

A WEAK LAW OF LARGE NUMBERS
FOR DEPENDENT RANDOM VARIABLES*I. KARATZAS[†] AND W. SCHACHERMAYER[‡]*Dedicated to the memory of A. N. Kolmogorov (1903–1987)
on the occasion of the 120th anniversary of his birth*

Abstract. Each sequence f_1, f_2, \dots of random variables satisfying $\lim_{M \rightarrow \infty} (M \sup_{k \in \mathbb{N}} \mathbf{P}(|f_k| > M)) = 0$ contains a subsequence f_{k_1}, f_{k_2}, \dots which, along with all its subsequences, satisfies the weak law of large numbers $\lim_{N \rightarrow \infty} ((1/N) \sum_{n=1}^N f_{k_n} - D_N) = 0$ in probability. Here, D_N is a “corrector” random variable with values in $[-N, N]$ for each $N \in \mathbb{N}$; these correctors are all equal to zero if, in addition, $\liminf_{n \rightarrow \infty} \mathbf{E}(f_n^2 \mathbf{1}_{\{|f_n| \leq M\}}) = 0$ for every $M \in (0, \infty)$.

Key words. weak law of large numbers, hereditary convergence, weak convergence, truncation, generalized expectation, nonlinear expectation

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1. Introduction. On a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, consider real-valued measurable functions f_1, f_2, \dots . If these are independent and have the same distribution with $\mathbf{E}|f_1| < \infty$, the celebrated Kolmogorov strong law of large numbers (SLLN [13], [12], [7, section 2.4]) states that the “sample average” $(f_1 + \dots + f_N)/N$ converges \mathbf{P} -a.e. to the “ensemble average” $\mathbf{E}(f_1) = \int_{\Omega} f_1 \, d\mathbf{P}$ as $N \rightarrow \infty$.

A deep result of Komlós [14], already 56 years old but always very striking, asserts that such “stabilization via averaging” occurs within *any* sequence f_1, f_2, \dots of measurable real-valued functions which is bounded in \mathbf{L}^1 , i.e., satisfies $\sup_{n \in \mathbb{N}} \mathbf{E}|f_n| < \infty$. More precisely, there exist then an integrable function f_* and a subsequence $\{f_{k_n}\}_{n \in \mathbb{N}}$ such that $(f_{k_1} + \dots + f_{k_N})/N$ converges to f_* , \mathbf{P} -a.e. as $N \rightarrow \infty$, and the same holds “hereditarily,” i.e., for any further subsequence of $\{f_{k_n}\}_{n \in \mathbb{N}}$.

We have also another celebrated result of Kolmogorov, the weak law of large numbers (WLLN [13], [5, section 5.2], [7, section 2.2.3]) for a sequence f_1, f_2, \dots of real-valued measurable functions which are independent. If these have the same distribution and satisfy the weak \mathbf{L}^1 -type condition

$$(1.1) \quad \lim_{M \rightarrow \infty} (M \cdot \mathbf{P}(|f_1| > M)) = 0$$

(rather than the stronger $\mathbf{E}|f_1| < \infty$), then the WLLN

$$(1.2) \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^N f_n - D_N \right) = 0 \quad \text{in probability}$$

holds for the sequence of “correctors”

$$(1.3) \quad D_N := \mathbf{E}(f_1 \mathbf{1}_{\{|f_1| \leq N\}}), \quad N \in \mathbb{N},$$

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whereas, if the independent functions f_1, f_2, \dots do not have the same distribution but satisfy

$$(1.4) \quad \lim_{N \rightarrow \infty} \sum_{n=1}^N \mathbf{P}(|f_n| > N) = 0, \quad \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{n=1}^N \mathbf{E}(f_n^2 \mathbf{1}_{\{|f_n| \leq N\}}) = 0,$$

then again the convergence in probability (WLLN) in (1.2) holds, though now with correctors

$$(1.5) \quad D_N := \frac{1}{N} \sum_{n=1}^N \mathbf{E}(f_n \mathbf{1}_{\{|f_n| \leq N\}}), \quad N \in \mathbf{N}.$$

It was shown in [9], [8], [5, Theorem 5.2.3] that, for independent f_1, f_2, \dots , the conditions in (1.4) are not only sufficient but also necessary for the existence of a sequence D_1, D_2, \dots of real numbers with property (1.2). Let us also note that the correctors in both (1.3) and (1.5) satisfy $|D_N| \leq N$ and that they are all equal to zero if the distribution of each of the f_1, f_2, \dots is symmetric.

1.1. Preview. The purpose of this note is to present a version of the weak law of large numbers which is valid for a sequence of *arbitrarily dependent* random variables (r.v.'s), and "hereditarily," i.e., along an appropriate subsequence of the given sequence, as well as along all further subsequences of this subsequence.

The result is formulated in the next section as Theorem 2.1 and proved in section 3. It can be construed as yet another manifestation of the "principle of subsequences." Motivated by the work of Komlós [14], this principle was enunciated by Chatterji [3] and was further clarified, buttressed, and extended by Aldous [1] and Berkes and Péter [2]; we refer the reader also to the excellent survey [4].

The proof of Theorem 2.1, considerably simpler than its counterpart for the strong law in [14], appears in section 3. It is based on truncation and weak convergence arguments, which provide sufficient conditions for the resulting correctors to be equal to zero. It does not seem possible to deduce Theorem 2.1 from the abovementioned general subsequence principle, as formulated on the first page of [4] (see also the first page of [2]): the result here is not cast in terms of a norm, as that principle requires. And although it might turn out to be possible to deduce this, or a related, result from the abstract considerations in Theorem 2 of [2], the directness, simplicity, and brevity of the approach adopted here have quite a bit going for them.

Ramifications are taken up in section 4, as are examples, which show that Theorem 2.1 cannot be subsumed by the abovementioned Komlós hereditary SLLN.

2. Result. We consider real-valued measurable functions f_1, f_2, \dots on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and introduce, for every $M \in (0, \infty)$, the quantities

$$(2.1) \quad \tau_n(M) := M \cdot \mathbf{P}(|f_n| > M), \quad \tau(M) := \sup_{n \in \mathbf{N}} \tau_n(M).$$

THEOREM 2.1 (a general, hereditary WLLN). *In the above context, we impose the weak \mathbf{L}^1 -type condition*

$$(2.2) \quad \lim_{M \rightarrow \infty} \tau(M) = 0.$$

There exist then a sequence of corrector r.v.'s D_1, D_2, \dots with $\mathbf{P}(|D_N| \leq N) = 1$ for every $N \in \mathbf{N}$, and a subsequence $\{f_{k_n}\}_{n \in \mathbf{N}}$ of the original sequence such that the WLLN

$$(2.3) \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^N f_{k_n} - D_N \right) = 0 \quad \text{in probability}$$

is satisfied hereditarily, i.e., not just along $\{f_{k_n}\}_{n \in \mathbf{N}}$ but also along all its subsequences.

As we shall see in the proof of Theorem 2.1, the correctors D_1, D_2, \dots correspond to the generalized mathematical expectations in Kolmogorov [13, section 6.4]; they are also related to the nonlinear expectations developed by Peng in [16]. The correctors can be chosen as $D_N = 0$ for every $N \in \mathbf{N}$ whenever, for each $M \in (0, \infty)$, we have

$$(2.4) \quad \liminf_{n \rightarrow \infty} \mathbf{E}(f_n^2 \mathbf{1}_{\{|f_n| \leq M\}}) = 0$$

or, more generally, $\liminf_{n \rightarrow \infty} \mathbf{E}(f_n \mathbf{1}_{\{|f_n| \leq M\}} \cdot \xi) = 0$ for every $\xi \in \mathbf{L}^2$.

The hereditary aspect of the convergence in (2.3) holds automatically under independence but requires attention in the present generality. The condition (2.2) can be thought of as an “omnibus,” in that it implies both conditions in (1.4). As shown in the examples of section 4, the condition (2.2) (or a suitable modification of it) is satisfied in contexts with $\mathbf{E}|f_n| = \infty$ for every $n \in \mathbf{N}$, as well as in contexts where $\mathbf{E}|f_n| < \infty$ holds for every $n \in \mathbf{N}$, but no subsequence exists which is bounded in \mathbf{L}^1 (and thus the Komlós [14] theorem cannot be applied). We note also that the requirement

$$\lim_{M \rightarrow \infty} \left(M \cdot \sup_{n \in \mathbf{N}} \mathbf{P}(|f_n| > M) \right) = 0$$

of (2.2) implies $\lim_{M \rightarrow \infty} \sup_{n \in \mathbf{N}} \mathbf{P}(|f_n| > M) = 0$ (boundedness in \mathbf{L}^0 , or tightness) and is implied by $\lim_{M \rightarrow \infty} \sup_{n \in \mathbf{N}} \mathbf{E}(|f_n| \cdot \mathbf{1}_{\{|f_n| > M\}}) = 0$ (uniform integrability).

3. Proof. We start with the simple but crucial idea of *truncation*. This goes back at least to the work of Khintchine and Kolmogorov ([9], [10], [11]), where it plays a major role in the proofs of laws of large numbers and of convergence results for series of r.v.'s.

LEMMA 3.1. *Under condition (2.2),*

$$(3.1) \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^N f_n - \frac{1}{N} \sum_{n=1}^N f_n \mathbf{1}_{\{|f_n| \leq N\}} \right) = 0 \quad \text{in probability.}$$

Proof. For every $\varepsilon > 0$, the expression

$$\begin{aligned} \mathbf{P} \left(\left| \frac{1}{N} \sum_{n=1}^N f_n \mathbf{1}_{\{|f_n| > N\}} \right| > \varepsilon \right) &\leq \mathbf{P} \left(\bigcup_{n=1}^N \{|f_n| > N\} \right) \\ &\leq \sum_{n=1}^N \mathbf{P}(|f_n| > N) \leq N \cdot \max_{1 \leq n \leq N} \mathbf{P}(|f_n| > N) \end{aligned}$$

is dominated by $N \sup_{n \in \mathbf{N}} \mathbf{P}(|f_n| > N) = \tau(N)$, which tends to zero as $N \uparrow \infty$ on the strength of (2.2). The lemma is proved.

It follows that, in order to establish (2.3), it is enough to prove

$$(3.2) \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^N f_n \mathbf{1}_{\{|f_n| \leq N\}} - D_N \right) = 0 \quad \text{in probability}$$

for a suitable sequence D_1, D_2, \dots of correctors, and along an appropriate subsequence of $\{f_n\}_{n \in \mathbf{N}}$ denoted by the same symbols for economy of exposition — as well as along all further subsequences of this subsequence.

Proof of Theorem 2.1. For each integer $N \in \mathbf{N}$, consider the truncated functions

$$(3.3) \quad f_n^{[-N, N]} := f_n \mathbf{1}_{\{|f_n| \leq N\}}, \quad n \in \mathbf{N},$$

that appear in (3.1), (3.2). These are bounded in \mathbf{L}^∞ (since they take values in $[-N, N]$), and thus are bounded in \mathbf{L}^2 as well. As a result we can extract, for each $N \in \mathbf{N}$, a subsequence of $\{f_n\}_{n \in \mathbf{N}}$ denoted by the same symbol for economy of exposition, such that the sequence in (3.3) converges weakly in \mathbf{L}^2 to some $D_N \in \mathbf{L}^2$,

$$(3.4) \quad \lim_{n \rightarrow \infty} \mathbf{E}(f_n^{[-N, N]} \cdot \xi) = \mathbf{E}(D_N \cdot \xi) \quad \forall \xi \in \mathbf{L}^2.$$

And by standard diagonalization arguments, we can extract then a further subsequence of $\{f_n\}_{n \in \mathbf{N}}$, denoted again by the same symbol such that the convergence in (3.4) is valid for every $N \in \mathbf{N}$. Clearly, the test function ξ in (3.4) can be taken $\sigma(f_1, f_2, \dots)$ -measurable.

It is fairly straightforward to check that these weak \mathbf{L}^2 -limits in (3.4) satisfy $\mathbf{P}(|D_N| \leq N) = 1$ for every $N \in \mathbf{N}$. On the other hand, the lower-semicontinuity of the \mathbf{L}^2 -norm under weak \mathbf{L}^2 -convergence, in this case

$$\|D_N\|_{\mathbf{L}^2} \leq \liminf_{n \rightarrow \infty} \|f_n^{[-N, N]}\|_{\mathbf{L}^2},$$

establishes that $\mathbf{P}(D_N = 0) = 1$ for every $N \in \mathbf{N}$, under (2.4); this also holds if $\liminf_{n \rightarrow \infty} \mathbf{E}(f_n \mathbf{1}_{\{|f_n| \leq M\}} \cdot \xi) = 0$ for each $M \in (0, \infty)$ and every $\sigma(f_1, f_2, \dots)$ -measurable $\xi \in \mathbf{L}^2$.

We introduce now, for each $M \in (0, \infty)$, the quantities

$$(3.5) \quad \sigma_n(M) := \frac{1}{M} \mathbf{E}(f_n^2 \mathbf{1}_{\{|f_n| \leq M\}}), \quad \sigma(M) := \sup_{n \in \mathbf{N}} \sigma_n(M).$$

As shown by Feller [8, p. 235] (see also [7, section 2.3.3]), these quantities are related to those in (2.1) via

$$(3.6) \quad 0 \leq \sigma_n(M) = \frac{2}{M} \int_0^M \tau_n(t) dt - \tau_n(M) \leq \frac{2}{M} \int_0^M \tau(t) dt$$

for every¹ $n \in \mathbf{N}$, $M \in (0, \infty)$; thus

$$(3.7) \quad 0 \leq \sigma(M) \leq \frac{2}{M} \int_0^M \tau(t) dt, \quad M \in (0, \infty).$$

¹In the integrand of this expression as it appears on page 235 of [8], there is a typographical error; this is here corrected. The identity in (3.6) is in fact a simple consequence of the Fubini theorem.

From bound (3.7) and assumption (2.2), it follows that we also have

$$(3.8) \quad \lim_{M \rightarrow \infty} \sigma(M) = 0.$$

Furthermore, we note

$$(3.9) \quad \mathbf{E}(f_n^{[-M,M]})^2 = \mathbf{E}(f_n^2 \mathbf{1}_{\{|f_n| \leq M\}}) = M \cdot \sigma_n(M) \leq M \cdot \sigma(M),$$

for all $n \in \mathbf{N}$, $M \in (0, \infty)$, and, therefore, on account of (3.8),

$$(3.10) \quad \mathbf{E}(D_M^2) \leq \sup_{n \in \mathbf{N}} \mathbf{E}(f_n^{[-M,M]})^2 \leq M \cdot \sigma(M) = o(M) \quad \text{as } M \rightarrow \infty.$$

We observe at this point that, in order to prove (3.2), and thus (2.3) as well, along a suitable subsequence, it is enough to show convergence along such a subsequence in \mathbf{L}^2 , namely,

$$(3.11) \quad \lim_{N \rightarrow \infty} \frac{1}{N^2} \cdot \mathbf{E} \left(\sum_{n=1}^N (f_n^{[-N,N]} - D_N) \right)^2 = 0.$$

And developing the square, we need to show that the expectations of both the sum of squares and the double sum of cross-products, i.e.,

$$(3.12) \quad \sum_{n=1}^N \mathbf{E}(f_n^{[-N,N]} - D_N)^2$$

and

$$(3.13) \quad 2 \sum_{n=1}^N \sum_{1 \leq j < n} \mathbf{E}[(f_j^{[-N,N]} - D_N)(f_n^{[-N,N]} - D_N)],$$

respectively, are of order $o(N^2)$ as $N \rightarrow \infty$, for the subsequence in question and for all its subsequences. Now, from (3.9), (3.10), the upper bound

$$\sum_{n=1}^N \mathbf{E}(f_n^{[-N,N]} - D_N)^2 \leq 2 \sum_{n=1}^N \mathbf{E}(f_n^{[-N,N]})^2 + 2N \cdot \mathbf{E}(D_N^2)$$

for the expression in (3.12) is already dominated by $4N^2 \cdot \sigma(N)$, which is of order $o(N^2)$ as $N \rightarrow \infty$ on account of (3.8).

It is instructive to recall what happens at this juncture, in the case of independent f_1, f_2, \dots : the correctors D_N are then the real constants in (1.5), so the differences $f_n^{[-N,N]} - D_N$, $n = 1, \dots, N$, are independent with zero mean, and thus uncorrelated. The expectations of their cross-products in (3.13) vanish, and the argument ends here.

In the general case, when *nothing* is assumed about the finite-dimensional distributions of the f_1, f_2, \dots (in particular, when these functions are not independent), we need to guarantee, by passing to a further subsequence if necessary, that the expression in (3.13) is also of order $o(N^2)$ as $N \rightarrow \infty$. One way to accomplish this is to select the terms f_1, f_2, \dots of the (relabelled) subsequence in such a way that the differences $f_n^{[-N,N]} - D_N$, $n = 1, \dots, N$, are *nearly uncorrelated*.

We do this by induction in the following manner: Suppose the terms f_1, \dots, f_{n-1} of the subsequence have been chosen. We select the next term f_n in such a way

that the difference $f_n^{[-N,N]} - D_N$, with $N \leq e^{n^2}$, is “almost orthogonal” to all of the preceding differences

$$f_1^{[-N,N]} - D_N, \dots, f_{n-1}^{[-N,N]} - D_N,$$

namely, that

$$(3.14) \quad |\mathbf{E}[(f_j^{[-N,N]} - D_N)(f_n^{[-N,N]} - D_N)]| \leq e^{-n^2} \leq \frac{1}{N}$$

holds for every $j = 1, \dots, n-1$, $N \leq e^{n^2}$. Such a choice of f_n is certainly possible on account of (3.4) and completes the induction step.

Returning to (3.13), we note that the double summation

$$2 \sum_{n=1}^{\lfloor \sqrt{\ln N} \rfloor} \sum_{1 \leq j < n} |\mathbf{E}[(f_j^{[-N,N]} - D_N)(f_n^{[-N,N]} - D_N)]|$$

is then straightforward to control: each summand is bounded by $N \cdot \sigma(N)$ on account of (3.9), (3.10), so the entire summation is of the order

$$N\sigma(N) \sum_{n=1}^{\lfloor \sqrt{\ln N} \rfloor} 2n \sim N\sigma(N) \cdot \ln N = o(N^2)$$

as $N \rightarrow \infty$. On the other hand, the validity of (3.14) for $j = 1, \dots, n-1$ and $N \leq e^{n^2}$ implies that the double summation

$$2 \sum_{n=1+\lfloor \sqrt{\ln N} \rfloor}^N \sum_{1 \leq j < n} |\mathbf{E}[(f_j^{[-N,N]} - D_N)(f_n^{[-N,N]} - D_N)]|$$

is of the order

$$2 \sum_{n=1+\lfloor \sqrt{\ln N} \rfloor}^N n e^{-n^2} \sim \int_{\sqrt{\ln N}}^N 2x e^{-x^2} dx = \frac{1}{N} - e^{-N^2}$$

as $N \rightarrow \infty$, and thus certainly of order $o(N^2)$.

Thus, it follows that the expression of (3.13) is of order $o(N^2)$ as well, and the argument is now complete. It is also straightforward to check that the argument works just as well for an arbitrary subsequence, of the subsequence just constructed. Theorem 2.1 is proved.

4. Ramifications and examples.

Condition (2.2), which reads

$$\lim_{M \rightarrow \infty} (\sup_{n \in \mathbf{N}} \tau_n(M)) = 0,$$

can be weakened to

$$(4.1) \quad \lim_{M \rightarrow \infty} \left(\liminf_{n \in \mathbf{N}} \tau_n(M) \right) = 0.$$

Indeed, by passing to a subsequence, this becomes

$$(4.2) \quad \lim_{M \rightarrow \infty} \left(\limsup_{n \in \mathbb{N}} \tau_n(M) \right) = 0,$$

and one checks relatively easily that (4.2) can replace (2.2) in the inductive construction of the subsequence (of) $\{f_n\}_{n \in \mathbb{N}}$. We note also that the condition (4.2) can be satisfied in situations where (2.2) fails.

Example 4.1. To illustrate this last point, take $g \in \mathbf{L}^0$ with

$$(4.3) \quad \limsup_{M \rightarrow \infty} (M \cdot \mathbf{P}(|g| > M)) > 0,$$

and thus $\mathbf{E}|g| = \infty$, e.g., with Cauchy distribution $\mathbf{P}(g \in A) = \int_A (\pi(1+x^2))^{-1} dx$, and define the functions

$$(4.4) \quad f_n := g \cdot \mathbf{1}_{\{|g| > n\}}, \quad n \in \mathbb{N},$$

also with $\mathbf{E}|f_n| = \infty$. We have then $\tau_n(M) = M \cdot \mathbf{P}(|g| > M \vee n)$, $\tau(M) = M \cdot \mathbf{P}(|g| > M)$, so (4.3) means that (2.2) fails. However, for every $M \in (0, \infty)$, we have $\lim_{n \rightarrow \infty} \tau_n(M) = M \cdot \lim_{n \rightarrow \infty} \mathbf{P}(|g| > n) = 0$, so (4.2) is satisfied.

Thus, the WLLN (2.3) follows for a suitable sequence of correctors D_1, D_2, \dots . It is also checked that the condition (2.4) is satisfied here, so all these correctors can actually be chosen equal to zero.

Example 4.2. To provide another illustration of Theorem 2.1 which highlights the role of condition (2.2) in a somewhat more substantial manner, let us revisit an old example from [11] (see also section 5.2 of [5]). Suppose that the functions f_1, f_2, \dots satisfy

$$(4.5) \quad \mathbf{P}(f_n = \pm k) = \frac{c}{k^2 \ln k}, \quad k = 2, 3, \dots,$$

with constant $2c = (\sum_{k \geq 2} k^{-2} (1/\ln k))^{-1}$ and thus $\mathbf{E}|f_n| = \infty$, for every $n \in \mathbb{N}$. We assume nothing about the finite-dimensional joint distributions of the f_1, f_2, \dots ; in particular, we do not require these functions to be independent.

In this setting,

$$\tau_n(M) = 2cM \sum_{k > M} \frac{1}{k^2 \ln k} \sim \frac{2c}{\ln M}$$

holds for integers $M \geq 2$ in the notation of (2.1). Thus,

$$\tau(M) = \sup_{n \in \mathbb{N}} \tau_n(M) \leq \frac{2c}{\ln M},$$

the condition (2.2) is satisfied, and there exists a sequence D_1, D_2, \dots of correctors such that (2.3) holds for a subsequence f_{k_1}, f_{k_2}, \dots of f_1, f_2, \dots and for all further subsequences.

These correctors are all equal to zero, and $\lim_{N \rightarrow \infty} (1/N) \sum_{n=1}^N f_n = 0$ holds in probability for the original sequence, when the f_1, f_2, \dots are also independent; cf. the example in section 5.2 of [5].

Remark 4.1. Theorem 2.1 has a direct extension, with only very obvious notational changes, to the case where f_1, f_2, \dots take values in some Euclidean space \mathbf{R}^d , rather than the real line.

In such an extension, it does not matter whether balls or cubes of \mathbf{R}^d are considered in the truncation scheme (3.3).

4.1. Equivalent change of measure; weak, but not strong, hereditary LLN. In both Examples 4.1 and 4.2, we have $\mathbf{E}|f_n| = \infty$ for every $n \in \mathbf{N}$. Let us consider now situations where $\mathbf{E}|f_n| < \infty$ holds for every $n \in \mathbf{N}$.

In the present context, this situation is actually the most important case for the following reason: It has been observed by Dellacherie–Meyer (cf. [6, VII:57]) that, given measurable functions h_1, h_2, \dots on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $[0, \infty)$, an equivalent probability measure $\mathbf{Q} \sim \mathbf{P}$ can be constructed on \mathcal{F} with \mathbf{P} -a.e. bounded density $d\mathbf{Q}/d\mathbf{P}$ and $\mathbf{E}^{\mathbf{Q}}(h_n) < \infty$ for all $n \in \mathbf{N}$. In light of this result, and of the fact that convergence in probability depends only on the equivalence class of the underlying probability measure \mathbf{P} , it follows that whenever there exists a subsequence f_{k_1}, f_{k_2}, \dots of f_1, f_2, \dots with $\sup_{n \in \mathbf{N}} \mathbf{E}|f_{k_n}| < \infty$ (we drop reference to the equivalent probability measure $\mathbf{Q} \sim \mathbf{P}$ from now on), the Komlós hereditary SLLN in [14] can be applied to this f_{k_1}, f_{k_2}, \dots and to all its subsequences.

The interesting question, then, is *whether the requirement $\mathbf{E}|f_n| < \infty$ for every $n \in \mathbf{N}$ can coexist with both (2.2) and*

$$(4.6) \quad \liminf_{n \rightarrow \infty} \mathbf{E}|f_n| = \sup_{N \in \mathbf{N}} \inf_{n \geq N} \mathbf{E}|f_n| = \infty,$$

thus precluding the applicability of the Komlós hereditary SLLN in [14] but allowing that of the hereditary WLLN in Theorem 2.1.

This question is answered affirmatively by the example that follows. We are greatly indebted to Andrew Lyasoff [15] for raising it, and for prompting us to construct such an example.

Example 4.3. Let us modify slightly the setting of Example 4.2, by considering functions f_1, f_2, \dots that satisfy

$$(4.7) \quad \mathbf{P}(f_n = \pm k) = \frac{c_n}{k^{2+(1/n)} \ln k}, \quad k = 2, 3, \dots,$$

with constant $2c_n = (\sum_{k \geq 2} k^{-(2+(1/n))} (1/\ln k))^{-1}$ for every $n \in \mathbf{N}$; once again, *nothing* is assumed about the finite-dimensional joint distributions of these functions.

Clearly,

$$\mathbf{E}|f_n| = 2c_n \sum_{k \geq 2} \frac{1}{k^{1+(1/n)} \ln k} < \infty, \quad \mathbf{E}(f_n) = 0 \quad \forall n \in \mathbf{N},$$

hold, as does

$$\sum_{k \geq 2} \frac{1}{k^{1+(1/N)} \ln k} \left(\sum_{k \geq 2} \frac{1}{k^2 \ln k} \right)^{-1} \leq \inf_{n \geq N} \mathbf{E}|f_n| < \infty,$$

for every $N \in \mathbf{N}$. The leftmost side in this inequality increases to infinity as $N \uparrow \infty$, so (4.6) is satisfied. On the other hand, it is checked readily that the quantity of (2.1) is here

$$\tau_n(M) = M \cdot \mathbf{P}(|f_n| > M) = M \sum_{k > M} \frac{1}{k^{2+(1/n)} \ln k} \left(\sum_{k \geq 2} \frac{1}{k^{2+(1/n)} \ln k} \right)^{-1},$$

and that (2.2) is satisfied as well: for some real constant $C > 0$, we have

$$\tau(M) = \sup_{n \in \mathbf{N}} \tau_n(M) \leq \frac{C}{\ln M} \left(\sum_{k \geq 2} \frac{1}{k^3 \ln k} \right)^{-1} \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

According to Theorem 2.1, there exists a sequence D_1, D_2, \dots of correctors, with the property that (2.3) holds for some subsequence f_{k_1}, f_{k_2}, \dots of f_1, f_2, \dots and for all its subsequences.

We note that in (4.7), and throughout this example, $1/n$ in the exponent of the denominator can be replaced by any $a_n \in (0, 1)$ which decreases to zero as $n \rightarrow \infty$.

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REFERENCES

- [1] D. J. ALDOUS, *Limit theorems for subsequences of arbitrarily-dependent sequences of random variables*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete, 40 (1977), pp. 59–82, <https://doi.org/10.1007/BF00535707>.
- [2] I. BERKES AND E. PÉTER, *Exchangeable random variables and the subsequence principle*, Probab. Theory Related Fields, 73 (1986), pp. 395–413, <https://doi.org/10.1007/BF00776240>.
- [3] S. D. CHATTERJI, *Un principe des sous-suites dans la théorie des probabilités*, in Séminaire des probabilités VI (Univ. Strasbourg, 1970–1971), Lecture Notes in Math. 258, Springer, Berlin, 1972, pp. 72–89, <https://doi.org/10.1007/BFb0059462>.
- [4] S. D. CHATTERJI, *A subsequence principle in probability theory*, Jahresber. Deutsch. Math.-Verein., 87 (1985), pp. 91–107.
- [5] K. L. CHUNG, *A Course in Probability Theory*, 2nd ed., Probab. Math. Statist. 21, Academic Press, New York, 1974.
- [6] C. DELLACHERIE AND P.-A. MEYER, *Probabilités et potentiel, Chapitres V à VIII: Théorie des martingales*, rev. ed., Actualités Sci. Indust. 1385, Inst. Math. Univ. Strasbourg XVII, Hermann, Paris, 1980.
- [7] R. DURRETT, *Probability. Theory and Examples*, 4th ed., Camb. Ser. Stat. Probab. Math. 31, Cambridge Univ. Press, Cambridge, 2010, <https://doi.org/10.1017/CBO9780511779398>.
- [8] W. FELLER, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd ed., John Wiley & Sons, New York, 1971.
- [9] B. V. GNEDENKO AND A. N. KOLMOGOROV, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley, Cambridge, MA, 1954.
- [10] A. KHINTCHINE AND A. KOLMOGOROFF, *Über Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden*, Mat. Sb., 32 (1925), pp. 668–677.
- [11] A. KOLMOGOROFF, *Über die Summen durch den Zufall bestimmter unabhängiger Größen*, Math. Ann., 99 (1928), pp. 309–319, <https://doi.org/10.1007/BF01459098>; *Bemerkungen zu meiner Arbeit “Über die Summen zufälliger Größen,”* 102 (1930), pp. 484–488, <https://doi.org/10.1007/BF01782357>.
- [12] A. KOLMOGOROFF, *Sur la loi forte des grandes nombres*, C.R. Acad. Sci. Paris, 191 (1930), pp. 910–912.
- [13] A. KOLMOGOROFF, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergeb. Math. Grenzgeb. 2, 3, J. Springer, Berlin, 1933; English translation: A. N. KOLMOGOROV, *Foundations of the Theory of Probability*, Chelsea Publishing, New York, 1950.
- [14] J. KOMLÓS, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar., 18 (1967), pp. 217–229, <https://doi.org/10.1007/BF02020976>.
- [15] A. LYASOFF, *private communication*, 2022.
- [16] S. PENG, *Nonlinear Expectations and Stochastic Calculus under Uncertainty: With Robust CLT and G-Brownian Motion*, Probab. Theory Stoch. Model. 95, Springer, Berlin, 2019, <https://doi.org/10.1007/978-3-662-59903-7>.