# NON-GALVIN FILTERS

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Abstract. We address the question of consistency strength of certain filters and ultrafilters which fail to satisfy the Galvin property. We answer questions [BG22a, Questions 7.8,7.9], [BGP23, Question 5] and improve theorem [BGP23, Theorem 2.3].

## 1. introduction

In this paper we continue the investigations on Galvin's property from [BG22a, BGS23, BGP23, BGP22]. Let $\kappa$ be a regular uncountable cardinal and $\mathscr{F}$ a $\kappa$-complete filter over it. We shall write $\mathrm{Gal}(\mathscr{F})$ as a shorthand for the following statement: Every $\langle A_\alpha \mid \alpha < \kappa^+\rangle \subseteq \mathscr{F}$ admits a subsequence $\langle A_{\alpha_\beta} \mid \beta < \kappa\rangle$ such that $\bigcap_{\beta<\kappa} A_{\alpha_\beta} \in \mathscr{F}$.[1] If $\mathrm{Gal}(\mathscr{F})$ holds we shall say that *Galvin's property holds for* $\mathscr{F}$ or, simply, that $\mathscr{F}$ *is Galvin*. This terminology is coined in homage to F. Galvin's discovery that if $\kappa^{<\kappa} = \kappa$ then the *club filter over* $\kappa$ ($\mathrm{Cub}_\kappa$) is Galvin [BHM75]. More generally, Galvin's proof shows that $\mathrm{Gal}(\mathscr{F})$ holds provided $\kappa^{<\kappa} = \kappa$ and $\mathscr{F}$ is normal.

The purpose of this paper is to present several constructions, both in the context of filters and ultrafilters, where Galvin's property fails. The first consistent example of a non-Galvin filter was provided by Abraham and Shelah [AS86]. In the said paper the authors exhibit a forcing poset producing a generic extension where $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ fails for a regular cardinal $\kappa$. By virtue of Galvin's theorem, $2^\kappa > \kappa^+$ in this latter model. An example of a ultrafilter $\mathscr{U} \subseteq \mathcal{P}(\kappa)$ for which $\mathrm{Gal}(\mathscr{U})$ fails was given by Benhamou, Garti and Shelah [BGS23]. Recently, in [BGP23] it was shown how to make $\mathrm{Cub}_{\kappa^+}$ non-Galvin for all singular cardinal $\kappa$, simultaneously.

The present manuscript is articulated in three blocks. In the first block (§2) we analyze the failure of Galvin's property for ultrafilters that extend the club filter. This issue was first raised in [BG22a] and subsequently answered in [BGS23] under the existence of a supercompact cardinal. Shortly after this was improved in [BG22b] using just a measurable cardinal. Here

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[1]In [BGP22, §1.1.] this is denoted by $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$. To streamline the presentation here we shall adopt the more compact notation $\mathrm{Gal}(\mathscr{F})$.

we modify the construction from [BGS23] aiming to produce ultrafilters $\mathscr{U}$ concentrating on the set of singular cardinals, $\{\alpha<\kappa \mid \mathrm{cf}(\alpha)<\alpha\}$. This method is flexible-enough to generate $\kappa$-complete ultrafilters $\mathscr{U}$ such that $\mathrm{Cub}_\kappa \subseteq \mathscr{U}$ and $\{\alpha<\kappa \mid \mathrm{cf}(\alpha)=\alpha\} \in \mathscr{U}$ (Theorem 2.4). These are the sort of ultrafilters constructed in [BG22b] using a completely different method. The advantage of the current strategy in front the one of [BG22b] is that the former, besides, adapts to handle the singular case. Following up with this issue, in Theorem 2.5 we give a lower bound for the consistency-strength of "There is a $\kappa$-complete ultrafilter $\mathscr{U} \supseteq \mathrm{Cub}_\kappa$ with $\{\alpha<\kappa \mid \mathrm{cf}(\alpha)<\alpha\} \in \mathscr{U}$" - this being $o(\kappa)\geq 2$. Later, in Theorem 2.6, we show starting from $o(\kappa)=2$ (i.e., from optimal assumptions) that it is possible to force a $\kappa$-complete ultrafilter $\mathscr{U}$ as above for which $\mathrm{Gal}(\mathscr{U})$ fails. The idea is to combine the Kurepa-tree-approach of [BGS23] and Gitik's construction of a $\kappa$-complete ultrafilter concentrating on singular cardinals [Git99].

In the second block of this paper (§3 and §4) we focus on failures of Galvin's property for filters. We commence with §3 showing the consistency of GCH with every regular cardinal $\kappa$ carrying a $\kappa$-complete non-Galvin filter. In particular, the normality assumption in Galvin's theorem is necessary. The key idea here is that the existence of $\kappa$-independent families $\mathscr{F} \subseteq \mathcal{P}(\kappa)$ (see page 14) yield such filters. It should be emphasized that we produce these configurations without bearing on any large-cardinal assumption. However, the disadvantage of this approach is that the filters generated do not contain the club filter. We address this issue in §3.2 where we prove the consistency of the GCH with the successor of every singular cardinal $\kappa$ carrying a $\kappa^+$-complete filter $\mathscr{F}$ such that $\mathrm{Cub}_{\kappa^+} \subseteq \mathscr{F}$ and $\neg\mathrm{Gal}(\mathscr{F})$. Unlike the previous approach, this latter consistency result uses large cardinals. Also, note that this differs from [BGP22, Theorem 2.3] in two aspects: first, the GCH holds; second, the filters $\mathscr{F}$ of interest are different from the club filter on $\kappa^+$ – in fact, they are non normal.

In §4, we describe how to produce $\kappa$-complete ultrafilters $\mathscr{U} \subseteq \mathcal{P}(\kappa)$ with $\mathrm{Cub}_\kappa \subseteq \mathscr{U}$ and $\{\alpha<\kappa \mid \mathrm{Gal}(\mathrm{Cub}_{\alpha^+}) \text{ fails}\} \in \mathscr{U}$. In particular, after Tree-Prikry-forcing with respect to $\mathscr{U}$ one gets a model where $\kappa$ is singular and there are cofinally many failures of Galvin's property below it. This can be used to illustrate a sort of failure of *compactness* at $\kappa$ relative to this property. In §4.2 we take a slightly different approach and show how to produce a similar configuration for the first singular cardinal, $\aleph_\omega$. The idea here is to introduce a Prikry sequence on a measurable cardinal and, simultaneously, force with the poset of Abraham and Shelah from [AS86]. The section ends indicating why Prikry-type forcings seem not useful to produce infinitely-many consecutive failures of Galvin's property.

The third and last block (§5) deals with the consistency-strength of the failure of Galvin's property at the successor of a singular cardinal. In Theorem 5.9 we show that $\neg\mathrm{Gal}(\mathrm{Cub}_{\aleph_{\omega+1}})$ is forceable starting with a cardinal $\kappa$ carrying a $(\kappa,\kappa^{++})$-extender. In particular, this pins down the consistency strength of this property to the optimal one; namely, $o(\kappa) = \kappa^{++}$. This answers a question from [BGP23, §5]. In addition, we get a close-to-optimal upper bound for the consistency strength of "$\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ fails for every singular cardinal $\kappa$". Specifically, we show that this is forceable starting with a $(\kappa + 3)$-strong cardinal. This improves [BGP23, Theorem 2.3].

1.1. **Notation.** Our notation is standard and mostly follows [BGP23, §1]. An important piece of notation is that referring to Galvin's property. For $\kappa = \mathrm{cf}(\kappa) > \omega$ and $\mu \leq \lambda \leq 2^\kappa$ we denote by $\mathrm{Gal}(\mathrm{Cub}_\kappa, \mu, \lambda)$ the statement:

"For every $\mathcal{C} \subseteq \mathrm{Cub}_\kappa$ with $|\mathcal{C}| = \lambda$ there is $\mathcal{D} \in [\mathcal{C}]^\mu$ such that $\cap\mathcal{D} \in \mathrm{Cub}_\kappa$."

We say that *Galvin's property holds at* $\kappa$ if $\mathrm{Gal}(\mathrm{Cub}_\kappa, \kappa, \kappa^+)$ holds. The above notation extends naturally to other filters $\mathscr{F}$ over $\kappa$. As explained earlier (see Footnote 1) in this paper we embrace the notation $\mathrm{Gal}(\mathscr{F})$ in lieu of the more cumbersome one $\mathrm{Gal}(\mathscr{F}, \kappa, \kappa^+)$. In some few parts of the paper, however, the more informative notation $\mathrm{Gal}(\cdot, \cdot, \cdot)$ will be required.

For two regular cardinals $\mu < \kappa$, $E^\kappa_\mu := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \mu\}$. The set of regulars below a cardinal $\kappa$ will be denoted by $\mathrm{Reg}_\kappa$. For ultrafilters $\mathscr{U}$ and $\mathscr{V}$ over $\kappa$ we write $\mathscr{U} \leq_{\mathrm{RK}} \mathscr{V}$ whenever $\mathscr{U}$ *is Rudin-Keisler below* $\mathscr{V}$; namely, if there is a function $f\colon \kappa \to \kappa$ such that for every $X \subseteq \kappa$,

$$X \in \mathscr{U} \text{ if and only if } f^{-1}[X] \in \mathscr{V}.$$

We force in the Israel style where $p \leq q$ means that $q \Vdash p \in G$ (i.e., $q$ *is stronger than* $p$). We write $p \parallel \varphi$ as a shorthand of "$p$ *decides* $\varphi$"; namely, either $p \Vdash \varphi$ or $p \Vdash \neg\varphi$. For cardinals $\mathrm{cf}(\kappa) = \kappa < \lambda$ we denote by $\mathbb{S}(\kappa, \lambda)$ the Abraham-Shelah poset from [AS86] (see also [BGP22, Definition 1.5]). This poset is $\kappa$-directed-closed and, assuming $2^\kappa = \kappa^+$, $\mathbb{S}(\kappa, \lambda)$ is a $\kappa^{++}$-cc. The two key properties of $\mathbb{S}(\kappa, \lambda)$ proved in [AS86] are: (1) $\mathbb{S}(\kappa, \lambda)$ is $\kappa^+$-distributive, hence it preserves $\kappa^+$; (2) $\mathbb{S}(\kappa, \lambda)$ forces $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+}, \kappa^+, \lambda)$.

## 2. The failure of Galvin's property for ultrafilters

2.1. **Non-Galvin ultrafilters.** Let $\kappa$ be a measurable cardinal. In [BGS23], the authors isolated a combinatorial consequence of the existence of a $\kappa$-complete ultrafilter $\mathscr{U}$ over $\kappa$ extending the club filter while concentrating on cofinality $\omega$. This was used to show that in some generic extension, where the negation of that combinatorial consequence holds, together with the existence of a $\kappa$-complete ultrafilter $\mathscr{U}$, extending the club filter and concentrating on cofinality $\omega$, $\mathrm{Gal}(\mathscr{U}, \kappa, \kappa^+)$ must fail. The argument generalizes straightforwardly to any cofinality $\omega < \theta < \kappa$ and breaks at $\theta = \kappa$. Namely, it was left open whether one can force the failure of Galvin's property at a $\kappa$-complete ultrafilter which concentrates on $\mathrm{Reg}_\kappa$.

Such ultrafilters were already constructed in [BG22b] from just a measurable cardinal, and our objective here is merely to expand the method of [BGS23] and to force $\neg\mathrm{Gal}(\mathscr{U})$ where $\kappa$ is measurable and $\mathscr{U}$ is a $\kappa$-complete ultrafilter over $\kappa$ which concentrates on $\mathrm{Reg}_\kappa$. We shall do it by modifying the forcing of [BGS23], but for our argument we need, first of all, a simple observation regarding the Rudin-Keisler order.

**Lemma 2.1.** *Assume that:*

(א) $\mathscr{U}, \mathscr{V}$ *are ultrafilters over* $\kappa$.
(ב) $\mathscr{U} \leq_{\rm RK} \mathscr{V}$.
(ג) ${\rm Gal}(\mathscr{V})$.

*Then* ${\rm Gal}(\mathscr{U})$.

*Proof.* Fix $\pi : \kappa \to \kappa$ witnessing the assumption $\mathscr{U} \leq_{\rm RK} \mathscr{V}$. Suppose that $\{C_i \mid i < \kappa^+\} \subseteq \mathscr{U}$. By definition, $\{\pi^{-1}[C_i] \mid i < \kappa^+\} \subseteq \mathscr{V}$, so one can find $I \in [\kappa^+]^\kappa$ and $B \in \mathscr{V}$ so that $B \subseteq \bigcap_{i \in I} \pi^{-1}[C_i]$. Let $A = \pi''B$. Notice that $A \subseteq C_i$ for every $i \in I$, thus ${\rm Gal}(\mathscr{U})$ is established. $\square$

From the above lemma we infer that if one forces $\neg{\rm Gal}(\mathscr{U})$ and $\mathscr{U} \leq_{\rm RK} \mathscr{V}$ then $\neg{\rm Gal}(\mathscr{V})$. Our strategy will be to force this situation where $\mathscr{V}$ concentrates on ${\rm Reg}_\kappa$. This will be done by adding one feature to the forcing construction of [BGS23]. We work with a slight simplification of the forcing $\mathbb{K}(S)$, where $S$ is a stationary subset of $\kappa$. This forcing notion adds an $S$-slim Kurepa tree.

**Definition 2.2.** The forcing notion $\mathbb{K}(S)$ consists of tuples $(t, f)$ such that $t$ is a normal tree of height $\beta+1$ for some $\beta \in \kappa$, and $|\mathcal{L}_\alpha(t)| \leq |\alpha|$ whenever $\alpha \in S \cap \beta + 1$. Finally, $f : \kappa^+ \to \mathcal{L}_\beta(t)$ is a partial function with $|f| \leq |\beta|$. If $(t, f), (s, g) \in \mathbb{K}(S)$ then we define $(t, f) \leq_{\mathbb{K}(S)} (s, g)$ iff $s \restriction (\beta + 1) = t, {\rm dom}(f) \subseteq {\rm dom}(g)$ and $f(\alpha) \leq_s g(\alpha)$ for every $\alpha \in {\rm dom}(f)$.

Let $G \subseteq \mathbb{K}(S)$ be generic, then a simple density argument shows that $\mathscr{T}_G = \bigcup\{t : \exists f, (t, f) \in G\}$ is the desired $S$-slim tree. For $\kappa$ is inaccessible, $\mathbb{K}(S)$ is $\kappa^+$-cc and $\kappa$ closed. Hence $\mathbb{K}(S)$ preserves cofinalities and preserves stationary subsets of $\kappa$.

*Remark* 2.3. The difference of this forcing with the one in [BGS23] is that we do not add a new stationary subset of $S$, and the slim Kurepa tree we are adding is $S$-slim $S$ being the stationary set we started with. The same argument as in [BGS23, Thm. 3.5] can be used to show that if $\kappa$ is supercompact, and we iterate with Easton support the forcing $\mathbb{K}(E^\alpha_\alpha)$ for inaccessibles $\alpha \leq \kappa$, then in the generic extension there is an $E^\kappa_\omega$-slim Kurepa tree and there is an ultrafilter $U$ extending ${\rm Cub}_\kappa \cup \{E^\kappa_\omega\}$. The reason the argument still works is that this iteration preserves cofinalities and therefore does not change $E^\kappa_\omega$.

**Theorem 2.4.** *Let* $\kappa$ *be supercompact. One can force* $\neg{\rm Gal}(\mathscr{V})$ *where* $\mathscr{V}$ *is a* $\kappa$*-complete ultrafilter over* $\kappa$ *such that* ${\rm Cub}_\kappa \subseteq \mathscr{V}$ *and* ${\rm Reg}_\kappa \in \mathscr{V}$.

*Proof.* Assume that $\kappa$ is supercompact and Let $E^\kappa_\omega$ be the set of ordinals less than $\kappa$ of cofinality $\omega$. We define a two-step iteration $\mathbb{S} = \mathbb{P} * \underset{\sim}{\mathbb{R}}$ as follows. The first component $\mathbb{P}$ is an Easton support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$, where $\underset{\sim}{\mathbb{Q}}_\beta$ is trivial unless $\beta$ is strongly inaccessible , in which case we let $\underset{\sim}{\mathbb{Q}}_\beta$ be a $\mathbb{P}_\beta$-name of the forcing notion $\mathbb{K}(E^\beta_\omega) \times \mathrm{Add}(\beta, 1)$. The second component $\underset{\sim}{\mathbb{R}}$ is a $\mathbb{P}$-name of the forcing notion $\mathbb{K}(E^\kappa_\omega) \times \mathrm{Add}(\kappa, 1)$.

Let $G \subseteq \mathbb{S}$ be $V$-generic, so $G$ factors into $G_{\mathbb{P}} * G_{\mathbb{R}}$ in a natural way. From [BGS23] we know that in $V[G]$ there is an $E^\kappa_\omega$-slim Kurepa tree. Moreover, there is a $\kappa$-complete ultrafilter $\mathscr{U}$ over $\kappa$ which extends $\mathrm{Cub}_\kappa \cup \{E^\kappa_\omega\}$ (by [BGS23, Cor. 3.4] $\mathscr{U}$ must satisfy $\neg\mathrm{Gal}(\mathscr{U})$). Let us briefly describe $\mathscr{U}$, and build another $\kappa$-complete ultrafilter $\mathscr{V}$ so that $\mathscr{U} \leq_{\mathrm{RK}} \mathscr{V}$ and $\mathrm{Reg}_\kappa \in \mathscr{V}$.

Choose $\lambda > 2^\kappa$ and a supercompact elementary embedding $\jmath : V \to M$ such that $\mathrm{crit}(\jmath) = \kappa, \jmath(\kappa) = \lambda$ and ${}^{2^\kappa}M \subseteq M$. Let $\mathbb{P}' = \jmath(\mathbb{P})$, so $\mathbb{P}' = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \leq \jmath(\kappa), \beta < \jmath(\kappa)\rangle$. Up to $\kappa$ we know that $\mathbb{P}'$ coincides with $\mathbb{P}$, since $\kappa = \mathrm{crit}(\jmath)$. Also, by elementarity and the definition of our iteration, $\mathbb{P}'_{\kappa+1} = \mathbb{P} * \underset{\sim}{\mathbb{R}}$, as $M$ compute $\mathbb{K}(E^\kappa_\omega) \times \mathrm{Add}(\kappa, 1)$ correctly.

In particular, one can form the generic extension $M[G]$ in $V[G]$. Observe that the rest of the iteration, that is, $\mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)} = \mathbb{P}'_{(\kappa+1,\jmath(\kappa))} * \jmath(\underset{\sim}{R})$, is $\theta$-closed where $\theta$ is the first $M[G]$-inaccessible above $\kappa$. In particular, it is $(2^\kappa)^+$-closed, where $(2^\kappa)^+$ is computed in $V$.

Working in $V$, let $\mathcal{C} = \{\underset{\sim}{C} : \underset{\sim}{C}$ is a nice $\mathbb{P}'_{\kappa+1}$-name for a club at $\kappa\}$, since $\mathbb{P}'_{\kappa+1}$ is $\kappa^+$-cc, we have that $|\mathcal{C}| = 2^\kappa$. Similarly, let $\mathcal{A}$ be $\{\underset{\sim}{A} : \underset{\sim}{A}$ is a nice $\mathbb{P}'_{\kappa+1}$-name for a subset of $\kappa\}$ and then $|\mathcal{A}| = 2^\kappa$. Since ${}^{2^\kappa}M \subseteq M$, we see that both $\{\jmath(\underset{\sim}{C}) : \underset{\sim}{C} \in \mathcal{C}\}$ and $\{\jmath(\underset{\sim}{A}) : \underset{\sim}{A} \in \mathcal{A}\}$ are elements of $M$.

As a first step towards the construction of $\mathscr{U}$ we claim that there exist an ordinal $\delta$ and a condition $p \in \mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)}$ such that $p$ forces in $\mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)}$ that $\delta \in \bigcap\{\jmath(\underset{\sim}{C}) : \underset{\sim}{C} \in \mathcal{C}\} \cap \jmath(E^\omega_\kappa)$. To see this, recall that each $\jmath(\underset{\sim}{C})$ is a name for a club at $\jmath(\kappa)$, and $\jmath(\kappa) = \lambda > 2^\kappa$. Thus, $\{\jmath(\underset{\sim}{C}) : \underset{\sim}{C} \in \mathcal{C}\}$ is a collection of $2^\kappa$ names of clubs of $\jmath(\kappa)$ and hence it is forced by the empty condition that $\bigcap\{\jmath(\underset{\sim}{C}) : \underset{\sim}{C} \in \mathcal{C}\}$ is a club at $\jmath(\kappa)$. In addition, $\jmath(E^\kappa_\omega)$ is a stationary subset of $\jmath(\kappa)$, so one can find $\delta$ and $p$ such that $p \Vdash_{\mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)}} \delta \in \bigcap\{\jmath(\underset{\sim}{C}) : \underset{\sim}{C} \in \mathcal{C}\} \cap \jmath(\underset{\sim}{S})$.

By the closure of $\mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)}$ there is $q \geq p$ such that $q$ decides the statement $\delta \in \jmath(\underset{\sim}{A})$ for every $\underset{\sim}{A} \in \mathcal{A}$. Indeed, enumerate $\mathcal{A}$ by $\{\underset{\sim}{A}_j : j \in 2^\kappa\}$ and create an increasing sequence of conditions $\langle q_j \mid j < 2^\kappa\rangle$ such that $q_j \parallel \delta \in \underset{\sim}{A}_j$ and $p \leq q_0$. At the end, let $q$ be an upper bound of every $q_j$.

Now in $V[G]$ we can define $\mathscr{U}$ as the set $\{(\underset{\sim}{A})_G : \underset{\sim}{A} \in \mathcal{A} \wedge q \Vdash \delta \in \jmath(\underset{\sim}{A})\}$. One can verify that $\mathscr{U}$ is a $\kappa$-complete ultrafilter over $\kappa$ (using the fact that $\kappa = \mathrm{crit}(\jmath)$ and the elementarity of $\jmath$). Moreover, $\mathrm{Cub}_\kappa \cup \{E^\omega_\kappa\} \subseteq \mathscr{U}$ by the choice of $\delta$. Our goal, therefore, is to construct $\mathscr{V}$.

Recall that $\underset{\sim}{\mathbb{R}}$ is $\mathbb{K}(E^\omega_\kappa) \times \mathrm{Add}(\kappa, 1)$, so let $f : \kappa \to \kappa$ be the Cohen part as interpreted by $G_{\mathbb{R}}$, and let $\underset{\sim}{f}$ be a $\mathbb{P}'_{\kappa+1}$-name for it. We claim that there are a condition $r \geq q$ and an $M[G]$-inaccessible $\rho \in \jmath(\kappa)$ such that:

$$r \Vdash_{\mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)}} \rho \in \bigcap\{\jmath(\underset{\sim}{C}) : \underset{\sim}{C} \in \mathcal{C}\} \wedge \jmath(\underset{\sim}{f})(\rho) = \delta$$

The claim is justified by the general fact that if $\mu$ is Mahlo in some model $W$ of $\mathsf{ZFC}$ and $\underset{\sim}{g}$ is a name for the $W$-generic function $g : \mu \to \mu$ for the forcing notion $\mathrm{Add}(\mu,1)$ then the empty condition forces that for every $\gamma<\mu$, there are stationarily many inaccessibles $\alpha<\mu$ such that $\underset{\sim}{g}(\alpha)=\gamma$.

Our $\jmath(\kappa)$ is forced to stay Mahlo after we force with $\mathbb{P}'_{(\kappa+1,\jmath(\kappa)+1)}$ over $M[G]$ , thus we may apply the general fact to $M[G]$ and $\jmath(\underset{\sim}{f})$, and deduce that for every $\gamma<\jmath(\kappa)$ the condition $q$ forces that there are stationarily many inaccessibles $\alpha<\jmath(\kappa)$ such that $\jmath(\underset{\sim}{f})(\alpha)=\gamma$. Taking $\gamma$ as our $\delta$ we see that there is an inaccessible $\rho<\jmath(\kappa)$, and $r\geq q$ such that

$$r\Vdash \jmath(\underset{\sim}{f})(\rho)=\delta\wedge\rho\in\bigcap\{\jmath(\underset{\sim}{C}):\underset{\sim}{C}\in\mathcal{C}\}.$$

Again, by closure we can assume that such a condition already determines all the statements $\rho\in\jmath(\underset{\sim}{A})$ for every $\underset{\sim}{A}\in\mathcal{A}$. This choice enables us to define, in $V[G]$, the following set:

$$\mathscr{V}=\{(\underset{\sim}{A})_G\mid\underset{\sim}{A}\in\mathcal{A},r\Vdash_{\mathbb{P}'_{(\kappa+1,\jmath(\kappa))}}\rho\in\jmath(\underset{\sim}{A})\}$$

It is routine to check that $\mathscr{V}$ is a $\kappa$-complete ultrafilter over $\kappa$ in $V[G]$. Let us show that $\mathscr{U}\leq_{\mathrm{RK}}\mathscr{V}$ as witnessed by $f$.

To prove this fact, fix $A\in\mathscr{U}$, so $A=\underset{\sim}{A}_G$ for some $\underset{\sim}{A}\in\mathcal{A}$. Let us show that $f^{-1}[A]\in\mathscr{V}$ (the opposite direction from $\mathscr{V}$ to $\mathscr{U}$ is similar). By definition, $A\in\mathscr{U}$ implies that $r\Vdash\delta\in\jmath(\underset{\sim}{A})$. Therefore, $r$ forces $\jmath(\underset{\sim}{f})(\rho)\in\jmath(\underset{\sim}{A})$ by the choice of $\rho$. This means that $r$ forces $\rho\in\jmath(\underset{\sim}{f}^{-1}[\underset{\sim}{A}])$. By the definition of $\mathscr{V}$ we conclude that $f^{-1}[\underset{\sim}{A}_G]\in\mathscr{V}$, thus $f^{-1}[A]\in\mathscr{V}$ as required.

Since $\rho$ is inaccessible in $M$, the set $\mathrm{Reg}_\kappa$ belongs to $\mathscr{V}$. From Lemma 2.1 we know that $\mathrm{Gal}(\mathscr{V})$ fails in $V[G]$, so we are done. $\square$

Let us indicate that stronger properties can be forced upon $\mathscr{V}$, due to the choice of $\rho$. Thus, since $\jmath(\kappa)$ is supercompact in $M$ one can choose $\rho$ to be measurable and then $\mathscr{V}$ concentrates on measurable cardinals.

2.2. **A non Galvin ultrafilter concentrating on singulars.** In this section we produce a $\kappa$-complete non-Galvin ultrafilter that concentrates on singulars and extends the club filter $\mathrm{Cub}_\kappa$. We will accomplish the construction starting from optimal large-cardinal assumptions, hence improving the main result of [BGS23]. The readers familiar with [BG22b] will note that the present context differs from the former in that all the ultrafilters considered in [BG22b] concentrated on the set of regular cardinals.

As the forthcoming theorem shows $o(\kappa)=2$ is the minimal large-cardinal assumption for the existence of a $\kappa$-complete ultrafilter $\mathscr{U}\supseteq\mathrm{Cub}_\kappa$ concentrating on the set of singular cardinals. The argument is due (basically) to W. Mitchell but we add the proof for the reader's convenience.

**Theorem 2.5.** *Suppose there is a $\kappa$-complete ultrafilter $U$ over $\kappa$ such that $\{\alpha<\kappa\mid\alpha\text{ is singular}\}\in U$ and $\mathrm{Cub}_\kappa\subseteq U$ or alternatively, that $[\mathrm{id}]_U$ is a generator of $j_U$ and $[\mathrm{id}]_U$ is singular. Then either there is an inner model with a cardinal $\lambda$ such that $o(\lambda)=\lambda^{++}$ or in the core model $\mathcal{K}$, $o^{\mathcal{K}}(\kappa)\geq 2$.*

*Proof.* Suppose that there is no inner model with a cardinal $\lambda$ such that $o(\lambda) = \lambda^{++}$ and let $\mathcal{K}$ be the Mitchell core model [JS13]. We shall prove that $o^{\mathcal{K}}(\kappa) \geq 2$. Towards a contradiction let us assume that $o^{\mathcal{K}}(\kappa) \leq 1$. Since $\kappa$ is measurable in $V$, by maximality of the core model $\mathcal{K}$, $\kappa$ is measurable in $\mathcal{K}$ as well. Consider $j_U : V \to M_U$ the ultrapower embedding. The elementary embedding $j_U \restriction \mathcal{K} \colon \mathcal{K} \to \mathcal{K}^{M_U}$ is a linear iterated ultrapower of $\mathcal{K}$ by its measures. For details on this see [Mit84] or a more general result due to Schindler [Sch06]. In addition, one may assume that this iteration is normal (see [Mit84, p. 241]); namely, that its critical points are increasing. Denote by $\langle i_{\alpha,\beta} \mid \alpha \leq \beta \leq \theta\rangle$ this iteration, hence $i_{0,\theta} = j_U \restriction \mathcal{K}$. Let $\langle \kappa_i \mid i \leq \lambda\rangle$ be the increasing enumeration of $\{i_{0,\alpha}(\kappa) \mid \alpha \leq \theta\}$. Since $o^{\mathcal{K}}(\kappa) = 1$ there is a single normal measure $W$ and by elementarity, for every $\alpha \leq \theta$, there is a single normal measure $i_{0,\alpha}(W)$. By [BGH21, Corollary 43], for each $i < \lambda$ there is a stage of the iteration $\alpha_i < \theta$, such that $crit(i_{\alpha_i,\alpha_i+1}) = \kappa_i = i_{0,\alpha_i}(\kappa)$, then $i_{\alpha_i,\alpha_{i+1}}$ must be the ultrapower by $i_{0,\alpha_i}(W)$. First note that $[id]_U \in \{\kappa_i \mid i < \lambda\}$. Just otherwise, $\kappa_\delta < [id]_U < \kappa_{\delta+1}$ and by [BGH21, Claim 44], there would be $f : \kappa \to \kappa$ such that $j_U(f)(\kappa_\delta) \geq [id]_U$. But then $C_f := \{\alpha < \kappa \mid f''\alpha \subseteq \alpha\}$ would be a club at $\kappa$ and $[id]_U \notin j_U(C_f)$. This is a contradiction to our assumption that $\mathrm{Cub}_\kappa \subseteq U$.

Next, as $[id]_U$ is singular in $M_U$, it follows that $[id]_U = \kappa_\delta$ for some limit $\delta$ since by [BGH21, Lemma 46] each successor element of the sequence of the form $\kappa_{i+1}$ is regular in $M_U$. We conclude that $\lambda \geq \omega$.

Consider the first $\omega$ many images of $\kappa$, $\{\kappa_n \mid n < \omega\}$. Since $M_U$ is closed under $\omega$-sequences $\langle \kappa_n \mid n < \omega\rangle \in M_U$. Recall that all the $\kappa_n$'s are critical points of the iteration, which we denoted by $\alpha_n$. Let $\alpha_\omega = \sup_{n<\omega} \alpha_n$. Then $i_{0,\alpha_\omega}(\kappa) < i_\theta(\kappa) = j_U(\kappa)$, otherwise $cf^{M_U}(j_U(\kappa)) = \omega$ which contradicts the elementarity of $j_U$. Hence by the normality of the iteration, $crit(i_{\alpha_\omega,\alpha_\omega+1}) = i_{0,\alpha_\omega}(\kappa)$ and since $o(i_{0,\alpha_\omega}(\kappa)) = 1$ it follows that $i_{\alpha_\omega,\alpha_\omega+1} : \mathcal{K}_{\alpha_\omega} \to \mathcal{K}_{\alpha_\omega+1}$ is the ultrapower by $i_{0,\alpha_\omega}(W)$. In particular, $i_{0,\alpha_\omega}(W) \notin \mathcal{K}_{\alpha_\omega+1}$. Working in $M_U$, using the sequence of $\kappa_n$'s which forms a Prikry sequence for $i_{0,\alpha_\omega}(W)$, we can reconstruct $i_{0,\alpha_\omega}(W) \in M_U$. Thus $i_{0,\alpha_\omega}(W) \in \mathcal{K}^{M_U}$ as any $\mathcal{K}^{M_U}$-measure in $M_U$ already belongs to $\mathcal{K}^{M_U}$. However, $i_{\alpha_\omega+1,\theta} : \mathcal{K}_{\alpha_\omega+1} \to \mathcal{K}^{M_U}$, and by normality of the iteration, $crit(i_{\alpha_\omega+1,\theta})$ is much above $i_{0,\alpha_\omega}(\kappa)$ which ensures that $i_{0,\alpha_\omega}(W) \in \mathcal{K}_{\alpha_\omega+1}$, contradiction. □

Our goal, which we have just established, was to deduce that in terms of consistency strength, the existence of an ultrafilter $U$ which extend the club filter and concentrates on singular is at least the consistency of $\exists\kappa\ o(\kappa) \geq 2$. However, it seems that one can prove a slightly stronger result, as the above argument also works under the assumption that there is no inner model with a Woodin cardinal. This requires a more delicate analysis of the iteration tree associated to $j_U \restriction \mathcal{K}$, the iteration over its main branch, and the appropriate covering lemma for this kind of anti large cardinal assumption.

To establish the equiconsistency result, let us prove that under the minimal assumption, it is possible to force a an ultrafilter as above for which the Galvin property fails.

**Theorem 2.6.** *Assume* GCH *and suppose* $o(\kappa) \geq 2$ *then it is consistent that there is a* $\kappa$*-complete ultrafilter* $W$ *such that* $\mathrm{Cub}_\kappa \subseteq W$ *and* $\{\alpha < \kappa \mid \alpha \text{ is singular }\} \in W$ *which fails to satisfy the Galvin property.*

The main Lemma is the following:

**Lemma 2.7.** *Suppose that* $U_0 \lhd U_1$ *are normal measures over* $\kappa$. *Then there is a forcing extension* $V[G]$ *such that in* $V[G]$ $U_0, U_1$ *extend to* $U_0^*, U_1^*$ *respectively,* $U_0^*$ *is normal,* $U_0^* \leq_{RK} U_1^*$, $U_1^*$ *concentrates on* $E_\omega^\kappa$ *and the* $\omega$*-iteration by* $U_0^*$ *denoted by* $\langle j_{n,m}, M_n \mid n \leq m < \omega\rangle$ *satisfies the following:*

(1) $\kappa = crit(j_{0,1})$, *and* $crit(j_{n,n+1}) = j_{0,n}(\kappa)$.
(2) $\langle j_{0,n}(\kappa) \mid n < \omega\rangle$ *is unbounded in* $[id]_{U_1^*}$.
(3) *There are factor maps for the embedding ultrapower* $j_{U_1^*}$, $k_n : M_n \to M_{U_1^*}$ *such that* $j_{U_1^*} = k_n \circ j_{0,n}$, $k_n = k_m \circ j_{n,m}$ *and* $crit(k_n) = j_{0,n}(\kappa)$.

Let us first conclude Theorem 2.6 from this lemma.

*Proof of Theorem 2.6.* We force over $N := V[G]$ with the same poset as in [BGS23]; namely, we force with the Easton-supported iteration that at each inaccessible cardinal $\alpha \leq \kappa$ forces with $\mathbb{K}(E^\alpha_\omega)$, the poset adding a $E^\alpha_\omega$-slim Kurepa tree. Denote this iteration by $\mathbb{P} * \mathbb{R}$. Let $H_\kappa * h$ be an $N$-generic filter for $\mathbb{P}*\mathbb{R}$. First we extend $j_{0,1} : N \to M_1$, note that by Easton support,

$$j_{0,1}(\mathbb{P} * \mathbb{R}) = \mathbb{P} * \mathbb{R} * \mathbb{P}_{(\kappa+1, j_{0,1}(\kappa))} * j_{0,1}(\mathbb{R})$$

and $j''_{0,1} H_\kappa = H_\kappa$. Let us define in $N[H_\kappa * h]$ an $M_1$-generic filter for $j_{0,1}(\mathbb{P}*\mathbb{R})$ by first taking $H_\kappa * h$. Note that the forcing $\mathbb{P}_{(\kappa+1, j_{0,1}(\kappa))}$ starts above $\kappa^+$ and by GCH, there are only $\kappa^+$-many dense subsets to meet. By standard arguments, exploiting the fact that $M_1$ is the ultrapower by a $\kappa$-complete measure, hence closed under $\kappa$-sequences of $V$, we construct an $M_1[H_\kappa * h]$-generic filter $T$ for $\mathbb{P}_{(\kappa+1, j_{0,1}(\kappa))}$.

Notice that the model $M_1[H_\kappa * h * T]$ is closed only under $\kappa$-sequences from $N[H_\kappa * h]$ and since $|h| = \kappa^+$, we cannot guarantee that $j''h \in M_1[H_\kappa * h * T]$. Instead, we start by constructing any $M_1[H_\kappa * h * T]$-generic $t'$ for $j_1(\mathbb{R})$ starting above the condition $\langle f_\kappa, T_\kappa\rangle$ where $T_\kappa$ is a tree of height $\kappa + 1$, $T_\kappa \restriction \kappa$ is constructed from $h$ and $Lev_\kappa(T_\kappa) = \{b_\kappa(\alpha) \mid \alpha < \kappa^+\}$ where $b_\kappa(\alpha)$ are the branches derived from $h$. It is crucial here that we can take $\kappa^+$-many elements in $Lev_\kappa(T_\kappa)$ since $\kappa \notin j_{0,1}(E^\kappa_\omega) = E_\omega^{j_{0,1}(\kappa)}$. The function $f_\kappa : j_{0,1}(\kappa)^+ \to Lev_\kappa(T_\kappa)$ is the partial function with $\mathrm{dom}(f_\kappa) = \kappa$ and $f_\kappa(\alpha) = b_\kappa(\alpha)$ for every $\alpha < \kappa$. Now in $N[H_\kappa * h]$ we define an additional filter $t$ which is obtained from $t'$ by changing the values of each branch of the form $b_{j_{0,1}(\kappa)}(j_{0,1}(\alpha))$ where $\alpha < \kappa^+$ so that

$$b_{j_{0,1}(\kappa)}(j_{0,1}(\alpha)) \restriction \kappa = b_\kappa(\alpha).$$

Formally, for every pair $\langle f_\kappa, T_\kappa\rangle \leq \langle g, S\rangle$ we define $\langle g^*, S\rangle$ where for every $j_{0,1}(\alpha) \in \mathrm{dom}(g) \cap j''_{0,1}\kappa^+$ we let $g^*(j_{0,1}(\alpha)) \restriction \kappa = b_\kappa(\alpha)$. Note that $g^* \in M_1[H_\kappa * h * T]$, since $\mathrm{dom}(g)$ is bounded in $j_{0,1}(\kappa)^+$, $|\mathrm{dom}(g) \cap j''\kappa^+| \leq \kappa$ and $M_1[H_\kappa * h * T]$ is closed under $\kappa$-sequences. Define $t = \{\langle b^*, B\rangle \mid \langle b, B\rangle \in t'\}$, then $t \subseteq j_{0,1}(\mathbb{R})$ and it is $M_1[H_\kappa * h * T]$-generic. To see this take any dense open set $D \in M_1[H_\kappa * h * T]$, Define $D^* \in M[H_\kappa * h * T]$ as the set of all pairs $\langle g, S\rangle$ such that $\langle g^*, S\rangle \in D$. Let us denote

$$H_\kappa = H_0,\ h = h_0,\ G_0 = H_0 * h_0 \text{ and}$$

$$H_1 = H_\kappa * h * T,\ h_1 = t,\ G_1 = H_1 * h_1.$$

In the same fashion, keep defining inductively the generic filters $G_n = H_n * h_n$ such that $G_{n+1} \restriction \kappa_n + 1 = G_n$. So far we have extended the embeddings $j^*_{n,m}$ to form a directed system $\langle j^*_{n,m}, M_n[G_n] \mid n \leq m < \omega\rangle$. Denote the direct limit of the models $M_\omega = \underrightarrow{Lim} M_n$, the direct limit embeddings $j_{n,\omega} : M_n \to M_\omega$, and the direct limit of the factor maps $k_\omega : M_\omega \to M_{U_1^*}$, which is defined by the equalities $k_\omega \circ j_{n,\omega} = k_n$. Finally, denote $\underrightarrow{Lim} M_n[G_n] = M_\omega[G_\omega]$ and the direct limit embeddings $j^*_{n,\omega} : M_n[G_n] \to M_\omega[G_\omega]$. Note that

$$G_\omega = H_\omega * h_\omega = j^*_{n,\omega}(H_n * h_n).$$

To extend $j_{U_1^*}$, we start the construction up to $[id]_{U_1^*}$ not including $[id]_{U_1^*}$. Recall that by the lemma, $crit(k_n) = j_{0,n}(\kappa)$ and $\langle j_{0,n}(\kappa) \mid n < \omega\rangle$ is unbounded in $[id]_{U_1^*}$. It follows that $k''_n H_n = H_n$ and $H_\omega = \cup_{n<\omega} H_n$. Since $crit(k_\omega) = \sup j_{0,n}(\omega) = [id]_{U_1^*}$, it follows that $H_\omega$ is $M_{U_1^*}$-generic for $\mathbb{P}_{[id]_{U_1^*}}$.

Next, at $[id]_{U_1^*}$ which is of cofinality $\omega$ in $M_{U_1^*}$ the iteration is defined to be trivial. Finally, above $[id]_{U_1^*}$ we construct the generic in a similar fashion to what we did for $j_{U_0^*}$ where in the construction of the tree at $j_{U_1^*}(\mathbb{R})$ we change the values of the generic at $j_{U_1^*}(\kappa)$ with respect to the point-wise image by $k_\omega$ of $h_\omega$.

We obtain an $M_{U_1^*}$-generic filter $G^* \in N[G_0]$ and extend $j_{U_1^*}$ to $j^*_{U_1^*}$ : $N[G_0] \to M_{U_1^*}[G^*]$. Note that by the construction of the generic filters $G_n$, also $k_n$ extends to commutative factor maps $k^*_n : M_n[G_n] \to M_{U_1^*}[G^*]$. Hence by the universal property of direct limits, there is $k^*_\omega : M_\omega[G_\omega] \to M_{U_1^*}[G^*]$ such that $k^*_\omega \circ j^*_{n,\omega} = k^*_n$. Since each $k^*_n$ extends $k_n$ we see that $k^*_\omega$ extends $k_\omega$.

In $N[G_0]$, derive the ultrafilter $W = \{X \subseteq \kappa \mid [id]_{U_1^*} \in j^*_{U_1^*}(X)\}$.

**Proposition 2.8.** $\mathrm{Cub}_\kappa \subseteq W$, $E^\omega_\kappa \in W$ *and* $W$ *fails to satisfy the Galvin property.*

*Proof of Proposition 2.8.* Since $M_{U_1^*}[G^*] \models cf([id]_{U_1^*}) = \omega$, $E^\kappa_\omega \in W$. Let $C \in \mathrm{Cub}_\kappa$, we would like to prove that $[id]_{U_1^*} \in j^*_{U_1^*}(C)$. By elementarity, $j^*_{U_1^*}(C)$ is closed and since $\langle j_{0,n}(\kappa) \mid n < \omega\rangle$ is unbounded in $[id]_{U_1^*}$, it suffices to prove the for every $n < \omega$, $j_{0,n}(\kappa) \in j^*_{U_1^*}(C)$. Let $n < \omega$, since $crit(k^*_n) = crit(k_n) = j_{0,n}(\kappa)$, and $k^*_n(j^*_{0,n}(C)) = j^*_{U_1^*}(C)$, $j^*_{U_1^*}(C) \cap j_{0,n}(\kappa) = j^*_{0,n}(C)$.

By elementarity, $j^*_{0,n}(C)$ is unbounded in $j_{0,n}(\kappa)$, and since $j^*_{U_1^*}(C)$ is closed, $j_{0,n}(\kappa) \in j^*_{U_1^*}(C)$. Recall that in $N[G_0]$ we have an $E^\omega_\kappa$-slim Kurepa tree, so by [BGS23], $W$ cannot have the Galvin property. $\square_{\text{Prop.2.8}}$ $\square_{\text{Thm.2.6}}$

*Proof of Lemma 2.7*: Let $U_0 \lhd U_1$ and $[\xi \mapsto U_0(\xi)]_{U_1} = U_0$. Let

$$E = \{\alpha < \kappa \mid \alpha \text{ is measurable with measure } U_0(\alpha)\} \in U_1 \setminus U_0.$$

Suppose also that $\forall \alpha \in E, E \cap \alpha \notin U_0(\alpha)$. Over the ground model $V$, we first force with $\mathbb{P}_\kappa$ which is an Easton support iteration of Prikry forcing as defined in [Git86] adding for each $\alpha \in E$ a Prikry sequence $\{\alpha_n \mid n < \omega\}$. Namely:

**Definition 2.9.** Let $\overline{E}$ be the closure of the set $E \cup \{\alpha+1 \mid \alpha \in E\} \cup \{\kappa\}$. For $\alpha \in \overline{E}$, we inductively define $\mathbb{P}_\alpha$, the conditions of $\mathbb{P}_\alpha$ are functions $p = \{\langle t^p_\gamma, A^p_\gamma\rangle \mid \gamma \in \mathrm{dom}(p)\}$ such that:

1. $\mathrm{dom}(p) \subseteq \overline{E} \cap \alpha$.
2. $p$ has Easton support, namely, for every inaccessible $\beta \leq \alpha$, $\mathrm{dom}(p) \cap \beta$ is bounded in $\beta$.
3. For every $\gamma \in \mathrm{dom}(p)$, $p \restriction \gamma := \{\langle t^p_\beta, A^p_\beta\rangle \mid \beta \in \mathrm{dom}(p) \cap \gamma\} \in \mathbb{P}_\gamma$ and if $\gamma \in E$ then $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \langle t^p_\gamma, A^p_\gamma\rangle \in \mathcal{P}(\underset{\sim}{U}^*_0(\gamma))$, where $\underset{\sim}{U}^*_0(\gamma)$ is a normal ultrafilter over $\gamma$ defined in Definition 2.13.

The order is $p \leq q$ if and only if:

1. $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$.
2. For every $\gamma \in \mathrm{dom}(p)$, $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \langle t^p_\gamma, A^p_\gamma\rangle \leq \langle t^q_\gamma, A^q_\gamma\rangle$.
3. There is a finite set $b$ such that for every $\gamma \in \mathrm{dom}(p) \setminus b$, $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} t^p_\gamma = t^q_\gamma$.

Moreover in clause 3. if $b = \emptyset$ then we say that $q$ is a direct extension of $p$ and denote it by $p \leq^* q$.

The following lemmas can be found in [Git86]:

**Lemma 2.10.** *Let $\langle p_\beta \mid \beta < \gamma < \alpha\rangle$ be a sequence of conditions in $\mathbb{P}_\alpha$ such that for every $\beta_1 \leq \beta_2$, $p_{\beta_1} \restriction \gamma+1 = p_{\beta_2} \restriction \gamma+1$ and $p_{\beta_1} \leq^* p_{\beta_2}$. Then there is $p \in \mathbb{P}_\alpha$ such that for every $\beta < \gamma$, $p \restriction \gamma+1 = p_\beta \restriction \gamma+1$ and $p_\beta \leq p$.*

**Lemma 2.11.** *Let $\alpha$ be a limit point of $E$ such that $\alpha$ is Mahlo. Then $\mathbb{P}_\alpha$ is $\alpha$-cc.*

**Lemma 2.12.** *For every $p \in \mathbb{P}_\alpha$ and every statement in the forcing language $\sigma$ there is $p \leq^* p^*$ such that $p^* || \sigma$.*

**Definition 2.13.** Fix some well ordering $W$ of $V_\lambda$ for some very large $\lambda$ such that for every $\beta < \lambda$ inaccessible $W \restriction V_\beta \leftrightarrow \beta$. Suppose that $\alpha \in E \cup \{\kappa\}$, let $\alpha' = \sup(E \cap \alpha)$ and consider the normal measure $U_0(\alpha)$ over $\alpha$. Let us define an ultrafilter $U^*_0(\alpha) \in V^{\mathbb{P}_\alpha}$:

1. If $\alpha' < \alpha$ then $U^*_0(\alpha) := \{X \subseteq \alpha \mid \exists Y \in U_0(\alpha), Y \subseteq X\}$.

(2) If $\alpha' = \alpha$ then by our assumption $E \cap \alpha \notin U_0(\alpha)$, consider $j_{U_0(\alpha)} : V \to M_{U_0(\alpha)}$, since $\alpha \notin j_{U_0(\alpha)}(E)$ it follows that $j_{U_0(\alpha)}(\mathbb{P}_\alpha) = \mathbb{P}_\alpha * \mathbb{P}_{(\alpha, j_{U_0(\alpha)}(\alpha))}$. Let $G_\alpha$ be $V$-generic. The model $M_{U_0(\alpha)}[G_\alpha]$ is closed under $\alpha$-sequences from $V[G_\alpha]$ and let $\langle \underset{\sim}{A}_\alpha \mid \alpha < \alpha^+ \rangle$ be the $j_{U_0(\alpha)}(W)$-minimal enumeration of all the nice names for subsets of $\alpha$. Also, construct a sequence of $\mathbb{P}_\alpha$-names for a $j_{U_0(\alpha)}(W)$-minimal *master sequence* $\langle \underset{\sim}{p}_\nu \mid \nu < \alpha^+ \rangle$, namely, a $\leq^*$-increasing sequence such that for each $\nu < \alpha^+$, $(\underset{\sim}{p}_\nu)_{G_\alpha} || \alpha \in j_{U_0(\alpha)}(\underset{\sim}{A}_\nu)$ and $\underset{\sim}{p}_\nu$ is the $j_{U_0(\alpha)}(W)$-minimal $\mathbb{P}_\alpha$-name for such a condition.

Since the forcing $\mathbb{P}_{(\alpha, j_{U_0(\alpha)}(\alpha))}/G_\alpha$ has more than $\alpha^+$-closure degree for $\leq^*$ with respect to $V[G_\alpha]$, such a sequence can be constructed recursively, exploiting the closure of $M_{U_0(\alpha)}[G_\alpha]$ to $\alpha$-sequences from $V[G_\alpha]$, making the forcing $\mathbb{P}_{(\alpha, j_{U_0(\alpha)}(\alpha))}/G_\alpha$ $\alpha^+$-closed with respect to $\leq^*$ (from the point of view of $V[G_\alpha]$). Define

$$U_0^*(\alpha) := \{(\underset{\sim}{A}_\nu)_{G_\alpha} \mid (\underset{\sim}{p}_\nu)_{G_\alpha} \Vdash \alpha \in j_{U_0(\alpha)}(\underset{\sim}{A}_\nu)\}$$

By [Git86], $U_0^*(\alpha)$ is a normal ultrafilter over $\alpha$. In particular this gives rise to the definition of the normal ultrafilter $U_0^* := U_0^*(\kappa)$ which extends $U_0$ after forcing with $\mathbb{P}_\kappa$. Now we extend $U_1$ to a non normal ultrafilter: Let $j_{U_1} : V \to M_{U_1}$ be the ultrapower by $U_1$. Then $\kappa \in j_{U_1}(E)$ and therefore

$$j_{U_1}(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * Q_\kappa * \mathbb{P}_{(\kappa, j_{U_1}(\kappa))}$$

Where $Q_\kappa$ is the Prikry forcing with the ultrafilter $U_0'$ where $U_0'$ was extended in the same fashion as in Definition 2.13 using the minimal witnesses with respect to $j_0(j_1(W))$. By [Git86, Lemma 2.1], $U_0' = U_0^*$. Let $\langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^+ \rangle$ be the $j_1(W)$-minimal enumeration of all the nice $\mathbb{P}_\kappa$-names for subsets of $\kappa$ and let us fix a sequence of $\mathbb{P}_{\kappa+1}$-names $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^+ \rangle$ of conditions in $\mathbb{P}_{j_{U_1}}(\kappa)/\mathbb{P}_{\kappa+1}$ such that for every $\alpha < \kappa^+$, $0 \Vdash_{\mathbb{P}_{\kappa+1}} \underset{\sim}{p}_\alpha || \kappa \in j_{U_1}(\underset{\sim}{A}_\alpha)$. Let $G \subseteq \mathbb{P}_\kappa$ be $V$-generic. In $V[G]$, define $A \in U_1^*$ if and only if $\exists p \in G \exists B \in U_0^* \exists \nu < \kappa^+$ such that

$$p^\frown \langle \langle \rangle, \underset{\sim}{B} \rangle^\frown \underset{\sim}{p}_\nu \Vdash \kappa \in j_{U_1}(\underset{\sim}{A})$$

where $\underset{\sim}{B}, \underset{\sim}{A}$ are any names interpreted by $G$ to be $B, A$ respectively. By [Git86, Lemma 2.2], $U_1^*$ is a $\kappa$-complete ultrafilter over $\kappa$ which extends $U_1$.

The intuition here is that $U_1^*$ concentrates on $\alpha$'s for which we have preformed the Prikry forcing using the $U_0(\alpha)$, hence $[id]_{U_1^*}$ has a Prikry sequence generated.

Indeed, consider $j_{U_1^*} : V[G] \to M_{U_1^*}$. Then $j_{U_1^*}(G)$ has a Prikry sequence for each $\alpha \in j_{U_1^*}(E)$. Since $E \in U_1$ and $U_1 \subseteq U_1^*$, it follows that $[id]_{U_1^*}$ has a Prikry sequence in $j_{U_1^*}(G)$. Denote by $\langle \kappa_n \mid n < \omega \rangle$ the Prikry sequence for $[id]_{U_1^*}$.

Denote $V^* = V[G]$ and consider for each $0 < n < \omega$ the function $\psi_n : \kappa \to [\kappa]^n$ defined by $\psi_n(\alpha) = \langle \alpha_0, ..., \alpha_{n-1} \rangle$ where $\langle \alpha_0, ..., \alpha_{n-1} \rangle$ are the first $n$-elements of the Prikry sequence for $\alpha$ in $G$. The function $\psi_n$ witnesses the

Rudin-Keisler projection of $U_1^*$ onto the product of $n$-copies of $U_0^*$ denoted by $U_0^{*n}$:

**Proposition 2.14.** *For each $n<\omega$ and every $X\subseteq[\kappa]^n$, $X\in U_0^{*n}$ if and only if $\psi_n^{-1''}X\in U_1^*$.*

*Proof.* Let $X\in U_0^{*n}$, by normality of $U_0^{*n}$, there is $A\in U_0^*$ such that $[A]^n\subseteq X$. Denote by $B:=\psi_n^{-1''}[A]^n=\{\alpha<\kappa\mid\alpha_0,...,\alpha_{n-1}\in A\}$ then there is $p\in G$, $Y\in U_0^*$ and $\nu<\kappa^+$ such that

$$p^\frown\langle\langle\rangle,\underset{\sim}{Y}\rangle^\frown p_\nu||\kappa\in j_{U_1}(\underset{\sim}{B})$$

Toward a contradiction suppose that

$$p^\frown\langle\langle\rangle,\underset{\sim}{Y}\rangle^\frown p_\nu\Vdash\kappa\notin j_{U_1}(\underset{\sim}{B})$$

Namely,

$$p^\frown\langle\langle\rangle,\underset{\sim}{Y}\rangle^\frown p_\nu\Vdash\underset{\sim}{\kappa}_0,...,\underset{\sim}{\kappa}_{n-1}\notin j_{U_1}(\underset{\sim}{A})\cap\kappa=\underset{\sim}{A}$$

To see the contradiction, note that $A\in U_0^*$, hence

$$p^\frown\langle\langle\rangle,\underset{\sim}{Y}\cap\underset{\sim}{A}\rangle^\frown p_\nu\Vdash\underset{\sim}{\kappa}_0,...,\underset{\sim}{\kappa}_{n-1}\notin\underset{\sim}{A}.$$

As for the other direction, suppose that $X\subseteq[\kappa]^n$ and $B:=\psi_n^{-1''}X\in U_1^*$, then there are $p\in G$, $Y\in U_0^*$ and $\nu<\kappa^+$ such that

$$p^\frown\langle\langle\rangle,\underset{\sim}{Y}\rangle^\frown p_\nu\Vdash\kappa\in j_{U_1}(\underset{\sim}{B})$$

It must be the case that for some extension $q\in G$ of $p$, $q\Vdash\underset{\sim}{Y}^n\subseteq\underset{\sim}{X}$ since otherwise, there would be $\vec{\nu}$ and $p\le q\in G$ such that $q\Vdash\vec{\nu}\in\underset{\sim}{Y}^n\setminus\underset{\sim}{X}$. Then $q^\frown\langle\vec{\nu},\underset{\sim}{Y}\setminus\max(\nu)\rangle^\frown p_\nu$ forces that $\vec{\nu}=\langle\underset{\sim}{\kappa}_0,...,\underset{\sim}{\kappa}_{n-1}\rangle$ and by definition of $B$, $\langle\underset{\sim}{\kappa}_0,...,\underset{\sim}{\kappa}_{n-1}\rangle\in\underset{\sim}{X}$, which is a contradiction. Hence $Y^n\subseteq X$, concluding that $X\in U_0^{*n}$ $\square$

Denote by $\langle j^*_{n,m}:M^*_n\to M^*_m\mid n\le m<\omega\rangle$ the $\omega$-th iteration by $U_0^*$. Then by the previous proposition there is a factor map $k_n:M_n^*\to M_{U_1^*}$ derived by the projection $\psi_n$ i.e. $k_n([f]_{U_0^{*n}})=[f\circ\psi_n]_{U_1^*}$.

**Proposition 2.15.** *For every $n<\omega$, $crit(k_n)=j^*_{0,n}(\kappa)=\kappa_n$, where $\kappa_n$ is the n-th element in the Prikry sequence for $[id]_{U_1^*}$.*

*Proof.* For every $n<\omega$, $\langle\kappa,j_{0,1}(\kappa),...,j_{0,n-1}(\kappa)\rangle=[id]_{U_0^{*n}}$ hence

$$k_n(\langle\kappa,j_{0,1}(\kappa),...,j_{0,n-1}(\kappa)\rangle)=[id\circ\psi_n]_{U_1^*}=\langle\kappa_0,...,\kappa_{n-1}\rangle.$$

Therefore for every $n<m<\omega$, $j_{0,n}(\kappa)\le k_m(j_{0,n}(\kappa))=\kappa_n$. We need to prove two separate statements, first that $crit(k_n)=\kappa_n$ and second that $crit(k_n)=j_{0,n}(\kappa)$. Once we prove that $crit(k_m)=\kappa_m$, we can deduce that $j_{0,n}(\kappa)\le\kappa_n<\kappa_m=crit(k_m)$ which in turn implies that $j_{0,n}(\kappa)=k_m(j_{0,n}(\kappa))=\kappa_n=crit(k_n)$.

Let $\gamma<crit(k_m)$, then $\gamma=[f]_{U_0^{*n}}$ and $\gamma=k_m(\gamma)=[f\circ\psi_m]_{U_1^*}$. Toward a contradiction, suppose that $\gamma\ge\kappa_m$. Thus

$$B=\{\alpha<\kappa\mid f(\alpha_0,...,\alpha_{m-1})\ge\alpha_m\}\in U_1^*$$

which by definition implies that there are $p \in G$, $X \in U_0^*$ and $\nu < \kappa^+$ such that

$$p^\frown\langle\langle\rangle, \underset{\sim}{X}\rangle^\frown p_\nu \Vdash \kappa \in j_{U_1}(\underset{\sim}{B}).$$

By definition of $B$ it follows that

$$p^\frown\langle\langle\rangle, \underset{\sim}{X}\rangle^\frown p_\nu \Vdash \underset{\sim}{f}(\underset{\sim}{\kappa}_0, ..., \underset{\sim}{\kappa}_{m-1}) = j_{U_1}(\underset{\sim}{f})(\underset{\sim}{\kappa}_0, ..., \underset{\sim}{\kappa}_{m-1}) \geq \underset{\sim}{\kappa}_m.$$

In $V[G]$ consider $C_f = \{\alpha < \kappa \mid f''[\alpha]^n \subseteq \alpha\}$. Then $C_f$ is a club and by normality $C_f \in U_0^*$. The condition

$$p^\frown\langle\langle\rangle, \underset{\sim}{X} \cap \underset{\sim}{C}_f\rangle^\frown p_\nu$$

forces that $\underset{\sim}{\kappa}_m \in \underset{\sim}{C_f}$ which implies that $\underset{\sim}{f}(\underset{\sim}{\kappa}_0, ..., \underset{\sim}{\kappa}_{m-1}) < \underset{\sim}{\kappa}_m$, contradiction.

As for the other direction, let us prove that $\kappa_m \subseteq Im(k_m)$. Let $[f]_{U_1^*} < \kappa_m$, then $B = \{\alpha < \kappa \mid f(\alpha) < \alpha_m\} \in U_1^*$. Hence there are $p \in G$, $X \in U_0^*$ and $\nu < \kappa^+$ such that

$$p^\frown\langle\langle\rangle, \underset{\sim}{X}\rangle^\frown p_\nu \Vdash \kappa \in j_{U_1}(\underset{\sim}{B}).$$

Work in $V[G]$, for every $\vec{\gamma} = \langle\gamma_0, ..., \gamma_{m-1}, \gamma_m\rangle \in [X]^{m+1}$,

$$\langle\vec{\gamma}, X \setminus \max(\vec{\gamma})\rangle^\frown p_\nu \Vdash j_{U_1}(\underset{\sim}{f})(\kappa) < \gamma_m.$$

Hence, by the $\kappa^+$-cc, for each $\delta < \gamma_m$ there is $\nu_\delta < \kappa^+$ such that $\langle\vec{\gamma}, X \setminus \max(\vec{\gamma})\rangle^\frown p_{\nu_\delta} || j_{U_1}(\underset{\sim}{f})(\kappa) = \delta$[2]. Let

$$X(\vec{\gamma}) = X \setminus \max(\vec{\gamma}) \in U_0^* \text{ and } \nu(\vec{\gamma}) = \sup_{\delta < \gamma_m} \nu_\delta < \kappa^+.$$

It follows that for some value $\alpha(\vec{\gamma}) < \gamma_m$,

$$\langle\vec{\gamma}, X(\vec{\gamma})\rangle^\frown p_{\nu(\vec{\gamma})} \Vdash j_{U_1}(\underset{\sim}{f})(\kappa) = \alpha(\vec{\gamma}) < \gamma_m$$

By varying $\gamma_m$ and pressing down, we can assume that $\alpha(\langle\gamma_0, ..., \gamma_{m-1}, \gamma_m\rangle)$ does not depend on the last coordinate and hence we can write

$$\alpha(\langle\gamma_0, ..., \gamma_{m-1}, \gamma_m\rangle) = \beta(\langle\gamma_0, ..., \gamma_{m-1}\rangle).$$

By normality and regularity respectively,

$$X^* := \Delta_{\vec{\gamma} \in [\kappa]^{m+1}} X(\vec{\gamma}) \in U_0^*, \ \nu^* := \sup_{\vec{\gamma} \in [\kappa]^{m+1}} \nu(\vec{\gamma}) < \kappa^+$$

In particular, there is $p \leq q \in G$ such that

$$q^\frown\langle\langle\rangle, \underset{\sim}{X}^*\rangle^\frown p_{\nu^*} \Vdash j_{U_1}(\underset{\sim}{f})(\kappa) = \underset{\sim}{\beta} \circ \underset{\sim}{\psi_m}(\kappa)$$

Hence $[f]_{U_1^*} = [\beta \circ \psi_m]_{U_1^*} = k_m([\beta]_{U_0^{*n}})$.

$\square$

[2]The point is that for each $\delta < \kappa$, we can find a high enough $\nu_\delta$ such that $p_{\nu_\delta} || \kappa \in j_{U_1}(\underset{\sim}{A}(\delta))$, where $\underset{\sim}{A}(\delta)$ is a name for $\{\alpha < \kappa \mid f(\alpha) = \delta\}$. Technically, $\nu_\delta$ can only be computed after forcing $\mathbb{P}_{\kappa+1}$, however, by $\kappa^+$-cc, we can find a single $\nu_\delta$ such that $\langle\vec{\gamma}, X \setminus \max(\vec{\gamma})\rangle \Vdash p_{\nu_\delta} || j_{U_1}(\underset{\sim}{f}) = \delta$.

## 3. The failure of Galvin's property for filters

Galvin's theorem asserts that if $2^\kappa = \kappa^+$ and $\mathscr{F}$ is a normal filter over $\kappa^+$ then $\mathrm{Gal}(\mathscr{F})$ holds. It is natural to ask to what extent this is sensitive to a modification of the above assumptions. As first noticed by Abraham and Shelah [AS86], the requirement $2^\kappa = \kappa^+$ is critical. Specifically, it is consistent with $\mathsf{ZFC}$ that $2^\kappa > \kappa^+$ and $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ fails.

In [BG22a, Question 7.8] the authors ask whether normality is also a critical requirement. More precisely, it is asked whether there is a $\kappa^+$-complete filter $\mathscr{F}$ over $\kappa^+$ for which $\mathrm{Gal}(\mathscr{F})$ fails, yet $2^\kappa = \kappa^+$ holds.[3] Our goal in this section is to give a positive answer to this question. The key observation here is that the existence of a $\kappa$-independent family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ gives rise to such filters, and every *strongly regular cardinal* $\kappa$ (i.e., $\kappa^{<\kappa} = \kappa$) carries such a family. Later we will prove the consistency of every regular cardinal $\kappa$ carrying a $\kappa$-independent family without affecting the power-set-function pattern (see Theorem 3.2). It is worth to stress that this result does not require any large cardinal whatsoever. Nevertheless, it is not clear that this construction provides filters containing the club filter. This issue, which refers to [BG22a, Question 7.9], will be addressed in §3.2. Specifically, we will show that it is consistent with the $\mathsf{GCH}$ that, for every singular cardinal $\kappa$, its successor $\kappa^+$ carries a non-Galvin filter extending $\mathrm{Cub}_{\kappa^+}$ (see Theorem 3.4).

3.1. **Many non-Galvin filters.** Let $\kappa$ be a regular cardinal. Recall that a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is called *$\kappa$-independent* if for every pair of disjoint sub-families $\mathcal{F}_1, \mathcal{F}_2 \in [\mathcal{F}]^{<\kappa}$ the following is true:

$$(\bigcap_{X \in \mathcal{F}_1} X) \cap (\bigcap_{Y \in \mathcal{F}_2} (\kappa \setminus Y)) \neq \emptyset.$$

One can show that if $\kappa^{<\kappa} = \kappa$ there is a $\kappa$-independent family of size $2^\kappa$ (see [Kun80, Exercise 8.10]). It should be noted that a $\kappa$-independent family does satisfy the $< \kappa$-intersection property, and therefore the set

$$\mathscr{W}_{\mathcal{F}} := \{X \subseteq \kappa \mid \exists \mathcal{A} \in [\mathcal{F}]^{<\kappa} \, (\bigcap \mathcal{A} \subseteq X)\}$$

defines a $\kappa$-complete filter which includes $\mathcal{F}$. Actually, it is the minimal such filter. The connection between this concept and non-Galvin filters is exemplified by the next theorem.

**Lemma 3.1.** *If $\mathcal{F}$ is a $\kappa$-independent family with $|\mathcal{F}| > \kappa$ then $\mathscr{W}_{\mathcal{F}}$ is a $\kappa$-complete filter over $\kappa$ and $\mathcal{F}$ witnesses the failure of* $\mathrm{Gal}(\mathscr{W}_{\mathcal{F}}, \kappa, |\mathcal{F}|)$.[4]

*Proof.* Assume toward contradiction that $\mathcal{F}$ is a $\kappa$-independent family and $\mathrm{Gal}(\mathscr{W}_{\mathcal{F}}, \kappa, |\mathcal{F}|)$ holds. Then, there is $\mathcal{A} \in [\mathcal{F}]^{\kappa}$ such that $\bigcap \mathcal{A} \in \mathscr{W}_{\mathcal{F}}$. By definition of $\mathscr{W}_{\mathcal{F}}$, there is $\mathcal{B} \in [\mathcal{F}]^{<\kappa}$ such that $\bigcap \mathcal{B} \subseteq \bigcap \mathcal{A}$. Since $|\mathcal{A}| = \kappa$ and $|\mathcal{B}| < \kappa$, one can pick $Y \in \mathcal{A} \setminus \mathcal{B}$. By $\kappa$-independence, there is $\nu \in \bigcap \mathcal{B}$

[3] Of course, if such a filter exists it cannot be normal.

[4] We invite our readers to revisit § 1.1 in regards to the meaning of $\mathrm{Gal}(\cdot, \cdot, \cdot)$.

such that $\nu \notin Y$ hence, in particular, $\nu \notin \bigcap \mathcal{A}$. This is a contradiction with the inclusion $\bigcap \mathcal{B} \subseteq \bigcap \mathcal{A}$. □

An outright consequence of the above lemma and [Kun80, Exercise 8.10] is that, in $L$, every regular cardinal $\kappa$ carries a $\kappa$-complete filter $\mathscr{F}$ over $\kappa$ such that $\mathrm{Gal}(\mathscr{F})$ fails. Next, we present an alternative argument which adapts to the case where the GCH does not hold.

**Theorem 3.2.** *Let $E\colon \mathrm{dom}(E) \subseteq \mathrm{Reg} \to \mathrm{Card}$ be an Easton-class function.*[5] *Then, in some generic extension the following holds:*

1. $2^\kappa = E(\kappa)$ *for all* $\kappa \in \mathrm{dom}(E)$;
2. *for every regular cardinal $\kappa$ there is a $\kappa$-complete filter $\mathscr{F}$ over $\kappa$ such that $\mathrm{Gal}(\mathscr{F})$ fails.*

*Proof.* Over $L$ force with $\mathbb{A}$, the class-Easton-supported iteration forcing with $\mathrm{Add}(\kappa, E(\kappa))$ at each regular cardinal $\kappa \in \mathrm{dom}(E)$. For each regular cardinal $\kappa \in \mathrm{dom}(E)$, $\mathbb{A}_{\kappa+1}$ forces a $\kappa$-independent family of size $E(\kappa)$. To see this let $\langle c_\alpha \mid \alpha \in E(\kappa)\rangle$ be the Cohen generics introduced by $\mathrm{Add}(\kappa, E(\kappa))$ over $V^{\mathbb{A}_\kappa}$. Next, define

$$\mathcal{F} := \{c_\alpha^{-1}\{1\} \mid \alpha \in E(\kappa)\}.$$

Using genericity, it is possible to show that $\mathcal{F}$ is $\kappa$-independent. In addition, the tail forcing $\mathbb{A}/\mathbb{A}_{\kappa+1}$ does preserve this fact, for it defines a $\kappa^+$-directed-closed poset. Thus, the purported result follows. □

Note that the filters described above do not necessarily extend the club filter – this seems to be quite a restrictive requirement. By the results in [BG22b], such a (ultra)filter can be forced together with GCH. For this one starts with a measurable cardinal. If one wishes to get such a filter on a successor cardinal one can collapse the cardinals below the measurable $\kappa$ makings this latter a successor cardinal. It is possible to prove now that the former non-Galvin ultrafilter remains so and also extends the club filter. However, it is unclear whether large cardinals are necessary in this situation.

**Question 3.3.** What is the consistency strength of $\neg\mathrm{Gal}(\mathscr{F})$ where $\mathscr{F}$ is a $\kappa^+$-complete filter over $\kappa^+$ extending $\mathrm{Cub}_{\kappa^+}$ and $2^\kappa = \kappa^+$? Does it require large cardinals?

3.2. **Many non-Galvin filters at successors of singulars containing the club filter.** The aim of this section is to prove the following result:

**Theorem 3.4.** *Assume the GCH holds and that there are a supercompact cardinal $\kappa$ together with a measurable cardinal $\lambda > \kappa$ carrying a $\lambda$-complete ultrafilter $\mathscr{U}$, containing $\mathrm{Cub}_\lambda$ and witnessing $\neg\mathrm{Gal}(\mathscr{U})$.*

*Then, there is a model of ZFC where:*

1. GCH *holds;*

[5] See [Kun80, Definition 4.1].

(2) *for every* $\xi \in \mathrm{Ord}$ *there is an* $\aleph_{3\cdot\xi+1}$*-complete filter* $\mathscr{F}_\xi$ *over* $\aleph_{3\cdot\xi+1}$ *that is not normal (yet extends the club filter* $\mathrm{Cub}_{\aleph_{3\cdot\xi+1}}$*) and for which* $\mathrm{Gal}(\mathscr{F}_\xi)$ *fails.*

*In particular, the* GCH *is consistent with the following: For every singular cardinal* $\kappa$*, there is a* $\kappa^+$*-complete non-normal filter* $\mathscr{F}$ *over* $\kappa^+$ *such that* $\mathrm{Cub}_{\kappa^+} \subseteq \mathscr{F}$ *and* $\mathrm{Gal}(\mathscr{F})$ *fails.*

*Remark* 3.5. Recall that our hypothesis on $\lambda$ are consistent with ZFC, as shown in Theorem 2.4. In [BG22b, Theorem 2.6] the same is proved by just requiring measurability upon $\lambda$. More recently, Gitik [Git23] showed the same employing an even simpler forcing construction. In all these three models the GCH is preserved after performing the relevant forcing iteration.

Let us begin the proof of the theorem. By preparing the universe *á-la-Laver* we can further assume that $\kappa$ is indestructible under $\kappa$-directed-closed forcing that preserve the GCH pattern (see e.g., [PRS21, Lemma 8.2]). Clearly, this does not affect the properties of our non-Galvin ultrafilter $\mathscr{U}$.

**Lemma 3.6.** *Working in* $V^{\mathrm{Col}(\kappa,<\lambda)}$*, there is a* $\kappa^+$*-complete filter* $\mathscr{F}$ *over* $\kappa^+$ *that extends* $\mathrm{Cub}_{\kappa^+}$ *and witnesses* $\neg\mathrm{Gal}(\mathscr{F})$*.*

*Proof.* Let $\mathcal{C} \subseteq \mathrm{Col}(\kappa,<\lambda)$ be a $V$-generic filter. Working in $V[\mathcal{C}]$, let $\mathscr{F}$ be the filter generated by $\mathscr{U}$. Namely,

$$\mathscr{F} := \{X \subseteq \lambda \mid \exists U \in \mathscr{U}\ (U \subseteq X)\}.$$

Since $\mathrm{Col}(\kappa,<\lambda)$ is $\lambda$-cc, every $C \in (\mathrm{Cub}_\lambda)^{V^{\mathrm{Col}(\kappa,<\lambda)}}$ contains some $V$-club $D \in (\mathrm{Cub}_\lambda)^V \subseteq \mathscr{U}$. It thus follows that $(\mathrm{Cub}_\lambda)^{V^{\mathrm{Col}(\kappa,<\lambda)}} \subseteq \mathscr{F}$.

We next show that $\mathscr{F}$ is $\lambda$-complete and $\mathrm{Gal}(\mathscr{F})$ fails.

**Claim 3.7.** $\mathrm{Gal}(\mathscr{F})$ *fails.*

*Proof of claim.* Let $\langle X_\alpha \mid \alpha < (\lambda^+)^V\rangle \in V$ be a sequence witnessing the failure of $\mathrm{Gal}(\mathscr{U})$. Notice that $\lambda = (\kappa^+)^{V[\mathcal{C}]}$ and $(\lambda^+)^V = (\kappa^{++})^{V[\mathcal{C}]}$. We claim that the above sequence witnesses $\neg\mathrm{Gal}(\mathscr{F})$ in $V[\mathcal{C}]$. Otherwise, let $\mathcal{I} \in [(\lambda^+)^V]^\lambda$ be in $V[\mathcal{C}]$ such that $\bigcap_{\alpha\in\mathcal{I}} X_\alpha \in \mathscr{F}$. By definition, there is a set $Y_\mathcal{I} \in \mathscr{U}$ contained in this intersection. By our assumption that $\mathrm{Gal}(\mathscr{U})$ fails, $\{\alpha < (\lambda^+)^V \mid Y_\mathcal{I} \subseteq X_\alpha\}$ has cardinality $<\lambda$ in $V$, hence cardinality $\leq\kappa$ in $V[\mathcal{C}]$. However, $\mathcal{I}$ is contained in this set, so in $V[\mathcal{C}]$ the cardinality of $\{\alpha < (\lambda^+)^V \mid Y_\mathcal{I} \subseteq X_\alpha\}$ must be at least $(\kappa^+)^{V[\mathcal{C}]}$. This is a contradiction. $\square$

**Claim 3.8.** *Let* $\mathbb{P}$ *be a* $\theta$*-cc forcing notion and* $\mathscr{W}$ *a* $\theta$*-complete filter over a set* $I$*. Then,* $\mathscr{F}_\mathscr{W} := \{X \subseteq I \mid \exists W \in \mathscr{W}\,(W \subseteq X)\}$ *is* $\theta$*-complete in* $V^\mathbb{P}$*.*

*In particular,* $\mathscr{F}$ *is* $\lambda$*-complete in* $V[\mathcal{C}]$*.*

*Proof of claim.* Let $p \in \mathbb{P}$, $\lambda < \theta$ and $\langle \underset{\sim}{X}_\alpha \mid \alpha < \lambda\rangle$ be a sequence of $\mathbb{P}$-names such that $p \Vdash_\mathbb{P} \langle \underset{\sim}{X}_\alpha \mid \alpha < \lambda\rangle \subseteq \underset{\sim}{\mathscr{F}}_\mathscr{W}$. For each $\alpha < \lambda$ let $\mathcal{A}_\alpha \subseteq \mathbb{P}/p$ be a maximal antichain such that for each $q \in \mathcal{A}_\alpha$ there is $W_{q,\alpha} \in \mathscr{W}$ with $q \Vdash_\mathbb{P} \underset{\sim}{X}_\alpha \supseteq W_{q,\alpha}$. In particular, $p \Vdash_\mathbb{P} \underset{\sim}{X}_\alpha \supseteq W_\alpha$, where $W_\alpha := \bigcap_{q\in\mathcal{A}_\alpha} W_{q,\alpha}$.

Note, however, that $\langle W_\alpha \mid \alpha < \lambda\rangle$ might not belong to $V$. To work around this we use yet again the $\theta$-ccness of $\mathbb{P}$: Let $\underset{\sim}{f}: \lambda \to \mathscr{W}$ be a $\mathbb{P}$-name for the above sequence and find $F \in V$, $F: \lambda \to \mathcal{P}_{<\theta}(\mathscr{W})$ such that $p \Vdash_{\mathbb{P}}$ "$\forall\alpha < \lambda\,(\underset{\sim}{f}(\alpha) \in \check{F}(\alpha))$". Set, $W := \bigcap_{\alpha<\lambda} \bigcap F(\alpha)$. Note that $W \in \mathscr{W}$, by $\theta$-completeness of $\mathscr{W}$. Clearly, $p \Vdash_{\mathbb{P}} W \subseteq \bigcap_{\alpha<\lambda} \underset{\sim}{X}_\alpha$. This shows that $p \Vdash_{\mathbb{P}} \bigcap_{\alpha<\lambda} \underset{\sim}{X}_\alpha \in \underset{\sim}{\mathscr{F}}_{\mathscr{W}}$, as wanted. □

We are done with the lemma. □

At this point we have produced a model of $\mathsf{GCH}$ where $\kappa$ is supercompact and there is a $\kappa^+$-complete filter $\mathscr{F}$ over $\kappa^+$ such that $\mathrm{Cub}_{\kappa^+} \subseteq \mathscr{F}$ and $\neg\mathrm{Gal}(\mathscr{F})$. Notice that, by Galvin's theorem, $\mathscr{F}$ cannot be normal. In a slight abuse of notation let us denote this model simply by $V$.

For the proof of Theorem 3.4 we will use a slight variation of the *Radin forcing with interleaved collapses* of [BGP22]. The following is a mild tweak of the notion of *weak constructing pair* appearing in [BGP22, Definition 1.7]:

**Definition 3.9** (Weak constructing pair)**.** A pair $(j, F)$ is called a *weak constructing pair* if it satisfies the following properties:

(1) $j: V \to M$ is an elementary embedding into a transitive inner model with $\mathrm{crit}(j) = \kappa$ and ${}^\kappa M \subseteq M$;
(2) $F$ is an $N$-generic filter for $\mathrm{Col}(\kappa^{+3}, i(\kappa))^N$, where the map $i: V \to N \simeq \mathrm{Ult}(V, \mathscr{U})$ is the ultrapower embedding derived from $j$;
(3) $F \in M$.

There is a minor discrepancy between Clause (2) of the above definition and the corresponding one in [BGP22, Definition 1.7]; namely, here we require $F$ to be a *guiding generic* for $\mathrm{Col}(\kappa^{+3}, i(\kappa))^N$ while in [BGP22] it was required to be for $\mathrm{Col}(\kappa^{+4}, i(\kappa))^N$. This had to do with the lack of $\mathsf{GCH}$ in the context of [BGP22]. Fortunately, this issue does not stand anymore. Since in our context $\mathsf{GCH}$ holds it is quite easy to produce a weak constructing pair $(j, F)$ as in Definition 3.9 for which $j: V \to M$ witnesses $\kappa^{++}$-supercompactness of $\kappa$. For details see either our proof in [BGP23, Lemma 2.4] or [Cum10, Lemma 8.5]. The reason we adopt this change in the guiding generic is that it leaves room for more cardinals to satisfy the pattern described in Clause (2) of Theorem 3.2.

Fix $(j, F)$ a weak constructing pair with the above property. The following instrumental notions are also borrowed from [BGP22]:

**Definition 3.10.** If $(j, F)$ is a weak constructing pair then put

$$C^* := \{f: \kappa \to V_\kappa \mid \mathrm{dom}(f) \in \mathscr{U} \ \wedge\ \forall\alpha \in \mathrm{dom}(f)\,(f(\alpha) \in \mathrm{Col}(\alpha^{+3}, \kappa))\},$$
$$F^* := \{f \in C^* \mid i(f)(\kappa) \in F\}.$$

**Definition 3.11.** Let $(j, F)$ be a weak constructing pair. A sequence $u$ is said to be *inferred from* $(j, F)$ iff

(1) $u \in M$;

(2) $u(0) := \mathrm{crit}(j)$;
(3) $u(1) := F^*$;
(4) $u(\alpha) := \{X \subseteq V_{u(0)} \mid u \restriction \alpha \in j(X)\}$, for $\alpha \in [2, \mathrm{len}(u))$;
(5) $M \models |\mathrm{len}(u)| \leq u(0)^{++}$.

We say that $(j, F)$ *constructs* $u$ if $u = w \restriction \alpha$, where $w$ is the sequence inferred from $(j, F)$ and $\alpha \in [1, \mathrm{len}(w))$.

**Definition 3.12.** If $u$ is constructed by some pair $(j, F)$ then $\kappa_u := u(0)$. Also, if $\mathrm{len}(u) \geq 2$ define

(1) $F^*_u := u(1)$;
(2) $\mu_u := \{X \subseteq \kappa_u \mid \exists f \in F^*_u \ \mathrm{dom}(f) = X\}$;
(3) $\bar{\mu}_u := \{X \subseteq V_{\kappa_u} \mid \{\alpha \mid \langle\alpha\rangle \in X\} \in \mu_u\}$,
(4) $F_u := \{i(f)(\kappa_u) \mid f \in F^*_u\}$,
(5) $\mathscr{F}_u := \bar{\mu}_u \cap \bigcap\{u(\alpha) \mid \alpha \in [2, \mathrm{len}(u))\}$.[6]

Otherwise, if $\mathrm{len}(u) = 1$, we put $F^*_u = F_u := \{\emptyset\}$ and $\mathscr{F}_u := \{\emptyset\}$.

**Definition 3.13.** The family of *measure sequences* $\mathcal{U}_\infty$ is defined as follows:

- $\mathcal{U}_0 := \{u \mid \exists(j, F)((j, F) \text{ constructs } u)\}$;
- $\mathcal{U}_{n+1} := \{u \in \mathcal{U}_n \mid \mathcal{U}_n \cap V_{\kappa_u} \in \mathscr{F}_u\}$;
- $\mathcal{U}_\infty := \bigcap_{n<\omega} \mathcal{U}_n$.

Let $u_*$ be the sequence inferred from $(j, F)$. Arguing as in [Cum92, Lemma 1], $u_* \restriction \alpha$ exists and belongs to $\mathcal{U}_\infty$, for all $\alpha < \kappa^{+++}$. In particular, there is $\alpha < \kappa^{+3}$ for which the sequence $u := u_* \restriction \alpha \in \mathcal{U}_\infty$ has a repeat point.[7] Also, since $j\colon V \to M$ is a $\kappa^{++}$-supercompact embedding, $\mathrm{Gal}(\mathscr{F})$ fails in the model $M$. Therefore,

$$A := \{w \in \mathcal{U}_\infty \mid \text{“}\exists \mathscr{F}_w \text{ witnessing Lemma 3.6 w.r.t. } \kappa^+_w\text{”}\} \in \mathscr{F}_u := \bigcap_{0<\beta<\alpha} u(\beta).$$

Let $\mathbb{R}_u$ be the Radin forcing with interleaved collapses as defined in [BGP23, §1] (considering the minor change explained right after Definition 3.9). Let $G \subseteq \mathbb{R}_u/p$ be $V$-generic, where $p := \langle(u, \omega, \emptyset, A, H)\rangle$ is a condition. Denote by $C_G$ the Radin club introduced by $G$ and let $\langle\kappa_\xi \mid \xi < \kappa\rangle$ be an increasing enumeration of it. The next can be proved as in [BGP23, Proposition 1.34]:

**Lemma 3.14** (Cardinal structure)**.** *The following holds in $V^{\mathbb{R}_u}$:*

(1) *Every $V$-cardinal $\geq \kappa^+$ remains a cardinal;*
(2) *The only cardinals $\leq \kappa$ are*

$$\{\aleph_0, \aleph_1, \aleph_2, \aleph_3\} \cup \{(\kappa_\xi^{+k})^V \mid 1 \leq k \leq 3,\ \xi < \kappa\} \cup \mathrm{Lim}(C_G) \cup \{\kappa\};$$

*Also, if $u$ has a repeat point then $\kappa$ remains measurable in $V^{\mathbb{R}_u}$.*

Everything is now in place to complete the proof of Theorem 3.4:

[6] If $\mathrm{len}(u) = 2$ we shall agree that $\mathscr{F}_u := \bar{\mu}_u$.

[7] I.e., an ordinal $\gamma < \mathrm{len}(u)$ such that $\bigcap_{0<\beta<\ell(u)} u(\beta) = \bigcap_{0<\beta<\gamma} u(\beta)$.

**Lemma 3.15.** *The following properties are true in $V[G]_\kappa$:*[8]

(1) $\mathsf{GCH}$ *holds;*
(2) *for every $\xi \in \mathrm{Ord}$ there is an $\aleph_{3\cdot\xi+1}$-complete filter $\mathscr{F}_\xi$ over $\aleph_{3\cdot\xi+1}$ that is is not normal (yet extends the club filter $\mathrm{Cub}_{\aleph_{3\cdot\xi+1}}$) and for which $\mathrm{Gal}(\mathscr{F}_\xi)$ fails.*

*Proof.* We divide the proof into a series of claims:

**Claim 3.16.** *Clause (1) holds.*

*Proof of claim.* This follows from an easy counting-nice-name argument involving the $\mathsf{GCH}$ from $V$ and the usual factoring of Radin forcing. □

**Claim 3.17.** *Clause (2) holds.*

*Proof of claim.* By Proposition 3.14, $\kappa$ remains inaccessible in $V[G]$.

For each $\xi < \kappa$ write

$$\Phi(\xi) \equiv \text{“}\mathscr{F}_\xi \text{ witnesses Lemma 3.6”},$$

where $\mathscr{F}_\xi$ is a filter witnessing $u_\xi \in A$.

Fix $\xi < \kappa$ and pick $q \in G$ mentioning both $\kappa_\xi$ and $\kappa_{\xi+1}$, say at coordinates $m$ and $m+1$, respectively. Note that $\Phi(\xi)$ holds in $V$ by virtue of our choice of $A$. The usual factoring arguments give:

$$\mathbb{R}_u/q \simeq \mathbb{R}_{u_\xi}/q^{\leq m} \times \mathrm{Col}(\kappa_\xi^{+3}, \kappa_{\xi+1}) \times \mathbb{R}_u/q^{>m+1}.$$

To not complicate the notations we shall tend to identify $\mathscr{F}_\xi$ with the filter generated by it in the different sub-generic extensions of $V^{\mathbb{R}_u}$.

► $\underline{\mathscr{F}_\xi \text{ is } \kappa_\xi^+\text{-complete:}}$ The first forcing is $\kappa_\xi^+$-cc, hence by Claim 3.8 it preserves $\kappa_\xi^+$-completeness of $\mathscr{F}_\xi$. The rest does not introduce $\kappa_\xi^+$-sequences so that it also preserves the property under consideration.

► $\underline{\mathscr{F}_\xi \text{ extends the club filter:}}$ Let $C \in \mathrm{Cub}_{\kappa_\xi^+}$ in $V[G]$. By the above factoring $C \in V[G_0]$, where $G_0$ is the projection of $G$ on $\mathbb{R}_{u_\xi}/q^{\leq m}$. Since this forcing is $\kappa_\xi^+$-cc there is $D \in \mathrm{Cub}^V_{\kappa_\xi^+}$ such that $D \subseteq C$. Finally, since $\mathrm{Cub}^V_{\kappa_\xi^+} \subseteq \mathscr{F}_\xi$ it follows that the filter generated by $\mathscr{F}_\xi$ has $C$ as an element.

► $\underline{\neg\mathrm{Gal}(\mathscr{F}_\xi)}$: Let $\mathcal{X} = \langle X_\alpha \mid \alpha < \kappa_\xi^{++}\rangle \subseteq \mathscr{F}_\xi$ in $V$ witnessing the failure of $\mathrm{Gal}(\mathscr{F}_\xi)$. Since $\kappa_\xi^+$ and $\kappa_\xi^{++}$ are preserved in $V[G_0]$ the argument of Claim 3.7 shows that $\mathcal{X}$ is still a witness for the failure of Galvin's property in $V[G_0]$. The other two posets do not introduce new subsets to $\kappa_\xi^{++}$ so $\mathcal{X}$ is going to be a witness for $\neg\mathrm{Gal}(\mathscr{F}_\xi)$ in $V[G]$.

Altogether, $\Phi(\xi)$ holds in $V[G]$. □

[8] Recall that our departing model $V$ had a $\kappa^+$-complete filter $\mathscr{F}$ on $\kappa^+$ such that $\mathrm{Cub}_{\kappa^+} \subseteq \mathscr{F}$ and $\neg\mathrm{Gal}(\mathscr{F})$.

Looking at Proposition 3.14(2) it is easy to check that

$$\kappa_\xi^+ := \begin{cases} \aleph_{3\cdot(\xi+1)+1}, & \text{if } \xi < \omega; \\ \aleph_{3\cdot\xi+1}, & \text{if } \xi \geq \omega. \end{cases}$$

By further forcing with $\mathrm{Col}(\aleph_0, \aleph_3)$ we get that, for every ordinal $\xi < \kappa$, $\kappa_\xi^+ = \aleph_{3\cdot\xi+1}$ holds in the resulting generic extension. Moreover, this forcing is $\aleph_4$-cc and, as a result, preserves $\Phi(\xi)$ for every $\xi < \kappa$ (see Lemma 3.6). Finally, $V[G]_\kappa$ has the desired properties. □

In the light of Theorem 3.2 and 3.4 the following becomes natural:

**Question 3.18.** What is the consistency strength of the configuration described in Theorem 3.4? Does it require large cardinals?

## 4. Failure of Galvin's property for the club filter

In this section we continue the study of [AS86, BGP23] and analyze the failure of Galvin's property relative to the club filter. The original motivation for this section was to produce infinitely-many consecutive failures of Galvin's property; e.g., $\neg\mathrm{Gal}(\mathrm{Cub}_{\aleph_{n+1}})$ for all $n < \omega$. Regrettably, we did not succeed in this enterprise (see Question 4.18). Instead, we will be analyzing the following failure of *pseudo-compactness*: $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ holds, yet $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ fails for *many* cardinals $\alpha < \kappa$.

In §4.1 we describe how to force a (non normal) measure $\mathscr{U}$ such that for $\mathscr{U}$-many $\alpha$'s, $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ fails. In particular, after forcing with the *Tree Prikry forcing* relative to this measure, one can produce a model where $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ fails for unboundedly many cardinals below a strong limit cardinal of countable cofinality. Later, in §4.2 we obtain a similar result at the level of the very first singular cardinal, $\aleph_\omega$. Both constructions are performed out of sharp assumptions; i.e., from a measurable cardinal.

### 4.1. A measure concentrating on failures of Galvin's property.

Assume the $\mathsf{GCH}$ holds and let $\mathscr{U}$ be a normal measure over $\kappa$. Let $\mathbb{P}_\kappa$ denote the Easton-supported iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ where, for each inaccessible cardinal $\beta < \kappa$, we force with the lottery sum of the trivial poset and $\mathbb{S}(\beta, \beta^{++})$ (i.e., *Abraham-Shelah poset* of §1.1). As customary, at non-inaccessible $\beta$'s, $\underset{\sim}{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for the trivial poset.

Let $G \subseteq \mathbb{P}_\kappa$ be a $V$-generic filter. Opting for the trivial forcing at the $\kappa^{\text{th}}$-stage of $j_{\mathscr{U}}(\mathbb{P}_\kappa)$ we can extend $j_{\mathscr{U}}$ to a $V[G]$-definable embedding $j_1^* : V[G] \to M[H]$. For this we use our $\mathsf{GCH}$-assumption and the usual lifting arguments. Next, let $\alpha \in (\kappa, j_{\mathscr{U}}(\kappa))$ be an ordinal such that $H$ opts for $\mathbb{S}(\alpha, \alpha^{++})$. Note that there is some of such $\alpha$'s by density. Define

$$\mathscr{W} := \{X \subseteq \kappa \mid \alpha \in j_1^*(X)\}$$

Note that $\mathscr{W}$ is a $\kappa$-complete ultrafilter over $\kappa$ concentrating on the collection of all $\beta$'s for which $\mathrm{Gal}(\mathrm{Cub}_{\beta^+})$ fails. In particular,

**Theorem 4.1.** *Starting from the assumptions of this section, the following is consistent: $\kappa$ is a strong limit cardinal, $\mathrm{cf}(\kappa)=\omega$, $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ holds, but $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ (strongly) fails for cofinaly many $\alpha<\kappa$.*[9]

*Proof.* Let $\mathscr{W}$ be the $\kappa$-complete measure defined above. Force with $\mathbb{P}_T(\mathscr{W})$, the corresponding Tree Prikry forcing. In the resulting extension $2^\kappa=\kappa^+$ holds, hence $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ also does. In addition, $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ strongly fails for all but finitely many members $\alpha$ in the Prikry sequence: this is because $\mathbb{P}_T(\mathscr{W})$ does not add bounded subsets to $\kappa$ and $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ strongly fails in $V$. □

By choosing an arbitrary $\alpha$ we lose control of the measure $\mathscr{W}$. For instance, it is unclear whether $\mathscr{W}$ contains the club filter $\mathrm{Cub}_\kappa$. To work around this we shall look at the second ultrapower by $\mathscr{U}$ and extract from there a different measure. So, put $\mathscr{U}_2:=j_{\mathscr{U}}(\mathscr{U})$ and look at $j_{\mathscr{U}_2}:M_{\mathscr{U}}\to M_{\mathscr{U}_2}$.

Let us extend the embedding $j_{\mathscr{U}_2}$. For this, we choose to force with $\mathbb{S}(j_{\mathscr{U}}(\kappa),j_{\mathscr{U}}(\kappa)^{++})_{M_{\mathscr{U}_2}}$ at the $j_{\mathscr{U}}(\kappa)^{\mathrm{th}}$-stage of the iteration $j_{\mathscr{U}_2}(j_{\mathscr{U}}(\mathbb{P}_\kappa))$. By our $\mathsf{GCH}$ assumption, $|j_{\mathscr{U}_2}(j_{\mathscr{U}}(\mathbb{P}_\kappa))|=\kappa^+$. Using this we can construct in $V[G]$ a generic for $j_{\mathscr{U}_2}(j_{\mathscr{U}}(\mathbb{P}_\kappa))$, where at the $j_{\mathscr{U}}(\kappa)^{\mathrm{th}}$-stage the iteration opts for $\mathbb{S}(j_{\mathscr{U}}(\kappa),j_{\mathscr{U}}(\kappa)^{++})_{M_{\mathscr{U}_2}}$. As in the previous argument, we extend the embedding $j_{\mathscr{U}_2}\circ j_{\mathscr{U}}$ to another embedding $j_2^*$. Define,

$$\mathscr{W}:=\{X\subseteq\kappa\mid j_{\mathscr{U}}(\kappa)\in j_2^*(X)\}.$$

Once again, $\mathscr{W}$ defines a $\kappa$-complete ultrafilter over $\kappa$ concentrating on those $\beta$'s for which $\mathrm{Gal}(\mathrm{Cub}_{\beta^+})$ fails. In addition, $\mathscr{W}$ contains the club filter $\mathrm{Cub}_\kappa$, hence there are stationarily-many $\beta$'s for which Galvin's property fails. Note, however, that $\mathscr{W}$ is not normal. Indeed, otherwise $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ would hold in the ultrapower by $\mathscr{W}$ and, as a result, also in the universe. This is impossible for it will imply the failure of the $\mathsf{SCH}$.[10]

The issue of getting a normal measure $\mathscr{W}$ exhibiting the above non-Galvin-like pattern will be revisited at the beginning of §5.

4.2. **A Prikry-type poset forcing the failure of Galvin's property.** Assume the $\mathsf{GCH}$ holds and let $\kappa$ be a measurable cardinal. Let $\mathcal{U}$ be a normal measure on $\kappa$ and $j\colon V\to M$ the corresponding ultrapower.

**Lemma 4.2.** *There is $K\in V$ that is $M$-generic for the poset*

$$(\mathbb{S}(\kappa^+,\kappa^{+3})\times\mathrm{Col}(\kappa^{+3},{<}j(\kappa))^M.$$

*Proof.* Denote by $\mathbb{Q}$ the above-displayed poset. Note that $\mathbb{Q}$ is $j(\kappa)$-cc in $M$ and, also, every condition in $\mathbb{Q}$ can be identified with a member of $(V_{j(\kappa)})^M$. Since $j(\kappa)$ is inaccessible in $M$, combining these two facts one infers that every maximal antichain (in $M$) for $\mathbb{Q}$ can be regarded as a member of

[9]The notion of strong failure was isolated in [BGP22, Definition 3.1]. In this particular context, it means that there is a family $\langle C_\beta\mid\beta<\alpha^{++}\rangle\subseteq\mathrm{Cub}_{\alpha^+}$ such that $|\bigcap_{\beta\in I}C_\beta|<\alpha$ for all $I\in[\alpha^{++}]^{\alpha^+}$.

[10]Recall that we just assume the existence of a measurable cardinal.

$(V_{j(\kappa)})^M$. In particular, there are at most $|j(\kappa)|^V$-many such objects in $V$. Using our assumption that $2^\kappa = \kappa^+$ we conclude that there are at most $\kappa^+$-many of such. Now, since $\mathbb{Q}$ is $\kappa^+$-directed-closed in $M$ and $M^\kappa \subseteq M$, so it is in $V$. From altogether we can easily produce the desired $M$-generic filter $K$ by diagonalizing over all the maximal antichains lying in $M$. $\square$

*Remark* 4.3. Note that the above argument is not available if we replace $\mathbb{S}(\kappa^+, \kappa^{+3})$ by, e.g., $\mathbb{S}(\kappa, \kappa^{++})$.

Hereafter we shall denote $\mathbb{Q} := (\mathbb{S}(\kappa^+, \kappa^{+3}) \times \mathrm{Col}(\kappa^{+3}, {<}j(\kappa)))^M$. For a measure one set $A^\star \in \mathcal{U}$ consisting of inaccessibles we have that

$$\mathbb{Q} := [\rho \in A^\star \mapsto \mathbb{Q}(\rho)].$$

where $\mathbb{Q}(\rho)$ stands for $\mathbb{S}(\rho^+, \rho^{+3}) \times \mathrm{Col}(\rho^{+3}, {<}\kappa)$. Likewise, for an inaccessible cardinal $\rho < \rho' \leq \kappa$ we shall denote $\mathbb{Q}(\rho, \rho') := \mathbb{S}(\rho^+, \rho^{+3}) \times \mathrm{Col}(\rho^{+3}, {<}\rho')$.

**Definition 4.4.** Let $\mathbb{P}$ the set of all conditions of the form

$$p = \langle \rho_0, a_0, \dots, \rho_{\ell(p)-1}, a_{\ell(p)-1}, A, H \rangle$$

such that the following requirements are met:

(1) $\rho_0 < \cdots < \rho_{\ell(p)-1}$ are inaccessible cardinals in $A^\star$;
(2) for each $i < \ell(p)$, $a_i \in \mathbb{Q}(\rho_i, \rho_{i+1})$, where we stipulate $\rho_{\ell(p)} := \kappa$.
(3) $A \in \mathcal{U}$ and $A \subseteq \{\rho \in A^\star \mid \rho > \mathrm{supp}(a_{\ell(p)-1})\}$;[11]
(4) $H$ is a function with $\mathrm{dom}(H) = A$, $H(\rho) \in \mathbb{Q}(\rho)$ and $[H]_\mathcal{U} \in K$.

We shall refer to $\mathrm{stem}(p) := \langle \rho_0, a_0, \cdots, \rho_{\ell(p)-1}, a_{\ell(p)-1} \rangle$ and $H$ as the *stem* and *supplier* of $p$, respectively. In addition, given two conditions

$$p = \langle \rho_0, a_0, \dots, \rho_{\ell(p)-1}, a_{\ell(p)-1}, A, H \rangle$$
$$q = \langle \rho'_0, b_0, \dots, \rho'_{\ell(q)-1}, b_{\ell(q)-1}, B, L \rangle$$

we shall write $q \leq^* p$ in case $\ell(p) = \ell(q)$ and the following hold:

(1) for each $i < \ell(p)$, $\rho_i = \rho'_i$ and $b_i \leq_{\mathbb{Q}(\rho_i, \rho_{i+1})} a_i$;
(2) $A \subseteq B$ and $L(\rho) \leq_{\mathbb{Q}(\rho)} H(\rho)$ for all $\rho \in A$.

Finally, we shall write $q \leq^{**} p$ iff $q \leq^* p$ and $a_i = b_i$ for all $i < \ell(p)$.

**Definition 4.5.** Given $p \in \mathbb{P}$ as above and $\rho \in A$ define

$$p^\frown\langle \rho \rangle := \langle \rho_0, a_0, \cdots, \rho_{\ell(p)-1}, a_{\ell(p)-1}, \rho, H(\rho), A_\rho, H_\rho \rangle,$$

where $A_\rho := \{\sigma \in A \mid \sigma > \mathrm{supp}(H(\rho))\}$ and $H_\rho := H \restriction A_\rho$.

In general, given $\vec{\rho} \in [A]^{<\omega}$ the sequence $p^\frown\langle \vec{\rho} \rangle$ is defined recursively.[12]

*Remark* 4.6. For every $p \in \mathbb{P}$ and $\rho \in A$, $p^\frown\langle \rho \rangle$ is a legitimate member of $\mathbb{P}$. In effect, since $H$ and $H_\varrho$ differ on a $\mathcal{U}$-negligible set, $[H_\rho]_\mathcal{U} = [H]_\mathcal{U} \in K$. Clearly, the same applies to $p^\frown\langle \vec{\rho} \rangle$ for all $\vec{\rho} = \langle \rho_0, \dots, \rho_{n-1} \rangle$ such that $\rho_0 \in A$ and $\rho_{i+1} \in A_{\rho_i}$ for $1 \leq i < n$.

[11] Here $\mathrm{supp}(a_i)$ is the smallest $\beta > \rho_i$ such that condition $a_i \in V_\beta$.

[12] Here, by convention, $p^\frown\langle \emptyset \rangle := p$.

**Definition 4.7.** For $p, q \in \mathbb{P}$ write $p \leq q$ iff there is $\vec{\rho} \in [A^p]^{<\omega}$ such that $p^\frown\vec{\rho} \leq^* q$. In addition, for a condition $p \in \mathbb{P}$, we say that $t$ *is a potential stem for* $p$ (in symbols, $t \in \mathcal{S}_p$) if there is $p \leq q$ such that $t = \mathrm{stem}(q)$.

The following is almost immediate.

**Lemma 4.8.** $\mathbb{P}$ *is* $\kappa^+$*-cc and even* $\kappa^+$*-Knaster.*
*In particular, in* $V^{\mathbb{P}} \models 2^\kappa = \kappa^+$*.*

*Proof.* Let $\mathcal{A} \subseteq \mathbb{P}$ be an antichain of size $\kappa^+$. Without loss of generality we may assume that both the length and the stem of the conditions in $\mathcal{A}$ are fixed. Note that this is possible in that any stem is a member of $V_\kappa$. Finally, observe that all members of $\mathcal{A}$ are compatible: this is thanks to the requirement that the suppliers come from $K$, which is a filter. $\square$

**Lemma 4.9.** *Let* $p = \langle \rho_0, a_0, \dots, \rho_{n-1}, a_{n-1}, A, H\rangle \in \mathbb{P}$*. Then, there is an isomorphism between* $\mathbb{P}/p$ *and* $\prod_{i<n-1} \mathbb{Q}(\rho_i, \rho_{i+1}) \times \mathbb{P}/\langle \rho_{n-1}, a_{n-1}, A, H\rangle$*.*

*Proof.* Let $\Phi\colon \mathbb{P}/p \to \prod_{i<n-1} \mathbb{Q}(\rho_i, \rho_{i+1}) \times \mathbb{P}/\langle \rho_{n-1}, a_{n-1}, A, H\rangle$ be the map

$$\Phi\colon q \mapsto \langle\langle a^q_0, \dots, a^q_{n-2}\rangle, \langle \rho_{n-1}, a_{n-1}, \rho^q_n, a^q_n, \dots, \rho_{\ell(q)-1}, a^q_{\ell(q)-1} A^q, H^q\rangle\rangle.$$

Clearly, $\Phi$ is bijective, and both it and its inverse are order-preserving. $\square$

The upcoming lemma provides the main technical tool. Prior to proving it let us first introduce some useful notation:

*Notation* 4.10. For $p \in \mathbb{P}$ and $t \in \mathcal{S}_p$ we denote by $p + t$ the sequence $p + t := t^\frown\langle A^p_{\max(t)}, H^p_{\max(t)}\rangle$. It is easy to check that $p + t \in \mathbb{P}$.

**Lemma 4.11.** $\mathbb{P}$ *has the Prikry property; namely, for each* $p \in \mathbb{P}$ *and a sentence* $\varphi$ *in the language of forcing of* $\mathbb{P}$ *there is* $p \leq^* q$ *such that* $q \parallel \varphi$*.*

*Proof.* Let $p \in \mathbb{P}$ and $\varphi$ be a sentence in the forcing language of $\mathbb{P}$. We shall separate the proof of the lemma into three claims:

**Claim 4.12.** *There is* $p \leq^{**} q$ *with the following property: for every* $q \leq r$ *such that* $r \parallel \varphi$ *then* $q + \mathrm{stem}(r) \parallel \varphi$*.*

*Proof of claim.* Let $t \in \mathcal{S}_p$ be an arbitrary stem. If there is some $p \leq r$ with $\mathrm{stem}(r) = t$ and $r \parallel \varphi$ then put $H_t := H^r$. Otherwise, put $H_t := H^p$. This generates a directed set of conditions $\{[H_t]_{\mathcal{U}} \mid t \in \mathcal{S}_p\}$ in $\mathbb{Q}$, which belongs to $M$.[13] In particular, we can let $[H^*]_{\mathcal{U}}$ be a $\leq_{\mathbb{Q}}$-upper bound for this collection.

For each $t \in \mathcal{S}_p$ let us consider

$$A_t := \{\rho < \kappa \mid H_t(\rho) \leq_{\mathbb{Q}(\rho)} H^*(\rho)\}.$$

Clearly, $A^* := \mathrm{dom}(H^*) \cap \triangle_{t \in \mathcal{S}_p} A_t \in \mathcal{U}$, where

$$\triangle_{t \in \mathcal{S}_p} A_t := \{\rho < \kappa \mid \forall t \in \mathcal{S}_p\, (\max(t) < \rho \Rightarrow \rho \in A_t)\}.$$

Put $q := \mathrm{stem}(p)^\frown\langle A^*, H^*\rangle$. Clearly, $q \in \mathbb{P}$ and $p \leq^{**} q$.

[13]Note the need here for closure of $M$ under $\kappa$-sequences.

We claim that $q$ has the desired property. To show this let $q \leq r$ be a condition that decides $\varphi$. Since $p \leq r$ there is a (possibly different) condition $p \leq u$ deciding $\varphi$ with $\mathrm{stem}(u) = \mathrm{stem}(r)$ and such that $H_{\mathrm{stem}(u)} = H^u$. Note that the very definition of diagonal intersection yields $u \leq q + \mathrm{stem}(r)$, hence $q + \mathrm{stem}(r) \parallel \varphi$, as well. This completes the verification of the claim. $\square$

Let $q = s^\frown\langle A^*, H^*\rangle$ be the condition obtained in the previous claim.

**Claim 4.13.** *There is a condition $q \leq^{**} q^*$ with the following property:*

*For each $t \in \mathcal{S}_q$, if $\rho \in A^{q^*}_{\max(t)}$ and $H^{q^*}_{\max(t)}(\rho) \leq_{\mathbb{Q}(\rho)} a$ are such that $t^\frown\langle\rho, a\rangle^\frown\langle A^{q^*}_\rho, H^{q^*}_\rho\rangle \parallel \varphi$ then $(q^* + t)^\frown\langle\eta\rangle \parallel \varphi$ for all $\eta \in A^{q^*}_{\max(t)}$.*

*Moreover, the decision made by the conditions $(q^* + t)^\frown\langle\eta\rangle$ is uniform.*[14]

*Proof.* For a stem $t \in \mathcal{S}_q$ consider

$$D^0_t := \{[L]_\mathcal{U} \in \mathbb{Q}/[H^*]_\mathcal{U} \mid (t^\frown\langle\kappa, [L]_\mathcal{U}\rangle^\frown\langle j(A^*)_\kappa, j(H^*)_\kappa\rangle \parallel j(\varphi))\}$$

and

$$D^1_t := \{[L]_\mathcal{U} \in \mathbb{Q}/[H^*]_\mathcal{U} \mid \forall [L']_\mathcal{U} \geq_\mathbb{Q} [L]_\mathcal{U}\ (t^\frown\langle\kappa, [L']_\mathcal{U}\rangle^\frown\langle j(A^*)_\kappa, j(H^*)_\kappa\rangle \nparallel j(\varphi))\}.$$

Clearly, $D_t := D^0_t \cup D^1_t$ is dense below $[H^*]_\mathcal{U} \in K$. Hence, we can choose $[H_t]_\mathcal{U} \in K \cap D_t$. Yet again, this yields a directed set $\{[H_t]_\mathcal{U} \mid t \in \mathcal{S}_q\}$ (in $M$) of conditions in $K$. Let $[\bar{H}]_\mathcal{U}$ be a $\leq_\mathbb{Q}$-upper bound in $K$.

For $t$ as before let $i(t) \in \{0, 1\}$ such that $[H_t]_\mathcal{U} \in D^{i(t)}_t$. If $i(t) = 0$, put

$$A^{i(t)}_t := \{\rho < \kappa \mid t^\frown\langle\rho, H_t(\rho)\rangle^\frown\langle A^*_\rho, H^*_\rho\rangle \parallel \varphi\}.$$

Otherwise, define

$$A^{i(t)}_t := \{\rho < \kappa \mid \forall a\, (H_t(\rho) \leq_{\mathbb{Q}(\rho)} a \Rightarrow t^\frown\langle\rho, a\rangle^\frown\langle A^*_\rho, H^*_\rho\rangle \nparallel \varphi)\}.$$

Set $B^{i(t)}_t := \{\rho < \kappa \mid H_t(\rho) \leq_{\mathbb{Q}(\rho)} \bar{H}(\rho)\} \cap A^{i(t)}_t$. Clearly, $B^{i(t)}_t \in \mathcal{U}$.

Let $A' := \mathrm{dom}(\bar{H}) \cap \triangle_{t \in \mathcal{S}_q} B^{i(t)}_t \in \mathcal{U}$ and set $q' := s^\frown\langle A', H'\rangle$, where $H' := \bar{H} \restriction A'$. We next show that $q'$ is almost the desired condition.

Suppose that $t \in \mathcal{S}_q$, $\rho \in A'_{\max(t)}$ and $H'_{\max(t)}(\rho) \leq_{\mathbb{Q}(\rho)} a$ are such that $t^\frown\langle\rho, a\rangle^\frown\langle A'_\rho, H'_\rho\rangle \parallel \varphi$. Since $\rho \in A'_{\max(t)}$ then $\rho \in B^{i(t)}_t \cap \mathrm{dom}(\bar{H})$, and so

$$H_t(\rho) \leq_{\mathbb{Q}(\rho)} \bar{H}(\rho) = H'(\rho) = H'_{\max(t)}(\rho) \leq_{\mathbb{Q}(\rho)} a.$$

By the definition of $A^{i(t)}_t$ the above yields $i(t) = 0$, hence $A'_{\max(t)} \subseteq A^0_t$.

Therefore,

$$t^\frown\langle\rho, H_t(\rho)\rangle^\frown\langle A^*_\rho, H^*_\rho\rangle \parallel \varphi, \text{ for all } \rho \in A'_{\max(t)}.$$

Finally, for each $\rho \in A'_{\max(t)}$,

$$t^\frown\langle\rho, H_t(\rho)\rangle^\frown\langle A^*_\rho, H^*_\rho\rangle \leq^* (q' + t)^\frown\langle\rho\rangle.$$

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[14] To wit, if for some $\rho_\star \in A^{q^*}_{\max(t)}$ the condition $(q^* + t)^\frown\langle\rho_\star\rangle$ forces $\varphi$ (resp. $\neg\varphi$) then $(q^* + t)^\frown\langle\eta\rangle$ forces $\varphi$ (resp. $\neg\varphi$) for all $\eta \in A^{q^*}_{\max(t)}$.

Thus $(q'+t)^\frown\langle\rho\rangle$ decides $\varphi$, as well.

Let us now $\leq^{**}$-extend $q'$ to $q^*$ to get a uniform decision about $\varphi$. For each $t\in\mathcal{S}_q$ put

$$A^{**}_{0,t} := \{\rho\in A'_{\max(t)} \mid (q^*+t)^\frown\langle\rho\rangle \Vdash \varphi\},$$

and

$$A^{**}_{1,t} := \{\rho\in A'_{\max(t)} \mid (q^*+t)^\frown\langle\rho\rangle \Vdash \neg\varphi\}.$$

Since this is a partition of $A'_{\max(t)}$ in two disjoint sets there is $i(t)\in\{0,1\}$ such that $A^{**}_{i(t),t}\in\mathcal{U}$. Put $A^{**} := A'\cap\triangle_{t\in\mathcal{S}_q}A^{**}_{i(t),t}$ and $q^* := s^\frown\langle A^{**},H^{**}\rangle$, where $H^{**} := H'\restriction A^{**}$. Arguing as above it is routine to check that $q^*$ has the desired property. $\square$

Let $q^* := s^\frown\langle A^{q^*},H^{q^*}\rangle$ be the condition provided by the above claim. Before proving the upcoming claim and completing the proof of the lemma let us note the following. If $u,v$ are conditions in $\mathbb{P}$ such that $u\parallel\varphi$ and $\mathrm{stem}(u)=\mathrm{stem}(v)$ then there is $v^*\in\mathbb{P}$ such that $v\leq^* v^*$ and $v^*\parallel\varphi$.

**Claim 4.14.** *There is a $\leq^*$-extension of $q^*$ deciding $\varphi$.*

*Proof.* Suppose otherwise. Let $q^*\leq r$ be a condition with minimal $\ell(r)$ such that $r\parallel\varphi$. By the previous comments, $\ell(r)>\ell(q^*)$, for otherwise there would be a $\leq^*$-extension of $q^*$ deciding $\varphi$.[15] Let $t\in\mathcal{S}_{q^*}$, $\rho\in A^{**}$ and $H^{**}(\rho)\leq_{\mathbb{Q}(\rho)} a$ be such that $r=t^\frown\langle\rho,a\rangle^\frown\langle B,L\rangle$. Since $q\leq^* q^*\leq r$ and $q$ satisfies the conclusion of Claim 4.12 it follows that $q+\mathrm{stem}(r)\parallel\varphi$, hence

$$q^*+\mathrm{stem}(r) = t^\frown\langle\rho,a\rangle^\frown\langle A^{q^*}_\rho,H^{q^*}_\rho\rangle\parallel\varphi.$$

By construction of $q^*$ (see Claim 4.13) it follows that, for each $\eta\in A^{q^*}_{\max(t)}$,

$$(q^*+t)^\frown\langle\eta\rangle\parallel\varphi.$$

Moreover, this decision is the same regardless of the choice of $\eta$. Say, for instance, that $(q^*+t)^\frown\langle\eta\rangle\Vdash\varphi$ for all such $\eta$. Since the set

$$\{(q^*+t)^\frown\langle\eta\rangle \mid \eta\in A^{q^*}_{\max(t)}\}$$

forms a maximal antichain below $q^*+t$ forcing $\varphi$ we infer that $q^*+t\Vdash\varphi$, as well. However, $q^*\leq q^*+t$ and $\ell(q^*+t)<\ell(r)$, which yields a contradiction with our minimality assumption upon $\ell(r)$. $\square$

We have accomplished the proof of the lemma. $\square$

The next simple lemma addresses the cardinal structure of $V^{\mathbb{P}}$.

**Lemma 4.15.** *Let $G\subseteq\mathbb{P}$ a generic filter and $\langle\rho_n\mid n<\omega\rangle$ be the induced Prikry sequence. Then the only infinite cardinals in $V[G]$ are $V$-cardinals in the set $[\aleph_0,\rho_0]\cup[\kappa,\infty)$ and*

$$\{\rho_n^{+k}\mid n<\omega,\ k\in\{0,1,2,3\}\}.$$

[15] Actually, it will decide $\varphi$ in the same way as $r$ does.

*By further forcing with* $\mathrm{Col}(\omega_1, {<}\rho_0)$ *over* $V[G]$ *we have that* $\rho_n := \aleph_{4\cdot n+2}$. *Furthermore, the* $\mathsf{GCH}$ *pattern is as follows:*

- $2^{\aleph_{4\cdot n+3}} = \aleph_{4\cdot n+5}$ *for every* $n < \omega$.
- *For every cardinal* $\lambda \notin \{\aleph_{4\cdot n+3} \mid n < \omega\}$, $2^\lambda = \lambda^+$.

*Proof.* Preservation of cardinals $>\kappa$ follows from the $\kappa^+$-cc of $\mathbb{P}$. Also, $\kappa$ will be preserved as a consequence of the preservation of the $\rho_n$'s. Cardinals $\leq\rho_0$ are preserved because the poset $\mathbb{P}/\langle\rho_0, a_0, A_0, H_0\rangle$ has the Prikry property and is $\rho_0^+$-closed with respect to $\leq^*$.[16] Finally, let $n < \omega$ and $p \in G$ be with $\ell(p) = n + 2$. By Lemma 4.9 we have that $\mathbb{P}/p$ is isomorphic to $\prod_{i\leq n} \mathbb{Q}(\rho_i, \rho_{i+1}) \times \mathbb{P}/\langle\rho_{n+1}, a_{n+1}, A^p, H^p\rangle$. The second of these factors preserves cardinals $\leq\rho_{n+1}$. Also, it does preserve $\mathsf{GCH}_{\leq\rho_{n+1}}$. Besides, since $\rho_n$ was inaccessible, $\prod_{i<n} \mathbb{Q}(\rho_i, \rho_{i+1})$ is a $\rho_n$-cc poset of size $\rho_n$. In particular, this forcing preserves both cardinals $\geq\rho_n$ and $\mathsf{GCH}_{\geq\rho_n}$. Finally, in the resulting generic extension the poset $\mathbb{Q}(\rho_n, \rho_{n+1})$ preserves $\rho_n^{+k}$ for $k \in \{0, 1, 2, 3\}$. This completes the verification and the $\mathbb{S}(\rho_n^+, \rho_n^{+3})$ makes $2^{\rho_n^+} = \rho_n^{+3}$. $\square$

**Theorem 4.16.** *Assume the* $\mathsf{GCH}$ *holds and that there is a measurable cardinal* $\kappa$. *Then, there is a* $\kappa^+$*-cc generic extension where:*

1. $\aleph_\omega$ *is strong limit;*
2. $\mathsf{GCH}_{\geq\aleph_\omega}$ *holds, hence* $\mathrm{Gal}(\mathrm{Cub}_{\lambda^+})$ *holds, for all* $\lambda \geq \aleph_\omega$*;*
3. $\mathrm{Gal}(\mathrm{Cub}_{\aleph_{4\cdot(n+1)}})$ *fails for* $n < \omega$.

*Proof.* Let us force with $\mathbb{P} * \widetilde{\mathrm{Col}}(\omega_1, {<}\widetilde{\rho}_0)$. Let $G \subseteq \mathbb{P}$ be a $V$-generic filter and $H \subseteq \mathrm{Col}(\omega_1, {<}\rho_0)$ be $V[G]$-generic. Clauses (1) and (2) are evident. The argument for (3) is mutatis mutandi the same as that of Lemma 4.15. Note that it suffices to show that $\mathrm{Gal}(\mathrm{Cub}_{\rho_n^{++}})$ fails in $V[G]$; indeed, $\mathrm{Col}(\omega_1, {<}\rho_0)$ is $\rho_0$-cc in $V[G]$ and thus has no effect upon this configuration. For details, see [BGP23, Lemma 2.1].

Denote by $\langle\rho_n \mid n < \omega\rangle$ the Prikry sequence inferred from $G$. Let $n < \omega$ and $p \in G$ be such that $\ell(p) = n + 2$. By Lemma 4.9 we have that $\mathbb{P}/p$ is isomorphic to $\prod_{i\leq n} \mathbb{Q}(\rho_i, \rho_{i+1}) \times \mathbb{P}/\langle\rho_{n+1}, a_{n+1}, A^p, H^p\rangle$. Observe that the second of these factors has the Prikry property and is $\rho_{n+1}^+$-closed with respect to the $\leq^*$-ordering. In particular, the failure of $\mathrm{Gal}(\mathrm{Cub}_{\rho_n^{++}})$ in $V[G]$ depends just on the effect of the first forcing. In this respect, since $\rho_n$ is an inaccessible cardinal, $\prod_{i<n} \mathbb{Q}(\rho_i, \rho_{i+1})$ has size $\rho_n$, so that $\mathsf{GCH}_{\rho_n^+}$ will still hold in the resulting generic extension, $W$. Hence, over $W$, the poset $\mathbb{Q}(\rho_n, \rho_{n+1})$ yields the failure of $\mathrm{Gal}(\mathrm{Cub}_{\rho_n^{++}})$. This completes the proof. $\square$

Abraham and Shelah method produces the so-called *ultimate failure*. Following [BGP23, §2], we say that Galvin's property *ultimately fails* (say) at $\aleph_{n+1}$ if $\mathrm{Gal}(\mathrm{Cub}_{\aleph_{n+1}}, \aleph_{n+1}, 2^{\aleph_{n+1}})$ fails. The next proposition suggests that combining Abraham and Shelah method with Prikry-type forcings seems unlikely to get infinitely-many consecutive failures of Galvin's property:

[16] Here $\langle\rho_0, a_0, A_0, H_0\rangle$ is a condition in $G$.

**Proposition 4.17.** *If for some $1 \le n < \omega$, $\mathrm{Gal}(\aleph_{m+1}, \aleph_{m+1}, 2^{\aleph_{m+1}})$ fails for all $m \in [n, \omega)$ then $2^{\aleph_n} > \aleph_\omega$. In particular, $\aleph_\omega$ is not strong limit.*

*Proof.* Let $n < \omega$ be such that $\mathrm{Gal}(\aleph_{m+1}, \aleph_{m+1}, 2^{\aleph_{m+1}})$ fails for all $m \ge n$. Suppose towards a contradiction that $2^{\aleph_n} < \aleph_\omega$. Arguing as in [BGP23, Claim 2.15] one can show that $2^\theta = 2^{\aleph_n}$ for all $\theta \in [\aleph_n, 2^{\aleph_n})$ (note that the GCH must fail as Galvin's property does). Indeed, if $\theta$ is the minimal such that $2^\theta > 2^{\aleph_n}$ then $\theta = \aleph_{m+1}$ and $2^{\aleph_m} = 2^{\aleph_n}$. Hence the weak diamond $\Phi_{\aleph_{m+1}}$ holds, and thus $\mathrm{Gal}(\aleph_{m+1}, \aleph_{m+1}, 2^{\aleph_{m+1}})$ holds, as well (see [Gar17]). By our departing assumption this latter is impossible. Thus, $2^{\aleph_n} = 2^{\aleph_{n+1}}$. Similarly, one can prove that, for each $m \ge n$, $2^{\aleph_m} = 2^{\aleph_{m+1}}$ and thus $2^{\aleph_n} \ge \aleph_\omega$. This yields the desired contradiction. □

**Question 4.18.** Is it consistent to have infinitely many consecutive failures of Galvin's property? If so, is the theory

$$\text{“}\mathsf{ZFC} + \forall n < \omega\ \neg\mathrm{Gal}(\aleph_{n+1}, \aleph_{n+1}, 2^{\aleph_{n+1}})\text{”}$$

consistent?

## 5. On consistency strength

In this section we compute the exact consistency strength of

$$(\star) \qquad \text{“}\exists\kappa\,(\kappa \text{ strong limit } \wedge\ \mathrm{cf}(\kappa) = \omega\ \wedge\ \mathrm{Gal}(\mathrm{Cub}_{\kappa^+}) \text{ fails})\text{”}$$

and give a close-to-optimal upper bound for

$$(\star\star) \qquad \text{“}\forall\kappa\,(\kappa \text{ singular } \Rightarrow\ \mathrm{Gal}(\mathrm{Cub}_{\kappa^+}) \text{ fails})\text{”}.$$

The first answers [BGP23, Question 5.12] and the second improves the large-cardinal assumptions used in [BGP23, Theorem 2.3].

Loosely speaking, the idea is to force a normal measure $\mathscr{U}$ on $\kappa$ concentrating on inaccessible cardinals $\alpha$ where $\mathrm{Gal}(\mathrm{Cub}_{\alpha^+})$ fails. As in §4.1, this will be accomplished by forcing with an Easton-supported iteration of $\mathbb{S}(\alpha, \alpha^{++})$'s at every inaccessible cardinal $\alpha \le \kappa$. As the reader may have noticed, the difference now is that we also need to force at $\kappa$ and, as a result, the lifting arguments of §4.1 are not longer straightforward. The crux of the matter is lifting the relevant embedding after forcing with $\mathbb{S}(\kappa, \kappa^{++})$. For this purpose, we use an improvement of *Woodin's surgery method* discovered by Ben-Shalom [BS17] (see page 30). The resulting lifting can be shown to be the ultrapower by a normal measure over $\kappa$, and $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ fails. Finally, one uses Prikry forcing with respect to this measure.

In Lemma 5.3 we spell out the details of this construction, also dealing with the additional caveat of preserving $(\kappa + 2)$-strongness of $\kappa$. This is the main ingredient to get a close-to-optimal bound for $(\star\star)$. Finally, in Theorem 5.9 we get the exact consistency strength for $(\star)$, even for $\aleph_\omega$.

For the rest of the section $\mathbb{P}$ will denote the Easton-support iteration forcing with $\mathbb{S}(\alpha, \alpha^{++})$ when $\alpha$ is an inaccessible cardinal $\le\kappa$, for a given (large) cardinal $\kappa$. In the other stages we simply force with the trivial poset.

5.1. **Improving the consistency strength of the global failure.** Before proceeding with the proof of the main lemma let us recall one important characteristic of elementary embeddings.

**Definition 5.1.** Let $k\colon M \to N$ be an elementary embedding between inner models and $\mu \in \mathrm{ORD}^M$. We say that $k$ *has width* $\leq\mu$ if every $x \in N$ can be written as $j(f)(a)$ for $a \in M$ and function $f \in M$, with $M \models |\mathrm{dom}(f)| \leq \mu$.

The notion of width will become instrumental in the proof of Lemma 5.3 as it permits to transfer generic filters from the initial inner model $M$ to the target $N$. More precisely, the following fact will be used repeatedly:

**Proposition 5.2** ([Cum10, Proposition 15.1])**.** *Suppose that $k\colon M \to N$ has width $\leq\mu$ and that $\mathbb{Q} \in M$ is a separative set-sized forcing such that*

$$M \models \text{“}\mathbb{P} \textit{ is } \mu^+\textit{-distributive”}.$$

*Let $G \subseteq \mathbb{Q}$ generic over $M$ and $H$ be the filter for $k(\mathbb{Q})$ induced by $k$“$G$. Then, $k$“$G$ generates an $N$-generic filter for $k(\mathbb{Q})$.*

Some comments are in order before addressing the proof of Lemma 5.3. First, since we will be working in a context where GCH fails we will not be able to ensure the existence of a weak constructing pair in the sense of Definition 3.9. This is because the Levy collapse used there (i.e. $\mathrm{Col}(\kappa^3, i(\kappa))^N$) is not closed enough to carry out the arguments. Instead, we show that a weak constructing pair exists when the Levy collapse is $\mathrm{Col}(\kappa^{+4}, {<}i(\kappa))^N$. This is more akin to the notion of weak constructing pair considered in [BGP22, §1.4] – in both cases the closure degree of the collapse is $\kappa^{+4}_N$. Nevertheless, here we just collapse cardinals below $i(\kappa)$ while in [BGP22] we allowed $i(\kappa)$ to be collapsed. This is a minor discrepancy with respect to [BGP22, Definition 1.7]. The digression has to do with the verification of Claims 5.5 and 5.7(c) where we use that our Levy collapses are $i(\kappa)$-cc. Fortunately, this variation does not create any trouble when verifying that the corresponding Radin forcing behaves as expected; to wit, that it has the Prikry property, collapses the right cardinals, etc. See for instance [Cum92, §3.1] where Cummings bears on similar assumptions.

Now, let us come back to the proof of the section's main lemma. Recall that $\mathbb{P}$ is the iteration described in page 27. The forthcoming result improves the large-cardinal assumptions of the key lemma [BGP23, Lemma 2.4].

**Lemma 5.3.** *Assume the* GCH *holds and that $\kappa$ is a $(\kappa+3)$-strong cardinal. Then, in $V^{\mathbb{P}}$ there is $(j, F)$ a weak constructing pair in the sense of* [BGP22, §1.4]*. Moreover $j$ witnessing that $\kappa$ is $(\kappa + 2)$-strong.*

*Proof.* Work in $V$. Let $j\colon V \to M$ be an elementary embedding witnessing that $\kappa$ is $(\kappa + 3)$-strong. Without loss of generality assume that $j$ is the ultrapower embedding by a $(\kappa, \kappa^{+3})$-extender $E$. Let $\ell\colon V \to \overline{M}$ be the ultrapower by the $(\kappa, \kappa^{++})$-extender induced by $E$. As usual, we denote by $k\colon \overline{M} \to M$ the factor map between $j$ and $\ell$; namely,

$$k(\ell(f)(a)) := j(f)(a) \text{ for each } a \in [\kappa^{++}]^{<\omega}.$$

**Claim 5.4.** $\mathrm{crit}(k) = \kappa^{+3}_{\overline{M}}$ *and $k$ has width* $\leq\kappa^{+3}_{\overline{M}}$.[17]

*Proof.* Clearly, $\mathrm{crit}(k) \geq \kappa^{++}_{\overline{M}}$ because $k(\ell(\mathrm{id})(\alpha)) = \alpha$ for all $\alpha < \kappa^{++}$. Also, $k(\kappa^{++}) = k(\kappa^{++}_{\overline{M}}) = \kappa^{++}_{M} = \kappa^{++}$. Combining both things we have $\mathrm{crit}(k) \geq \kappa^{+3}_{\overline{M}}$. Moreover, this is a strict inequality because

$$k(\kappa^{+3}_{\overline{M}}) = \kappa^{+3}_{M} = \kappa^{+3} \text{ and } \kappa^{+3}_{\overline{M}} < \ell(\kappa) < \kappa^{+3}.$$

Here the latest inequality follows from our $\mathsf{GCH}$ assumption.

Regarding the width of $k$. Note that $M$ is the model described by

$$\{j(f)(a) \mid a \in [\kappa^{+3}]^{<\omega} \wedge \mathrm{dom}(f) = [\kappa]^{|a|}\},$$

which in turn is the same as

$$\{k(h)(a) \mid a \in [\kappa^{+3}]^{<\omega}, \mathrm{dom}(f) = [\kappa^{+3}_{\overline{M}}]^{|a|}\}.$$

This shows that $k$ has width $\leq\kappa^{+3}_{\overline{M}}$. □

Let $i\colon V \to N$ be the standard ultrapower induced by $j$, and $\overline{k}$ be

$$\bar{k}(i(f)(\kappa)) := j(f)(\kappa).$$

In other words, $\bar{k}$ is the factor map between $\ell$ and $i$. Arguing as in the previous claim $\mathrm{crit}(\overline{k}) = \kappa^{++}_{N}$ and $\overline{k}$ has width $\leq\kappa^{++}_{N}$.

Let $G * g$ be generic for $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{S}}(\kappa, \kappa^{++})$ over $V$. Working over $V[G]$,

$$\mathbb{S}(\kappa, \kappa^{++})_{N[G]} = \mathbb{S}(\kappa, \kappa^{++}_{N})_{V[G]},$$

as $N[G]$ is closed under $\kappa$-sequences in $V[G]$. Put $g_0 := g \cap \mathbb{S}(\kappa, \kappa^{++}_{N})_{V[G]}$. This yields a generic for $\mathbb{S}(\kappa, \kappa^{++}_{N})_{V[G]}$ over $V[G]$.

By standard lifting arguments the embedding $i$ lifts to another embedding $i\colon V[G] \to N[G * g_0 * H]$, where $H \in V[G * g_0]$ is generic for the tail forcing $i(\mathbb{P})/(G * g_0)$. Similarly, $\overline{k}$ lifts to $\overline{k}\colon N[G * g_0 * H] \to \overline{M}[G * g * \overline{k}``H]$: In effect, on one hand, $\mathrm{crit}(\overline{k}) = \kappa^{++}_{N}$ and thus $\overline{k}``g_0 \subseteq g$; on the other hand, the width of $\overline{k}$ is $\kappa^{++}_{N}$ and $i(\mathbb{P})/(G*g_0)$ is $\kappa^{+3}_{N}$-closed in $N[G*g_0]$, hence $\overline{k}``H$ induces a generic filter for $\ell(\mathbb{P})/(G * g)$ over $\overline{M}[G * g]$ (see Proposition 5.2). A similar argument shows that $k\colon \overline{M} \to M$ lifts to

$$k\colon \overline{M}[G * g * \overline{k}``H] \to M[G * g * (k \circ \overline{k})``H].$$

All in all, in $V[G * g]$, we have a commutative diagram of embeddings given by $j\colon V[G] \to M[j(G)]$, $\ell\colon V[G] \to \overline{M}[\ell(G)]$, $i\colon V[G] \to N[i(G)]$, $k\colon \overline{M}[\ell(G)] \to M[j(G)]$ and $\overline{k}\colon N[i(G)] \to \overline{M}[\ell(G)]$.

**Claim 5.5.** *$V[G * g]$ contains the following objects:*

(1) $F \subseteq \mathrm{Col}(\kappa^{+4}, {<}i(\kappa))_{N[i(G)]}$ *a generic filter over* $N[i(G)]$*;*
(2) $K \subseteq \mathrm{Add}(i(\kappa), i(\kappa))_{N[i(G)]}$ *a generic filter over* $N[i(G)]$*.*

[17] In addition, $\kappa^{+i}_{\overline{M}} = \kappa^{+i} = \kappa^{+i}_{M}$, for $i \leq 2$.

*Proof of claim.* We just prove (1) as (2) can be verified in the very same way. Let us begin noticing that $i(\kappa)$ is an inaccessible cardinal in $N[i(G)]$. In particular, the poset $\mathrm{Col}(\kappa^{+4}, {<}i(\kappa))_{N[i(G)]}$ is $i(\kappa)$-cc and so there are at most $i(\kappa)^{<i(\kappa)} = i(\kappa)$-many maximal antichains. Also,

$$|i(\kappa)| = |\{i(f)(\kappa) \mid f \colon \kappa \to \kappa,\ f \in V[G]\}|,$$

so that $|i(\kappa)| = |\kappa^\kappa|^{V[G]} = \kappa^+$. Finally, standard diagonalization arguments yield the desired generic filter $F$ in $V[G * g]$. $\square$

Since the width of $\overline{k}$ is $\kappa_N^{++}$, both $F$ and $K$ can be transferred to generic filters for $\mathrm{Col}(\kappa^{+4}, {<}\ell(\kappa))_{\overline{M}[\ell(G)]}$ and $\mathrm{Add}(\ell(\kappa), \ell(\kappa))_{\overline{M}[\ell(G)]}$, respectively. For simplicity, let us denote these generics also by $F$ and $K$.

**Claim 5.6.** *$\ell$ lifts to $\ell \colon V[G * g] \to \overline{M}[\ell(G * g)]$ in $V[G * g]$.*

*Proof.* Work in $V[G]$. Factor $\mathbb{S}(\kappa, \kappa^{++})$ as $\mathrm{Add}(\kappa, \kappa^{++}) * \underset{\sim}{\mathbb{Q}}$, where the latter stands for the quotient forcing. By arguments of Abraham and Shelah ([AS86, Lemma 1.7]) the weakest condition of $\mathrm{Add}(\kappa, \kappa^{++})$ forces $\underset{\sim}{\mathbb{Q}}$ to be $\kappa^+$-distributive. For convenience, let us split the generic filter $g$ as $g^c * g^q$.

We begin lifting $\ell$ via $\mathrm{Add}(\kappa, \kappa^{++})_{V[G]}$. For this, let $K$ be as before. Appealing to arguments of Ben Shalom [BS17] we can find a one-to-one map $\varphi \colon \ell(\kappa) \to \ell(\kappa^{++})$ in $V[G * g]$ such that the function $K \diamond \varphi$ defined as

$$K \diamond \varphi := \{\langle\langle\varphi(\alpha), \beta\rangle, \gamma\rangle \mid \langle\langle\alpha, \beta\rangle, \gamma\rangle \in K \text{ and } \alpha \in \mathrm{dom}(\varphi)\}$$

defines a generic for $\mathrm{Add}(\ell(\kappa), \ell(\kappa^{++}))_{\overline{M}[\ell(G)]}$. Evidently, $K \diamond \varphi \in V[G * g]$.

Specifically, we use [BS17, Lemma 2.18] and Ben Shalom's arguments in the last paragrapgh of p.8 and p.9 of [BS17] showing that a one-to-one map witnessing the assumptions of [BS17, Lemma 2.18] exists.

Using the so-called *Woodin's surgery method* (see [Cum10, §25, p.879]) one can alter (within $V[G * g]$) $K \diamond \varphi$ to a $\overline{M}[\ell(G)]$-generic filter $\overline{g}^c$ such that $\ell``g^c \subseteq \overline{g}^c$. Thus, $\ell$ lifts to

$$\ell \colon V[G * g^c] \to \overline{M}[\ell(G) * \overline{g}^c].$$

To complete the argument observe that $\ell$ is an embedding with width $\leq\kappa$ and that $g^q$ is a generic for a $\kappa^+$-distributive forcing over $V[G * g^c]$. In particular, $\ell``g^q$ induces a generic filter for $\ell(\underset{\sim}{\mathbb{Q}})_{\overline{M}[\ell(G) * \overline{g}^c]}$ (Proposition 5.2). From altogether we conclude that $\ell$ lifts to $\ell \colon V[G * g] \to \overline{M}[\ell(G * g)]$. $\square$

In addition, $k$ lifts to $k \colon \overline{M}[\ell(G * g)] \to M[j(G * g)]$, for $k$ has width $\leq\kappa_{\overline{M}}^{+3}$ and $\mathbb{S}(\ell(\kappa), \ell(\kappa^{++}))_{\overline{M}[\ell(G)]}$ is $\ell(\kappa)$-closed in $\overline{M}[\ell(G)]$. Consequently, by commutativity of the diagram, $j$ lifts to $j \colon V[G * g] \to M[j(G * g)]$.

Write $V^* := V[G * g]$, $M^* := M[j(G * g)]$ and $\overline{M}^* := \overline{M}[\ell(G * g)]$.

**Claim 5.7.**

(1) *$j \colon V^* \to M^*$ witnesses that $\kappa$ is $(\kappa + 2)$-strong;*
(2) *$\ell \colon V^* \to \overline{M}^*$ is the ultrapower embedding derived from $j$;*

(3) *$F$ is $\overline{M}^*$-generic for* $\mathrm{Col}(\kappa^{+4}, {<}\ell(\kappa))_{\overline{M}^*}$.

*In particular, $(j, F)$ is a weak constructing pair.*

*Proof of claim.* (1) It suffices to check the following two bullets:

▶ $M^*$ is closed under $\kappa$-sequences in $V^*$: First, $\mathbb{P}_\kappa * \underset{\sim}{\mathrm{Add}}(\kappa, \kappa^{++})$ is $\kappa^+$-cc and so $M[G * g^c]$ is closed under $\kappa$-sequences in $V[G * g^c]$. Second, the quotient forcing $\mathbb{Q}$ is $\kappa^+$-distributive in $V[G * g^c]$ and so $M[G * g]$ is closed under $\kappa$-sequences in $V[G * g]$. Finally, the tail forcing $\ell(\mathbb{P})/(G * g)$ is $\kappa^+$-closed in $V[G * g]$ and thus $M^* = M[\ell(G * g)]$ is closed under $\kappa$-sequences in $V[G * g]$.

▶ $V^*_{\kappa+2} \subseteq M^*$: Let $x \in V^*_{\kappa+2}$. Since $\mathbb{P}$ is a $\kappa^{++}$-cc forcing notion then there is $\tau \in V_{\kappa+2} \subseteq M$ such that $x = \tau_{G*g}$. Hence $x \in M^*$, as wanted.

(2) Let $\iota\colon V^* \to \mathcal{N}$ be the ultrapower embedding inferred from $\ell$ and $\chi\colon \mathcal{N} \to \overline{M}^*$ be the corresponding factor map. Since $\mathrm{crit}(k \circ \chi) > \kappa$, $\mathcal{N}$ is the ultrapower derived from $j$. Thus, it suffices to argue that $\chi = \mathrm{id}$, which amounts to show that $\chi$ is surjective.[18] Before checking this observe that $\mathrm{crit}(\chi) > \kappa^{++}$: In effect, $\kappa^+_{\mathcal{N}} = \kappa^+$ and also, since $\mathcal{N} \models$ "$2^\kappa = \kappa^{++}$" and $\mathcal{P}(\kappa) \subseteq \mathcal{N}$, $\kappa^{++}_{\mathcal{N}} = (2^\kappa)_{\mathcal{N}} \geq 2^\kappa = \kappa^{++}$. Altogether, $\mathrm{crit}(\chi) > \kappa^{++}$.

Let $x \in \overline{M}^*$. Since $\overline{M}$ was the ultrapower by a $(\kappa, \kappa^{++})$-extender it follows from standard forcing arguments (see e.g., [Cum10, Proposition 9.4]) that

$$\overline{M}^* = \{\ell(f)(a) \mid f \in V[G * g],\ f\colon [\kappa]^{|a|} \to V[G * g],\ a \in [\kappa^{++}]^{<\omega}\}.$$

Thus, $x$ is of the form $\ell(f)(a)$ for some $f$ and $a$. Since $\mathrm{crit}(\chi) > \kappa^{++}$, it follows that $\chi(a) = a$. All in all,

$$x = \ell(f)(a) = \chi(\iota(f))(\chi(a)) = \chi(\iota(f)(a))$$

and thus $x \in \mathrm{ran}(\chi)$, as wanted.

(3) Since $\ell(\mathbb{P})/\ell(G)$ is $\ell(\kappa)$-closed in $\overline{M}[\ell(G)]$ the poset $\mathrm{Col}(\kappa^{+4}, {<}\ell(\kappa))$ is computed both by $\overline{M}^*$ and $\overline{M}[\ell(G)]$ in the same manner. Besides, this is a $\ell(\kappa)$-cc forcing hence any maximal antichain in $\overline{M}^*$ belongs to $\overline{M}[\ell(G)]$. Thereby, $F$ is $\overline{M}^*$-generic for the poset $\mathrm{Col}(\kappa^{+4}, {<}\ell(\kappa))_{\overline{M}^*}$ □

This completes the proof of the lemma. □

The following is an immediate consequence of the above lemma:

**Theorem 5.8.** *Assume that* ZFC *is consistent with the existence of a* $(\kappa+3)$*-strong cardinal. Then* ZFC *is also consistent with*

"$\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ *fails at every limit cardinal* $\kappa$*".*

*Moreover,* ZFC *is consistent with*

"$\mathrm{Gal}(\mathrm{Cub}_{\aleph_{4\cdot\xi+1}})$ *fails for every* $\xi \in \mathrm{Ord}$*".*

---

[18] Recall that the identity is the unique isomorphism between transitive models of ZFC.

*Proof.* Let $\kappa$ be a $(\kappa+3)$-strong cardinal. Appealing to Lemma 5.3 we get a model of $\mathsf{ZFC}$ where $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ fails and there is a weak constructing pair $(j, F)$ witnessing that $\kappa$ is $(\kappa+2)$-strong. Denote this model by $V$ and let $u_*$ be the sequence inferred from $(j, F)$ (see p.18). Arguing as in [Cum92, Lemma 1], for each $\alpha < \kappa^{+3}$ the sequence $u_* \restriction \alpha$ exists and belongs to $\mathcal{U}_\infty$ (Definition 3.13). In particular, there is $\alpha < \kappa^{+3}$ such that the sequence $u := u_* \restriction \alpha \in \mathcal{U}_\infty$ has a repeat point. From this point on argues exactly as in [BGP23, Theorem 2.3]. $\square$

5.2. **The exact consistency strength for the local failure.** In this section we compute the exact consistency strength of the failure of Galvin's property at the successor of a strong limit singular cardinal.

**Theorem 5.9.** *Assume that there is a cardinal $\kappa$ carrying a $(\kappa, \kappa^{++})$-extender. Then, there is a generic extension of the set-theoretic universe where $\aleph_\omega$ is strong limit and $\mathrm{Gal}(\mathrm{Cub}_{\aleph_{\omega+1}})$ fails.*

*Proof.* Let $j\colon V \to M$ be the ultrapower induced by a $(\kappa, \kappa^{++})$-extender, and $i\colon V \to N$ be the natural ultrapower embedding inferred from $j$. Arguing as in Lemma 5.3 lift $j$ to $j\colon V[G * g] \to M[j(G * g)]$. In addition, find $F \subseteq \mathrm{Col}(\kappa^{+3}, {<}j(\kappa))_{M[j(G)]}$ a generic filter over $M[j(G)]$ living in $V[G * g]$. Since $\mathbb{S}(j(\kappa), j(\kappa^{++}))_{M[j(G)]}$ is $j(\kappa)$-closed we can argue as in Claim 5.7(3) that $F$ is generic for $\mathrm{Col}(\kappa^{+3}, {<}j(\kappa))_{M[j(G*g)]}$ over $M[j(G * g)]$. By Claim 5.7(2), $M[j(G * g)]$ is just the ultrapower by the normal measure inferred from $j$.

Working in $V[G * g]$, let $\mathcal{U}$ denote this measure and $\mathbb{Q}$ be the Prikry forcing with interleaved collapses relative to $\mathcal{U}$ and the *guiding generic* $F$ (see [Cum10, Example 8.6]). Since in $V[G * g]$ the principle $\mathrm{Gal}(\mathrm{Cub}_{\kappa^+})$ fails and $\mathbb{Q}$ is $\kappa^+$-cc, it also fails in any generic extension of $V[G * g]$ by $\mathbb{Q}$. Finally, observe that in this latter model $\aleph_\omega$ is strong limit and $\kappa = \aleph_\omega$. $\square$

**Corollary 5.10.** "$\mathsf{ZFC}+\aleph_\omega$ *is strong limit* $+\mathrm{Gal}(\mathrm{Cub}_{\aleph_{\omega+1}})$ *fails" and* "$\mathsf{ZFC}+\exists\kappa\,(o(\kappa) = \kappa^{++})$" *are equiconsistent theories.*

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