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Local Times for Continuous Paths of Arbitrary Regularity

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Abstract
We study a pathwise local time of even integer order $p \geq 2$, defined as an occupation density, for continuous functions with finite pth variation along a sequence of time partitions. With this notion of local time and a new definition of the Föllmer integral, we establish Tanaka-type change-of-variable formulas in a pathwise manner. We also derive some identities involving this high-order pathwise local time, each of which generalizes a corresponding identity from the theory of semimartingale local time. We then use collision local times between multiple functions of arbitrary regularity to study the dynamics of ranked continuous functions.

Keywords Pathwise Itô calculus · Pathwise local time · Pathwise Tanaka–Meyer formulas · Ranked functions of arbitrary regularity · Fractional Brownian motion

Mathematics Subject Classification (2020) 60G17 · 60G22 · 60H05

1 Introduction

Hans Föllmer [9] showed almost 40 years ago that Itô's change-of-variable formula can be established in a pathwise manner, devoid of any probabilistic structure. This result gave rise to the development of pathwise approaches to stochastic calculus. Föllmer used a notion of pathwise quadratic variation $[S]$ for a given continuous function S, along a fixed sequence of partitions of a given time interval. For integrands of the form $f'(S)$ with $f \in C^2(\mathbb{R})$, he defined a pathwise integral $\int f'(S(u))dS(u)$ as a limit of Riemann sums and derived Itô's change-of-variable formula in terms of this integral. Then, Föllmer's student Würmli introduced in her unpublished diploma thesis a corresponding concept of pathwise local time, as a weak L^2-limit of "discrete local times" in the space variable, and showed that the Itô-Tanaka formula involving this

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local time holds for less regular functions f. This notion of pathwise local time for functions of finite quadratic variation has been further studied recently [8, 16].

An important generalization of the pathwise approach to Itô calculus was developed by Cont and Perkowski [6]. These authors extended Föllmer's Itô formula to rougher functions admitting finite pth variation with an even integer $p \geq 2$, instead of quadratic variation ($p = 2$). They extended Würmli's definition of second order, L^2-local time to a higher-order local time as a L^q-limit in weak topology for $q \in [1, \infty]$, appropriate for these rougher functions. They also derived a higher-order version of the Tanaka-type change-of-variable formula for functions $f \in C^{p-1}(\mathbb{R})$.

In this paper, we use a different, more classical definition of local time, via the occupation density formula for continuous functions S with arbitrary regularity. As in Cont and Perkowski [6], "arbitrary regularity" means that the functions admit finite pth variation along a given sequence of time partitions, for any even integer $p \geq 2$. Since the local time of standard Brownian motion, as well as of fractional Brownian motion, is defined as the density of the occupation measure, this notion of pathwise local time for S seems more appropriate for the generic paths of fractional Brownian motion than other concepts of local time. We then show that this local time enables an Itô-Tanaka formula for $f(S)$ to apply to less regular functions f than before, namely to $f \in C^{(p-2)}(\mathbb{R})$, with a slightly generalized construction of the Föllmer integral.

Establishing the Itô-Tanaka formula for $C^{(p-2)}(\mathbb{R})$ functions is important, because we can derive from it pth-order Tanaka–Meyer formulas as corollaries. These formulas provide explicit representations of local time, leading to a lot of identities that generalize those in semimartingale local time theory. Using these results as building blocks, we derive expressions for the rankings (in descending order) among N given functions admitting finite pth-order variation, in terms of the original ones and of appropriate local times. More specifically, we represent the Föllmer integral with respect to the kth ranked function $X_{(k)}$ among N continuous functions $X_1, \cdots, X_N$, as the sum of similar integrals of the original functions, collision local time terms generated whenever these functions collide, and some cross-terms inevitably produced due to the roughness of the X_i's. This representation helps us understand the dynamics of ranked particles, whose motions fluctuate over time more irregularly than standard Brownian motions or semimartingales.

Outline Section 2 provides a review of basic concepts related to the pathwise Itô theory. The definition of pathwise local time of order p is then given, and pathwise Itô-Tanaka formulas are derived with a new definition of the Föllmer integral. Section 3 presents several identities involving pathwise local times. In Sect. 4, we study ranked functions with finite pth variation, using pathwise collision local times.

2 Pathwise Local Time and Itô-Tanaka Formulas

2.1 Pathwise Local Time of Order *p*

For a fixed real number $T > 0$, we define and fix a nested sequence of partitions $\pi_n = \{t_0^n, \cdots, t_{N(\pi_n)}^n\}$ with $0 = t_0^n < \cdots < t_k^n < \cdots < t_{N(\pi_n)}^n = T$, for each $n \in \mathbb{N}$. We consider a continuous function $S : [0, T] \to \mathbb{R}$, which we denote by

$S \in C([0,T])$, and define the oscillation of the function S along the partition π_n as

$$osc(S, \pi_n) := \max_{t_j \in \pi_n} \max_{u,r \in [t_j, t_{j+1}]} \big| S(u) - S(r) \big|. \tag{2.1}$$

Here and below, t_j and t_{j+1} are consecutive elements of π_n, i.e., $t_j < t_{j+1}$, $\pi_n \cap (t_j, t_{j+1}) = \emptyset$.

Definition 2.1 (Variation of order p along a sequence of partitions) For a given real number $p > 0$, a function $S \in C([0,T])$ is said to have pth variation along a given sequence of partitions $\pi = (\pi_n)_{n \in \mathbb{N}}$, if $\lim_{n\to\infty} osc(S, \pi_n) = 0$, and the sequence of measures

$$\sum_{t_j \in \pi_n} \big| S(t_{j+1}) - S(t_j) \big|^p \cdot \delta_{t_j}, \qquad n \in \mathbb{N}$$

converges vaguely to a σ-finite measure μ_π on $\mathscr{B}(\mathbb{R})$ without atoms.

Here δ_t denotes the Dirac measure at $t \in [0,T]$. We write $V_p(\pi)$ for the collection of all continuous functions having finite pth variation along the sequence of partitions $\pi = (\pi_n)_{n \in \mathbb{N}}$. We call the continuous function $[0,T] \ni t \mapsto [S]^p_\pi(t) := \mu_\pi([0,t]) \in [0,\infty)$, the *pth variation of* S *along* π, and the number $\mu_\pi([0,t])$ the *pth variation of* S *on the interval* $[0,t]$ *along* π.
We note that $V_p(\pi)$ depends in general on the specific sequence of partitions $\pi = (\pi_n)_{n \in \mathbb{N}}$; the pth variations along two different sequences of partitions of the same function S can be different, even when both exist. In particular, Proposition 70 of Freedman [10] states that one can construct a sequence of partitions π, such that $[S]^2_\pi \equiv 0$ for every $S \in C([0,T])$. Further discussion regarding the dependence of quadratic variation on the sequence of partitions can be found in Cont and Das [5]. In what follows, we shall write $[S]^p$ instead of $[S]^p_\pi$ to simplify the notation, as we fix the sequence of partitions π throughout this paper.

Here are some examples of functions admitting finite pth variation: Almost every path of a fractional Brownian motion $B^{(H)}$ with Hurst index $H \in (0,1)$ is known to have finite variation of order $1/H$ (see Sect. 2.3 for further discussion); a signed Takagi–Landsberg function with Hurst parameter $H \in (0,1)$ also has finite variation of order $1/H$ along the sequence of dyadic partitions [13]; the function $F(\cdot) := u(\cdot, x)$ for a solution $u(t,x)$ of the stochastic heat equation with white noise on $[0,T] \times \mathbb{R}$ for any $x \in \mathbb{R}$, belongs to $V_4(\pi)$ for any sequence π of partitions of $[0,T]$ whenever the mesh size of π_n goes to zero as $n \to \infty$ [19].

The following result gives a simple and useful characterization of the above definition. The vague convergence of the sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ on $\mathscr{B}([0,T])$ is equivalent to the pointwise convergence of their cumulative distribution functions at all continuity points of the limiting cumulative distribution function. If this limiting distribution function is continuous, the convergence is uniform.

Lemma 2.2 *A function $S \in C([0,T])$ belongs to $V_p(\pi)$ if, and only if, there exists a continuous, nondecreasing function $[S]^p_\pi : [0,T] \to [0,\infty)$ such that*

$$[S]^p_{\pi_n}(t) := \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \big|S(t_{j+1}) - S(t_j)\big|^p \xrightarrow{n\to\infty} [S]^p_\pi(t) \tag{2.2}$$

holds for every $t \in [0,T]$. If this is the case, the convergence in (2.2) is uniform.

In the following, we denote by $C^k(\mathbb{R})$ the space of functions $f : \mathbb{R} \to \mathbb{R}$ which are k-times differentiable with continuous kth derivative. Föllmer's pathwise Itô formula [9] for a class of real-valued $C^2(\mathbb{R})$ functions of $S \in V_2(\pi)$ can be generalized to any even natural number p. This was done in Theorem 1.5 of Cont and Perkowski [6], as follows.

Theorem 2.3 (Change-of-variable formula for paths of finite pth variation) *Fix a sequence of partitions $\pi = (\pi_n)_{n\in\mathbb{N}}$, an even integer $p \in \mathbb{N}$, and a continuous function $S \in V_p(\pi)$. Then for every function $F \in C^p(\mathbb{R})$, the pathwise change of variable formula*

$$F\big(S(t)\big) - F\big(S(0)\big) = \int_0^t F'\big(S(u)\big)dS(u) + \frac{1}{p!}\int_0^t F^{(p)}\big(S(u)\big)d[S]^p(u) \tag{2.3}$$

holds for every $t \in [0,T]$. Here, the "Föllmer integral of order p", namely

$$\int_0^t F'\big(S(u)\big)dS(u) := \lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \sum_{k=1}^{p-1} \frac{F^{(k)}\big(S(t_j)\big)}{k!}\big(S(t_{j+1}) - S(t_j)\big)^k, \tag{2.4}$$

is defined as a pointwise limit of compensated Riemann sums.

We present now a definition of pathwise local time of order p, in the spirit of Paul Lévy's classical notion of local time for Brownian motion. As we only consider even positive orders p, denote $2\mathbb{N}$ the set of even natural numbers and fix $p \in 2\mathbb{N}$. This definition applies to a function $S \in V_p(\pi)$ and uses the occupation density formula (2.5) as its starting point.

Definition 2.4 (*Pathwise local time of order p*) Let $p \in 2\mathbb{N}$. We say that $S \in V_p(\pi)$ has a pathwise local time of order p along the given sequence of partitions $\pi = (\pi_n)_{n\in\mathbb{N}}$, if there exists a jointly continuous mapping $[0,T] \times \mathbb{R} \ni (t,x) \mapsto L^{S,p}_t(x) \in [0,\infty)$ satisfying

$$\int_A L^{S,p}_t(x)dx = \frac{1}{p!}\int_0^t \mathbb{1}_A\big(S(u)\big)d[S]^p(u), \tag{2.5}$$

for every $t \in [0,T]$ and Borel set A. We write $\mathscr{L}_p(\pi)$ for the collection of all functions $S \in V_p(\pi)$ admitting this local time of order p along the given sequence π of partitions.

In what follows, we denote the minimum and the maximum of $S \in C([0, T])$ up to t by $\underline{S}_t := \min_{0 \le u \le t} S(u)$ and $\overline{S}_t := \max_{0 \le u \le t} S(u)$ for any $t \in [0, T]$. From (2.5), we emphasize that $L_t(\cdot)$ has compact support $[\underline{S}_t, \overline{S}_t]$ for any $t \in [0, T]$. A general and comprehensive review of local time as a density of occupation measure appears in Geman and Horowitz [11]. Conditions under which a function has "jointly continuous" local time are developed mostly in the case of almost every path of a Gaussian stochastic process; we refer the interested reader to Berman [2] and Cuzick and DuPreez [7].

The following result shows that for a given function $S \in C[0, T]$, there is at most one "proper order" p such that $S \in V_p(\pi)$ (or $\mathscr{L}_p(\pi)$) with nontrivial $[S]^p$ (or $L^{S,p}$, respectively).

Lemma 2.5 (Proper order of function S) *Suppose that S belongs to $V_p(\pi)$ for a fixed $p \in 2\mathbb{N}$. Then, $S \in V_q(\pi)$ with $[S]^q \equiv 0$ for $q > p$; $S \notin V_q(\pi)$ for $q < p$. Furthermore, $S \in \mathscr{L}_p(\pi)$ implies that $S \in \mathscr{L}_q(\pi)$ with $L^{S,q} \equiv 0$ for $q > p$; $S \notin \mathscr{L}_q(\pi)$ for $q < p$.*

Proof For any $t \in [0, T]$, we have

$$\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} |S(t_{j+1}) - S(t_j)|^q \begin{cases} \le \Big(\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} |S(t_{j+1}) - S(t_j)|^p \Big) \cdot \\ \quad osc(S, \pi_n)^{q-p} \xrightarrow{n \to \infty} 0, & \text{if} \quad q > p, \\ \ge \Big(\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} |S(t_{j+1}) - S(t_j)|^p \Big) \cdot \\ \quad osc(S, \pi_n)^{q-p} \xrightarrow{n \to \infty} \infty, & \text{if} \quad q < p. \end{cases} \tag{2.6}$$

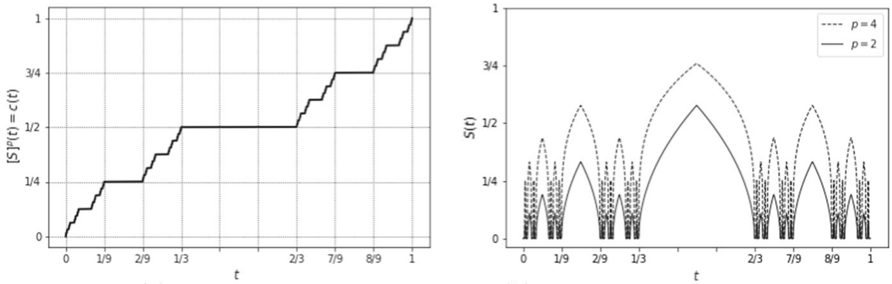
The last claim also follows from Definition 2.4. □

In order to simplify notation, we shall write throughout $L^S_t(\cdot)$ or even $L_t(\cdot)$, instead of $L^{S,p}_t(\cdot)$ whenever the order p and the function $S \in V_p(\pi)$ are fixed and apparent from the context. Taking $A = \mathbb{R}$ in (2.5), we have $\int_{\mathbb{R}} L_t(x)dx = [S]^p(t)/p!$ so that $L_t(\cdot) \in L^1(\mathbb{R})$; with the choice $A = [a - \epsilon, a + \epsilon]$ for a fixed $a \in \mathbb{R}$, letting $\epsilon \downarrow 0$ yields an explicit expression for $L_t(a)$, namely, the occupation density formula

$$L_t(a) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon p!} \int_0^t \mathbb{1}_{[a-\epsilon, a+\epsilon]}\big(S(u)\big) d[S]^p(u).$$

When the pathwise local time exists for a function $S \in C([0, T])$, it can be used to characterize the finiteness of the integral functional $\int_0^t f(S(u))d[S]^p(u)$, in the spirit of the Engelbert–Schmidt zero–one law. The following result can be proved in the same manner as in Proposition 3.6.27 of Karatzas and Shreve [12], if we replace du by $d[S]^p(u)$.

Proposition 2.6 *For $S \in \mathscr{L}_p(\pi)$, let I be an interval in $\mathbb{R}$ satisfying $S(t) \in I$ for $t \in [0, T]$. Suppose for every $a \in I$, that, there exists a positive number $T_a > 0$ such that $L_{T_a}(a) > 0$ holds. Then, for a Borel measurable $f : \mathbb{R} \to [0, \infty)$, the following are equivalent:*



(a) Cantor function $c(t)$ **(b)** Function $S(t)$ for $p = 2$, $p = 4$

Fig. 1 A function in Example 2.7

(i) $\int_0^t f\big(S(u)\big) d[S]^p(u) < \infty$, *for every* $t \in [0, T]$.
(ii) f *is locally integrable on* I.

The following example shows that the inclusion $\mathscr{L}_p(\pi) \subset V_p(\pi)$ can be strict for any $p \in 2\mathbb{N}$. The example is inspired from Example 3.6 of Davis et al. [8] in the case of $p = 2$; the original construction is from Bertoin [3].

Example 2.7 For any $p \in 2\mathbb{N}$, there exists a function which admits pathwise pth variation but no pathwise local time of order p.

For a fixed even integer p, we construct in the following a continuous function $S : [0, 1] \to \mathbb{R}$ whose pth variation $[S]^p$ along an appropriate Lebesgue partition is equal to the Cantor function c, depicted in Figure 1(a).

The Cantor ternary set C is obtained by iteratively deleting the open middle third from the closed interval $[0, 1]$. By deleting the open middle subinterval $I_1^1 := (\frac{1}{3}, \frac{2}{3})$ of $[0, 1]$, we obtain the two line segments $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next, the open middle subintervals $I_1^2 := (\frac{1}{3^2}, \frac{2}{3^2})$ and $I_2^2 := (\frac{7}{3^2}, \frac{8}{3^2})$ of each segment are removed, leaving four line segments. This process is continued to infinity to define the Cantor set as

$$C := [0, 1] \setminus \big(\cup_{i=1}^{\infty} \cup_{j=1}^{2^{i-1}} I_j^i \big).$$

We then define the continuous function S on $[0, 1]$ as

$$S(t) := \big(2 \min_{u \in C} |u - t|\big)^{\frac{1}{p}}.$$

Figure 1b describes the function S for $p = 2$ and $p = 4$. Since S first increases, attains the maximum $3^{-\frac{i}{p}}$ at the midpoint and then decreases toward 0 on each interval I_j^i of length 3^{-i}, we consider a refining sequence $(\pi_{n,j}^i)_{n=1}^{\infty}$ of dyadic Lebesgue partitions on each interval I_j^i, by setting $\pi_{n,j}^i = \{\inf I_j^i = t_{n,j}^{0,i} < t_{n,j}^{1,i} < \cdots < t_{n,j}^{2^{ni+1},i} = \sup I_j^i\}$ such that $S(t_{n,j}^{k,i}) \in (2^{-ni} 3^{-\frac{i}{p}})\mathbb{Z}$ for every $k = 0, \cdots, 2^{ni+1}$. Note that on each interval I_j^i, $|S(t_{n,j}^{k+1,i}) - S(t_{n,j}^{k,i})| = 2^{-ni} 3^{-\frac{i}{p}}$ holds by construction and that the pth variation along $\pi^i_{n,j}$ is computed as

$$[S]^{p,\bar{I}^i_j}_{\pi^i_{n,j}} := \sum_{k=0}^{2^{ni+1}-1} \left|S(t^{k+1,i}_{n,j}) - S(t^{k,i}_{n,j})\right|^p = 2^{1-ni(p-1)}3^{-i}, \tag{2.7}$$

which converges to 0 as $n \to \infty$. Finally, we set the refining sequence $\pi = (\pi_n)_{n=1}^{\infty}$ of partitions of $[0, 1]$ as $\pi_n := \{0, 1\} \cup \left(\cup_{i=1}^n \cup_{j=1}^{2^{i-1}} \pi^i_{n,j}\right)$ and show in the following that $[S]^p_\pi$ exists and coincides with the Cantor function c on $[0, 1]$.

Because the Cantor function c is continuous, nondecreasing function which is constant on every interval $\bar{I}^i_j$ with value $\frac{2j-1}{2^i}$, it is enough to show the convergence $[S]^p_{\pi_n}(t) \to c(t)$ for every t in the dense subset $D := \{0, 1\} \cup \left(\cup_{i=1}^\infty \cup_{j=1}^{2^{i-1}} \bar{I}^i_j\right)$. We first compute the pth variation $[S]^p_\pi$ at $t = 1$:

$$\begin{aligned}
[S]^p_\pi(1) &= \lim_{n\to\infty} \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} \sum_{k=0}^{2^{ni+1}-1} \left|S(t^{k+1,i}_{n,j}) - S(t^{k,i}_{n,j})\right|^p \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(2^{-n(p-1)+1}\, 3^{-1}\right)^i \\
&= \lim_{n\to\infty} \frac{1 - \left(2^{-n(p-1)+1}\, 3^{-1}\right)^{n+1}}{1 - \left(2^{-n(p-1)+1}\, 3^{-1}\right)} = 1 = c(1).
\end{aligned} \tag{2.8}$$

Since $[S]^{p,\bar{I}^1_1}_{\pi_n} \equiv [S]^{p,[\frac{1}{3},\frac{2}{3}]}_{\pi_n} \to 0$ from (2.7) and $[S]^{p,[0,\frac{1}{3}]}_{\pi_n} = [S]^{p,[\frac{2}{3},1]}_{\pi_n}$ by the symmetry, we obtain $[S]^p_{\pi_n}(t) \to \frac{1}{2} = c(t)$ as $n \to \infty$ for each $t \in \bar{I}^1_1$ from (2.8). Similarly, we have $[S]^{p,\bar{I}^2_1}_{\pi_n} \to 0$ and $[S]^{p,[0,\frac{1}{3^2}]}_{\pi_n} = [S]^{p,[\frac{2}{3^2},\frac{1}{3}]}_{\pi_n}$, which implies $[S]^p_{\pi_n}(t) \to \frac{1}{4} = c(t)$ as $n \to \infty$ for each $t \in \bar{I}^2_1$. Continuing this argument shows that $[S]^p_{\pi_n}(t) \to c(t)$ for each $t \in D$, thus $[S]^p_\pi \equiv c$ on $[0, 1]$.

We now assume that S admits the local time L of order p in Definition 2.4. For each i, the set $\{u \in [0, 1] : S(u) \in (3^{-\frac{i+1}{p}}, 3^{-\frac{i}{p}}]\}$ is a subset of $\cup_{j=1}^{2^i-1} I^i_j$ (recall the fact that S has the maximum $3^{-\frac{i}{p}}$ on each I^i_j), but $[S]^p_\pi$ is flat on $\cup_{j=1}^{2^i-1} I^i_j$. From the occupation density formula, we have

$$0 = \frac{1}{p!}\int_0^1 \mathbb{1}_{(3^{-\frac{i+1}{p}},3^{-\frac{i}{p}}]}\big(S(u)\big)d[S]^p_\pi(u) = \int_{(3^{-\frac{i+1}{p}},3^{-\frac{i}{p}}]} L_1(x)dx,$$

for each $i \in \mathbb{N}$, thus $L_1(x) = 0$ almost everywhere on the interval $(0, 3^{-\frac{1}{p}}]$. The continuity of $L_1(\cdot)$ gives $L_1 \equiv 0$. However, again from the occupation density formula (2.5) with $A = \mathbb{R}$, and (2.8), we obtain the contradiction

$$\frac{1}{p!} = \frac{1}{p!}[S]^p_\pi(1) = \int_{\mathbb{R}} L_1(x)dx = 0,$$

and we conclude that S admits no pathwise local time. □

We conclude this subsection with the following lemmas which will be useful in Sect. 3. We denote $X \vee Y := \max(X, Y)$ and $X \wedge Y := \min(X, Y)$ to represent the maximum and the minimum of two functions X and Y defined on $[0, T]$.

Lemma 2.8 *For any X, Y in $V_p(\pi)$, the functions $X \vee Y$ and $X \wedge Y$ also belong to $V_p(\pi)$, and the equations*

$$[X \vee Y]^p(t) = \int_0^t \mathbb{1}_{\{X(u)>Y(u)\}} d[X]^p(u) + \int_0^t \mathbb{1}_{\{X(u)\le Y(u)\}} d[Y]^p(u), \tag{2.9}$$

$$[X \wedge Y]^p(t) = \int_0^t \mathbb{1}_{\{X(u)<Y(u)\}} d[X]^p(u) + \int_0^t \mathbb{1}_{\{X(u)\ge Y(u)\}} d[Y]^p(u), \tag{2.10}$$

hold for every $t \in [0, T]$.

Proof Let $Z := X \vee Y$ and fix an arbitrary t in $[0, T]$. Consider the open set $U := \{s \in [0, t] : X(s) > Y(s)\}$, which we express as a countable union $U = \bigcup_{\ell=1}^{\infty} (\alpha_\ell, \beta_\ell)$ of disjoint open intervals. We also consider $V := [0, t] \setminus U$ and define a continuous, nondecreasing function $[Z]^p(\cdot)$ on $[0, t]$ such that for every $a < b$

$$\int_a^b d[Z]^p(u) = \begin{cases} \int_a^b d[X]^p(u), & \text{if } [a, b] \subset (\alpha_\ell, \beta_\ell), \\ \int_a^b d[Y]^p(u), & \text{if } [a, b] \cap U = \emptyset. \end{cases}$$

By virtue of Lemma 2.2,

$$\begin{aligned} [Z]^p(t) &= \int_U d[X]^p(u) + \int_V d[Y]^p(u) \\ &= \lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \cap U \\ t_j \le t}} \left|X(t_{j+1}) - X(t_j)\right|^p + \lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \cap V \\ t_j \le t}} \left|Y(t_{j+1}) - Y(t_j)\right|^p \end{aligned} \tag{2.11}$$

We claim that the first limit in (2.11) is equal to

$$\lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j)>Y(t_j)\}} \left|Z(t_{j+1}) - Z(t_j)\right|^p,$$

by showing that

$$\limsup_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j)>Y(t_j),\, X(t_{j+1})\le Y(t_{j+1})\}} \left|Z(t_{j+1}) - Z(t_j)\right|^p = 0. \tag{2.12}$$

For each $n,\ \ell \in \mathbb{N}$, we select an element t_ℓ^n of π_n satisfying $\alpha_\ell < t_\ell^n < \beta_\ell \le t_\ell^n + 1$; then we have

$$\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j) > Y(t_j),\, X(t_{j+1}) \le Y(t_{j+1})\}} \big| Z(t_{j+1}) - Z(t_j) \big|^p \le \sum_{\substack{\ell=1 \\ t_\ell^n \le t}}^{\infty} \big| Z(t_\ell^n + 1) - Z(t_\ell^n) \big|^p$$

for every $n \in \mathbb{N}$. Fix an arbitrary $\epsilon > 0$ and set $n_0 = 0$. For each $\ell \in \mathbb{N}$, we find a large enough $n_\ell \in \mathbb{N}$ such that $n_\ell > n_{\ell-1}$ and

$$\big| Z(t_\ell^{n_\ell} + 1) - Z(t_\ell^{n_\ell}) \big|^p \le \big| X(t_\ell^{n_\ell} + 1) - X(t_\ell^{n_\ell}) \big|^p + \big| Y(t_\ell^{n_\ell} + 1) - Y(t_\ell^{n_\ell}) \big|^p \le \frac{\epsilon}{2^\ell},$$

on the set $\{X(t_\ell^{n_\ell}) > Y(t_\ell^{n_\ell}),\, X(t_\ell^{n_\ell} + 1) \le Y(t_\ell^{n_\ell} + 1)\}$, from the continuity of $X(\cdot)$ and $Y(\cdot)$. Therefore, we deduce that Eq. (2.12) holds, since

$$\limsup_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j) > Y(t_j),\, X(t_{j+1}) \le Y(t_{j+1})\}} \big| Z(t_{j+1}) - Z(t_j) \big|^p \le \sum_{\ell=1}^{\infty} \frac{\epsilon}{2^\ell} = \epsilon.$$

Similar argument yields that

$$\limsup_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j) \le Y(t_j),\, X(t_{j+1}) > Y(t_{j+1})\}} \big| Z(t_{j+1}) - Z(t_j) \big|^p = 0,$$

thus the second limit of (2.11) has the same value as

$$\lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j) \le Y(t_j)\}} \big| Z(t_{j+1}) - Z(t_j) \big|^p.$$

Equation (2.11) is now

$$[Z]^p(t) = \int_U d[X]^p(u) + \int_V d[Y]^p(u) = \lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \big| Z(t_{j+1}) - Z(t_j) \big|^p,$$

thus $[Z]^p(\cdot)$ is indeed the pth variation of $Z = X \vee Y$ from Lemma 2.2 and the identity (2.9) holds.

The statement $X \wedge Y \in V_p(\pi)$ and the identity (2.10) can be proved in a similar manner. □

Lemma 2.9 *For any X, Y in $\mathscr{L}_p(\pi)$, the functions $X \vee Y$ and $X \wedge Y$ also belong to $\mathscr{L}_p(\pi)$, and the equations*

$$L_t^{X\vee Y}(x) = \int_0^t \mathbb{1}_{\{X(u) > Y(u)\}} dL_u^X(x) + \int_0^t \mathbb{1}_{\{X(u) \le Y(u)\}} dL_u^Y(x), \tag{2.13}$$

$$L_t^{X\wedge Y}(x) = \int_0^t \mathbb{1}_{\{X(u)<Y(u)\}} dL_u^X(x) + \int_0^t \mathbb{1}_{\{X(u)\geq Y(u)\}} dL_u^Y(x), \tag{2.14}$$

hold for every $(t, x) \in [0, T] \times \mathbb{R}$.

Proof Let $Z := X \vee Y$, and consider a decomposition of the integral for any Borel set A

$$\int_0^t \mathbb{1}_A\big(Z(u)\big) d[Z]^p(u) = \int_0^t \mathbb{1}_{\{X(u)>Y(u)\}} \mathbb{1}_A\big(X(u)\big) d[X]^p(u) + \int_0^t \mathbb{1}_{\{X(u)\leq Y(u)\}} \mathbb{1}_A\big(Y(u)\big) d[Y]^p(u). \tag{2.15}$$

Recall the open set $\{u \in [0, t] : X(u) > Y(u)\}$ and its representation $\bigcup_{\ell=1}^{\infty}(\alpha_\ell, \beta_\ell)$ in terms of disjoint open intervals. The first integral on the right-hand side is then equal to

$$\sum_{\ell=1}^{\infty} \int_{(\alpha_\ell,\beta_\ell)} \mathbb{1}_A\big(X(u)\big) d[X]^p(u) = p! \sum_{\ell=1}^{\infty} \int_A \big(L_{\beta_\ell}^X(x) - L_{\alpha_\ell}^X(x)\big) dx = p! \int_A \int_0^t \mathbb{1}_{\{X(u)>Y(u)\}} dL_u^X(x) dx,$$

by the monotone convergence theorem and the occupation density formula (2.5).

For the last integral of (2.15), the same argument applied to each open set $\{u \in [0, t] : X(u) < Y(u) + 1/m\}$ yields the identity for each $m \in \mathbb{N}$

$$\int_0^t \mathbb{1}_{\{X(u)<Y(u)+1/m\}} \mathbb{1}_A\big(Y(u)\big) d[Y]^p(u) = p! \int_A \int_0^t \mathbb{1}_{\{X(u)<Y(u)+1/m\}} dL_u^Y(x) dx. \tag{2.16}$$

Taking $m \to \infty$ in conjunction with the bounded convergence theorem, the last integral of (2.15) is seen to be equal to $p! \int_A \int_0^t \mathbb{1}_{\{X(u)\leq Y(u)\}} dL_u^Y(x) dx$. Therefore, the right-hand side of Eq. (2.15) is rewritten as

$$p! \int_A \left\{ \int_0^t \mathbb{1}_{\{X(u)>Y(u)\}} dL_u^X(x) + \int_0^t \mathbb{1}_{\{X(u)\leq Y(u)\}} dL_u^Y(x) \right\} dx,$$

which shows the existence of the pathwise local time for $Z = X \vee Y$ and the relationship (2.13). The joint continuity of $(t, x) \mapsto L_t^{X\vee Y}(x)$ follows from those of $L_t^X(x)$ and $L_t^Y(x)$, with the inequalities

$$\begin{aligned} \left|L_s^{X\vee Y}(x) - L_t^{X\vee Y}(y)\right| &\leq \left|L_s^{X\vee Y}(x) - L_t^{X\vee Y}(x)\right| + \left|L_t^{X\vee Y}(x) - L_t^{X\vee Y}(y)\right| \\ &\leq \left|L_s^X(x) - L_t^X(x)\right| + \left|L_s^Y(x) - L_t^Y(x)\right| \\ &\quad + \left|L_t^X(x) - L_t^X(y)\right| + \left|L_t^Y(x) - L_t^Y(y)\right| \end{aligned}$$

for $s, t \in [0, T]$ and $x, y \in \mathbb{R}$. The last inequality follows from (2.13). The statement regarding $X \wedge Y$ can be proved in a similar way. ◻

2.2 New Föllmer Integral and Pathwise Itô-Tanaka Formulas

Föllmer's pathwise change-of-variable formula (2.3) requires the function F to be in $C^p(\mathbb{R})$. In order to establish a pathwise Tanaka-type formula which holds for F with weaker differentiability, we generalize the definition of the Föllmer integral as follows.

First, we fix a locally integrable function f and let F be a pth antiderivative of f, i.e., $F^{(p)} = f$. We then consider the sequence of smooth functions $(F_m)_{m=1}^{\infty}$ approximating F such that F_m and its derivatives $F_m^{(k)}$ converge pointwise to F and to the corresponding derivatives $F^{(k)}$ of F up to order $p-1$, respectively, and that $F_m^{(p)}$ converges to f Lebesgue-a.e. and locally in L^1, as $m \to \infty$:

$$F_m \to F, \qquad F_m^{(k)} \to F^{(k)} \quad \text{for} \quad k = 1, \cdots, p-1, \quad F_m^{(p)} \to f \\ a.e. \text{ and locally in } L^1 \tag{2.17}$$

For a standard example of such $(F_m)_{m=1}^{\infty}$, we consider the mollifiers $\phi_m(x) := m\phi(mx)$ for each $m \in \mathbb{N}$, and the mollification

$$F_m(x) := (F * \phi_m)(x) = \int_{\mathbb{R}} F(x-y)\phi_m(y)dy \tag{2.18}$$

of F, with the standard mollifier

$$\phi(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where C is a constant satisfying $\int_{\mathbb{R}} \phi(x)dx = 1$.

By the change-of-variable formula (2.3) applied to F_m with the occupation density formula (2.5), we obtain

$$F_m\big(S(t)\big) - F_m\big(S(0)\big) - \int_{\mathbb{R}} L_t(x) F_m^{(p)}(x)dx = \int_0^t F_m'\big(S(u)\big)dS(u), \tag{2.19}$$

where the right-hand side represents the Föllmer integral defined as the limit in (2.4). If we take the limit on both sides of (2.19) as $m \to \infty$, we obtain

$$F\big(S(t)\big) - F\big(S(0)\big) - \int_{\mathbb{R}} L_t(x) f(x)dx = \lim_{m\to\infty} \int_0^t F_m'\big(S(u)\big)dS(u), \tag{2.20}$$

because each term on the left-hand side of (2.19) converges to the corresponding term in (2.20). The left-hand side of (2.20) does not depend on the sequence $(F_m)_{m=1}^{\infty}$ of approximating functions, so the limit on the right-hand side of (2.20) should also converge to the same quantity, regardless of the choice of functions F_m approximating F. This gives rise to the following new definition.

Definition 2.10 (Modified Föllmer integral) Fix a sequence of partitions $\pi = (\pi_n)_{n\in\mathbb{N}}$ of $[0, T]$, an integer $p \in 2\mathbb{N}$, and a continuous function S in $\mathscr{L}_p(\pi)$. For a given function F, assume that there exists a sequence of smooth functions $(F_m)_{m=1}^{\infty}$ such that F_m converges to F pointwise as $m \to \infty$, and the limit

$$\lim_{m\to\infty} \int_{\mathbb{R}} L_t(x) F_m^{(p)}(x) dx =: I \tag{2.21}$$

exists. Then, the double limit

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \sum_{k=1}^{p-1} \frac{F_m^{(k)}\big(S(t_j)\big)}{k!} \big(S(t_{j+1}) - S(t_j)\big)^k \tag{2.22}$$

exists and is equal to $F\big(S(t)\big) - F\big(S(0)\big) - I$ from (2.20).

If the limit I in (2.21) has the same value regardless of the choice of the sequence $(F_m)_{m=1}^{\infty}$ approximating F, we denote the double limit of (2.22) by $\int_0^t F'\big(S(u)\big)dS(u)$ and call it *the modified Föllmer integral of order p of the function F along π.*

With this new definition of the (modified) Föllmer integral, the pathwise Itô change of variable formula can be easily extended to the following Tanaka-type formula using the classical approach.

Theorem 2.11 ("Itô-Tanaka" formula) *For $p \in 2\mathbb{N}$, let $F \in C^{p-2}(\mathbb{R})$ be a function with absolutely continuous derivative $F^{(p-2)}$, and assume that the weak derivative $F^{(p-1)}$ of this latter function is right continuous and of bounded variation.*

Then for any function $S \in \mathscr{L}_p(\pi)$, the change-of-variable formula

$$F\big(S(t)\big) - F\big(S(0)\big) = \int_0^t F'\big(S(u)\big)dS(u) + \int_{\mathbb{R}} L_t(x) dF^{(p-1)}(x) \tag{2.23}$$

holds, where the first integral on the right-hand side is the modified Föllmer integral, defined as in (2.22).

Proof We first assume without loss of generality that F has compact support $[\underline{S}_T, \overline{S}_T]$. We consider the mollification F_m of F as in (2.18), then F_m and its $(p-1)$th derivative $F_m^{(p-1)}$ converge pointwise to F and $F^{(p-1)}$, respectively, on the compact support of F and these functions are uniformly bounded on the compact support. We apply Theorem 2.3 to each smooth function F_m to obtain the equation

$$F_m\big(S(t)\big) - F_m\big(S(0)\big) - \int_{\mathbb{R}} L_t(x) F_m^{(p)}(x) dx = \int_0^t F_m'\big(S(u)\big)dS(u). \tag{2.24}$$

For any function $g \in C^1(\mathbb{R})$ with compact support, integration by parts gives

$$\lim_{m\to\infty}\int_{\mathbb{R}} g(x)F_m^{(p)}(x)dx = -\lim_{m\to\infty}\int_{\mathbb{R}} g'(x)F_m^{(p-1)}(x)dx$$
$$= -\int_{\mathbb{R}} g'(x)F^{(p-1)}(x)dx = \int_{\mathbb{R}} g(x)dF^{(p-1)}(x).$$

Because the continuous function $L_t(\cdot)$ with compact support can be uniformly approximated by functions in $C^1(\mathbb{R})$, we also have

$$\lim_{m\to\infty}\int_{\mathbb{R}} L_t(x)F_m^{(p)}(x)dx = \int_{\mathbb{R}} L_t(x)dF^{(p-1)}(x).$$

We take $m \to \infty$ to the both sides of (2.24), then we have

$$F\big(S(t)\big) - F\big(S(0)\big) - \int_{\mathbb{R}} L_t(x)dF^{(p-1)}(x) = \lim_{m\to\infty}\int_0^t F'_m\big(S(u)\big)dS(u).$$

The left-hand side does not depend on F_m, so by virtue of Definition 2.10 we can write the right-hand side as $\int_0^t F'\big(S(u)\big)dS(u)$, representing the double limit of (2.22). ◻

A pathwise, pth-order extension of the classical Tanaka–Meyer formula follows directly from Theorem 2.11, applied to the function $F(x) = \big((x-a)^+\big)^{p-1}$ which satisfies the conditions in Theorem 2.11.

Corollary 2.12 (“Tanaka–Meyer” formula) *For an integer $p \in 2\mathbb{N}$ and a function $S \in \mathscr{L}_p(\pi)$, the pathwise pth-order Tanaka–Meyer formula*

$$L_t(a) = \big((S(t)-a)^+\big)^{p-1} - \big((S(0)-a)^+\big)^{p-1} - \int_0^t F'\big(S(u)\big)dS(u) \quad (2.25)$$

holds for all $(t,a) \in [0,T]\times\mathbb{R}$. Here, the last term represents the modified Föllmer integral of the function $F(x) := \big((x-a)^+\big)^{p-1}$.

Remark 2.13 (*Additional Tanaka–Meyer formulas*) We obtain formulas analogous to (2.25) by applying Theorem 2.11 to the choices $G(x) = \big((x-a)^-\big)^{p-1}$ and $H(x) = |x-a|^{p-1}$, respectively:

$$L_t(a) = \big((S(0)-a)^-\big)^{p-1} - \big((S(t)-a)^-\big)^{p-1} + \int_0^t G'\big(S(u)\big)dS(u), \quad (2.26)$$

and

$$2L_t(a) = |S(t)-a|^{p-1} - |S(0)-a|^{p-1} - \int_0^t H'\big(S(u)\big)dS(u). \quad (2.27)$$

2.3 Application to Fractional Brownian Motion

We show now that almost every path of fractional Brownian motion (fBM) admits the pathwise local time of Definition 2.4 along a specific sequence of partitions of $[0, T]$ so that the previous results can be applied to the paths of fBM.

Consider a fractional Brownian motion $(B_t^{(H)})_{t\geq 0}$ of Hurst index $H \in (0, 1)$ on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. For the exact definition of fBM and its basic properties, see, for example, Chapter 1 of Biagini et al. [4]. With the dyadic-rational sequence of partitions $\pi = (\pi_n)_{n\in\mathbb{N}}$ of the form

$$\pi_n = \{kT/2^n : k \in \mathbb{N}_0\} \cap [0, T], \tag{2.28}$$

Rogers [18] proved that $B^{(H)}$ has finite pth variation with $p = 1/H$, and the pth variation of order $p = 1/H$ converges in probability, namely

$$\sum_{\substack{t_j \in \pi_n \\ t_j \leq t}} \big|B^{(H)}(t_{j+1}) - B^{(H)}(t_j)\big|^p \xrightarrow{\mathbb{P}} t\, \mathbb{E}\big[|B_1^{(H)}|^p\big]$$

as $n \to \infty$. Thus, from now on, we fix the Hurst index H to be the reciprocal of a positive even integer p. There exists then a subsequence $\tilde{\pi}$ of $\pi = (\pi_n)_{n\in\mathbb{N}}$ in (2.28) such that almost every path of $B^{(H)}$ belongs to $V_{1/H}(\tilde{\pi})$ and the pth variation along $\tilde{\pi}$ in the sense of Definition 2.1 is given as

$$[B^{(H)}]^p(t) = t\, \mathbb{E}\big[|B_1^{(H)}|^p\big]. \tag{2.29}$$

In the case of standard Brownian motion, $B^{(1/2)}$ belongs to $V_2(\pi)$ and $[B^{(1/2)}]^2(t) = t$ for any refining sequence $\pi = (\pi_n)_{n\in\mathbb{N}}$ of $[0, T]$ whose mesh size $|\pi_n| \to 0$; see Proposition 2.12 of Chapter II of Revuz and Yor [17].

On the other hand, Berman [2] introduced the local time $\ell_t(x)$ of $B^{(H)}$ as the density of the occupation measure $\mathscr{B}(\mathbb{R}) \ni \Gamma \mapsto \int_0^t \mathbb{1}_\Gamma\big(B^{(H)}(s)\big)ds$ and proved that this local time $[0, T] \times \mathbb{R} \ni (t, x) \mapsto \ell_t(x)$ has a jointly continuous version. Berman's local time $\ell_t(x)$ of fBM $B^{(H)}$ coincides with our pathwise local time $L_t(x)$ for a.e. $x \in \mathbb{R}$ along the sequence $\tilde{\pi}$, up to a constant:

$$\begin{aligned}\int_A \ell_t(x)dx &= \int_0^t \mathbb{1}_A\big(B^{(H)}(u)\big)du \\ &= \frac{1}{c_p}\int_0^t \mathbb{1}_A\big(B^{(H)}(u)\big)d[B^{(H)}]^p(u) = \frac{p!}{c_p}\int_A L_t(x)dx,\end{aligned}$$

with $c_p := \mathbb{E}\big[|B_1^{(H)}|^p\big]$ from (2.29). Therefore, almost every path of the fBM $B^{(H)}$ admits a pathwise local time of order p along the specific subsequence $\tilde{\pi}$ of the sequence π of dyadic-rational partitions.

3 Some Identities for Local Times

Using the Tanaka–Meyer formulas (2.25), (2.26), and (2.27), we can establish several useful identities involving the pathwise local time of the previous section. Each of these identities is reminiscent of familiar properties of local time for continuous semimartingales.

The first result shows that the pathwise local time $L_t^X(a)$ is flat off $\{u \geq 0 : X(u) = a\}$. Note that $L_t^X(a)$ for fixed $a \in \mathbb{R}$, is nondecreasing in t, thus of finite first variation in this variable, and the integrals in this result should be understood as Lebesgue–Stieltjes integrals with respect to this temporal variable.

Proposition 3.1 *For $X \in \mathscr{L}_p(\pi)$, we have the following identities for every fixed $a \in \mathbb{R}$*

$$\int_0^t \mathbb{1}_{\{X(u)=a\}} dL_u(a) = L_t(a),$$
$$\int_0^t \mathbb{1}_{\{X(u)<a\}} dL_u(a) = \int_0^t \mathbb{1}_{\{X(u)>a\}} dL_u(a) = 0. \tag{3.1}$$

Proof Consider the function $F(x) := \big((x-a)^+\big)^{p-1}$ which appeared in Corollary 2.12. Note that F has derivatives $F^{(k)}(x) = (p-1)(p-2)\cdots(p-k)(x-a)^{p-k-1}\mathbb{1}_{(a,\infty)}(x)$ for $k = 1, \cdots, p-2$ and $F^{(p-2)}$ has a weak derivative $F^{(p-1)}(x) = \mathbb{1}_{[a,\infty)}(x)(p-1)!$ of bounded variation. Let $(F_m)_{m=1}^{\infty}$ be a sequence of smooth functions approximating F, i.e., satisfies (2.17).

Suppose $X(u) < a$ holds for every $u \in [v, w]$ for some $0 \leq v < w \leq T$. Then, there exists $\delta > 0$ such that $X(u) < a - \delta$ for every $u \in [v, w]$, from the continuity of X. There also exists large enough $M \in \mathbb{N}$ such that $F_m^{(k)}\big(X(u)\big) = 0$ holds for all $k = 0, \cdots, p-1$ for every $m \geq M$, thus we have $\int_v^w F'\big(X(u)\big)dX(u) = 0$ from the representation (2.22) of the modified Föllmer integral. The Tanaka–Meyer formula (2.25) now gives $L_v(a) = L_w(a)$, provided that $X(u) < a$ holds for every $u \in [v, w]$. The open set $\{u : X(u) < a\}$ can be represented as a countable union of disjoint open intervals and $\int_{(\alpha,\beta)} dL_u(a) = 0$ on each interval (α, β). Therefore, we obtain $\int_0^t \mathbb{1}_{\{X(u)<a\}} dL_u(a) = 0$. Similar argument using (2.26) yields the last equality of (3.1) and the result follows. □

Lemma 3.2 *For any nonnegative function $X \in \mathscr{L}_p(\pi)$, we have the identity*

$$L_t^X(0) = \lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \leq t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\big(X(t_{j+1}) - X(t_j)\big)^{p-1}. \tag{3.2}$$

Proof Let us recall the function $F(x) := \big(x^+\big)^{p-1}$, its derivatives up to order $p-2$, and the weak derivative $F^{(p-1)}(x) = \mathbb{1}_{[0,\infty)}(x)(p-1)!$, which appeared in the proof of Proposition 3.1 for $a = 0$. We consider an approximation of the functions $F \equiv F^{(0)}, F^{(1)}, \cdots, F^{(p-1)}$ from the left; a sequence $(\delta_m)_{m=1}^{\infty}$ of positive real numbers satisfying $\delta_m \downarrow 0$ as $m \to \infty$, and sequences $(g_m^k(x))_{m=1}^{\infty}$ of smooth functions such that each function of the form

$$F_m^{(k)}(x) := \begin{cases} (p-1)\cdots(p-k)x^{p-k-1} \quad \text{for} \quad k \geq 1, \qquad x^{p-1} \quad \text{for} \quad k = 0, & \text{if} \quad x \geq 0, \\ g_m^k(x) & \text{if} \quad x \in (-\delta_m, 0), \\ 0 & \text{if} \quad x \leq -\delta_m \end{cases}$$

is smooth and satisfies the conditions of (2.17).

From the Itô-Tanaka formula (2.23), we have

$$\int_{\mathbb{R}} L_t(x)dF_m^{(p-1)}(x) = F_m\big(X(t)\big) - F_m\big(X(0)\big) - \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \sum_{k=1}^{p-1} \frac{F_m^{(k)}\big(X(t_j)\big)}{k!}\big(X(t_{j+1}) - X(t_j)\big)^k. \tag{3.3}$$

Since X is nonnegative, the last term on the right-hand side, which represents Föllmer integral $\int_0^t F_m'(X(u))dX(u)$, is equal to

$$\lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \sum_{k=1}^{p-1} \mathbb{1}_{(0,\infty)}\big(X(t_j)\big)\binom{p-1}{k}\big(X(t_j)\big)^{p-k-1}\big(X(t_{j+1}) - X(t_j)\big)^k$$
$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \mathbb{1}_{(0,\infty)}\big(X(t_j)\big)\Big\{\big(X(t_{j+1})\big)^{p-1} - \big(X(t_j)\big)^{p-1}\Big\},$$

where the last equality follows from the binomial theorem. Using the telescoping sum representation, the right-hand side of (3.3) is given as

$$\sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \Big[F_m\big(X(t_{j+1})\big) - F_m\big(X(t_j)\big)\Big]$$
$$- \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \mathbb{1}_{(0,\infty)}\big(X(t_j)\big)\Big\{\big(X(t_{j+1})\big)^{p-1} - \big(X(t_j)\big)^{p-1}\Big\}$$
$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \Big(1 - \mathbb{1}_{(0,\infty)}\big(X(t_j)\big)\Big)\Big\{\big(X(t_{j+1})\big)^{p-1} - \big(X(t_j)\big)^{p-1}\Big\}$$
$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\leq t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\Big\{\big(X(t_{j+1})\big)^{p-1} - \big(X(t_j)\big)^{p-1}\Big\}$$

$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\big(X(t_{j+1})\big)^{p-1}.$$

Taking $m \to \infty$ on the left-hand side of (3.3) yields the desired result. □

Next, we offer another representation of the local time for general functions $X \in \mathscr{L}_p(\pi)$. In what follows, $X^+ := \max(X, 0)$ and $X^- := -\min(X, 0)$, respectively, and we know from Lemma 2.9 that $X^+, X^- \in \mathscr{L}_p(\pi)$ for any $X \in \mathscr{L}_p(\pi)$.

Lemma 3.3 *For any $X \in \mathscr{L}_p(\pi)$, we have*

$$L_t^X(0) = L_t^{X^+}(0) = \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(X(t_j)\big)\big(X^+(t_{j+1})\big)^{p-1}. \tag{3.4}$$

Proof As in the proof of Lemma 3.2, we consider the smooth approximation $F_m^{(k)}(x)$ of the function $F(x) := (x^+)^{p-1}$ and its derivatives from the right; a sequence $(\delta_m)_{m=1}^\infty$ of positive real numbers satisfying $\delta_m \downarrow 0$ as $m \to \infty$, and sequences $(g_m^k(x))_{m=1}^\infty$ of smooth functions such that each function of the form

$$F_m^{(k)}(x) := \begin{cases} (p-1)\cdots(p-k)x^{p-k-1} \quad \text{for} \quad k\ge 1, \qquad x^{p-1} \quad \text{for} \quad k=0, & \text{if} \quad x \ge \delta_m, \\ g_m^k(x) & \text{if} \quad x\in(0,\delta_m), \\ 0 & \text{if} \quad x\le 0 \end{cases}$$

is smooth, $F_m^{(p)} \to \delta_{\{0\}}$ in L^1, and satisfies the conditions of (2.17).

For any $m, n \in \mathbb{N}$ and $t_j \in \pi_n$, from the Taylor expansion with the occupation density formula, there exist real numbers ξ_{t_j} between X_{t_j} and $X_{t_{j+1}}$ such that the identity

$$\begin{aligned}\int_{\mathbb{R}} \big(L_{t_{j+1}}^X(x) - L_{t_j}^X(x)\big)F_m^{(p)}(x)dx &= F_m\big(X(t_{j+1})\big) - F_m\big(X(t_j)\big) \\ &\quad - \sum_{k=1}^{p-1} \frac{F_m^{(k)}\big(X(t_j)\big)}{k!}\big(X(t_{j+1}) - X(t_j)\big)^k \\ &\quad - \frac{F_m^{(p+1)}\big(\xi(t_j)\big)}{(p+1)!}\big(X(t_{j+1}) - X(t_j)\big)^{p+1}\end{aligned}$$

holds. Multiplying both sides by $\mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)$ and summing over all t_j's satisfying $t_j \in \pi_n$ and $t_j \le t$, we obtain

$$\sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big) \int_{\mathbb{R}} \big(L_{t_{j+1}}^X(x) - L_{t_j}^X(x)\big)F_m^{(p)}(x)dx \tag{3.5}$$

$$= \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\Big[F_m\big(X(t_{j+1})\big) - \frac{F_m^{(p+1)}\big(\xi(t_j)\big)}{(p+1)!}\big(X(t_{j+1}) - X(t_j)\big)^{p+1}\Big].$$

Since $F_m^{(p+1)}$ converges pointwise to 0, by sending $m \to \infty$ we have the identity

$$\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\big(L^X_{t_{j+1}}(0) - L^X_{t_j}(0)\big) = \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{(-\infty,0]}\big(X(t_j)\big)\big(X^+(t_{j+1})\big)^{p-1}, \tag{3.6}$$

Now, as $n \to \infty$, the left-hand side of (3.6) is $L^X_t(0)$ by virtue of Proposition 3.1, and we arrive at the identity (3.4). The last limit in (3.4) coincides with $L^{X^+}_t(0)$ from Lemma 3.2. ◻

Remark 3.4 Equations (3.1), (3.2), and (3.4) are pathwise generalizations of order p of the well-known identities in semimartingale theory

$$\int_0^t \mathbb{1}_{\{Z(u)=a\}} dL^Z_u(a) = L^Z_t(a),$$
$$\int_0^t \mathbb{1}_{\{Z(u)<a\}} dL^Z_u(a) = \int_0^t \mathbb{1}_{\{Z(u)>a\}} dL^Z_u(a) = 0,$$
$$L^Z_t(0) = \int_0^t \mathbb{1}_{\{Z(u)=0\}} dZ(u), \qquad \text{if } Z \ge 0,$$
$$L^Z_t(0) = L^{Z^+}_t(0) = \int_0^t \mathbb{1}_{\{Z(u)=0\}} dZ^+(u),$$

where Z is a semimartingale on a probability space, L^Z is semimartingale local time of Z, and the last two integrals are standard Itô integrals [15].

The following result provides an expression for the pathwise local time of the maximum of two given continuous functions; Lemma 3.3 plays an essential role in its proof.

Lemma 3.5 *For $X, Y \in \mathscr{L}_p(\pi)$, we have*

$$L^{X\vee Y}_t(0) = \int_0^t \mathbb{1}_{\{Y(u)<0\}} dL^X_u(0) + \int_0^t \mathbb{1}_{\{X(u)<0\}} dL^Y_u(0) + \lim_{n\to\infty} \sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \mathbb{1}_{\{X(t_j)=Y(t_j)=0\}} \big((X^+ \vee Y^+)(t_{j+1})\big)^{p-1}, \tag{3.7}$$

as well as

$$L_t^{X\wedge Y}(0) = \int_0^t \mathbb{1}_{\{Y(u)>0\}} dL_u^X(0) + \int_0^t \mathbb{1}_{\{X(u)>0\}} dL_u^Y(0) + \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{X(t_j)=Y(t_j)=0\}} \big((X^+\wedge Y^+)(t_{j+1})\big)^{p-1}. \tag{3.8}$$

Proof Consider the function $Z = X \vee Y$, and use Lemma 3.3 to obtain the decomposition of its local time at the origin as follows:

$$\begin{aligned} L_t^Z(0) &= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{0\}}\big(Z(t_j)\big)\big(Z^+(t_{j+1})\big)^{p-1} \\ &= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \Big[\mathbb{1}_{\{Y(t_j)<X(t_j)=0\}}\big(Z^+(t_{j+1})\big)^{p-1} \\ &\quad + \mathbb{1}_{\{X(t_j)<Y(t_j)=0\}}\big(Z^+(t_{j+1})\big)^{p-1} \\ &\quad + \mathbb{1}_{\{X(t_j)=Y(t_j)=0\}}\big(Z^+(t_{j+1})\big)^{p-1}\Big]. \end{aligned} \tag{3.9}$$

Since the two paths Z^+ and X^+ coincide on the set $\{u \in [0, T] : Y(u) < X(u)\}$, the first term on the right-most side of (3.9) is

$$\lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{Y(t_j)<0\}}\mathbb{1}_{\{X(t_j)=0\}}\big(X^+(t_{j+1})\big)^{p-1} = \int_0^t \mathbb{1}_{\{Y(u)<0\}} dL_u^X(0),$$

on the strength of (3.4) in Lemma 3.3. By the same token, the second term on the right-most side of (3.9) is equal to $\int_0^t \mathbb{1}_{\{X(u)<0\}} dL_u^Y(0)$. For the last term, we use the fact $Z^+ = X^+ \vee Y^+$, and the result (3.7) follows. The same argument leads to (3.8). □

Combining two equations in Lemma 3.5, we have the following algebraic identity. This generalizes the results of Yan [20, 21], valid for continuous semimartingales; see also Ouknine [14, 15].

Theorem 3.6 *For $X, Y \in \mathscr{L}_p(\pi)$, we have the identity*

$$L_t^{X\vee Y}(0) + L_t^{X\wedge Y}(0) = L_t^X(0) + L_t^Y(0). \tag{3.10}$$

Proof The elementary identity $(a \vee b)^{p-1} + (a \wedge b)^{p-1} = a^{p-1} + b^{p-1}$ holds for arbitrary numbers a and b. Therefore, the sum of the last terms of (3.7) and (3.8) can be expressed as

$$\lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{X(t_j)=Y(t_j)=0\}}\Big[\big((X^+\vee Y^+)(t_{j+1})\big)^{p-1} + \big((X^+\wedge Y^+)(t_{j+1})\big)^{p-1}\Big]$$

$$= \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \mathbb{1}_{\{X(t_j)=Y(t_j)=0\}} \Big[\big(X^+(t_{j+1})\big)^{p-1} + \big(Y^+(t_{j+1})\big)^{p-1} \Big]$$

$$= \int_0^t \mathbb{1}_{\{Y(u)=0\}} dL_u^X(0) + \int_0^t \mathbb{1}_{\{X(u)=0\}} dL_u^Y(0),$$

where the last equality holds from Lemma 3.3. Then, the result follows by adding two Eqs. (3.7) and (3.8). □

4 Descending Ranks

For given N continuous functions $X_1, \cdots, X_N \in \mathscr{L}_p(\pi)$, we define their "ranked" versions

$$X_{(k)}(t) := \max_{1\le i_1<\cdots<i_k\le N} \min\{X_{i_1}(t), \cdots, X_{i_k}(t)\}, \qquad \text{for any } t \in [0, T] \quad (4.1)$$

in descending order. More explicitly, for any $t \in [0, T]$, we have

$$\max_{1\le i\le N} X_i(t) = X_{(1)}(t) \ge X_{(2)}(t) \ge \cdots \ge X_{(N-1)}(t) \ge X_{(N)}(t)$$
$$= \min_{1\le i\le N} X_i(t). \quad (4.2)$$

From Lemma 2.9 the ranked functions $X_{(k)}$ belong to $\mathscr{L}_p(\pi)$ for $k = 1, \cdots, N$. We assume in this subsection that their differences $X_{(k)} - X_{(h)}$ also belong to $\mathscr{L}_p(\pi)$ for $k, h = 1, \cdots, N$. We then introduce the notation

$$S_k(t) := \{i : X_i(t) = X_{(k)}(t)\} \quad \text{and} \quad N_k(t) := |S_k(t)|,$$
$$\text{for any } t \in [0, T]. \quad (4.3)$$

Here, $N_k(t)$ is the number of functions which are at rank k at time t. We present next the following extension of Theorem 3.6, which can be proved using induction, as in Theorem 2.2 of Banner and Ghomrasni [1].

Proposition 4.1 *For any $t \in [0, T]$, we have the identity*

$$\sum_{k=1}^N L_t^{X_{(k)}}(0) = \sum_{i=1}^N L_t^{X_i}(0).$$

Our next and final aim is to derive expressions of the descending ranked functions $X_{(k)}$ for $k = 1, \cdots, N$ in terms of the original functions $X_1, \cdots, X_N$ and appropriate local times, in the spirit of Theorem 2.3 in Banner and Ghomrasni [1]. In this result, expressions such as "$dX_i(t)$" appear and represent Itô integration with respect to a semimartingale integrator.

In our setting, however, such expression "$dX_i(t)$" makes sense only when a certain type of integrand, namely $F'(X_i(t))$, is given, as the Föllmer integral $\int_0^t F'\big(X_i(u)\big)dX_i(u)$ is defined as in (2.4) or (2.22). This integral has different representations depending on the regularity of the test-function $F : \mathbb{R} \to \mathbb{R}$ as in Theorems 2.3 and 2.11. We choose less regular functions F satisfying the conditions of Theorem 2.11 to deal with the modified Föllmer integral and present the following "integration along ranks" formula.

Theorem 4.2 *Let F be a function in $C^{p-2}(\mathbb{R})$ with absolutely continuous derivative $F^{(p-2)}$ which admits a right continuous weak derivative $F^{(p-1)}$ of bounded variation. Then, the modified Föllmer integral of order p, defined as in Definition 2.10, of the kth rank function $X_{(k)}(\cdot)$ among N given functions $X_1, \cdots, X_N$, is expressed as*

$$\begin{aligned}
&\int_0^t F'(X_{(k)}(u))dX_{(k)}(u) \qquad (4.4)\\
&= \sum_{i=1}^{N} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(u)=X_i(u)\}}}{N_k(u)} F'(X_i(u))dX_i(u)\\
&\quad + \sum_{h=k+1}^{N} \int_0^t \frac{F^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(k)}-X_{(h)})}(0)\\
&\quad - \sum_{h=1}^{k-1} \int_0^t \frac{F^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(h)}-X_{(k)})}(0)\\
&\quad + R_t^k,
\end{aligned}$$

for $k = 1, \cdots, N$ and any $t \in [0, T]$. Here, the last term represents the limit

$$\begin{aligned}
R_t^k := \lim_{m\to\infty} \lim_{n\to\infty} &\sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \qquad (4.5)\\
&\sum_{\ell=1}^{p-2} \sum_{r=\ell}^{p-1} \frac{F_m^{(r)}\big(X_{(k)}(t_j)\big)}{\ell!(r-\ell)!} \big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell},
\end{aligned}$$

for any sequence $(F_m)_{m=1}^{\infty}$ of smooth functions approximating F and satisfying (2.17).

Proof For any sequence $(F_m)_{m=1}^{\infty}$ approximating F and for any $m \in \mathbb{N}$, we start with an expression

$$\sum_{r=1}^{p-1} \frac{F_m^{(r)}\big(X_{(k)}(t_j)\big)}{r!} \Big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\Big)^r, \qquad (4.6)$$

whose sum over all t_j's satisfying $t_j \in \pi_n$ and $t_j \le t$ will converge as $n \to \infty$ to the Föllmer integral $\int_0^t F_m'\big(X_{(k)}(u)\big)dX_{(k)}(u)$, which in turn converges as $m \to \infty$ to the modified Föllmer integral

$$\int_0^t F'\big(X_{(k)}(u)\big)dX_{(k)}(u). \tag{4.7}$$

For any integer $r \in \{1, \cdots, p-1\}$, definition (4.3) and the fact

$$\sum_{i=1}^{N} N_k(t_j)^{-1}\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} = 1, \tag{4.8}$$

lead to

$$\begin{aligned}
\big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big)^r &= \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)}\big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big)^r \\
&= \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)}\big(X_i(t_{j+1}) - X_i(t_j)\big)^r \\
&\quad + \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)}\Big\{\big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big)^r \\
&\quad - \big(X_i(t_{j+1}) - X_i(t_j)\big)^r\Big\}.
\end{aligned} \tag{4.9}$$

When $X_{(k)}(t_j) = X_i(t_j)$ holds, by the binomial theorem, the expression in the curly brackets in the last sum is rewritten as

$$\begin{aligned}
&\big(X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big)^r - \big(X_i(t_{j+1}) - X_i(t_j)\big)^r \\
&= \sum_{\ell=1}^{r} \frac{r!}{\ell!(r-\ell)!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell} \\
&\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell}.
\end{aligned} \tag{4.10}$$

Then, by plugging (4.10), (4.9) into (4.6), the Expression (4.6) becomes

$$\sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \sum_{r=1}^{p-1} \frac{F_m^{(r)}\big(X_i(t_j)\big)}{r!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^r \tag{4.11}$$

$$\begin{aligned}
&+ \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \sum_{r=1}^{p-1}\sum_{\ell=1}^{r} \frac{F_m^{(r)}\big(X_{(k)}(t_j)\big)}{\ell!(r-\ell)!} \\
&\big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell}\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell}.
\end{aligned} \tag{4.12}$$

Since

$$\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \sum_{r=1}^{p-1} \frac{F_m^{(r)}\big(X_i(t_j)\big)}{r!} \big(X_i(t_{j+1}) - X_i(t_j)\big)^r \quad \xrightarrow{n\to\infty,\ m\to\infty} \quad \int_0^t F'(X_i(u))dX_i(u),$$

for each i, the sum

$$\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \sum_{r=1}^{p-1} \frac{F_m^{(r)}\big(X_i(t_j)\big)}{r!} \big(X_i(t_{j+1}) - X_i(t_j)\big)^r$$

of (4.11) over t_j's also converges by virtue of (4.8) as $n \to \infty$ followed by $m \to \infty$, and we denote this double limit as

$$\sum_{i=1}^{N} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(u)=X_i(u)\}}}{N_k(u)} F'(X_i(u))dX_i(u). \tag{4.13}$$

Next, we need to deal with the expression (4.12); we change the order of the last two summation and take out the term of $\ell = r = p-1$ to obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \sum_{\ell=1}^{p-1} \sum_{r=\ell}^{p-1} \frac{F_m^{(r)}\big(X_{(k)}(t_j)\big)}{\ell!(r-\ell)!} \\
&\quad \big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell} \\
&\quad = \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \sum_{\ell=1}^{p-2} \sum_{r=\ell}^{p-1} \frac{F_m^{(r)}\big(X_{(k)}(t_j)\big)}{\ell!(r-\ell)!} \\
&\quad \big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell} && (4.14) \\
&\quad + \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \frac{F_m^{(p-1)}\big(X_{(k)}(t_j)\big)}{(p-1)!} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{p-1}. && (4.15)
\end{aligned}$$

For (4.15), we decompose the expression $\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{p-1}$ into

$$\Big(\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{+}\Big)^{p-1} - \Big(\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{-}\Big)^{p-1}$$

to obtain

$$\sum_{i=1}^{N} \frac{F_m^{(p-1)}\big(X_{(k)}(t_j)\big)}{(p-1)!N_k(t_j)} \mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} \Big(\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{+}\Big)^{p-1} \tag{4.16}$$

$$-\sum_{i=1}^{N} \frac{F_m^{(p-1)}\big(X_{(k)}(t_j)\big)}{(p-1)!N_k(t_j)} \mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} \Big(\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{-} \Big)^{p-1}. \quad (4.17)$$

The sum of the expression in (4.16) over the t_j's, namely,

$$\sum_{\substack{t_j \in \pi_n \\ t_j \le t}} \sum_{i=1}^{N} \frac{F_m^{(p-1)}\big(X_{(k)}(t_j)\big)}{(p-1)!N_k(t_j)} \mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}} \Big(\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{+} \Big)^{p-1}$$

converges as $n \to \infty$ to

$$\sum_{i=1}^{N} \int_0^t \frac{F_m^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(k)}-X_i)^+}(0), \quad (4.18)$$

as a result of Lemma 3.3. Similarly, the sum of (4.17) over the t_j's converges to

$$\sum_{i=1}^{N} \int_0^t \frac{F_m^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(k)}-X_i)^-}(0). \quad (4.19)$$

By the identities

$$(X_{(k)} - X_{(h)})^+ = \begin{cases} X_{(k)} - X_{(h)}, & \text{if } h > k \\ 0, & \text{if } h \le k, \end{cases}$$
$$(X_{(k)} - X_{(h)})^- = \begin{cases} X_{(h)} - X_{(k)}, & \text{if } h < k \\ 0, & \text{if } h \ge k, \end{cases}$$

with Proposition 4.1, the integrals (4.18) and (4.19) become

$$\sum_{h=k+1}^{N} \int_0^t \frac{F_m^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(k)}-X_{(h)})}(0),$$
$$\sum_{h=1}^{k-1} \int_0^t \frac{F_m^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(h)}-X_{(k)})}(0), \quad (4.20)$$

respectively. Here, the local time $L_u^{(X_{(k)}-X_{(h)})}(0)$ in (4.20) is called a "collision local time" among ranked functions $X_{(1)}, \cdots, X_{(m)}$. Thus, the sum of the expression (4.15) over t_j's converges as $n \to \infty$ and $m \to \infty$ to

$$\sum_{h=k+1}^{N} \int_0^t \frac{F^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(k)}-X_{(h)})}(0)$$
$$-\sum_{h=1}^{k-1} \int_0^t \frac{F^{(p-1)}\big(X_{(k)}(u)\big)}{(p-1)!N_k(u)} dL_u^{(X_{(h)}-X_{(k)})}(0). \tag{4.21}$$

To sum up, (4.6) is represented as the sum of (4.11), (4.14), and (4.15). The sums of (4.6), (4.11), and (4.15) over t_j's converge to the integrals (4.7), (4.13), and (4.21), respectively, as $n \to \infty$, $m \to \infty$. Therefore, the sum of (4.14) over t_j's should also converge, and we denote this limit by R_t^k. Since all other terms of (4.4) do not depend on the choice of $(F_m)_{m=1}^{\infty}$, R_t^k also does not depend on $(F_m)_{m=1}^{\infty}$. □

We conclude with a few remarks in Theorem (4.2).

Remark 4.3 For functions $F \in C^{p-1}(\mathbb{R})$, the approximation $(F_m)_{m=1}^{\infty}$ is not needed and the same formula (4.4) holds, where the modified Föllmer integrals are now replaced by the original Föllmer integrals.

Remark 4.4 (*Integral Representation for Ranked Processes of quadratic variation*) The last term R_t^k, including all of the extraneous cross-terms, is quite a nuisance, but inevitable due to the roughness of the functions. However, when $p = 2$, this term vanishes and we have simpler formula

$$\int_0^t F'(X_{(k)}(u))dX_{(k)}(u) = \sum_{i=1}^{N} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(u)=X_i(u)\}}}{N_k(u)} F'(X_i(u))dX_i(u) \tag{4.22}$$
$$+ \sum_{h=k+1}^{N} \int_0^t \frac{F'\big(X_{(k)}(u)\big)}{N_k(u)} dL_u^{(X_{(k)}-X_{(h)})}(0)$$
$$-\sum_{h=1}^{k-1} \int_0^t \frac{F'\big(X_{(k)}(u)\big)}{N_k(u)} dL_u^{(X_{(h)}-X_{(k)})}(0).$$

This representation for the descending order statistics of (4.2), in terms of (modified) Föllmer integrals with respect to the original functions $X_1, \cdots, X_N$ and the collision local times, is a pathwise generalization of the result of Banner and Ghomrasni [1] in the semimartingale context. In particular, the choice $F(x) = x$ yields the exact same formula with the one in Theorem 2.3 of Banner and Ghomrasni [1].

Remark 4.5 In (4.4), the presence of the pth-order local times is essential. If we try to remove the integrals involving local times by choosing F such that $F^{(p-1)} \equiv 0$, the formula becomes somewhat trivial. For example, if we set $F(x) = x$ for the functions X which are rougher than generic paths of semimartingales ($p > 2$), we have

$$X_{(k)}(t) - X_{(k)}(0) = \sum_{i=1}^{N} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(u)=X_i(u)\}}}{N_k(u)} dX_i(u)$$
$$+ \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big).$$

If we recall the expression (2.4) of the Föllmer integral, the right-hand side is just a rearrangement of the left-hand side.

Remark 4.6 We present now a way to simplify R_t^k by imposing a condition on the function F. Recall the expression R_t^k of (4.5) which is the limit of the sum over the t_j's. We rewrite this sum over the t_j's as

$$\sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \sum_{i=1}^{N} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \sum_{\ell=1}^{p-2}\sum_{r=\ell}^{p-1} \frac{F^{(r)}\big(X_i(t_j)\big)}{\ell!(r-\ell)!}$$
$$\big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell}\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell}$$
$$= \sum_{i=1}^{N}\sum_{\ell=1}^{p-2}\sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{\ell!\, N_k(t_j)} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell}$$
$$\sum_{r=\ell}^{p-1} \frac{F^{(r)}\big(X_i(t_j)\big)}{(r-\ell)!} \big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell}. \tag{4.23}$$

In (4.23), for each t_j and i,

$$\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell} \quad \longrightarrow \quad 0 \quad \text{as} \quad n \to \infty, \tag{4.24}$$

because, when $X_{(k)}(t_j) = X_i(t_j)$,

$$\big|X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big|^{\ell} = \big|X_{(k)}(t_{j+1}) - X_{(k)}(t_j) + X_i(t_j) - X_i(t_{j+1})\big|^{\ell}$$
$$\le \Big(\big|X_{(k)}(t_{j+1}) - X_{(k)}(t_j)\big| + \big|X_i(t_{j+1}) - X_i(t_j)\big|\Big)^{\ell}$$
$$\le 2^{\ell}\{osc(X, \pi_n)\}^{\ell},$$

and $osc(X, \pi_n) := \max_{i=1,\cdots,N}\{osc(X_i, \pi_n)\}$ which converges to zero as $n \to \infty$. On the other hand, the last part of (4.23) can be rewritten as

$$\sum_{r=\ell}^{p-1} \frac{F^{(r)}\big(X_i(t_j)\big)}{(r-\ell)!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^{r-\ell}$$
$$= f^{(\ell)}\big(X_i(t_j)\big) + \sum_{q=1}^{p-1-\ell} \frac{G^{(q)}\big(X_i(t_j)\big)}{q!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^{q}, \qquad (4.25)$$

where we used the substitution $q := r - \ell$ and $G(\cdot) := F^{(\ell)}(\cdot)$. If F satisfies

$$F^{(p)}(\cdot) = F^{(p+1)}(\cdot) = \cdots \equiv 0, \qquad (4.26)$$

or, equivalently, $G^{(p-\ell)}(\cdot) = G^{(p-\ell+1)}(\cdot) = \cdots \equiv 0$, then

$$\sum_{q=1}^{p-1-\ell} \frac{G^{(q)}\big(X_i(t_j)\big)}{q!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^{q} = \sum_{q=1}^{p-1} \frac{G^{(q)}\big(X_i(t_j)\big)}{q!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^{q}$$

and we have the convergence

$$\sum_{\substack{t_j\in\pi_n\\ t_j\le t}} \sum_{q=1}^{p-1} \frac{G^{(q)}\big(X_i(t_j)\big)}{q!}\big(X_i(t_{j+1}) - X_i(t_j)\big)^{q}$$
$$\xrightarrow{n\to\infty} \int_0^t G'(X_i(u))dG_i(u) \qquad (4.27)$$

in the spirit of (2.4).

Therefore, using the facts (4.24), (4.25), and (4.27), R_t^k takes simpler form

$$R_t^k = \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n\\ t_j\le t}} \sum_{i=1}^{m} \sum_{\ell=1}^{p-2} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \frac{F^{(\ell)}\big(X_i(t_j)\big)}{\ell!}$$
$$\big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell}, \qquad (4.28)$$

provided that the condition (4.26) holds.

A standard example satisfying the condition (4.26) is $F(x) := x^{(p-1)}$. The pathwise Itô's formula (2.3) applied to this choice of F gives

$$X_{(k)}^{(p-1)}(t) - X_{(k)}^{(p-1)}(0) = \int_0^t F'\big(X_{(k)}(u)\big)dX_{(k)}(u),$$

and combining with the expression (4.28), we have the dynamics of $(p-1)$th power of ranked functions

$$
\begin{aligned}
X_{(k)}^{(p-1)}(t) - X_{(k)}^{(p-1)}(0) &= \sum_{i=1}^{m} \int_0^t \frac{\mathbb{1}_{\{X_{(k)}(u)=X_i(u)\}}}{N_k(u)} F'\big(X_i(u)\big) dX_i(u) \\
&+ \sum_{h=k+1}^{m} \int_0^t \frac{(p-1)!}{N_k(u)} dL_u^{(X_{(k)}-X_{(h)})}(0) \\
&- \sum_{h=1}^{k-1} \int_0^t \frac{(p-1)!}{N_k(u)} dL_u^{(X_{(h)}-X_{(k)})}(0) \\
&+ \lim_{n\to\infty} \sum_{\substack{t_j\in\pi_n \\ t_j\le t}} \sum_{i=1}^{m} \sum_{\ell=1}^{p-2} \frac{\mathbb{1}_{\{X_{(k)}(t_j)=X_i(t_j)\}}}{N_k(t_j)} \binom{p-1}{\ell} \\
&\big(X_i(t_j)\big)^{p-1-\ell} \big(X_{(k)}(t_{j+1}) - X_i(t_{j+1})\big)^{\ell}.
\end{aligned}
$$

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