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Implicit Representation of Sparse Hereditary Families

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Abstract

For a hereditary family of graphs $\mathcal{F}$, let $\mathcal{F}_n$ denote the set of all members of $\mathcal{F}$ on n vertices. The speed of $\mathcal{F}$ is the function $f(n) = |\mathcal{F}_n|$. An implicit representation of size $\ell(n)$ for $\mathcal{F}_n$ is a function assigning a label of $\ell(n)$ bits to each vertex of any given graph $G \in \mathcal{F}_n$, so that the adjacency between any pair of vertices can be determined by their labels. Bonamy, Esperet, Groenland, and Scott proved that the minimum possible size of an implicit representation of $\mathcal{F}_n$ for any hereditary family $\mathcal{F}$ with speed $2^{\Omega(n^2)}$ is $(1 + o(1)) \log_2 |\mathcal{F}_n|/n \ (= \Theta(n))$. A recent result of Hatami and Hatami shows that the situation is very different for very sparse hereditary families. They showed that for every $\delta > 0$ there are hereditary families of graphs with speed $2^{O(n \log n)}$ that do not admit implicit representations of size smaller than $n^{1/2-\delta}$. In this note we show that even a mild speed bound ensures an implicit representation of size $O(n^c)$ for some $c < 1$. Specifically we prove that for every $\varepsilon > 0$ there is an integer $d \geq 1$ so that if $\mathcal{F}$ is a hereditary family with speed $f(n) \leq 2^{(1/4-\varepsilon)n^2}$ then $\mathcal{F}_n$ admits an implicit representation of size $O(n^{1-1/d} \log n)$. Moreover, for every integer $d > 1$ there is a hereditary family for which this is tight up to the logarithmic factor.

Keywords Hereditary properties · Implicit representation · VC-dimension · Shatter function · (Induced)-universal graphs

Mathematics Subject Classification 05C62 · 05C78 · 52C10

Dedicated to the memory of Eli Goodman.

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1 Introduction

A family of graphs $\mathcal{F}$ is hereditary if it is closed under taking induced subgraphs. Let $\mathcal{F}_n$ denote the set of all members of $\mathcal{F}$ with n vertices. The speed of $\mathcal{F}$ is the function $f(n) = |\mathcal{F}_n|$. An implicit representation of size $\ell(n)$ of $\mathcal{F}_n$ is a function assigning a label of $\ell(n)$ bits to each vertex of any given graph $G \in \mathcal{F}_n$, so that the adjacency between any pair of vertices can be determined by their labels. It is easy and well known (see [14]) that the existence of such a function is equivalent to the existence of a graph on $2^{\ell(n)}$ vertices which contains every member of $\mathcal{F}_n$ as an induced subgraph (here we do not assume that the function assigning labels has to be efficiently computable). Such a graph is called an induced-universal graph for $\mathcal{F}_n$. Since we consider here only induced-universal graphs we simply write, throughout the paper, universal graphs. To see the equivalence observe that given a function corresponding to an implicit representation of size $\ell(n)$ the graph whose vertices are all possible labels in which two are adjacent iff the corresponding labels determine adjacency in a graph of $\mathcal{F}_n$ is a universal graph for $\mathcal{F}_n$. The converse follows by assigning to each vertex of a graph $G \in \mathcal{F}_n$ the number of the vertex of the universal graph that plays its role in a copy of G in this graph.

There is a vast literature dealing with universal graphs for various families, see, e.g., [4, 9, 12] and the many references therein. By the above remark, the minimum possible size $\ell(n)$ of labels for a family $\mathcal{F}_n$ has to satisfy $[2^{\ell(n)}]^n \geq |\mathcal{F}_n|$, that is, $\ell(n) \geq \log_2 |\mathcal{F}_n|/n$, and it is known that this is essentially tight in many interesting cases. In particular, this is the case for the family of all graphs (see [4, 19]). It is also nearly tight for many additional examples, including all hereditary families satisfying $|\mathcal{F}_n| = 2^{\Omega(n^2)}$. By known results [1, 8], if $|\mathcal{F}_n| = 2^{\Omega(n^2)}$ then $|\mathcal{F}_n| = 2^{(1-1/k)n^2/2+o(n^2)}$ for some integer $k > 1$. Bonamy et al. [9] proved that in all these cases there is an implicit representation with labels of length $(1-1/k)n/2 + o(n)$. On the other hand, a recent result of Hatami and Hatami [12], settling a problem raised by Kannan et al. [14], shows that there are very sparse hereditary families for which any implicit representation requires labels of size nearly $\sqrt{n}$. Specifically it is shown in [12] that for every $\delta > 0$ there is a hereditary family $\mathcal{F}$ satisfying $|\mathcal{F}_n| = 2^{O(n \log n)}$ so that the size of any implicit representation for $\mathcal{F}_n$ is at least $\Omega(n^{1/2-\delta})$. It is not clear if the exponent $1/2$ can be improved, and it is also not known what happens for families $\mathcal{F}$ with speed $f(n)$ exceeding $2^{n \log n}$ which is $2^{o(n^2)}$. It is known that in this range the speed is at most $2^{n^{2-\varepsilon}}$ for some fixed $\varepsilon > 0$ (see [5]). Our contribution here is to show that in all these cases there is an implicit representation of size at most $O(n^{1-\varepsilon})$.

Theorem 1.1 *For any $\varepsilon > 0$ and any integer n_0 there is an integer $d \geq 1$ so that the following holds. Let $\mathcal{F}$ be a hereditary family of graphs with speed $f(n) = |\mathcal{F}_n| \leq 2^{(1/4-\varepsilon)n^2}$ for all $n \geq n_0$ (and hence $f(n) = 2^{o(n^2)}$). Then there is an implicit representation of size at most $O(n^{1-1/d} \log n)$ for $\mathcal{F}_n$. In addition, for any such integer $d > 1$ there is a hereditary family for which this is tight up to the $\log n$ factor.*

The proof of this theorem is presented in the next section. The final section contains some concluding remarks, including a description of several natural hereditary families with slowly growing speed functions.

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2 Shatter Functions and Implicit Representations

The theorem can be proved using the notion of the VC-dimension of graphs and some of its properties, but we prefer to describe a proof using the related notion of shatter functions. This version provides better quantitative bounds in some explicit cases. We proceed with the details.

For any two integers $k, d \geq 1$ let $U(k, d)$ denote the bipartite graph with two vertex classes A, B satisfying $|A| = d$ and $|B| = k \cdot 2^d$, where for each subset $C \subset A$ there are exactly k vertices in B whose set of neighbors in A is exactly C. If X, Y are disjoint sets of vertices of a graph G, let $G[X, Y]$ denote the bipartite graph induced by the sets X and Y (ignoring the edges inside X and inside Y). Call a graph $U(k, d)$-free if it contains no two disjoint sets of vertices X, Y so that $G[X, Y]$ is a copy of $U(k, d)$. Note that the graph $U(d, d)$ contains every bipartite graph with two classes of vertices, each of size d, as an induced subgraph. Therefore, if a graph contains a copy of $U(d, d)$ then it contains at least 2^{d^2} distinct labelled induced subgraphs on $2d$ vertices. It thus follows that if the speed of a hereditary family $\mathcal{F}$ satisfies $f(n) \leq 2^{(1/4-\varepsilon)n^2}$ for some fixed $\varepsilon > 0$ and all $n \geq n_0$ then there is a finite $d = d(\varepsilon, n_0)$ so that every graph in the family is $U(d, d)$-free. We proceed to show that the family of all $U(d, d)$-free graphs admits an implicit representation of size at most $O(n^{1-1/d} \log n)$.

A set I of coordinates is shattered by a family of binary vectors if the projections of these vectors on I includes all $2^{|I|}$ possible binary vectors of length $|I|$. We need the following lemma.

Lemma 2.1 *Let* $\mathbf{T}$ *be a family of at least*

$$1 + (k + d - 1) \cdot 2^d \cdot \binom{t}{d} + \sum_{i=0}^{d-1} \binom{t}{i}$$

distinct binary vectors of length t*. Then there is a set* I *of* d *coordinates shattered* $k + d$ *times, namely, every binary function from* I *to* $\{0, 1\}$ *is a projection of at least* $k + d$ *distinct vectors in* $\mathbf{T}$ *on* I.

Proof As long as $\mathbf{T}$ contains more than $\sum_{i=0}^{d-1} \binom{t}{i}$ vectors there is a shattered set of d coordinates, by the Sauer–Perles–Shelah Lemma [20]. Removing the 2^d shattering vectors from $\mathbf{T}$ and repeating the argument $(k+d) \cdot \binom{t}{d}$ times we get, by the pigeonhole principle, the same d-set shattered $k + d$ times. □

For a binary vector v let $c(v)$ denote the number of indices i so that $v_i \neq v_{i+1}$. Note that these indices partition the set of all indices into $c(v) + 1$ intervals, so that v is constant on each interval. The primal shatter function of a family of binary vectors is the function $g(t)$ whose value is the largest number of distinct projections of the vectors on a set of t coordinates. The following lemma is proved in [22] (after its optimization in [13]), see also [10, 16]. The formulation in these references is in terms of the notion of spanning trees with low crossing number. The (equivalent) formulation we use here appears in [6].

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Lemma 2.2 *Let $\mathcal{G}$ be a family of binary vectors of length n with primal shatter function $g(t) \le ct^d$ for some constant $c > 0$ and integer $d \ge 1$. Then there is a fixed permutation of the coordinates of the vectors so that for each permuted vector v, $c(v) \le O(n^{1-1/d})$.*

Proof of Theorem 1.1 Let $\mathcal{F}$ be a hereditary family with speed $f(n) \le 2^{(1/4-\varepsilon)n^2}$ for all $n \ge n_0$. By the assumption and the remark in the first paragraph of this section there is a finite integer $d \ge 1$ so that every member of $\mathcal{F}_n$ is $U(d, d)$-free. For a graph $G \in \mathcal{F}_n$ let $\mathcal{G}$ be the set of rows of the adjacency matrix of G. These are binary vectors of length n. We claim that the primal shatter function of these family of vectors satisfies $g(t) \le 10t^d$ for all $t > d$. Indeed, otherwise by Lemma 2.1 with $k = d$ there is a set I of d-coordinates which is shattered $2d$ times by these vectors. This gives a set A of d vertices of G and another set B' of $2d \cdot 2^d$ vertices so that for every subset C of A there are $2d$ vertices in B' whose set of neighbors in A is exactly C. Let B be a subset of $B' - A$ containing exactly d vertices for each such subset C. This gives a copy of $U(d, d)$ contradicting the fact that G contains no such copy. This proves the claim. Therefore by Lemma 2.2 there is a numbering of the vertices so that according to this numbering the set of all neighbors of each vertex consists of at most $O(n^{1-1/d})$ intervals. Assign to each vertex a label consisting of its number and the endpoints of the corresponding intervals. This is clearly a valid implicit representation, establishing the required upper bound.

The (near) tightness follows by using the projective norm graphs described in [7]. These are graphs on n vertices with $\Omega(n^{2-1/d})$ edges that contain no copy of the complete bipartite graph $K_{d,k}$ with $k = (d-1)!+1$. Our hereditary family $\mathcal{F}$ consists of all these graphs (for all values of n for which they exist) and all their (not necessarily induced) subgraphs. This is a hereditary family, in fact even a monotone one. It does not contain an induced copy of $U(k, d)$ and hence, by the argument above which works for $U(k, d)$ just as done for $U(d, d)$, admits an implicit representation of size $O(n^{1-1/d} \log n)$. Here, in fact, there is a simpler way to get the existence of such an implicit representation. By the Kővári–Sós–Turán theorem [15] every graph in $\mathcal{F}_n$ is $p = O(n^{1-1/d})$-degenerate, hence there is an ordering of the vertices so that every vertex has at most p neighbors following it. One can thus assign to each vertex a label consisting of its number in this ordering and the numbers of its neighbors following it to get the required representation. On the other hand the speed of $\mathcal{F}$ satisfies $f(n) \ge 2^{\Omega(n^{2-1/d})}$ for every n for which our family contains one of the projective norm graphs. Therefore each implicit representation for $\mathcal{F}_n$ requires labels of length at least $\log |\mathcal{F}_n|/n = \Omega(n^{1-1/d})$. This completes the proof. □

3 Concluding Remarks and Open Problems

1. The proof of Theorem 1.1 is closely related to the known proof [5] that bounds the speed of hereditary families which are $U(1, d)$-free. The crucial additional argument here is the application of the results of [13, 22] about spanning trees with low crossing numbers, as formulated in Lemma 2.2 here, which supplies the desired implicit

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representation. The proof in [5] bounds the speed of the families, but provides no economical implicit representation.

2. In view of Theorem 1.1 one may suspect that for any sparse hereditary family $\mathcal{F}$ like the ones considered here there is an integer d so that the shortest implicit representation for $\mathcal{F}_n$ is of order $n^{1-1/d}$ up to logarithmic factors. This, however, is not the case. Indeed, for any small $\varepsilon > 0$ there is a hereditary family $\mathcal{F}$ such that for infinitely many values of n, $\mathcal{F}_n$ admits an implicit representation of size $O((\log n)/\varepsilon)$, whereas for infinitely many values of n any implicit representation for $\mathcal{F}_n$ is of size at least $\Omega(\varepsilon n^{1-2\varepsilon} \log n)$. We proceed with a sketch of the proof of this fact. Recall that a graph is called k-degenerate if any induced subgraph of it contains a vertex of degree at most k. Equivalently this means that its vertices can be ordered so that each vertex has at most k neighbors preceding it in this order. The family of all k-degenerate graphs on n vertices admits an implicit representation of length $O(k \log n)$, since one can assign each vertex a label consisting of its number in an ordering as above together with the numbers of all its neighbors that precede it in this order. We need the following simple lemma.

Lemma 3.1 *For every small $\varepsilon > 0$ and any $n \geq n_0(\varepsilon)$ there is a family $\mathbf{T}_n$ of at least $n^{0.5\varepsilon n^{2-2\varepsilon}}$ distinct labelled graphs on n vertices, so that every induced subgraph of each of these graphs on a set of at most n^{ε} vertices is $(4/\varepsilon)$-degenerate.*

Proof Put $p = n^{-2\varepsilon}$ and let $G = G(n, p)$ be a binomial random graph on a set of n labelled vertices. If the number of edges of any induced subgraph of G on any set of $c \leq n^{\varepsilon}$ vertices contains less than $2c/\varepsilon$ edges, then any such induced subgraph is $(4/\varepsilon)$-degenerate. The probability that there is a set of c vertices violating the condition above is smaller than

$$\binom{n}{c} \cdot \binom{c^2/2}{2c/\varepsilon} \cdot p^{2c/\varepsilon} \leq n^c \cdot c^{2c/\varepsilon} \cdot n^{-4c} \leq n^c \cdot n^{2c} \cdot n^{-4c} = n^{-c}.$$

Summing over all values of c, $2/\varepsilon \leq c \leq n^{\varepsilon}$ it follows that the probability that there is such a dense subset is (much) smaller than, say, 0.1. Therefore the total probability of the graphs G that satisfy the desired condition is at least 0.9. This implies that their number is larger than the number of all graphs on n vertices with at most $p\binom{n}{2}$ edges (as these are the graphs with the largest probability in the model considered, and their total probability is $1/2 + o(1) < 0.9$). The desired result follows by taking the set $\mathbf{T}_n$ to be the set of all the graphs satisfying the required condition. □

Define, next, the following fast growing sequence of integers. $a_1 = n_0(\varepsilon)$, where $n_0(\varepsilon)$ is taken from the previous lemma, and $a_{k+1} = \lceil a_k^{2/\varepsilon} \rceil$ for all $k \geq 1$. Let $\mathcal{F}$ be the hereditary family of graphs consisting of the union of all graphs in the union of the families $\mathbf{T}_{a_k}$ from the previous lemma for all $k \geq 1$, and all their induced subgraphs. For every n which equals a_k for some k, the number of graphs in $\mathcal{F}_n$ is at least $|\mathbf{T}_{a_k}| \geq n^{0.5\varepsilon n^{2-\varepsilon}}$. Therefore any implicit representation for $\mathcal{F}_n$ is of length at least $\log_2 |\mathcal{F}_n|/n = \Omega(\varepsilon n^{1-2\varepsilon} \log n)$ (since every member of $\mathcal{F}_n$ can be reconstructed from the n labels of its vertices). On the other hand, for any value of n satisfying

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$a_k < n < a_k^2$ $(\le a_{k+1}^{\varepsilon})$ for some k, every graph in $\mathcal{F}_n$ is $(4/\varepsilon)$-degenerate, and hence for any such n the family $\mathcal{F}_n$ admits an implicit representation of length $O((\log n)/\varepsilon)$.

3. By Theorem 1.1, if $\mathcal{F}$ is a hereditary family with speed $f(n) = 2^{o(n^2)}$ then $\mathcal{F}_n$ admits an implicit representation of size at most $O(n^{1-1/d} \log n)$ for some integer $d \ge 1$. It would be interesting to decide if tighter bounds hold when the growth rate of the speed $f(n)$ is slower. A particularly interesting case is $f(n) \le 2^{O(n \log n)}$, as this holds for many interesting hereditary families including all the ones in which every vertex is a point in a real space of bounded dimension, and the adjacency of two vertices is determined by the signs of a finite set of bounded degree polynomials in the coordinates of the corresponding points. Such families, which are hereditary by definition, include many intersection graphs of simple geometric objects of a prescribed shape. By a theorem of Warren from real algebraic geometry that deals with sign patterns of real polynomials [21] (following earlier work of Milnor [18]) the speed of any such family is at most $2^{O(n \log n)}$, and in many cases it is possible to obtain nearly tight bounds for the speed. The argument, which is similar to the one used by Goodman and Pollack in [11], see also [2], in order to estimate the number of configurations and polytopes in R^d, found a significant number of applications following their work. Their initial paper using this approach appears in the very first volume of the journal Discrete and Computational Geometry they founded in the mid. 80 s. See also [3] and the references therein for several additional early applications of the method. However, there are quite a few families of this type for which the existence of economic implicit representations is not known. Simple examples include intersection graphs of segments or discs in the plane studied in [17].

4. By the main result of [12] for any $\delta > 0$ there are hereditary families with speed $f(n) \le 2^{O(n \log n)}$ so that $\mathcal{F}_n$ does not admit an implicit representation of size smaller than $n^{1/2-\delta}$, and the authors of [12] raise the natural question if the constant $1/2$ can be improved. Is it possible that such families always admit an implicit representation of size $O(n^{1/2} \log n)$? Similarly, if the speed is smaller than $2^{n^{1+\varepsilon}}$ for a sufficiently small fixed $\varepsilon > 0$, is there always an implicit representation of size at most $O(n^{2/3} \log n)$?

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