to $P_{\mathcal{X}}^{\beta,m}(t)$. Hence, by Brenier's theorem [12], [56, Theorem 2.12], we have to show that $\mathcal{X}_t^{\beta,m}$ is the *gradient of a convex function*, for all $t > t_0$ sufficiently close to t_0.

Indeed, from the definitions in (6.16) and (6.17) we see that the functions $\mathcal{X}_t^{\beta,m}$ are gradients for all $m \in \mathbf{N}$ and $t \geqslant t_0$. More precisely, we have

$$\mathcal{X}_t^{\beta,m}(x) = \nabla\Big(\frac{1}{2}\,|x|^2 + (t - t_0)\cdot \varphi_m^{\beta}(t_0, x)\Big), \qquad x \in \mathbf{R}^n. \tag{6.19}$$

As the Hessian matrix of $\varphi_m^{\beta}(t_0, \cdot\,)$ is uniformly bounded, the function in the bracket of (6.19) is a convex function of x for every $m \in \mathbf{N}$ and $t \in (t_0, t_0 + \varepsilon_m)$ for $\varepsilon_m > 0$ small enough. We also note for later use that $\mathcal{X}_t^{\beta,m}$ defines a Lipschitz bijection on $\mathbf{R}^n$ again for every $m \in \mathbf{N}$ and $t \in (t_0, t_0 + \varepsilon_m)$.

Step 4. From Step 3 we know that, for every $m \in \mathbf{N}$, there exists some $\varepsilon_m > 0$ such that for all $t \in (t_0, t_0 + \varepsilon_m)$ the localized map $\mathcal{X}_t^{\beta,m}$ is the optimal transport from $P^{\beta}(t_0)$ to $P_{\mathcal{X}}^{\beta,m}(t)$ with respect to quadratic Wasserstein costs. Therefore, we can apply the considerations of Step 1 to the optimal map $\mathcal{X}_t^{\beta,m}$ in (6.17) and deduce that

$$\lim_{t \downarrow t_0} \frac{W_2\big(P_{\mathcal{X}}^{\beta,m}(t), P^{\beta}(t_0)\big)}{t - t_0} = \big(\mathbf{E}_{\mathbf{P}}\big[\big|v_m^{\beta}\big(t_0, X(t_0)\big)\big|^2\big]\big)^{1/2} \tag{6.20}$$

holds for every $m \in \mathbf{N}$. Invoking (6.15) and (6.16), we obtain from this

$$\lim_{m \to \infty} \lim_{t \downarrow t_0} \frac{W_2\big(P_{\mathcal{X}}^{\beta,m}(t), P^{\beta}(t_0)\big)}{t - t_0} = \big(\mathbf{E}_{\mathbf{P}}\big[\big|v^{\beta}\big(t_0, X(t_0)\big)\big|^2\big]\big)^{1/2} = \frac{1}{2}\,\|a + 2b\|_{L^2(\mathbf{P})}. \tag{6.21}$$

From the inequalities (6.13) and (6.14) of Step 1 (with $P_{\mathcal{X}}^{\beta,m}(t)$ instead of $P_{\mathcal{X}}^{\beta}(t)$) it follows that, in order to conclude (6.7), it remains to establish the analogue of the identity (6.12):

$$\lim_{m \to \infty} \lim_{t \downarrow t_0} \frac{W_2\big(P^{\beta}(t), P_{\mathcal{X}}^{\beta,m}(t)\big)}{t - t_0} = 0. \tag{6.22}$$

Step 5. The time-dependent velocity field $(v^{\beta}(t, \cdot\,))_{t \geqslant t_0}$ induces a *curved flow* $(\mathcal{Y}_t^{\beta})_{t \geqslant t_0}$, which is characterized by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y}_t^{\beta} = v^{\beta}(t, \mathcal{Y}_t^{\beta}) \quad \text{for all } t \geqslant t_0, \qquad \mathcal{Y}_{t_0}^{\beta} = \mathrm{Id}_{\mathbf{R}^n}\,. \tag{6.23}$$

Then, for every $t \geqslant t_0$, the map $\mathcal{Y}_t^{\beta} \colon \mathbf{R}^n \to \mathbf{R}^n$ transports the measure $P^{\beta}(t_0) = P(t_0)$ to $P^{\beta}(t)$, i.e., $(\mathcal{Y}_t^{\beta})_{\#} P^{\beta}(t_0) = P^{\beta}(t)$.

The localized linear mappings $\mathcal{X}_t^{\beta,m} \colon \mathbf{R}^n \to \mathbf{R}^n$ of (6.17) transport $P^{\beta}(t_0)$ to $P_{\mathcal{X}}^{\beta,m}(t)$, as in (6.18). As mentioned at the end of Step 3, the inverse mappings $(\mathcal{X}_t^{\beta,m})^{-1} \colon \mathbf{R}^n \to \mathbf{R}^n$ are well defined for all $m \in \mathbf{N}$ and $t \in (t_0, t_0 + \varepsilon_m)$; they satisfy

$$\big((\mathcal{X}_t^{\beta,m})^{-1}\big)_{\#} P_{\mathcal{X}}^{\beta,m}(t) = P^{\beta}(t_0), \qquad t \in (t_0, t_0 + \varepsilon_m). \tag{6.24}$$

From Step 4, our remaining task is to prove (6.22). To this end, we have to construct maps $\mathcal{Z}_t^{\beta,m} \colon \mathbf{R}^n \to \mathbf{R}^n$ that transport $P_{\mathcal{X}}^{\beta,m}(t)$ to $P^\beta(t)$, i.e., $(\mathcal{Z}_t^{\beta,m})_\# P_{\mathcal{X}}^{\beta,m}(t) = P^\beta(t)$, and satisfy

$$\lim_{m\to\infty} \lim_{t\downarrow t_0} \frac{1}{t-t_0} \Big(\mathbf{E}_{\mathbf{P}_{\mathcal{X}}^{\beta,m}} \big[\big| \mathcal{Z}_t^{\beta,m}\big(X(t)\big) - X(t) \big|^2 \big] \Big)^{1/2} = 0, \tag{6.25}$$

where $\mathbf{P}_{\mathcal{X}}^{\beta,m}$ denotes a probability measure on the path space under which the r.v. $X(t)$ has distribution $P_{\mathcal{X}}^{\beta,m}(t)$ as in (6.18). We define for this job the candidate maps

$$\mathcal{Z}_t^{\beta,m} := \mathcal{Y}_t^\beta \circ \big(\mathcal{X}_t^{\beta,m}\big)^{-1}, \qquad t \in (t_0, t_0+\varepsilon_m); \tag{6.26}$$

recall that $(\mathcal{X}_t^{\beta,m})^{-1}$ transports $P_{\mathcal{X}}^{\beta,m}(t)$ to $P^\beta(t_0)$, while $\mathcal{Y}_t^\beta$ transports $P^\beta(t_0)$ to $P^\beta(t)$; and conclude that $\mathcal{Z}_t^{\beta,m}$ of (6.26) transports $P_{\mathcal{X}}^{\beta,m}(t)$ to $P^\beta(t)$. Thus, we obtain

$$\mathbf{E}_{\mathbf{P}_{\mathcal{X}}^{\beta,m}} \big[\big| \mathcal{Z}_t^{\beta,m}\big(X(t)\big) - X(t) \big|^2 \big] = \mathbf{E}_{\mathbf{P}} \big[\big| \mathcal{Y}_t^\beta\big(X(t_0)\big) - \mathcal{X}_t^{\beta,m}\big(X(t_0)\big) \big|^2 \big]. \tag{6.27}$$

Combining (6.25) with (6.27), we see that we have to establish

$$\lim_{m\to\infty} \lim_{t\downarrow t_0} \frac{1}{(t-t_0)^2} \mathbf{E}_{\mathbf{P}} \big[\big| \mathcal{Y}_t^\beta\big(X(t_0)\big) - \mathcal{X}_t^{\beta,m}\big(X(t_0)\big) \big|^2 \big] = 0. \tag{6.28}$$

Using (6.17) and the elementary inequality $|x+y|^2 \leqslant 2(|x|^2+|y|^2)$ for $x, y \in \mathbf{R}^n$, we derive the estimate

$$\frac{1}{2} \, |\mathcal{Y}_t^\beta(x) - \mathcal{X}_t^{\beta,m}(x)|^2 \leqslant (t-t_0)^2 \cdot |v^\beta(t_0,x) - v_m^\beta(t_0,x)|^2 \tag{6.29}$$

$$+ \big| \big(\mathcal{Y}_t^\beta(x) - x\big) - (t-t_0)\cdot v^\beta(t_0,x) \big|^2. \tag{6.30}$$

Therefore, in order to establish (6.28), it suffices to show the limiting assertions (6.31) and (6.32) below; these correspond to (6.29) and (6.30), respectively. We already have the first limiting identity from (6.15) and (6.16) of Step 2, namely,

$$\lim_{m\to\infty} \mathbf{E}_{\mathbf{P}} \big[\big| v^\beta\big(t_0, X(t_0)\big) - v_m^\beta\big(t_0, X(t_0)\big) \big|^2 \big] = 0. \tag{6.31}$$

Step 6. Our final task is to justify that

$$\lim_{t\downarrow t_0} \mathbf{E}_{\mathbf{P}} \left[\left| \frac{1}{t-t_0} \big(\mathcal{Y}_t^\beta\big(X(t_0)\big) - X(t_0) \big) - v^\beta\big(t_0, X(t_0)\big) \right|^2 \right] = 0. \tag{6.32}$$

To this end, we first note that by (6.23) we have for all $t \geqslant t_0$ the identity

$$\mathcal{Y}_t^\beta(x) = x + \int_{t_0}^t v^\beta\big(u, \mathcal{Y}_u^\beta(x)\big) \, \mathrm{d}u, \qquad x \in \mathbf{R}^n, \tag{6.33}$$

on whose account the expectation in (6.32) is equal to

$$\mathbf{E}_{\mathbf{P}} \left[\left| \frac{1}{t-t_0} \int_{t_0}^t v^\beta\big(u, \mathcal{Y}_u^\beta\big(X(t_0)\big)\big) \, \mathrm{d}u - v^\beta\big(t_0, X(t_0)\big) \right|^2 \right]. \tag{6.34}$$

As $\mathcal{Y}_t^\beta$ transports $P^\beta(t_0) = P(t_0)$ to $P^\beta(t)$, and because the r.v. $X(t_0)$ has the same distribution under $\mathbf{P}^\beta$ as it does under $\mathbf{P}$, this expectation can also be expressed with respect to the probability measure $\mathbf{P}^\beta$, and it thus suffices to show the limiting assertion

$$\lim_{t \downarrow t_0} \mathbf{E}_{\mathbf{P}^\beta}\left[\left|\frac{1}{t-t_0}\int_{t_0}^t v^\beta\big(u,X(u)\big)\,\mathrm{d}u - v^\beta\big(t_0,X(t_0)\big)\right|^2\right] = 0. \tag{6.35}$$

For this purpose, we first observe that by the continuity of the paths of the canonical coordinate process $(X(t))_{t\geqslant 0}$, the family of r.v.'s

$$\left(\left|\frac{1}{t-t_0}\int_{t_0}^t v^\beta\big(u,X(u)\big)\,\mathrm{d}u - v^\beta\big(t_0,X(t_0)\big)\right|^2\right)_{t\geqslant t_0} \tag{6.36}$$

converges $\mathbf{P}^\beta$-a.s. to zero, as $t \downarrow t_0$. In order to show that their expectations also converge to zero, i.e., that (6.35) does hold, we have to verify that the family of (6.36) is uniformly integrable with respect to $\mathbf{P}^\beta$. As the r.v. $|v^\beta(t_0,X(t_0))|^2$ belongs to $L^1(\mathbf{P}^\beta)$, and we have

$$\left|\frac{1}{t-t_0}\int_{t_0}^t v^\beta\big(u,X(u)\big)\,\mathrm{d}u\right|^2 \leqslant \frac{1}{t-t_0}\int_{t_0}^t \left|v^\beta\big(u,X(u)\big)\right|^2\mathrm{d}u, \qquad t \geqslant t_0, \tag{6.37}$$

by Jensen's inequality, it is sufficient to prove the uniform integrability of the family

$$\left(\frac{1}{t-t_0}\int_{t_0}^t \left|v^\beta\big(u,X(u)\big)\right|^2\mathrm{d}u\right)_{t\geqslant t_0}. \tag{6.38}$$

Invoking the definition of the time-dependent velocity field $(v^\beta(t,\,\cdot\,))_{t\geqslant t_0}$ in (6.4) and the fact that the perturbation β is smooth and compactly supported, the uniform integrability of the family in (6.38) above is equivalent to the uniform integrability of the family

$$\left(\frac{1}{t-t_0}\int_{t_0}^t \frac{\left|\nabla \ell^\beta\big(u,X(u)\big)\right|^2}{\ell^\beta\big(u,X(u)\big)^2}\,\mathrm{d}u\right)_{t\geqslant t_0}. \tag{6.39}$$

Now by continuity, the family (6.39) converges $\mathbf{P}^\beta$-a.s. to $|\nabla \ln \ell(t_0,X(t_0))|^2$. Thus, to establish this uniform integrability, it suffices to show that the family of r.v.'s in (6.39) converges in $L^1(\mathbf{P}^\beta)$. Hence, in view of *Scheffé's lemma* (Lemma A.1), it remains to check that the corresponding expectations also converge. But at this point we use for the first time our choice of $t_0 \in \mathbf{R}_+ \setminus N$ and recall (4.22) and (4.24) from the proof of Corollary 4.4, which gives us

$$\lim_{t \downarrow t_0} \mathbf{E}_{\mathbf{P}^\beta}\left[\frac{1}{t-t_0}\int_{t_0}^t \frac{\left|\nabla \ell^\beta\big(u,X(u)\big)\right|^2}{\ell^\beta\big(u,X(u)\big)^2}\,\mathrm{d}u\right] = \mathbf{E}_{\mathbf{P}}\left[\frac{\left|\nabla \ell\big(t_0,X(t_0)\big)\right|^2}{\ell\big(t_0,X(t_0)\big)^2}\right], \tag{6.40}$$

as required. This completes the proof of the claim made at the beginning of Step 6.

Summing up, in light of (6.29) and (6.30) from Step 5, the limiting assertions (6.31) and (6.32) imply the limiting behavior (6.28). According to the results of Steps 4 and 5, the latter also entails the validity of the limiting identity (6.22), which completes the proof of Theorem 6.2.

Equipped with Theorem 6.2, we can now easily deduce Theorem 6.1.

Proof of Theorem 6.1. The second equality in (6.3) follows from the representation of the relative Fisher information in (3.6) and the definition of the time-dependent velocity field $(v(t,\cdot))_{t \geqslant t_0}$ in (6.1). The first equality in (6.3) follows from Theorem 6.2 if we set $\beta \equiv 0$. However, the limit in (6.7) is only from the right, while the limit in (6.3) is two-sided. But the only reason for considering right-sided limits in Theorem 6.2 was the presence of the perturbation β at time $t \geqslant t_0$. If there is no such perturbation, one can replace all limits from the right by two-sided ones. This completes the proof of Theorem 6.1.

Open question. Derive the main results of the present section, Theorems 6.1 and 6.2, using probabilistic, rather than analytical, tools.

Appendix A. Some measure-theoretic results. In the proofs of Propositions 4.5 and 4.7, we have used a result about conditional expectations, which we formulate below as Proposition A.2; we refer the reader to Proposition D.2 in Appendix D of [34] for its proof. We place ourselves on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a left-continuous filtration $(\mathcal{F}(t))_{t \geqslant 0}$. We first recall the following result, which is well known under the name of *Scheffé's lemma* [58, 5.10].

LEMMA A.1 (Scheffé's lemma). *For a sequence of integrable r.v.'s* $(X_n)_{n \in \mathbf{N}}$ *which converges* $\mathbf{P}$*-a.s. to another integrable r.v.* X*, the convergence of the* $L^1(\mathbf{P})$*-norms (i.e.,* $\lim_{n\to\infty} \mathbf{E}[|X_n|] = \mathbf{E}[|X|]$*) is equivalent to the convergence in* $L^1(\mathbf{P})$ *(i.e.,* $\lim_{n\to\infty} \mathbf{E}[|X_n - X|] = 0$*).*

PROPOSITION A.2. *Let* $(B(t))_{0 \leqslant t \leqslant T}$ *and* $(C(t))_{0 \leqslant t \leqslant T}$ *be adapted continuous processes which are nonnegative and uniformly bounded, respectively. Define the process*

$$A(t) := \int_0^t \big(B(u) + C(u)\big)\,\mathrm{d}u, \qquad 0 \leqslant t \leqslant T, \tag{A.1}$$

and assume that $\mathbf{E}\big[\int_0^T B(u)\,\mathrm{d}u\big]$ *is finite. By the Lebesgue differentiation theorem, for Lebesgue-a.e.* $t_0 \in [0,T]$,

$$\lim_{t \uparrow t_0} \frac{\mathbf{E}[A(t) - A(t_0)]}{t - t_0} = \lim_{t \uparrow t_0} \frac{1}{t - t_0}\,\mathbf{E}\left[\int_{t_0}^t \big(B(u) + C(u)\big)\,\mathrm{d}u\right] = \mathbf{E}[B(t_0) + C(t_0)]. \tag{A.2}$$

Now fix a "Lebesgue point" $t_0 \in [0,T]$ *for which* (A.2) *does hold. Then we have the analogous limiting assertion for the conditional expectations, i.e.,*

$$\lim_{t \uparrow t_0} \frac{\mathbf{E}[A(t_0) - A(t) \,|\, \mathcal{F}(t)]}{t_0 - t} = B(t_0) + C(t_0), \tag{A.3}$$

where the limit exists in $L^1(\mathbf{P})$.

In the proof of Theorem 4.1 we invoked the following result. For its proof, we apply Lemma 2.48 in [32] to the continuous local martingale $\tilde{N}(t) = N(t)/N(0)$, $t \geqslant 0$.

PROPOSITION A.3. *Suppose* $(N(t))_{t \geqslant 0}$ *is a strictly positive local martingale with continuous paths. Let* τ *be a* $[0,\infty)$*-valued stopping time such that* $\ln N(\tau)$ *is integrable and* $\mathbf{E}[(\ln N(0))^+] < \infty$*. Then* $\ln N(0)$ *is integrable, and*

$$\mathbf{E}[\ln N(\tau)] - \mathbf{E}[\ln N(0)] = -\frac{1}{2}\,\mathbf{E}[\langle \ln N \rangle(\tau)], \tag{A.4}$$

where $\langle \ln N \rangle$ *denotes the quadratic variation of* $(\ln N(t))_{t \geqslant 0}$.

Appendix B. The proof of Lemma 4.10. In order to show (4.44), we recall the notation of (4.42) and consider the time-dependent velocity field

$$[0,1] \times \mathbf{R}^n \ni (t,\xi) \longmapsto v_t(\xi) := \gamma\big((T_t^\gamma)^{-1}(\xi)\big) \in \mathbf{R}^n, \tag{B.1}$$

which is well defined P_t-a.s. for every $t \in [0,1]$. Then $(v_t)_{0 \leqslant t \leqslant 1}$ is the velocity field associated with $(T_t^\gamma)_{0 \leqslant t \leqslant 1}$, i.e.,

$$T_t^\gamma(x) = x + \int_0^t v_\theta\big(T_\theta^\gamma(x)\big)\,\mathrm{d}\theta, \tag{B.2}$$

on account of (4.42). Let $p_t(\,\cdot\,)$ be the probability density function of the probability measure P_t in (4.42). Then, according to [56, Theorem 5.34], the function $p_t(\,\cdot\,)$ satisfies the continuity equation

$$\partial_t p_t(x) + \operatorname{div}\big(v_t(x)\,p_t(x)\big) = 0, \qquad (t,x) \in (0,1) \times \mathbf{R}^n, \tag{B.3}$$

which can be written equivalently as

$$-\,\partial_t p_t(x) = \operatorname{div}\big(v_t(x)\big)\,p_t(x) + \langle v_t(x), \nabla p_t(x)\rangle, \quad (t,x) \in (0,1) \times \mathbf{R}^n. \tag{B.4}$$

Recall that X_0 is an r.v. with probability distribution P_0 on the probability space $(S, \mathcal{S}, \nu)$. Then the integral equation

$$X_t = X_0 + \int_0^t v_\theta(X_\theta)\,\mathrm{d}\theta, \qquad 0 \leqslant t \leqslant 1, \tag{B.5}$$

defines r.v.'s X_t with probability distributions $P_t = (T_t^\gamma)_\#(P_0)$ for $t \in [0,1]$, as in (4.42). We have

$$\mathrm{d}p_t(X_t) = \partial_t p_t(X_t)\,\mathrm{d}t + \langle \nabla p_t(X_t), \mathrm{d}X_t\rangle = -p_t(X_t)\operatorname{div}\big(v_t(X_t)\big)\,\mathrm{d}t \tag{B.6}$$

on account of (B.4) and (B.5); thus also

$$\mathrm{d}\ln p_t(X_t) = -\operatorname{div}\big(v_t(X_t)\big)\,\mathrm{d}t, \qquad 0 \leqslant t \leqslant 1. \tag{B.7}$$

Recall the function $q(x) = \mathrm{e}^{-2\Psi(x)}$, for which

$$\mathrm{d}\ln q(X_t) = -\langle 2\nabla\Psi(X_t), \mathrm{d}X_t\rangle = -\langle 2\nabla\Psi(X_t), v_t(X_t)\rangle\,\mathrm{d}t. \tag{B.8}$$

For the likelihood ratio function $\ell_t(\,\cdot\,)$ of (4.41) we get from (B.7) and (B.8) that

$$\mathrm{d}\ln \ell_t(X_t) = \langle 2\nabla\Psi(X_t), v_t(X_t)\rangle\,\mathrm{d}t - \operatorname{div}\big(v_t(X_t)\big)\,\mathrm{d}t, \qquad 0 \leqslant t \leqslant 1. \tag{B.9}$$

Taking expectations in the integral version of (B.9), we obtain that the difference

$$H(P_t\,|\,\mathrm{Q}) - H(P_0\,|\,\mathrm{Q}) = \mathbf{E}_\nu[\ln \ell_t(X_t)] - \mathbf{E}_\nu[\ln \ell_0(X_0)] \tag{B.10}$$

is equal to

$$\mathbf{E}_\nu\left[\int_0^t \Big(\langle 2\nabla\Psi(X_\theta), v_\theta(X_\theta)\rangle - \operatorname{div}\big(v_\theta(X_\theta)\big)\Big)\,\mathrm{d}\theta\right] \tag{B.11}$$

for $t \in [0,1]$. Consequently,

$$\lim_{t \downarrow 0} \frac{H(P_t\,|\,\mathrm{Q}) - H(P_0\,|\,\mathrm{Q})}{t} = \mathbf{E}_\nu\big[\langle 2\nabla\Psi(X_0), v_0(X_0)\rangle - \operatorname{div}\big(v_0(X_0)\big)\big]. \tag{B.12}$$

Integrating by parts, we see that

$$\mathbf{E}_\nu\big[\operatorname{div}\big(v_0(X_0)\big)\big] = \int_{\mathbf{R}^n} \operatorname{div}\big(v_0(x)\big)p_0(x)\,\mathrm{d}x = -\int_{\mathbf{R}^n} \langle v_0(x), \nabla p_0(x)\rangle\,\mathrm{d}x \tag{B.13}$$

$$= -\langle \nabla \ln p_0(X_0), v_0(X_0)\rangle_{L^2(\nu)}. \tag{B.14}$$

Recalling (B.12) and combining it with the relation $\nabla \ln \ell_t(x) = \nabla \ln p_t(x) + 2\nabla\Psi(x)$, as well as with (B.13) and (B.14), we get

$$\lim_{t\downarrow 0} \frac{H(P_t\,|\,\mathrm{Q}) - H(P_0\,|\,\mathrm{Q})}{t} = \langle \nabla \ln \ell_0(X_0), v_0(X_0)\rangle_{L^2(\nu)}. \tag{B.15}$$

Since $v_0 = \gamma$, this leads to (4.44). Lemma 4.10 is proved.

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REFERENCES

[1] B. Acciaio, M. Beiglböck, F. Penkner, W. Schachermayer, and J. Temme, *A trajectorial interpretation of Doob's martingale inequalities*, Ann. Appl. Probab., 23 (2013), pp. 1494–1505, https://doi.org/10.1214/12-AAP878.

[2] S. Adams, N. Dirr, M. Peletier, and J. Zimmer, *Large deviations and gradient flows*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 371 (2013), 20120341, https://doi.org/10.1098/rsta.2012.0341.

[3] L. Ambrosio and N. Gigli, *A user's guide to optimal transport*, in Modelling and Optimisation of Flows on Networks, Lecture Notes in Math. 2062, Fond. CIME/CIME Found. Subser., Springer, Heidelberg, 2013, pp. 1–155, https://doi.org/10.1007/978-3-642-32160-3_1.

[4] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures Math. ETH Zürich, Birkhäuser Verlag, Basel, 2008, https://doi.org/10.1007/978-3-7643-8722-8.

[5] L. Bachelier, *Théorie de la spéculation*, Ann. Sci. École Norm. Sup. (3), 17 (1900), pp. 21–86, https://doi.org/10.24033/asens.476.

[6] L. Bachelier, *Louis Bachelier's Theory of Speculation: The Origins of Modern Finance*, translated and with commentary by M. Davis and A. Etheridge, Princeton Univ. Press, Princeton, 2006, https://doi.org/10.1515/9781400829309.

[7] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de Probabilités XIX (Univ. Strasbourg, 1983/84), Lecture Notes in Math. 1123, Springer, Berlin, 1985, pp. 177–206, https://doi.org/10.1007/BFb0075847.

[8] M. Beiglböck and P. Siorpaes, *Pathwise versions of the Burkholder–Davis–Gundy inequality*, Bernoulli, 21 (2015), pp. 360–373, https://doi.org/10.3150/13-BEJ570.

[9] L. Boltzmann, *Vorlesungen über Gastheorie, Vol.* I: *Theorie der Gase mit ein-atomigen Molecülen, deren Dimensionen gegen die mittlere Weglänge verschwinden*, J. A. Barth, Leipzig, 1896.

[10] L. Boltzmann, *Ueber die sogenannte H-curve*, Math. Ann., 50 (1898), pp. 325–332, https://doi.org/10.1007/BF01448073.

[11] L. Boltzmann, *Vorlesungen über Gastheorie, Vol.* II: *Theorie van der Waals'; Gase mit Zusammengesetzten Molecülen, Gasdissociation; Schlussbemerkungen*, J. A. Barth, Leipzig, 1898.

[12] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math., 44 (1991), pp. 375–417, https://doi.org/10.1002/cpa.3160440402.

[13] E. A. Carlen and A. Soffer, *Entropy production by block variable summation and central limit theorems*, Comm. Math. Phys., 140 (1991), pp. 339–371, https://doi.org/10.1007/BF02099503.

[14] D. Cordero-Erausquin, *Some applications of mass transport to Gaussian-type inequalities*, Arch. Ration. Mech. Anal., 161 (2002), pp. 257–269, https://doi.org/10.1007/s002050100185.

[15] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed., Wiley Ser. Telecommun. Signal Process., Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2006, https://doi.org/10.1002/047174882X.

[16] P. Dai Pra and M. Pavon, *Variational path-integral representations for the density of a diffusion process*, Stochastics Stochastics Rep., 26 (1989), pp. 205–226, https://doi.org/10.1080/17442508908833558.

[17] M. H. A. Davis and I. Karatzas, *A deterministic approach to optimal stopping*, in Probability, Statistics and Optimization Wiley Ser. Probab. Math. Statist. Probab. Math. Statist. 104, John Wiley & Sons, Chichester, NY, 1994, pp. 455–466.

[18] M. Fathi, *A gradient flow approach to large deviations for diffusion processes*, J. Math. Pures Appl. (9), 106 (2016), pp. 957–993, https://doi.org/10.1016/j.matpur.2016.03.018.

[19] H. Föllmer, *An entropy approach to the time reversal of diffusion processes*, in Stochastic Differential Systems (Marseille–Luminy, 1984), Lect. Notes Control Inf. Sci. 69, Springer, Berlin, 1985, pp. 156–163, https://doi.org/10.1007/BFb0005070.

[20] H. Föllmer, *Time reversal on Wiener space*, in Stochastic Processes — Mathematics and Physics (Bielefeld, 1984), Lecture Notes in Math. 1158, Springer, Berlin, 1986, pp. 119–129, https://doi.org/10.1007/BFb0080212.

[21] H. Föllmer, *Random fields and diffusion processes*, in École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87, Lecture Notes in Math. 1362, Springer, Berlin, 1988, pp. 101–203, https://doi.org/10.1007/BFb0086180.

[22] J. Fontbona and B. Jourdain, *A trajectorial interpretation of the dissipations of entropy and Fisher information for stochastic differential equations*, Ann. Probab., 44 (2016), pp. 131–170, https://doi.org/10.1214/14-AOP969.

[23] A. Friedman, *Stochastic Differential Equations and Applications*, Vol. 1, Probab. Math. Statist. 28, Academic Press, New York, 1975, https://doi.org/10.1016/C2013-0-07332-X.

[24] C. Gardiner, *Stochastic Methods. A Handbook for the Natural and Social Sciences*, 4th rev. ed., Springer Ser. Synergetics 13, Springer-Verlag, Berlin, 2009.

[25] I. Gentil, C. Léonard, and L. Ripani, *Dynamical aspects of the generalized Schrödinger problem via Otto calculus — a heuristic point of view*, Rev. Mat. Iberoam., 36 (2020), pp. 1071–1112, https://doi.org/10.4171/rmi/1159.

[26] I. Gentil, C. Léonard, L. Ripani, and L. Tamanini, *An entropic interpolation proof of the HWI inequality*, Stochastic Process. Appl., 130 (2020), pp. 907–923, https://doi.org/10.1016/j.spa.2019.04.002.

[27] N. Gozlan and C. Léonard, *Transport inequalities. A survey*, Markov Process. Related Fields, 16 (2010), pp. 635–736.

[28] U. G. Haussmann and E. Pardoux, *Time reversal of diffusions*, Ann. Probab., 14 (1986), pp. 1188–1205, https://doi.org/10.1214/aop/1176992362.

[29] O. Johnson, *Information Theory and the Central Limit Theorem*, Imperial College Press, London, 2004, https://doi.org/10.1142/p341.

[30] R. Jordan and D. Kinderlehrer, *An extended variational principle*, in Partial Differential Equations and Applications, Lecture Notes in Pure and Appl. Math. 177, Marcel Dekker, New York, 1996, pp. 187–200.

[31] R. Jordan, D. Kinderlehrer, and F. Otto, *The variational formulation of the Fokker–Planck equation*, SIAM J. Math. Anal., 29 (1998), pp. 1–17, https://doi.org/10.1137/S0036141096303359.

[32] I. Karatzas and C. Kardaras, *Portfolio Theory and Arbitrage: A Course in Mathematical Finance*, Grad. Stud. Math. 214, Amer. Math. Soc., Providence, RI, 2021.

[33] I. Karatzas, J. Maas, and W. Schachermayer, *Trajectorial dissipation and gradient flow for the relative entropy in Markov chains*, Commun. Inf. Syst., 21 (2021), pp. 481–536, https://doi.org/10.4310/CIS.2021.v21.n4.a1.

[34] I. Karatzas, W. Schachermayer, and B. Tschiderer, *Trajectorial Otto Calculus*, preprint, https://arxiv.org/abs/1811.08686, 2020.

[35] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Grad. Texts in Math. 113, Springer-Verlag, New York, 1998, https://doi.org/10.1007/978-1-4612-0949-2.

[36] A. Kolmogoroff, *Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung*, Math. Ann., 104 (1931), pp. 415–458, https://doi.org/10.1007/BF01457949.

[37] A. Kolmogoroff, *Zur Umkehrbarkeit der statistischen Naturgesetze*, Math. Ann., 113 (1937), pp. 766–772, https://doi.org/10.1007/BF01571664.

[38] C. Léonard, *Some properties of path measures*, in Séminaire de Probabilités XLVI, Lecture Notes in Math. 2123, Springer, Cham, 2014, pp. 207–230, https://doi.org/10.1007/978-3-319-11970-0_8.

[39] C. Léonard, *On the convexity of the entropy along entropic interpolations*, in Measure Theory in Non-Smooth Spaces, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017, pp. 194–242, https://doi.org/10.1515/9783110550832-006.

[40] P. A. Markowich and C. Villani, *On the trend to equilibrium for the Fokker–Planck equation: An interplay between physics and functional analysis*, in VI Workshop on Partial Differential Equations, Part II (Rio de Janeiro, 1999), Mat. Contemp. 19, Rio de Janeiro, 2000, pp. 1–29.

[41] R. J. McCann, *A convexity principle for interacting gases*, Adv. Math., 128 (1997), pp. 153–179, https://doi.org/10.1006/aima.1997.1634.

[42] P. A. Meyer, *Sur une transformation du mouvement brownien due à Jeulin et Yor*, in Séminaire de Probabilités XXVIII, Lecture Notes in Math. 1583, Springer, Berlin, 1994, pp. 98–101, https://doi.org/10.1007/BFb0073836.

[43] E. Nelson, *Dynamical Theories of Brownian Motion*, 2nd ed., Math. Notes, Princeton Univ. Press, Princeton, NJ, 2001.

[44] F. Otto, *The geometry of dissipative evolution equations: The porous medium equation*, Comm. Partial Differential Equations, 26 (2001), pp. 101–174, https://doi.org/10.1081/PDE-100002243.

[45] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal., 173 (2000), pp. 361–400, https://doi.org/10.1006/jfan.1999.3557.

[46] É. Pardoux, *Grossissement d'une filtration et retournement du temps d'une diffusion*, in Séminaire de Probabilités XX, Univ. Strasbourg, 1984/85, Lecture Notes in Math. 1204, Springer, Berlin, 1986, pp. 48–55, https://doi.org/10.1007/BFb0075711.

[47] M. Pavon, *Stochastic control and nonequilibrium thermodynamical systems*, Appl. Math. Optim., 19 (1989), pp. 187–202, https://doi.org/10.1007/BF01448198.

[48] H. Risken, *The Fokker–Planck Equation. Methods of Solution and Applications*, 2nd ed., Springer Ser. Synergetics 18, Springer-Verlag, Berlin, 1996, https://doi.org/10.1007/978-3-642-61544-3.

[49] L. C. G. Rogers, *Smooth transition densities for one-dimensional diffusions*, Bull. London Math. Soc., 17 (1985), pp. 157–161, https://doi.org/10.1112/blms/17.2.157.

[50] E. Schrödinger, *Über die Umkehrung der Naturgesetze*, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl., 1931 (1931), pp. 144–153.

[51] E. Schrödinger, *Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique*, Ann. Inst. H. Poincaré, 2 (1932), pp. 269–310.

[52] Z. Schuss, *Singular perturbation methods in stochastic differential equations of mathematical physics*, SIAM Rev., 22 (1980), pp. 119–155, https://doi.org/10.1137/1022024.

[53] A. J. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Information and Control, 2 (1959), pp. 101–112, https://doi.org/10.1016/S0019-9958(59)90348-1.

[54] K.-T. Sturm, *On the geometry of metric measure spaces* I, Acta Math., 196 (2006), pp. 65–131, https://doi.org/10.1007/s11511-006-0002-8.

[55] K.-T. Sturm, *On the geometry of metric measure spaces* II, Acta Math., 196 (2006), pp. 133–177, https://doi.org/10.1007/s11511-006-0003-7.

[56] C. Villani, *Topics in Optimal Transportation*, Grad. Stud. Math. 58, Amer. Math. Soc., Providence, RI, 2003, https://doi.org/10.1090/gsm/058.

[57] C. Villani, *Optimal Transport. Old and New*, Grundlehren Math. Wiss. 338, Springer-Verlag, Berlin, 2009, https://doi.org/10.1007/978-3-540-71050-9.

[58] D. Williams, *Probability with Martingales*, Cambridge Math. Textbooks, Cambridge Univ. Press, Cambridge, 1991, https://doi.org/10.1017/CBO9780511813658.