

1                   **INVERTIBILITY OF DIGRAPHS AND TOURNAMENTS\***

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4                   **Abstract.** For an oriented graph  $D$  and a set  $X \subseteq V(D)$ , the *inversion* of  $X$  in  $D$  is the  
5                   digraph obtained by reversing the orientations of the edges of  $D$  with both endpoints in  $X$ . The  
6                   *inversion number* of  $D$ ,  $\text{inv}(D)$ , is the minimum number of inversions which can be applied in  
7                   turn to  $D$  to produce an acyclic digraph. Answering a recent question of Bang-Jensen, da Silva,  
8                   and Havet we show that, for each  $k \in \mathbb{N}$  and tournament  $T$ , the problem of deciding whether  
9                    $\text{inv}(T) \leq k$  is solvable in time  $O_k(|V(T)|^2)$ , which is tight for all  $k$ . In particular, the problem is  
10                  fixed-parameter tractable when parameterised by  $k$ . On the other hand, we build on their work to  
11                  prove their conjecture that for  $k \geq 1$  the problem of deciding whether a general oriented graph  $D$  has  
12                   $\text{inv}(D) \leq k$  is NP-complete. We also construct oriented graphs with inversion number equal to twice  
13                  their cycle transversal number, confirming another conjecture of Bang-Jensen, da Silva, and Havet,  
14                  and we provide a counterexample to their conjecture concerning the inversion number of so-called  
15                  ‘dijoin’ digraphs while proving that it holds in certain cases. Finally, we asymptotically solve the  
16                  natural extremal question in this setting, improving on previous bounds of Belkhechine, Bouaziz,  
17                  Boudabbous, and Pouzet to show that the maximum inversion number of an  $n$ -vertex tournament is  
18                   $(1 + o(1))n$ .

19                  **Key words.** inversion, digraphs, tournaments

20                  **MSC codes.** 05C20, 05C76

21                  **1. Introduction.** In this paper we only consider digraphs without loops, digons,  
22                  or parallel edges, for which we use the terms *digraph* and *oriented graph* interchangeably.  
23                  For such a digraph  $D = (V, E)$  and a set  $X \subseteq V$ , the *inversion* of  $X$  in  $D$  is the  
24                  digraph obtained from  $D$  by reversing the direction of the edges with both endpoints  
25                  in  $X$ ; we say that we *invert*  $X$  in  $D$ . Given a family of sets  $X_1, \dots, X_k \subseteq V$ , we can  
26                  invert  $X_1$  in  $D$ , then  $X_2$  in the resulting digraph, and so on, noting that the final  
27                  digraph produced by these inversions is independent of the order in which we perform  
28                  them. If inverting  $X_1, \dots, X_k$  in turn transforms  $D$  into an acyclic digraph, then we  
29                  say that these sets form a *decycling family* of  $D$ . We will refer to a set  $X \subseteq V$  which  
30                  forms a decycling family by itself as a *decycling set*. The *inversion number* of  $D$ ,  
31                  denoted  $\text{inv}(D)$ , is defined to be the minimum size of a decycling family of  $D$ , and for  
32                   $k \in \mathbb{N}_0$  we say that  $D$  is  $k$ -*invertible* if  $\text{inv}(D) \leq k$ .

33                  The study of inversions began in Houmem Belkhechine’s PhD thesis [5] and con-  
34                  tinued in [6, 7, 26], in which many foundational results were established. The present  
35                  work is inspired by a recent paper of Bang-Jensen, da Silva, and Havet [4] which  
36                  studied a wide range of questions about invertibility, with an emphasis on those of  
37                  an algorithmic or extremal nature. They also posed a host of interesting conjectures  
38                  and problems, some of which we answer in this paper.

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39 **1.1. The inversion number of  $k$ -joins.** The cornerstone of many of the conjectures made by Bang-Jensen, da Silva, and Havet in [4] is the following ‘dijoin 40 conjecture’. For oriented graphs  $L$  and  $R$ , the *dijoin*  $L \rightarrow R$  from  $L$  to  $R$  is the 41 oriented graph consisting of vertex-disjoint copies of  $L$  and  $R$ , with an edge  $\vec{uv}$  for all 42  $u \in V(L)$  and  $v \in V(R)$ .

43 **CONJECTURE 1.1** ([4]). *For oriented graphs  $L$  and  $R$  we have  $\text{inv}(L \rightarrow R) = \text{inv}(L) + \text{inv}(R)$ .*

44 Noting that the left-hand side is certainly at most the right-hand side for all  $L$  45 and  $R$  and that the conjecture holds trivially if  $\text{inv}(L) = 0$  or  $\text{inv}(R) = 0$ , Bang- 46 Jensen, da Silva, and Havet showed it to be true when  $\text{inv}(L) + \text{inv}(R) \leq 3$ , and when 47  $\text{inv}(L) = \text{inv}(R) = 2$  and  $L$  and  $R$  are both strongly connected.<sup>1</sup> They also observed 48 (see our section 3) that the conjecture is equivalent to its restriction to tournaments. 49 We disprove Conjecture 1.1 by exhibiting a tournament  $R$  with  $\text{inv}(R) = \text{inv}(\vec{C}_3 \rightarrow 50 R)$ , where  $\vec{C}_3$  is the directed cycle on three vertices.

51 **THEOREM 1.2.** *There exists a tournament  $R$  with  $\text{inv}(R) = \text{inv}(\vec{C}_3 \rightarrow R) = 3$ .*

52 While Theorem 1.2 shows that the dijoin conjecture is false in general, we prove 53 it in the case where  $\text{inv}(L) = \text{inv}(R) = 2$ .

54 **THEOREM 1.3.** *If  $L$  and  $R$  are digraphs with  $\text{inv}(L) = \text{inv}(R) = 2$ , then  $\text{inv}(L \rightarrow 55 R) = 4$ .*

56 The proof of Theorem 1.3 relies on the strongly connected case and our next result, 57 which concerns the following generalisation of dijoins to arbitrarily many digraphs. 58 For  $k \in \mathbb{N}$  the  *$k$ -join* of digraphs  $D_1, \dots, D_k$ , written  $[D_1, \dots, D_k]$ , is the digraph 59 consisting of vertex-disjoint copies of  $D_1, \dots, D_k$  with an additional edge  $\vec{uv}$  whenever 60  $u \in V(D_i), v \in V(D_j)$  for  $i < j$ . We write  $[D]_k = [D, \dots, D]$  for the  $k$ -join of  $k$  copies 61 of the same oriented graph  $D$ . The following result can be viewed as a  $k$ -join analogue 62 of the dijoin conjecture holding under certain conditions. It generalises a theorem of 63 Pouzet, Kaddour, and Thatte [26] which states that  $\text{inv}([\vec{C}_3]_k) = k$  for all  $k$ .

64 **THEOREM 1.4.** *Let  $k \in \mathbb{N}$  and let  $D_1, \dots, D_k$  be oriented graphs. Assume that 65  $\text{inv}(D_i) \leq 2$  for all  $i$ , with equality for at most one  $i$ . Then*

$$66 \quad (1.1) \quad \text{inv}([D_1, \dots, D_k]) = \sum_{i=1}^k \text{inv}(D_i).$$

67 We will use Theorem 1.4 to confirm another conjecture from [4] which was made 68 based on the dijoin conjecture (see Theorem 1.6 below). Theorem 1.4 and, in turn, 69 Theorem 1.3 follow from a characterisation of the decycling families of size  $k$  of arbitrary  $k$ -joins of oriented graphs with inversion number 1. We will need some further 70 terminology to state this result: for a digraph  $D$ , sets  $X_1, \dots, X_k \subseteq V(D)$ , and 71 a vertex  $v \in V(D)$ , we define the *characteristic vector* of  $v$  in  $X_1, \dots, X_k$  to be 72  $(I_{\{v \in X_i\}} : i \in [k]) \in \mathbb{F}_2^k$ , where  $I_{\{v \in X_i\}}$  is the indicator function of the event  $v \in X_i$ . 73 For vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^k$  we write  $\mathbf{u} \cdot \mathbf{v}$  for the usual scalar product of  $\mathbf{u}$  and  $\mathbf{v}$  over 74  $\mathbb{F}_2$ . This is not a genuine inner product, but we say nevertheless that a collection 75  $\mathbf{u}_1, \dots, \mathbf{u}_\ell \in \mathbb{F}_2^k$  is *orthonormal* if  $\mathbf{u}_i \cdot \mathbf{u}_i = 1$  for all  $i$  and  $\mathbf{u}_i \cdot \mathbf{u}_j = 0$  for all  $i \neq j$ . 76 Finally, we refer to the canonical copy of  $D_i$  in  $D = [D_1, \dots, D_k]$  as the  *$i$ th factor* of 77

<sup>1</sup>The case where  $\text{inv}(L) = 2$  and  $\text{inv}(R) = 1$  is not explicitly mentioned in [4], but follows easily from the case where  $\text{inv}(L) = 1$  and  $\text{inv}(R) = 2$  by inverting  $V(L \rightarrow R)$ .

81  $D$ . We are now ready to state our characterisation theorem, the case  $k = 2$  of which  
 82 was shown by Bang-Jensen, da Silva, and Havet [4]. Its proof is based on an approach  
 83 used by Pouzet, Kaddour, and Thatte [26].

84 **THEOREM 1.5.** *Let  $D_1, \dots, D_k$  be oriented graphs with  $\text{inv}(D_i) = 1$  for all  $i$  and  
 85 let  $\widehat{D} = [D_1, \dots, D_k]$  be their  $k$ -join. Then sets  $X_1, \dots, X_k \subseteq V(\widehat{D})$  form a decycling  
 86 family of  $\widehat{D}$  if and only if there are orthonormal vectors  $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}_2^k$  and for  
 87 each  $i$  a decycling set  $V_i \subseteq V(D_i)$  of the  $i$ th factor of  $\widehat{D}$  such that, for each  $i$ , the  
 88 vertices in  $V_i$  have characteristic vector  $\mathbf{u}_i$  (in  $X_1, \dots, X_k$ ), and all other vertices have  
 89 characteristic vector  $\mathbf{0}$  (in  $X_1, \dots, X_k$ ).*

90 *In particular, any acyclic digraph obtained from  $\widehat{D}$  by  $k$  inversions can also be  
 91 obtained by inverting a decycling set for each factor in turn.*

92 **1.2. Computational complexity.** One focus of Bang-Jensen, da Silva, and  
 93 Havet's paper [4] was on the computational complexity of deciding whether an oriented  
 94 graph is  $k$ -invertible. More formally, they considered, for fixed  $k \in \mathbb{N}_0$ , the problem  
 95 of  $k$ -INVERSION:

INPUT: an oriented graph  $D$ .  
PROBLEM:  $\text{inv}(D) \leq k$ ?

96 A first observation is that 0-INVERSION is equivalent to checking whether a digraph  
 97  $D$  is acyclic, which is well known to be possible in time  $O(|V(D)|^2)$  (see [12, p. 612]),  
 98 so we need only consider  $k \geq 1$ .

99 Bang-Jensen, da Silva, and Havet [4] showed that 1-INVERSION is NP-complete  
 100 using a reduction from MONOTONE 3-IN-1 SAT. Then, using the special cases of the  
 101 dijoin conjecture proved in that paper, they observed that for a digraph  $D$  we have  
 102  $\text{inv}(D \rightarrow D) = 2$  if and only if  $\text{inv}(D) = 1$ , from which it follows that 2-INVERSION is  
 103 also NP-complete. They conjectured that NP-completeness extends to  $k$ -INVERSION  
 104 for all  $k \geq 3$ , noting that this would follow from a similar argument if the dijoin  
 105 conjecture were true. Of course, the full dijoin conjecture is not required, and indeed  
 106 it is easy to see that Theorem 1.4 is enough: it implies that  $\text{inv}(D) = 1$  if and only  
 107 if  $\text{inv}([D]_k) = k$ , which reduces 1-INVERSION to  $k$ -INVERSION and hence shows the  
 108 following (see section 4).

109 **THEOREM 1.6.**  *$k$ -INVERSION is NP-complete for all  $k \in \mathbb{N}$ .*

110 Bang-Jensen, da Silva, and Havet also considered the computational complexity  
 111 of the same problem when the input is restricted to tournaments. For fixed  $k \in \mathbb{N}$  the  
 112 problem of  $k$ -TOURNAMENT-INVERSION is:

INPUT: a tournament  $T$ .  
PROBLEM:  $\text{inv}(T) \leq k$ ?

113 One way of analysing the complexity of this problem is to use  $k$ -inversion-critical  
 114 tournaments: we say that a tournament  $T$  is a  $k$ -inversion-critical tournament if  
 115  $\text{inv}(T) = k$  but  $\text{inv}(T - \{v\}) < k$  for all  $v \in V(T)$ , and denote by  $\mathcal{IC}_k$  the set  
 116 of  $k$ -inversion-critical tournaments. It is not difficult to see that a tournament has  
 117 inversion number at most  $k$  if and only if it contains no element of  $\mathcal{IC}_{k+1} \cup \mathcal{IC}_{k+2}$   
 118 as a subtournament. Indeed, for any digraph  $D$  and vertex  $v \in V(D)$  with out-  
 119 120

121 neighbourhood  $A \subseteq V(D)$ , adding  $A$  and  $A \cup \{v\}$  to a decycling family of  $D - \{v\}$   
 122 gives a decycling family of  $D$ . We deduce that  $\text{inv}(D) \leq \text{inv}(D - \{v\}) + 2$ . Hence, if  
 123  $\text{inv}(T) > k$ , then by arbitrarily deleting vertices from  $T$  one by one, we can obtain a  
 124 subtournament  $T'$  of  $T$  with  $\text{inv}(T') \in \{k+1, k+2\}$ . This  $T'$  contains a member of  
 125  $\mathcal{IC}_{k+1} \cup \mathcal{IC}_{k+2}$  as a subtournament.

126 Belkhechine, Bouaziz, Boudabbous, and Pouzet [7] showed that  $\mathcal{IC}_k$  is finite for  
 127 all  $k \in \mathbb{N}$ . Writing  $m_k$  for the maximum number of vertices of an element of  $\mathcal{IC}_k$ , it fol-  
 128 lows that  $k$ -TOURNAMENT-INVERSION can be solved in time  $O(|V(T)|^{\max(m_{k+1}, m_{k+2})})$ .  
 129 Thus, in particular,  $k$ -TOURNAMENT-INVERSION can be solved in polynomial time  
 130 for any fixed  $k$ . Plainly  $\mathcal{IC}_1 = \{\vec{C}_3\}$ , so  $m_1 = 3$ , and  $\mathcal{IC}_2$  was explicitly described  
 131 in [26], giving  $m_2 = 6$ . However, no upper bound on  $m_k$  is known for  $k \geq 3$ , so for  
 132 no  $k \geq 1$  does the above give a concrete polynomial bound on the complexity of  $k$ -  
 133 TOURNAMENT-INVERSION. Note also that this approach does not identify a decycling  
 134 family of size  $k$  given a  $k$ -invertible tournament, it can only confirm the existence of  
 135 one.

136 Bang-Jensen, da Silva, and Havet [4] used an alternative approach to show that  
 137 1-TOURNAMENT-INVERSION can be solved in time  $O(|V(T)|^3)$  while 2-TOURNAMENT-  
 138 INVERSION can be solved in time  $O(|V(T)|^6)$ . The idea behind their algorithm for 1-  
 139 TOURNAMENT-INVERSION is to check whether the tournament contains a vertex which  
 140 can be made into a source, and for 2-TOURNAMENT-INVERSION they check whether  
 141 it contains a pair of vertices which can be made into a source and a sink respec-  
 142 tively. They went on to ask for the least real numbers  $r_k$  such that  $k$ -TOURNAMENT-  
 143 INVERSION can be solved in time  $O(|V(T)|^{r_k})$ . We answer this question by showing  
 144 that, perhaps surprisingly, for each fixed  $k \in \mathbb{N}$  there is an algorithm solving  $k$ -  
 145 TOURNAMENT-INVERSION in time  $O(|V(T)|^2)$ . In the language of complexity theory,  
 146 this means that the likely NP-hard problem of determining whether  $\text{inv}(T) \leq k$  for  
 147 inputs  $k$  and  $T$  (see Conjecture 8.1) is fixed-parameter tractable when parameterised  
 148 by  $k$ .<sup>2</sup>

149 **THEOREM 1.7.** *For fixed  $k \in \mathbb{N}$ ,  $k$ -TOURNAMENT-INVERSION can be solved for  $n$ -  
 150 vertex tournaments in time  $O(n^2)$ . Moreover, if the input tournament is  $k$ -invertible,  
 151 then our algorithm finds a decycling family of size at most  $k$ .*

152 Note that the exponent of  $n$  in this running time is optimal, since any algorithm  
 153 solving  $k$ -TOURNAMENT-INVERSION needs to inspect the orientation of every edge  
 154 in the input tournament. However, the implied constant in the running time of our  
 155 algorithm is doubly exponential in  $k$ , so it is unlikely to be of practical use for large  $k$ .

156 **1.3. Relation to other parameters.** Bang-Jensen, da Silva, and Havet [4]  
 157 also considered the relationship between the inversion number and other digraph pa-  
 158 rameters. Two well studied parameters of particular interest are the cycle transversal  
 159 number and the cycle edge-transversal number, defined as follows. A *cycle transversal*  
 160 (or *feedback vertex set*) in a digraph  $D$  is a set of vertices of  $D$  whose removal from  $D$   
 161 leaves an acyclic digraph and the *cycle transversal number* of  $D$ , denoted  $\tau(D)$ , is the  
 162 minimum size of a cycle transversal in  $D$ . Analogously, a *cycle edge-transversal* (or  
 163 *feedback arc set*) in  $D$  is a set of edges of  $D$  whose removal leaves an acyclic digraph  
 164 and the *cycle edge-transversal number* of  $D$ ,  $\tau'(D)$ , is the minimum size of a cycle  
 165 edge-transversal in  $D$ . Note that the inequality  $\tau(D) \leq 2\tau'(D)$  always holds, since  
 166 the endpoints of the edges in a cycle edge-transversal of  $D$  form a cycle transversal of

<sup>2</sup>See [14] for the definition of fixed-parameter tractability and an exposition of the surrounding theory.

167  $D$ .

168 Bang-Jensen, da Silva, and Havet [4] made the following observations concerning  
 169 the relationships between  $\text{inv}(D)$ ,  $\tau(D)$ , and  $\tau'(D)$  for a digraph  $D$ . Firstly, we  
 170 have  $\text{inv}(D) \leq \tau'(D)$ . This follows from the fact that if  $F \subseteq E(D)$  is a cycle edge-  
 171 transversal of  $D$ , then since  $(V(D), E(D) \setminus F)$  is acyclic, there is a labelling  $v_1, \dots, v_n$   
 172 of  $V(D)$  such that  $\overrightarrow{v_j v_i} \notin E(D) \setminus F$  if  $i < j$ . Applying the family of inversions  
 173  $\{\{v_i, v_j\} : i < j, \overrightarrow{v_j v_i} \in F\}$  transforms  $D$  into an acyclic digraph and hence  $\text{inv}(D) \leq$   
 174  $\tau'(D)$  as claimed. They also observed that this inequality is tight for all values of  
 175  $\tau'(D)$  as exhibited by  $[C_3]_k$ , which clearly has cycle edge-transversal number  $k$ , and  
 176 as mentioned above was shown in [26] to have inversion number  $k$ .

177 Turning to  $\tau(D)$ , the inequality  $\text{inv}(D) \leq 2\tau(D)$  was obtained in [4] as follows.  
 178 After observing that  $\tau(D) = 0$  implies  $\text{inv}(D) = 0$ , we may assume that  $\tau(D) \geq 1$ .  
 179 Let  $S \subseteq V(D)$  be a cycle transversal in  $D$  of size  $\tau(D)$  and pick  $v \in S$ . Then observe  
 180 that  $D - \{v\}$  has cycle transversal number  $\tau(D) - 1$ , with  $S \setminus \{v\}$  a cycle transversal.  
 181 Moreover, as noted in Section 1.2 we have  $\text{inv}(D) \leq \text{inv}(D - \{v\}) + 2$ , from which it  
 182 follows by induction that  $\text{inv}(D) \leq 2\tau(D)$ .

183 Bang-Jensen, da Silva, and Havet conjectured that this inequality is tight for all  
 184 values of  $\tau(D)$ . Indeed, they considered the graph  $V_5$  obtained by adding a vertex  $v$   
 185 and edges  $\overrightarrow{v1}, \overrightarrow{2v}, \overrightarrow{v3}, \overrightarrow{4v}$  to the (transitive) tournament on vertex set  $\{1, 2, 3, 4\}$  with  
 186 edges  $\overrightarrow{ij}$  for  $i < j$ , which can easily be seen to have  $\tau(V_5) = 1$  and  $\text{inv}(V_5) = 2$ . They  
 187 noted that if the dijoin conjecture holds, then  $\tau([V_5]_k) = k$  and  $\text{inv}([V_5]_k) = 2k$  for all  
 188  $k$  (in fact, since  $V_5$  is strongly connected, the case  $k = 2$  follows from the special cases  
 189 for which they proved the dijoin conjecture). We construct digraphs with a similar  
 190 character to  $V_5$  which confirm their conjecture.

191 **THEOREM 1.8.** *For all  $k \in \mathbb{N}$  there exists an oriented graph  $D$  with  $\text{inv}(D) =$   
 192  $2\tau(D) = 2k$ .*

193 **1.4. The extremal problem.** Finally, we consider  $\text{inv}(n)$ , defined for each  
 194  $n \in \mathbb{N}$  as the maximum inversion number of an oriented graph (or, equivalently, a  
 195 tournament) on  $n$  vertices. Belkhechine, Bouaziz, Boudabbous, and Pouzet [7] were  
 196 the first to study this parameter, obtaining bounds of the form<sup>3</sup>

$$197 \quad \frac{n}{2} - \log(n) + O(1) \leq \text{inv}(n) \leq n + O(1).$$

198 Their lower and upper bounds follow from counting and inductive arguments respec-  
 199 tively (see section 7 for details), and they conjectured that  $\text{inv}(n) \geq \lfloor \frac{n-1}{2} \rfloor$  for all  $n$ .  
 200 Bounds of the form above previously remained the best known, with Bang-Jensen, da  
 201 Silva, and Havet [4] noting that the  $O(1)$  term in the upper bound can be improved  
 202 very slightly.

203 Using a random construction, we show that  $\text{inv}(n) = (1 + o(1))n$ .

204 **THEOREM 1.9.** *For sufficiently large  $n$  we have*

$$205 \quad \text{inv}(n) \geq n - \sqrt{2n \log(n)}.$$

206 *Moreover, a uniformly random labelled  $n$ -vertex tournament has at least this inversion  
 207 number with probability tending to 1.*

208 In section 7 we also show that  $\text{inv}(n) \leq n - \log(n + 1)$ .

<sup>3</sup>All logarithms in this paper are taken base 2.

209     **1.5. Outline of the paper.** The remainder of the paper is organised as follows.  
 210    In section 2 we introduce some further notation, definitions, and preliminary obser-  
 211    vations which will be useful in the rest of the paper. In the very short section 3 we  
 212    prove Theorem 1.2, constructing a counterexample to the dijoin conjecture. Our re-  
 213    sults on the inversion number of  $k$ -joins, Theorem 1.3, Theorem 1.4, and Theorem 1.5,  
 214    are proved in section 4, along with Theorem 1.6. Section 5 concerns the complexity  
 215    of  $k$ -TOURNAMENT-INVERSION and contains the proof of Theorem 1.7. We give the  
 216    proof of Theorem 1.8 in section 6. In section 7 we discuss the existing bounds on  
 217     $\text{inv}(n)$  before proving Theorem 1.9 and giving an improved upper bound. Finally,  
 218    in section 8 we restate some conjectures and questions from previous papers which  
 219    remain open and pose some new ones of our own.

220    **Note added.** Almost simultaneously with the initial release of this paper, Aubian,  
 221    Havet, Hörsch, Klingelhoefer, Nisse, Rambaud, and Vermande announced indepen-  
 222    dent work [2] on some of the problems we address here. Specifically, they prove a  
 223    stronger version of Theorem 1.2 (in fact, they prove that a strong version of our Con-  
 224    jecture 8.9 holds provided at least one of  $\ell$  and  $r$  is odd and at least 3) and they show  
 225    upper and lower bounds on  $\text{inv}(n)$  of forms similar to those we give in subsection 1.4.

226    **2. Notation and preliminaries.** In this section we detail some of the defini-  
 227    tions, observations, and notation to be used in the rest of the paper. As noted above,  
 228    all digraphs will be oriented graphs, that is, loopless directed graphs with at most one  
 229    edge between each pair of vertices. An *acyclic digraph* is a digraph with no directed  
 230    cycles. In the case where the digraph is a tournament, we use the term *transitive*  
 231    instead of acyclic. Note that for each  $n \in \mathbb{N}$  there is a unique unlabelled transitive  
 232    tournament on  $n$  vertices. To a transitive tournament  $T$  we associate the total order  
 233     $<$  on  $V(T)$  where  $u < v$  for all  $u, v \in V(T)$  such that  $\overrightarrow{uv} \in E(T)$ . We write  $[n]$  for  
 234    the set  $\{1, 2, \dots, n\}$ . For a digraph  $D$  and a set  $S \subseteq V(D)$  we write  $D - S$  for the  
 235    digraph produced by deleting the vertices in  $S$  from  $D$ . We now give the following  
 236    key definitions.

237    **DEFINITION 2.1.** Recall that for a digraph  $D$ , sets  $X_1, \dots, X_k \subseteq V(D)$ , and a  
 238    vertex  $v \in V(D)$ , the characteristic vector of  $v$  in  $X_1, \dots, X_k$  is  $(I_{\{v \in X_i\}} : i \in [k]) \in$   
 239     $\mathbb{F}_2^k$ , where  $I_{\{v \in X_i\}}$  is the indicator function of the event  $v \in X_i$ . Define an equivalence  
 240    relation  $\sim$  on  $V(D)$  by setting  $u \sim v$  if  $u$  and  $v$  have the same characteristic vector in  
 241     $X_1, \dots, X_k$ . The atoms of  $X_1, \dots, X_k$  in  $D$  are the equivalence classes of this relation.

242    Note that, equivalently, the atoms of  $X_1, \dots, X_k$  in  $D$  are the atoms of the set  
 243    algebra on  $V(D)$  generated by  $X_1, \dots, X_k$ , and that there are at most  $2^k$  atoms for  
 244    given  $D$  and  $X_1, \dots, X_k$ . The next observation will be useful throughout the paper.

245    **OBSERVATION 2.2.** Let  $D$  be a digraph and suppose that  $u, v \in V(D)$  are joined by  
 246    an edge in  $D$ . Let  $X_1, \dots, X_k \subseteq V(D)$ . Write  $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^k$  for the characteristic vectors  
 247    of  $u$  and  $v$  in  $X_1, \dots, X_k$  respectively. Then the edge between  $u$  and  $v$  undergoes a net  
 248    change in orientation when  $X_1, \dots, X_k$  are inverted in  $D$  if and only if  $\mathbf{u} \cdot \mathbf{v} = 1$ .

249    This follows from the fact that  $\mathbf{u} \cdot \mathbf{v}$  is the parity of the number of  $X_1, \dots, X_k$   
 250    which contain both  $u$  and  $v$ . An obvious implication of Observation 2.2 is that given  
 251     $D$  and  $X_1, \dots, X_k$ , for every pair of (not necessarily distinct) atoms  $A$  and  $B$ , either  
 252    all edges  $\{ab : a \in A, b \in B\}$  undergo a net orientation change when  $X_1, \dots, X_k$  are  
 253    inverted, or none of them do. In particular, for every vertex  $v$  and atom  $A$ , either all  
 254    edges  $\{va : a \in A\}$  change orientation or none of them do.

255    Finally, we note some simple observations which will be used freely in what follows.

256    (i) If  $D'$  is a subdigraph of an oriented graph  $D$ , then  $\text{inv}(D') \leq \text{inv}(D)$ .

257 (ii) For every oriented graph  $D$  and every non-negative integer  $k \leq \text{inv}(D)$ , there  
 258 exists a spanning subdigraph of  $D$  with inversion number  $k$ .  
 259 (iii) If  $X_1, \dots, X_k$  is a decycling family of an oriented graph  $D$ , then  $D$  can be  
 260 extended to a tournament  $T$  for which  $X_1, \dots, X_k$  is still a decycling family.  
 261 In particular  $\text{inv}(T) = \text{inv}(D)$ .  
 262 For (ii), delete edges of  $D$  one by one, noting that the inversion number drops by at  
 263 most 1 at each step. For (iii), invert the decycling family in  $D$ , extend the resulting  
 264 acyclic digraph to a transitive tournament, then invert the decycling family again.

265 **3. A counterexample to the dijoin conjecture.** In this short section we give  
 266 a counterexample to the dijoin conjecture of Bang-Jensen, da Silva, and Havet [4],  
 267 that is, the conjecture that  $\text{inv}(L \rightarrow R) = \text{inv}(L) + \text{inv}(R)$  for all oriented graphs  $L$   
 268 and  $R$ . As noted in the introduction, this conjecture is equivalent to its restriction  
 269 to tournaments. Indeed, suppose that  $L$  and  $R$  are digraphs with  $\text{inv}(L \rightarrow R) <$   
 270  $\text{inv}(L) + \text{inv}(R)$ . Extend  $L \rightarrow R$  to a tournament of the same inversion number and  
 271 observe that this tournament is  $L' \rightarrow R'$  for some tournaments  $L'$  and  $R'$  extending  $L$   
 272 and  $R$  respectively. These clearly satisfy  $\text{inv}(L') \geq \text{inv}(L)$  and  $\text{inv}(R') \geq \text{inv}(R)$ , so we  
 273 have tournaments  $L'$  and  $R'$  with  $\text{inv}(L' \rightarrow R') < \text{inv}(L) + \text{inv}(R) \leq \text{inv}(L') + \text{inv}(R')$ .

274 *Proof of Theorem 1.2.* Let  $L$  be a copy of  $\vec{C}_3$ . Suppose that  $R$  is a tournament  
 275 with  $\text{inv}(R) = 3$  for which there exist disjoint  $A, B, C \subseteq V(R)$  such that  $A \cup B$ ,  $A \cup C$   
 276 and  $B \cup C$  form a decycling family of  $R$ . Then for distinct vertices  $u, v \in V(L)$  the  
 277 sets  $A \cup B \cup \{u, v\}$ ,  $A \cup C \cup \{u, v\}$  and  $B \cup C \cup \{u, v\}$  form a decycling family of  
 278  $L \rightarrow R$ , demonstrating that

$$279 \quad \text{inv}(L \rightarrow R) = 3 < 4 = \text{inv}(L) + \text{inv}(R).$$

280 One way to construct such an  $R$  is as follows: let  $R$  be the tournament with  
 281 vertex set  $[9]$ , let  $A = \{1, 3\}$ ,  $B = \{4, 6\}$ , and  $C = \{7, 9\}$ , and let the edge  $ij$  be  
 282 directed backwards (that is, from  $j$  to  $i$  when  $i < j$ ) if and only if  $i$  and  $j$  are both in  
 283  $A \cup B \cup C$ , but not both in  $A$ ,  $B$ , or  $C$ . It is clear that inverting  $A \cup B$ ,  $A \cup C$  and  
 284  $B \cup C$  transforms  $R$  into a transitive tournament, and a computer check shows that  
 285  $\text{inv}(R) = 3$ , as required.  $\square$

286 **4. Decycling families of  $k$ -joins.** In this section we prove Theorem 1.5, which  
 287 characterises the decycling families of size  $k$  of  $k$ -joins of digraphs each with inversion  
 288 number 1. We will then deduce Theorem 1.4 from this characterisation, and use  
 289 Theorem 1.4 to obtain Theorem 1.3 and Theorem 1.6. The bulk of the work in our  
 290 proof of Theorem 1.5 is put towards proving Lemma 4.1, which deals with the case  
 291  $\widehat{D} = [\vec{C}_3]_k$ .

292 **LEMMA 4.1.** *Let  $k \in \mathbb{N}$ , let  $\widehat{D} = [\vec{C}_3]_k$ , and let  $X_1, \dots, X_k \subseteq V(\widehat{D})$  be a decycling  
 293 family of  $\widehat{D}$ . Then there exist orthonormal vectors  $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}_2^k$  such that in the  
 294  $i$ th factor of  $[\vec{C}_3]_k$ , one vertex has characteristic vector  $\mathbf{0}$  and the other two have  
 295 characteristic vector  $\mathbf{u}_i$ .*

296 We will use the setup that Pouzet, Kaddour, and Thatte [26] introduced in their  
 297 proof that  $\text{inv}(\widehat{D}) = k$ . The first part of our argument is essentially a reformulation  
 298 of theirs, but we include it for completeness and to build intuition.

299 *Proof of Lemma 4.1.* Let  $T$  be the transitive tournament obtained by inverting  
 300 the sets  $X_1, \dots, X_k$  in  $\widehat{D}$ , and let  $<$  be the total order on  $V(\widehat{D})$  associated to  $T$ .  
 301 Note that for all  $i$ , after inverting  $X_1, \dots, X_k$  the  $i$ th factor has one vertex that has  
 302 out-edges to the other two vertices in the factor and exactly one of these edges has

303 undergone a net reversal. Thus we can label the vertices in the  $i$ th factor as  $u_i, v_i, w_i$   
 304 where  $\bar{u}_i v_i \bar{w}_i$  is a directed 3-cycle in  $\bar{D}$ , and the edge between  $u_i$  and  $w_i$  undergoes a  
 305 net reversal under  $X_1, \dots, X_k$  while the edge between  $u_i$  and  $v_i$  does not. In particular,  
 306 we will use throughout that  $u_i < v_i, w_i$  and that, by Observation 2.2,  $\mathbf{u}_i \cdot \mathbf{v}_i = 0$  and  
 307  $\mathbf{u}_i \cdot \mathbf{w}_i = 1$  where  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in \mathbb{F}_2^k$  are the respective characteristic vectors of  $u_i, v_i, w_i$   
 308 in  $X_1, \dots, X_k$ . We have the following claim, originally proved in [26].

309 **CLAIM 4.2** ([26]). *The vectors  $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}_2^k$  are linearly independent.*

310 *Proof.* The statement is equivalent to the claim that for all non-empty  $I \subseteq [k]$   
 311 we have  $\sum_{i \in I} \mathbf{u}_i \neq \mathbf{0}$ . Fix such an  $I$  and note that it is sufficient to show that  
 312 there exists some  $\mathbf{x} \in \mathbb{F}_2^k$  such that  $(\sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{x} \neq 0$ . Let  $m \in I$  be such that  
 313  $u_i < u_m$  for all  $i \in I \setminus \{m\}$ . Note that  $u_m < v_m, w_m$ , so by the transitivity of  $T$   
 314 we have  $u_i < v_m, w_m$  for all  $i \in I$ . It is straightforward to deduce from this that  
 315 for all  $i \in I \setminus \{m\}$ , the orientations of the edges  $u_i v_m$  and  $u_i w_m$  are either both  
 316 unchanged after  $X_1, \dots, X_k$  are inverted, or both reversed. By Observation 2.2, in  
 317 other words we have  $\mathbf{u}_i \cdot \mathbf{v}_m = \mathbf{u}_i \cdot \mathbf{w}_m$  for all  $i \in I \setminus \{m\}$ . On the other hand we  
 318 have  $\mathbf{u}_m \cdot \mathbf{v}_m = 0$  while  $\mathbf{u}_m \cdot \mathbf{w}_m = 1$ , so it follows by linearity of the dot product that  
 319  $(\sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{v}_m \neq (\sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{w}_m$ . One of these two dot products is thus non-zero, and  
 320 we deduce that  $\sum_{i \in I} \mathbf{u}_i \neq \mathbf{0}$ , as required.  $\square$

321 We now build on Claim 4.2 as follows.

322 **CLAIM 4.3.** *Let  $\ell \in [k]$  and suppose that the vectors  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$  for  $\ell \leq i \leq k$  all lie  
 323 in a subspace  $V$  of  $\mathbb{F}_2^k$  of dimension  $k - \ell + 1$ . Then  $\mathbf{u}_\ell, \dots, \mathbf{u}_k$  are orthonormal, and  
 324 for all  $\ell \leq i \leq k$  we have  $\mathbf{u}_i = \mathbf{w}_i$  and  $\mathbf{v}_i = \mathbf{0}$ .*

325 *Proof.* We will prove the claim by reverse induction on  $\ell$ . In the  $\ell = k$  case  
 326 the claim follows easily from the fact that  $\mathbf{u}_k \cdot \mathbf{w}_k = 1$  while  $\mathbf{u}_k \cdot \mathbf{v}_k = 0$ . Thus,  
 327 let  $\ell \leq k - 1$  and write  $[\ell, k]$  for  $\{\ell, \ell + 1, \dots, k\}$ . Let  $z$  be the  $<$ -minimal vertex  
 328 among  $v_\ell, \dots, v_k, w_\ell, \dots, w_k$ . Write  $\mathbf{z} \in V \subseteq \mathbb{F}_2^k$  for the characteristic vector of  $z$  in  
 329  $X_1, \dots, X_k$  and let  $t \in [\ell, k]$  be such that  $z \in \{v_t, w_t\}$ . By Claim 4.2, the vectors  
 330  $\mathbf{u}_\ell, \dots, \mathbf{u}_k$  form a basis of  $V$  so there exists  $I \subseteq [\ell, k]$  such that  $\mathbf{z} + \sum_{i \in I} \mathbf{u}_i = \mathbf{0}$ .

331 First suppose that  $I \notin \{\emptyset, \{t\}\}$  and let  $m \in I$  be such that  $u_i < u_m$  for all  
 332  $i \in I \setminus \{m\}$ . If  $m \neq t$ , then we have  $z < v_m, w_m$ , so  $\mathbf{z} \cdot \mathbf{v}_m = \mathbf{z} \cdot \mathbf{w}_m$  by Observation 2.2.  
 333 As in the proof of Claim 4.2, we have  $\mathbf{u}_i \cdot \mathbf{v}_m = \mathbf{u}_i \cdot \mathbf{w}_m$  for all  $i \in I \setminus \{m\}$ , but  $\mathbf{u}_m \cdot \mathbf{v}_m \neq$   
 334  $\mathbf{u}_m \cdot \mathbf{w}_m$ , so  $(\mathbf{z} + \sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{v}_m \neq (\mathbf{z} + \sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{w}_m$ , and hence  $\mathbf{z} + \sum_{i \in I} \mathbf{u}_i \neq \mathbf{0}$ . If  $m = t$ ,  
 335 then let  $j \in I \setminus \{t\}$  and note that  $z < v_j, w_j$  by the minimality of  $z$ . Consequently,  
 336  $\mathbf{z} \cdot \mathbf{v}_j = \mathbf{z} \cdot \mathbf{w}_j$ . Moreover, since  $u_m = u_t < z$ , we have  $u_m < v_j, w_j$ . From this it follows  
 337 that  $u_i < v_j, w_j$  for all  $i \in I$ . Thus,  $\mathbf{u}_i \cdot \mathbf{v}_j = \mathbf{u}_i \cdot \mathbf{w}_j$  for all  $i \in I \setminus \{j\}$ , while  $\mathbf{u}_j \cdot \mathbf{v}_j \neq$   
 338  $\mathbf{u}_j \cdot \mathbf{w}_j$ . Hence, similarly to above, we have  $(\mathbf{z} + \sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{v}_j \neq (\mathbf{z} + \sum_{i \in I} \mathbf{u}_i) \cdot \mathbf{w}_j$ ,  
 339 so  $\mathbf{z} + \sum_{i \in I} \mathbf{u}_i \neq \mathbf{0}$ .

340 The remaining cases are  $I = \emptyset$  and  $I = \{t\}$ , so we have  $\mathbf{z} \in \{\mathbf{0}, \mathbf{u}_t\}$ . Suppose that  
 341  $\mathbf{z} = \mathbf{u}_t$ . If  $z = v_t$ , then we have  $\mathbf{v}_t = \mathbf{z} = \mathbf{u}_t$ , so  $\mathbf{v}_t \cdot \mathbf{w}_t = \mathbf{u}_t \cdot \mathbf{w}_t = 1$ , i.e. the edge  
 342 between  $v_t$  and  $w_t$  undergoes a net reversal under  $X_1, \dots, X_k$ . This would imply that  
 343  $w_t < v_t = z$ , which contradicts the minimality of  $z$ . Similarly, if  $z = w_t$ , then since  
 344 the edge between  $u_t$  and  $v_t$  is not inverted, neither is the edge between  $w_t$  and  $v_t$ , so  
 345  $v_t < w_t = z$ , another contradiction. Therefore  $\mathbf{z} = \mathbf{0}$ . This means no edges incident  
 346 to  $z$  are reversed when  $X_1, \dots, X_k$  are inverted so by the minimality of  $z$  we have  
 347  $z = v_\ell$ .

348 We have shown that  $\mathbf{v}_\ell = \mathbf{0}$ , so the only vertex among the  $u_i, v_i, w_i$  with  $i \geq \ell$   
 349 which precedes  $v_\ell$  in  $<$  is  $u_\ell$ . It follows that  $u_\ell$  is the least element among the  $u_i, v_i, w_i$   
 350 with  $i \geq \ell$ . Hence, by Observation 2.2 we have  $\mathbf{u}_\ell \cdot \mathbf{u}_i = \mathbf{u}_\ell \cdot \mathbf{v}_i = \mathbf{u}_\ell \cdot \mathbf{w}_i = 0$  for

351 all  $i \geq \ell + 1$ , so if  $V'$  is the subspace of  $V$  spanned by the  $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$  with  $i \geq \ell + 1$ ,  
 352 then  $\mathbf{u}_\ell \cdot \mathbf{x} = 0$  for all  $\mathbf{x} \in V'$ . We have  $\mathbf{u}_\ell \cdot \mathbf{w}_\ell = 1$ , so  $V'$  is a proper subspace of  $V$ ,  
 353 but  $\mathbf{u}_{\ell+1}, \dots, \mathbf{u}_k \in V'$  are linearly independent, so we deduce that  $V'$  has dimension  
 354  $k - \ell$ . Therefore by the induction hypothesis  $\mathbf{u}_{\ell+1}, \dots, \mathbf{u}_k$  are orthonormal, and we  
 355 have  $\mathbf{u}_i = \mathbf{w}_i$  and  $\mathbf{v}_i = \mathbf{0}$  for  $i \geq \ell + 1$ .

356 To complete the induction step it remains to show that  $\mathbf{u}_\ell = \mathbf{w}_\ell$  and  $\mathbf{u}_\ell \cdot \mathbf{u}_\ell = 1$ .  
 357 The latter follows from the fact that  $\mathbf{u}_\ell, \dots, \mathbf{u}_k$  is a basis for  $V$  with  $\mathbf{u}_\ell \cdot \mathbf{u}_i = 0$  for  
 358 all  $i \geq \ell + 1$ , but  $\mathbf{w}_\ell \in V$  has  $\mathbf{u}_\ell \cdot \mathbf{w}_\ell = 1$ . For the former, note that  $\mathbf{w}_\ell = \sum_{i \in I} \mathbf{u}_i$  for  
 359 some  $I \subseteq [\ell, k]$  and by the established properties of the  $\mathbf{u}_i$  this set  $I$  contains exactly  
 360 those  $i$  for which  $\mathbf{u}_i \cdot \mathbf{w}_\ell = 1$ . Thus, we certainly have  $\ell \in I$ . Suppose that  $\mathbf{u}_i \cdot \mathbf{w}_\ell = 1$   
 361 for some  $i \geq \ell + 1$ . Since  $\mathbf{w}_i = \mathbf{u}_i$  and  $\mathbf{v}_i = \mathbf{0}$ , by Observation 2.2 we find that the  
 362 cycle  $\overrightarrow{w_\ell v_i w_i}$  appears in  $T$ , which is a contradiction. Hence  $I = \{\ell\}$  and  $\mathbf{u}_\ell = \mathbf{w}_\ell$ , as  
 363 required.  $\square$

364 The lemma now follows from the  $\ell = 1$  case of Claim 4.3.  $\square$

365 We will now deduce Theorem 1.5 from the lemma. In the proof, we will use the  
 366 easy fact that every family of orthonormal vectors in  $\mathbb{F}_2^k$  is linearly independent.

367 *Proof of Theorem 1.5.* The sufficiency of the given conditions for  $X_1, \dots, X_k$  to  
 368 be a decycling family of  $\widehat{D}$  is straightforward to verify using Observation 2.2. This  
 369 observation also allows the ‘in particular’ part of the theorem statement to be easily  
 370 deduced from the preceding part. It remains to prove that the given conditions are  
 371 necessary.

372 Given a decycling family  $X_1, \dots, X_k$  of  $\widehat{D}$ , extend  $\widehat{D}$  to a tournament  $T$  for which  
 373  $X_1, \dots, X_k$  is still a decycling family. For each  $i$ , let  $T_i$  be the subtournament of  $T$   
 374 induced on the vertex set of the  $i$ th factor of  $\widehat{D}$ . Since  $D_i$  contains a directed cycle,  
 375 so does  $T_i$ , and hence the latter contains a copy of  $\overrightarrow{C_3}$ . We can thus find a copy of  
 376  $[\overrightarrow{C_3}]_k$  in  $T$  whose  $i$ th factor is contained in  $T_i$ . It follows by Lemma 4.1 that there are  
 377 orthonormal vectors  $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{F}_2^k$  and for each  $i$  a triangle  $\overrightarrow{u_i v_i w_i}$  in  $T_i$  such that  $u_i$   
 378 and  $w_i$  have characteristic vector  $\mathbf{u}_i$  and  $v_i$  has characteristic vector  $\mathbf{0}$  in  $X_1, \dots, X_k$ .

379 We next show that for all  $i$ , all vertices in  $T_i$  have characteristic vector either  
 380  $\mathbf{u}_i$  or  $\mathbf{0}$  in  $X_1, \dots, X_k$ . Let  $z \in V(T_i)$  and let  $\mathbf{z}$  be its characteristic vector. Since  
 381  $\mathbf{u}_1, \dots, \mathbf{u}_k$  form a basis of  $\mathbb{F}_2^k$ , there exists  $J \subseteq [k]$  such that  $\mathbf{z} = \sum_{j \in J} \mathbf{u}_j$ . If there  
 382 exists  $\ell \in J \setminus \{i\}$ , then  $\mathbf{z} \cdot \mathbf{u}_\ell = \mathbf{u}_\ell \cdot \mathbf{u}_\ell = 1$  and hence the directions of the edge between  
 383  $z$  and  $u_\ell$  and the edge between  $z$  and  $w_\ell$  are reversed under  $X_1, \dots, X_k$ . If  $\ell < i$ , then  
 384 the cycle  $\overrightarrow{u_\ell v_\ell z}$  appears in  $T$  and if  $i < \ell$ , then the cycle  $\overrightarrow{z v_\ell w_\ell}$  appears in  $T$ . We have  
 385 a contradiction in both cases, so  $J = \emptyset$  or  $J = \{i\}$  as desired.

386 We have shown that all vertices in the  $i$ th factor of  $\widehat{D}$  have characteristic vector  
 387 either  $\mathbf{u}_i$  or  $\mathbf{0}$  in  $X_1, \dots, X_k$ . The effect on this copy of  $D_i$  of inverting these sets in  
 388  $\widehat{D}$  is therefore the same as inverting the set of vertices with characteristic vector  $\mathbf{u}_i$ ,  
 389 which we call  $V_i$ . The latter is therefore a decycling set for the  $i$ th factor of  $\widehat{D}$ . This  
 390 completes the proof of the theorem.  $\square$

391 Theorem 1.4 now follows easily.

392 *Proof of Theorem 1.4.* It is clear that the left-hand side of equation (1.1) is at  
 393 most the right-hand side. For the reverse inequality, let  $\widehat{D} = [D_1, \dots, D_k]$  and  
 394 note that we may assume that none of the  $D_i$  have inversion number 0. Indeed,  
 395 if  $\text{inv}(D_i) = 0$  for some  $i \geq 2$ , then view  $\widehat{D}$  as the  $(k-1)$ -join  $[D_1, \dots, D_{i-2}, D_{i-1} \rightarrow$   
 396  $D_i, D_{i+1}, \dots, D_k]$  and, since  $\text{inv}(D_{i-1} \rightarrow D_i) = \text{inv}(D_{i-1})$ , the result follows by in-  
 397 duction on  $k$ . The case where  $i = 1$  can be handled similarly.

398 Thus, consider the case where  $\text{inv}(D_i) = 1$  for all  $i$  and suppose for a contradiction

399 that  $X_1, \dots, X_k$  is a decycling family of  $\widehat{D}$  with  $X_k = \emptyset$ . By Theorem 1.5 there exist  
 400  $k$  orthonormal, and hence linearly independent, vectors in  $\mathbb{F}_2^k$  each of which occurs as  
 401 the characteristic vector of some vertex of  $\widehat{D}$  in  $X_1, \dots, X_k$ . This contradicts the fact  
 402 that all such vectors have a 0 in their final coordinate. Hence, in this case,  $\text{inv}(\widehat{D}) = k$ .

403 It remains to check the case where  $\text{inv}(D_j) = 2$  for some  $j$  and  $\text{inv}(D_i) = 1$  for  
 404 all  $i \neq j$ . Start by letting  $D'_j$  be a spanning subdigraph of the  $j$ th factor of  $\widehat{D}$  with  
 405  $\text{inv}(D'_j) = 1$ , then define  $\widehat{D}'$  to be the digraph obtained by replacing the  $j$ th factor of  
 406  $\widehat{D}$  by  $D'_j$ . Assume for a contradiction that  $X_1, \dots, X_k$  is a decycling family of  $\widehat{D}$ , in  
 407 which case it is also a decycling family of  $\widehat{D}'$ . Theorem 1.5 thus yields a vector  $\mathbf{u}_j \in \mathbb{F}_2^k$   
 408 with  $\mathbf{u}_j \cdot \mathbf{u}_j = 1$  such that all the vertices in the  $j$ th factor of  $\widehat{D}'$  (and hence also the  
 409  $j$ th factor of  $\widehat{D}$ ) have characteristic vector either  $\mathbf{0}$  or  $\mathbf{u}_j$  in  $X_1, \dots, X_k$ . Inverting  
 410  $X_1, \dots, X_k$  in  $\widehat{D}$  therefore has the same effect on its  $j$ th factor as inverting the set of  
 411 vertices with characteristic vector  $\mathbf{u}_j$ . It follows that this set of vertices is a decycling  
 412 set for  $D_j$ , contradicting  $\text{inv}(D_j) = 2$ .  $\square$

413 As mentioned in the introduction, it follows from Theorem 1.4 that for any di-  
 414 graph  $D$  we have  $\text{inv}(D) = 1$  if and only if  $\text{inv}([D]_k) = k$ , which in turn implies  
 415 Theorem 1.6 (which states that  $k$ -INVERSION is NP-complete for all  $k \in \mathbb{N}$ ). Indeed,  
 416 Theorem 1.4 directly gives  $\text{inv}([D]_k) = k$  in the case  $\text{inv}(D) = 1$ , and if  $\text{inv}(D) = 0$   
 417 then clearly  $\text{inv}([D]_k) = 0$ . If  $\text{inv}(D) > 1$ , then there are subdigraphs  $D'$  and  $D''$  of  
 418  $D$  with  $\text{inv}(D') = 1$  and  $\text{inv}(D'') = 2$ . The  $k$ -join  $D'' \rightarrow [D']_{k-1}$ , which has inversion  
 419 number  $k+1$  by Theorem 1.4, is a subdigraph of  $[D]_k$  and thus  $\text{inv}([D]_k) \geq k+1$  as  
 420 required.

421 Finally, we deduce Theorem 1.3 (which states that  $\text{inv}(L \rightarrow R) = 4$  for all di-  
 422 graphs  $L$  and  $R$  with inversion number 2) from Theorem 1.4. We will use the fact,  
 423 shown in [4], that if  $L$  and  $R$  are strongly connected digraphs with  $\text{inv}(L), \text{inv}(R) \geq 2$ ,  
 424 then  $\text{inv}(L \rightarrow R) \geq 4$ .

425 *Proof of Theorem 1.3.* Let  $L$  and  $R$  be digraphs with  $\text{inv}(L) = \text{inv}(R) = 2$ . It  
 426 is immediate that  $\text{inv}(L \rightarrow R) \leq 4$ , so it is sufficient to prove the lower bound. For  
 427 this, extend  $L \rightarrow R$  to a tournament  $T$  of the same inversion number and let the  
 428 tournaments to which  $L$  and  $R$  are extended be  $L'$  and  $R'$  respectively. Note that  
 429  $\text{inv}(L'), \text{inv}(R') \geq 2$  and  $T$  is  $L' \rightarrow R'$ .

430 Every tournament can be written as the  $k$ -join of its strongly connected com-  
 431 ponents, so let  $L'$  be  $[L_1, \dots, L_{k_1}]$  and  $R'$  be  $[R_1, \dots, R_{k_2}]$  for some  $k_1, k_2 \in \mathbb{N}$  and  
 432 strongly connected tournaments  $L_1, \dots, L_{k_1}, R_1, \dots, R_{k_2}$ . Since  $\text{inv}(L') \geq 2$ , either  
 433 there is some  $L_i$  with  $\text{inv}(L_i) \geq 2$ , or there are  $i < j$  such that  $\text{inv}(L_i) = \text{inv}(L_j) = 1$ .  
 434 An analogous condition holds for  $R'$ . If there are  $i$  and  $j$  such that  $\text{inv}(L_i), \text{inv}(R_j) \geq$   
 435 2, then since  $T$  contains  $L_i \rightarrow R_j$ , we have  $\text{inv}(T) \geq \text{inv}(L_i \rightarrow R_j) \geq 4$  by the above  
 436 result of [4]. Otherwise, either there exist  $i < j$  such that  $\text{inv}(L_i) = \text{inv}(L_j) = 1$ , in  
 437 which case  $\text{inv}(T) \geq \text{inv}([L_i, L_j, R]) = 4$  by Theorem 1.4, or there exist  $i < j$  with  
 438  $\text{inv}(R_i) = \text{inv}(R_j) = 1$ , in which case it follows similarly that  $\text{inv}(T) \geq 4$ .  $\square$

439 **5. Complexity of  $k$ -Tournament-Inversion.** In this section we prove The-  
 440 rem 1.7 by constructing, for each fixed  $k \in \mathbb{N}$ , an algorithm solving  $k$ -TOURNAMENT-  
 441 INVERSION in time  $O(|V(T)|^2)$ . Our proof uses a technique known as iterative com-  
 442 pression; see [14] for a description of this method and other applications of it. The  
 443 most involved part of our proof concerns the ‘compression step’ of the algorithm.  
 444 This step is handled by the following lemma, which roughly says that for constant  $k$ ,  
 445 given an  $n$ -vertex tournament  $T_0$  and a decycling family of  $T_0$  of constant size, in time

446 linear in  $n$  we can find a decycling family of  $T_0$  of size  $k$  if one exists. Throughout  
 447 this section, we represent a total order  $<$  on a finite set  $S = \{s_1, \dots, s_m\}$  by the tuple  
 448  $(s_1, \dots, s_m)$  where  $s_1 < \dots < s_m$ .

449 **LEMMA 5.1.** *Fix  $k, s \in \mathbb{N}$ . There is an algorithm which solves the following prob-  
 450 lem for  $n$ -vertex tournaments in time  $O(n)$ :*

INPUTS:

- a tournament  $T_0$ ;
- a decycling family  $X_1, \dots, X_s$  of  $T_0$  (transforming  $T_0$  into  $T$ , say);
- the order on  $V(T_0)$  associated to  $T$ .

OUTPUTS:

EITHER

- that  $T_0$  is not  $k$ -invertible;

OR

- a decycling family  $Y_1, \dots, Y_k$  of  $T_0$  (transforming  $T_0$  into  $T'$ , say);
- the order on  $V(T_0)$  associated to  $T'$ .

451

452 We now use iterative compression to prove Theorem 1.7 before returning to  
 453 Lemma 5.1.

454 *Proof of Theorem 1.7.* Fix  $k \geq 1$ . We will induct on  $n$  to define an algorithm solv-  
 455 ing the following problem for  $n$ -vertex tournaments in time  $C_k \cdot n^2$  for some constant  
 456  $C_k$ :

INPUT:

- a tournament  $T_0$ .

OUTPUTS:

EITHER

- that  $T_0$  is not  $k$ -invertible;

OR

- a decycling family  $Y_1, \dots, Y_k$  of  $T_0$  (transforming  $T_0$  into  $T$ , say);
- the order on  $V(T_0)$  associated to  $T$ .

457

458 In particular, this algorithm solves  $k$ -TOURNAMENT-INVERSION.

459 Fix  $n \geq 2$  and assume that we have defined such an algorithm for all smaller  
 460 tournaments. Let  $T_0$  be an  $n$ -vertex tournament and pick some  $v \in V(T_0)$ . Applying  
 461 the induction hypothesis, in time  $C_k \cdot (n-1)^2$  we either find that  $T_0 - \{v\}$  is not  
 462  $k$ -invertible or we obtain a decycling family  $X_1, \dots, X_k$  of  $T_0 - \{v\}$  and the order on  
 463  $V(T_0) \setminus \{v\}$  associated to the transitive tournament obtained by inverting these sets  
 464 in  $T_0$ . In the former case, it follows that  $T_0$  is also not  $k$ -invertible and we can output  
 465 that fact. In the latter case, let  $A$  be the out-neighbourhood of  $v$  in  $T_0$ , and define  
 466  $X_{k+1} = A \cup \{v\}$  and  $X_{k+2} = A$ . Then  $X_1, \dots, X_{k+2}$  is a decycling family of  $T_0$ , and we  
 467 can obtain the order associated to the resulting transitive tournament by adding  $v$  to  
 468 the previous order as the maximal element. By Lemma 5.1 we can now, in linear time,  
 469 either find that  $T_0$  is not  $k$ -invertible or obtain a decycling family  $Y_1, \dots, Y_k$  of  $T_0$   
 470 of size  $k$  and the order associated to the transitive tournament obtained by inverting  
 471 these sets in  $T_0$ . As required, this algorithm runs in time  $C_k \cdot (n-1)^2 + O(n)$ , which  
 472 is at most  $C_k \cdot n^2$  if  $C_k$  is large enough.  $\square$

473 It is left to prove Lemma 5.1. To this end, we describe an algorithm which explores  
 474 what happens if, starting from  $T$ , we invert  $X_1, \dots, X_s$  and  $k$  further sets  $Y_1, \dots, Y_k$   
 475 to obtain a tournament  $T_Y$ , where  $Y = (Y_1, \dots, Y_k)$ . Since  $T_Y$  is the tournament  
 476 obtained by inverting  $Y_1, \dots, Y_k$  in  $T_0$ , these  $k$  sets are a decycling family of  $T_0$  if  
 477 and only if  $T_Y$  is transitive. If we were to examine each possibility individually there  
 478 would be too many for this exploration process to be tractable. However, the fact that  
 479 we are starting from a transitive tournament  $T$  makes it possible to identify cycles  
 480 in the final tournament  $T_Y$  without fully specifying the sets  $Y_1, \dots, Y_k$ . This means  
 481 there are far fewer cases to consider, indeed few enough that the exploration process  
 482 is linear in  $n$  for fixed  $k$  and  $s$ .

483 *Proof of Lemma 5.1.* Fix  $k, s \in \mathbb{N}$  and let  $T_0, X_1, \dots, X_s$ , and  $T$  be as in the  
 484 statement of the lemma. Let  $n = |V(T_0)|$  and label the vertices of  $T_0$  as  $u_1, \dots, u_n$  in  
 485  $T$ -increasing order. With notation as above, we wish to investigate for which  $Y$  the  
 486 tournament  $T_Y$  is transitive. For each  $Y$  we write  $\mathbf{u}_i \in \mathbb{F}_2^{s+k}$  for the characteristic vector  
 487 of  $u_i$  in  $X_1, \dots, X_s, Y_1, \dots, Y_k$  (suppressing the dependence on  $Y$  in the notation)  
 488 and then let  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ . There is a bijective correspondence between  $Y$  and  $\mathbf{u}$   
 489 and it will be more convenient to work with the latter, so let  $T_{\mathbf{u}} = T_Y$  and write  $\mathcal{U}$   
 490 for the set of all possible  $\mathbf{u}$ . Our first aim is to determine in linear time whether there  
 491 exists  $\mathbf{u} \in \mathcal{U}$  such that  $T_{\mathbf{u}}$  is transitive, and to identify such a  $\mathbf{u}$  if so.

492 The tournament  $T_{\mathbf{u}}$  is transitive exactly when it contains no cyclic triples. It is  
 493 straightforward to use Observation 2.2 to show that this is equivalent to the condition  
 494 that there are no  $a < b < c$  in  $[n]$  such that  $\mathbf{u}_a \cdot \mathbf{u}_b = \mathbf{u}_b \cdot \mathbf{u}_c$  but  $\mathbf{u}_a \cdot \mathbf{u}_b \neq \mathbf{u}_a \cdot \mathbf{u}_c$ .  
 495 We describe the triple  $(\mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c)$  as *bad* if this occurs. Thus,  $T_{\mathbf{u}}$  is transitive if and  
 496 only if  $B(\mathbf{u}) = \{(\mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c) : a < b < c\}$  contains no bad triples, and  $T_0$  is  $k$ -invertible  
 497 if and only if  $\mathcal{B} = \{B(\mathbf{u}) : \mathbf{u} \in \mathcal{U}\}$  contains a set which is free of bad triples. Our  
 498 algorithm will construct this set  $\mathcal{B}$  and check whether any of its elements are free of  
 499 bad triples. If one of these sets is free of bad triples, then we need to be able to  
 500 output a corresponding decycling family of  $T_0$ , so for each  $B \in \mathcal{B}$  we will also record  
 501 some  $\mathbf{u} \in \mathcal{U}$  for which  $B = B(\mathbf{u})$ .

502 We will now explain how the above can be achieved in linear time. First note that  
 503 we may assume that  $n \geq 4$ . Let  $\mathcal{U}'$  be the set of all possible vectors  $\mathbf{u}' = (\mathbf{u}_1, \dots, \mathbf{u}_{n-1})$   
 504 of characteristic vectors of  $u_1, \dots, u_{n-1}$  in  $X_1, \dots, X_s, Y_1, \dots, Y_k$ . For  $\mathbf{u}' \in \mathcal{U}'$ , let  
 505  $B'(\mathbf{u}') = \{(\mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c) : 1 \leq a < b < c \leq n-1\}$  and let  $\mathcal{B}' = \{B'(\mathbf{u}') : \mathbf{u}' \in \mathcal{U}'\}$ . We  
 506 may assume inductively that there is a constant  $C$  depending only on  $k$  and  $s$  such  
 507 that in time  $C \cdot (n-1)$  we can construct  $\mathcal{B}'$  and associate to each  $B' \in \mathcal{B}'$  some  $\mathbf{u}' \in \mathcal{U}'$   
 508 such that  $B' = B'(\mathbf{u}')$ . For the induction step, we need to show that we can use this  
 509 to obtain in time  $C$  the set  $\mathcal{B}$  and for each  $B \in \mathcal{B}$  some  $\mathbf{u} \in \mathcal{U}$  such that  $B = B(\mathbf{u})$ .

510 The key observation is that there are only  $2^{s+k}$  possible characteristic vectors for  
 511 each of  $u_1, \dots, u_n$ , so the number of triples of characteristic vectors is at most  $2^{3(s+k)}$   
 512 and the sizes of  $\mathcal{B}$  and  $\mathcal{B}'$  are at most  $2^{2^{3(s+k)}}$ . In particular, there are only boundedly  
 513 many pairs  $(B', \mathbf{u}_n)$  where  $B' \in \mathcal{B}'$  and  $\mathbf{u}_n$  is a possible characteristic vector for  $u_n$ .  
 514 For each such pair, we can construct in bounded time the set  $S(B', \mathbf{u}_n)$  consisting  
 515 of all triples in  $B'$ , and all triples of the form  $(\mathbf{v}_i, \mathbf{v}_j, \mathbf{u}_n)$  for  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in B'$  and  
 516  $1 \leq i < j \leq 3$ . It is not hard to see that  $\mathcal{B}$  equals the set of all sets  $S(B', \mathbf{u}_n)$  and  
 517 that each  $S(B', \mathbf{u}_n)$  can be associated with the  $\mathbf{u} \in \mathcal{U}$  formed by appending  $\mathbf{u}_n$  to the  
 518  $\mathbf{u}' \in \mathcal{U}'$  associated with  $B'$ . Indeed, given  $B'$  and  $\mathbf{u}_n$  and defining  $\mathbf{u}$  as in the previous  
 519 sentence, since  $n \geq 4$ , we have  $S(B', \mathbf{u}_n) = B(\mathbf{u})$ . For the other direction, given  
 520  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_n) \in \mathcal{U}$  and letting  $\mathbf{u}' = (\mathbf{u}_1, \dots, \mathbf{u}_{n-1})$ , we have  $B(\mathbf{u}) = S(B'(\mathbf{u}'), \mathbf{u}_n)$ .

521 We can construct this set in bounded time and then forget about all but one of

522 the elements of  $\mathcal{U}$  associated to each  $B \in \mathcal{B}$ . Thus, in constant time we have obtained  
 523  $\mathcal{B}$  and for each  $B \in \mathcal{B}$  some  $\mathbf{u} \in \mathcal{U}$  such that  $B = B(\mathbf{u})$ , and the induction continues.

524 Once we have constructed  $\mathcal{B}$  in linear time, since it has bounded size we can check  
 525 whether any of its members is free of bad triples in bounded time. If not, then  $T_0$  is  
 526 not  $k$ -invertible. If so, then pick  $B \in \mathcal{B}$  with no bad triples and use the  $\mathbf{u}$  associated to  
 527 it to construct a decycling family  $Y_1, \dots, Y_k$  of  $T_0$ . Let  $T'$  be the transitive tournament  
 528 obtained by inverting these sets in  $T_0$ .

529 It remains to show that we can obtain the order on  $V(T_0)$  associated to  $T'$  in  
 530 linear time. Inverting the sets  $X_1, \dots, X_s, Y_1, \dots, Y_k$  transforms  $T$  into  $T'$ , and we  
 531 have the characteristic vector of each vertex in these sets as well as the order on the  
 532 vertices associated to  $T$ . We can therefore in linear time obtain the atoms of these  
 533  $s+k$  inversions and for each atom  $A$  the restriction to  $A$  of the order associated to  
 534  $T$ . By reversing the order on each atom whenever the edges within it undergo a net  
 535 reversal under the inversions, we obtain the order on that atom associated to  $T'$ . The  
 536  $T'$ -minimal vertex is now the minimal vertex of one of the atoms under their current  
 537 orderings. There are at most  $2^{s+k}$  atoms so we can identify the  $T'$ -minimal vertex in  
 538 constant time. After deleting this vertex from its atom, the second smallest vertex  
 539 according to  $T'$  is one of the new minimal vertices of the atoms so can be found in  
 540 constant time again. Continuing in this way we can obtain the full ordering in linear  
 541 time, as required.  $\square$

542 Note that the implicit constant in the running time given by this proof is doubly  
 543 exponential in  $s+k$ .

544 **6. Cycle transversals.** In this section we will prove Theorem 1.8, constructing  
 545 for each  $k \in \mathbb{N}$  a digraph  $D$  with  $\tau(D) = k$  and  $\text{inv}(D) = 2k$ . We will use the so-called  
 546 Eventown theorem, proved independently by Berlekamp [8] and Graver [17].

547 **THEOREM 6.1** (Eventown [8], [17]). *Let  $n \in \mathbb{N}$  and let  $\mathcal{F} \subseteq \mathcal{P}([n])$  be a family of  
 548 subsets of  $[n]$  such that  $|F_1 \cap F_2|$  is even for all  $F_1, F_2 \in \mathcal{F}$ . Then  $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$ .*

549 For a digraph  $D$  and vertices  $u, v, w \in V(D)$ , we will say that  $u$  and  $v$  differ on  $w$   
 550 if either  $\overrightarrow{uw}, \overrightarrow{vw} \in E(D)$  or  $\overrightarrow{vw}, \overrightarrow{wu} \in E(D)$ . We are now ready to prove the theorem.

551 *Proof of Theorem 1.8.* Fix  $k \in \mathbb{N}$  and let  $n \in \mathbb{N}$  be large and divisible by  $2^k$ .  
 552 We will define a digraph  $D$  on vertex set  $\{u_0, \dots, u_{k-1}, v_0, \dots, v_{n-1}\}$  and then show  
 553 that it satisfies the conditions of the theorem. Start by including all directed edges  
 554  $\overrightarrow{v_i v_j}$  for  $i < j$ , so that the subdigraph of  $D$  induced on  $\{v_0, \dots, v_{n-1}\}$  is a transitive  
 555 tournament. For  $i \in \{0, \dots, k-1\}$  and  $j \in \{0, \dots, n-1\}$ , add the edge  $\overrightarrow{u_i v_j}$  if in the  
 556 binary expansion of  $j$ , the digit in the  $2^i$  place is a 0, and add the edge  $\overrightarrow{v_j u_i}$  otherwise.  
 557 For ease of exposition we will not include any edges among the  $u_i$  (though including  
 558 any combination of such edges would still give a valid construction), so this completes  
 559 the definition of  $D$ . As noted above, the removal of the vertices  $u_0, \dots, u_{k-1}$  from  $D$   
 560 leaves an acyclic digraph, so  $\tau(D) \leq k$ .

561 It remains to show that  $\text{inv}(D) \geq 2k$ , as then  $\text{inv}(D) = 2\tau(D) = 2k$  follows from  
 562  $\text{inv}(D) \leq 2\tau(D)$ . Suppose for a contradiction that  $X_1, \dots, X_{2k-1} \subseteq V(D)$  form a  
 563 decycling family of  $D$  and let  $D'$  be the acyclic digraph obtained by inverting these  
 564 sets in  $D$ . Consider the characteristic vectors of  $v_0, \dots, v_{n-1}$  in  $X_1, \dots, X_{2k-1}$ , which  
 565 we will denote by  $\mathbf{v}_0, \dots, \mathbf{v}_{n-1} \in \mathbb{F}_2^{2k-1}$  respectively. Let  $K = 2^k$ . By the pigeonhole  
 566 principle, if  $n$  is large enough, then there exist distinct  $i, i' \in \{0, \dots, n/K-1\}$  such  
 567 that

$$(v_{iK}, v_{iK+1}, \dots, v_{(i+1)K-1}) = (v_{i'K}, v_{i'K+1}, \dots, v_{(i'+1)K-1}).$$

568 We may assume that  $i = 0$  and  $i' = 1$ .

We will show that  $\mathbf{v}_0, \dots, \mathbf{v}_{K-1}$  are pairwise distinct and that  $\mathbf{v}_i \cdot \mathbf{v}_j$  is constant as  $i, j \in \{0, \dots, K-1\}$  vary. We claim that these conditions force a contradiction. Indeed, in the case where  $\mathbf{v}_i \cdot \mathbf{v}_j = 0$  for all  $i, j$ , we have that the  $\mathbf{v}_i$  are indicator vectors of pairwise distinct subsets of  $[2k-1]$  which each have even size, and each pair of which have even intersection. By Eventown, every such collection has at most  $2^{(2k-1)/2} < 2^k = K$  members, giving the required contradiction. On the other hand, if  $\mathbf{v}_i \cdot \mathbf{v}_j = 1$  for all  $i, j$ , then consider the ‘complement’ vectors  $\mathbf{w}_0, \dots, \mathbf{w}_{K-1}$ , which have 1’s where the  $\mathbf{v}_i$  have 0’s and 0’s where the  $\mathbf{v}_i$  have 1’s. It is straightforward to use the fact that the vectors have odd length to show that these  $\mathbf{w}_i$  are pairwise distinct and satisfy  $\mathbf{w}_i \cdot \mathbf{w}_j = 0$  for all  $i, j$ , from which we can derive a contradiction as above.

We continue by showing that the vectors  $\mathbf{v}_0, \dots, \mathbf{v}_{K-1}$  are pairwise distinct, which is equivalent to showing that each of  $v_0, \dots, v_{K-1}$  is in a different atom. Suppose for a contradiction that  $v_i$  and  $v_j$  are in the same atom for some  $i < j$  in  $\{0, \dots, K-1\}$ , and note that  $v_{K+i}$  and  $v_{K+j}$  are in this atom too by assumption. By the construction of  $D$  there is some  $\ell \in \{0, \dots, k-1\}$  such that  $v_i$  and  $v_j$  differ on  $u_\ell$  in  $D$ . Since they are in the same atom as each other, they also differ on  $u_\ell$  in  $D'$ . If  $\overrightarrow{v_i u_\ell}, \overrightarrow{u_\ell v_j} \in E(D')$ , then to avoid a cyclic triple in  $D'$  we have  $\overrightarrow{v_i v_j} \in E(D')$ . This means that the edges within  $v_i$  and  $v_j$ ’s atom have the same orientations in  $D$  as they do in  $D'$ , so in particular we have  $\overrightarrow{v_j v_{K+i}} \in E(D')$ . Moreover  $v_i$  and  $v_{K+i}$  are in the same atom and are either both in-neighbours of  $u_\ell$  in  $D$  or both out-neighbours, so since  $\overrightarrow{v_i u_\ell} \in E(D')$  we also have  $\overrightarrow{v_{K+i} u_\ell} \in E(D')$ . Hence, the cycle  $\overrightarrow{v_j v_{K+i} u_\ell}$  appears in  $D'$ . Similarly if  $\overrightarrow{v_j u_\ell}, \overrightarrow{u_\ell v_i} \in E(D')$ , then we have  $\overrightarrow{v_j v_i} \in E(D')$ . In this case the edges within the atom of  $v_i$  and  $v_j$  switch orientation between  $D$  and  $D'$ , so the cycle  $\overrightarrow{v_j u_\ell v_{K+i}}$  appears in  $D'$ . In both cases we have the desired contradiction, and we deduce that the vertices  $v_0, \dots, v_{K-1}$  are all in different atoms.

It remains to show that  $\mathbf{v}_i \cdot \mathbf{v}_j$  is constant as  $i, j \in \{0, \dots, K-1\}$  vary. Suppose for a contradiction that this is not the case, then there exists  $i \in \{0, \dots, K-1\}$  such that  $\mathbf{v}_i \cdot \mathbf{v}_j$  is not constant as  $j \in \{0, \dots, K-1\}$  varies. For such  $i$  we can pick  $j \in \{0, \dots, K-1\}$  such that  $\mathbf{v}_i \cdot \mathbf{v}_i \neq \mathbf{v}_i \cdot \mathbf{v}_j$ . Now if  $\mathbf{v}_i \cdot \mathbf{v}_i = 0$ , then  $\mathbf{v}_i \cdot \mathbf{v}_j = 1$  so by Observation 2.2,  $D'$  contains the cycle  $\overrightarrow{v_{K+i} v_j v_i}$  if  $i < j$  or the cycle  $\overrightarrow{v_{K+i} v_{K+j} v_i}$  if  $i > j$ . Similarly if  $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ , then  $D'$  contains one of the cycles  $\overrightarrow{v_i v_j v_{K+i}}$  and  $\overrightarrow{v_i v_{K+j} v_{K+i}}$ . We have a contradiction in all cases, so the value of  $\mathbf{v}_i \cdot \mathbf{v}_j$  is constant as  $i, j \in \{0, \dots, K-1\}$  vary, as required.  $\square$

## 7. Bounds on $\text{inv}(n)$ .

**7.1. Lower bounds.** In this section we discuss the previous known lower bound on  $\text{inv}(n)$  and give the proof of Theorem 1.9. As noted in the introduction, Belkhechine, Bouaziz, Boudabbous, and Pouzet [7] used a counting argument to lower bound  $\text{inv}(n)$ . They observed that since there are  $n!$  labelled transitive tournaments on  $n$  vertices, there are at most  $n! \cdot 2^{n(k-1)}$  labelled  $(k-1)$ -invertible tournaments on  $n$  vertices. There are a total of  $2^{n(n-1)/2}$  labelled  $n$ -vertex tournaments, so for any  $k$  such that  $2^{n(n-1)/2} > n! \cdot 2^{n(k-1)}$  we have  $\text{inv}(n) \geq k$ . Taking logarithms base 2 and rearranging, this condition becomes  $k < (n-1)/2 - \log(n!)/n$ , so we have

$$\text{inv}(n) \geq \left\lfloor \frac{n-1}{2} - \frac{\log(n!)}{n} \right\rfloor \geq \left\lfloor \frac{n-1}{2} - \log(n) \right\rfloor,$$

where for the final inequality we used  $n! \leq n^n$ . Lower bounds on  $\text{inv}(n)$  of this form were the best known (disregarding very slight tightenings of the argument).

616 The proof of Theorem 1.9 uses the following lemma which gives a bound on the  
 617 probability that a random symmetric binary matrix has at most a certain rank. In  
 618 fact, these probabilities are known exactly [23], but we will use a simpler bound which  
 619 is essentially tight for our purposes and for which we include a short proof.

620 **LEMMA 7.1.** *The probability that a uniformly random  $n \times n$  symmetric matrix  
 621 over  $\mathbb{F}_2$  has rank at most  $n - s$  (over  $\mathbb{F}_2$ ) is at most  $2^{s \log(n) - \binom{s}{2}}$ .*

622 *Proof.* Construct the random matrix in  $n$  steps, in the  $i$ th step choosing the first  
 623  $i$  entries of the  $i$ th row of the matrix (and also, by symmetry, the  $i$ th column). For  
 624 each  $i \in [n]$ , let  $M_i$  be the random symmetric  $i \times i$  matrix obtained after step  $i$ .

625 Note that for each  $i$  the nullity increases by at most 1 between  $M_i$  and  $M_{i+1}$ .  
 626 It follows that if the nullity of  $M_n$  is at least  $s$ , then for all  $1 \leq j \leq s-1$  we  
 627 can define  $k_j$  to be the smallest  $i$  such that the nullity of  $M_i$  is  $j+1$ , and we have  
 628  $2 \leq k_1 < k_2 < \dots < k_{s-1} \leq n$ . For each  $j$ , the ranks of  $M_{k_j-1}$  and  $M_{k_j}$  are equal, so  
 629 the first  $k_j - 1$  entries of the  $k_j$ th row of  $M_{k_j}$  lie in the  $(k_j - 1 - j)$ -dimensional row  
 630 space of  $M_{k_j-1}$ , which happens with probability  $2^{-j}$ . There are  $\binom{n}{s-1}$  ways to choose  
 631  $k_1, \dots, k_{s-1}$  as above, so the probability that  $M_n$  has rank at most  $n - s$  is at most

$$632 \quad \binom{n}{s-1} \prod_{j=1}^{s-1} 2^{-j} \leq 2^{s \log(n) - \binom{s}{2}},$$

633 as required. □

634 We are now ready to prove Theorem 1.9.

635 *Proof of Theorem 1.9.* Let  $T$  be a uniformly random tournament on vertex set  
 636  $[n]$  and let  $M_T = (m_{ab})$  be the  $n \times n$  matrix over  $\mathbb{F}_2$  defined as follows. For  $a < b$ ,  
 637 let  $m_{ab}$  be 0 if  $\vec{ab}$  is an edge of  $T$  and 1 otherwise, then define  $m_{ba} = m_{ab}$ , and finally  
 638 choose each diagonal entry uniformly at random. Note that the  $\binom{n}{2}$  entries of  $M_T$   
 639 above the diagonal determine  $T$ , and the other entries are defined such that  $M_T$  is a  
 640 uniformly random symmetric binary matrix.

641 Let  $s = \lfloor \sqrt{2n \log(n)} \rfloor$  and write  $k = n - s$ . Suppose that  $\text{inv}(T) \leq k$  and let  
 642  $X_1, \dots, X_k$  be a decycling family of  $T$ . For each  $X_i$ , let  $M_i$  be the  $n \times n$  binary  
 643 matrix whose  $(a, b)$  entry is 1 if and only if  $a, b \in X_i$ . Observe that, working over  
 644  $\mathbb{F}_2$ , we have  $\text{rank}(M_i) \leq 1$  for all  $i$ , and thus  $\text{rank}(\sum_i M_i) \leq k$ . By construction,  
 645  $M_T + \sum_i M_i$  is a matrix whose entries above the diagonal correspond to a transitive  
 646 tournament on  $[n]$  (its diagonal entries can be anything). Let  $\mathcal{M}$  be the set of binary  
 647 matrices corresponding in this manner to a transitive tournament on  $[n]$ , and note  
 648 that  $|\mathcal{M}| = n!2^n$ .

649 Putting all of this together, we have that if  $\text{inv}(T) \leq k$ , then there exists  $M \in \mathcal{M}$   
 650 such that  $\text{rank}(M_T + M) \leq k$ . For each fixed  $M$ , we have that  $M_T + M$  is a uniformly  
 651 random symmetric binary matrix and hence has rank at most  $k$  with probability at  
 652 most  $2^{s \log(n) - \binom{s}{2}}$  by Lemma 7.1. Taking a union bound over all  $M \in \mathcal{M}$  we obtain

$$653 \quad \mathbb{P}(\text{inv}(T) \leq k) \leq n!2^n 2^{s \log(n) - \binom{s}{2}}.$$

654 Since  $n! = O(\sqrt{n}(n/e)^n)$ , the right-hand side is  $O(2^{f(n)})$  where

$$655 \quad f(n) = \frac{\log(n)}{2} + n \log(n) - n \log(e) + n + s \log(n) - \binom{s}{2} \\ 656 \quad = -n(\log(e) - 1) + o(n),$$

658 and thus  $\mathbb{P}(\text{inv}(T) \leq k) \rightarrow 0$  as  $n \rightarrow \infty$  as desired. □

659     **7.2. Upper bounds.** The only approach which has been used to prove upper  
660     bounds on  $\text{inv}(n)$ , introduced in [7], is to ‘solve’ one vertex at a time, as follows.  
661     Given a tournament  $T$ , pick a vertex  $v$  and invert the set consisting of  $v$  and its  
662     out-neighbourhood. In the resulting tournament  $T_1$ ,  $v$  is a sink. Using a further  
663      $\text{inv}(n-1)$  inversions we can transform  $T_1 - \{v\}$  into a transitive tournament, so  
664      $\text{inv}(n) \leq \text{inv}(n-1) + 1$  for all  $n \geq 2$ . The authors of [7] observed that  $\text{inv}(4) = 1$ ,  
665     so  $\text{inv}(n) \leq n-3$  for  $n \geq 4$ . For  $n \geq 6$  this was improved by 1 in [4] using the fact  
666     that  $\text{inv}(6) = 2$  (which they attribute to [6] and which we have verified by a computer  
667     check). We introduce a slightly different approach to prove the following.

668     PROPOSITION 7.2. *For all  $n \in \mathbb{N}$ ,*

$$669 \quad \text{inv}(n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor + \text{inv} \left( \left\lceil \frac{n-1}{2} \right\rceil \right).$$

670     *Proof.* Let  $n \in \mathbb{N}$  and let  $T$  be an  $n$ -vertex tournament. Pick  $v \in V(T)$  and  
671     write  $A$  and  $B$  for the in- and out-neighbourhoods of  $v$  respectively. We may assume  
672     that  $|A| \geq \lceil (n-1)/2 \rceil$  (the case where  $B$  is the larger of the two is similar). By  
673     ‘solving’ each vertex in  $B$  one after another, we can find at most  $|B|$  inversions which  
674     transform  $T$  into a tournament  $T'$  such that the subtournament of  $T'$  induced on  
675      $B \cup \{v\}$  is transitive (with  $v$  as the minimal element) and every edge of  $T'$  between  
676      $A$  and  $B \cup \{v\}$  is oriented away from  $A$ . With a further  $\text{inv}(A) \leq \text{inv}(|A|)$  inversions  
677     we can transform  $T'$  into a transitive tournament. Thus,  $\text{inv}(T) \leq |B| + \text{inv}(|A|)$ .

678     We have  $\text{inv}(k) \leq \text{inv}(k-1)+1$  for all  $k \in \mathbb{N}$  and we can apply this  $|A| - \lceil (n-1)/2 \rceil$   
679     times to obtain  $\text{inv}(|A|) \leq \text{inv}(\lceil (n-1)/2 \rceil) + |A| - \lceil (n-1)/2 \rceil$ . Using the fact that  
680      $|A| + |B| = n-1$ , this yields

$$681 \quad \text{inv}(T) \leq |B| + \text{inv} \left( \left\lceil \frac{n-1}{2} \right\rceil \right) + |A| - \left\lceil \frac{n-1}{2} \right\rceil = \left\lfloor \frac{n-1}{2} \right\rfloor + \text{inv} \left( \left\lceil \frac{n-1}{2} \right\rceil \right),$$

682     and the claim follows.  $\square$

683     We can use this result to improve (for large  $n$ ) the upper bound on  $\text{inv}(n)$ .

684     COROLLARY 7.3. *For all  $n \in \mathbb{N}_0$ ,  $\text{inv}(n) \leq n - \log(n+1)$ .*

685     *Proof.* We prove the statement by induction on  $n$ , with the case  $n = 0$  clear. If  
686      $n \geq 1$  and the claim holds for all smaller values, then we have

$$687 \quad \begin{aligned} \text{inv}(n) &\leq \left\lfloor \frac{n-1}{2} \right\rfloor + \text{inv} \left( \left\lceil \frac{n-1}{2} \right\rceil \right), \\ &\leq \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lceil \frac{n-1}{2} \right\rceil - \log \left( \left\lceil \frac{n-1}{2} \right\rceil + 1 \right), \\ &\leq n-1 - \log \left( \frac{n+1}{2} \right), \\ &= n - \log(n+1). \end{aligned} \quad \square$$

688     **8. Conclusion.** In this paper we have answered several of the questions posed  
689     in [4]. We have shown that their ‘dijoin conjecture’, that  $\text{inv}(L \rightarrow R) = \text{inv}(L) +$   
690      $\text{inv}(R)$ , is false in general, but have verified it in the case where  $\text{inv}(L) = \text{inv}(R) = 2$   
691     and have also shown that a  $k$ -join analogue holds under certain conditions. In addition,  
692     we have confirmed their related conjectures that  $k$ -INVERSION is NP-complete

693 for all  $k \geq 1$ , and that the inequality  $\text{inv}(D) \leq 2\tau(D)$  is tight. We have answered  
 694 their question concerning the minimal  $r_k$  such that  $k$ -TOURNAMENT-INVERSION can  
 695 be solved in time  $O(|V(T)|^{r_k})$ , showing that  $r_k = 2$  for all  $k$ , and have improved the  
 696 lower bound on  $\text{inv}(n)$  to show that  $\text{inv}(n) = (1 + o(1))n$ . There are, however, still  
 697 many interesting open problems in this area. Before discussing some of them, we will  
 698 touch on two operations similar to inversion.

699 **8.1. Similar operations.** We first consider an operation on permutations which  
 700 is used by molecular biologists as a model for genetic mutations, and could loosely  
 701 be seen as a vertex analogue of inversions in tournaments. Given a permutation  
 702  $\pi = (\pi_1 \ \pi_2 \ \dots \ \pi_n)$  of  $[n]$ , for  $1 \leq i < j \leq n$ , the *reversal of the interval*  $[i, j]$  is the  
 703 permutation obtained by reversing the order of  $\pi_i, \dots, \pi_j$  in  $\pi$ . The *reversal distance*,  
 704  $d(\pi)$ , of a permutation  $\pi$  is the minimum number of reversals required to transform  $\pi$   
 705 into the identity permutation. For a survey of reversals and the reversal distance (and  
 706 many other combinatorial models of genome rearrangements) see [15]. We highlight  
 707 some results of particular relevance to our work. With regards to computational  
 708 complexity, Caprara [9] showed that the problem of SORTING BY REVERSALS, that  
 709 is, determining whether  $d(\pi) \leq k$  for inputs of a permutation  $\pi$  and  $k \in \mathbb{N}$ , is NP-  
 710 complete, while Hannenhalli and Pevzner [18, 19] showed that it is fixed-parameter  
 711 tractable when parameterised by  $k$ . The natural extremal problem was solved by  
 712 Bafna and Pevzner [3] who proved that for a permutation  $\pi$  of  $[n]$ , we have  $d(\pi) \leq n-1$   
 713 with equality if and only if  $\pi \in \{\gamma_n, \gamma_n^{-1}\}$  for an explicit  $\gamma_n$ .

714 Inversions in digraphs can also be thought of as generalisations of *edge reversals*,  
 715 i.e. the operations which reverse the orientation of a single edge. It is not difficult to see  
 716 (using an argument from subsection 1.3) that the minimum number of such operations  
 717 required to transform a digraph  $D$  into an acyclic digraph is equal to  $\tau'(D)$ , the cycle  
 718 edge-transversal number of  $D$ . Determining this quantity is the famous feedback arc  
 719 set problem, which has been widely studied (see [21] for an overview). In particular  
 720 the problem of determining for inputs  $D$  and  $k$  whether  $\tau'(D) \leq k$  was one of the  
 721 first shown to be NP-complete [20] and it remains NP-complete when the input is  
 722 restricted to tournaments [1, 10]. However, Chen, Liu, Lu, O'Sullivan, and Razgon  
 723 [11] showed that this problem is again fixed-parameter tractable when parameterised  
 724 by  $k$ . On the extremal side, it was shown by Spencer [27, 28] that the maximum  
 725 cycle edge-transversal number of an  $n$ -vertex tournament is  $\frac{1}{2} \binom{n}{2} - \Theta(n^{3/2})$  and that  
 726 a random labelled  $n$ -vertex tournament has this cycle edge-transversal number with  
 727 probability tending to 1. Bounds of this form remain the best known (see also [13, 25]).

728 **8.2. Open problems.** We have shown (in Theorem 1.7) that the problem which  
 729 takes as inputs a tournament  $T$  and an integer  $k \in \mathbb{N}$ , and asks whether  $\text{inv}(T) \leq k$ , is  
 730 fixed-parameter tractable when parameterised by  $k$ . In keeping with the pattern ex-  
 731 hibited in the settings discussed in subsection 8.1, Bang-Jensen, da Silva, and Havet [4]  
 732 conjectured that the full problem is NP-complete.

733 **CONJECTURE 8.1** ([4]). *The problem of deciding whether  $\text{inv}(T) \leq k$  for inputs  
 734 of  $k \in \mathbb{N}$  and a tournament  $T$  is NP-complete.*

735 Note that Theorem 1.7 does not make any progress towards disproving this be-  
 736 cause the implied constant in the  $O(n^2)$  running time is not polynomial in  $k$ . In fact,  
 737 as noted above, the constant arising from our algorithm is doubly exponential in  $k$ .  
 738 However, again in keeping with both settings discussed in subsection 8.1 (and indeed  
 739 many natural fixed-parameter tractable problems), we conjecture that this constant  
 740 can be taken to be singly exponential in  $k$ , perhaps with a higher power of  $n$ .

741 CONJECTURE 8.2. *There exist constants  $c_1, c_2 > 0$  such that, for any  $k \in \mathbb{N}$ ,  
742  $k$ -TOURNAMENT-INVERSION can be solved in time  $O(2^{k^{c_1}} |V(T)|^{c_2})$ .*

743 As discussed in the introduction, the set  $\mathcal{IC}_k$  of  $k$ -inversion-critical tournaments  
744 was shown to be finite for all  $k$  in [7]. They explicitly described  $\mathcal{IC}_1$  and  $\mathcal{IC}_2$ , for the  
745 latter using results of Gallai [16] (see [24] for an English translation) and Latka [22],  
746 but for  $k \geq 3$  very little is known about these sets. In particular, it would be interesting  
747 to determine  $m_k$ , the maximum number of vertices in a tournament in  $\mathcal{IC}_k$ , for  
748  $k \geq 3$ .

749 QUESTION 8.3 ([4]). *What is the value of  $m_k$  for  $k \geq 3$ ?*

750 Finding the minimum possible size of a  $k$ -inversion-critical tournament is equivalent  
751 to the problem of determining  $\text{inv}(n)$ . The best known bounds on  $\text{inv}(n)$  for  
752 large  $n$  are now

$$753 n - \sqrt{2n \log(n)} \leq \text{inv}(n) \leq n - \log(n + 1),$$

754 and it would be interesting to tighten these further.

755 QUESTION 8.4. *What is the asymptotic behaviour of  $n - \text{inv}(n)$ ?*

756 In light of our improved lower bound on  $\text{inv}(n)$ , the lack of an explicit construction  
757 for a tournament of large inversion number is even more apparent: no  $n$ -vertex  
758 construction with inversion number more than about  $n/3$  (as given by the  $(n/3)$ -join  
759  $[\vec{C}_3]_{n/3}$ ) is known.

760 PROBLEM 8.5. *Construct  $n$ -vertex tournaments with inversion number closer to  
761  $\text{inv}(n)$ .*

762 Belkhechine, Bouaziz, Boudabbous, and Pouzet ([6]; see [4]) defined for each  
763  $n \in \mathbb{N}$  a tournament  $Q_n$  on vertex set  $[n]$  in which for  $i < j$  the edge  $ij$  is oriented  
764 towards  $j$ , except if  $j = i + 1$ , in which case it is oriented towards  $i$ , and conjectured  
765 that these graphs satisfy  $\text{inv}(Q_n) = \lfloor \frac{n-1}{2} \rfloor$ .

766 CONJECTURE 8.6 ([6]). *For all  $n \in \mathbb{N}$  we have  $\text{inv}(Q_n) = \lfloor \frac{n-1}{2} \rfloor$ .*

767 The conjecture is known to hold for  $n \leq 8$  [7, 4], and it is certainly true that  
768  $\text{inv}(Q_n) \leq \lfloor \frac{n-1}{2} \rfloor$  for all  $n$  since the sets

$$769 \{2, 3\}, \{4, 5\}, \{6, 7\}, \dots, \{2 \lfloor (n-1)/2 \rfloor, 2 \lfloor (n-1)/2 \rfloor + 1\}$$

770 form a decycling family of  $Q_n$ .

771 Defining the *inversion distance*,  $\text{inv}(T, T')$ , between two labelled tournaments  $T$   
772 and  $T'$  on the same vertex set to be the minimum number of inversions required  
773 to transform  $T$  into  $T'$ , we remark that the matrix rank techniques developed in  
774 subsection 7.1 can be used to show that the maximum inversion distance between  
775 two  $n$ -vertex tournaments is exactly  $n - 1$ . Moreover, combining these ideas with  
776 Lemma 7.1 gives an upper bound of  $2^{\binom{n}{2} + n - \binom{s}{2} + s \log(n)}$  on the number of labelled  
777 tournaments within inversion distance  $n - s$  of a given labelled tournament.

778 It is natural in this context to study the random walk  $\mathcal{W}$  on the space of labelled  
779 tournaments on  $[n]$  where each step in the walk consists of picking a uniform random  
780 subset of  $[n]$  and inverting that set in the current tournament. In particular, we ask  
781 the following.

782 QUESTION 8.7. *What is the mixing time of  $\mathcal{W}$ ? Does it satisfy the cutoff phe-  
783 nomenon?*

784 Returning to the dijoin conjecture, Theorem 1.3 completes the work of Bang-  
 785 Jensen, da Silva, and Havet in showing that the conjecture holds in the cases where  
 786  $\text{inv}(L), \text{inv}(R) \leq 2$ . We have also shown (Theorem 1.4) a  $k$ -join analogue of the dijoin  
 787 conjecture for collections of 2-invertible digraphs  $D_1, \dots, D_k$  at most one of which has  
 788 inversion number 2. We conjecture that this final condition can be removed.

789 **CONJECTURE 8.8.** *Let  $k \in \mathbb{N}$  and let  $D_1, \dots, D_k$  be oriented graphs satisfying  
 790  $\text{inv}(D_i) \leq 2$  for all  $i$ . Then*

$$791 \quad \text{inv}([D_1, \dots, D_k]) = \sum_{i=1}^k \text{inv}(D_i).$$

792 On the other hand, Theorem 1.2 gives a counterexample to the dijoin conjecture  
 793 where  $\text{inv}(L) = 1$  and  $\text{inv}(R) = 3$ . From this, we can obtain counterexamples with  
 794  $\text{inv}(L) = k$  and  $\text{inv}(R) = 3$  for any  $k \in \mathbb{N}$ : let  $L = [\vec{C}_3]_k$  and let  $R$  be as in the proof  
 795 of Theorem 1.2. The tournaments obtained from these by inverting the whole vertex  
 796 set give counterexamples in which  $\text{inv}(L) = 3$  and  $\text{inv}(R) = k$ . We conjecture that  
 797 here 3 can be replaced with any larger integer, or in other words that the only values  
 798 of  $\text{inv}(L)$  and  $\text{inv}(R)$  for which the dijoin conjecture always holds are those where  
 799  $\text{inv}(L), \text{inv}(R) \leq 2$  or where one of  $\text{inv}(L)$  or  $\text{inv}(R)$  is 0.

800 **CONJECTURE 8.9.** *For all  $\ell, r \in \mathbb{N}$  with  $\ell \geq 3$  or  $r \geq 3$  there exist oriented graphs  
 801  $L$  and  $R$  with  $\text{inv}(L) = \ell$  and  $\text{inv}(R) = r$ , but  $\text{inv}(L \rightarrow R) < \ell + r$ .*

802 This conjecture is equivalent to the claim that for all  $r \geq 3$  there exists a tour-  
 803 nament  $R$  with  $\text{inv}(R) = \text{inv}(\vec{C}_3 \rightarrow R) = r$ . To see that this follows from the  
 804 conjecture, note that for  $r \geq 3$ , if  $\text{inv}(L) = 1$  and  $\text{inv}(R) = \text{inv}(L \rightarrow R) = r$ ,  
 805 then we can extend  $L \rightarrow R$  to a tournament  $T = L' \rightarrow R'$  with inversion num-  
 806 ber  $r$ . Clearly  $\text{inv}(R') = r$  and  $\text{inv}(L') \geq 1$ , so  $L'$  contains a copy of  $\vec{C}_3$ . Thus,  
 807  $r = \text{inv}(R') \leq \text{inv}(\vec{C}_3 \rightarrow R') \leq \text{inv}(T) = r$ , as required. The converse follows from  
 808 the arguments of the previous paragraph.

809 Finally, we noted in subsection 1.2 that  $\text{inv}(D) \leq \text{inv}(D - \{v\}) + 2$  for all di-  
 810 graphs  $D$  and vertices  $v \in V(D)$ . It is certainly the case that this inequality is tight  
 811 for some  $D$  and  $v$ . Indeed, a reformulation of Theorem 1.8 yields the stronger state-  
 812 ment that for all  $k \in \mathbb{N}$  there exists a digraph  $D$  and a set  $S \subseteq V(D)$  with  $|S| = k$   
 813 such that for all  $T \subseteq S$  we have  $\text{inv}(D - T) = \text{inv}(D) - 2|T|$ . We conjecture, however,  
 814 that the inequality  $\text{inv}(D) \leq \text{inv}(D - \{v\}) + 2$  cannot be tight for all vertices  $v$  in a  
 815 given digraph  $D$ .

816 **CONJECTURE 8.10.** *Let  $D$  be a digraph with at least one vertex. Then there exists  
 817  $v \in V(D)$  such that  $\text{inv}(D - \{v\}) \geq \text{inv}(D) - 1$ .*

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821

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