ARTICLE COMMENTARY | OCTOBER 04 2023

Comment on “Modification of Lie's transform perturbation theory for charged particle motion in a magnetic field” [Phys. Plasmas 30, 042515 (2023)]

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Phys. Plasmas 30, 104701 (2023)
https://doi.org/10.1063/5.0167236

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Cite as: Phys. Plasmas **30**, 104701 (2023); doi: 10.1063/5.0167236
Submitted: 11 July 2023 · Accepted: 21 September 2023 ·
Published Online: 4 October 2023

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ABSTRACT

A recent paper by L. Zheng [Phys. Plasmas **30**, 042515 (2023)] presented a critical analysis of standard Lie-transform perturbation theory and suggested that its application to the problem of charged-particle motion in a magnetic field suffered from ordering inconsistencies. In the present Comment, we suggest that this criticism is unjustified and that standard Lie-transform perturbation theory does not need to be modified in its application to guiding-center theory.

In a recent paper, Zheng[1] suggested that when standard Lie-transform perturbation theory[2] is applied to the guiding-center theory of charged-particle motion in a magnetic field,[3–5] ordering inconsistencies arise. Unfortunately, Zheng never defined an ordering parameter (denoted ϵ in the present Comment) in his critique of standard Lie-transform perturbation theory and, here, it is argued that his proposed modification of Lie-transform perturbation theory is completely unnecessary.

In guiding-center theory,[6,8] the mathematical construction of the magnetic moment relies on the space-time scales (L_B, ω^{-1}) of the confining magnetic field $\mathbf{B} = B\,\hat{\mathsf{b}}$ to be long compared to the characteristic gyroradius ρ and the gyroperiod $\Omega^{-1} = mc/eB$, respectively, leading to the small dimensionless small parameter[6]

$$\epsilon_B \equiv \rho/L_B \sim \omega/\Omega \ll 1, \tag{1}$$

which is also used in Zheng's paper. While this dimensional parameter makes physical sense, it is not an ordering parameter *per se* to be used in a perturbation expansion.

In early formulations of guiding-center theory,[7–11] the dimensional ratio m/e was proposed as an ordering parameter in deriving guiding-center equations of motion, which is consistent with Eq. (1), i.e., $\epsilon_B \propto m/e$. In previous Hamiltonian guiding-center models,[6,12–14] on the other hand, a dimensionless ordering parameter ϵ was introduced either as a mass renormalization $m \to \epsilon\, m$[12] or as a charge renormalization $e \to e/\epsilon$,[6,13,14] so that the dimensional ratio $m/e \to \epsilon\, m/e$ is indeed considered small in both cases (i.e., $\epsilon_B \sim \epsilon$). These renormalization orderings can then form the basis for a well-defined perturbation-expansion analysis of charged-particle motion in a magnetic field by Lie-transform perturbation methods.[2]

Depending on the renormalization ordering used, we can begin our guiding-center perturbation analysis with the particle Lagrangian, either expressed according to the charge renormalization as

$$L(\mathbf{x}, \mathbf{p}) = \left(\frac{e}{\epsilon c}\,\mathbf{A} + \mathbf{p}\right) \cdot \dot{\mathbf{x}} - \left(\epsilon^{-1}\, e\,\Phi + \frac{|\mathbf{p}|^2}{2m}\right), \tag{2}$$

or, according to the mass renormalization, as

$$L'(\mathbf{x}, \mathbf{p}) = \left(\frac{e}{c}\,\mathbf{A} + \epsilon\,\mathbf{p}\right) \cdot \dot{\mathbf{x}} - \left(e\,\Phi + \epsilon\,\frac{|\mathbf{p}|^2}{2m}\right), \tag{3}$$

which are simply related as $L(\mathbf{x}, \mathbf{p}) \equiv \epsilon^{-1} L'(\mathbf{x}, \mathbf{p})$. We note that since the time-dependence of the electromagnetic potentials $(\Phi, \mathbf{A})$ is not relevant to our discussion, it will, therefore, be ignored in what follows. In addition, while $\epsilon_B \sim \epsilon$, these dimensionless parameters play very different roles, i.e., the particle Lagrangians (2) and (3) are still meaningful in the case of a uniform magnetic field (where $\epsilon_B = 0$) or time-independent electromagnetic fields. Moreover, the ordering parameter ϵ is the same dimensionless ordering parameter that appears in the dimensionless equation of motion $\epsilon\,\bar{\mathbf{x}}'' = \bar{\mathbf{x}}' \times \bar{\mathbf{B}}$ initially studied by Kruskal.[7,9]

As a result of the Lie-transform perturbation analysis, once again based on a definite choice for ϵ (independent of ϵ_B), the guiding-center

Lagrangian can also either be expressed according to the charge-renormalization ordering[4–6,13,14] as

$$L_{gc}(\mathbf{X}, p_{||}, \mu, \zeta) = \left(\frac{e}{\epsilon c}\mathbf{A} + p_{||}\hat{b}\right)\cdot\dot{\mathbf{X}} - H_{gc}(\mathbf{X}, p_{||}, \mu) + \epsilon\,(mc/e)\,\mu\big(\dot{\zeta} - \boldsymbol{\mathcal{R}}^*\cdot\dot{\mathbf{X}}\big), \tag{4}$$

where the explicit expression for the guiding-center Hamiltonian H_{gc} is not important in what follows, or, according to the mass-renormalization ordering,[12] as

$$L'_{gc}(\mathbf{X}, p_{||}, \mu, \zeta) = \left(\frac{e}{c}\mathbf{A} + \epsilon\,p_{||}\hat{b}\right)\cdot\dot{\mathbf{X}} - \epsilon\,H_{gc}(\mathbf{X}, p_{||}, \mu) + \epsilon^2\,(mc/e)\,\mu\big(\dot{\zeta} - \boldsymbol{\mathcal{R}}^*\cdot\dot{\mathbf{X}}\big), \tag{5}$$

where $L_{gc}(\mathbf{X}, p_{||}, \mu, \zeta) \equiv \epsilon^{-1}L'_{gc}(\mathbf{X}, p_{||}, \mu, \zeta)$. In both cases, the guiding-center Lagrangian is independent of the gyroangle ζ (up to a specified truncated order in ϵ), and, according to Noether's theorem,[15] the canonically conjugate gyroaction $\partial L_{gc}/\partial\dot{\zeta} = \epsilon\,(mc/e)\,\mu$ is a guiding-center invariant (up to that specified truncated order). Once an ordering choice is made (i.e., using either the renormalizations e/ϵ or $\epsilon\,m$), the ϵ-expansion of the guiding-center Lagrangian has to be consistent with this choice. We note that, with the choice $e = m = c = 1$ used by Littlejohn,[4] the guiding-center Lagrangian (4) corresponds exactly to Eq. (29) obtained by Littlejohn[4] by Lie-transform perturbation method, with the substitution $\Phi \to \epsilon\,\Phi$.

In the guiding-center Lagrangians (4) and (5), the higher-order correction $-(mc/e)\mu\,\boldsymbol{\mathcal{R}}^*\cdot\dot{\mathbf{X}}$ involves the vector field,[13,14]

$$\boldsymbol{\mathcal{R}}^* \equiv \boldsymbol{\mathcal{R}} + \frac{1}{2}\nabla\times\hat{b}, \tag{6}$$

which includes the gyrogauge vector field $\boldsymbol{\mathcal{R}} \equiv \nabla\hat{e}_1\cdot\hat{e}_2$[3,4] that is defined in terms of the local fixed unit-vector basis $(\hat{e}_1, \hat{e}_2, \hat{b} \equiv \hat{e}_1\times\hat{e}_2)$, as well as the term $\frac{1}{2}\nabla\times\hat{b}$ that modifies the standard correction $\frac{1}{2}(\hat{b}\cdot\nabla\times\hat{b})\,\hat{b}$[5] in order to take into account guiding-center polarization.[13] While this higher-order correction is absent from Zheng's work,[1] the gyrogauge vector field $\boldsymbol{\mathcal{R}}$ is needed in the guiding-center Lagrangians (4) and (5) in order to ensure the gyrogauge invariance of the guiding-center equations of motion[3–5] (i.e., the guiding-center Lagrangian dynamics should not only be independent of the gyroangle ζ, but it should also be independent of how the gyroangle is measured in terms of the local perpendicular unit vectors $\hat{e}_1$ and $\hat{e}_2$). Hence, a proper guiding-center Lagrangian must, at least, include the combination $\dot{\zeta} - \boldsymbol{\mathcal{R}}\cdot\dot{\mathbf{X}}$, which is gyrogauge-invariant[4] under the transformation $\zeta \to \zeta + \psi(\mathbf{X})$, where $\psi(\mathbf{X})$ denotes a locally defined gyrogauge angle, with $\boldsymbol{\mathcal{R}} \to \boldsymbol{\mathcal{R}} + \nabla\psi$ and $\dot{\zeta} \to \dot{\zeta} + \dot{\mathbf{X}}\cdot\nabla\psi$. The case for time-dependent fields is further discussed in Refs. 3 and 16, while the importance of the vector field (6) in establishing the faithfulness of the guiding-center representation of particle orbits in nonuniform magnetic fields was recently demonstrated for the case of axisymmetric magnetic geometries.[14]

In his critique of standard Lie-transform perturbation analysis, and without explicitly displaying the dimensionless ordering parameter ϵ upon which it is to be based, Zheng[1] mistakenly proceeds to compare the guiding-center Lagrangians (4) and (5), derived with different renormalization orderings, and concludes that, when the guiding-center Lagrangian (5) is truncated at first order, the term $\epsilon^2(mc/e)\mu\,\dot{\zeta}$ disappears, while the term $\epsilon\,(mc/e)\mu\,\dot{\zeta}$ remains in the guiding-center Lagrangian (4), although it is still a second-order term compared to the lowest order term appearing at ϵ^{-1}. However, Zheng seems to be unaware that the guiding-center Lagrangian (4), which was derived without Lie-transform perturbation method by Cary and Brizard[6] in what Zheng calls the direct method, was also derived by Lie-transform perturbation method by Littlejohn,[4] Brizard,[5] and Tronko and Brizard.[13]

More importantly, Zheng argues that, in contrast to the ϵ-ordering scalings displayed in the guiding-center Lagrangians (4) and (5), the terms $p_{||}\hat{b}\cdot\dot{\mathbf{X}}$ and $(mc/e)\,\mu\,\dot{\zeta}$ must appear at the same order in a modified guiding-center perturbation expansion, which leads him to construct a completely unnecessary (and nonsensical) modification of Lie-transform perturbation theory. However, this modified ordering is clearly inconsistent with the property of gyrogauge invariance based on the following argument. First, by momentarily hiding the ϵ-ordering scalings in Eqs. (4) and (5), the guiding-center Lagrangian can be written as

$$L_{gc}(\mathbf{X}, p_{||}, \mu, \zeta) = \left[\frac{e}{c}\mathbf{A} + p_{||}\hat{b} - (mc/e)\,\mu\,\boldsymbol{\mathcal{R}}^*\right]\cdot\dot{\mathbf{X}} + (mc/e)\,\mu\,\dot{\zeta} - H_{gc}(\mathbf{X}, p_{||}, \mu), \tag{7}$$

where we have combined the gyrogauge-correction term $-(mc/e)\,\mu\boldsymbol{\mathcal{R}}^*$, omitted in Zheng's work,[1] with the spatial components $(e/c)\mathbf{A} + p_{||}\hat{b}$. Here, we clearly see that these spatial components satisfy the following ordering $\epsilon^{-1} \gg 1 \gg \epsilon$.[17] Hence, after restoring the ϵ-ordering scalings of the spatial components in Eq. (7), we obtain

$$L_{gc}(\mathbf{X}, p_{||}, \mu, \zeta) = \left[\frac{e}{\epsilon c}\mathbf{A} + p_{||}\hat{b} - \epsilon\,(mc/e)\,\mu\,\boldsymbol{\mathcal{R}}^*\right]\cdot\dot{\mathbf{X}} + \delta\,(mc/e)\,\mu\,\dot{\zeta} - H_{gc}(\mathbf{X}, p_{||}, \mu), \tag{8}$$

where we have also introduced a dimensionless ordering parameter δ for the gyromotion term $(mc/e)\,\mu\,\dot{\zeta}$. Next, we note that the guiding-center Lagrangian (8) now contains the gyrogauge combination

$$(mc/e)\,\mu\big(\delta\,\dot{\zeta} - \epsilon\,\boldsymbol{\mathcal{R}}\cdot\dot{\mathbf{X}}\big),$$

which is gyrogauge invariant only if $\delta = \epsilon$ (and not $\delta = 1$ as proposed by Zheng[1]), i.e., the term $(mc/e)\,\mu\,\dot{\zeta}$ must appear at one order higher than $p_{||}\hat{b}\cdot\dot{\mathbf{X}}$ in a perturbation expansion leading to a gyrogauge-invariant guiding-center Lagrangian theory, based on either Eq. (4) or Eq. (5). The ordering $\delta = \epsilon$ in Eq. (8) is, therefore, entirely consistent with the renormalization $m/e \to \epsilon\,m/e$ of the mass-to-charge ratio used (either implicitly[4,5] or explicitly[6,13]) in previous works as the consistent basis for applications of the standard Lie-transform perturbation analysis.

In conclusion, the standard Lie-transform perturbation method[2] does not need to be modified in its applications to guiding-center theory[4,6] and, fortunately, the modified Lie-transform perturbation method proposed by Zheng[1] will not be needed in deriving a modified nonlinear gyrokinetic theory.[18] The paper by Zheng[1] reminds us that perturbation theory relies on a well-defined dimensionless ordering parameter $\epsilon \ll 1$, followed by a rigorous algorithm (e.g., Lie-transform perturbation theory) that allows terms to be computed at arbitrary order.

The present work was supported by the National Science Foundation, Grant No. PHY-2206302.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

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[17]In general magnetic geometry, the unit vector $\hat{\mathsf{b}} \equiv \partial \mathbf{x}/\partial s$ is defined as the rate of change of the position $\mathbf{x}$ of a point as it moves along a magnetic-field line (where the spatial coordinate s measures distance along that line). Using the Frenet–Serret formulas,[15] with $\hat{\mathsf{e}}_1 = \kappa^{-1}\partial\hat{\mathsf{b}}/\partial s$ (where κ denotes the Frenet–Serret curvature) and $\hat{\mathsf{e}}_2 = \kappa^{-1}\hat{\mathsf{b}} \times \partial\hat{\mathsf{b}}/\partial s$, we find $\hat{\mathsf{b}} \cdot \boldsymbol{\mathcal{R}} = \kappa^{-2}(\hat{\mathsf{b}} \times \partial\hat{\mathsf{b}}/\partial s) \cdot \partial^2\hat{\mathsf{b}}/\partial s^2 \equiv \tau$, expressed in terms of the Frenet–Serret torsion τ (with units of m^{-1}). Hence, in Eq. (7), the guiding-center parallel momentum $p_{||}$ can be compared with $(mc/e)\mu\,\hat{\mathsf{b}} \cdot \boldsymbol{\mathcal{R}} = \left(\frac{1}{2}\rho_\perp \tau\right) p_\perp$, where we used $(mc/e)\,\mu \simeq \frac{1}{2}\,p_\perp\,\rho_\perp$ at the lowest guiding-center order. Assuming that the guiding-center spatial ordering (1) holds, so that $\frac{1}{2}\rho_\perp|\tau| \ll 1$, we then find that $(mc/e)\mu\,|\hat{\mathsf{b}} \cdot \boldsymbol{\mathcal{R}}| \ll |p_{||}|$, if $|p_{||}| \sim p_\perp$ (i.e., away from possible turning points) as is assumed by Zheng.[1] For example, in simple axisymmetric tokamak geometry,[14] we find $\hat{\mathsf{b}} \cdot \boldsymbol{\mathcal{R}} \simeq 1/qR_0$, where q denotes the safety factor, and R_0 denotes the major radius of the magnetic axis measured from the vertical z-axis.
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